MA1101R Cheatsheet 19/20 Semester 1 Final

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Matrices

Theorem 1.2.7. If **augmented matrices** of two systems of linear equations are row equivalent, then the two systems have the same set of solutions. (* Even for two homogeneous linear systems, we still need to say that $(\boldsymbol{A} \mid \boldsymbol{0})$ is row equivalent to $(\boldsymbol{B} \mid \boldsymbol{0})$, not that $\boldsymbol{A}$ is row equivalent to $\boldsymbol{B}$.)

Example 1.4.10. Suppose augmented matrix $\boldsymbol{R}$ is in (R)REF:

1. LS has no solution
   $\Leftrightarrow$ Last column of $\boldsymbol{R}$ is pivot.
2. LS has one unique solution
   $\Leftrightarrow$ **Only** last column of $\boldsymbol{R}$ is non-pivot.
3. LS has infinite number of solution
   $\Leftrightarrow$ At least one column other than the last one is non-pivot
   $\Leftrightarrow$ Number of variables $>$ Number of non-zero rows in $\boldsymbol{R}$
   (* # non-pivot columns in (R)REF $-1 =$ # unique solutions)

Theorem 6.1.8. $\boldsymbol{A}$ is invertible when:

1. $\exists \boldsymbol{B}$ s.t. $\boldsymbol{AB} = \boldsymbol{I} \vee \boldsymbol{BA} = \boldsymbol{I}$
2. Refer to **Theorem 2.4.7.2** below
3. $\operatorname{rref}(\boldsymbol{A}) = \boldsymbol{I}$
4. $\boldsymbol{A}$ is a product of elementary matrices
5. $\det(\boldsymbol{A}) \neq 0$
6. Rows of $\boldsymbol{A}$ is a basis of $\mathbb{R}^n$
7. Columns of $\boldsymbol{A}$ is a basis of $\mathbb{R}^n$
8. 0 is not an eigenvalue of $\boldsymbol{A}$

Remark 2.3.4 (Cancellation Laws for Matrices). Let $\boldsymbol{A}$ be an invertible $m \times m$ matrix,

(a) If $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ are $m \times n$ matrices with $\boldsymbol{AB_1} = \boldsymbol{AB_2}$, then $\boldsymbol{B}_1 = \boldsymbol{B}_2$

(b) If $\boldsymbol{C}_1$ and $\boldsymbol{C}_2$ are $n \times m$ matrices with $\boldsymbol{C_1A} = \boldsymbol{C_2A}$, then $\boldsymbol{C}_1 = \boldsymbol{C}_2$

Theorem 2.4.7.2 (generalised). Relationship between singularity of $\boldsymbol{A}$ and the number of solutions of a linear system $\boldsymbol{Ax} = \boldsymbol{b}$:

1. $\boldsymbol{A}$ is singular $\Leftrightarrow$ $\boldsymbol{Ax} = \boldsymbol{b}$: has ∞ solutions (only case for homogeneous LS) or no solutions
2. $\boldsymbol{A}$ is invertible $\Leftrightarrow$ $\boldsymbol{Ax} = \boldsymbol{b}$: has one unique solution (trivial solution for homogeneous LS)

Definition 2.5.2. Let $\boldsymbol{A} = (a_{ij})$ be an $n \times n$ matrix. Let $\boldsymbol{M}_{ij}$ be an $(n-1) \times (n-1)$ matrix obtained from $\boldsymbol{A}$ by deleting the ith row and the jth column. Then the *determinant* of $\boldsymbol{A}$ is defined as

$$\det(\boldsymbol{A}) = \begin{cases} a_{11} & \text{if } n = 1 \\ a_{11}A_{11} + \cdots + a_{1n}A_{1n} & \text{if } n > 1 \end{cases}$$

where

$$A_{ij} = (-1)^{i+j} \det\left(\boldsymbol{M_{ij}}\right)$$

The number A_{ij} is called the (i,j)-*cofactor* of $\boldsymbol{A}$.

Theorem 2.5.8. The determinant of a triangular matrix is equal to the product of its diagonal entries.

Theorem 2.5.12 (added-on). The determinant of a square matrix is 0 when:

1. it has two identical rows, or
2. it has two identical columns
3. any row/column of its (R)REF is zero

Theorem 2.5.15. Let $\boldsymbol{A}$ be a square matrix. k is a non-zero constant.

