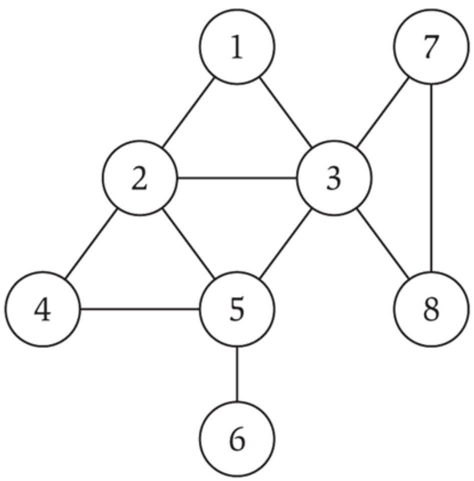
Graphs

Slides based on Kevin Wayne / Pearson-Addison Wesley

Undirected Graphs

Undirected graph. G = (V, E)

- V = nodes (or vertices).
- E = edges between pairs of nodes.
- Captures pairwise relationship between objects.
- Graph size parameters: $n = |V|$, $m = |E|$.



V = { 1, 2, 3, 4, 5, 6, 7, 8 }

E = { 1-2, 1-3, 2-3, 2-4, 2-5, 3-5, 3-7, 3-8, 4-5, 5-6 }

n = 8

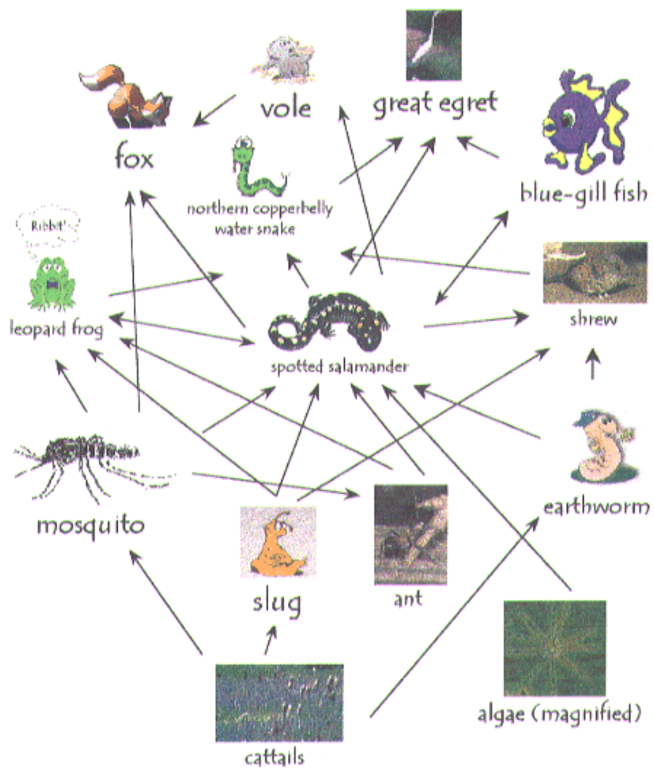
m = 11

Some Graph Applications

Graph	*Nodes*	*Edges*
transportation	street intersections	highways
communication	computers	fiber optic cables
World Wide Web	web pages	hyperlinks
social	people	relationships
food web	species	predator-prey
software systems	functions	function calls
scheduling	tasks	precedence constraints
circuits	gates	wires

Ecological Food Web

Food web graph.
- Node = species.
- Edge = from prey to predator.

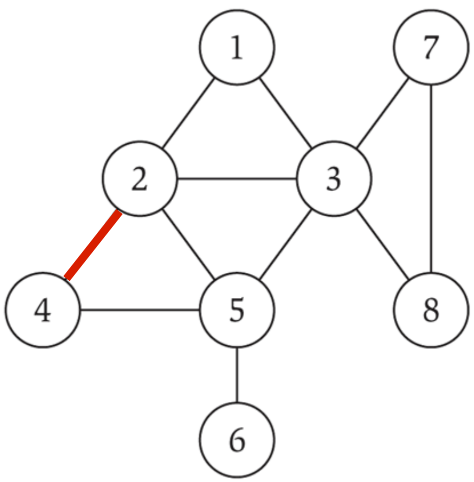


Reference: http://www.twingroves.district96.k12.il.us/Wetlands/Salamander/SalGraphics/salfoodweb.giff

Graph Representation: Adjacency Matrix

Adjacency matrix. n-by-n matrix with $A_{uv} = 1$ if (u, v) is an edge.

