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- 2022-04-11 2022-05-08 $
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Julia :

- Julia

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4

4.1 1 2

4.2 : 2

4.3 : 2

4.4 : 1

4.5 : 1

4.6 : ^{2}

4.7 :

4.8 : ^{2}

4.9 :

4.10 :

5

5.1

5.2

5.3 Poisson

5.4

5.5 ^{2}

5.6

5.7

5.8 t

5.9

5.10 F

5.11

5.12

5.13

```
ENV["LINES"], ENV["COLUMNS"] = 100, 100
using BenchmarkTools
using Distributions
using Printf
```

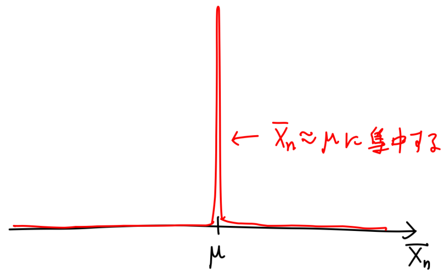
大数の法則と中心極限定理のイメージ

← $\mu = E[X_i]$, $\sigma = \sqrt{E[(X_i-\mu)^2]}$ とおく。

$X_1, X_2, X_3, \ldots$ は独立同分布確率変数列であるとし、$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ とおく。

平均 μ, 標準偏差 $\frac{\sigma}{\sqrt{n}} \to 0$

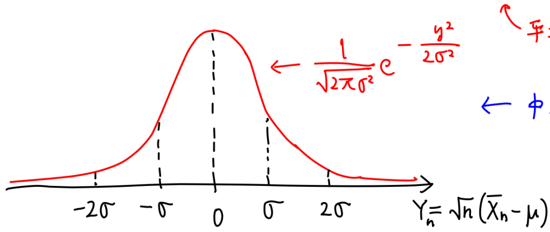
サイズnの標本の平均 $\bar{X}_n$ の分布の $n \to \infty$ での様子：



$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ の分布は $n \to \infty$ で $\mu = E[X_i]$ に集中する。

$\bar{X}_n$ と μ の差を $\sqrt{n}$ 倍して拡大して見るとこうなる：$Y_n = \sqrt{n}(\bar{X}_n - \mu) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(X_i - \mu)$.

平均0, 標準偏差 σ

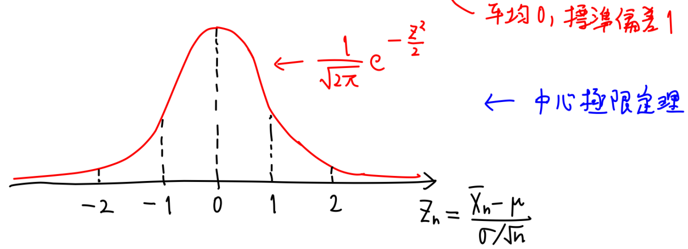


$\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$ と μ の差の大きさはだいたい $\frac{\sigma}{\sqrt{n}}$ の大きさであり、

差を $\sqrt{n}$ 倍拡大して分布を見ると標準偏差 σ の正規分布になっている。

さらに σ 分の1倍するとこうなる：$Z_n = \frac{\sqrt{n}}{\sigma}(\bar{X}_n - \mu) = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{X_i - \mu}{\sigma}$

