Math 239 Reference

Theorem 1 *For every $n \geq 1$, the number of lists of an n-element set S is*

$$n(n-1)(n-2)...3 \cdot 2 \cdot 1$$

Theorem 2 *For every $n \geq 0$, the number of subsets of an n-element set S is 2^n.*

Theorem 3 *For $n, k \geq 0$, the number of partial lists of length k of an n-element set is*

$$n(n-1)...(n-k+2)(n-k+1)$$

Theorem 4 *For $0 \leq k \leq n$, the number of k-element subsets of an n-element set is*

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Definition 1 (Multiset) *Let $n \geq 0$ and $t \geq 1$ be integers. A multiset of size n with elements of t types is a sequence of nonnegative integers $(m_1, ..., m_t)$ such that*

$$m_1 + m_2 + ... + m_t = n$$

Theorem 5 *For any $n \geq 0$ and $t \geq 1$, the number of n-element multisets with elements of t types is*

$$\binom{n+t-1}{t-1}$$

Definition 2 *Let $f : A \to B$ be a function from set A to set B.*

- *The function f is surjective if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$.*
- *The function f is injective if for every $a, a' \in A$, if $f(a) = f(a')$, then $a = a'$.*
- *The function f is bijective if it is both surjective and injective.*
- *The notation $A \rightleftharpoons B$ indicates there is a bijection between sets A and B.*

Theorem 6 *Let $f : A \to B$ and $g : B \to A$ be functions between two sets A and B. Assume the following:*

- $\forall a \in A$, $g(f(a)) = a$
- $\forall b \in B$, $f(g(b)) = b$

Then both f and g are bijections. Moreover, for $a \in A$ and $b \in B$, we have $f(a) = b$ iff $g(b) = a$.

Theorem 7 (Inclusion/Exclusion) *Let $A_1, A_2, ..., A_m$ be finite sets. Then*

$$|A_1 \cup A_2 \cup ... \cup A_m| = \sum_{\emptyset \neq S \subseteq \{1,...,m\}} (-1)^{|S|-1} |A_S|$$

Theorem 8 (Binomial Theorem) *For any natural number $n \in \mathbb{N}$,*

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Theorem 9 (Binomial Series) *For any positive integer $t \geq 1$,*

$$\frac{1}{(1-x)^t} = \sum_{n=0}^{\infty} \binom{n+t-1}{t-1} x^n$$

Definition 3 (Weight Function) *Let $\mathcal{A}$ be a set. A function $\omega : \mathcal{A} \to \mathbb{N}$ is a weight function provided that for all $n \in \mathbb{N}$,*

$$\mathcal{A}_n = \omega^{-1}(n) = \{\alpha \in \mathcal{A} : \omega(\alpha) = n\}$$

is finite.

Definition 4 (Generating Series) *Let $\mathcal{A}$ be a set with a weight function $\omega : \mathcal{A} \to \mathbb{N}$. The generating series of $\mathcal{A}$ with respect to ω is*

$$A(x) = \Phi_{\mathcal{A}}^{\omega}(x) = \sum_{\alpha \in \mathcal{A}} x^{\omega(\alpha)}$$

Theorem 10 *Let $\mathcal{A}$ be a set with a weight function $\omega : \mathcal{A} \to \mathbb{N}$, and let*

$$\Phi_{\mathcal{A}}(x) = \sum_{n=0}^{\infty} a_n x^n$$

For every $n \in \mathbb{N}$, the number of elements of $\mathcal{A}$ of weight n is $a_n = |\mathcal{A}_n|$.

Definition 5 *Let $G(x) = \sum_{n=0}^{\infty} g_n x^n$ be any power series. Then for any $k \in \mathbb{N}$,*

$$[x^k]G(x) = g_k$$

is the coefficient of x^k in the power series $G(x)$.