- Two representations of each edge.
- Space proportional to n^2.
- Checking if (u, v) is an edge takes $\Theta(1)$ time.
- Identifying all edges takes $\Theta(n^2)$ time.



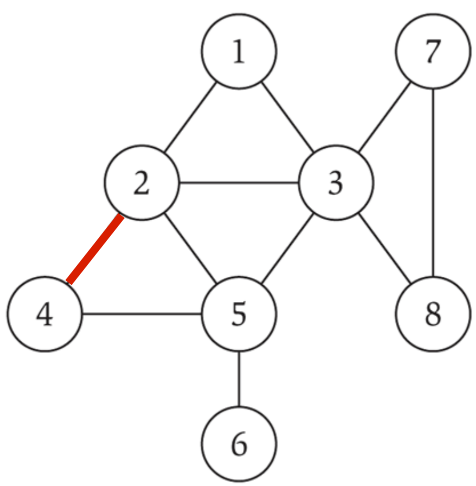
	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

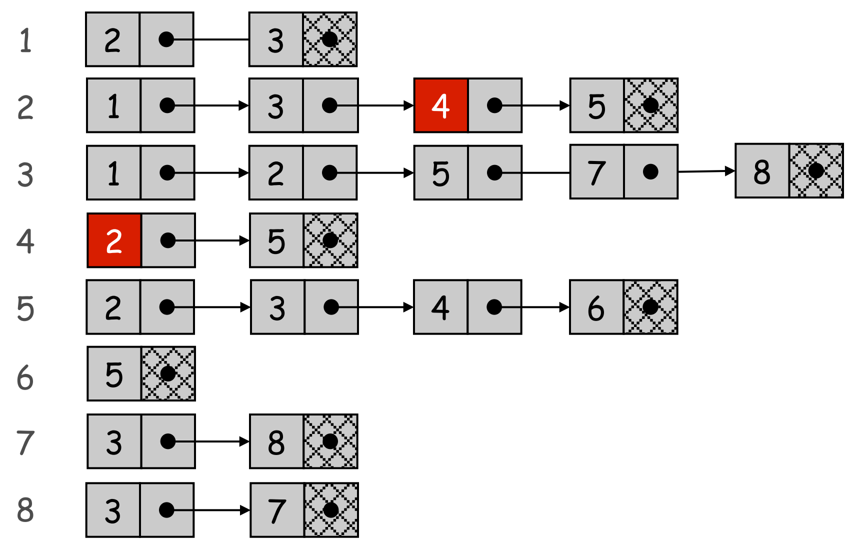
Graph Representation: Adjacency List

Adjacency list. Node indexed array of lists.

- Two representations of each edge.
- Space proportional to m + n.
- Checking if (u, v) is an edge takes O(deg(u)) time.
- Identifying all edges takes Θ(m + n) time.

degree = number of neighbors of u



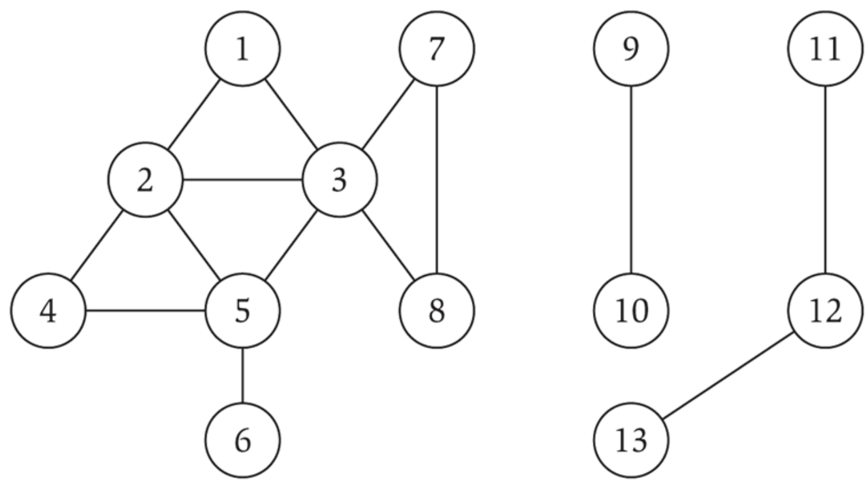


Paths and Connectivity

Def. A path in an undirected graph $G = (V, E)$ is a sequence P of nodes $v_1, v_2, \ldots, v_{k-1}, v_k$ with the property that each consecutive pair v_i, v_{i+1} is joined by an edge in E.

