正規分布モデルの共役事前分布によるベイズ統計

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In [1]:

```julia
ENV["COLUMNS"] = 120

using Distributions
using LinearAlgebra
using Random
Random.seed!(4649373)
using StatsPlots
default(fmt=:png, size=(500, 350),
    titlefontsize=10, tickfontsize=6, guidefontsize=9,
    plot_titlefontsize=10)
using SymPy
using Turing
```

In [2]:

```julia
# Override the Base.show definition of SymPy.jl:
# https://github.com/JuliaPy/SymPy.jl/blob/29c5bfd1d10ac53014fa7fef468bc8deccadc2fc/src/types.

@eval SymPy function Base.show(io::IO, ::MIME"text/latex", x::SymbolicObject)
    print(io, as_markdown("\\displaystyle " *
            sympy.latex(x, mode="plain", fold_short_frac=false)))
end
@eval SymPy function Base.show(io::IO, ::MIME"text/latex", x::AbstractArray{Sym})
    function toeqnarray(x::Vector{Sym})
        a = join(["\\displaystyle " *
                sympy.latex(x[i]) for i in 1:length(x)], "\\\\")
        """\\left[ \\begin{array}{r}$a\\end{array} \\right]"""
    end
    function toeqnarray(x::AbstractArray{Sym,2})
        sz = size(x)
        a = join([join("\\displaystyle " .* map(sympy.latex, x[i,:]), "&")
                for i in 1:sz[1]], "\\\\")
        "\\left[ \\begin{array}{" * repeat("r",sz[2]) * "}" * a * "\\end{array}\\right]"
    end
    print(io, as_markdown(toeqnarray(x)))
end
```

In [3]:
```julia
# One sample t-test

function pvalue_ttest(x̄, s², n, μ)
    t = (x̄ - μ)/√(s²/n)
    2ccdf(TDist(n-1), abs(t))
end

function pvalue_ttest(x, μ)
    x̄, s², n = mean(x), var(x), length(x)
    pvalue_ttest(x̄, s², n, μ)
end

function confint_ttest(x̄, s², n; α = 0.05)
    c = quantile(TDist(n-1), 1-α/2)
    [x̄ - c*√(s²/n), x̄ + c*√(s²/n)]
end

function confint_ttest(x; α = 0.05)
    x̄, s², n = mean(x), var(x), length(x)
    confint_ttest(x̄, s², n; α)
end
```

Out[3]: confint_ttest (generic function with 2 methods)

In [4]:
```julia
# Bayesian analogue of one sample t-test

posterior_μ_ttest(n, x̄, s²) = x̄ + √(s²/n)*TDist(n-1)
posterior_μ_ttest(x) = posterior_μ_ttest(length(x), mean(x), var(x))

preddist_ttest(n, x̄, s²) = x̄ + √(s²*(1 + 1/n))*TDist(n-1)
preddist_ttest(x) = preddist_ttest(length(x), mean(x), var(x))
```

Out[4]: preddist_ttest (generic function with 2 methods)

In [5]:

```julia
# Jeffreys事前分布などのimproper事前分布を定義するために以下が使われる.

"""
    PowerPos(p::Real)

The *positive power distribution* with real-valued parameter `p` is the improper distribution
of real numbers that has the improper probability density function

```math
f(x) = \\begin{cases}
0 & \\text{if } x \\leq 0, \\\\
x^p & \\text{otherwise}.
\\end{cases}
```
"""
struct PowerPos{T<:Real} <: ContinuousUnivariateDistribution
    p::T
end
PowerPos(p::Integer) = PowerPos(float(p))

Base.minimum(d::PowerPos{T}) where T = zero(T)
Base.maximum(d::PowerPos{T}) where T = T(Inf)

Base.rand(rng::Random.AbstractRNG, d::PowerPos) = rand(rng) + 0.5
function Distributions.logpdf(d::PowerPos, x::Real)
    T = float(eltype(x))
    return x ≤ 0 ? T(-Inf) : d.p*log(x)
end

Distributions.pdf(d::PowerPos, x::Real) = exp(logpdf(d, x))

For vec support
function Distributions.loglikelihood(d::PowerPos, x::AbstractVector{<:Real})
    T = float(eltype(x))
    return any(xi ≤ 0 for xi in x) ? T(-Inf) : d.p*log(prod(x))
end

@doc PowerPos
```

Out[5]: `PowerPos(p::Real)`

The *positive power distribution* with real-valued parameter `p` is the improper distribution of real numbers that has the improper probability density function

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x^p & \text{otherwise}. \end{cases}$$

In [6]:

```julia
# 以下は使わないが,
# Flat() や PowerPos(p) と正規分布や逆ガンマ分布の関係は次のようになっている.

MyNormal(μ, σ) = σ == Inf ? Flat() : Normal(μ, σ)
MyInverseGamma(κ, θ) = θ == 0 ? PowerPos(-κ-1) : InverseGamma(κ, θ)
```

Out[6]: `MyInverseGamma (generic function with 1 method)`

1 正規分布モデルの共役事前分布とその応用

1.1 逆ガンマ正規分布

平均 $\mu \in \mathbb{R}$, 分散 $v = \sigma^2 \in \mathbb{R}_{>0}$ の正規分布の確率密度函数を次のように表す:

$$p_{\mathrm{Normal}}(y|\mu, v) = \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{1}{2v}(y-\mu)^2\right) \quad (y \in \mathbb{R}).$$

分散パラメータ σ^2 を v に書き直している理由は, σ^2 を1つの変数として扱いたいからである.

パラメータ $\kappa, \theta > 0$ の逆ガンマ分布の確率密度函数を次のように書くことにする:

$$p_{\mathrm{InverseGamma}}(v|\kappa, \theta) = \frac{\theta^\kappa}{\Gamma(\kappa)} v^{-\kappa-1} \exp\left(-\frac{\theta}{v}\right) \quad (v > 0).$$

v がこの逆ガンマ分布に従う確率変数だとすると,

$$
\frac{1}{v} \sim \mathrm{Gamma}\left(\kappa, \frac{1}{\theta}\right) = \frac{1}{2\theta}\mathrm{Gamma}\left(\frac{2\kappa}{2}, 2\right) = \frac{1}{2\theta}\mathrm{Chisq}(2\kappa),
$$
$$
E[v] = \frac{\theta}{\kappa - 1}, \quad \mathrm{var}(v) = \frac{E[v]^2}{\kappa - 2}.
$$

A と B が μ, v に関する定数因子の違いを除いて等しいことを $A \propto B$ と書くことにする.

逆ガンマ正規分布の密度函数を次のように定義する:

$$
\begin{aligned}
p_{\mathrm{InverseGammaNormal}}(\mu, v|\mu_*, v_*, \kappa, \theta) &= p_{\mathrm{Normal}}(\mu|\mu_*, v_* v) p_{\mathrm{InverseGamma}}(v|\kappa, \theta) \\
&\propto v^{-(\kappa+1/2)-1} \exp\left(-\frac{1}{v}\left(\theta + \frac{1}{2v_*}(\mu - \mu_*)^2\right)\right).
\end{aligned}
$$

この逆ガンマ正規分布の密度函数に従う確率変数を μ, v と書くと,

$$
E[v] = \frac{\theta}{\kappa - 1}, \quad \mathrm{var}(v) = \frac{E[v]^2}{\kappa - 2}, \quad \mathrm{cov}(\mu, v) = 0, \quad E[\mu] = \mu_*, \quad \mathrm{var}(\mu) = v_* E[v].
$$

この逆ガンマ正規分布が正規分布の共役事前分布になっていることを次の節で確認する.

1.2 共役事前分布のBayes更新

データの数値 $y_1, \dots, y_n$ が与えられたとき, 正規分布モデルの尤度函数は

$$
\prod_{i=1}^n p_{\mathrm{Normal}}(y_i|\mu, v) \propto v^{-n/2} \exp\left(-\frac{1}{2v} \sum_{i=1}^n (y_i - \mu)^2\right)
$$

の形になる. このとき,

$$
\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i, \quad \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^2.
$$

とおくと,

$$
\sum_{i=1}^n (y_i - \mu)^2 = n(\mu - \bar{y})^2 + n\hat{\sigma}^2
$$

なので, 尤度を最大化する μ, v は $\mu = \bar{y}, v = \hat{\sigma}^2$ になることがわかる.