Lemma 11 (Sum Lemma) *Let $\mathcal{A}$ and $\mathcal{B}$ be disjoint sets, so that $\mathcal{A} \cap \mathcal{B} = \emptyset$. Assume that $\omega : (\mathcal{A} \cup \mathcal{B}) \to \mathbb{N}$ is a weight function on the union of $\mathcal{A}$ and $\mathcal{B}$. We may regard ω as a weight function on each of $\mathcal{A}$ or $\mathcal{B}$ separately (by restriction). Under these conditions,*

$$\Phi_{\mathcal{A} \cup \mathcal{B}}(x) = \Phi_{\mathcal{A}}(x) + \Phi_{\mathcal{B}}(x)$$

Lemma 12 (Infinite Sum Lemma) *Let $\mathcal{A}_0, \mathcal{A}_1, \ldots$ be pairwise disjoint sets (so that $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$ if $i \neq j$), and let $\mathcal{B} = \bigcup_{j=0}^{\infty} \mathcal{A}_j$. Assume that $\omega : \mathcal{B} \to \mathbb{N}$ is a weight function. We may regard ω as a weight function on each of the sets $\mathcal{A}_j$ separately (by restriction). Under these conditions,*

$$\Phi_{\mathcal{B}}(x) = \sum_{j=0}^{\infty} \Phi_{\mathcal{A}_j}(x)$$

Lemma 13 (Product Lemma) *Let $\mathcal{A}$ and $\mathcal{B}$ be sets with weight functions $\omega : \mathcal{A} \to \mathbb{N}$ and $v : \mathcal{B} \to \mathbb{N}$, respectively. Define $\eta : \mathcal{A} \times \mathcal{B} \to \mathbb{N}$ by putting $\eta(\alpha, \beta) = \omega(\alpha) + v(\beta)$ for all $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$. Then η is a weight function on $\mathcal{A} \times \mathcal{B}$, and*

$$\Phi^{\eta}_{\mathcal{A}\times\mathcal{B}}(x) = \Phi^{\omega}_{\mathcal{A}}(x) \cdot \Phi^{v}_{\mathcal{B}}(x)$$

Lemma 14 *Let $\mathcal{A}$ be a set with weight function $\omega : \mathcal{A} \to \mathbb{N}$, and define*

$$\mathcal{A}^* = \bigcup_{k=0}^{\infty} \mathcal{A}^k$$

with $\omega^* : \mathcal{A}^* \to \mathbb{N}$ where $\omega^* = \omega_k$ for each $k \in \mathbb{N}$. Then ω^* is a weight function on $\mathcal{A}^*$ if and only if there are no elements in $\mathcal{A}$ of weight zero (that is, $\mathcal{A}_0 = \emptyset$).

Lemma 15 (String Lemma) *Let $\mathcal{A}$ be a set with a weight function $\omega : \mathcal{A} \to \mathbb{N}$ such that there are no elements of $\mathcal{A}$ of weight zero. Then*

$$\Phi_{\mathcal{A}^*}(x) = \frac{1}{1 - \Phi_{\mathcal{A}}(x)}$$

Definition 6 (Composition) *A composition is a finite sequence of positive integers*

$$\gamma = (c_1, c_2, ..., c_k)$$

in which $k \in \mathbb{N}$ and each $c_i \geq 1$. The entries c_i are called the parts of the composition. The length of the composition is $l(\gamma) = k$, the number of parts. The size of the composition is

$$|\gamma| = c_1 + ... + c_k,$$

the sum of the parts.

