

Math 138 Reference

Definition 1 Riemann Sum Given a bounded function f on $[a, b]$, a partition P

$$a = t_0 < t_1 < t_2 < \dots < t_{i-1} < t_i < \dots < t_{n-1} < t_n = b$$

of $[a, b]$, and a set $\{c_1, c_2, \dots, c_n\}$ where $c_i \in [t_{i-1}, t_i]$, then a Riemann sum for f with respect to P is a sum of the form

$$S = \sum_{i=1}^n f(c_i) \Delta t_i$$

Definition 2 Regular n -Partition Given an interval $[a, b]$ and an $n \in \mathbb{N}$, the regular n -partition of $[a, b]$ is the partition $P^{(n)}$ with

$$a = t_0 < t_1 < t_2 < \dots < t_{i-1} < t_i < \dots < t_{n-1} < t_n = b$$

of $[a, b]$ where each subinterval has the same length $\Delta t_i = \frac{b-a}{n}$

Definition 3 Right-hand Riemann Sum The right-hand Riemann sum for f with respect to the partition P is the Riemann Sum R obtained from P by choosing c_i to be t_i , the right-hand endpoint of $[t_{i-1}, t_i]$. That is

$$R = \sum_{i=1}^n f(t_i) \Delta t_i$$

If $P^{(n)}$ is the regular n -partition, we denote the right-hand Riemann sum by

$$R_n = \sum_{i=1}^n f(t_i) \Delta t_i = \sum_{i=1}^n f(t_i) \frac{b-a}{n} = \sum_{i=1}^n f\left(a + i\left(\frac{b-a}{n}\right)\right) \left(\frac{b-a}{n}\right)$$

Definition 4 Left-hand Riemann Sum The left-hand Riemann sum for f with respect to the partition P is the Riemann Sum L obtained from P by choosing c_i to be t_{i-1} , the left-hand endpoint of $[t_{i-1}, t_i]$. That is

$$L = \sum_{i=1}^n f(t_{i-1}) \Delta t_i$$

If $P^{(n)}$ is the regular n -partition, we denote the left-hand Riemann sum by

$$L_n = \sum_{i=1}^n f(t_{i-1}) \Delta t_i = \sum_{i=1}^n f(t_{i-1}) \frac{b-a}{n} = \sum_{i=1}^n f\left(a + (i-1)\left(\frac{b-a}{n}\right)\right) \left(\frac{b-a}{n}\right)$$

Definition 5 Definite Integral We say that a bounded function f is integrable on $[a, b]$ if there exists a unique number $I \in \mathbb{R}$ such that if whenever $\{P_n\}$ is a sequence of partitions with $\lim_{n \rightarrow \infty} \|P_n\| = 0$ and $\{S_n\}$ is any sequence of Riemann sums associated with the P_n 's, we have

$$\lim_{n \rightarrow \infty} S_n = I$$

In this case, we call I the integral of f over $[a, b]$ and denote it by

$$\int_a^b f(t) dt$$

The points a and b are called the limits of integration and the function $f(t)$ is called the integrand. The variable t is called the variable of integration.

Theorem 1 Integrability Theorem for Continuous Functions Let f be continuous on $[a, b]$. Then f is integrable on $[a, b]$. Moreover,

$$\int_a^b f(t) dt = \lim_{n \rightarrow \infty} S_n$$

where

$$S_n = \sum_{i=1}^n f(c_i) \Delta t_i$$

is any Riemann sum associated with the regular n -partitions. In particular,

$$\int_a^b f(t) dt = \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(t_i) \frac{b-a}{n}$$

$$\int_a^b f(t) dt = \lim_{n \rightarrow \infty} L_n = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(t_{i-1}) \frac{b-a}{n}$$

Theorem 2 Properties of Integrals Assume that f and g are integrable on the interval $[a, b]$. Then:

1. For any $c \in \mathbb{R}$, $\int_a^b c f(t) dt = c \int_a^b f(t) dt$
2. $\int_a^b (f + g)(t) dt = \int_a^b f(t) dt + \int_a^b g(t) dt$
3. If $m \leq f(t) \leq M$ for all $t \in [a, b]$, then $m(b-a) \leq \int_a^b f(t) dt \leq M(b-a)$
4. If $0 \leq f(t)$ for all $t \in [a, b]$, then $0 \leq \int_a^b f(t) dt$
5. If $g(t) \leq f(t)$ for all $t \in [a, b]$, then $\int_a^b g(t) dt \leq \int_a^b f(t) dt$
6. The function $|f|$ is integrable on $[a, b]$ and $|\int_a^b f(t) dt| \leq \int_a^b |f(t)| dt$