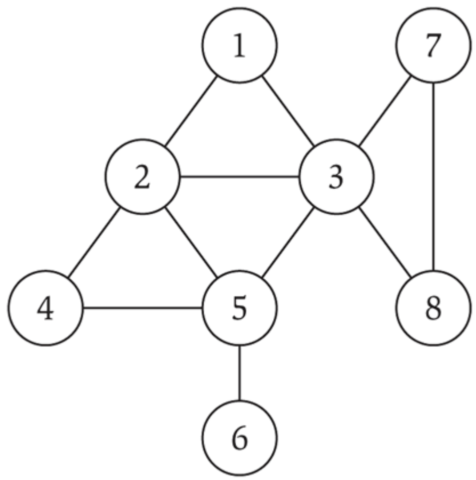
Def. A path is simple if all nodes are distinct.

Def. An undirected graph is connected if for every pair of nodes u and v, there is a path between u and v.



Cycles

Def. A cycle is a path $v_1, v_2, \ldots, v_{k-1}, v_k$ in which $v_1 = v_k$, $k > 2$, and the first k-1 nodes are all distinct.



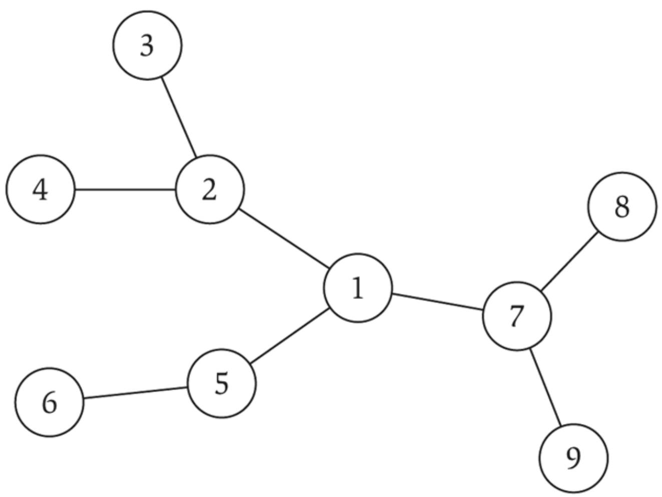
cycle C = 1-2-4-5-3-1

Trees

Def. An undirected graph is a tree if it is connected and does not contain a cycle.

Theorem. Let G be an undirected graph on n nodes. Any two of the following statements imply the third.

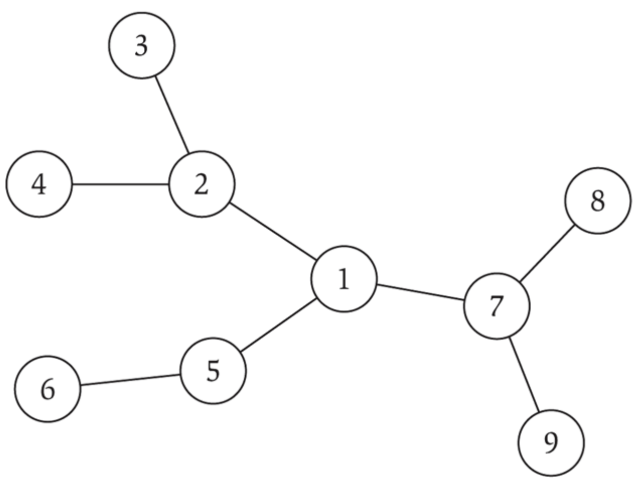
- G is connected.
- G does not contain a cycle.
- G has n-1 edges.



