Math 237 Reference

Definition 1 ***Scalar Function*** *A scalar function $f(x_1, ..., x_n)$ of n variables is a function whose domain is a subset of $\mathbb{R}^n$ and whose range is a subset of $\mathbb{R}$.*

Definition 2 ***Level Curves*** *The level curves of a function $f(x, y)$ are the curves*

$$f(x, y) = k$$

where $k \in R(f)$.

Definition 3 ***Level Surfaces*** *A level surface of a scalar function $f(x, y, z)$ is defined by*

$$f(x, y, z) = k$$

where $k \in R(f)$.

Definition 4 ***Level Sets*** *A level set of a scalar function $f(\vec{x})$, $\vec{x} \in \mathbb{R}^n$ is defined by*

$$\{\vec{x} \in \mathbb{R}^n : f(\vec{x}) = k, k \in R(f)\}$$

Definition 5 ***Neighbourhood*** *An r-neighbourhood of a point $(a, b) \in \mathbb{R}^2$ is a set*

$$N_r(a, b) = \{(x, y) \in \mathbb{R}^2 : ||(x, y) - (a, b)|| < r, r \in \mathbb{R}\}$$

Definition 6 ***Limit*** *Assume $f(x, y)$ is defined in a neighbourhood of (a, b), except possibly at (a, b). If, for every $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$0 < ||(x, y) - (a, b)|| < \delta \Rightarrow |f(x, y) - L| < \epsilon$$

then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

Theorem 1 ***Limits 1*** *If $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ and $\lim\limits_{(x,y)\to(a,b)} g(x, y)$ both exist,*

1. $\lim\limits_{(x,y)\to(a,b)} [f(x, y) + g(x, y)] = \lim\limits_{(x,y)\to(a,b)} f(x, y) + \lim\limits_{(x,y)\to(a,b)} g(x, y)$

2. $\lim\limits_{(x,y)\to(a,b)} [f(x, y)g(x, y)] = \left[\lim\limits_{(x,y)\to(a,b)} f(x, y)\right] \left[\lim\limits_{(x,y)\to(a,b)} g(x, y)\right]$

3. $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x, y)}{g(x, y)} = \dfrac{\lim\limits_{(x,y)\to(a,b)} f(x, y)}{\lim\limits_{(x,y)\to(a,b)} g(x, y)}$ *provided* $\lim\limits_{(x,y)\to(a,b)} g(x, y) \neq 0$

Theorem 2 ***Limits 2*** *If $\lim\limits_{(x,y)\to(a,b)} f(x, y)$ exists, then the limit is unique.*

Theorem 3 ***Squeeze Theorem*** *If there exists a function $B(x,y)$ such that*

$$|f(x,y) - L| \leq B(x,y) \text{ for all } (x,y) \neq (a,b)$$

in some neighbourhood of (a,b) and $\lim_{(x,y)\to(a,b)} B(x,y) = 0$, then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

Theorem 4 ***Continuity 1*** *If f and g are both continuous at (a,b), then $f+g$ and fg are continuous at (a,b).*

Theorem 5 ***Continuity 2*** *If f and g are both continuous at (a,b) and $g(a,b) \neq 0$, then the quotient $\frac{f}{g}$ is continuous at (a,b).*

Theorem 6 ***Continuity 3*** *If $f(x,y)$ is continuous at (a,b) and $g(t)$ is continuous at $f(a,b)$, then the composition $g \circ f$ is continuous at (a,b).*

Definition 7 ***Partial Derivatives*** *The partial derivatives of $f(x,y)$ are defined by*

$$\frac{\partial f}{\partial x}(x,y) = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h}$$
$$\frac{\partial f}{\partial y}(x,y) = f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Theorem 7 ***Clairaut's Theorem*** *If f_{xy} and f_{yx} are defined in some neighbourhood of (a,b) and are both continuous at (a,b), then*

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Definition 8 ***Tangent Plane*** *The tangent plane to $z = f(x,y)$ at the point $(a,b,f(a,b))$ is*

$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

Definition 9 ***Linearization and Linear Approximation*** *For a function $f(x,y)$ we define the linearization $L_{(a,b)}(x,y)$ of f at (a,b) by*

$$L_{(a,b)}(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

We call the approximation $f(x,y) \approx L_{(a,b)}(x,y)$ the linear approximation of $f(x,y)$ at (a,b).