Definition 6 Identical Limits of Integration Let $f(t)$ be defined at $t = a$. Then we define

$$\int_a^a f(t)dt = 0$$

Definition 7 Switching the Limits of Integration Let f be integrable on the interval $[a, b]$ where $a < b$. Then we define

$$\int_b^a f(t)dt = - \int_a^b f(t)dt$$

Theorem 3 Integrals over Subintervals Assume that f is integrable on an interval I containing a , b , and c . Then

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$$

Definition 8 Average Value of f If f is continuous on $[a, b]$, the average value of f on $[a, b]$ is defined as

$$\frac{1}{b-a} \int_a^b f(t)dt$$

Theorem 4 Average Value Theorem (MVT for Integrals) Assume that f is continuous on $[a, b]$. Then there exists $a \leq c \leq b$ such that

$$f(c) = \frac{1}{b-a} \int_a^b f(t)dt$$

Theorem 5 Fundamental Theorem of Calculus (Part 1) Assume that f is continuous on an open interval I containing point a . Let

$$G(x) = \int_a^x f(t)dt$$

Then $G(x)$ is differentiable at each $x \in I$ and $G'(x) = f(x)$. Equivalently,

$$G'(x) = \frac{d}{dx} \int_a^x f(t)dt = f(x)$$

Theorem 6 Extended Fundamental Theorem of Calculus Assume that f is continuous and that g and h are differentiable. Let

$$H(x) = \int_{g(x)}^{h(x)} f(t)dt$$

Then $H(x)$ is differentiable and

$$H'(x) = f(h(x))h'(x) - f(g(x))g'(x)$$

Definition 9 Antiderivative Given a function f , an antiderivative is a function F such that $F'(x) = f(x)$. If for an interval I , $\forall x \in I, F'(x) = f(x)$, then we say F is an antiderivative for f on I .

Theorem 7 Power Rule for Antiderivatives If $\alpha \neq -1$, then

$$\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1} + C$$

Theorem 8 Fundamental Theorem of Calculus (Part 2) Assume that f is continuous and that F is any antiderivative of f . Then

$$\int_a^b f(t)dt = F(b) - F(a)$$

Theorem 9 Change of Variables Assume that $g'(x)$ is continuous on $[a, b]$ and $f(u)$ is continuous on $g([a, b])$. Then

$$\int_{x=a}^{x=b} f(g(x))g'(x)dx = \int_{u=g(a)}^{u=g(b)} f(u)du$$

Definition 10 Integration by Parts

$$\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx$$

Assume that f and g are such that both f' and g' are continuous on an interval containing a and b . Then

$$\int_a^b f(x)g'(x)dx = f(x)g(x)|_a^b - \int_a^b f'(x)g(x)dx$$

Definition 11 Type 1 Partial Fraction Decomposition Assume that

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials such that $\text{Deg}(p(x)) < \text{Deg}(q(x)) = k$ and $q(x)$ can be factored into the product of linear terms each with distinct roots. That is

$$q(x) = a(x - a_1)(x - a_2)\dots(x - a_k)$$

where the a_i 's are unique and none of the a_i 's are roots of $p(x)$. Then there exists constants A_1, A_2, \dots, A_k such that

$$f(x) = \frac{1}{a} \left[\frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \dots + \frac{A_k}{x - a_k} \right]$$

We say that f admits Type 1 Partial Fraction Decomposition

Theorem 10 Integration of Type 1 Partial Fractions Assume that $f(x) = \frac{p(x)}{q(x)}$ admits a Type 1 Partial Fraction Decomposition of the form

$$f(x) = \frac{1}{a} \left[\frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \dots + \frac{A_k}{x - a_k} \right]$$

Then

$$\int f(x)dx = \frac{1}{a} \left[\int \frac{A_1}{x - a_1} dx + \int \frac{A_2}{x - a_2} dx + \dots + \int \frac{A_k}{x - a_k} dx \right]$$

Definition 12 Type 2 Partial Fraction Decomposition Assume that

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials such that $\text{Deg}(p(x)) < \text{Deg}(q(x)) = k$ and $q(x)$ can be factored into the product of linear terms with non-distinct roots. That is

$$q(x) = a(x - a_1)^{m_1}(x - a_2)^{m_2} \dots (x - a_l)^{m_l}$$

where at least one of the m_j 's is greater than 1. Each expression $(x - a_j)^{m_j}$ in the factorization of $q(x)$ will contribute m_j terms to the decomposition, one for each power of $(x - a_j)$ from 1 to m_j , which when combined will be of the form

$$\frac{p(x)}{q(x)} = \sum_{j=1}^l \frac{A_{j,1}}{x - a_j} + \frac{A_{j,2}}{(x - a_j)^2} + \dots + \frac{A_{j,m_j}}{(x - a_j)^{m_j}}$$