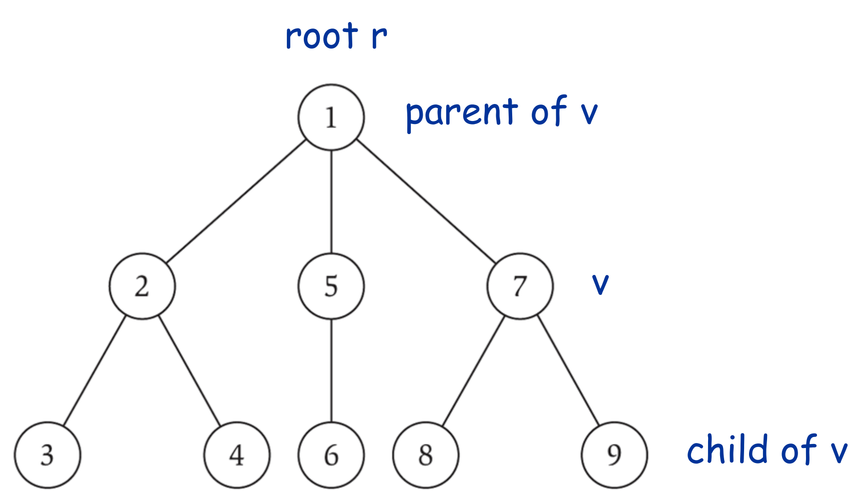
Rooted Trees

Rooted tree. Given a tree T, choose a root node r and orient each edge away from r.

Importance. Models hierarchical structure.



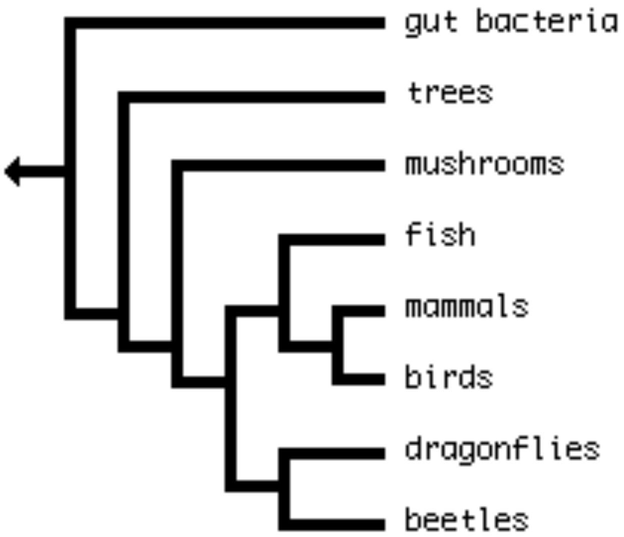
a tree



the same tree, rooted at 1

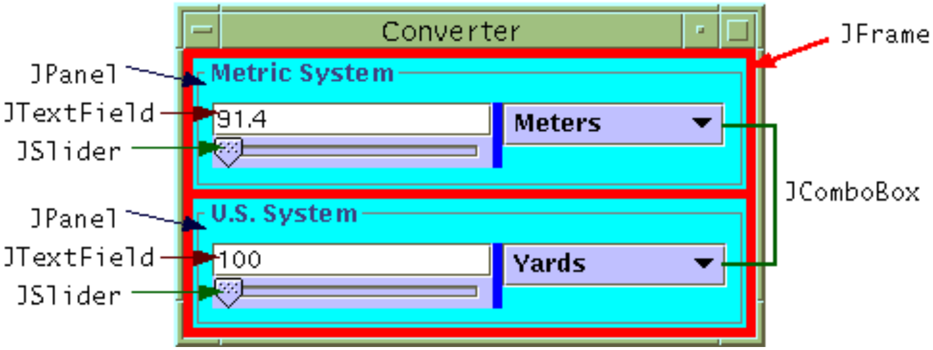
Phylogeny Trees

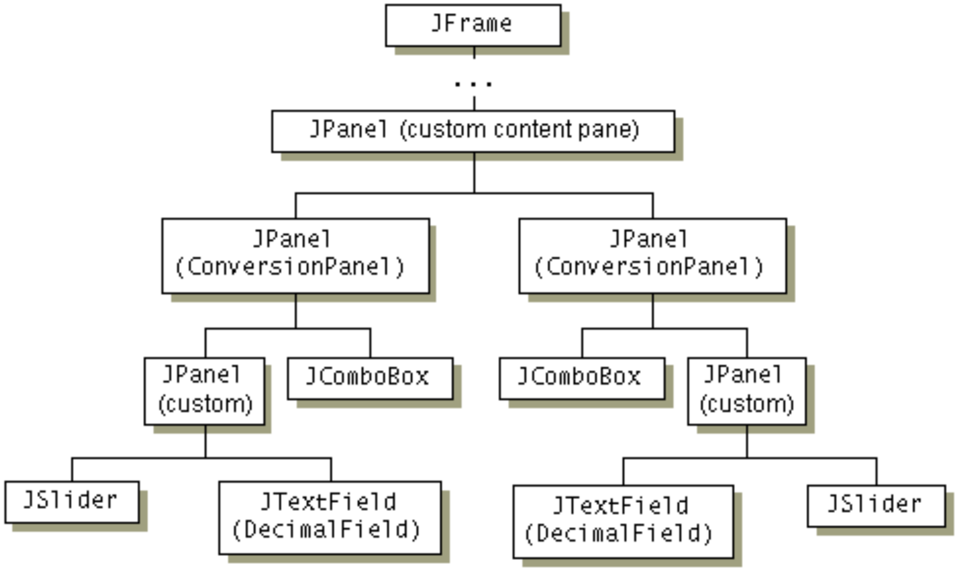
Phylogeny trees. Describe evolutionary history of species.