平均0, 標準偏差1



以上のように、大数の法則によって1点に集中する標本平均の分布を

$\sqrt{n}$ 倍（もしくは $\sigma/\sqrt{n}$ 分の1倍）することによってより精密に見ると中心極限定理が得られる。

Figure 1: CLT.PNG

```
using QuadGK
using Random
Random.seed!(4649373)
using Roots
using SpecialFunctions
using StaticArrays
using StatsBase
using StatsFuns
using StatsPlots
default(fmt = :png, titlefontsize = 10, size = (400, 250))
using SymPy
```

```
# Override the Base.show definition of SymPy.jl:
# https://github.com/JuliaPy/SymPy.jl/blob/29c5bfd1d10ac53014fa7fef468bc8deccadc2fc/src/ty

@eval SymPy function Base.show(io::IO, ::MIME"text/latex", x::SymbolicObject)
    print(io, as_markdown("\\displaystyle " * sympy.latex(x, mode="plain", fold_short_frac
end
@eval SymPy function Base.show(io::IO, ::MIME"text/latex", x::AbstractArray{Sym})
    function toeqnarray(x::Vector{Sym})
        a = join(["\\displaystyle " * sympy.latex(x[i]) for i in 1:length(x)], "\\\\")
        """\\left[ \\begin{array}{r}$a\\end{array} \\right]"""
    end
    function toeqnarray(x::AbstractArray{Sym,2})
        sz = size(x)
        a = join([join("\\displaystyle " .* map(sympy.latex, x[i,:]), "&") for i in 1:sz[1
        "\\left[ \\begin{array}{" * repeat("r",sz[2]) * "}" * a * "\\end{array}\\right]"
    end
    print(io, as_markdown(toeqnarray(x)))
end
```

```
x   y = x < y || x   y

mypdf(dist, x) = pdf(dist, x)
mypdf(dist::DiscreteUnivariateDistribution, x) = pdf(dist, round(x))

distname(dist::Distribution) = replace(string(dist), r"{.*}" => "")
myskewness(dist) = skewness(dist)
mykurtosis(dist) = kurtosis(dist)
function standardized_moment(dist::ContinuousUnivariateDistribution, m)
     ,   = mean(dist), std(dist)
```

```
    quadgk(x -> (x -  )^m * pdf(dist, x), extrema(dist)...)[1] /  ^m
end
myskewness(dist::MixtureModel{Univariate, Continuous}) = standardized_moment(dist, 3)
mykurtosis(dist::MixtureModel{Univariate, Continuous}) = standardized_moment(dist, 4) - 3
```

```
mykurtosis (generic function with 2 methods)
```

(law of large numbers, LLN) , , $X_1, X_2, \ldots$ n $\bar{X}_n = (1/n)\sum_{i=1}^n X_i$, $\bar{X}_n$ X_i . (, , , . , , .)

.

Markov

X $a > 0$

$$P(|X| \geq a) \leq \frac{1}{a} E[|X|].$$

: $1_{|x|\geq a}(x)$

$$1_{|x|\geq a}(x) = \begin{cases} 1 & (|x| \geq a) \\ 0 & (|x| < a) \end{cases}$$

, $|x| \geq a$ $1 \leq |x|/a$ $1_{|x|\geq a}(x) \leq |x|/a$. ,

$$P(|X| \geq a) = E[1_{|x|\geq a}(x)] \leq E\left[\frac{|X|}{a}\right] = \frac{1}{a} E[|X|].$$

: Markov , Chebyshev , Jensen , Gibbs Bernoulli .

Chebyshev

Markov X $(X-\mu)^2$ $(\mu = E[X])$ Chebyshev .

X $E[|X|] < \infty$, X $\mu = E[X]$. , X $\sigma^2 = E[(X-\mu)^2]$, $\varepsilon > 0$

$$P(|X-\mu| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}.$$

: Markov X, a $(X-\mu)^2$, ε^2 ,

$$P(|X-\mu| \geq \varepsilon) = P((X-\mu)^2 \geq \varepsilon^2) \leq \frac{1}{\varepsilon^2} E[(X-\mu)^2] = \frac{\sigma^2}{\varepsilon^2}.$$

$X_1, X_2, X_3, ...$, $\mu = E[X_i]$, $\sigma^2 = E[(X_i - \mu)^2]$. , n

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$$

, $\varepsilon > 0$,

$$\lim_{n \to \infty} P(|\bar{X}_n - \mu| \geq \varepsilon) = 0.$$

:

$$\lim_{n \to \infty} P(|\bar{X}_n - \mu| < \varepsilon) = 1$$

$\bar{X}_n$ $n \to \infty$ μ ($\varepsilon > 0$ μ ε) . $\bar{X}_n$ μ , $\bar{X}_n$ $n \to \infty$ μ , . , .