The number m_j is called the multiplicity of the root a_j . We say that f admits Type 2 Partial Fraction Decomposition

Definition 13 Type 3 Partial Fraction Decomposition Let $f(x) = \frac{p(x)}{q(x)}$

be a rational function with $\text{Deg}(p(x)) < \text{Deg}(q(x))$, but $q(x)$ does not factor into linear terms. We say that f admits a Type 3 Partial Fraction Decomposition. Suppose that $q(x)$ has an irreducible factor $x^2 + bx + c$ with multiplicity m . Then this factor will contribute terms of the form

$$\frac{B_1x + C_1}{x^2 + bx + c} + \frac{B_2x + C_2}{(x^2 + bx + c)^2} + \dots + \frac{B_mx + C_m}{(x^2 + bx + c)^m}$$

Definition 14 Type 1 Improper Integral

1. Let f be integrable on $[a, b]$ for each $a \leq b$. We say the Type 1 Improper Interval

$$\int_a^\infty f(x)dx \text{ converges if } \lim_{b \rightarrow \infty} \int_a^b f(x)dx \text{ exists.}$$

In this case, we write

$$\int_a^\infty f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx$$

Otherwise, we say $\int_a^\infty f(x)dx$ diverges.

2. Let f be integrable on $[b, a]$ for each $b \leq a$. We say that the Type I Improper Integral

$$\int_{-\infty}^a f(x)dx \text{ converges if } \lim_{b \rightarrow -\infty} \int_b^a f(x)dx \text{ exists.}$$

In this case, we write

$$\int_{-\infty}^a f(x)dx = \lim_{b \rightarrow -\infty} \int_b^a f(x)dx \text{ exists.}$$

Otherwise, we say $\int_{-\infty}^a f(x)dx$ diverges.

3. Assume that f is integrable on $[a, b]$ for each $a, b \in \mathbb{R}$ with $a < b$. We say that the Type I Improper Integral

$$\int_{-\infty}^\infty f(x)dx \text{ converges if both } \lim_{c \rightarrow \infty} \int_{-\infty}^c f(x)dx \text{ and } \lim_{c \rightarrow \infty} \int_c^\infty f(x)dx$$

exist for some $c \in \mathbb{R}$. In this case, we write

$$\int_{-\infty}^\infty f(x)dx = \lim_{c \rightarrow \infty} \int_{-\infty}^c f(x)dx + \lim_{c \rightarrow \infty} \int_c^\infty f(x)dx$$

Otherwise, we say $\int_{-\infty}^\infty f(x)dx$ diverges.

Theorem 11 p -Test for Type 1 Improper Integrals The improper integral

$$\int_1^\infty \frac{1}{x^p}dx$$

converges if and only if $p > 1$. If $p > 1$, then

$$\int_1^\infty \frac{1}{x^p}dx = \frac{1}{p-1}$$

Theorem 12 Properties of Type 1 Improper Integrals Assume that $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ both converge.

1. $\int_a^\infty cf(x)dx$ converges for each $c \in \mathbb{R}$ and

$$\int_a^\infty cf(x)dx = c \int_a^\infty f(x)dx$$

2. $\int_a^\infty (f(x) + g(x))dx$ converges and

$$\int_a^\infty (f(x) + g(x))dx = \int_a^\infty f(x)dx + \int_a^\infty g(x)dx$$

3. If $f(x) \leq g(x)$ for all $a \leq x$, then

$$\int_a^\infty f(x)dx \leq \int_a^\infty g(x)dx$$

4. If $a < c < \infty$, then $\int_c^\infty f(x)dx$ converges and

$$\int_a^\infty f(x)dx = \int_a^c f(x)dx + \int_c^\infty f(x)dx$$

Theorem 13 The Monotone Convergence Theorem for Functions Assume that f is non-decreasing on $[a, \infty)$.

1. If $\{f(x)|x \in [a, \infty)\}$ is bounded above, then $\lim_{x \rightarrow \infty} f(x)$ exists and

$$\lim_{x \rightarrow \infty} f(x) = L = \text{lub}(\{f(x)|x \in [a, \infty)\})$$

2. If $\{f(x)|x \in [a, \infty)\}$ is not bounded above, then

$$\lim_{x \rightarrow \infty} f(x) = \infty$$

Theorem 14 Comparison Test for Type 1 Improper Integrals Assume that $0 \leq g(x) \leq f(x)$ for all $x \geq a$ and that both f and g are continuous on $[a, \infty)$.