Theorem 16 *Let $P = \{1, 2, 3, ...\}$.*

1. *The set $\mathcal{C}$ of all compositions is $\mathcal{C} = P^*$*
2. *The generating series for $\mathcal{C}$ with respect to size is*

$$\Phi_{\mathcal{C}}(x) = 1 + \frac{x}{1 - 2x}$$

3. *For each $n \in \mathbb{N}$, the number of compositions of size n is*

$$|\mathcal{C}_n| = \begin{cases} 1 & n = 0 \\ 2^{n-1} & n \geq 1 \end{cases}$$

Definition 7 (Binary String) *A binary string is a finite sequence $\sigma = b_1 b_2 ... b_n$ in which each bit b_i is either 0 or 1. The number of bits is the length of the string, denoted $l(\sigma) = n$. Thus, a binary string of length n is an element of the Cartesian power $\{0, 1\}^n$. A binary string of arbitrary length is an element of the set $\{0, 1\}^*$. There is exactly one binary string $\epsilon = ()$ of length zero, the empty string with no bits.*

Definition 8 (Regular Expression) *A regular expression is defined recursively, as follows.*

- *All of* $\epsilon, 0,$ *and* 1 *are regular expressions.*
- *If* R *and* S *are regular expressions, then so is* $\mathsf{R} \smile \mathsf{S}$
- *If* R *and* S *are regular expressions, then so is* RS
- *If* R *is a regular expression, then so is* R^*

Definition 9 (Concatenation Product) *Let* $\alpha = a_1 a_2 ... a_m, \beta = b_1 b_2 ... b_n \in \{0,1\}^*$ *be binary strings. The concatenation of* α *and* β *is*

$$\alpha\beta = a_1 a_2 ... a_m b_1 b_2 ... b_n$$

Let $\mathcal{A}, \mathcal{B} \subseteq \{0,1\}^*$ *be sets of binary strings. The concatenation product* $\mathcal{AB}$ *is the set*

$$\mathcal{AB} = \{\alpha\beta : \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$$

Definition 10 (Rational Language) *A rational language is a set* $\mathcal{R} \subseteq \{0,1\}^*$ *of binary strings that is produced by a regular expression; this is defined recursively as follows.*

- ϵ *produces* $\{\epsilon\}$*,* 1 *produces* $\{1\}$*, and* 0 *produces* $\{0\}$
- *If* R *produces* $\mathcal{R}$ *and* S *produces* $\mathcal{S}$*, then* $\mathsf{R} \smile \mathsf{S}$ *produces* $\mathcal{R} \cup \mathcal{S}$
- *If* R *produces* $\mathcal{R}$ *and* S *produces* $\mathcal{S}$*, then* RS *produces* $\mathcal{RS}$
- *If* R *produces* $\mathcal{R}$*, then* R^* *produces* $\mathcal{R}^*$

Definition 11 (Unambiguous Expression) *Let* R *be a regular expression that produces a rational language* $\mathcal{R}$*. Then* R *is unambiguous if every string in* $\mathcal{R}$ *is produced exactly once by* R*. If an expression is not unambiguous then it is ambiguous.*

Lemma 17 *Let* R *and* S *be unambiguous expressions producing the sets* $\mathcal{R}$ *and* $\mathcal{S}$*, respectively.*

- *The expressions* 0*,* 1*, and* ϵ *are unambiguous*
- *The expression* $\mathsf{R} \smile \mathsf{S}$ *is unambiguous if and only if* $\mathcal{R} \cap \mathcal{S} = \emptyset$
- *The expression* RS *is unambiguous if and only if there is a bijection* $\mathcal{RS} \rightleftharpoons \mathcal{R} \times \mathcal{S}$
- *The expression* R^* *is unambiguous if and only if each of the concatenation products* R^k *are unambiguous and the union* $\bigcup_{k=0}^{\infty} \mathcal{R}^k$ *is a disjoint union of sets.*

Definition 12 *A regular expression leads to a rational function; this is defined recursively, as follows. Assume that* R *and* S *are regular expressions that lead to* $R(x)$ *and* $S(x)$*, respectively.*

- ϵ *leads to 1,* 1 *leads to* x*, and* 0 *leads to* x
- *The expression* $\mathsf{R} \smile \mathsf{S}$ *leads to* $R(x) + S(x)$
- *The expression* RS *leads to* $R(x)S(x)$
- *The expression* R^* *leads to* $\frac{1}{1-R(x)}$

