Math 138 Reference

Definition 1 ***Riemann Sum*** *Given a bounded function f on $[a,b]$, a partition P*

$$a = t_0 < t_1 < t_2 < ... < t_{i-1} < t_i < ... < t_{n-1} < t_n = b$$

of $[a,b]$, and a set $\{c_1, c_2, ..., c_n\}$ where $c_i \in [t_{i-1}, t_i]$, then a Riemann sum for f with respect to P is a sum of the form

$$S = \sum_{i=1}^{n} f(c_i) \triangle t_i$$

Definition 2 ***Regular n-Partition*** *Given an interval $[a,b]$ and an $n \in \mathbb{N}$, the regular n-partition of $[a,b]$ is the partition $P^{(n)}$ with*

$$a = t_0 < t_1 < t_2 < ... < t_{i-1} < t_i < ... < t_{n-1} < t_n = b$$

of $[a,b]$ where each subinterval has the same length $\triangle t_i = \dfrac{b-a}{n}$

Definition 3 ***Right-hand Riemann Sum*** *The right-hand Riemann sum for f with respect to the partition P is the Riemann Sum R obtained from P by choosing c_i to be t_i, the right-hand endpoint of $[t_{i-1}, t_i]$. That is*

$$R = \sum_{i=1}^{n} f(t_i) \triangle t_i$$

If $P^{(n)}$ is the regular n-partition, we denote the right-hand Riemann sum by

$$R_n = \sum_{i=1}^{n} f(t_i) \triangle t_i = \sum_{i=1}^{n} f(t_i) \frac{b-a}{n} = \sum_{i=1}^{n} f(a + i(\frac{b-a}{n}))(\frac{b-a}{n})$$

Definition 4 ***Left-hand Riemann Sum*** *The left-hand Riemann sum for f with respect to the partition P is the Riemann Sum L obtained from P by choosing c_i to be t_{i-1}, the left-hand endpoint of $[t_{i-1}, t_i]$. That is*

$$L = \sum_{i=1}^{n} f(t_{i-1}) \triangle t_i$$

If $P^{(n)}$ is the regular n-partition, we denote the left-hand Riemann sum by

$$L_n = \sum_{i=1}^{n} f(t_{i-1}) \triangle t_i = \sum_{i=1}^{n} f(t_{i-1}) \frac{b-a}{n} = \sum_{i=1}^{n} f(a + (i-1)(\frac{b-a}{n}))(\frac{b-a}{n})$$

Definition 5 ***Definite Integral*** *We say that a bounded function f is integrable on $[a, b]$ if there exists a unique number $I \in \mathbb{R}$ such that if whenever $\{P_n\}$ is a sequence of partitions with $\lim_{n\to\infty} ||P_n|| = 0$ and $\{S_n\}$ is any sequence of Riemann sums associated with the P_n's, we have*

$$\lim_{n\to\infty} S_n = I$$

In this case, we call I the integral of f over $[a, b]$ and denote it by

$$\int_a^b f(t)dt$$

The points a and b are called the limits of integration and the function $f(t)$ is called the integrand. The variable t is called the variable of integration.

Theorem 1 ***Integrability Theorem for Continuous Functions*** *Let f be continuous on $[a, b]$. Then f is integrable on $[a, b]$. Moreover,*

$$\int_a^b f(t)dt = \lim_{n\to\infty} S_n$$

where

$$S_n = \sum_{i=1}^{n} f(c_i)\triangle t_i$$

is any Riemann sum associated with the regular n-partitions. In particular,

$$\int_a^b f(t)dt = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\frac{b-a}{n}$$

$$\int_a^b f(t)dt = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\frac{b-a}{n}$$

Theorem 2 ***Properties of Integrals*** *Assume that f and g are integrable on the interval $[a, b]$. Then:*

1. *For any $c \in \mathbb{R}$, $\int_a^b cf(t)dt = c\int_a^b f(t)dt$*
2. *$\int_a^b (f + g)(t)dt = \int_a^b f(t)dt + \int_a^b g(t)dt$*
3. *If $m \leq f(t) \leq M$ for all $t \in [a, b]$, then $m(b - a) \leq \int_a^b f(t)dt \leq M(b - a)$*
4. *If $0 \leq f(t)$ for all $t \in [a, b]$, then $0 \leq \int_a^b f(t)dt$*
5. *If $g(t) \leq f(t)$ for all $t \in [a, b]$, then $\int_a^b g(t)dt \leq \int_a^b f(t)dt$*
6. *The function $|f|$ is integrable on $[a, b]$ and $|\int_a^b f(t)dt| \leq \int_a^b |f(t)|dt$*