1. If $\int_a^\infty f(x)dx$ converges, then so does $\int_a^\infty g(x)dx$
2. If $\int_a^\infty g(x)dx$ diverges, then so does $\int_a^\infty f(x)dx$

Definition 15 Absolute Convergence for Type 1 Improper Integrals Let f be integrable on $[a, b]$ for all $b \geq a$. We say the improper integral $\int_a^\infty f(x)dx$ converges absolutely if

$$\int_a^\infty |f(x)|dx$$

converges.

Theorem 15 Absolute Convergence Theorem for Improper Integrals
Let f be integrable on $[a, b]$ for all $b > a$. Then $|f|$ is also integrable on $[a, b]$ for all $b > a$. Moreover, if we assume that

$$\int_a^\infty |f(x)|dx \text{ converges, then so does } \int_a^\infty f(x)dx$$

In particular, if $0 \leq |f(x)| \leq g(x)$ for all $x \geq a$, both f and g are integrable on $[a, b]$ for all $b \geq a$, and if $\int_a^\infty g(x)dx$ converges, then so does

$$\int_a^\infty f(x)dx$$

Definition 16 The Gamma Function For each $x \in \mathbb{R}$, define

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

Definition 17 Type 2 Improper Integral

1. Let f be integrable on $[t, b]$ for every $t \in (a, b]$ with either $\lim_{x \rightarrow a^+} f(x) = \infty$ or $\lim_{x \rightarrow a^+} f(x) = -\infty$. We say the Type 2 Improper Integral

$$\int_a^b f(x)dx \text{ converges if } \lim_{t \rightarrow a^+} \int_t^b f(x)dx \text{ exists.}$$

In this case, we write

$$\int_a^b f(x)dx = \lim_{t \rightarrow a^+} \int_t^b f(x)dx$$

Otherwise, we say $\int_a^b f(x)dx$ diverges.

2. Let f be integrable on $[a, t]$ for every $t \in [a, b)$ with either $\lim_{x \rightarrow b^-} f(x) = \infty$ or $\lim_{x \rightarrow b^-} f(x) = -\infty$. We say the Type 2 Improper Integral

$$\int_a^b f(x)dx \text{ converges if } \lim_{t \rightarrow b^-} \int_a^t f(x)dx \text{ exists.}$$

In this case, we write

$$\int_a^b f(x)dx = \lim_{t \rightarrow b^-} \int_a^t f(x)dx$$

Otherwise, we say $\int_a^b f(x)dx$ diverges.

3. If f has an infinite discontinuity at $x = c$ where $a < c < b$, then we say that the Type 2 Improper Integral

$$\int_a^b f(x)dx \text{ converges if both } \int_a^c f(x)dx \text{ and } \int_c^b f(x)dx \text{ converge}$$

In this case, we write

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

If one or both of these integrals diverge, then we say that $\int_a^b f(x)dx$

Theorem 16 p -Test for Type 2 Improper Integrals The improper integral

$$\int_0^1 \frac{1}{x^p} dx$$

converges if and only if $p < 1$. If $p < 1$, then

$$\int_0^1 \frac{1}{x^p} dx = \frac{1}{1-p}$$

Definition 18 Differential Equation A differential equation is an equation involving an independent variable such as x , a function $y = f(x)$ and various derivatives of y . In general, we will write

$$F(x, y, y'', \dots, y^{(n)}) = 0$$

A solution to the differential equation is a function φ such that

$$F(x, \varphi(x), \varphi'(x), \dots, \varphi^{(n)}(x)) = 0$$

The highest order of a derivative appearing in the equation is called the order of the differential equation.

Definition 19 Separable Differential Equation A first-order differentiable equation is separable if there exists functions $f = f(x)$ and $g = g(y)$ such that the differentiable equation can be written in the form

$$y' = f(x)g(y)$$

Definition 20 Constant (Equilibrium) Solution to a Separable Differential Equation If

$$y' = f(x)g(y)$$

is a separable differential equation and if $y_0 \in \mathbb{R}$ is such that $g(y_0) = 0$, then

$$\phi(x) = y_0$$

is called a constant or equilibrium solution to the differential equation.

Definition 21 First-Order Linear Differentiable Equation A first-order differential equation is said to be linear if it can be written in the form

$$y' = f(x)y + g(x)$$

Theorem 17 Solving First-Order Linear Differential Equations Let f and g be continuous and let

$$y' = f(x)y + g(x)$$

be a first-order linear differential equation. Then the solutions to this equation are of the form

$$y = \frac{\int g(x)I(x)dx}{I(x)}$$

where $I(x) = e^{-\int f(x)dx}$.