GUI Containment Hierarchy

GUI containment hierarchy. Describe organization of GUI widgets.





Reference: http://java.sun.com/docs/books/tutorial/uiswing/overview/anatomy.html

Graph Traversal

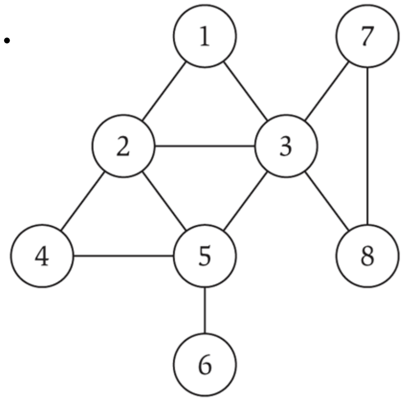
Connectivity

s-t connectivity problem. Given two node s and t, is there a path between s and t?

s-t shortest path problem. Given two node s and t, what is the length of the shortest path between s and t?

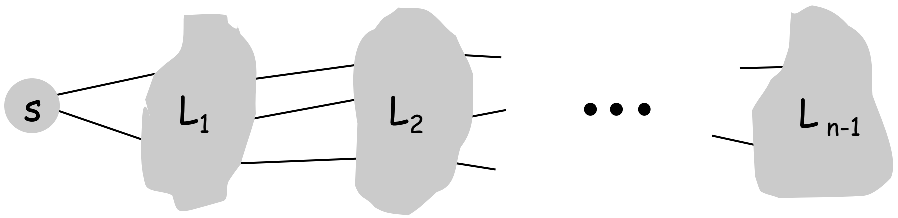
Applications.

- Facebook.
- Maze traversal.
- Kevin Bacon number.
- Fewest number of hops in a communication network.
- Erdos number.



Breadth First Search

BFS intuition. Explore outward from s in all possible directions, adding nodes one "layer" at a time.



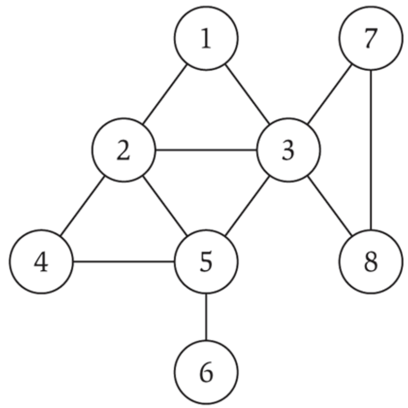
BFS algorithm.

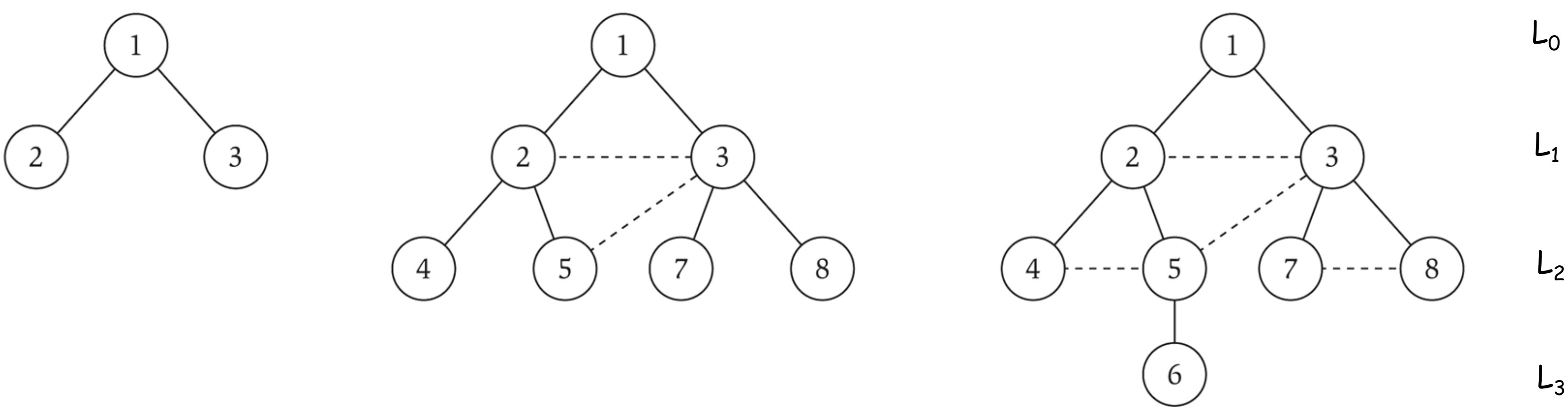
- $L_0 = \{ s \}$.
- L_1 = all neighbors of L_0.
- L_2 = all nodes that do not belong to L_0 or L_1, and that have an edge to a node in L_1.
- L_{i+1} = all nodes that do not belong to an earlier layer, and that have an edge to a node in L_i.