Definition 6 ***Identical Limits of Integration*** *Let $f(t)$ be defined at $t = a$. Then we define*

$$\int_a^a f(t)dt = 0$$

Definition 7 ***Switching the Limits of Integration*** *Let f be integrable on the interval $[a, b]$ where $a < b$. Then we define*

$$\int_b^a f(t)dt = -\int_a^b f(t)dt$$

Theorem 3 ***Integrals over Subintervals*** *Assume that f is integrable on an interval I containing a, b, and c. Then*

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$$

Definition 8 ***Average Value of*** f *If f is continuous on $[a, b]$, the average value of f on $[a, b]$ is defined as*

$$\frac{1}{b-a}\int_a^b f(t)dt$$

Theorem 4 ***Average Value Theorem (MVT for Integrals)*** *Assume that f is continuous on $[a, b]$. Then there exists $a \leq c \leq b$ such that*

$$f(c) = \frac{1}{b-a}\int_a^b f(t)dt$$

Theorem 5 ***Fundamental Theorem of Calculus (Part 1)*** *Assume that f is continuous on an open interval I containing point a. Let*

$$G(x) = \int_a^x f(t)dt$$

Then $G(x)$ is differentiable at each $x \in I$ and $G'(x) = f(x)$. Equivalently,

$$G'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Theorem 6 ***Extended Fundamental Theorem of Calculus*** *Assume that f is continuous and that g and h are differentiable. Let*

$$H(x) = \int_{g(x)}^{h(x)} f(t)dt$$

Then $H(x)$ is differentiable and

$$H'(x) = f(h(x))h'(x) - f(g(x))g'(x)$$

Definition 9 ***Antiderivative*** *Given a function f, an antiderivative is a function F such that $F'(x) = f(x)$. If for an interval I, $\forall x \in I, F'(x) = f(x)$, then we say F is an antiderivative for f on I.*

Theorem 7 ***Power Rule for Antiderivatives*** *If $\alpha \neq 1$, then*

$$\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C$$

Theorem 8 ***Fundamental Theorem of Calculus (Part 2)*** *Assume that f is continuous and that F is any antiderivative of f. Then*

$$\int_a^b f(t)dt = F(b) - F(a)$$

Theorem 9 ***Change of Variables*** *Assume that $g'(x)$ is continuous on $[a,b]$ and $f(u)$ is continuous on $g([a,b])$. Then*

$$\int_{x=a}^{x=b} f(g(x))g'(x)dx = \int_{u=g(a)}^{u=g(b)} f(u)du$$

Definition 10 ***Integration by Parts***

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Assume that f and g are such that both f' and g' are continuous on an interval containing a and b. Then

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx$$

Definition 11 ***Type 1 Partial Fraction Decomposition*** *Assume that*

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials such that $Deg(p(x)) < Deg(q(x)) = k$ and $q(x)$ can be factored into the product of linear terms each with distinct roots. That is

$$q(x) = a(x - a_1)(x - a_2)...(x - a_k)$$

where the a_i's are unique and none of the a_i's are roots of $p(x)$. Then there exists constants $A_1, A_2, ..., A_k$ such that

$$f(x) = \frac{1}{a}\left[\frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + ... + \frac{A_k}{x - a_k}\right]$$

We say that f admits Type 1 Partial Fraction Decomposition

Theorem 10 ***Integration of Type 1 Partial Fractions*** *Assume that* $f(x) = \frac{p(x)}{q(x)}$ *admits a Type 1 Partial Fraction Decomposition of the form*

$$f(x) = \frac{1}{a}\left[\frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + ... + \frac{A_k}{x-a_k}\right]$$

Then

$$\int f(x)dx = \frac{1}{a}\left[\int \frac{A_1}{x-a_1}dx + \int \frac{A_2}{x-a_2}dx + ... + \int \frac{A_k}{x-a_k}dx\right]$$

Definition 12 ***Type 2 Partial Fraction Decomposition*** *Assume that*

$$f(x) = \frac{p(x)}{q(x)}$$

where p *and* q *are polynomials such that* $Deg(p(x)) < Deg(q(x)) = k$ *and* $q(x)$ *can be factored into the product of linear terms with non-distinct roots. That is*

$$q(x) = a(x-a_1)^{m_1}(x-a_2)^{m_2}...(x-a_l)^{m_l}$$

where at least one of the m_j*'s is greater than 1. Each expression* $(x-a_j)^{m_j}$ *in the factorization of* $q(x)$ *will contribute* m_j *terms to the decomposition, one for each power of* $(x-a_j)$ *from 1 to* m_j*, which when combined will be of the form*

$$\frac{p(x)}{q(x)} = \sum_{j=1}^{l} \frac{A_{j,1}}{x-a_j} + \frac{A_{j,2}}{(x-a_j)^2} + ... + \frac{A_{j,m_j}}{(x-a_j)^{m_j}}$$

The number m_j *is called the multiplicity of the root* a_j*. We say that* f *admits Type 2 Partial Fraction Decomposition*