Definition 10 ***Gradient*** *Suppose that $f(x,y,z)$ has partial derivatives at $\vec{a} \in \mathbb{R}^3$. The gradient of f at $\vec{a}$ is defined by*

$$\nabla f(\vec{a}) = (f_x(\vec{a}), f_y(\vec{a}), f_z(\vec{a}))$$

Definition 11 ***Linearization and Linear Approximation*** *Suppose that $f(\vec{x})$, $\vec{x} \in \mathbb{R}^3$, has partial derivatives at $\vec{a} \in \mathbb{R}^3$. The linearization of f at $\vec{a}$ is defined by*

$$L_{\vec{a}}(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a})(\vec{x} - \vec{a})$$

The linear approximation of f at $\vec{a}$ is

$$f(\vec{x}) \approx f(\vec{a}) + \nabla f(\vec{a})(\vec{x} - \vec{a})$$

Theorem 8 ***Differentiability 1*** *If $g'(a)$ exists, then $\lim_{x \to a} \frac{|R_{1,a}(x)|}{|x-a|} = 0$ where*

$$R_{1,a}(x) = g(x) - L_a(x) = g(x) - g(a) - g'(a)(x-a)$$

Definition 12 ***Differentiable*** *A function $f(x, y)$ is differentiable at (a, b) if*

$$\lim_{(x,y)\to(a,b)} \frac{|R_{1,(a,b)}(x, y)|}{||(x, y) - (a, b)||} = 0$$

where

$$R_{1,(a,b)}(x, y) = f(x, y) - L_{(a,b)}(x, y)$$

Theorem 9 ***Differentiability 2*** *If a function $f(x, y)$ satisfies*

$$\lim_{(x,y)\to(a,b)} \frac{|f(x, y) - f(a, b) - c(x-a) - d(y-b)|}{||(x, y) - (a, b)||} = 0$$

for some constants c and d, then $c = f_x(a, b)$ and $d = f_y(a, b)$.

Theorem 10 ***Continuity*** *If $f(x, y)$ is differentiable at (a, b), then f is continuous at (a, b).*

Theorem 11 *If the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are both continuous at (a, b), then $f(x, y)$ is differentiable at (a, b).*

Definition 13 ***Differentiability for $f : \mathbb{R}^n \to \mathbb{R}$*** *A function $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at a point $\vec{a} = (a_1, ..., a_n)$ if*

$$\lim_{\vec{x}\to\vec{a}} \frac{|f(\vec{x}) - f(\vec{a}) - L_{\vec{a}}(\vec{x} - \vec{a})|}{||\vec{x} - \vec{a}||} = 0$$

where $L : \mathbb{R}^n \to \mathbb{R}$ is a linear transformation.

Theorem 12 *If $f(x_1, ..., x_n)$ is differentiable at $\vec{a} = (a_1, ..., a_n)$, then f is continuous at $\vec{a}$.*

Theorem 13 *If $\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}$ are continuous at $\vec{a} = (a_1, ..., a_n)$, then $f(x_1, ..., x_n)$ is differentiable at $\vec{a}$.*

Theorem 14 ***Chain Rule*** *Let $G(t) = f(x(t), y(t))$ and let $a = x(t_0)$ and $b = y(t_0)$. If f is differentiable at (a, b) and $x'(t_0)$ and $y'(t_0)$ exist, then $G'(t_0)$ exists and is given by*

$$G'(t_0) = f_x(a, b)x'(t_0) + f_y(a, b)y'(t_0)$$

Definition 14 ***Directional Derivative*** *The directional derivative of $f(x, y)$ at point (a, b) in the direction of a unit vector $\vec{u} = (u_1, u_2)$ where $||\vec{u}|| = 1$ is defined by*

$$D_{\vec{u}}f(a, b) = \frac{d}{ds} f(a + su_1, b + su_2)\bigg|_{s=0} = \lim_{t \to 0} \frac{f((a, b) + t\vec{u}) - f(a, b)}{t}$$

