MA1521 Cheat Sheet
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AY2019/20 Semester 1

# 1 MF26 Magic

### Trigo

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$
$$\sin P + \sin Q = 2 \sin \frac{1}{2}(P+Q) \cos \frac{1}{2}(P-Q)$$
$$\sin P - \sin Q = 2 \cos \frac{1}{2}(P+Q) \sin \frac{1}{2}(P-Q)$$
$$\cos P + \cos Q = 2 \cos \frac{1}{2}(P+Q) \cos \frac{1}{2}(P-Q)$$
$$\cos P - \cos Q = -2 \sin \frac{1}{2}(P+Q) \sin \frac{1}{2}(P-Q)$$
$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$
$$a^2 = b^2 + c^2 - 2bc \cos A$$

### Derivatives

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx} \tan^{-1} x = \frac{1}{1+x^2}$$
$$\frac{d}{dx} \csc x = -\csc x \cot x$$
$$\frac{d}{dx} \sec x = \sec x \tan x$$
$$\frac{d}{dx} \tan x = \sec^2 x$$
$$\frac{d}{dx} \cot x = -\csc^2 x$$

### Integrals

Take note of the absolute sign, and always remember to $+c$

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + c$$
$$\int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}(\frac{x}{a}) + c$$
$$\int \frac{1}{x} dx = \ln|x| + c$$
$$\int \frac{x}{\sqrt{a^2-x^2}} dx = -\sqrt{a^2-x^2} + c$$
$$\int x^n e^x dx = x^n e^x - nx^{n-1}e^x + n(n-1)x^{n-2}e^x - \cdots \pm n! e^x + c$$

# 2 Basics

## Extreme Values

Points where $f$ can have an extreme value:

- Interior point where $f'(x) = 0$
- Interior points where $f'(x)$ doesn't exist
- End points of the domain of $f$

## L'Hospital's Rule

The $\frac{0}{0}$ form: (1) $f$ and $g$ are differentiable in a neighborhood of $x_0$, (2) $f(x_0) = g(x_0) = 0$, (3) $g'(x) \neq 0$ except possibly at $x_0$

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

E.g. $\lim_{x \to 0} \frac{3x - \sin x}{x} = \frac{3 - \cos x}{1}|_{x=0} = 2$
The $\frac{\infty}{\infty}$ form: when $x \to a$, $f(x), g(x) \to \infty$, and both differentiable,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Else, change to these two forms. (e.g $\lim_{x \to 0^+} x \cot x = \lim_{x \to 0^+} \frac{x}{\tan x} = \lim_{x \to 0^+} \frac{1}{\sec^2 x} = 1$ )

## Fundamental Theorem of Calculus

$$\frac{d}{d\Box} \int_c^{\Box} f(t) dt = f(\Box)$$

# 3 Series

## Geometric Series

Sum: $S_n = a\frac{1-r^n}{1-r}$, $r \neq 1$

Ratio test: $\lim_{n \to \infty} = \left|\frac{a_{n+1}}{a_n}\right| = \rho$;

1. $\rho < 1$: converge;
2. $\rho > 1$: diverge;
3. $\rho = 1$, no conclusion;

For convergent series: $S_n \to \frac{a}{1-r}$

## Power Series

$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + \ldots$, where $a$ is the center of the power series
Convergence: $n \to \infty, S_n \to k$

1. $\sum c_n (x-a)^n$ converges at $x = a$ and diverges elsewhere
2. $h \in \mathbb{Z}$ that the series only converges in $(a-h, a+h)$
3. converges for every $x$

## Finding Radius of Convergence

Apply ratio test and find

$$M = \lim_{n \to \infty} \left|\frac{u_{n+1}}{u_n}\right| M < 1$$

and transform it to the form of $|x - a| < b$; $a$ is the center, $b$ is the RoC
Or, if the series converges for all $x$, the RoC is $\infty$; if it only converges at $a$, the RoC is 0;
Some magic:

$$\frac{1}{1-\Box} = \sum_{n=0}^{\infty} \Box^n, |\Box| < 1$$

## Taylor Series

of $f$ at $a$:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \ldots$$
$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + \ldots$$
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$
$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$
$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

## Finding a specific high order derivative

1. given $\int f dx$
2. evaluate f in polynomial form and integrate the polynomial form
3. Compare the coefficient with the item that contains $f^{(100)}(0)$ in the Taylor expansion

## Rules of Series

$$\int \sum_{n=0}^{\infty} \Box dx = \sum_{n=0}^{\infty} \int \Box dx$$
$$\frac{d}{dx} \sum_{n=0}^{\infty} \Box = \sum_{n=0}^{\infty} \frac{d}{dx} \Box$$
$$x^k e^x = x^k \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+k}}{n!} \text{ (Note: } x^k \text{ is a constant in the series)}$$

# 4 Vectors

Angle between two vectors: $\cos\theta = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{\|\vec{v_1}\| \|\vec{v_2}\|}$
Perpendicular vectors: $\vec{v_1} \cdot \vec{v_2} = 0$

# 5 Partial Differentiation

$$f_{xy}(a,b) = f_{yx}(a,b)$$
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

## Directional Derivative

For **unit vector** $u = u_1\vec{i} + u_2\vec{j}$, we have:

$$D_{\vec{u}}f(a,b) = f_x(a,b)\cdot u_1 + f_y(a,b)\cdot u_2$$

Gradient Vector:

$$\nabla f = f_x\vec{i} + f_y\vec{j}$$

Relation between $D_{\vec{u}}f(a,b)$ and $\nabla f$:

$$D_{\vec{u}}f(a,b) = \nabla f(a,b)\cdot\vec{u} = \|\nabla f(a,b)\|\cos\theta$$

Some characteristics:

- $f$ increases most rapidly in $\nabla f(a,b)$ and decreases most rapidly in $-\nabla f(a,b)$.
- Max value of $D_{\vec{u}}f(a,b) = \|\nabla f(a,b)\|$ when $\vec{u}$ and $\nabla f$ in the same direction, since $\cos\theta = 0$
- Increment in $f$ (approx.): $\Delta f \approx [D_{\vec{u}}f(\vec{p})](\Delta t)$, where $p$ is the point to measure the increment and $u$ is the unit vector of direction.