さらに, 次が成立することもわかる:

$$
\begin{aligned}
&\prod_{i=1}^n p_{\mathrm{Normal}}(y_i|\mu, v) \times p_{\mathrm{InverseGammaNormal}}(\mu, v|\mu_*, v_*, \kappa, \theta) \\
&\propto v^{-n/2} \exp\left(-\frac{n}{2v}\left((\mu - \bar{y})^2 + \hat{\sigma}^2\right)\right) \times v^{-(\kappa+1/2)-1} \exp\left(-\frac{1}{v}\left(\theta + \frac{1}{2v_*}(\mu - \mu_*)^2\right)\right) \\
&= v^{-(\kappa+n/2+1/2)-1} \exp\left(-\frac{1}{v}\left(\theta + \frac{n}{2}\left(\hat{\sigma}^2 + \frac{(\bar{y} - \mu_*)^2}{1 + nv_*}\right) + \frac{1 + nv_*}{2v_*}\left(\mu - \frac{\mu_* + nv_*\bar{y}}{1 + nv_*}\right)^2\right)\right).
\end{aligned}
$$

ゆえに共役事前分布から得られる事後分布のパラメータは次のようになる:

$$
\begin{aligned}
\tilde{\kappa} &= \kappa + \frac{n}{2} = \frac{n}{2}\left(1 + \frac{2\kappa}{n}\right), \\
\tilde{\theta} &= \theta + \frac{n}{2}\left(\hat{\sigma}^2 + \frac{(\bar{y} - \mu_*)^2}{1 + nv_*}\right) = \frac{n\hat{\sigma}^2}{2}\left(1 + \frac{2\theta}{n\hat{\sigma}^2} + \frac{(\bar{y} - \mu_*)^2}{(1 + nv_*)\hat{\sigma}^2}\right), \\
\tilde{\mu}_* &= \frac{\mu_* + nv_*\bar{y}}{1 + nv_*} = \bar{y}\frac{1 + \mu_*/(nv_*\bar{y})}{1 + 1/(nv_*)}, \\
\tilde{v}_* &= \frac{v_*}{1 + nv_*} = \frac{1}{n}\frac{1}{1 + 1/(nv_*)}.
\end{aligned}
$$

In [7]:
```julia
function bayesian_update(μstar, vstar, κ, θ, n, ȳ, σ̂²)
    μstar_new = (μstar/vstar + n*ȳ)/(1/vstar + n)
    vstar_new = 1/(1/vstar + n)
    κ_new = κ + n/2
    θ_new = θ + (n/2)*(σ̂² + ((ȳ - μstar)^2/vstar)/(1/vstar + n))
    μstar_new, vstar_new, κ_new, θ_new
end

function bayesian_update(μstar, vstar, κ, θ, y)
    n, ȳ, σ̂² = length(y), mean(y), var(y; corrected=false)
    bayesian_update(μstar, vstar, κ, θ, n, ȳ, σ̂²)
end
```

Out[7]:
```
bayesian_update (generic function with 2 methods)
```

In [8]:
```julia
@vars n ȳ v̂ μ v μ0 v0 κ θ
```

Out[8]:
```
(n, ȳ, v̂, μ, v, μ0, v0, κ, θ)
```

In [9]:
```julia
negloglik = n/2*log(v) + n/(2v)*((μ - ȳ)^2 + v̂)
```

Out[9]:

$$\frac{n\log(v)}{2}+\frac{n\left(\hat{v}+\left(-\bar{y}+\mu\right)^2\right)}{2v}$$

In [10]:
```julia
neglogpri = (κ + 1//2 + 1)*log(v) + 1/v*(θ + 1/(2v0)*(μ-μ0)^2)
```

Out[10]:

$$\left(\kappa+\frac{3}{2}\right)\log(v)+\frac{\theta+\frac{(\mu-\mu_0)^2}{2v_0}}{v}$$

In [11]:
```julia
neglogpost = (κ + n/2 + 1//2 + 1)*log(v) +  1/v*(
    θ + n/2*(v̂ + (ȳ - μ0)^2/(1+n*v0)) +
    (1 + n*v0)/(2v0)*(μ - (μ0 + n*v0*ȳ)/(1 + n*v0))^2)
```

Out[11]:

$$\left(\frac{n}{2}+\kappa+\frac{3}{2}\right)\log(v)+\frac{\frac{n\left(\hat{v}+\frac{(\bar{y}-\mu_0)^2}{nv_0+1}\right)}{2}+\theta+\frac{\left(\mu-\frac{nv_0\bar{y}+\mu_0}{nv_0+1}\right)^2(nv_0+1)}{2v_0}}{v}$$

In [12]:
```julia
simplify(negloglik + neglogpri - neglogpost)
```

Out[12]: 0

In [13]:
```julia
bayesian_update(μ0, v0, κ, θ, n, ȳ, v̂) ▷ collect
```

Out[13]:

$$\left[\begin{array}{c}\frac{n\bar{y}+\frac{\mu_0}{v_0}}{n+\frac{1}{v_0}}\\ \frac{1}{n+\frac{1}{v_0}}\\ \frac{n}{2}+\kappa\\ \frac{n\left(\hat{v}+\frac{(\bar{y}-\mu_0)^2}{v_0\left(n+\frac{1}{v_0}\right)}\right)}{2}+\theta\end{array}\right]$$

1.3 μの周辺事前・事後分布および事前・事後予測分布

確率密度函数

$$p(\mu|\mu_*,v_*,\kappa,\theta)=\int_{\mathbb{R}_{>0}}p_{\mathrm{InverseGammaNormal}}(\mu,v|\mu_*,v_*,\kappa,\theta)\,dv$$

で定義される μ の周辺事前分布は次になる:

$$\mu\sim\mu_*+\sqrt{\frac{\theta}{\kappa}v_*}\,\mathrm{TDist}(2\kappa).$$

なぜならば, $v \sim \mathrm{InverseGamma}(\kappa, \theta)$ のとき,

$$\frac{1}{v} \sim \mathrm{Gamma}\left(\kappa, \frac{1}{\theta}\right) = \frac{1}{2\theta}\mathrm{Gamma}\left(\frac{2\kappa}{2}, 2\right) = \frac{1}{2\theta}\mathrm{Chisq}(2\kappa)$$

より,

$$\begin{aligned}
\mu &\sim \mu_* + \sqrt{v_* v}\ \mathrm{Normal}(0,1) \\
&\sim \mu_* + \sqrt{\frac{2\theta v_*}{\mathrm{Chisq}(2\kappa)}}\ \mathrm{Normal}(0,1) \\
&= \mu_* + \sqrt{\frac{2\theta v_*}{2\kappa}}\ \frac{\mathrm{Normal}(0,1)}{\sqrt{\mathrm{Chisq}(2\kappa)/(2\kappa)}} \\
&= \mu_* + \sqrt{\frac{\theta}{\kappa} v_*}\ \mathrm{TDist}(2\kappa).
\end{aligned}$$

y_{new} の事前予測分布は, 確率密度函数

$$p_*(y_{\mathrm{new}}|\mu_*, v_*, \kappa, \theta) = \iint_{\mathbb{R}\times\mathbb{R}_{>0}} p_{\mathrm{Normal}}(y_{\mathrm{new}}|\mu, v) p_{\mathrm{InverseGammaNormal}}(\mu, v|\mu_*, v_*, \kappa, \theta)\, d\mu\, dv$$

によって定義される. このとき

$$\begin{aligned}
\int_{\mathbb{R}} p_{\mathrm{Normal}}(y_{\mathrm{new}}|\mu, v) p_{\mathrm{Normal}}(\mu|\mu_*, v_* v)\, d\mu &= p_{\mathrm{Normal}}(y_{\mathrm{new}}|\mu_*, v + v * v) \\
&= p_{\mathrm{Normal}}(y_{\mathrm{new}}|\mu_*, v(1+v_*))
\end{aligned}$$

であることより,

$$p_*(y_{\mathrm{new}}|\mu_*, v_*, \kappa, \theta) = \int_{\mathbb{R}_{>0}} p_{\mathrm{InverseGammaNormal}}(y_{\mathrm{new}}, v(1+v_*)|\mu_*, v_*, \kappa, \theta)\, dv.$$

ゆえに, μ の周辺事前分布の場合の計算より,

$$y_{\mathrm{new}} \sim \mu_* + \sqrt{\frac{\theta}{\kappa}(1+v_*)}\ \mathrm{TDist}(2\kappa).$$

パラメータをBayes更新後のパラメータ

$$\begin{aligned}
\tilde{\kappa} &= \kappa + \frac{n}{2} = \frac{n}{2}\left(1 + \frac{2\kappa}{n}\right), \\
\tilde{\theta} &= \theta + \frac{n}{2}\left(\hat{\sigma}^2 + \frac{(\bar{y}-\mu_*)^2}{1+nv_*}\right) = \frac{n\hat{\sigma}^2}{2}\left(1 + \frac{2\theta}{n\hat{\sigma}^2} + \frac{(\bar{y}-\mu_*)^2}{(1+nv_*)\hat{\sigma}^2}\right), \\
\tilde{\mu}_* &= \frac{\mu_* + nv_*\bar{y}}{1+nv_*} = \bar{y}\frac{1+\mu_*/(nv_*\bar{y})}{1+1/(nv_*)}, \\
\tilde{v}_* &= \frac{v_*}{1+nv_*} = \frac{1}{n}\frac{1}{1+1/(nv_*)}.
\end{aligned}$$

に置き換えればこれは μ の周辺事後分布および事後予測分布になる.

その事後分布を使った区間推定の幅は

- n が大きいほど狭くなる.
- κ が大きいほど狭くなる.
- θ が大きいほど広くなる.
- $|\bar{y} - \mu_*|/\hat{\sigma}$ が大きいほど広くなる.
- $|\bar{y} - \mu_*|/\hat{\sigma}$ が大きくても, v_* がさらに大きければ狭くなる.

In [14]:
```julia
posterior_μ(μstar, vstar, κ, θ) = μstar + √(θ/κ*vstar)*TDist(2κ)
preddist(μstar, vstar, κ, θ) = μstar + √(θ/κ*(1 + vstar))*TDist(2κ)
```

Out[14]: `preddist (generic function with 1 method)`

1.4 Jeffreys事前分布の場合

パラメータ空間が $\{(\mu, v) = (\mu, \sigma^2) \in \mathbb{R}\times\mathbb{R}_{>0}\}$ の 2 次元の正規分布モデルのJeffreys事前分布 $p_{\mathrm{Jeffreys}}(\mu, v)$ は

$$p_{\text{Jeffreys}}(\mu, v) \propto v^{-3/2}$$

になることが知られている. ただし, 右辺の $(\mu, v) \in \mathbb{R} \times \mathbb{R}_{>0}$ に関する積分は ∞ になるので, この場合のJeffreys事前分布はimproperである.