Theorem 15 ***Directional Derivative Theorem*** *If $f(x, y)$ is differentiable at (a, b) and $\vec{u} = (u_1, u_2)$ where $||\vec{u}|| = 1$, then*

$$D_{\vec{u}}f(a, b) = \nabla f(a, b) \cdot \vec{u}$$

Theorem 16 ***Greatest Rate of Change*** *If $f(x, y)$ is differentiable at (a, b) and $\nabla f(a, b) \neq (0, 0)$, then the largest value of $D_{\vec{u}}f(a, b)$ is $||\nabla f(a, b)||$, and occurs when $\vec{u}$ is in the direction of $\nabla f(a, b)$.*

Theorem 17 ***Orthogonality Theorem*** *If $f(x, y) \in C^1$ in a neighbourhood of (a, b) and $\nabla f(a, b) \neq (0, 0)$, then $\nabla f(a, b)$ is orthoganol to the level curve $f(x, y) = k$ through (a, b).*

This theorem extends to three dimensions for $f(x, y, z)$ and (a, b, c).

Definition 15 ***2nd Degree Taylor Polynomial*** *Let f be a function of two variables. The second degree Taylor Polynomial $P_{2,(a,b)}$ of $f(x, y)$ at (a, b) is given by*

$$\begin{aligned} P_{2,(a,b)}(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &= \frac{1}{2}\left[f_{xx}(a, b)(x - a)^2 + 2f_{xy}(a, b)(x - a)(y - b) + f_{yy}(a, b)(y - b)^2\right] \end{aligned}$$

Definition 16 ***Hessian Matrix*** *The Hessian matrix of $f(x, y)$, denoted by $Hf(x, y)$, is defined as*

$$\begin{bmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{bmatrix}$$

Theorem 18 ***Taylor's Theorem for Functions of 2 Variables*** *If $f(x, y) \in C^2$ in some neighbourhood $N(a, b)$ of (a, b), then for all $(x, y) \in N(a, b)$ there exists a point (c, d) on the line segment joining (a, b) and (x, y) such that*

$$f(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) + R_{1,(a,b)}(x, y)$$

where

$$R_{1(a,b)}(x, y) = \frac{1}{2}\left[f_{xx}(c, d)(x - a)^2 + 2f_{xy}(c, d)(x - a)(y - b) + f_{yy}(c, d)(y - b)^2\right]$$

Corollary 18.1 *If $f(x, y) \in C^2$ in some closed neighbourhood $N(a, b)$ of (a, b), then there exists a positive constant M such that*

$$|R_{1,(a,b)}(x, y)| \le M||(x, y) - (a, b)||^2$$

for all $(x, y) \in N(a, b)$.

Definition 17 ***k-th Degree Taylor Polynomial*** *The k-th degree Taylor polynomial of a function $f(x, y)$ is*

$$P_{k,(a,b)}(x, y) = \sum_{|\alpha| \le k} \partial^\alpha f(a, b) \frac{[(x, y) - (a, b)]^\alpha}{\alpha!}$$

Definition 18 ***Taylor's Theorem of Order k*** *If $f(x, y) \in C^{k+1}$ in some neighbourhood $N(a, b)$ of (a, b), then for all $(x, y) \in N(a, b)$ there exists a point (c, d) on the line segment between (a, b) and (x, y) such that*

$$f(x, y) = P_{k,(a,b)}(x, y) + R_{k,(a,b)}(x, y)$$

where

$$R_{k,(a,b)}(x, y) = \sum_{|\alpha| \le k+1} \partial^\alpha f(c, d) \frac{[(x, y) - (a, b)]^\alpha}{\alpha!}$$

Corollary 18.2 *If $f(x, y) \in C^k$ in some neighbourhood of (a, b), then*

$$\lim_{(x,y)\to(a,b)} \frac{|f(x, y) - P_{k,(a,b)}(x, y)|}{||(x, y) - (a, b)||^k} = 0$$

Corollary 18.3 *If $f(x, y) \in C^{k+1}$ in some closed neighbourhood $N(a, b)$ of (a, b), then there exists a constant $M > 0$ such that*

$$|f(x, y) - P_{k,(a,b)}(x, y)| \le M||(x, y) - (a, b)||^{k+1}$$

for all $(x, y) \in N(a, b)$.