Theorem 18 Uniqueness and Existence Theorem for First-Order Linear Differential Equations Assume that f and g are continuous functions on an interval I . Then for each $x_0 \in I$ and for all $y_0 \in \mathbb{R}$, the initial value problem

$$y' = f(x)y + g(x)$$

$$y(x_0) = y_0$$

has exactly one solution $y = \varphi(x)$ on the interval I .

Definition 22 Series Given a sequence $\{a_n\}$, the formal sum

$$a_1 + a_2 + a_3 + a_4 + \dots + a_n + \dots$$

is called a series. The series is called formal because we have not yet given it a meaning numerically. The a_n 's are called the terms of the series. For each term a_n , the index of the term is n . We will denote the series by

$$\sum_{n=1}^{\infty} a_n$$

Definition 23 Convergence of a Series Given a series

$$\sum_{n=1}^{\infty} a_n$$

for each $k \in \mathbb{N}$, we will define the k -th partial sum S_k by

$$S_k = \sum_{n=1}^k a_n$$

We say that the series $\sum_{n=1}^{\infty} a_n$ converges if the sequence $\{S_k\}$ of partial sums converges. In this case, if $L = \lim_{k \rightarrow \infty} S_k$, then we write

$$\sum_{n=1}^{\infty} a_n = L$$

and assign the sum this value. Otherwise, we say that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Definition 24 Geometric Series A geometric series is a series of the form

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + r^3 + r^4 + \dots$$

The number r is called the ratio of the series.

Theorem 19 Geometric Series Test The geometric series $\sum_{n=0}^{\infty} r^n$ converges if $|r| < 1$ and diverges otherwise. If $|r| < 1$, then

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Theorem 20 Divergence Test Assume that $\sum_{n=1}^{\infty} a_n$ converges. Then

$$\lim_{n \rightarrow \infty} a_n = 0$$

Equivalently, if $\lim_{n \rightarrow \infty} a_n \neq 0$ or if $\lim_{n \rightarrow \infty} a_n$ does not exist, then $\sum_{n=1}^{\infty} a_n$ diverges.

Theorem 21 Arithmetic for Series I Assume that $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge.

1. The series $\sum_{n=1}^{\infty} ca_n$ converges for every $c \in \mathbb{R}$ and

$$\sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$$

2. The series $\sum_{n=1}^{\infty} (a_n + b_n)$ converges and

$$\sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$$

Theorem 22 Arithmetic for Series II

1. If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=j}^{\infty} a_n$ also converges for each j .
2. If $\sum_{n=j}^{\infty} a_n$ converges for some j , then $\sum_{n=1}^{\infty} a_n$ converges.

In either of these two cases,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_{j-1} + \sum_{n=j}^{\infty} a_n$$

Definition 25 Monotonic Sequences Given a sequence $\{a_n\}$, we say that the sequence is

1. non-decreasing if $a_{n+1} \geq a_n$ for every $n \in \mathbb{N}$
2. increasing if $a_{n+1} > a_n$ for every $n \in \mathbb{N}$
3. non-increasing if $a_{n+1} \leq a_n$ for every $n \in \mathbb{N}$
4. decreasing if $a_{n+1} < a_n$ for every $n \in \mathbb{N}$

We say that $\{a_n\}$ is monotonic if it satisfies one of these four conditions.

Theorem 23 Monotone Convergence Theorem Let $\{a_n\}$ be a non-decreasing sequence.

1. If $\{a_n\}$ is bounded above, then $\{a_n\}$ converges to $L = \text{lub}(\{a_n\})$.
2. If $\{a_n\}$ is not bounded above, then $\{a_n\}$ diverges to ∞ .

In particular, $\{a_n\}$ converges if and only if it is bounded above.

Definition 26 Positive Series We call a series $\sum_{n=1}^{\infty} a_n$ positive if the terms $a_n \geq 0$ for all $n \in \mathbb{N}$

Theorem 24 Comparison Test for Series Assume that $0 \leq a_n \leq b_n$ for each $n \in \mathbb{N}$.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

Theorem 25 Limit Convergence Test Assume that $a_n > 0$ and $b_n > 0$ for each $n \in \mathbb{N}$. Assume also that

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$$

where either $L \in \mathbb{R}$ or $L = \infty$.