## Finding $D_{\vec{u}}f$

1. Find the direction $\vec{p}$
2. Find the unit vector $\vec{u} = \frac{\vec{p}}{\|\vec{p}\|}$
3. Find $\nabla f$, then find $D_{\vec{u}}f = \nabla f\cdot\vec{u}$

## Critical Points

A point of $f$ that satisfies either is a critical point:

1. $f_x(a,b) = 0$ and $f_y(a,b) = 0$
2. $f_x(a,b)$ or $f_y(a,b)$ doesn't exist

Perform Second Derivative Test: let $f_x(a,b) = 0$ and $f_y(a,b) = 0$

$$D = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}(a,b)^2$$

- $D > 0, f_{xx} > 0$, f has a local minimum at (a,b)
- $D > 0, f_{xx} < 0$, f has a local maximum at (a,b)
- $D < 0$, f has a saddle point at (a,b)
- $D = 0$, no conclusion

# 6 Double Integrals

For a region $R$ s.t. $a \le x \le b$ and $c \le y \le d$, volume is given by:

$$\iint_R f(x,y)\,\mathrm{d}A = \int_c^d\int_a^b f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_a^b\int_c^d f(x,y)\,\mathrm{d}y\mathrm{d}x$$

if $f(x,y) = g(x)h(y)$, then

$$\iint_R f(x,y)\,\mathrm{d}A = (\int_a^b g(x)\,\mathrm{d}x)(\int_c^d h(y)\,\mathrm{d}y)$$

## Rectangular Regions

Express horizontal/vertical bounds as a function $g(x)$ or $h(y)$
Type A(top and bottom are curves)

$$\int_a^b\left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,\mathrm{d}y\right]\mathrm{d}x$$

Type B(left and right are curves)

$$\int_c^d\left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,\mathrm{d}x\right]\mathrm{d}y$$

## Polar Coordinates

$R$: $a \le r \le b$, $\alpha \le \theta \le \beta$

$$\iint_R f(x,y)\,\mathrm{d}A = \int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)r\,\mathrm{d}r\mathrm{d}\theta$$

## Surface Area

$$S = \iint_R\sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}\,\mathrm{d}A$$

# 7 Ordinary Differential Equation

## Separable Equations

$$M(x) - N(y)y' = 0 \implies \int M(x)dx = \int N(y)dy + c$$

## Reduction to Separable Form

Let $v = y/x \implies y = xv \to y' = v + xv'$, transform equations of $y' = g(\frac{y}{x})$ to $v + xv' = g(v)$ such that

$$\frac{dv}{g(v) - v} = \frac{dx}{x}$$

Similarly, $y' = f(ax + by + c)$ can be solved by $u = ax + by + c$

## Linear First Order ODE

To solve $y' + Py = Q$: find integration factor

$$R = e^{\int P dx}$$

Then, answer

$$y = \frac{1}{R}\int RQdx$$

## Reduction to Linear Form

A Bernoulli equation: $y' + P(x)y = Q(x)y^n$, where $n \in \mathbb{R}$;
(When $n = -1$, try Reduction to Separable Form) To solve it, let $v = y^{1-n}$;
Find and express $dv/dx$ in $dy/dx$; find $dy/dx$ and sub that in original equation; transform into

$$v' + (1-n)Pv = Q(1-n)$$

and solve the linear ODE.

## Homogeneous Linear Second Order DE

For $y'' + ay' + by = 0$, the characteristic equation is $\lambda^2 + a\lambda + b = 0$
Find $\Delta = a^2 - 4b$:

1. $\Delta > 0$, $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$
2. $\Delta = 0$, $y = (c_1 + c_2x)e^{-\frac{ax}{2}}$
3. $\Delta < 0$, it has two complex roots;$\lambda_1 = \alpha + \beta i, \lambda_2 = \alpha - \beta i$;
   $y = c_1e^{\alpha x}\cos\beta x + c_2e^{\alpha x}\sin\beta x$

where,

$$\lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b})$$
$$\lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b})$$

# 8 Modelling

## Population Growth

Malthus's Model: not an accurate representation

$$\frac{dN}{dt} = kN, k = B - D$$
$$N(t) = N_0e^{kt}$$

## Logistic Model

Let $D = sN$, where $s$ is a constant and $B$ is birth rate per capita:

$$\frac{dN}{dt} = BN - DN = BN - sN^2$$

The curve approaches carrying capacity $N = B/S$; point of inflection is at $N = B/2s$

$$N = \frac{N_\infty}{1 + (\frac{N_\infty}{N_0} - 1)e^{-Bt}}, N_\infty = \frac{B}{s}$$

## Harvesting

Let $E$ is fish caught per year, similar to the above model:

$$\frac{dN}{dt} = BN - sN^2 - E$$

Desirable result: $E < \frac{B^2}{4s}$, approaches the second root $\beta_2 = \frac{B + \sqrt{B^2 - 4Es}}{2s}$, when $\frac{dN}{dt} = 0$

## Strategies

- When given $dx/dt$, find $x$ that $dx/dt = 0$, draw out the axis, determine the sign of $dx/dt$ within each region, and find the flow (+ to the right, - to the left)
- To find E, draw the graph without E and find the line of symmetry; use the product of the roots to find E;