Theorem 18 *Let* R *be a regular expression producing the rational language* $\mathcal{R}$ *and leading to the rational function* $R(x)$*. If* R *is an unambiguous regular expression for* $\mathcal{R}$ *then* $R(x) = \Phi_{\mathcal{R}}(x)$*, the generating series for* $\mathcal{R}$ *with respect to length.*

Definition 13 (Block Decompositions) *The regular expressions*

$$0^*(1^*10^*0)^*1^* \text{ and } 1^*(0^*01^*1)0^*$$

are unambiguous expressions for the set $\{0,1\}^*$ *of all binary strings.*

Theorem 19 *Let* $\kappa \in \{0,1\}^*$ *be a non-empty string of length* n*, and let* $\mathcal{A} = \mathcal{A}_\kappa$ *be the set of binary strings that avoid* κ*. Let* $\mathcal{C}$ *be the set of all nonempty suffixes* γ *of* κ *such that* $\kappa\gamma = \eta\kappa$ *for some non-empty prefix* η *of* κ*. Let* $C(x) = \sum_{\gamma \in \mathcal{C}} x^{l(\gamma)}$*. Then,*

$$A(x) = \frac{1 + C(x)}{(1-2x)(1+C(x)) + x^n}$$

Definition 14 (Homogeneous Linear Recurrence Relation) *Let* $\mathbf{g} = (g_0, g_1, ...)$ *be an infinite sequence of complex numbers. Let* $a_1, a_2, ..., a_d$ *be in* $\mathbb{C}$*, and let* $N \geq d$ *be an integer. We say that* $\mathbf{g}$ *satisfies a homogeneous linear recurrence relation provided that*

$$g_n + a_1 g_{n-1} + ... + a_d g_{n-d} = 0$$

for all $n \geq N$*. The values* $g_0, ..., g_{N-1}$ *are initial conditions of the recurrence. The relation is linear because the LHS is a linear combination of the entries of the sequence* $\mathbf{g}$*; it is homogeneous because the RHS of the equation is zero.*

Theorem 20 *Let* $\mathbf{g} = (g_0, g_1, ...)$ *be an infinite sequence of complex numbers, and let* $G(x) = \sum_{n\geq 0} g_n x^n$ *be the corresponding generating series. The following are equivalent.*

- *The sequence* $\mathbf{g}$ *satisfies a homogeneous linear recurrence relation*

$$g_n + a_1 g_{n-1} + ... + a_d g_{n-d} = 0$$

  for all $n \geq N$ *with initial conditions* $g_0, g_1, ..., g_{N-1}$

- *The series* $G(x) = P(x)/Q(x)$ *is a quotient of two polynomials. The denominator is*

$$Q(x) = 1 + a_1 x + ... + a_d x^d$$

  and the numerator is

$$P(x) = b_0 + b_1 x + ... + b_{N-1} x^{N-1}$$

  in which $b_k = g_k + a_1 g_{k-1} + ... + a_d g_{k-d}$ *for all* $0 \leq k \leq N-1$*, with the convention that* $g_n = 0$ *for all* $n < 0$*.*

Theorem 21 *Let* $G(x) = P(x)/Q(x)$ *be a rational function in which* $\deg P < \deg Q$ *and the constant term of* $Q(x)$ *is 1. Factor the denominator to obtain its inverse roots:*

$$Q(x) = (1 - \lambda_1 x)^{d_1} ... (1 - \lambda_s x)^{d_s}$$

in which $\lambda_1, ..., \lambda_s$ *are distinct nonzero complex numbers and* $d_1 + ... + d_s = d = \deg Q$*. Then, there are* d *complex numbers*

$$C_1^{(1)}, ..., C_1^{(d_1)}; ...; C_s^{(1)}, ..., C_s^{(d_s)}$$

such that

$$G(x) = \frac{P(x)}{Q(x)} = \sum_{i=1}^{s} \sum_{j=1}^{d_s} \frac{C_i^{(j)}}{(1 - \lambda_i x)^j}$$