1. If $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=1}^{\infty} b_n$ converges.
2. If $L = 0$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges. Equivalently, if $\sum_{n=1}^{\infty} a_n$ diverges, then so does $\sum_{n=1}^{\infty} b_n$.
3. If $L = \infty$ and $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} b_n$ converges. Equivalently, if $\sum_{n=1}^{\infty} b_n$ diverges, then so does $\sum_{n=1}^{\infty} a_n$.

Theorem 26 Integral Test for Convergence Assume that $f(x)$ is decreasing and continuous with $f(x) > 0$ on $[1, \infty)$. Let $a_k = f(k)$. For each $n \in \mathbb{N}$, let $S_n = \sum_{k=1}^n a_k$. Then

1. For all $n \in \mathbb{N}$,

$$\int_1^{n+1} f(x)dx \leq S_n \leq a_1 + \int_1^n f(x)dx$$

2. $\sum_{k=1}^{\infty} a_k$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.

3. In the case $\sum_{k=1}^{\infty} a_k$ converges, then

$$\int_1^{\infty} f(x)dx \leq \sum_{k=1}^{\infty} a_k \leq a_1 + \int_1^{\infty} f(x)dx$$

and

$$\int_{n+1}^{\infty} f(x)dx \leq S - S_n \leq \int_n^{\infty} f(x)dx$$

where $S = \sum_{k=1}^{\infty} a_k$. Note that by 2, $\int_n^{\infty} f(x)dx$ exists.

Theorem 27 *p-Series Test* The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Definition 27 *Alternating Series* A series of the form

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$$

or of the form

$$\sum_{n=1}^{\infty} (-1)^n a_n = -a_1 + a_2 - a_3 + a_4 - \dots$$

is said to be alternating provided that $a_n > 0$ for all n .

Theorem 28 *Alternating Series Test* Assume that $a_n > 0$ for all n , $a_{n+1} \leq a_n$ for all n , and $\lim_{n \rightarrow \infty} a_n = 0$. Then the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

converges. If $S_k = \sum_{n=1}^k (-1)^{n-1} a_n$, then S_k approximates the sum $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ with an error that is at most a_{k+1} . That is,

$$|S_k - S| \leq a_{k+1}$$

Definition 28 *Absolute vs Conditional Convergence* A series $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if $\sum_{n=1}^{\infty} |a_n|$ converges. A series $\sum_{n=1}^{\infty} a_n$ is said to converge conditionally if $\sum_{n=1}^{\infty} |a_n|$ diverges while $\sum_{n=1}^{\infty} a_n$ converges.

Theorem 29 *Absolute Convergence Theorem* If $\sum_{n=1}^{\infty} |a_n|$ converges, then so does $\sum_{n=1}^{\infty} a_n$. The sums of $\sum_{n=1}^{\infty} |a_n|$ and $\sum_{n=1}^{\infty} a_n$ will converge to different values unless $a_n \geq 0$ for all n .

Definition 29 *Rearrangement of a Series* Given a series $\sum_{n=1}^{\infty} a_n$ and a 1-1 and onto function $\phi : \mathbb{N} \rightarrow \mathbb{N}$, if we let $b_n = a_{\phi(n)}$, then the series $\sum_{n=1}^{\infty} b_n$ is called a rearrangement of $\sum_{n=1}^{\infty} a_n$.

Theorem 30 *Rearrangement Theorem*

1. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series. If $\sum_{n=1}^{\infty} b_n$ is any rearrangement of $\sum_{n=1}^{\infty} a_n$, then $\sum_{n=1}^{\infty} b_n$ also converges and

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n$$

2. Let $\sum_{n=1}^{\infty} a_n$ be a conditionally convergent series. Let $\alpha \in \mathbb{R}$ or $\alpha = \pm\infty$.

Then there exists a rearrangement $\sum_{n=1}^{\infty} b_n$ of $\sum_{n=1}^{\infty} a_n$ such that

$$\sum_{n=1}^{\infty} b_n = \alpha$$

Theorem 31 *Ratio Test* Given a series $\sum_{n=1}^{\infty} a_n$, assume that

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

where $L \in \mathbb{R}$ or $L = \infty$.

1. If $0 \leq L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, then no conclusion is possible.

Theorem 32 *Polynomial vs Factorial Growth* For any $x \in \mathbb{R}$,

$$\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$$

Theorem 33 Root Test Given a series $\sum_{n=1}^{\infty} a_n$, assume that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$$

where $L \in \mathbb{R}$ or $L = \infty$.

1. If $0 \leq L < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. If $L > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$, then no conclusion is possible.

