MA1101R Cheatsheet 19/20 Semester 1 Mid-term
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Matrices

Theorem 1.2.7. If **augmented matrices** of two systems of linear equations are row equivalent, then the two systems have the same set of solutions. (* Even for two homogeneous linear systems, we still need to say that $(\boldsymbol{A} \mid \boldsymbol{0})$ is row equivalent to $(\boldsymbol{B} \mid \boldsymbol{0})$, not that $\boldsymbol{A}$ is row equivalent to $\boldsymbol{B}$.)

Example 1.4.10. Suppose augmented matrix $\boldsymbol{R}$ is in (R)REF:

1. LS has no solution
   $\iff$ Last column of $\boldsymbol{R}$ is pivot.
2. LS has one unique solution
   $\iff$ **Only** last column of $\boldsymbol{R}$ is non-pivot.
3. LS has infinite number of solution
   $\iff$ At least one column other than the last one is non-pivot
   $\iff$ Number of variables > Number of non-zero rows in $\boldsymbol{R}$
   (* # non-pivot columns in (R)REF -1 = # unique solutions)

Definition 2.3.2, Theorem 2.4.7 & 2.5.19. $\boldsymbol{A}$ is invertible when:

1. $\exists \boldsymbol{B}$ s.t. $\boldsymbol{AB} = \boldsymbol{I} \vee \boldsymbol{BA} = \boldsymbol{I}$
2. Refer to **Theorem 2.4.7.2** below
3. $\text{rref}(\boldsymbol{A}) = \boldsymbol{I}$
4. $\det(\boldsymbol{A}) \neq 0$
5. $\boldsymbol{A}$ is a product of elementary matrices
6. Rows of $\boldsymbol{A}$ is a basis of $\mathbb{R}^n$
7. Columns of $\boldsymbol{A}$ is a basis of $\mathbb{R}^n$

Remark 2.3.4 (Cancellation Laws for Matrices). Let $\boldsymbol{A}$ be an invertible $m \times m$ matrix,

(a) If $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are $m \times n$ matrices with $\boldsymbol{AB_1} = \boldsymbol{AB_2}$, then $\boldsymbol{B}_1 = \boldsymbol{B}_2$

(b) If $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are $n \times m$ matrices with $\boldsymbol{C_1A} = \boldsymbol{C_2A}$, then $\boldsymbol{C}_1 = \boldsymbol{C}_2$

Theorem 2.4.7.2 (generalised). Relationship between singularity of $\boldsymbol{A}$ and the number of solutions of a linear system $\boldsymbol{Ax} = \boldsymbol{b}$:

1. $\boldsymbol{A}$ is singular $\iff$ $\boldsymbol{Ax} = \boldsymbol{b}$: has ∞ solutions (only case for homogeneous LS) or no solutions
2. $\boldsymbol{A}$ is invertible $\iff$ $\boldsymbol{Ax} = \boldsymbol{b}$: has one unique solution (trivial solution for homogeneous LS)

Definition 2.5.2. Let $\boldsymbol{A} = (a_{ij})$ be an $n \times n$ matrix. Let $\boldsymbol{M_{ij}}$ be an $(n-1) \times (n-1)$ matrix obtained from $\boldsymbol{A}$ by deleting the ith row and the jth column. Then the *determinant* of $\boldsymbol{A}$ is defined as

$$\det(\boldsymbol{A}) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + \cdots + a_{1n}A_{1n} & \text{if } n > 1 \end{cases}$$

where

$$A_{ij} = (-1)^{i+j} \det(\boldsymbol{M_{ij}})$$

The number A_{ij} is called the (i,j)-*cofactor* of $\boldsymbol{A}$.

Theorem 2.5.8. The determinant of a triangular matrix is equal to the product of its diagonal entries.

Theorem 2.5.12 (added-on). The determinant of a square matrix is 0 when:

1. it has two identical rows, or
2. it has two identical columns
3. any row/column of its (R)REF is zero

Theorem 2.5.15. Let $\boldsymbol{A}$ be a square matrix. k is a non-zero constant.

1. $\boldsymbol{A} \xrightarrow{k\boldsymbol{R}_i} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = k\det(\boldsymbol{A})$
2. $\boldsymbol{A} \xrightarrow{\boldsymbol{R}_i \leftrightarrow \boldsymbol{R}_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = -\det(\boldsymbol{A})$
3. $\boldsymbol{A} \xrightarrow{\boldsymbol{R}_i + k\boldsymbol{R}_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = \det(\boldsymbol{A})$
4. Let $\boldsymbol{E}$ be an elementary matrix of the same size as $\boldsymbol{A}$. Then $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

Remark 2.5.18. Since $\det(\boldsymbol{A}$ $= \det(A^T)$, theorem 2.5.15 holds if "rows" are changed to "columns".