1. $\boldsymbol{A} \xrightarrow{k\boldsymbol{R}_i} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = k\det(\boldsymbol{A})$
2. $\boldsymbol{A} \xrightarrow{\boldsymbol{R}_i \leftrightarrow \boldsymbol{R}_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = -\det(\boldsymbol{A})$
3. $\boldsymbol{A} \xrightarrow{\boldsymbol{R}_i + k\boldsymbol{R}_j} \boldsymbol{B} \Rightarrow \det(\boldsymbol{B}) = \det(\boldsymbol{A})$
4. Let $\boldsymbol{E}$ be an elementary matrix of the same size as $\boldsymbol{A}$. Then $\det(\boldsymbol{EA}) = \det(\boldsymbol{E})\det(\boldsymbol{A})$.

Remark 2.5.18. Since $\det(\boldsymbol{A}) = \det\left(A^T\right)$, theorem 2.5.15 holds if "rows" are changed to "columns".

Theorem 2.5.22. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ are two square matrices of order n and c is a scalar. Then

1. $\det(c\boldsymbol{A}) = c^n \det(\boldsymbol{A})$
2. $\det(\boldsymbol{AB}) = \det(\boldsymbol{A})\det(\boldsymbol{B})$
3. if $\boldsymbol{A}$ is invertible, $\det\left(\boldsymbol{A}^-1\right) = \frac{1}{\det(\boldsymbol{A})}$

Definition 2.5.24. Let $\boldsymbol{A}$ be a square matrix of order n. Then the *(classical) adjoint* of $\boldsymbol{A}$ is the $n \times n$ matrix

$$\mathbf{adj}(\boldsymbol{A}) = (A_{ij})^T_{n \times n}$$

where A_{ij} is the (i,j)-cofactor of $\boldsymbol{A}$.

Theorem 2.5.27 (Cramer's Rule). Suppose $\boldsymbol{Ax} = \boldsymbol{b}$ is a linear system where $\boldsymbol{A}$ is an $n \times n$ matrix. Let $\boldsymbol{A_i}$ be the matrix obtained from $\boldsymbol{A}$ be replacing the ith column of $\boldsymbol{A}$ by $\boldsymbol{b}$. If $\boldsymbol{A}$ is invertible, then the system has only one solution

$$\boldsymbol{x} = \frac{1}{\det(\boldsymbol{A})}\begin{pmatrix} \det(\boldsymbol{A_1}) \\ \vdots \\ \det(\boldsymbol{A_n}) \end{pmatrix}$$

Mixed Notes 1. $\boldsymbol{A}^{-1}$ is able to be computed by:

1. Find $\boldsymbol{B}$ s.t. $\boldsymbol{AB} = \boldsymbol{I} \vee \boldsymbol{BA} = \boldsymbol{I}$
2. Find using **Theorem 2.5.25**: $\boldsymbol{A}^{-1} = \frac{1}{\det(\boldsymbol{A})}\mathbf{adj}(\boldsymbol{A})$
3. Find using: $(\boldsymbol{A} \mid \boldsymbol{I}) \xrightarrow{GJE} (\boldsymbol{I} \mid \boldsymbol{A}^{-1})$

Mixed Notes 2. $\det(\boldsymbol{A})$ is able to be computed by:

1. Using **Theorem 2.5.2**
2. Using cross multiplication (for 2×2 and 3×3 matrices only)
3. Doing some ERO (e.g. GE, consider **Thoerem 2.5.15**) and making it triangular then using **Theorem 2.5.8** or making it have properties in **Theorem 2.5.12**
4. Using **Theorem 2.5.22**

Mixed Notes 3. Some random notes:

1. In $\mathbb{R}^n$ where $n \geq 2$, a set with 1 parameter is a line and that with 2 parameters is a space.
2. $\boldsymbol{M}^2 + \boldsymbol{M} = \boldsymbol{I} \Rightarrow \boldsymbol{M}(\boldsymbol{M} + \boldsymbol{I}) = \boldsymbol{I}$ (Don't put that $\boldsymbol{I}$ to be scalar 1!)
3. Two matrices have same RREF $\Leftrightarrow$ They are row equivalent
4. In exam, express a matrix in the form $\boldsymbol{A} = (a_{ij})_{m \times n}$. **DO NOT** use dots form
5. When using ERO $\boldsymbol{R}_i = \frac{1}{k}\boldsymbol{R}_j$, discuss whether k is 0 when necessary