Definition 19 ***Generalized Taylor Polynomial*** *The Taylor polynomial of degree k for functions of n variables is*

$$P_{k,\vec{a}}(\vec{x}) = \sum_{|\alpha| \le k} \partial^\alpha f(\vec{a}) \frac{(\vec{x} - \vec{a})^\alpha}{\alpha!}$$

Definition 20 ***Local Maximum and Minimum*** *A point (a, b) is a local maximum point of f if $f(x, y) \le f(a, b)$ for all (x, y) in some neighbourhood of (a, b). A point (a, b) is a local minimum point of f if $f(x, y) \ge f(a, b)$ for all (x, y) in some neighbourhood of (a, b).*

Theorem 19 *If (a, b) is a local maximum or minimum point of f, then each partial derivative is either equal to zero or does not exist.*

Definition 21 ***Critical Point*** *A point (a, b) in the domain of $f(x, y)$ is called a critical point of f if*

$$\frac{\partial f}{\partial x}(a, b) = 0 \text{ or } \frac{\partial f}{\partial x}(a, b) \text{ does not exist}$$

and

$$\frac{\partial f}{\partial y}(a, b) = 0 \text{ or } \frac{\partial f}{\partial y}(a, b) \text{ does not exist}$$

Definition 22 ***Saddle Point*** *A critical point (a, b) of $f(x, y)$ is called a saddle point of f if in every neighbourhood of (a, b) there exists points (x_1, y_1) and (x_2, y_2) such that*

$$f(x_1, y_1) > f(a, b) \text{ and } f(x_2, y_2) < f(a, b)$$

Definition 23 ***Quadratic Form*** *A function Q of the form*

$$Q(u, v) = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = a_{11}u^2 + 2a_{12}uv + a_{22}v^2$$

where a_{11}, a_{12}, and a_{22} are constants, is called a quadratic form on $\mathbb{R}^2$.

Definition 24 ***Determinant and Quadratic Forms*** *A quadratic form $Q(u, v) = a_{11}u^2 + 2a_{12}uv + a_{22}v^2$ on $\mathbb{R}^2$ is*

1. *Positive definite if* $\det(A) > 0$ *and* $a_{11} > 0$
2. *Negative definite if* $\det(A) > 0$ *and* $a_{11} < 0$
3. *Indefinite if* $\det(A) < 0$
4. *Semidefinite if* $\det(A) = 0$

Theorem 20 ***Second Partial Derivatives Test*** *Suppose that $f(x, y) \in C^2$ in some neighbourhood of (a, b) and that*

$$f_x(a, b) = 0 = f_y(a, b)$$

1. *If $Hf(a, b)$ is positive definite, then (a, b) is a local minimum point of f*
2. *If $Hf(a, b)$ is negative definite, then (a, b) is a local maximum point of f*
3. *If $Hf(a, b)$ is indefinite, then (a, b) is a saddle point of f*
4. *If $Hf(a, b)$ is semidefinite, then the test is inconclusive*

Definition 25 ***Convex and Strictly Convex*** *Let $f(x, y) \in C^2$. We say that f is convex if $Hf(x, y)$ is positive semi-definite for all (x, y) and that f is strictly convex if $Hf(x, y)$ is positive definite for all (x, y).*

Theorem 21 *If $f(x,y) \in C^2$ and is strictly convex, then*

1. $f(x,y) > L_{(a,b)}(x,y)$ *for all* $(x,y) \neq (a,b)$
2. $f(a_1 + t(b_1 - a_1), a_2 + t(b_2 - a_2)) < f(a_1,a_2) + t[f(b_1,b_2) - f(a_1,a_2)]$ *for* $0 < t < 1$ *and* $(a_1,a_2) \neq (b_1,b_2)$