逆ガンマ正規分布の密度函数

$$p_{\text{InverseGammaNormal}}(\mu, v|\mu_*, v_*, \kappa, \theta) \propto v^{-(\kappa+1/2)-1} \exp\left(-\frac{1}{v}\left(\theta + \frac{1}{2v_*}(\mu-\mu_*)^2\right)\right).$$

と比較すると, Jeffreys事前分布に対応する共役事前分布のパラメータ値は形式的に次になることがわかる:

$$\kappa \to 0, \quad \theta \to 0, \quad v_* \to \infty.$$

そのとき, Bayes更新後のパラメータの公式は次のようにシンプルになる:

$$\tilde{\kappa} = \frac{n}{2}, \quad \tilde{\theta} = \frac{n\hat{\sigma}^2}{2}, \quad \tilde{\mu}_* = \bar{y}, \quad \tilde{v}_* = \frac{1}{n}.$$

さらに, 前節の公式から, $n \to \infty$ のとき, 一般のパラメータ値に関するBayes更新の結果は, $n \to \infty$ のとき漸近的にこのJeffreys事前分布の場合に一致する.

さらに, Jeffreys事前分布の場合には

$$\frac{\tilde{\theta}}{\tilde{\kappa}} = \hat{\sigma}^2, \quad \tilde{v}_* = \frac{1}{n}, \quad 2\tilde{\kappa} = n.$$

ゆえに, μ に関する周辺事後分布は

$$\mu \sim \bar{y} + \frac{\hat{\sigma}}{\sqrt{n}}\ \text{TDist}(n)$$

になり, 事後予測分布は次になる:

$$y_{\text{new}} \sim \bar{y} + \hat{\sigma}\sqrt{1+\frac{1}{n}}\ \text{TDist}(n).$$

In [15]:
```julia
prior_jeffreys() = 0.0, Inf, 0.0, 0.0

posterior_μ_jeffreys(n, ȳ, σ̂²) = ȳ + √(σ̂²/n)*TDist(n)

function posterior_μ_jeffreys(y)
    n, ȳ, σ̂² = length(y), mean(y), var(y; corrected=false)
    posterior_μ_jeffreys(n, ȳ, σ̂²)
end

preddist_jeffreys(n, ȳ, σ̂²) = ȳ + √(σ̂²*(1+1/n))*TDist(n)

function preddist_jeffreys(y)
    n, ȳ, σ̂² = length(y), mean(y), var(y; corrected=false)
    preddist_jeffreys(n, ȳ, σ̂²)
end
```

Out[15]:
```
preddist_jeffreys (generic function with 2 methods)
```

In [16]:
```julia
μ_true, σ_true, n = 10, 3, 5
@show dist_true = Normal(μ_true, σ_true) n
y = rand(Normal(μ_true, σ_true), n)
```

```
dist_true = Normal(μ_true, σ_true) = Normal{Float64}(μ=10.0, σ=3.0)
n = 5
```

Out[16]:
```
5-element Vector{Float64}:
 12.53108543491323
 10.859953860136164
  9.647916540030597
 14.29645219454655
  5.808917945819022
```

In [17]:
```julia
n, ȳ, σ̂² = length(y), mean(y), var(y; corrected=false)
```

Out[17]:
```
(5, 10.62886519508911, 8.263437992021275)
```

In [18]:
```julia
post_μ = posterior_μ(bayesian_update(prior_jeffreys()..., y)...)
```

Out[18]:
```
LocationScale{Float64, Continuous, TDist{Float64}}(
μ: 10.62886519508911
σ: 1.285568978469944
ρ: TDist{Float64}(ν=5.0)
)
```

In [19]:
```julia
posterior_μ_jeffreys(y) ≈ post_μ
```

Out[19]:
```
true
```

1.5 Jeffreys事前分布の場合の結果の数値的確認

In [20]:
```julia
# プロット用函数

function plot_posterior_μ(chn, y, postμ_theoretical;
        xlim = quantile.(postμ_theoretical, (0.0001, 0.9999)), kwargs...)
    postμ_ttest = posterior_μ_ttest(y)
    plot(legend=:outertop)
    if !isnothing(chn)
        stephist!(vec(chn[:μ]); norm=true,
            label="MCMC posterior of μ", c=1)
    end
    plot!(postμ_theoretical, xlim...;
        label="theoretical posterior of μ", c=2, ls=:dash)
    plot!(postμ_ttest, xlim...;
        label="ȳ+√(s²/n)TDist(n-1)", c=3, ls=:dashdot)
    plot!(; xlim, kwargs...)
end

function plot_preddist(chn, y, pred_theoretical;
        xlim = quantile.(pred_theoretical, (0.0001, 0.9999)), kwargs...)
    pdf_pred(y_new) = mean(pdf(Normal(μ, √σ²), y_new)
        for (μ, σ²) in zip(vec(chn[:μ]), vec(chn[:σ²])))
    pred_ttest = preddist_ttest(y)

    plot(legend=:outertop)
    if !isnothing(chn)
        plot!(pdf_pred, xlim...; label="MCMC prediction", c=1)
    end
    plot!(pred_theoretical, xlim...;
        label="theoretical prediction", c=2, ls=:dash)
    plot!(pred_ttest, xlim...;
        label="ȳ+√(s²(1+1/n))TDist(n-1)", c=3, ls=:dashdot)
    plot!(; kwargs...)
end
```

Out[20]:
```
plot_preddist (generic function with 1 method)
```

In [21]:
```julia
@model function normaldistmodel_jeffreys(y)
    σ² ~ PowerPos(-3/2)
    μ ~ Flat()
    y ~ MvNormal(fill(μ, length(y)), σ²*I)
end
```

Out[21]:
```
normaldistmodel_jeffreys (generic function with 2 methods)
```

In [22]:
```julia
μ_true, σ_true, n = 1e4, 1e2, 5
@show dist_true = Normal(μ_true, σ_true) n
y = rand(Normal(μ_true, σ_true), n)
```

```
dist_true = Normal(μ_true, σ_true) = Normal{Float64}(μ=10000.0, σ=100.0)
n = 5
```

Out[22]:
```
5-element Vector{Float64}:
  9871.967706849937
  9979.570383530274
 10101.128050729112
 10191.06138205461
  9903.360736211474
```

In [23]:

```julia
L = 10^5
n_threads = min(Threads.nthreads(), 10)
chn = sample(normaldistmodel_jeffreys(y), NUTS(), MCMCThreads(), L, n_threads);
```

```
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
```

In [24]:

```julia
chn
```

Out[24]:

```
Chains MCMC chain (100000×14×10 Array{Float64, 3}):

Iterations        = 1001:1:101000
Number of chains  = 10
Samples per chain = 100000
Wall duration     = 30.2 seconds
Compute duration  = 259.5 seconds
parameters        = σ², μ
internals         = lp, n_steps, is_accept, acceptance_rate, log_density, hamiltonian_energy, ha
miltonian_energy_error, max_hamiltonian_energy_error, tree_depth, numerical_error, step_size, no
m_step_size

Summary Statistics
  parameters         mean          std   naive_se      mcse           ess      rhat   ess_per_se
c
      Symbol      Float64      Float64    Float64   Float64       Float64   Float64      Float6
4

          σ²   24032.5834   31054.6236    31.0546   54.8659   302335.5550    1.0000     1165.065
1
           μ   10009.4698      69.0766     0.0691    0.1128   377614.2969    1.0000     1455.155
5

Quantiles
  parameters        2.5%        25.0%        50.0%        75.0%        97.5%
      Symbol     Float64      Float64      Float64      Float64      Float64

          σ²   5626.9990   10937.4676   16624.8592   27029.5096   86415.2164
           μ   9871.6226    9970.3075   10009.4220   10048.4509   10147.3769
```

In [25]:

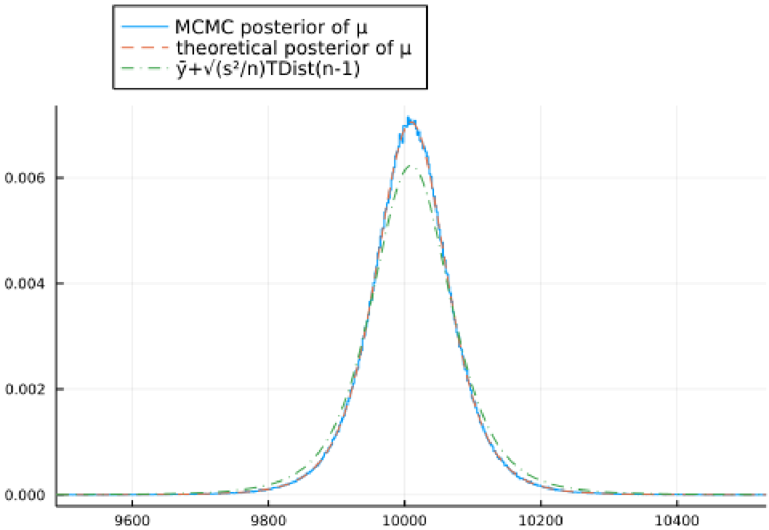
```julia
@show confint_ttest(y);
```

```
confint_ttest(y) = [9842.326560237438, 10176.508743512724]
```