Theorem 22 *Let* $\mathbf{g} = (g_0, g_1, ...)$ *be an infinite sequence of complex numbers, and let* $G(x) = \sum_{n\geq 0} g_n x^n$ *be the corresponding generating series. Assume that the equivalent conditions of Theorem 20 hold, and that*

$$G(x) = R(x) + \frac{P(x)}{Q(x)}$$

for some polynomials $P(x)$*,* $Q(x)$*, and* $R(x)$ *with* $\deg P < \deg Q$ *and* $Q(0) = 1$*. Factor* $Q(x)$ *to obtain its inverse roots and their multiplicities:*

$$Q(x) = (1 - \lambda_1 x)^{d_1} ... (1 - \lambda_s x)^{d_s}$$

Then there are polynomials $p_i(n)$ *for* $1 \leq i \leq s$ *with* $\deg p_i < d_i$ *such that for all* $n > \deg R(x)$*,*

$$g_n = p_1(n)\lambda_1^n + ... + p_s(n)\lambda_s^n$$

Theorem 23 *Let* $\mathbf{g} = (g_0, g_1, ...)$ *be an infinite sequence of complex numbers. The following are equivalent.*

- *The sequence* $\mathbf{g}$ *satisfies a homogeneous linear recurrence relation (with initial conditions)*
- *The sequence* $\mathbf{g}$ *satisfies a possibly inhomogeneous linear recurrence relation (with initial conditions) in which the RHS is an eventually polyexp function*
- *The generating series* $G(x) = \sum_{n \geq 0} g_n x^n$ *is a rational function*
- *The sequence* $\mathbf{g}$ *is an eventually polyexp function*

Definition 15 *The sequence* $\mathbf{g}$ *satisfies a quadratic recurrence if its generating series* $G(x)$ *satisfies a quadratic equation:*

$$A(x)G(x)^2 + B(x)G(x) + C(x) = 0$$

where $A, B,$ *and* C *are power series in* x*.*

Definition 16 (Graph) *A graph* G *is a finite non-empty set,* $V(G)$*, of vertices, together with a set,* $E(G)$*, of unordered pairs of distinct vertices called edges.*

Definition 17 (Isomorphism) *Two graphs* G_1 *and* G_2 *are isomorphic if there exists a bijection* $f : V(G_1) \to V(G_2)$ *such that vertices* $f(u)$ *and* $f(v)$ *are adjacent in* G_2 *if and only if* u *and* v *are adjacent in* G_1*.*

Theorem 24 (Handshaking Lemma) *For any graph* G *we have*

$$\sum_{v \in V(G)} \deg v = 2|E(G)|$$

Corollary 24.1 *The number of vertices of odd degree in a graph is even.*

Corollary 24.2 *The average degree of a vertex in the graph* G *is*

$$\frac{2|E(G)|}{|V(G)|}$$

Definition 18 (Complete Graph) *A complete graph is one in which all pairs of distinct vertices are adjacent. The complete graph with* p *vertices is denoted by* K_p*,* $p \geq 1$*.*

Definition 19 (Bipartite Graph) *A graph in which the vertices can be partitioned into two sets* A *and* B*, so that all edges join a vertex in* A *to a vertex in* B*, is called a bipartite graph, with bipartition* (A, B)*.*

Definition 20 (Complete Bipartite Graph) *The complete bipartite graph* $K_{m,n}$ *has all vertices in* A *adjacent to all vertices in* B*, with* $|A| = m$*,* $|B| = n$*.*

Definition 21 (N-cube) *For $n \geq 0$, the n-cube is the graph whose vertices are the $\{0,1\}$-strings of length n, and two strings are adjacent iff they differ in exactly one position.*

