Math 138 Reference

Definition 1 ***Riemann Sum*** *Given a bounded function f on $[a, b]$, a partition P*

$$a = t_0 < t_1 < t_2 < ... < t_{i-1} < t_i < ... < t_{n-1} < t_n = b$$

of $[a, b]$, and a set $\{c_1, c_2, ..., c_n\}$ where $c_i \in [t_{i-1}, t_i]$, then a Riemann sum for f with respect to P is a sum of the form

$$S = \sum_{i=1}^{n} f(c_i) \triangle t_i$$

Definition 2 ***Regular n-Partition*** *Given an interval $[a, b]$ and an $n \in \mathbb{N}$, the regular n-partition of $[a, b]$ is the partition $P^{(n)}$ with*

$$a = t_0 < t_1 < t_2 < ... < t_{i-1} < t_i < ... < t_{n-1} < t_n = b$$

of $[a, b]$ where each subinterval has the same length $\triangle t_i = \dfrac{b-a}{n}$

Definition 3 ***Right-hand Riemann Sum*** *The right-hand Riemann sum for f with respect to the partition P is the Riemann Sum R obtained from P by choosing c_i to be t_i, the right-hand endpoint of $[t_{i-1}, t_i]$. That is*

$$R = \sum_{i=1}^{n} f(t_i) \triangle t_i$$

If $P^{(n)}$ is the regular n-partition, we denote the right-hand Riemann sum by

$$R_n = \sum_{i=1}^{n} f(t_i) \triangle t_i = \sum_{i=1}^{n} f(t_i) \frac{b-a}{n} = \sum_{i=1}^{n} f(a + i(\frac{b-a}{n}))(\frac{b-a}{n})$$

Definition 4 ***Left-hand Riemann Sum*** *The left-hand Riemann sum for f with respect to the partition P is the Riemann Sum L obtained from P by choosing c_i to be t_{i-1}, the left-hand endpoint of $[t_{i-1}, t_i]$. That is*

$$L = \sum_{i=1}^{n} f(t_{i-1}) \triangle t_i$$

If $P^{(n)}$ is the regular n-partition, we denote the left-hand Riemann sum by

$$L_n = \sum_{i=1}^{n} f(t_{i-1}) \triangle t_i = \sum_{i=1}^{n} f(t_{i-1}) \frac{b-a}{n} = \sum_{i=1}^{n} f(a + (i-1)(\frac{b-a}{n}))(\frac{b-a}{n})$$

Definition 5 ***Definite Integral*** *We say that a bounded function f is integrable on $[a, b]$ if there exists a unique number $I \in \mathbb{R}$ such that if whenever $\{P_n\}$ is a sequence of partitions with $\lim_{n\to\infty} ||P_n|| = 0$ and $\{S_n\}$ is any sequence of Riemann sums associated with the P_n's, we have*

$$\lim_{n\to\infty} S_n = I$$

In this case, we call I the integral of f over $[a, b]$ and denote it by

$$\int_a^b f(t)dt$$

The points a and b are called the limits of integration and the function $f(t)$ is called the integrand. The variable t is called the variable of integration.

Theorem 1 ***Integrability Theorem for Continuous Functions*** *Let f be continuous on $[a, b]$. Then f is integrable on $[a, b]$. Moreover,*

$$\int_a^b f(t)dt = \lim_{n\to\infty} S_n$$

where

$$S_n = \sum_{i=1}^{n} f(c_i)\triangle t_i$$

is any Riemann sum associated with the regular n-partitions. In particular,

$$\int_a^b f(t)dt = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i)\frac{b-a}{n}$$

$$\int_a^b f(t)dt = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_{i-1})\frac{b-a}{n}$$

Theorem 2 ***Properties of Integrals*** *Assume that f and g are integrable on the interval $[a, b]$. Then:*

1. *For any $c \in \mathbb{R}$, $\int_a^b cf(t)dt = c\int_a^b f(t)dt$*
2. *$\int_a^b (f + g)(t)dt = \int_a^b f(t)dt + \int_a^b g(t)dt$*
3. *If $m \le f(t) \le M$ for all $t \in [a, b]$, then $m(b - a) \le \int_a^b f(t)dt \le M(b - a)$*
4. *If $0 \le f(t)$ for all $t \in [a, b]$, then $0 \le \int_a^b f(t)dt$*
5. *If $g(t) \le f(t)$ for all $t \in [a, b]$, then $\int_a^b g(t)dt \le \int_a^b f(t)dt$*
6. *The function $|f|$ is integrable on $[a, b]$ and $|\int_a^b f(t)dt| \le \int_a^b |f(t)|dt$*

Definition 6 ***Identical Limits of Integration*** *Let $f(t)$ be defined at $t = a$. Then we define*

$$\int_a^a f(t)dt = 0$$

Definition 7 ***Switching the Limits of Integration*** *Let f be integrable on the interval $[a, b]$ where $a < b$. Then we define*

$$\int_b^a f(t)dt = -\int_a^b f(t)dt$$

Theorem 3 ***Integrals over Subintervals*** *Assume that f is integrable on an interval I containing a, b, and c. Then*

$$\int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt$$

Definition 8 ***Average Value of*** f *If f is continuous on $[a, b]$, the average value of f on $[a, b]$ is defined as*

$$\frac{1}{b-a}\int_a^b f(t)dt$$

Theorem 4 ***Average Value Theorem (MVT for Integrals)*** *Assume that f is continuous on $[a, b]$. Then there exists $a \leq c \leq b$ such that*

$$f(c) = \frac{1}{b-a}\int_a^b f(t)dt$$

Theorem 5 ***Fundamental Theorem of Calculus (Part 1)*** *Assume that f is continuous on an open interval I containing point a. Let*

$$G(x) = \int_a^x f(t)dt$$

Then $G(x)$ is differentiable at each $x \in I$ and $G'(x) = f(x)$. Equivalently,

$$G'(x) = \frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Theorem 6 ***Extended Fundamental Theorem of Calculus*** *Assume that f is continuous and that g and h are differentiable. Let*

$$H(x) = \int_{g(x)}^{h(x)} f(t)dt$$

Then $H(x)$ is differentiable and

$$H'(x) = f(h(x))h'(x) - f(g(x))g'(x)$$

Definition 9 ***Antiderivative*** *Given a function f, an antiderivative is a function F such that $F'(x) = f(x)$. If for an interval I, $\forall x \in I, F'(x) = f(x)$, then we say F is an antiderivative for f on I.*

Theorem 7 ***Power Rule for Antiderivatives*** *If $\alpha \neq 1$, then*

$$\int x^{\alpha} dx = \frac{x^{\alpha+1}}{\alpha+1} + C$$

Theorem 8 ***Fundamental Theorem of Calculus (Part 2)*** *Assume that f is continuous and that F is any antiderivative of f. Then*

$$\int_a^b f(t)dt = F(b) - F(a)$$

Theorem 9 ***Change of Variables*** *Assume that $g'(x)$ is continuous on $[a, b]$ and $f(u)$ is continuous on $g([a, b])$. Then*

$$\int_{x=a}^{x=b} f(g(x))g'(x)dx = \int_{u=g(a)}^{u=g(b)} f(u)du$$