In [26]:

```
postμ_theoretical = posterior_μ_jeffreys(y)
plot_posterior_μ(chn, y, postμ_theoretical)
```

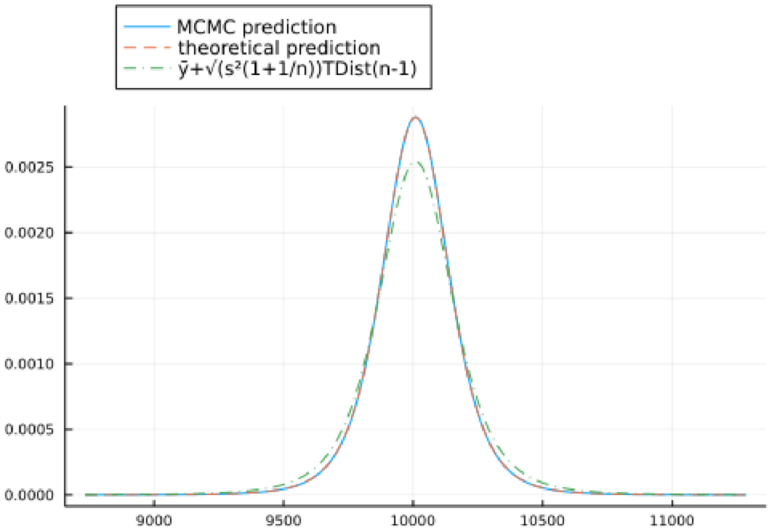
Out[26]:



In [27]:

```
pred_theoretical = preddist_jeffreys(y)
plot_preddist(chn, y, pred_theoretical)
```

Out[27]:



1.6 平均と対数分散について一様な事前分布の場合

平均 μ と分数の対数 $\log v = \log \sigma^2$ に関する一様な事前分布は

$$p_{\text{flat}}(\mu, v) \propto v^{-1}$$

になる. ただし, 右辺の $(\mu, v) \in \mathbb{R} \times \mathbb{R}_{>0}$ に関する積分は ∞ になるので, この事前分布はimproperである.

逆ガンマ正規分布の密度函数

$$p_{\mathrm{InverseGammaNormal}}(\mu, v | \mu_*, v_*, \kappa, \theta) \propto v^{-(\kappa+1/2)-1} \exp\left(-\frac{1}{v}\left(\theta + \frac{1}{2v_*}(\mu - \mu_*)^2\right)\right).$$

と比較すると, 平均と対数分散について一様な事前分布に対応する共役事前分布のパラメータ値は形式的に次になることがわかる:

$$\kappa \to -\frac{1}{2}, \quad \theta \to 0, \quad v_* \to \infty.$$

このとき, Bayes更新後のパラメータの公式は次のようになる:

$$\tilde{\kappa} = \frac{n-1}{2}, \quad \tilde{\theta} = \frac{n\hat{\sigma}^2}{2}, \quad \tilde{\mu}_* = \bar{y}, \quad \tilde{v}_* = \frac{1}{n}.$$

この場合には

$$\frac{\tilde{\theta}}{\tilde{\kappa}} = \frac{n\hat{\sigma}^2}{n-1} = s^2, \quad \tilde{v}_* = \frac{1}{n}, \quad 2\tilde{\kappa} = n - 1.$$

ここで, s^2 はデータの数値 $y_1, \dots, y_n$ の不偏分散

$$s^2 = \frac{1}{n-1}\sum_{i=1}^n (y_i - \bar{y})^2 = \frac{n\hat{\sigma}^2}{n-1} > \hat{\sigma}^2$$

であり, s はその平方根である.

ゆえに, μ に関する周辺事後分布は

$$\mu \sim \bar{y} + \frac{s}{\sqrt{n}}\,\mathrm{TDist}(n-1)$$

になり, y_{new} に関する事後予測分布は次になる:

$$y_{\mathrm{new}} \sim \bar{y} + s\sqrt{1 + \frac{1}{n}}\,\mathrm{TDist}(n-1).$$

したがって, 前節の結果と比較すると, Jeffreys事前分布の事後分布と予測分布による区間推定よりもこの場合の区間推定は少し広くなる.

In [28]:
```
prior_flat() = 0.0, Inf, -1/2, 0.0

posterior_μ_flat(n, ȳ, s²) = ȳ + √(s²/n)*TDist(n-1)

function posterior_μ_flat(y)
    n, ȳ, s² = length(y), mean(y), var(y)
    posterior_μ_flat(n, ȳ, s²)
end

preddist_flat(n, ȳ, s²) = ȳ + √(s²*(1+1/n))*TDist(n-1)

function preddist_flat(y)
    n, ȳ, s² = length(y), mean(y), var(y)
    preddist_flat(n, ȳ, s²)
end
```

Out[28]:
```
preddist_flat (generic function with 2 methods)
```

In [29]:
```
y = rand(Normal(10, 3), 5)
@show dist_true = Normal(μ_true, σ_true) n
n, ȳ, s² = length(y), mean(y), var(y)
```

```
dist_true = Normal(μ_true, σ_true) = Normal{Float64}(μ=10000.0, σ=100.0)
n = 5
```

Out[29]:
```
(5, 11.1106880662252, 21.104519098898074)
```

In [30]:
```
post_μ = posterior_μ(bayesian_update(prior_flat()..., y)...)
```

Out[30]:
```
LocationScale{Float64, Continuous, TDist{Float64}}(
μ: 11.1106880662252
σ: 2.0544838329321586
ρ: TDist{Float64}(ν=4.0)
)
```

In [31]:
```julia
posterior_μ_flat(y) ≈ post_μ
```

Out[31]:
```
true
```

1.7 平均と対数分散について一様な事前分布の場合の結果の数値的確認

In [32]:
```julia
@model function normaldistmodel_flat(y)
    σ² ~ PowerPos(-1)
    μ ~ Flat()
    y ~ MvNormal(fill(μ, length(y)), σ²*I)
end
```

Out[32]:
```
normaldistmodel_flat (generic function with 2 methods)
```

In [33]:
```julia
μ_true, σ_true, n = 1e4, 1e2, 5
@show dist_true = Normal(μ_true, σ_true) n
y = rand(Normal(μ_true, σ_true), n)
```

```
dist_true = Normal(μ_true, σ_true) = Normal{Float64}(μ=10000.0, σ=100.0)
n = 5
```

Out[33]:
```
5-element Vector{Float64}:
 10108.020838164251
 10155.518276574256
 10025.63825824536
  9873.923285032452
  9922.184234799222
```

In [34]:
```julia
L = 10^5
n_threads = min(Threads.nthreads(), 10)
chn = sample(normaldistmodel_flat(y), NUTS(), MCMCThreads(), L, n_threads);
```

```
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
```

In [35]:
```
chn
```

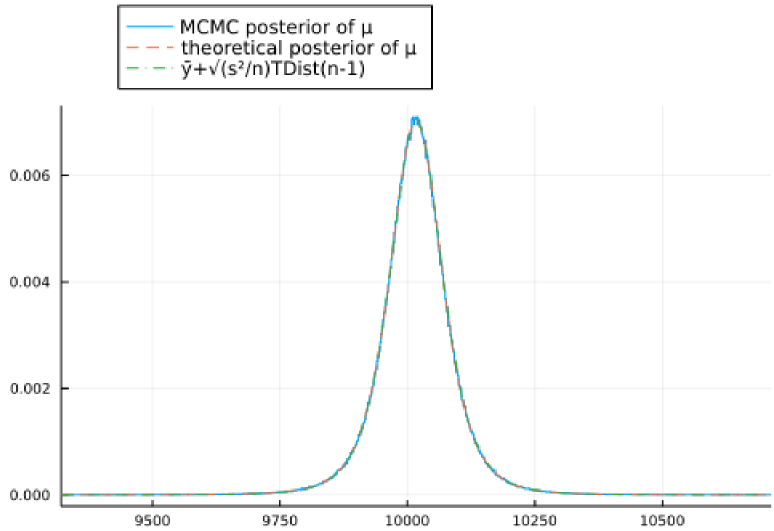
Out[35]:
```
Chains MCMC chain (100000×14×10 Array{Float64, 3}):

Iterations        = 1001:1:101000
Number of chains  = 10
Samples per chain = 100000
Wall duration     = 17.9 seconds
Compute duration  = 163.47 seconds
parameters        = σ², μ
internals         = lp, n_steps, is_accept, acceptance_rate, log_density, hamiltonian_energy, ha
miltonian_energy_error, max_hamiltonian_energy_error, tree_depth, numerical_error, step_size, no
m_step_size

Summary Statistics
  parameters         mean           std   naive_se       mcse           ess      rhat   ess_per_
sec
      Symbol      Float64       Float64    Float64    Float64       Float64   Float64      Floa
t64

          σ²   29227.7665   180014.2432   180.0142   532.8373   114731.9787    1.0001      701.8
448
           μ   10016.9103       77.8501     0.0779     0.1765   182480.9275    1.0000     1116.2
825

Quantiles
  parameters        2.5%       25.0%       50.0%       75.0%       97.5%
      Symbol     Float64     Float64     Float64     Float64     Float64

          σ²   5117.1201  10598.1820  16998.3379  29745.5589  118351.6627
           μ   9868.0050   9977.5443  10017.0374  10056.6491   10165.3243
```