Definition 22 (Adjacency Matrix) *The adjacency matrix of graph G having vertices $v_1, ..., v_p$ is the $p \times p$ matrix $A = [a_{ij}]$ where*

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

Definition 23 (Incidence Matrix) *The incidence matrix of graph G having vertices $v_1, ..., v_p$ and edges $e_1, ..., e_q$ is the $p \times q$ matrix $B = [b_{ij}]$ where*

$$b_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is incident with } e_j \\ 0 & \text{otherwise} \end{cases}$$

Definition 24 (Subgraph) *A subgraph of G is a graph whose vertex set is a subset U of $V(G)$ and whose edge set is a subset of those edges of G that have both vertices in U. If subgraph H has all vertices of graph G, then H is a spanning subgraph of G. If subgraph H is not equal to G, then H is a proper subgraph of G.*

Definition 25 (Walk) *A walk in graph G from v_0 to v_n for $n \geq 0$ is an alternating sequence of vertices and edges of G. The length of a walk is the number of edges in it. We say a walk is closed if $v_0 = v_n$.*

Definition 26 (Path) *A path is a walk in which all the vertices (and edges) are distinct.*

Theorem 25 *If there is a walk from vertex x to vertex y in G, then there is also a path from x to y in G.*

Theorem 26 *Let x, y, and z be vertices of G. If there is a path from x to y in G and a path from y to z in G, then there is a path from x to z in G.*

Definition 27 (Cycle) *A cycle in graph G is a subgraph with n distinct vertices $v_0, v_1, ..., v_{n-1}$ for $n \geq 1$, and n distinct edges $\{v_0, v_1\}$, $\{v_1, v_2\}$, ..., $\{v_{n-1}, v_0\}$. Equivalently, a cycle is a connected graph that is regular of degree two. A spanning cycle is called a Hamiltonian cycle.*

Theorem 27 *If every vertex in G has degree at least 2, then G contains a cycle.*

Definition 28 (Girth) *The girth of a graph G is the length of the shortest cycle in G, and is denoted by $g(G)$.*

Definition 29 (Connected) *A graph G is connected if, for each two vertices x and y, there is a path from x to y.*

Theorem 28 *Let G be a graph and let v be a vertex in G. If for each vertex w in G there is a path from v to w in G, then G is connected.*

Definition 30 (Component) *A component of G is a subgraph C of G such that C is connected and is maximal subject to being connected.*

Definition 31 (Cut) *Given a subset X of the vertices of graph G, the cut induced by X is the set of edges that have exactly one end in X.*

Theorem 29 *A graph G is not connected iff there exists a proper non-empty subset X of $V(G)$ such that the cut induced by X is empty.*

Definition 32 (Eulerian Circuit) *An Eulerian circuit of a graph G is a closed walk that contains every edge of G exactly once.*

Theorem 30 *Let G be a connected graph. Then G has an Eulerian circuit iff every vertex has even degree.*

Definition 33 (Bridge) *An edge e of G is a bridge if $G - e$ has more components than G.*

Theorem 31 *If $e = \{x, y\}$ is a bridge of a connected graph G, then $G - e$ has precisely two components; furthermore, x and y are in different components.*

Theorem 32 *An edge e is a bridge of a graph G iff it is not contained in any cycle of G.*

Theorem 33 *If there are two distinct paths from vertex u to vertex v in G, then G contains a cycle.*

Definition 34 (Tree) *A tree is a connected graph with no cycles.*

Definition 35 (Forest) *A forest is a graph with no cycles.*

Theorem 34 *If u and v are vertices in a tree T, then there is a unique u, v-path in T.*

Theorem 35 *Every edge of a tree T is a bridge.*

Theorem 36 *If T is a tree, then $|E(T)| = |V(T)| - 1$.*

Theorem 37 *If G is a forest with k components, then $|E(G)| = |V(G)| - k$.*

Definition 36 (Leaf) *A leaf in a tree is a vertex of degree 1.*

Theorem 38 *A tree with at least two vertices has at least two leaves.*

Theorem 39 *A graph G is connected iff it has a spanning tree.*

Theorem 40 *If G is connected, with p vertices and $q = p - 1$ edges, then G is a tree.*