Theorem 2.5.22. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ are two square matrices of order n and c is a scalar. Then

1. $\det(c\boldsymbol{A}) = c^n \det(\boldsymbol{A})$
2. $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$
3. if $\boldsymbol{A}$ is invertible, $\det(\boldsymbol{A}^-1) = \frac{1}{\det(\boldsymbol{A})}$

Definition 2.5.24. Let $\boldsymbol{A}$ be a square matrix of order n. Then the *(classical) adjoint* of $\boldsymbol{A}$ is the $n \times n$ matrix

$$\textbf{adj}(\boldsymbol{A}) = (A_{ij})^T_{n \times n}$$

where A_{ij} is the (i,j)-cofactor of $\boldsymbol{A}$.

Theorem 2.5.25. If $\boldsymbol{A}$ is invertible, then $\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\textbf{adj}(\boldsymbol{A})$ (or written as: $\boldsymbol{A}[\textbf{adj}(\boldsymbol{A})] = \det(\boldsymbol{A})\boldsymbol{I}$).

Theorem 2.5.27 (Cramer's Rule). Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ is a linear system where $\boldsymbol{A}$ is an $n \times n$ matrix. Let $\boldsymbol{A_i}$ be the matrix obtained from $\boldsymbol{A}$ be replacing the ith column of $\boldsymbol{A}$ by $\boldsymbol{b}$. If $\boldsymbol{A}$ is invertible, then the system has only one solution

$$\boldsymbol{x} = \frac{1}{\det(\boldsymbol{A})} \begin{pmatrix} \det(\boldsymbol{A_1}) \\ \vdots \\ \det(\boldsymbol{A_n}) \end{pmatrix}$$

Mixed Notes 1. $\boldsymbol{A}^{-1}$ is able to be computed by:

1. Find $\boldsymbol{B}$ s.t. $\boldsymbol{AB} = \boldsymbol{I} \vee \boldsymbol{BA} = \boldsymbol{I}$
2. Find using **Theorem 2.5.25**
3. Find using: $(\boldsymbol{A} \mid \boldsymbol{I}) \xrightarrow{GJE} (\boldsymbol{I} \mid \boldsymbol{A}^{-1})$

Mixed Notes 2. $\det(\boldsymbol{A})$ is able to be computed by:

1. Using **Theorem 2.5.2**
2. Using cross multiplication (for 2×2 and 3×3 matrices only)
3. Doing some ERO (e.g. GE, consider **Thoerem 2.5.15**) and making it triangular then using **Theorem 2.5.8** or making it have properties in **Theorem 2.5.12**
4. Using **Theorem 2.5.22**

Mixed Notes 3. Some random notes:

1. In $\mathbb{R}^n$ where $n \geq 2$, a set with 1 parameter is a line and that with 2 parameters is a space.
2. $\boldsymbol{M}^2 + \boldsymbol{M} = \boldsymbol{I} \Rightarrow \boldsymbol{M}(\boldsymbol{M} + \boldsymbol{I}) = \boldsymbol{I}$ (Don't put that $\boldsymbol{I}$ to be scalar 1!)
3. Two matrices have same RREF $\Leftrightarrow$ They are row equivalent
4. In exam, express a matrix in the form $\boldsymbol{A} = (a_{ij})_{m \times n}$. **DO NOT** use dots form
5. When using ERO $\boldsymbol{R}_i = \frac{1}{k}\boldsymbol{R}_j$, discuss whether k is 0 when necessary

Mixed Notes 4. When we are asked to use Gaussian Elimination or Gauss-Jordan Elimination, steps in presentation is important and only these elementary row operations should be used:

1. (For GE) $\boldsymbol{R}_i \leftrightarrow \boldsymbol{R}_j$, where $i > j$.
2. (For GE) $\boldsymbol{R}_i + k\boldsymbol{R}_j$, where $k \in \mathbb{R} \wedge i > j$.
3. (For GJE) $\boldsymbol{R}_i + k\boldsymbol{R}_j$, where $k \in \mathbb{R} \wedge i < j$.

Mixed Notes 5. Generally, for (square) matrices $\boldsymbol{A}$ and $\boldsymbol{B}$,

1. $\boldsymbol{AB} \neq \boldsymbol{BA}$
2. $(\boldsymbol{AB})^2 \neq \boldsymbol{A}^2\boldsymbol{B}^2$
3. $\boldsymbol{AB} = 0 \nRightarrow \boldsymbol{A} = 0 \vee \boldsymbol{B} = 0$
4. $\boldsymbol{A}^2 = I \nRightarrow \boldsymbol{A} = \pm\boldsymbol{I}$ (For example: 2 EMs of 2nd type ERO)

Mixed Notes 5. When expanding a row/column with cofactors of the other row/column, 0 will be yielded:

$$\sum_{m=1}^{n} a_{im}A_{jm} = \sum_{m=1}^{n} a_{mi}A_{mj} = 0, \text{ for some } i \neq j$$

This can be proven by the following steps:

1. Consider $X = \sum_{m=1}^{n} a_{im}A_{jm}$, known value of A_{jm} and X does not depend on values of row j.
2. Create a new matrix by replacing j-th row of $\boldsymbol{A}$ with its i-th row, named it $\boldsymbol{A'}$. We then have $a'_{im} = a_{im}$ and $a'_{jm} = a'_{im}$. At the same time, by (1), $A'_{jm} = A_{jm}$
3. Then $X = \sum_{m=1}^{n} a'_{im}A'_{jm} = \sum_{m=1}^{n} a'_{jm}A'_{jm} = \det(\boldsymbol{A'}) = 0$ since two of the rows of (A') are the same, by **Theorem 2.5.12.1**.
4. Consider $\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$ and the above steps $\sum_{m=1}^{n} a_{mi}A_{mj} = 0$.