In [36]:
```
@show confint_ttest(y);
```

```
confint_ttest(y) = [9868.825378827085, 10165.288578299132]
```

In [37]:
```
postμ_theoretical = posterior_μ_flat(y)
plot_posterior_μ(chn, y, postμ_theoretical)
```

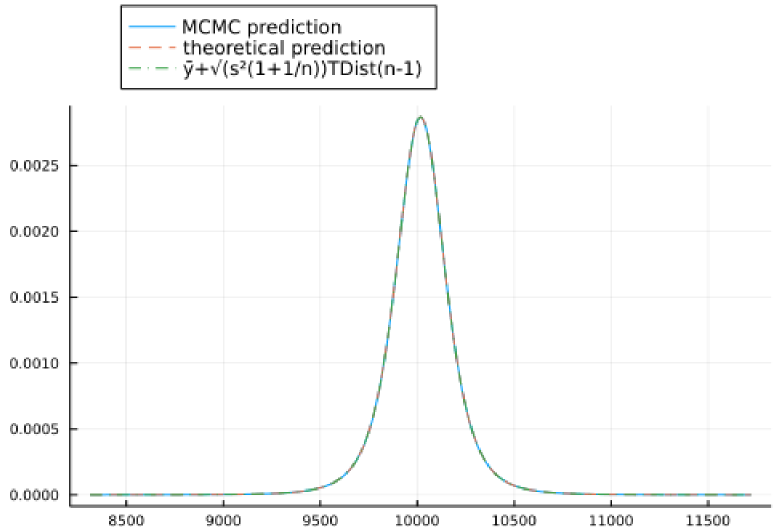
Out[37]:



In [38]:

```
pred_theoretical = preddist_flat(y)
plot_preddist(chn, y, pred_theoretical)
```

Out[38]:



1.8 通常の信頼区間と予測区間との比較

通常の t 分布を使う平均の信頼区間と次の値の予測区間の構成では以下を使う:

$$\frac{\bar{y}-\mu}{s/\sqrt{n}} \sim \mathrm{TDist}(n-1), \qquad \frac{y_{\mathrm{new}}-\bar{y}}{s\sqrt{1+1/n}} \sim \mathrm{TDist}(n-1).$$

ここで, s^2 はデータの数値の不偏分散であり, s はその平方根である.

したがって, 前節の結果と比較すると, 通常の信頼区間と予測区間は, 平均と対数分散に関する一様事前分布に関する事後分布と予測分布を用いた区間推定に一致する.

1.9 データの数値から事前分布を決めた場合

$a, b > 0$ であると仮定する.

データの数値から共役事前分布のパラメータを次の条件によって決めたと仮定する:

$$E[\mu] = \mu_* = \bar{y}, \quad E[v] = \frac{\theta}{\kappa-1} = \hat{\sigma}^2, \quad \mathrm{var}(\mu) = v_* E[v] = a\hat{\sigma}^2, \quad \mathrm{var}(v) = \frac{E[v]^2}{\kappa-2} = b\hat{\sigma}^4.$$

これは次と同値である:

$$\mu_* = \bar{y}, \quad v_* = a, \quad \kappa = 2 + \frac{1}{b}, \quad \theta = \hat{\sigma}^2\left(1+\frac{1}{b}\right).$$

このパラメータ値に対応する共役事前分布を以下では **適応事前分布** (adaptive prior)と呼ぶことにする(注意: ここだけの用語).

これのBayes更新の結果は以下のようになる:

$$
\begin{aligned}
\tilde{\kappa} &= 2 + \frac{1}{b} + \frac{n}{2} = \frac{n}{2}\left(1 + \frac{2(2+1/b)}{n}\right) && \to 2 + \frac{n}{2},\\
\tilde{\theta} &= \hat{\sigma}^2\left(1 + \frac{1}{b} + \frac{n}{2}\right) + \frac{n}{2}\frac{(\bar{y}-\bar{y})^2}{1+na} = \frac{n\hat{\sigma}^2}{2}\left(1 + \frac{2(1+1/b))}{n}\right) && \to \hat{\sigma}^2\left(1 + \frac{n}{2}\right),\\
\tilde{\mu}_* &= \frac{\bar{y} + nv_*\bar{y}}{1+nv_*} = \bar{y} && \to \bar{y},\\
\tilde{v}_* &= \frac{a}{1+na} = \frac{1}{n}\frac{1}{1+1/(na)} && \to \frac{1}{n}.
\end{aligned}
$$

以上における $\to$ は $a \to \infty$, $b \to \infty$ での極限を意味する.

適応事前分布の構成のポイントは, $\mu_* = \bar{y}$ となっているおかげで, $\tilde{\mu}_*$ も $\tilde{\mu}_* = \bar{y}$ となってバイアスが消え, さらに, $\tilde{\theta}$ の中の $\dfrac{n}{2}\dfrac{(\bar{y}-\mu_*)^2}{1+na}$ の項が消えて, 区間推定の幅が無用に広くならずに済むことである.

ただし, 適応事前分布の場合には

$$
\frac{\tilde{\theta}}{\tilde{\kappa}} = \hat{\sigma}^2\frac{1+2(1+1/b)/n}{1+2(2+1/b)/n} < \hat{\sigma}^2, \quad v_* = \frac{1}{n}\frac{1}{1+1/(na)} < \frac{1}{n}
$$

なので, 区間推定の幅はJeffreys事前分布の場合よりも少し狭くなる.

しかし, n が大きければそれらの違いは小さくなる.

In [39]:

```julia
function prior_adaptive(n, ȳ, σ̂²; a = 2.5, b = 2.5)
    μstar = ȳ
    vstar = a
    κ = 2 + 1/b
    θ = σ̂²*(1 + 1/b)
    μstar, vstar, κ, θ
end

function prior_adaptive(y; a = 2.5, b = 2.5)
    n, ȳ, σ̂² = length(y), mean(y), var(y; corrected=false)
    prior_adaptive(n, ȳ, σ̂²; a, b)
end

function posterior_adaptive(n, ȳ, σ̂²; a = 2.5, b = 2.5)
    μstar = ȳ
    vstar = 1/(1/a + n)
    κ = 2 + 1/b + n/2
    θ = σ̂²*(1 + 1/b + n/2)
    μstar, vstar, κ, θ
end

function posterior_adaptive(y; a = 2.5, b = 2.5)
    n, ȳ, σ̂² = length(y), mean(y), var(y; corrected=false)
    posterior_adaptive(n, ȳ, σ̂²; a, b)
end
```

Out[39]:

```
posterior_adaptive (generic function with 2 methods)
```

In [40]:

```julia
μ_true, σ_true, n = 1e4, 1e2, 5
@show dist_true = Normal(μ_true, σ_true) n
y = rand(Normal(μ_true, σ_true), n)
```

```
dist_true = Normal(μ_true, σ_true) = Normal{Float64}(μ=10000.0, σ=100.0)
n = 5
```

Out[40]:

```
5-element Vector{Float64}:
 10139.744551661583
 10060.228608645126
 10072.121209420195
  9760.871797333557
 10019.941184983956
```

In [41]:

```julia
n, ȳ, σ̂² = length(y), mean(y), var(y; corrected=false)
```

Out[41]:

```
(5, 10010.581470408884, 17075.520891126696)
```

In [42]:
```
μstar, vstar, κ, θ = prior_adaptive(y)
a, b = 2.5, 2.5
@show ȳ, σ̂², a*σ̂², b*σ̂²^2
(ȳ, σ̂², a*σ̂², b*σ̂²^2) .≈ (μstar, θ/(κ - 1), (θ/(κ - 1))*vstar, (θ/(κ - 1))^2/(κ - 2))
```

```
(ȳ, σ̂², a * σ̂², b * σ̂² ^ 2) = (10010.581470408884, 17075.520891126696, 42688.80222781674, 7.2893
35342582606e8)
```

Out[42]:
```
(true, true, true, true)
```

In [43]:
```
posterior_adaptive(n, ȳ, σ̂²)
```

Out[43]:
```
(10010.581470408884, 0.18518518518518517, 4.9, 66594.53147539412)
```

In [44]:
```
bayesian_update(prior_adaptive(y)..., y)
```

Out[44]:
```
(10010.581470408884, 0.18518518518518517, 4.9, 66594.53147539412)
```

In [45]:
```
posterior_adaptive(y)
```

Out[45]:
```
(10010.581470408884, 0.18518518518518517, 4.9, 66594.53147539412)
```

In [46]:
```
posterior_adaptive(y) .≈ bayesian_update(prior_adaptive(y)..., y)
```

Out[46]:
```
(true, true, true, true)
```

1.10 $n = 5$ では適応事前分布の場合と無情報事前分布の場合の結果が結構違う.