Definition 13 ***Type 3 Partial Fraction Decomposition*** *Let* $f(x) = \frac{p(x)}{q(x)}$ *be a rational function with* $Deg(p(x)) < Deg(q(x))$*, but* $q(x)$ *does not factor into linear terms. We say that* f *admits a Type 3 Partial Fraction Decomposition. Suppose that* $q(x)$ *has an irreducible factor* x^2+bx+c *with multiplicity* m*. Then this factor will contribute terms of the form*

$$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + ... + \frac{B_mx+C_m}{(x^2+bx+c)^m}$$

Definition 14 ***Type 1 Improper Integral***

1. *Let f be integrable on $[a, b]$ for each $a \leq b$. We say the Type 1 Improper Interval*
$$\int_a^\infty f(x)dx \text{ converges if } \lim_{b\to\infty} \int_a^b f(x)dx \text{ exists.}$$
In this case, we write
$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$
Otherwise, we say $\int_a^\infty f(x)dx$ diverges.

2. *Let f be integrable on $[b, a]$ for each $b \leq a$. We say that the Type I Improper Integral*
$$\int_{-\infty}^a f(x)dx \text{ converges if } \lim_{b\to-\infty} \int_b^a f(x)dx \text{ exists.}$$
In this case, we write
$$\int_{-\infty}^a f(x)dx = \lim_{b\to-\infty} \int_b^a f(x)dx \text{ exists.}$$
Otherwise, we say $\int_{-\infty}^a f(x)dx$ diverges.

3. *Assume that f is integrable on $[a, b]$ for each $a, b \in \mathbb{R}$ with $a < b$. We say that the Type I Improper Integral*
$$\int_{-\infty}^\infty f(x)dx \text{ converges if both } \lim_{c\to\infty} \int_{-\infty}^c f(x)dx \text{ and } \lim_{c\to\infty} \int_c^\infty f(x)dx$$
exist for some $c \in \mathbb{R}$. In this case, we write
$$\int_{-\infty}^\infty f(x)dx = \lim_{c\to\infty} \int_{-\infty}^c f(x)dx + \lim_{c\to\infty} \int_c^\infty f(x)dx$$
Otherwise, we say $\int_{-\infty}^\infty f(x)dx$ diverges.

Theorem 11 ***p-Test for Type 1 Improper Integrals*** *The improper integral*
$$\int_1^\infty \frac{1}{x^p}dx$$
converges if and only if $p > 1$. If $p > 1$, then
$$\int_1^\infty \frac{1}{x^p}dx = \frac{1}{p-1}$$

Theorem 12 ***Properties of Type 1 Improper Integrals*** *Assume that* $\int_a^\infty f(x)dx$ *and* $\int_a^\infty g(x)dx$ *both converge.*

1. $\int_a^\infty cf(x)dx$ *converges for each* $c \in \mathbb{R}$ *and*
$$\int_a^\infty cf(x)dx = c\int_a^\infty f(x)dx$$
2. $\int_a^\infty (f(x)+g(x))dx$ *converges and*
$$\int_a^\infty (f(x)+g(x))dx = \int_a^\infty f(x)dx + \int_a^\infty g(x)dx$$
3. *If* $f(x) \leq g(x)$ *for all* $a \leq x$, *then*
$$\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$$
4. *If* $a < c < \infty$, *then* $\int_c^\infty f(x)dx$ *converges and*
$$\int_a^\infty f(x)dx = \int_a^c f(x)dx + \int_c^\infty f(x)dx$$

Theorem 13 ***The Monotone Convergence Theorem for Functions*** *Assume that* f *is non-decreasing on* $[a,\infty)$.

1. *If* $\{f(x)|x \in [a,\infty)\}$ *is bounded above, then* $\lim_{x\to\infty} f(x)$ *exists and*
$$\lim_{x\to\infty} f(x) = L = lub(\{f(x)|x \in [a,\infty)\})$$
2. *If* $\{f(x)|x \in [a,\infty)\}$ *is not bounded above, then*
$$\lim_{x\to\infty} f(x) = \infty$$

Theorem 14 ***Comparison Test for Type 1 Improper Integrals*** *Assume that* $0 \leq g(x) \leq f(x)$ *for all* $x \geq a$ *and that both* f *and* g *are continuous on* $[a,\infty)$.

1. *If* $\int_a^\infty f(x)dx$ *converges, then so does* $\int_a^\infty g(x)dx$
2. *If* $\int_a^\infty g(x)dx$ *diverges, then so does* $\int_a^\infty f(x)dx$

Definition 15 ***Absolute Convergence for Type 1 Improper Integrals*** *Let* f *be integrable on* $[a,b)$ *for all* $b \geq a$. *We say the improper integral* $\int_a^\infty f(x)dx$ *converges absolutely if*
$$\int_a^\infty |f(x)|dx$$
converges.