Theorem 41 *If T is a spanning tree of G and e is an edge not in T, then $T+e$ contains exactly one cycle C. Moreover, if e' is any edge on C, then $T + e - e'$ is also a spanning tree of G.*

Theorem 42 *If T is a spanning tree of G and e is an edge in T, then $T - e$ has 2 components. If e' is in the cut induced by one of the components, then $T - e + e'$ is also a spanning tree of G.*

Theorem 43 *An odd cycle is not bipartite.*

Theorem 44 *A graph is bipartite iff it contains no odd cycles.*

Definition 37 (Planar) *A graph G is planar if it has a drawing in the plane so that its edges intersect only at their ends, and so that no two vertices coincide. The actual drawing is called a planar embedding of G, or a planar map.*

Theorem 45 *If we have a planar embedding of a connected graph G with faces $f_1, ..., f_s$, then*

$$\sum_{i=1}^{s} \deg(f_i) = 2|E(G)|$$

Theorem 46 *If the connected graph G has a planar embedding with f faces, the average degree of a face in the embedding is*

$$\frac{2|E(G)|}{f}$$

Theorem 47 (Euler's Formula) *Let G be a connected graph with p vertices and q edges. If G has a planar embedding with f faces, then*

$$p - q + f = 2$$

Theorem 48 *A graph is planar if and only if it can be drawn on the surface of a sphere.*

Theorem 49 *There are exactly 5 platonic graphs.*

Theorem 50 *Let G be a planar embedding with p vertices, q edges, and s faces, in which each vertex has degree $d \geq 3$ and each face has degree $d^* \geq 3$. Then (d, d^*) is one of the 5 pairs*

$$\{(3,3), (3,4), (4,3), (3,5), (5,3)\}$$

Theorem 51 *If G is a platonic graph with p vertices, q edges, and f faces, where each vertex has degree d and each face has degree d^*, then*

$$q = \frac{2dd^*}{2d + 2d^* - dd^*}$$

Theorem 52 *If G contains a cycle, then in a planar embedding of G, the boundary of each face contains a cycle.*

Theorem 53 *Let G be a planar embedding with p vertices and q edges. If each face of G has degree at least d^*, then*

$$(d^* - 2)q \leq d^*(p - 2)$$

Theorem 54 *In a planar graph G with $p \geq 3$ vertices and q edges, we have*

$$q \leq 3p - 6$$

Theorem 55 *K_5 is not planar.*

Theorem 56 *A planar graph has a vertex of degree at most five.*

Theorem 57 *In a bipartite planar graph G with $p \geq 3$ vertices and q edges, we have*

$$2 \leq 2p - 4$$

Theorem 58 *$K_{3,3}$ is not planar.*

Definition 38 (Edge Subdivision) *An edge subdivision of graph G is obtained by replacing each edge e of G:*

- *Replace e with a path of length $m \geq 1$*
- *If $m = 1$, then e is unchanged.*
- *If $m > 1$, then there are $m - 1$ new vertices and $m - 1$ new edges.*

Theorem 59 (Kuratowski's Theorem) *A graph is not planar iff it has a subgraph that is an edge subdivision of K_5 or $K_{3,3}$.*

Definition 39 (k-colouring) *A k-colouring of graph G is a function from $V(G)$ to a set of size k whose elements are called colours. Adjacent vertices have different colours. A graph with a k-colouring is called a k-colourable graph.*