In [47]:
```
@model function normaldistmodel_adaptive(y; a = 2.5, b = 2.5)
    μstar, vstar, κ, θ = prior_adaptive(y; a, b)
    σ² ~ InverseGamma(κ, θ)
    μ ~ Normal(μstar, √(vstar * σ²))
    y ~ MvNormal(fill(μ, length(y)), σ²*I)
end
```

Out[47]:
```
normaldistmodel_adaptive (generic function with 2 methods)
```

In [48]:
```
μ_true, σ_true, n = 1e4, 1e2, 5
@show dist_true = Normal(μ_true, σ_true) n
y = rand(Normal(μ_true, σ_true), n)
```

```
dist_true = Normal(μ_true, σ_true) = Normal{Float64}(μ=10000.0, σ=100.0)
n = 5
```

Out[48]:
```
5-element Vector{Float64}:
  9933.804443506962
  9928.461031727456
 10090.805438322303
  9961.11410617526
 10159.51959338753
```

In [49]:

```julia
L = 10^5
n_threads = min(Threads.nthreads(), 10)
chn = sample(normaldistmodel_adaptive(y), NUTS(), MCMCThreads(), L, n_threads);
```

```
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
```

In [50]:

```julia
chn
```

Out[50]:

```
Chains MCMC chain (100000×14×10 Array{Float64, 3}):

Iterations        = 1001:1:101000
Number of chains  = 10
Samples per chain = 100000
Wall duration     = 19.27 seconds
Compute duration  = 169.31 seconds
parameters        = σ², μ
internals         = lp, n_steps, is_accept, acceptance_rate, log_density, hamiltonian_energy, ha
miltonian_energy_error, max_hamiltonian_energy_error, tree_depth, numerical_error, step_size, no
m_step_size

Summary Statistics
  parameters         mean         std   naive_se      mcse           ess      rhat   ess_per_sec
      Symbol      Float64     Float64    Float64   Float64       Float64   Float64       Float64

          σ²    8721.2039   5089.0278     5.0890    7.0544   493919.9453    1.0000     2917.2521
           μ   10014.7144     40.2007     0.0402    0.0509   636961.0189    1.0000     3762.0992

Quantiles
  parameters        2.5%       25.0%        50.0%        75.0%        97.5%
      Symbol     Float64     Float64      Float64      Float64      Float64

          σ²   3365.9838   5525.6441    7442.9868   10367.6196   21678.5083
           μ   9934.5279   9989.6082   10014.6436   10039.7659   10095.0421
```

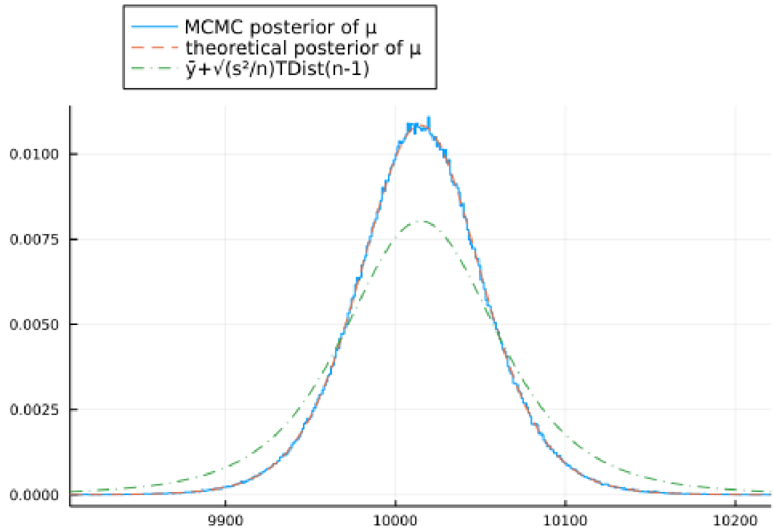
In [51]:

```julia
@show confint_ttest(y);
```

```
confint_ttest(y) = [9885.081467238768, 10144.400378009035]
```

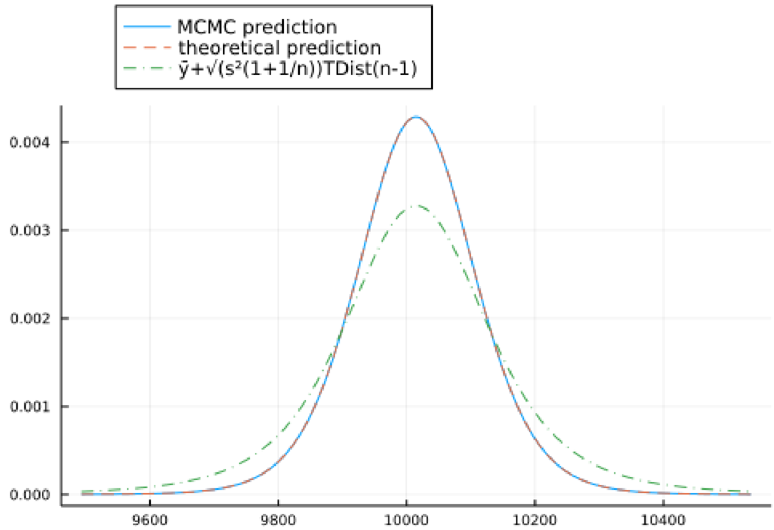
In [52]:
```
postμ_theoretical = posterior_μ(posterior_adaptive(y)...)
plot_posterior_μ(chn, y, postμ_theoretical)
```

Out[52]:



In [53]:
```
pred_theoretical = preddist(posterior_adaptive(y)...)
plot_preddist(chn, y, pred_theoretical)
```

Out[53]:



以上のように $n = 5$ の場合には, 適応事前分布の場合の結果は無情報事前分布の場合の結果(緑のdashdotライン)とかなり違う.

1.11 n = 20 ではデフォルト事前分布の場合と無情報事前分布の場合の結果が近付く.

In [54]:
```
μ_true, σ_true, n = 1e4, 1e2, 20
@show dist_true = Normal(μ_true, σ_true)
y = rand(dist_true, n)
@show length(y) mean(y) var(y);
```

```
dist_true = Normal(μ_true, σ_true) = Normal{Float64}(μ=10000.0, σ=100.0)
length(y) = 20
mean(y) = 9987.869164116411
var(y) = 10349.825335803103
```

In [55]:

```julia
L = 10^5
n_threads = min(Threads.nthreads(), 10)
chn = sample(normaldistmodel_adaptive(y), NUTS(), MCMCThreads(), L, n_threads);
```

```
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
```

In [56]:

```julia
chn
```

Out[56]:

```
Chains MCMC chain (100000×14×10 Array{Float64, 3}):

Iterations        = 1001:1:101000
Number of chains  = 10
Samples per chain = 100000
Wall duration     = 18.42 seconds
Compute duration  = 148.11 seconds
parameters        = σ², μ
internals         = lp, n_steps, is_accept, acceptance_rate, log_density, hamiltonian_energy, ha
miltonian_energy_error, max_hamiltonian_energy_error, tree_depth, numerical_error, step_size, no
m_step_size

Summary Statistics
  parameters       mean         std   naive_se      mcse           ess      rhat   ess_per_sec
      Symbol    Float64     Float64    Float64   Float64       Float64   Float64       Float64

          σ²  9828.4697   3038.8520     3.0389    3.6317   712632.4480    1.0000     4811.4755
           μ  9987.8811     21.9403     0.0219    0.0250   791733.7360    1.0000     5345.5431

Quantiles
  parameters       2.5%       25.0%      50.0%       75.0%       97.5%
      Symbol    Float64     Float64    Float64     Float64     Float64

          σ²  5550.7652   7703.6085  9287.0791  11336.5365  17241.9121
           μ  9944.3997   9973.4862  9987.8736  10002.2964  10031.1712
```

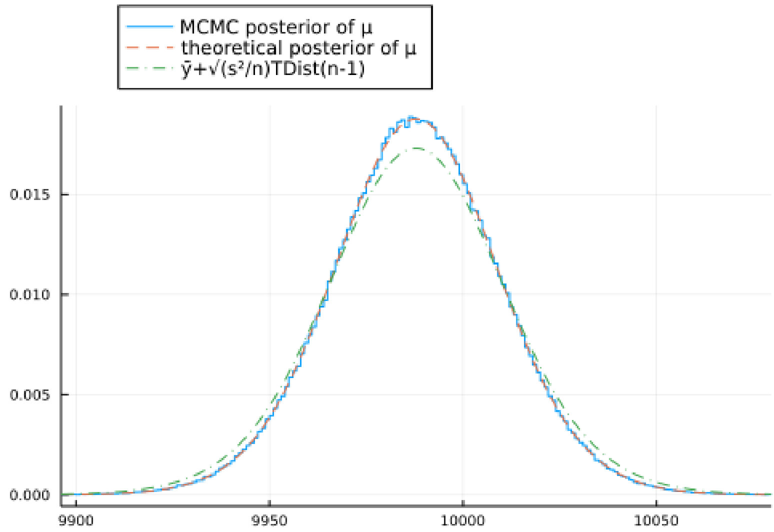
In [57]:

```julia
@show confint_ttest(y);
```

```
confint_ttest(y) = [9940.25614375669, 10035.482184476134]
```