Theorem 15 ***Absolute Convergence Theorem for Improper Integrals*** *Let f be integrable on $[a,b]$ for all $b > a$. Then $|f|$ is also integrable on $[a,b]$ for all $b > a$. Moreover, if we assume that*

$$\int_a^\infty |f(x)|dx \text{ converges, then so does } \int_a^\infty f(x)dx$$

In particular, if $0 \leq |f(x)| \leq g(x)$ for all $x \geq a$, both f and g are integrable on $[a,b]$ for all $b \geq a$, and if $\int_a^\infty g(x)dx$ converges, then so does

$$\int_a^\infty f(x)dx$$

Definition 16 ***The Gamma Function*** *For each $x \in \mathbb{R}$, define*

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

Definition 17 ***Type 2 Improper Integral***

1. *Let f be integrable on $[t,b]$ for every $t \in (a,b]$ with either $\lim_{x\to a^+} f(x) = \infty$ or $\lim_{x\to a^+} f(x) = -\infty$. We say the Type 2 Improper Integral*

   $$\int_a^b f(x)dx \text{ converges if } \lim_{t\to a^+}\int_t^b f(x)dx \text{ exists.}$$

   In this case, we write

   $$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$$

   Otherwise, we say $\int_a^b f(x)dx$ diverges.

2. *Let f be integrable on $[a,t]$ for every $t \in [a,b)$ with either $\lim_{x\to b^-} f(x) = \infty$ or $\lim_{x\to b^-} f(x) = -\infty$. We say the Type 2 Improper Integral*

   $$\int_a^b f(x)dx \text{ converges if } \lim_{t\to b^-}\int_a^t f(x)dx \text{ exists.}$$

   In this case, we write

   $$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$$

   Otherwise, we say $\int_a^b f(x)dx$ diverges.

3. *If f has an infinite discontinuity at $x = c$ where $a < c < b$, then we say that the Type 2 Improper Integral*

$$\int_a^b f(x)dx \text{ converges if both } \int_a^c f(x)dx \text{ and } \int_c^b f(x)dx \text{ converge}$$

In this case, we write

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

If one or both of these integrals diverge, then we say that $\int_a^b f(x)dx$

Theorem 16 ***p-Test for Type 2 Improper Integrals*** *The improper integral*

$$\int_0^1 \frac{1}{x^p}dx$$

converges if and only if $p < 1$. If $p < 1$, then

$$\int_0^1 \frac{1}{x^p}dx = \frac{1}{1-p}$$

Definition 18 ***Differential Equation*** *A differential equation is an equation involving an independent variable such as x, a function $y = f(x)$ and various derivatives of y. In general, we will write*

$$F(x, y, y'', ..., y^{(n)}) = 0$$

A solution to the differential equation is a function φ such that

$$F(x, \varphi(x), \varphi'(x), ...\varphi^{(n)}(x)) = 0$$

The highest order of a derivative appearing in the equation is called the order of the differential equation.

Definition 19 ***Separable Differential Equation*** *A first-order differentiable equation is separable if there exists functions $f = f(x)$ and $g = g(y)$ such that the differentiable equation can be written in the form*

$$y' = f(x)g(y)$$

Definition 20 ***Constant (Equilibrium) Solution to a Separable Differential Equation*** *If*

$$y' = f(x)g(y)$$

is a separable differential equation and if $y_0 \in \mathbb{R}$ is such that $g(y_0) = 0$, then

$$\phi(x) = y_0$$

is called a constant or equilibrium solution to the differential equation.

Definition 21 ***First-Order Linear Differentiable Equation*** *A first-order differential equation is said to be linear if it can be written in the form*

$$y' = f(x)y + g(x)$$

Theorem 17 ***Solving First-Order Linear Differential Equations*** *Let f and g be continuous and let*

$$y' = f(x)y + g(x)$$

be a first-order linear differential equation. Then the solutions to this equation are of the form

$$y = \frac{\int g(x)I(x)dx}{I(x)}$$

where $I(x) = e^{-\int f(x)dx}$.

Theorem 18 ***Uniqueness and Existence Theorem for First-Order Linear Differential Equations*** *Assume that f and g are continuous functions on an interval I. Then for each $x_0 \in I$ and for all $y_0 \in \mathbb{R}$, the initial value problem*

$$y' = f(x)y + g(x)$$

$$y(x_0) = y_0$$

has exactly one solution $y = \varphi(x)$ on the interval I.