Mixed Notes 4. Generally, for (square) matrices $\boldsymbol{A}$ and $\boldsymbol{B}$,

1. $\boldsymbol{AB} \neq \boldsymbol{BA}$
2. $(\boldsymbol{AB})^2 \neq \boldsymbol{A}^2\boldsymbol{B}^2$
3. $\boldsymbol{AB} = 0 \nRightarrow \boldsymbol{A} = 0 \vee \boldsymbol{B} = 0$
4. $\boldsymbol{A}^2 = I \nRightarrow \boldsymbol{A} = \pm\boldsymbol{I}$ (For example: 2 EMs of 2nd type ERO)

Mixed Notes 5. When expanding a row/column with cofactors of the other row/column, 0 will be yielded:

$$\sum_{m=1}^{n} a_{im}A_{jm} = \sum_{m=1}^{n} a_{mi}A_{mj} = 0, \text{ for some } i \neq j$$

Euclidean Spaces

Discussion 3.2.5. Given $S = \{\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_m}\} \subseteq \mathbb{R}^n\}$, show $\operatorname{span}(S) = \mathbb{R}^n$:

Consider $\boldsymbol{v_i} = (v_{i1}, \ldots, v_{in})$,

$$\begin{pmatrix} \boldsymbol{v_{11}} & \cdots & \boldsymbol{v_{m1}} \\ \vdots & \ddots & \vdots \\ \boldsymbol{v_{1n}} & \cdots & \boldsymbol{v_{mn}} \end{pmatrix} \xrightarrow{GE} \boldsymbol{R}$$

$\operatorname{span}(S) = \mathbb{R}^n \Leftrightarrow \boldsymbol{R}$ has no zero rows

Theorem 3.2.7. If $|S| < n$, $\operatorname{span}(S) \neq \mathbb{R}^n$.

Theorem 3.2.10. Let $S_1 = \{\boldsymbol{u_1}, \ldots, \boldsymbol{u_k}\}$ and $S_2 = \{\boldsymbol{v_1}, \ldots, \boldsymbol{v_m}\}$ be subsets of $\mathbb{R}^n$. Then, $\operatorname{span}(S_1) \subseteq \operatorname{span}(S_2) \Leftrightarrow \forall i = 1, 2, \ldots, k,\ u_i \in \operatorname{span}\{\boldsymbol{v_1}, \ldots, \boldsymbol{v_m}\}$.

Definition 3.3.2. Let V be a subset of $\mathbb{R}^n$. Then V is called a *subspace* of $\mathbb{R}^n$ if $V = \operatorname{span}(S)$ where $S = \{\boldsymbol{u_1}, \ldots, \boldsymbol{u_k}\}$ for some vectors $\boldsymbol{u_1}, \ldots, \boldsymbol{u_k} \in \mathbb{R}^n$.

More precisely, V is called the *subspace spanned* by S (or the *subspace spanned* by $\boldsymbol{u_1}, \ldots, \boldsymbol{u_k}$). We also say that S *spans* (or $\boldsymbol{u_1}, \ldots, \boldsymbol{u_k}$ *span*) the subspace V.

By contraposition, $V = \operatorname{span}(S) \Rightarrow \boldsymbol{0} \in V \equiv \boldsymbol{0} \notin V \Rightarrow V \neq \operatorname{span}(S)$. (* i.e., If $\boldsymbol{0}$ is not in V, V is not a subspace of $\mathbb{R}^n$)

Theorem 3.3.6. If $V = \{\boldsymbol{x} | \boldsymbol{Ax} = \boldsymbol{0}\}$, V is a subspace of $\mathbb{R}^n$.

Remark 3.3.8. Let V be a non-empty subset of $\mathbb{R}^n$. Then V is a subspace of $\mathbb{R}^n$ if and only if

$$\text{for all } \boldsymbol{u}, \boldsymbol{v} \in V \text{ and } c, d \in \mathbb{R}, \;\; c\boldsymbol{u} + d\boldsymbol{v} \in V$$

(* This checks whether V is **closed** under addition and scalar multiplication)

Definition 3.4.2/4. Consider $\boldsymbol{u_1}, \boldsymbol{u_2}, ..., \boldsymbol{u_k}$ which are column vectors, set $S = \boldsymbol{u_1}, \boldsymbol{u_2}, ..., \boldsymbol{u_k}$ is **Linear Indepedent** iff. any of:

1. $(\boldsymbol{u_1}\boldsymbol{u_2}...\boldsymbol{u_k})\boldsymbol{x} = \boldsymbol{0}$ has only trivial solution.
2. No vectors in S can be written as a linear combination of other vectors in S.
3. S is a subset of a **Linear Independent** set.