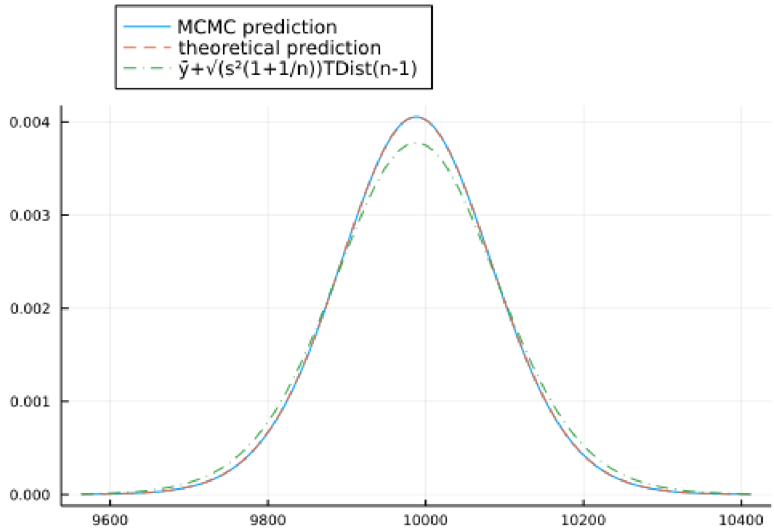
In [58]:
```julia
postμ_theoretical = posterior_μ(posterior_adaptive(y)...)
plot_posterior_μ(chn, y, postμ_theoretical)
```

Out[58]:



In [59]:
```julia
pred_theoretical = preddist(posterior_adaptive(y)...)
plot_preddist(chn, y, pred_theoretical)
```

Out[59]:



1.12 n = 20 で事前分布とデータの数値の相性が悪い場合

In [60]:
```julia
@model function normaldistmodel(y, μstar, vstar, κ, θ)
    σ² ~ InverseGamma(κ, θ)
    μ ~ Normal(μstar, √(vstar * σ²))
    y ~ MvNormal(fill(μ, length(y)), σ²*I)
end
```

Out[60]: normaldistmodel (generic function with 2 methods)

In [61]:
```
# 固定された事前分布の設定

a, b = 5.0^2, 5.0^2
μstar, vstar, κ, θ = 0.0, a, 2 + 1/b, 1 + 1/b
@show μstar vstar κ θ
println()

Eμ, Ev = μstar, θ/(κ - 1)
var_μ, var_v = vstar*Ev, Ev^2/(κ - 2)
@show Eμ Ev var_μ var_v;
```

```
μstar = 0.0
vstar = 25.0
κ = 2.04
θ = 1.04

Eμ = 0.0
Ev = 1.0
var_μ = 25.0
var_v = 24.99999999999998
```

以下では以上のようにして定めた事前分布を使う.

この事前分布における μ の平均と分散はそれぞれ 0 と 5^2 であり, $v = \sigma^2$ の平均と分散はそれぞれ 1 と 5^2 である.

In [62]:
```
μ_true, σ_true, n = 1e4, 1e2, 20
@show dist_true = Normal(μ_true, σ_true)
y = rand(dist_true, n)
@show length(y) mean(y) var(y);
```

```
dist_true = Normal(μ_true, σ_true) = Normal{Float64}(μ=10000.0, σ=100.0)
length(y) = 20
mean(y) = 10018.197822009017
var(y) = 17010.479215472027
```

平均と分散がそれぞれ 10000, 100^2 の正規分布でサイズ 20 のサンプルを生成している.

平均 10000 と分散 100^2 は上で定めた事前分布と極めて相性が悪い.

In [63]:
```
L = 10^5
n_threads = min(Threads.nthreads(), 10)
chn = sample(normaldistmodel(y, μstar, vstar, κ, θ), NUTS(), MCMCThreads(), L, n_threads);
```

```
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
│   isfinite.((θ, r, ℓπ, ℓκ)) = (true, false, false, false)
└ @ AdvancedHMC D:\.julia\packages\AdvancedHMC\iWHPQ\src\hamiltonian.jl:47
┌ Warning: The current proposal will be rejected due to numerical error(s).
```

In [64]:
```
chn
```

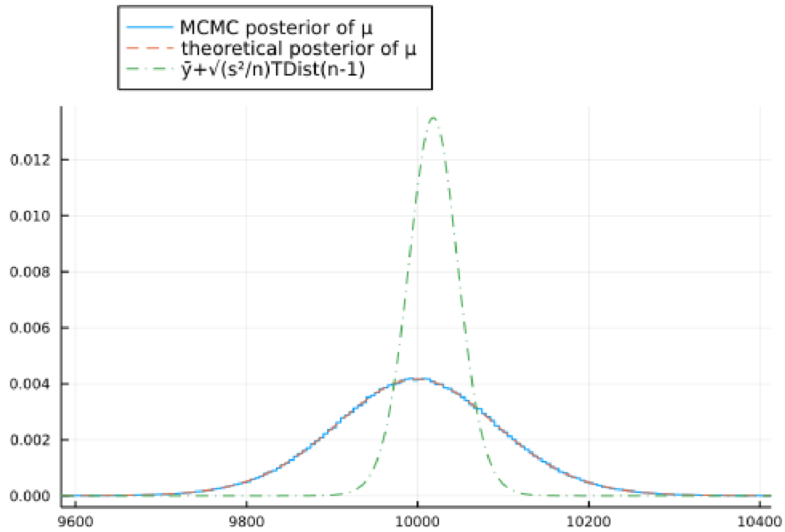
Out[64]:
```
Chains MCMC chain (100000×14×10 Array{Float64, 3}):

Iterations        = 1001:1:101000
Number of chains  = 10
Samples per chain = 100000
Wall duration     = 16.31 seconds
Compute duration  = 158.74 seconds
parameters        = σ², μ
internals         = lp, n_steps, is_accept, acceptance_rate, log_density, hamiltonian_energy, ha
miltonian_energy_error, max_hamiltonian_energy_error, tree_depth, numerical_error, step_size, no
m_step_size

Summary Statistics
  parameters          mean          std    naive_se       mcse           ess      rhat   ess_per_s
ec
      Symbol       Float64      Float64     Float64    Float64       Float64   Float64       Float
64

          σ²   196157.5206   61791.6263     61.7916    74.2261   765686.0192    1.0000     4823.37
09
           μ     9998.4619      98.7598      0.0988     0.1105   855541.1202    1.0000     5389.40
51

Quantiles
  parameters          2.5%         25.0%         50.0%         75.0%         97.5%
      Symbol       Float64       Float64       Float64       Float64       Float64

          σ²   109753.9613   152862.6346   184995.6745   226723.7485   347574.4600
           μ     9803.6408     9933.5464     9998.4054    10063.3379    10193.2900
```

In [65]:
```
@show confint_ttest(y);
```

```
confint_ttest(y) = [9957.157404419688, 10079.238239598346]
```

In [66]:
```
postμ_theoretical = posterior_μ(bayesian_update(μstar, vstar, κ, θ, y)...)
plot_posterior_μ(chn, y, postμ_theoretical)
```

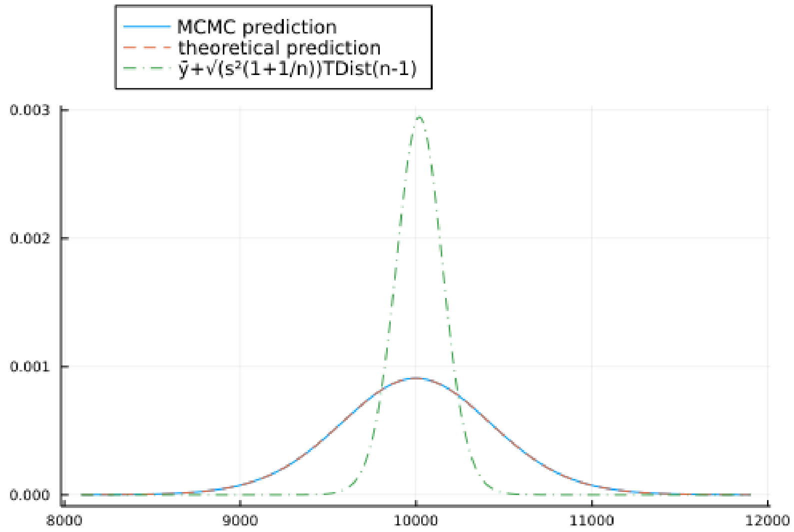
Out[66]:



In [67]:

```
pred_theoretical = preddist(bayesian_update(μstar, vstar, κ, θ, y)...)
plot_preddist(chn, y, pred_theoretical)
```

Out[67]:



1.13 n = 200 で事前分布とデータの数値の相性が悪い場合

前節の続き

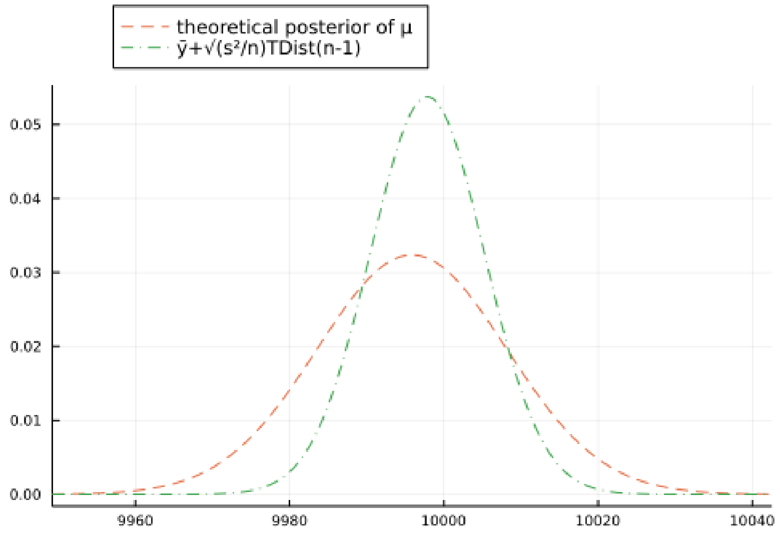
In [68]:

```
μ_true, σ_true, n = 1e4, 1e2, 200
@show dist_true = Normal(μ_true, σ_true)
y = rand(dist_true, n)
@show length(y) mean(y) var(y);
```

```
dist_true = Normal(μ_true, σ_true) = Normal{Float64}(μ=10000.0, σ=100.0)
length(y) = 200
mean(y) = 9997.8544461325
var(y) = 10989.56551724728
```