Definition 22 ***Series*** *Given a sequence $\{a_n\}$, the formal sum*

$$a_1 + a_2 + a_3 + a_4 + ... + a_n + ...$$

is called a series. The series is called formal because we have not yet given it a meaning numerically. The a_n's are called the terms of the series. For each term a_n, the index of the term is n. We will denote the series by

$$\sum_{n=1}^{\infty} a_n$$

Definition 23 ***Convergence of a Series*** *Given a series*

$$\sum_{n=1}^{\infty} a_n$$

for each $k \in \mathbb{N}$, we will define the k-th partial sum S_k by

$$S_k = \sum_{n=1}^{k} a_n$$

We say that the series $\sum_{n=1}^{\infty} a_n$ converges if the sequence $\{S_k\}$ of partial sums converges. In this case, if $L = \lim_{k\to\infty} S_k$, then we write

$$\sum_{n=1}^{\infty} a_n = L$$

and assign the sum this value. Otherwise, we say that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Definition 24 ***Geometric Series*** *A geometric series is a series of the form*

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + \ldots$$

The number r is called the ratio of the series.

Theorem 19 ***Geometric Series Test*** *The geometric series $\sum_{n=0}^{\infty} r^n$ converges if $|r| < 1$ and diverges otherwise. If $|r| < 1$, then*

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Theorem 20 ***Divergence Test*** *Assume that $\sum_{n=1}^{\infty} a_n$ converges. Then*

$$\lim_{n\to\infty} a_n = 0$$

Equivalently, if $\lim_{n\to\infty} a_n \neq 0$ or if $\lim_{n\to\infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 21 ***Arithmetic for Series I*** *Assume that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge.*

1. *The series $\sum_{n=1}^{\infty} ca_n$ converges for every $c \in \mathbb{R}$ and*

$$\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$$

2. *The series $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and*

$$\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

Theorem 22 ***Arithmetic for Series II***

1. *If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\sum_{n=j}^{\infty} a_n$ *also converges for each* j.

2. *If* $\sum_{n=j}^{\infty} a_n$ *converges for some* j, *then* $\sum_{n=1}^{\infty} a_n$ *converges.*

In either of these two cases,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + ... + a_{j-1} + \sum_{n=j}^{\infty} a_n$$

Definition 25 ***Monotonic Sequences*** *Given a sequence* $\{a_n\}$, *we say that the sequence is*

1. *non-decreasing if* $a_{n+1} \geq a_n$ *for every* $n \in \mathbb{N}$
2. *increasing if* $a_{n+1} > a_n$ *for every* $n \in \mathbb{N}$
3. *non-increasing if* $a_{n+1} \leq a_n$ *for every* $n \in \mathbb{N}$
4. *decreasing if* $a_{n+1} < a_n$ *for every* $n \in \mathbb{N}$

We say that $\{a_n\}$ *is monotonic if it satisfies one of these four conditions.*

Theorem 23 ***Monotone Convergence Theorem*** *Let* $\{a_n\}$ *be a non-decreasing sequence.*

1. *If* $\{a_n\}$ *is bounded above, then* $\{a_n\}$ *converges to* $L = lub(\{a_n\})$.
2. *If* $\{a_n\}$ *is not bounded above, then* $\{a_n\}$ *diverges to* ∞.

In particular, $\{a_n\}$ *converges if and only if it is bounded above.*

Definition 26 ***Positive Series*** *We call a series* $\displaystyle\sum_{n=1}^{\infty} a_n$ *positive if the terms* $a_n \geq 0$ *for all* $n \in \mathbb{N}$

Theorem 24 ***Comparison Test for Series*** *Assume that* $0 \leq a_n \leq b_n$ *for each* $n \in \mathbb{N}$.

1. *If* $\displaystyle\sum_{n=1}^{\infty} b_n$ *converges, then* $\displaystyle\sum_{n=1}^{\infty} a_n$ *converges.*
2. *If* $\displaystyle\sum_{n=1}^{\infty} a_n$ *diverges, then* $\displaystyle\sum_{n=1}^{\infty} b_n$ *diverges.*

Theorem 25 ***Limit Convergence Test*** *Assume that $a_n > 0$ and $b_n > 0$ for each $n \in \mathbb{N}$. Assume also that*

$$\lim_{n\to\infty} \frac{a_n}{n_n} = L$$

where either $L \in \mathbb{R}$ or $L = \infty$.

1. *If $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.*
2. *If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. Equivalently, if $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$.*
3. *If $L = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges. Equivalently, if $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$.*

Theorem 26 ***Integral Test for Convergence*** *Assume that $f(x)$ is decreasing and continuous with $f(x) > 0$ on $[1, \infty)$. Let $a_k = f(k)$. For each $n \in \mathbb{N}$, let $S_n = \sum_{k=1}^{n} a_k$. Then*

1. *For all $n \in \mathbb{N}$,*

$$\int_1^{n+1} f(x)dx \leq S_n \leq a_1 + \int_1^n f(x)dx$$

2. *$\sum_{k=1}^{\infty} a_k$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.*
3. *In the case $\sum_{k=1}^{\infty} a_k$ converges, then*

$$\int_1^{\infty} f(x)dx \leq \sum_{k=1}^{\infty} a_k \leq a_1 + \int_1^{\infty} f(x)dx$$

and

$$\int_{n+1}^{\infty} f(x)dx \leq S - S_n \leq \int_n^{\infty} f(x)dx$$

where $S = \sum_{k=1}^{\infty} a_k$. Note that by 2, $\int_n^{\infty} f(x)dx$ exists.