Definition 3.5.4/Theorem 3.6.7. A set S is a basis of a vector space if:

1. $S \subseteq V$
2. Any 2 of the 3 below:
    2.1. S is Linear Independent
    2.2. S spans V
    2.3. $|S| = \dim(V)$

Definition 3.5.8. Let $S = \boldsymbol{u_1}, \boldsymbol{u_2}, ..., \boldsymbol{u_k}$ be a basis for a vector space V and $\boldsymbol{v}$ is a vector in V. By T3.5.7, $\boldsymbol{v}$ is expressed uniquely as a LC:

$$\boldsymbol{v} = c_1\boldsymbol{u_1} + c_2\boldsymbol{u_2} + \cdots + c_k\boldsymbol{u_k}$$

Then we shall have the **coordinate vector** of $\boldsymbol{v}$ relative to the basis S : $(\boldsymbol{v})_S = (c_1, c_2, \ldots, c_k) \in \mathbb{R}^k$ (assuming vectors in S are in fixed order).

Remark 3.5.10/Theorem 3.5.11. Let S be a basis for a vector space V,

1. $\forall \boldsymbol{u}, \boldsymbol{v} \in V, \boldsymbol{u} = \boldsymbol{v} \Leftrightarrow (\boldsymbol{u})_S = (\boldsymbol{v})_S$
2. Coordinate vectors are closed under scalar multiplication and addition
3. Let $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_r} \in V$, they are LI iff. $(\boldsymbol{v_1})_S, (\boldsymbol{v_2})_S, \ldots, (\boldsymbol{v_k})_S$ are LI
4. $\operatorname{span} \boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_r} = V \Leftrightarrow \operatorname{span} (\boldsymbol{v_1})_S, (\boldsymbol{v_2})_S, \ldots, (\boldsymbol{v_k})_S = \mathbb{R}^{|S|}$

Theorem 3.6.9. Let U be a subspace of V, then $\dim(U) \le \dim(V)$. Furthermore, if $U \ne V$, then $\dim(U) < \dim(V)$.

Definition 3.7.3. Let $S = \boldsymbol{u_1}, \boldsymbol{u_2}, ..., \boldsymbol{u_k}$ and T be two bases for a vector space. The square matrix $\boldsymbol{P} = \begin{pmatrix}[\boldsymbol{u_1}]_T & [\boldsymbol{u_2}]_T & \ldots & [\boldsymbol{u_k}]_T\end{pmatrix}$ is called the **transition matrix** from S to T.

Mixed Theorem 6. Consider S and T are two bases for vector space V and $\boldsymbol{P}$ is the transition matrix from S to T. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices with elements of S and T respectively as columns, we have $\boldsymbol{BP} = \boldsymbol{A}$.

Mixed Theorem 7. ERO preserves row space **(T4.1.7)**, and we have:

- **(R4.1.9)** $\boldsymbol{R}$ is RREF of $\boldsymbol{A}$. Non-empty rows in $\boldsymbol{R}$ forms the basis of row space of $\boldsymbol{A}$.
- **(T4.2.1)** Row space and column space of a matrix have the same dimension.

Remark 4.2.5. Regarding $\operatorname{rank}(\boldsymbol{A})$:

1. For $m * n$ matrix $\boldsymbol{A}$, $\operatorname{rank}(\boldsymbol{A}) \le \min m, n$. If $\operatorname{rank}(\boldsymbol{A}) = \min m, n$, $\boldsymbol{A}$ is said to have **full rank**.
2. A square matrix $\boldsymbol{A}$ have full rank iff. it is invertible.
3. $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A^T})$.