Definition 30 Power Series A power series centered at $x = a$ is a series of the form

$$\sum_{n=0}^{\infty} a_n(x - a)^n$$

where x is considered a variable and the value a_n is called the coefficient of the term $(x - a)^n$

Definition 31 Interval and Radius of Convergence Given a power series of the form $\sum_{n=0}^{\infty} a_n(x - a)^n$, the set

$$I = \{x_0 \mid \sum_{n=0}^{\infty} |a_n(x - a)^n| \text{ converges}\}$$

is an interval centered at $x = a$ which we call the interval of convergence for the power series. Let

$$R := \begin{cases} \text{lub}(\{|x_0 - a| \mid x_0 \in I\}) & \text{if } I \text{ is bounded} \\ \infty & \text{if } I \text{ is not bounded} \end{cases}$$

Then R is called the radius of convergence of the power series.

Theorem 34 Fundamental Convergence Theorem for Power Series

Given a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ centered at $x = a$, let R be the radius of convergence.

1. If $R = 0$, then $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges for $x = a$ but it diverges for all other values of x .
2. If $0 < R < \infty$, then the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges absolutely for every $x \in (a-R, a+R)$ and diverges if $|x-a| < R$.
3. If $R = \infty$, then the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges absolutely for every $x \in \mathbb{R}$.

In particular, $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges on an interval that is centered at $x = a$ which may or may not include one or both of the endpoints.

Theorem 35 Test for the Radius of Convergence Let $\sum_{n=0}^{\infty} a_n(x-a)^n$ be a power series for which

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

where $0 \leq L < \infty$ or $L = \infty$. Let R be the radius of convergence of the power series.

1. If $0 < L < \infty$, then $R = \frac{1}{L}$
2. If $L = 0$, then $R = \infty$
3. If $L = \infty$, then $R = 0$.

Theorem 36 Equivalence of Radius of Convergence Let p and q be non-zero polynomials where $q(n) \neq 0$ for $n \geq k$. Then the following series have the same radius of convergence:

1. $\sum_{n=k}^{\infty} a_n(x-a)^n$
2. $\sum_{n=k}^{\infty} \frac{a_n p(n)(x-a)^n}{q(n)}$

However, they may have different intervals of convergence.

Definition 32 Functions Represented by a Power Series Let $\sum_{n=0}^{\infty} a_n(x-a)^n$ be a power series with radius of convergence $R > 0$. Let I be the interval of convergence for $\sum_{n=0}^{\infty} a_n(x-a)^n$. Let f be the function defined on the interval I by the formula

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for each $x \in I$. We say that the function $f(x)$ is represented by the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ on I .

Theorem 37 Abel's Theorem - Continuity of Power Series Assume that the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has interval of convergence I . Let

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for each $x \in I$. Then $f(x)$ is continuous on I .

Theorem 38 Addition of Power Series Assume that f and g are represented by power series centered at $x = a$ with

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

and

$$g(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$$

respectively. Assume also that the radii of convergence of these series are R_f and R_g with intervals of convergence I_f and I_g . Then

$$(f+g)(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x-a)^n$$

Moreover, if $R_f \neq R_g$, then the radius of convergence of the power series representing $f+g$ is $R = \min\{R_f, R_g\}$ and the interval of convergence is $I = I_f \cap I_g$. If $R_f = R_g$, then $R \geq R_f$.

Theorem 39 *Multiplication of Power Series* Assume that f is represented by a power series centered at $x = a$ as

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

with radius of convergence R_f and interval of convergence I_f . Assume that $h(x) = (x-a)^m f(x)$ where $m \in \mathbb{N}$. Then $h(x)$ can also be represented by a power series centered at $x = a$ with

$$h(x) = \sum_{n=0}^{\infty} a_n(x-a)^{n+m}$$

Moreover, the series that represents h has the same radius of convergence and the same interval of convergence as the series that represents f .

Theorem 40 *Power Series of Composite Functions* Assume that f has a power series representation

$$f(u) = \sum_{n=0}^{\infty} a_n u^n$$

centered at $u = 0$ with radius of convergence R_f and interval of convergence I_f . Let $h(x) = f(c \times x^m)$ where c is a non-zero constant. Then h has a power series representation centered at $x = 0$ of the form

$$h(x) = f(c \times x^m) = \sum_{n=0}^{\infty} (a_n \times c^n) x^{mn}$$

The interval of convergence is

$$I_h = \{x \in \mathbb{R} | c \times x^m \in I_f\}$$

and the radius of convergence is $R_h = \sqrt[m]{\frac{R_f}{|c|}}$ if $R < \infty$ and $R_h = \infty$ otherwise.