Theorem 27 ***p-Series Test*** *The series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if and only if* $p > 1$.

Definition 27 ***Alternating Series*** *A series of the form*

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n = a_1 - a_2 + a_3 - a_4 + ...$$

or of the form

$$\sum_{n=1}^{\infty}(-1)^{n}a_n = -a_1 + a_2 - a_3 + a_4 - ...$$

is said to be alternating provided that $a_n > 0$ *for all* n.

Theorem 28 ***Alternating Series Test*** *Assume that* $a_n > 0$ *for all* n, $a_{n+1} \leq a_n$ *for all* n, *and* $\lim_{n\to\infty} a_n = 0$. *Then the alternating series*

$$\sum_{n=1}^{\infty}(-1)^{n-1}a_n$$

converges. If $S_k = \sum_{n=1}^{k}(-1)^{n-1}a_n$, *then* S_k *approximates the sum* $S = \sum_{n=1}^{\infty}(-1)^{n-1}a_n$ *with an error that is a t most* a_{k+1}. *That is,*

$$|S_k - S| \leq a_{k+1}$$

Definition 28 ***Absolute vs Conditional Convergence*** *A series* $\sum_{n=1}^{\infty} a_n$ *is said to converge absolutely if* $\sum_{n=1}^{\infty} |a_n|$ *converges. A series* $\sum_{n=1}^{\infty} a_n$ *is said to converge conditionally if* $\sum_{n=1}^{\infty} |a_n|$ *diverges while* $\sum_{n=1}^{\infty} a_n$ *converges.*

Theorem 29 ***Absolute Convergence Theorem*** *If* $\sum_{n=1}^{\infty} |a_n|$ *converges, then so does* $\sum_{n=1}^{\infty} a_n$. *The sums of* $\sum_{n=1}^{\infty} |a_n|$ *and* $\sum_{n=1}^{\infty} a_n$ *will converge to different values unless* $a_n \geq 0$ *for all* n.

Definition 29 ***Rearrangement of a Series*** *Given a series* $\sum_{n=1}^{\infty} a_n$ *and a* $1-1$ *and onto function* $\phi : \mathbb{N} \to \mathbb{N}$, *if we let* $b_n = a_{\phi(n)}$, *then the series* $\sum_{n=1}^{\infty} b_n$ *is called a rearrangement of* $\sum_{n=1}^{\infty} a_n$.

Theorem 30 ***Rearrangement Theorem***

1. *Let* $\sum_{n=1}^{\infty} a_n$ *be an absolutely convergent series. If* $\sum_{n=1}^{\infty} b_n$ *is any rearrangement of* $\sum_{n=1}^{\infty} a_n$*, then* $\sum_{n=1}^{\infty} b_n$ *also converges and*

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$$

2. *Let* $\sum_{n=1}^{\infty} a_n$ *be a conditionally convergent series. Let* $\alpha \in \mathbb{R}$ *or* $\alpha = \pm\infty$*. Then there exists a rearrangement* $\sum_{n=1}^{\infty} b_n$ *of* $\sum_{n=1}^{\infty} a_n$ *such that*

$$\sum_{n=1}^{\infty} b_n = \alpha$$

Theorem 31 ***Ratio Test*** *Given a series* $\sum_{n=1}^{\infty} a_n$*, assume that*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

where $L \in \mathbb{R}$ *or* $L = \infty$*.*

1. *If* $0 \leq L < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*

2. *If* $L > 1$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*

3. *If* $L = 1$*, then no conclusion is possible.*

Theorem 32 ***Polynomial vs Factorial Growth*** *For any* $x \in \mathbb{R}$*,*

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

Theorem 33 ***Root Test*** *Given a series* $\sum_{n=1}^{\infty} a_n$*, assume that*

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

where $L \in \mathbb{R}$ *or* $L = \infty$.

1. *If* $0 \leq L < 1$*, then* $\sum_{n=1}^{\infty} a_n$ *converges absolutely.*
2. *If* $L > 1$*, then* $\sum_{n=1}^{\infty} a_n$ *diverges.*
3. *If* $L = 1$*, then no conclusion is possible.*

Definition 30 ***Power Series*** *A power series centered at* $x = a$ *is a series of the form*

$$\sum_{n=0}^{\infty} a_n (x-a)^n$$

where x *is considered a variable and the value* a_n *is called the coefficient of the term* $(x-a)^n$