Theorem 4.3.6. Suppose linear system $\boldsymbol{Ax} = \boldsymbol{b}$ has solution $\boldsymbol{v}$, then the solution set of this system is given by:

$$M = \{\boldsymbol{u} + \boldsymbol{v} | \boldsymbol{u} \in \operatorname{nullspace}(\boldsymbol{A})\}$$

Orthogonality

Definition 5.1.2.3/4. For two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$:
$d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|$.
Angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ is:

$$\cos^{-1}(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{\|\boldsymbol{u}\|\|\boldsymbol{v}\|})$$

Theorem 5.2.4. If S is an orthogonal set of non-zero vectors in a vector space, S is **LI**.

Theorem 5.2.8. Consider $S = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}\}$ is a basis for a vector space V, then for any vector $\boldsymbol{w}$ in V:

1. If S is orthogonal, we have
$$(\boldsymbol{w})_S = (\frac{\boldsymbol{w} \cdot \boldsymbol{u_1}}{\boldsymbol{u_1} \cdot \boldsymbol{u_1}}\boldsymbol{u_1}, \frac{\boldsymbol{w} \cdot \boldsymbol{u_2}}{\boldsymbol{u_2} \cdot \boldsymbol{u_2}}\boldsymbol{u_2}, \ldots, \frac{\boldsymbol{w} \cdot \boldsymbol{u_k}}{\boldsymbol{u_k} \cdot \boldsymbol{u_k}}\boldsymbol{u_k})$$
2. If S is orthonomal, we have
$$(\boldsymbol{w})_S = (\boldsymbol{w} \cdot \boldsymbol{u_1}, \boldsymbol{w} \cdot \boldsymbol{u_2}, \ldots, \boldsymbol{w} \cdot \boldsymbol{u_k})$$

T5.2.15: $(\boldsymbol{w})_S$ is the projection of $\boldsymbol{w}$ onto V if $\boldsymbol{w} \in \mathbb{R}^n \wedge V$ is a subspace of $\mathbb{R}^n$ (condition of $\boldsymbol{w}$ changed but same formula applies).

Theorem 5.2.19 (Gram-Schmidt Process). Let $\boldsymbol{u_1}, \boldsymbol{u_2}, \ldots, \boldsymbol{u_k}$ be a basis for a vector space V. Let

$$\begin{aligned} \boldsymbol{v_1} &= \boldsymbol{u_1}, \\ \boldsymbol{v_2} &= \boldsymbol{u_2} - \frac{\boldsymbol{u_2} \cdot \boldsymbol{v_1}}{\boldsymbol{v_1} \cdot \boldsymbol{v_1}}\boldsymbol{v_1}, \\ \boldsymbol{u_3} &= \boldsymbol{u_3} - \frac{\boldsymbol{u_3} \cdot \boldsymbol{v_1}}{\boldsymbol{v_1} \cdot \boldsymbol{v_1}}\boldsymbol{v_1} - \frac{\boldsymbol{u_3} \cdot \boldsymbol{v_2}}{\boldsymbol{v_2} \cdot \boldsymbol{v_2}}\boldsymbol{v_2}, \\ &\vdots \end{aligned}$$

Then $\boldsymbol{v_1}, \boldsymbol{v_2}, \ldots, \boldsymbol{v_k}$ is an orthogonal basis for V. Normalize all vectors in it then we have a orthonormal basis for V.

Definition 5.3.6. Let $\boldsymbol{Ax} = \boldsymbol{b}$ be a linear system where $\boldsymbol{A}$ is an $m * n$ matrix. A vector $\boldsymbol{u} \in \mathbb{R}^n$ is called a **least squares solution** to the linear system if $\forall \boldsymbol{u} \in \mathbb{R}^n, \|\boldsymbol{b} - \boldsymbol{Au}\| \le \|\boldsymbol{b} - \boldsymbol{Av}\|$.

Theorem 5.3.8. Continuing **D5.3.6**, let $\boldsymbol{p}$ be the projection of $\boldsymbol{b}$ onto the column space of $\boldsymbol{A}$. $\boldsymbol{u}$ is the least squares solution iff. $\boldsymbol{Au} = \boldsymbol{p}$.

Theorem 5.3.10. Continuing **D5.3.6**, $\boldsymbol{u}$ is the least squares solution iff. $\boldsymbol{u}$ is a solution to $\boldsymbol{A}^T\boldsymbol{Ax} = \boldsymbol{A}^T\boldsymbol{b}$.