Euclidean Spaces

Definition 3.2.3. Let $S = \{\boldsymbol{u_1}, \dots, \boldsymbol{u_k}\}$ be a set of vectors in $\mathbb{R}^n$. Then the set of all linear combinations of $\boldsymbol{u_1}, \dots, \boldsymbol{u_k}$,

$$\{c_1\boldsymbol{u_1} + \cdots + c_k\boldsymbol{u_k} \mid c_1, \dots, c_k \in \mathbb{R}\}$$

is called the *linear span* of S (or the *linear span* of $\boldsymbol{u_1}, \dots, \boldsymbol{u_k}$) and is denoted by $\text{span}(S)$ (or $\text{span}\{\boldsymbol{u_1}, \dots, \boldsymbol{u_k}\}$).

Discussion 3.2.5. Given $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \dots, \boldsymbol{v_m}\} \subseteq \mathbb{R}^n\}$, show $\text{span}(S) = \mathbb{R}^n$:

Consider $\boldsymbol{v_i} = (v_{i1}, \dots, v_{in})$,

$$\begin{pmatrix} \boldsymbol{v_{11}} & \dots & \boldsymbol{v_{m1}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{v_{1n}} & \dots & \boldsymbol{v_{mn}} \end{pmatrix} \xrightarrow{GE} \boldsymbol{R}$$

$\text{span}(S) = \mathbb{R}^n \iff \boldsymbol{R}$ has no zero rows

Theorem 3.2.7. If $|S| < n$, $\text{span}(S) \neq \mathbb{R}^n$.

Theorem 3.2.10. Let $S_1 = \{\boldsymbol{u_1}, \dots, \boldsymbol{u_k}\}$ and $S_2 = \{\boldsymbol{v_1}, \dots, \boldsymbol{v_m}\}$ be subsets of $\mathbb{R}^n$. Then, $\text{span}(S_1) \subseteq \text{span}(S_2) \iff \forall i = 1, 2, \dots, k$, $u_i \in \text{span}\{\boldsymbol{v_1}, \dots, \boldsymbol{v_m}\}$.

In other words, consider $\boldsymbol{u_i} = (u_{i1}, \dots, u_{in})$ and $\boldsymbol{v_i} = (v_{i1}, \dots, v_{in})$,

$$\left(\begin{array}{ccc|c|c|c} \boldsymbol{v_{11}} & \dots & \boldsymbol{v_{m1}} & \boldsymbol{u_{11}} & \dots & \boldsymbol{u_{k1}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{v_{1n}} & \dots & \boldsymbol{v_{mn}} & \boldsymbol{u_{1n}} & \dots & \boldsymbol{u_{kn}} \end{array}\right) \xrightarrow{GE} \boldsymbol{R}$$

$\text{span}(S_1) \subseteq \text{span}(S_2) \iff \boldsymbol{R}$ has its last **k** columns non-pivot.

Definition 3.3.2. Let V be a subset of $\mathbb{R}^n$. Then V is called a *subspace* of $\mathbb{R}^n$ if $V = \text{span}(S)$ where $S = \{\boldsymbol{u_1}, \dots, \boldsymbol{u_k}\}$ for some vectors $\boldsymbol{u_1}, \dots, \boldsymbol{u_k} \in \mathbb{R}^n$.

More precisely, V is called the *subspace spanned* by S (or the *subspace spanned* by $\boldsymbol{u_1}, \dots, \boldsymbol{u_k}$). We also say that S *spans* (or $\boldsymbol{u_1}, \dots, \boldsymbol{u_k}$ *span*) the subspace V.

By contraposition, $V = \text{span}(S) \Rightarrow \boldsymbol{0} \in V \equiv \boldsymbol{0} \notin V \Rightarrow V \neq \text{span}(S)$. ($*$ i.e., If $\boldsymbol{0}$ is not in V, V is not a subspace of $\mathbb{R}^n$)

Theorem 3.3.6. If $V = \{\boldsymbol{x} | \boldsymbol{Ax} = \boldsymbol{0}\}$, V is a subspace of $\mathbb{R}^n$.

Remark 3.3.8. Let V be a non-empty subset of $\mathbb{R}^n$. Then V is a subspace of $\mathbb{R}^n$ if and only if

$$\text{for all } \boldsymbol{u}, \boldsymbol{v} \in V \text{ and } c, d \in \mathbb{R}, \ \ c\boldsymbol{u} + d\boldsymbol{v} \in V$$

($*$ This checks whether V is **closed** under addition and scalar multiplication)