Theorem 60 *A graph is 2-colourable iff it is bipartite.*

Theorem 61 *K_n is n-colourable and not k-colourable for any $k < n$.*

Theorem 62 *Every planar graph is 6-colourable.*

Definition 40 (Contracting) *Let G be a graph and let $e = \{x, y\}$ be an edge of G. The graph G/e obtained from G by contracting the edge e is the graph with vertex set $V(G) \setminus \{x, y\} \cup \{z\}$ where z is a new vertex, and edge set*

$$\{\{u, v\} \in E(G) : \{u, v\} \cap \{x, y\} = \emptyset\} \cup \{\{u, z\} : u \notin \{x, y\}, \{u, w\} \in E(G) \text{ for some } w \in \{x, y\}\}$$

Intuitively, we can think of the operation of contracting e as allowing the "length" of e to decrease to 0, so that the vertices x and y are identified into a new vertex z. Any other vertex that was adjacent to one (or both) of x and y is adjacent to z in the new graph G/e.

Theorem 63 *Every planar graph is* 5*-colourable.*

Theorem 64 *Every planar graph is* 4*-colourable.*

Definition 41 (Matching) *A matching of graph G is a set of M edges of G such that no two edges in M have a common end. We say that vertex $v \in V(G)$ is saturated by M if v is incident to an edge in M. A perfect matching saturates every vertex.*

Definition 42 (Alternating Path) *A path is alternating with respect to matching M of graph G if edges alternate between being in M and not being in M.*

Definition 43 (Augmenting Path) *An augmenting path is an alternating path that joins two distinct vertices, neither of them being saturated by M.*

Theorem 65 *If M has an augmenting path, it is not a maximum matching.*

Definition 44 (Cover) *A cover of graph G is a set C of vertices such that every edge of G has at least one end in C.*

Theorem 66 *If M is a matching of graph G and C is a cover of G, then $|M| \leq |C|$.*

Theorem 67 *If M is a matching and C is a cover and $|M| = |C|$, then M is a maximum matching and C is a minimum cover.*

Theorem 68 (Konig's Theorem) *In a bipartite graph the maximum size of a matching is the minimum size of a cover.*

Theorem 69 *Let X_0 be the set of vertices in A not saturated by M and let Z denote the set of vertices in G that are joined by to a vertex in X_0 by an alternating path. Let M be a matching of bipartite graph G with bipartition A, B, and let $X = A \cap Z$ and $Y = B \cap Z$. Then,*

1. *There is no edge of G from X to $B - Y$*
2. *$C = Y \cup (A - X)$ is a cover of G*
3. *There is no edge of M from Y to $A - X$*
4. *$|M| = |C| - |U|$ where U is the set of unsaturated vertices in Y*
5. *There is an augmenting path to each vertex in U*

Theorem 70 (Hall's Theorem) *A bipartite graph G with bipartition A, B has a matching saturating every vertex in A, if and only if every subset D of A satisfies*

$$|N(D)| \geq |D|.$$

Theorem 71 (Hall's SDR Theorem) *The collection $Q_1, Q_2, ..., Q_n$ of subsets of the finite set Q has an SDR if and only if, for every subset J of $\{1, 2, ..., n\}$, we have*

$$\left| \bigcup_{i \in J} Q_i \right| \geq |J|$$

Theorem 72 *A bipartite graph G with bipartition A, B has a perfect matching if and only if $|A| = |B|$ and every subset D of A satisfies*

$$|N(D)| \geq |D|$$

Theorem 73 *If G is a k-regular bipartite graph with $k \geq 1$, then G has a perfect matching.*

Theorem 74 *A bipartite graph with maximum degree Δ has an edge Δ-colouring.*

Theorem 75 *Let G be a bipartite graph having at least one edge. Then G has a matching saturating each vertex of maximum degree.*

Theorem 76 *Let G be a graph with q edges, and suppose k, m are positive integers such that G has an edge k-colouring and $q \leq km$. Then G has an edge k-colouring in which every colour is used at most m times.*

Theorem 77 *In a bipartite graph G, there is an edge k-colouring in which each colour is used at most m times if and only if $\Delta \leq k$ and $q \leq km$.*