Definition 33 *The Formal Derivative of a Power Series* Given a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, the formal derivative is the series

$$\sum_{n=0}^{\infty} n a_n (x-a)^{n-1} = \sum_{n=1}^{\infty} n a_n (x-a)^{n-1}$$

Theorem 41 Term-by-Term Differentiation of Power Series Assume that the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $R > 0$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for all $x \in (a-R, a+R)$. Then f is differentiable on $(a-R, a+R)$ and for each $x \in (a-R, a+R)$,

$$f'(x) = \sum_{n=1}^{\infty} na_n(x-a)^{n-1}$$

Theorem 42 Uniqueness of Power Series Representation Suppose that

$$f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$$

for all $x \in (a-R, a+R)$ where $R > 0$. Then

$$a_n = \frac{f^{(n)}(a)}{n!}$$

In particular, if

$$f(x) = \sum_{n=0}^{\infty} b_n(x-a)^n$$

then $b_n = a_n$ for each $n = 0, 1, 2, 3, \dots$

Definition 34 Formal Antiderivative of a Power Series Given a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$, we define the formal antiderivative to be the power series

$$\sum_{n=0}^{\infty} \int a_n(x-a)^n dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1}$$

where C is an arbitrary constant.

Theorem 43 Term-by-Term Integration of Power Series Assume that the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ has radius of convergence $R > 0$. Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ for every $x \in (a-R, a+R)$. Then the series

$$\sum_{n=0}^{\infty} \int a_n(x-a)^n dx = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1}$$

also has radius of convergence R and if

$$F(x) = C + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-a)^{n+1}$$

then $F'(x) = f(x)$. Furthermore, if $[c, b] \subset (a-R, a+R)$, then

$$\begin{aligned} \int_c^b f(x) dx &= \int_c^b \sum_{n=0}^{\infty} a_n(x-a)^n dx = \sum_{n=0}^{\infty} \int_c^b a_n(x-a)^n dx \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n+1} ((b-a)^{n+1} - (c-a)^{n+1}) \end{aligned}$$

Definition 35 Taylor Polynomials Assume that f is n -times differentiable at $x = a$. The n -th degree Taylor polynomial for f centered at $x = a$ is the polynomial

$$T_{n,a}(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

Definition 36 Taylor Remainder Assume that f is n times differentiable at $x = a$. Let

$$R_{n,a}(x) = f(x) - T_{n,a}(x)$$

$R_{n,a}(x)$ is called the n -th degree Taylor remainder function centered at $x = a$.

Definition 37 Taylor's Theorem Assume that f is $n+1$ times differentiable on an interval I containing $x = a$. Let $x \in I$. Then there exists a point c between x and a such that

$$f(x) - T_{n,a}(x) = R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$$

Definition 38 Taylor's Approximation Theorem I Assume that $f^{(k+1)}$ is continuous on $[-1, 1]$. Then there exists a constant $M > 0$ such that

$$|f(x) - T_{k,0}(x)| \leq M|x|^{k+1}$$

or equivalently that

$$-M|x|^{k+1} \leq f(x) - T_{k,0}(x) \leq M|x|^{k+1}$$

for each $x \in [-1, 1]$.

Definition 39 Taylor Series Assume that f has derivatives of all orders at $a \in \mathbb{R}$. The series $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$ is called the Taylor series for f centered at $x = a$. We write

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

In the special case where $a = 0$, the series is referred to as the Maclaurin series for f .

Theorem 44 Convergence Theorem for Taylor Series Assume that $f(x)$ has derivatives of all orders on an interval I containing $x = a$. Assume also that there exists an M such that

$$|f^{(k)}(x)| \leq M$$

for all k and for all $x \in I$. Then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

for all $x \in I$.

Theorem 45 Binomial Theorem Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$. Then for each $x \in \mathbb{R}$ we have that

$$(a+x)^n = \sum_{k=0}^n \binom{n}{k} a^{n-k} x^k$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

In particular, where $a = 1$ we have

$$(1+x)^n = 1 + \sum_{k=1}^n \frac{n(n-1)(n-2)\dots(n-k+1)}{k!} x^k$$

Definition 40 Generalized Binomial Coefficients and Binomial Series Let $\alpha \in \mathbb{R}$ and let $k \in \{0, 1, 2, 3, \dots\}$. Then we define the generalized binomial coefficient

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!}$$

if $k \neq 0$ and

$$\binom{\alpha}{0} = 1$$

We also define the generalized binomial series for α to be the power series

$$1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

Theorem 46 Generalized Binomial Theorem Let $\alpha \in \mathbb{R}$. Then for each $x \in (-1, 1)$ we have that

$$(1+x)^\alpha = 1 + \sum_{k=1}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!} x^k = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$