Definition 31 ***Interval and Radius of Convergence*** *Given a power series of the form* $\sum_{n=0}^{\infty} a_n (x-a)^n$*, the set*

$$I = \{x_0 | \sum_{n=0}^{\infty} |a_n (x-a)^n \ \textit{converges}\}$$

is an interval centered at $x = a$ *which we call the interval of convergence for the power series. Let*

$$R := \begin{cases} lub(\{|x_0 - a| | x_0 \in I\}) & \textit{if } I \textit{ is bounded} \\ \\ \infty & \textit{if } I \textit{ is not bounded} \end{cases}$$

Then R *is called the radius of convergence of the power series.*

Theorem 34 ***Fundamental Convergence Theorem for Power Series*** *Given a power series* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *centered at* $x = a$*, let* R *be the radius of convergence.*

1. *If* $R = 0$*, then* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *converges for* $x = a$ *but it diverges for all other values of* x*.*

2. *If* $0 < R < \infty$*, then the series* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *converges absolutely for every* $x \in (a-R, a+R)$ *and diverges if* $|x-a| < R$*.*

3. *If* $R = \infty$*, then the series* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *converges absolutely for every* $x \in \mathbb{R}$*.*

In particular, $\sum_{n=0}^{\infty} a_n(x-a)^n$ *converges on an interval that is centered at* $x = a$ *which may or may not include one or both of the endpoints.*

Theorem 35 ***Test for the Radius of Convergence*** *Let* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *be a power series for which*

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

where $0 \le L < \infty$ *or* $L = \infty$*. Let* R *be the radius of convergence of the power series.*

1. *If* $0 < L < \infty$*, then* $R = \dfrac{1}{L}$

2. *If* $L = 0$*, then* $R = \infty$

3. *If* $L = \infty$*, then* $R = 0$*.*

Theorem 36 ***Equivalence of Radius of Convergence*** *Let* p *and* q *be non-zero polynomials where* $q(n) \neq 0$ *for* $n \ge k$*. Then the following series have the same radius of convergence:*

1. $\sum_{n=k}^{\infty} a_n(x-a)^n$

2. $\sum_{n=k}^{\infty} \dfrac{a_n p(n)(x-a)^n}{q(n)}$

However, they may have different intervals of convergence.

Definition 32 ***Functions Represented by a Power Series*** *Let* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *be a power series with radius of convergence* $R > 0$*. Let* I *be the interval of convergence for* $\sum_{n=0}^{\infty} a_n(x-a)^n$*. Let* f *be the function defined on the interval* I *by the formula*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for each $x \in I$*. We say that the function* $f(x)$ *is represented by the power series* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *on* I*.*

Theorem 37 ***Abel's Theorem - Continuity of Power Series*** *Assume that the power series* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *has interval of convergence* I*. Let*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for each $x \in I$*. Then* $f(x)$ *is continuous on* I*.*

Theorem 38 ***Addition of Power Series*** *Assume that* f *and* g *are represented by power series centered at* $x = a$ *with*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

and

$$g(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$$

respectively. Assume also that the radii of convergence of these series are R_f *and* R_g *with intervals of convergence* I_f *and* I_g*. Then*

$$(f+g)(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x-a)^n$$

Moreover, if $R_f \neq R_g$*, then the radius of convergence of the power series representing* $f+g$ *is* $R = min\{R_f, R_g\}$ *and the interval of convergence is* $I = I_f \cap I_g$*. If* $R_f = R_g$*, then* $R \geq R_f$*.*

Theorem 39 ***Multiplication of Power Series*** *Assume that f is represented by a power series centered at $x = a$ as*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-a)^n$$

with radius of convergence R_f and interval of convergence I_f. Assume that $h(x) = (x-a)^m f(x)$ where $m \in \mathbb{N}$. Then $h(x)$ can also be represented by a power series centered at $x = a$ with

$$h(x) = \sum_{n=0}^{\infty} a_n (x-a)^{n+m}$$

Moreover, the series that represents h has the same radius of convergence and the same interval of convergence as the series that represents f.

Theorem 40 ***Power Series of Composite Functions*** *Assume that f has a power series representation*

$$f(u) = \sum_{n=0}^{\infty} a_n u^n$$

centered at $u = 0$ with radius of convergence R_f and interval of convergence I_f. Let $h(x) = f(c \times x^m)$ where c is a non-zero constant. Then h has a power series representation centered at $x = 0$ of the form

$$h(x) = f(c \times x^m) = \sum_{n=0}^{\infty} (a_n \times c^n) x^{mn}$$

The interval of convergence is

$$I_h = \{x \in \mathbb{R} | c \times x^m \in I_f\}$$

and the radius of convergence is $R_h = \sqrt[m]{\dfrac{R_f}{|c|}}$ if $R < \infty$ and $R_h = \infty$ otherwise.