Theorem 22 *If $f(x,y) \in C^2$ is convex, then every critical point (c,d) satisfies $f(x,y) \geq f(c,d)$ for all $(x,y) \neq (c,d)$. If $f(x,y) \in C^2$ is strictly convex and has a critical point (c,d), then $f(x,y) > f(c,d)$ for all $(x,y) \neq (c,d)$ and f has no other critical point.*

Theorem 23 *Let $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$ be a positive definite matrix. If $|\tilde{a} - a|$, $|\tilde{b} - b|$, and $|\tilde{c} - c|$ are sufficiently small, then $\begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{c} \end{bmatrix}$ is positive definite.*

Definition 26 ***Absolute Maximum and Minimum*** *Given a function $f(x,y)$ and a set $S \subseteq \mathbb{R}^2$,*

1. *A point $(a,b) \in S$ is an absolute maximum point of f on S if*

$$f(x,y) \leq f(a,b) \text{ for all } (x,y) \in S$$

   The value $f(a,b)$ is called the absolute maximum value of f on S.

2. *A point $(a,b) \in S$ is an absolute minimum point of f on S if*

$$f(x,y) \geq f(a,b) \text{ for all } (x,y) \in S$$

   The value $f(a,b)$ is called the absolute minimum value of f on S.

Definition 27 ***Bounded Set*** *A set $S \subset \mathbb{R}^2$ is said to be bounded if it is contained in some neighbourhood of the origin.*

Definition 28 ***Boundary Point*** *Given a set $S \subset \mathbb{R}^2$, a point $(a,b) \in \mathbb{R}^2$ is said to be a boundary point of S if every neighbourhood of (a,b) contains at least one point in S and one point not in S.*

Definition 29 ***Boundary of S*** *The set $B(S)$ of all boundary points of S is called the boundary of S.*

Definition 30 ***Closed Set*** *A set $S \subseteq \mathbb{R}^2$ is said to be closed if S contains all of its boundary points.*

Theorem 24 ***Extreme Value Theorem*** *If $f(x,y)$ is continuous on a closed and bounded set $S \subseteq \mathbb{R}^2$, then there exists points $(a,b),(c,d) \in S$ such that*

$$f(a,b) \leq f(x,y) \leq f(c,d)$$

for all $(x,y) \in S$.

Theorem 25 *For continuous function $f(x,y)$ on closed and bounded set $S \subset \mathbb{R}^2$, to find the maximum and minimum values of f:*

1. *Find all critical points of f contained in S*
2. *Evaluate f at each critical point*
3. *Find the maximum and minimum points of f on boundary $B(S)$*

Theorem 26 ***Lagrange Multiplier Algorithm*** *Assume that $f(x,y)$ is a differentiable function and $g \in C^1$. To find the maximum value and minimum value of f subject to the constraint $g(x,y) = k$, evaluate $f(x,y)$ at all points (a,b) which satisfy one of the following conditions.*

1. *$\nabla f(a,b) = \lambda \nabla g(a,b)$ and $g(a,b) = k$*
2. *$\nabla g(a,b) = (0,0)$ and $g(a,b) = k$*
3. *(a,b) is an endpoint of the curve $g(x,y) = k$*

Theorem 27 ***Lagrange Multipliers Generalization*** *Let $f(\vec{x})$ be a function of n variables with r constraints of the form*

$$g_1(\vec{x}) = 0, g_2(\vec{x}) = 0, ..., g_r(\vec{x}) = 0$$

To find all possible maximum and minimum points of f subject to the constraints, find all points $\vec{a}$ such that

$$\nabla f(\vec{a}) = \lambda_1 \nabla g_1(\vec{a}) + ... + \lambda_r \nabla g_r(\vec{a})$$

and $g_i(\vec{a}) = 0$ for all $1 \leq i \leq r$.