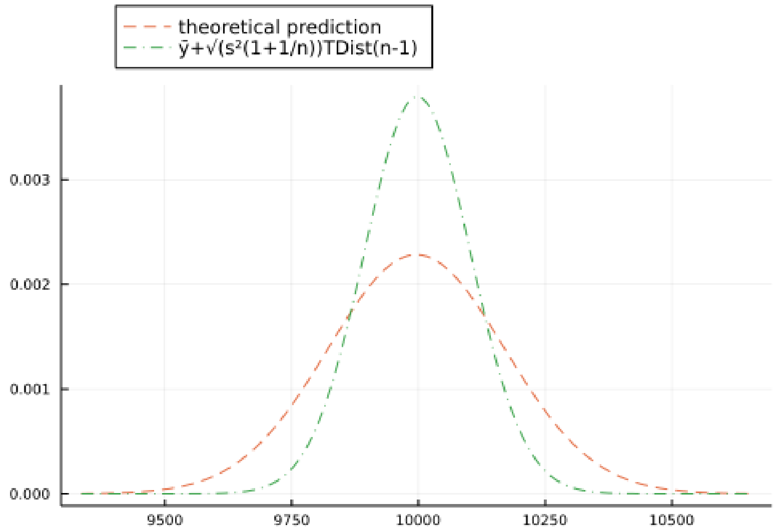
In [69]:
```julia
postμ_theoretical = posterior_μ(bayesian_update(μstar, vstar, κ, θ, y)...)
plot_posterior_μ(nothing, y, postμ_theoretical)
```

Out[69]:



In [70]:
```julia
pred_theoretical = preddist(bayesian_update(μstar, vstar, κ, θ, y)...)
plot_preddist(nothing, y, pred_theoretical)
```

Out[70]:



1.14 n = 2000 で事前分布とデータの数値の相性が悪い場合

前節の続き

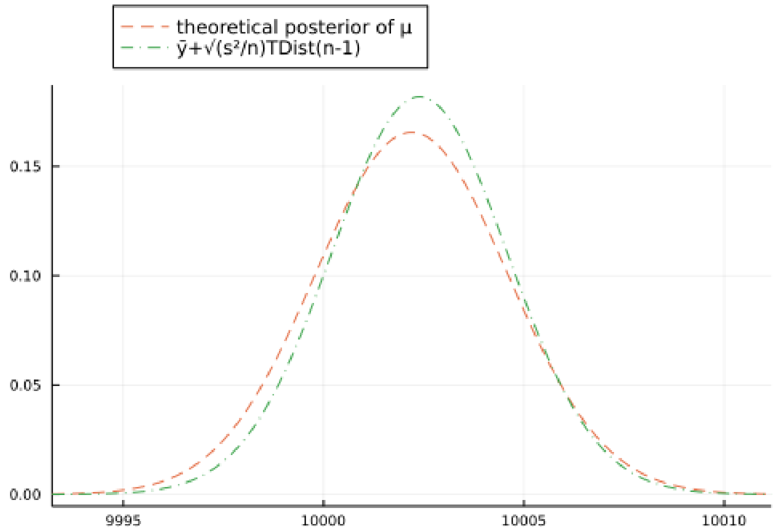
In [71]:
```julia
μ_true, σ_true, n = 1e4, 1e2, 2000
@show dist_true = Normal(μ_true, σ_true)
y = rand(dist_true, n)
@show length(y) mean(y) var(y);
```

```
dist_true = Normal(μ_true, σ_true) = Normal{Float64}(μ=10000.0, σ=100.0)
length(y) = 2000
mean(y) = 10002.394067284347
var(y) = 9627.127072443118
```

In [72]:

```
postμ_theoretical = posterior_μ(bayesian_update(μstar, vstar, κ, θ, y)...)
plot_posterior_μ(nothing, y, postμ_theoretical)
```

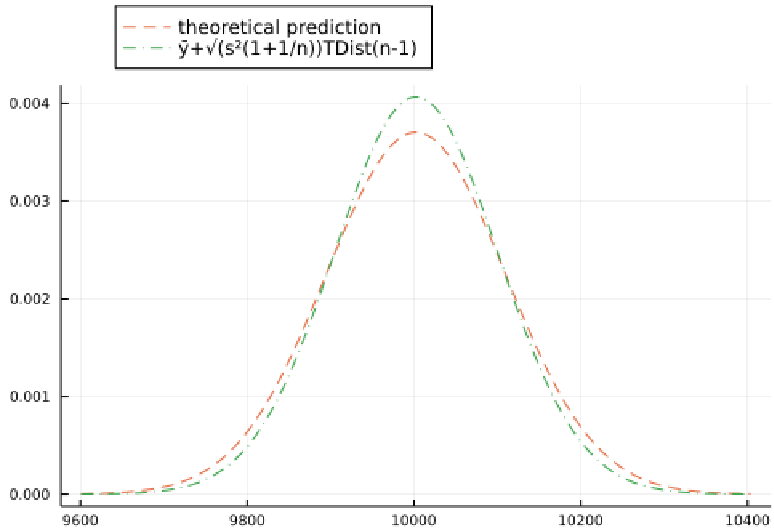
Out[72]:



In [73]:

```
pred_theoretical = preddist(bayesian_update(μstar, vstar, κ, θ, y)...)
plot_preddist(nothing, y, pred_theoretical)
```

Out[73]:



1.15 n = 20000 で事前分布とデータの数値の相性が悪い場合

前節の続き

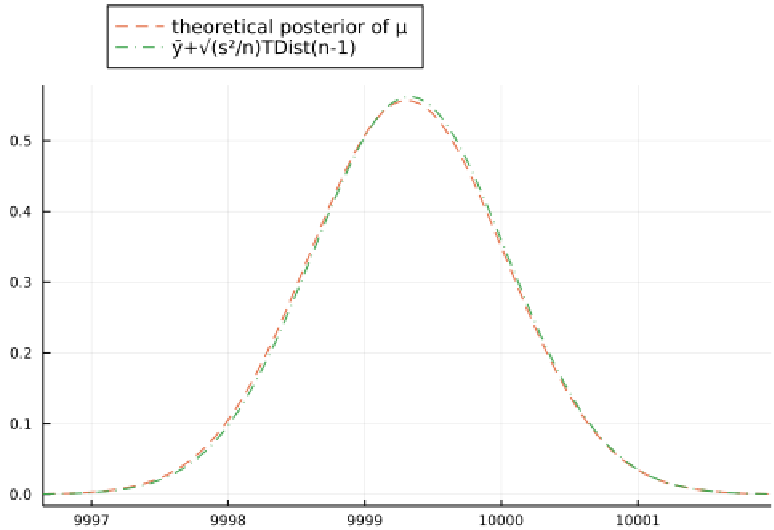
In [74]:

```
μ_true, σ_true, n = 1e4, 1e2, 20000
@show dist_true = Normal(μ_true, σ_true)
y = rand(dist_true, n)
@show length(y) mean(y) var(y);
```

```
dist_true = Normal(μ_true, σ_true) = Normal{Float64}(μ=10000.0, σ=100.0)
length(y) = 20000
mean(y) = 9999.328496504879
var(y) = 10061.296670416541
```

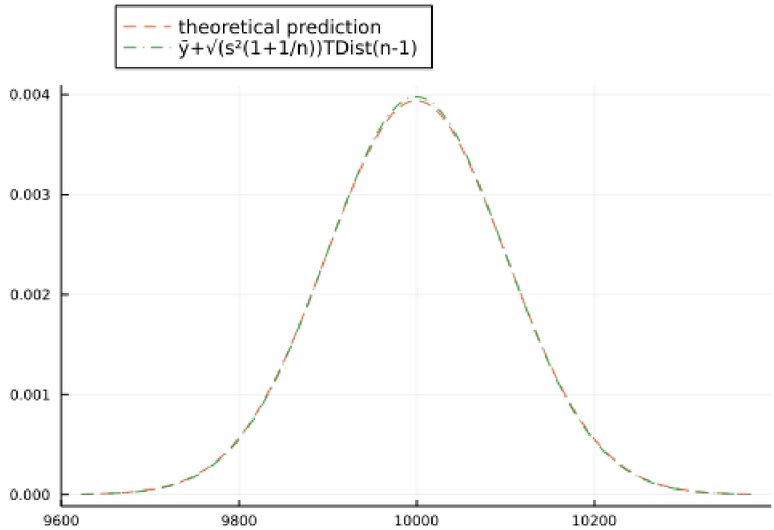
In [75]:
```
postμ_theoretical = posterior_μ(bayesian_update(μstar, vstar, κ, θ, y)...)
plot_posterior_μ(nothing, y, postμ_theoretical)
```

Out[75]:



In [76]:
```
pred_theoretical = preddist(bayesian_update(μstar, vstar, κ, θ, y)...)
plot_preddist(nothing, y, pred_theoretical)
```

Out[76]:



In [ ]:
```

```