Definition 33 ***The Formal Derivative of a Power Series*** *Given a power series $\displaystyle\sum_{n=0}^{\infty} a_n (x-a)^n$, the formal derivative is the series*

$$\sum_{n=0}^{\infty} n a_n (x-a)^{n-1} = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$$

Theorem 41 ***Term-by-Term Differentiation of Power Series*** *Assume that the power series* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *has radius of convergence* $R > 0$*. Let*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for all $x \in (a-R, a+R)$*. Then* f *is differentiable on* $(a-R, a+R)$ *and for each* $x \in (a-R, a+R)$*,*

$$f'(x) = \sum_{n=1}^{\infty} n a_n(x-a)^{n-1}$$

Theorem 42 ***Uniqueness of Power Series Representation*** *Suppose that*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for all $x \in (a-R, a+R)$ *where* $R > 0$*. Then*

$$a_n = \frac{f^{(n)}(a)}{n!}$$

In particular, if

$$f(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$$

then $b_n = a_n$ *for each* $n = 0, 1, 2, 3, ...$

Definition 34 ***Formal Antiderivative of a Power Series*** *Given a power series* $\sum_{n=0}^{\infty} a_n(x-a)^n$*, we define the formal antiderivative to be the power series*

$$\sum_{n=0}^{\infty} \int a_n(x-a)^n dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$

where C *is an arbitrary constant.*

Theorem 43 ***Term-by-Term Integration of Power Series*** *Assume that the power series* $\sum_{n=0}^{\infty} a_n(x-a)^n$ *has radius of convergence* $R > 0$. *Let* $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ *for every* $x \in (a-R, a+R)$. *Then the series*

$$\sum_{n=0}^{\infty} \int a_n(x-a)^n dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$

also has radius of convergence R *and if*

$$F(x) = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-a)^{n+1}$$

then $F'(x) = f(x)$. *Furthermore, if* $[c,b] \subset (a-R, a+R)$, *then*

$$\int_c^b f(x)dx = \int_c^b \sum_{n=0}^{\infty} a_n(x-a)^n dx = \sum_{n=0}^{\infty} \int_c^b a_n(x-a)^n dx$$

$$= \sum_{n=0}^{\infty} \frac{a_n}{n+1}((b-a)^{n+1} - (c-a)^{n+1})$$

Definition 35 ***Taylor Polynomials*** *Assume that* f *is* n*-times differentiable at* $x = a$. *The* n*-th degree Taylor polynomial for* f *centered at* $x = a$ *is the polynomial*

$$T_{n,a}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Definition 36 ***Taylor Remainder*** *Assume that* f *is* n *times differentiable at* $x = a$. *Let*

$$R_{n,a}(x) = f(x) - T_{n,a}(x)$$

$R_{n,a}(x)$ *is called the* n*-th degree Taylor remainder function centered at* $x = a$.

Definition 37 ***Taylor's Theorem*** *Assume that* f *is* $n+1$ *times differentiable on an interval* I *containing* $x = a$. *Let* $x \in I$. *Then there exists a point* c *between* x *and* a *such that*

$$f(x) - T_{n,a}(x) = R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Definition 38 ***Taylor's Approximation Theorem I*** *Assume that* $f^{(k+1)}$ *is continuous on* $[-1,1]$. *Then there exists a constant* $M > 0$ *such that*

$$|f(x) - T_{k,0}(x)| \leq M|x|^{k+1}$$

or equivalently that

$$-M|x|^{k+1} \leq f(x) - T_{k,0}(x) \leq M|x|^{k+1}$$

for each $x \in [-1,1]$.

Definition 39 ***Taylor Series*** *Assume that f has derivatives of all orders at $a \in \mathbb{R}$. The series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$ is called the Taylor series for f centered at $x = a$. We write*

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

In the special case where $a = 0$, the series is referred to as the Maclaurin series for f.

Theorem 44 ***Convergence Theorem for Taylor Series*** *Assume that $f(x)$ has derivatives of all orders on an interval I containing $x = a$. Assume also that there exists an M such that*

$$|f^{(k)}(x)| \leq M$$

for all k and for all $x \in I$. Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for all $x \in I$.

Theorem 45 ***Binomial Theorem*** *Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then for each $x \in \mathbb{R}$ we have that*

$$(a+x)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} x^k$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

In particular, where $a = 1$ we have

$$(1+x)^n = 1 + \sum_{k=1}^{n} \frac{n(n-1)(n-2)...(n-k+1)}{k!} x^k$$

Definition 40 ***Generalized Binomial Coefficients and Binomial Series*** *Let $\alpha \in \mathbb{R}$ and let $k \in \{0, 1, 2, 3, ...\}$. Then we define the generalized binomial coefficient*

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-k+1)}{k!}$$

if $k \neq 0$ and

$$\binom{\alpha}{0} = 1$$

We also define the generalized binomial series for α to be the power series

$$1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-k+1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

Theorem 46 ***Generalized Binomial Theorem*** *Let* $\alpha \in \mathbb{R}$. *Then for each* $x \in (-1,1)$ *we have that*

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)...(\alpha-k+1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$