Definition 31 ***Vector-Valued Function*** *A function whose domain is a subset of $\mathbb{R}^n$ and whose codomain is $\mathbb{R}^m$ is called a vector-valued function.*

Definition 32 ***Mapping*** *A vector-valued function whose domain is a subset of $\mathbb{R}^n$ and whose codomain is a subset of $\mathbb{R}^n$ is called a mapping.*

Definition 33 ***Derivative Matrix*** *The derivative matrix of a mapping defined by*

$$F(x,y) = (f(x,y), g(x,y))$$

is denoted DF and is defined by

$$DF = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

Definition 34 ***General Mapping and Linear Approximation*** *A mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $\vec{u} = F(\vec{x}) = (f_1(\vec{x}), ..., f_m(\vec{x}))$ for $\vec{x} \in \mathbb{R}^n$. If $F \in C^1$, the $m \times n$ derivative matrix of F is*

$$DF(\vec{x}) = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_m}{\partial x_1} & \cdots & \dfrac{\partial f_m}{\partial x_n} \end{bmatrix}$$

and the linear approximation of F at $\vec{a}$ is

$$F(\vec{x}) \approx F(\vec{a}) + DF(\vec{a})\Delta\vec{x}$$

where

$$\Delta\vec{u} = \begin{bmatrix} \Delta u_1 \\ \vdots \\ \Delta u_m \end{bmatrix} \in \mathbb{R}^m, \Delta\vec{x} = \begin{bmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{bmatrix} \in \mathbb{R}^n$$

Theorem 28 ***Chain Rule in Matrix Form*** *Let F and G be mappings from $\mathbb{R}^2$ to $\mathbb{R}^2$. If $G \in C^1$ at (x, y) and $F \in C^1$ at $(u, v) = G(x, y)$, then the composite mapping $F \circ G$ is C^1 (x, y) and*

$$D(F \circ G)(x, y) = DF(u, v)DG(x, y)$$

Definition 35 ***Invertible Mapping and Inverse Mapping*** *Let F be a mapping from a set D_{xy} to a set D_{uv}. If there exists a mapping F^{-1}, called the inverse of F which maps D_{uv} onto D_{xy} such that*

$$(x, y) = F^{-1}(u, v) \text{ iff } (u, v) = F(x, y)$$

then F is invertible on D_{xy}.

Theorem 29 ***One-to-One Implies Invertible*** *Let F be a mapping from a set D_{xy} onto a set D_{uv}. If F is one-to-one on D_{xy}, then F is invertible on D_{xy}.*

Theorem 30 ***Inverse of the Derivative Matrix*** *Let F be a mapping from a set D_{xy} onto a set D_{uv}. If $F \in C^1$ at $\vec{x} \in D_{xy}$, and there exists an inverse mapping F^{-1} of F which has continuous partial derivatives at $\vec{u} = F(\vec{x}) \in D_{uv}$, then*

$$DF^{-1}(\vec{u})DF(\vec{x}) = I$$

Definition 36 ***The Jacobian*** *The Jacobian of a mapping*

$$(u, v) = F(x, y) = (u(x, y), v(x, y))$$

is denoted $\dfrac{\partial(u, v)}{\partial(x, y)}$ and is defined by

$$\frac{\partial(u, v)}{\partial(x, y)} = \det[DF(x, y)] = \det \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}$$

Corollary 30.1 *Consider a mapping defined by*

$$(u, v) = F(x, y) = (u(x, y), v(x, y))$$

which maps a subset D_{xy} onto a subset D_{uv}. Suppose $f, g \in C^1$ on D_{xy}. If F has an inverse mapping F^{-1}, with continuous partial derivatives on D_{uv}, then the Jacobian of F is non-zero:

$$\frac{\partial(u, v)}{\partial(x, y)} \neq 0 \text{ on } D_{xy}$$

Corollary 30.2 ***Inverse Property of the Jacobian*** *Consider a mapping F which maps D_{xy} onto D_{uv}. If $F \in C^1$ at $\vec{x} \in D_{xy}$, and there exists an inverse mapping F^{-1} of F which is C^1 at $\vec{u} = F(\vec{x}) \in D_{uv}$, then*