D5.4.3/R5.4.4/T5.4.6. $\boldsymbol{A}$ is a square matrix of order n. The following are equivalent:

1. $\boldsymbol{A}$ is orthogonal
2. $\boldsymbol{A}^{-1} = \boldsymbol{A}^T$
3. $\boldsymbol{AA}^T = \boldsymbol{A}^T\boldsymbol{A} = \boldsymbol{I}$
4. The rows of $\boldsymbol{A}$ form an **orthonormal** basis for $\mathbb{R}^n$
5. The columns of $\boldsymbol{A}$ form an **orthonormal** basis for $\mathbb{R}^n$

Theorem 5.4.7. Let S and T be two **orthonormal** bases for a vector space and let $\boldsymbol{P}$ be the transition matrix from S to T. Then $\boldsymbol{P}$ is orthogonal and $\boldsymbol{P}^T$ is the transition matrix from T to S.

Diagonalization

Definition 6.1.3. $\boldsymbol{A}$ is a square matrix of order n. $\boldsymbol{u} \in \mathbb{R}^n$ is an non-zero column vector that satisfies:

$$\boldsymbol{Au} = \lambda\boldsymbol{u}$$

for some scalar λ. λ is called an **eigenvalue** of $\boldsymbol{A}$. $\boldsymbol{u}$ is said to be an **eigenvector** of $\boldsymbol{A}$ **associated** with the eigenvalue λ.

Theorem 6.1.9. If $\boldsymbol{A}$ is triangular, the eigenvalues of $\boldsymbol{A}$ are the diagonal entries of $\boldsymbol{A}$.

Remark 6.2.5. Suppose the characteristic polynomial of the matrix $\boldsymbol{A}$ can be factorized as

$$\det(\lambda\boldsymbol{I} - \boldsymbol{A}) = (\lambda - \lambda_1)^{r_1}(\lambda - \lambda_2)^{r_2} \ldots (\lambda - \lambda_k)^{r_k}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of $\boldsymbol{A}$. Then for each eigenvalue λ_i,

$$\dim(E_{\lambda_i}) \le r_i$$

Furthermore, $\boldsymbol{A}$ is diagonalizable iff. $\forall 1 \le i \le k, \dim(E_{\lambda_i}) = r_i$.

Definition 6.3.2/T*.4. A square matrix $\boldsymbol{A}$ is said to be orthogonally diagonalizable iff. there exists an orthogonal matrix $\boldsymbol{P}$ such that $\boldsymbol{P^T AP}$ is diagonal.

A square matrix is orthogonally diagonalizable iff. it is **symmetric**.

Algorithm 6.3.5. Similar to the process for the normal matrix, orthogonal matrix $\boldsymbol{P}$ can be found by using vectors of T as **its columns** where $T = T_{\lambda_1} \cup T_{\lambda_2} \cup \cdots \cup T_{\lambda_k}$ and T_{λ_i} is transformed from S_{λ_1} using Gram-Schmidt Process.

Linear Transformation

Theorem 7.1.4. Let T be a linear transformation, we have:

1. $T(\boldsymbol{0}) = \boldsymbol{0}$
2. T is closed under scalar multiplication and addition

Discussion 7.1.8. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with the standard matrix $\boldsymbol{A}$. Let $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \ldots, \boldsymbol{e_n}\}$ be the standard basis for $\mathbb{R}^n$. We then have:

$$\boldsymbol{A} = \begin{pmatrix}T(\boldsymbol{e_1}) & T(\boldsymbol{e_2}) & \ldots & T(\boldsymbol{e_n})\end{pmatrix}$$

Theorem 7.2.4. Continuing **D7.1.8**. We have:

$$R(T) = \operatorname{span}\{T(\boldsymbol{e_1}), T(\boldsymbol{e_2}), \ldots, T(\boldsymbol{e_n})\} = \text{the column space of } \boldsymbol{A}$$

which is a subspace of $\mathbb{R}^m$

D7.2.5/T7.2.9/D7.2.10/T7.2.12. Continuing **T7.2.4**. We have:

- $\operatorname{rank}(T) = \dim(R(T)) = \operatorname{rank}(\boldsymbol{A})$
- $\operatorname{nullity}(T) = \operatorname{nullity}(\boldsymbol{A})$
- $\operatorname{rank}(T) + \operatorname{nullity}(T) = n$
- $\ker(T) = \text{the nullspace of } \boldsymbol{A}$