: $\bar{X}_n$:

$$
\begin{aligned}
E[\bar{X}_n] &= \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\sum_{i=1}^{n} \mu = \mu,\\
(\bar{X}_n - \mu)^2 &= \left(\frac{1}{n}\sum_{i=1}^{n}(X_i - \mu)\right)^2 = \frac{1}{n^2}\sum_{i,j=1}^{n}(X_i-\mu)(X_j-\mu),\\
\operatorname{var}(\bar{X}_n) &= E[(\bar{X}_n-\mu)^2] = \frac{1}{n^2}\sum_{i,j=1}^{n} E[(X_i-\mu)(X_j-\mu)] = \frac{1}{n^2}\sum_{i,j=1}^{n}\delta_{ij}\sigma^2 = \frac{\sigma^2}{n}.
\end{aligned}
$$

2 X_i ( ) . , Chebyshev $X,\ \mu,\ \sigma^2$ $\bar{X}_n,\ \mu,\ \operatorname{var}(\bar{X}_n) = \sigma^2/n$, $\varepsilon > 0$,

$$
P(|\bar{X}_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2/n}{\varepsilon^2} \to 0 \quad \text{as } n \to \infty.
$$

$n \to \infty$

$X_1, X_2, X_3, \ldots$, n :

$$
\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad S_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2.
$$

$\mu = E[X_i]$, $\sigma^2 = E[(X_i-\mu)^2] < \infty$, $E[|X_i-\mu|^3] < \infty$, $E[(X_i-\mu)^4] < \infty$ . . X_i (, skewness) (, kurtosis)

$$
\bar{\kappa}_3 = E\left[\left(\frac{X_i-\mu}{\sigma}\right)^3\right], \quad \bar{\kappa}_4 = E\left[\left(\frac{X_i-\mu}{\sigma}\right)^4\right] - 3
$$

,

$$
\begin{aligned}
&E[\bar{X}_n] = \mu, \quad E[S_n^2] = \sigma^2,\\
&\operatorname{var}(\bar{X}_n) = \frac{\sigma^2}{n}, \quad \operatorname{cov}(\bar{X}_n, S_n^2) = \sigma^3\frac{\bar{\kappa}_3}{n}, \quad \operatorname{var}(S_n^2) = \sigma^4\left(\frac{\bar{\kappa}_4}{n} + \frac{2}{n-1}\right).
\end{aligned}
$$

, Chebyshev $X,\ \mu,\ \sigma^2$ $S_n^2,\ E[S_n^2] = \sigma^2,\ \operatorname{var}(S_n^2) = \sigma^4(\bar{\kappa}_4/n + 2/(n-1))$, $\varepsilon > 0$,

$$P(|S_n^2-\sigma^2|\geq\varepsilon)\leq\frac{\sigma^4}{\varepsilon^2}\left(\frac{\bar{\kappa}_4}{n}+\frac{2}{n-1}\right)\to 0\quad\text{as } n\to\infty.$$

, S_n^2 , S_n^2 $n\to\infty$ σ^2 .

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, . , , .
σ-algebra ($=$ σ-field) $\mathcal{F}$ $\mathcal{F}\ni A\mapsto P(A)\in\mathbb{R}_{\geq 0}$ Ω . Ω ω . Ω
.

Borel-Cantelli

Ω :

$$\sum_{k=1}^{\infty}P(A_k)<\infty\implies P\left(\bigcap_{n=1}^{\infty}\bigcup_{k\geq n}A_k\right)=0.$$

: $\sum_{k=1}^{\infty}P(A_k)<\infty$, $B_n=\bigcup_{k\geq n}A_k$. $P\left(\bigcap_{n=1}^{\infty}B_n\right)=0$. $\sum_{k=1}^{\infty}P(A_k)<\infty$,

$$P(B_b)\leq\sum_{k\geq n}P(A_k)=\sum_{k=1}^{\infty}P(A_k)-\sum_{k=1}^{n-1}P(A_k)\to 0\text{as } n\to\infty.$$

$B_1\supset B_2\supset B_3\supset\cdots$

$$P\left(\bigcap_{n=1}^{\infty}B_n\right)=\lim_{n\to\infty}P(B_n)=0.$$