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\frac{\partial(u, v)}{\partial(x, y)}}$$

Theorem 31 ***Inverse Mapping Theorem*** *If a mapping $(u, v) = F(x, y)$ has continuous partial derivatives in some neighbourhood of (a, b) and $\frac{\partial(u,v)}{\partial(x,y)} \neq 0$ at (a, b), then there is a neighbourhood of (a, b) in which F has an inverse mapping $(x, y) = F^{-1}(u, v)$ which is C^1.*

Definition 37 ***General Jacobian Form*** *For a mapping defined by*

$$\vec{u} = F(\vec{x}) = (f_1(\vec{x}), ..., f_n(\vec{x}))$$

where $\vec{u} = (u_1, ..., u_n)$ and $\vec{x} = (x_1, ..., x_n)$, the Jacobian of F is

$$\frac{\partial(u_1, ..., u_n)}{\partial(x_1, ..., x_n)} = \det[DF(\vec{x})] = \det \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

Definition 38 ***Integrable Function*** *Let $D \subset \mathbb{R}^2$ be closed and bounded. Let P be a partition of D, and let $|\Delta P|$ denote the length of the longest side of all rectangles in the partition P. A function $f(x, y)$ which is bounded on D is integrable on D if all Riemann sums approach the same value as $|\Delta P| \to 0$.*

Definition 39 ***Double Integral*** *If $f(x, y)$ is integrable on a closed bounded set D, then we define the double integral of f on D as*

$$\iint_D f(x, y) dA = \lim_{\Delta P \to 0} \sum_{i=1}^{n} f(x_i, y_i) \Delta A_i$$

Theorem 32 ***Double Integral Linearity*** *If $D \subset \mathbb{R}^2$ is a closed and bounded set and f and g are two integrable functions on D, then for any constant c:*

$$\iint_D (f + cg)dA = \iint_D f dA + c \iint_D g dA$$

Theorem 33 ***Double Integral Inequality*** *If $f(x, y)$ is integrable on a closed bounded set D such that $f(x, y) \leq g(x, y)$ for all $(x, y) \in D$,*

$$\iint_D f(x, y) \leq \iint_D g(x, y)$$

Theorem 34 ***Double Integral Absolute Value Inequality*** *If $D \subset \mathbb{R}^2$ is a closed and bounded set and f is an integrable function on D, then*

$$\left| \iint_D f dA \right| \leq \iint_D |f| dA$$

Theorem 35 ***Double Integral Decomposition*** *Let $D \subset \mathbb{R}^2$ be a closed and bounded set and f be an integrable on D. If D is decomposed into two closed and bounded subsets D_1 and D_2 by a piecewise smooth curve C, then*

$$\iint_D f dA = \iint_{D_1} f dA + \iint_{D_2} f dA$$

Theorem 36 ***Iterated Double Integrals*** *Let $D \subset \mathbb{R}^2$ be defined by*

$$y_l(x) \leq y \leq y_u(x) \text{ and } x_l \leq x \leq x_u$$

where $y_l(x)$ and $y_u(x)$ are continuous for $x_l \leq x \leq x_u$. If $f(x, y)$ is continuous on D, then

$$\iint_D f(x, y) dA = \int_{x_l}^{x_u} \int_{y_l(x)}^{y_u(x)} f(x, y) dy dx$$

Theorem 37 *Let each of D_{uv} and D_{xy} be a closed bounded set whose boundary is a piecewise-smooth closed curve. Let*

$$(x, y) = F(u, v) = (f(u, v), g(u, v))$$

be a one-to-one mapping of D_{uv} onto D_{xy} with $f, g \in C^1$ and $\frac{\partial(x,y)}{\partial(u,v)} \neq 0$ except for possibly on a finite collection of piece-wise smooth curves in D_{uv}. If $G(x, y)$ is continuous on D_{xy}, then

$$\iint_{D_{xy}} G(x, y) dx dy = \iint_{D_{uv}} G(f(u, v), g(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du dv$$

The double integral theorems can be extended to triple integrals. Due to repetition, they are omitted.