Theorem. For each i, L_i consists of all nodes at distance exactly i from s. There is a path from s to t iff t appears in some layer.

Breadth First Search

Property. Let T be a BFS tree of G = (V, E), and let (x, y) be an edge of G. Then the level of x and y differ by at most 1.





(a) (b) (c)

Breadth First Search Algorithm

Property. Finds all nodes reachable from a starting node, s.

Byproduct. Computes distances from s to all other vertices.

Breadth first search implemented with a queue data structure

```
BFS ( G=(V,E), s )
seen[v]=false, dist[v]= ∞  for every vertex v
beg=1; end=2; Q[1]=s; seen[s]=true; dist[s]=0;
while (beg<end) do
    head=Q[beg];
    for every u s.t. (head,u) is an edge and
                          not seen[u] do
        Q[end]=u; dist[u]=dist[head]+1;
        seen[u]=true; end++;
    beg++;
```

Breadth First Search: Analysis

Theorem. The above implementation of BFS runs in $O(m + n)$ time if the graph is given by its adjacency list representation.

Pf.

- Easy to prove $O(n^2)$ running time:
  - at most n iterations in the while loop (each one considering a different node)
  - when we consider node u, there are $\leq n$ incident edges (u, v), and we spend $O(1)$ processing each edge

- Actually runs in $O(m + n)$ time:
  - when we consider node u, there are deg(u) incident edges (u, v)
  - total time processing edges is $\Sigma_{u \in V} \deg(u) = 2m$ ▪

each edge (u, v) is counted exactly twice in sum: once in deg(u) and once in deg(v)

Depth First Search Algorithm

Property. Finds all nodes reachable from a starting node, s, in a different order than breadth first search.

Complexity. O(m+n) for same reason as breadth first search

depth first search implemented either recursively or with a stack data structure

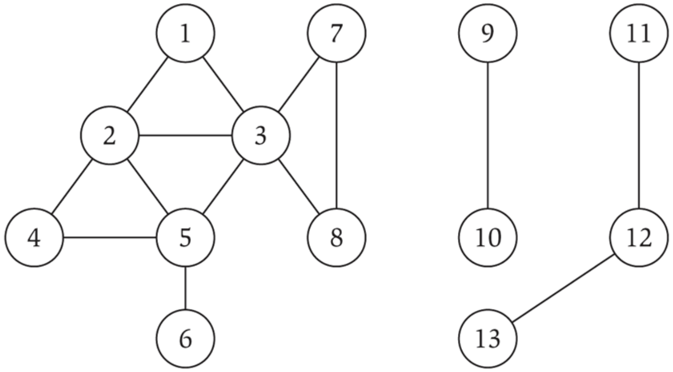
```
DFS-RUN ( G=(V,E), s )
1. seen[v]=false for every vertex v
2. DFS(s)

DFS(v)
1. seen[v]=true
2. for every neighbor u of v
3.     if not seen[u] then DFS(u)
```

Note – DFS will visit nodes in different order based on implementation.

- What order the neighbors are added to stack or called recursively
- Whether nodes are marked as visited as they are put on the stack or only when they are processed

Connected Component

Connected component. Find all nodes reachable from s.



Connected component containing node 1 = { 1, 2, 3, 4, 5, 6, 7, 8 }.