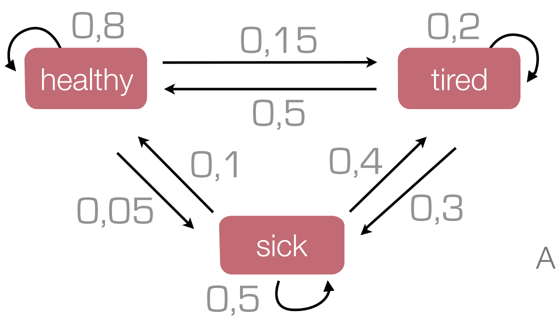
# Hidden Markov Models

Thierry Parmentelat — INRIA
thierry.parmentelat@inria.fr

## Markov Models




- model : $\lambda=(A,\pi)$
- A: Transitions
- $\pi$: p(initial state=i)

A

| | | | |
|---|---|---|---|
| 0,8 | 0,15 | 0,05 | $\Sigma=1$ |
| 0,5 | 0,2 | 0,3 | |
| 0,1 | 0,4 | 0,5 | |

$\pi$

| | | |
|---|---|---|
| 0,7 | 0,2 | 0,1 |

## Markov Models

- Markov hypothesis
  - behaviour depends only on current state (**not** on history)
- Observation
  - $E_1E_2..E_t..$ : sequence of states, $E_i$ in {1..N}
- Problems
  - (I) Probability of a given sequence
  - (II) Probability to observe state i at time t

## Notations

- Mathematical: $A_{ij}$
- Computer Science: A[i][j]
- Example Problem (I)
  - Maths: sequence $E_1E_2..E_t$
    proba = $\pi(E_1)*\prod_{i=2..t} A_{(Ei-1)(Ei)}$
  - CS: sequence seq[t]
    proba = pi [seq[0]] $* \prod_{i=2..t}$ ( A [seq[i-1]] [seq[i]] )

From now on,
we will have
all indices start at 0

## Python features, and modules of interest

- range — range(5) -> [0, 1, 2, 3, 4] — range(1,5) -> [1, 2, 3, 4]
- [ function(x) for x in inputs ] — ou [ function(x) for x in inputs if condition(x) ]
- built-in sum — sum ( range(5) ) -> 10
- module operator — from operator import add, mul
- reduce — reduce (add, range(5),0 ) -> 10 — reduce (mul, range(1,5), 1) -> 24

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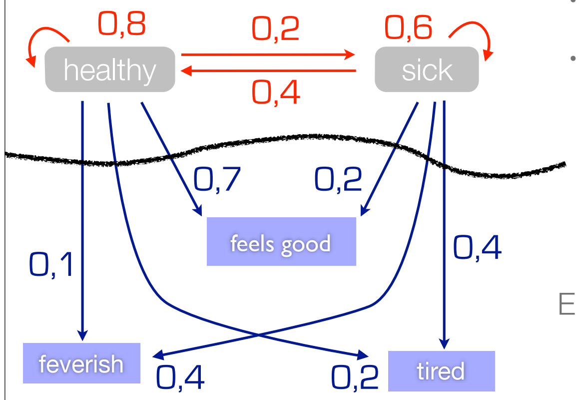
- numpy — http://wiki.scipy.org/Tentative_NumPy_Tutorial
- matplotlib — http://matplotlib.org

## Hidden Markov Models




- Model : $\lambda$=(A,$\pi$,E)
- decoupling state-output

| E | | | | |
|---|---|---|---|---|
| | 0,1 | 0,2 | 0,7 | $\Sigma$=1 |
| | 0,4 | 0,2 | 0,4 | |

## Two major kinds of Hidden Markov Model

- Discrete: Emission among a **finite** set of observations
- Continuous: Emission in $R^d$ - second chapter

## More on notations

- ***sequence*** : a list of observations (a.k.a. ***seq***)
- ***path*** : a list of states
- indices:
  - ***i,j*** : for denoting states (***N***: nb of states)
  - ***t*** : for denoting time (***T***: max time for a given seq/path)
  - ***o*** : for denoting outputs/observations (***O***: nb of signals)

## Classical problems in HMM's

- Reference paper - Lawrence Rabiner
  A tutorial on Hidden Markov Models…
  http://www.cs.ubc.ca/~murphyk/Bayes/rabiner.pdf
- (I) Probability to observe a given sequence
  algos : **forward** and **backward** (a.k.a. alpha and beta)
- (II) Decoding: estimate (most probable) sequence of states that produce a given sequence - algo: **Viterbi**
- (III) Learning: given observed sequences, and from a given HMM, find an HMM that is more likely to produce them
  algo: **Baum-Welch**

## Probability for a given sequence of observations

- Naive algorithm:
  - probability for observing sequence along a given path
  - sum on all state paths

$$P(seq/path) = \pi(path_0) * \prod_{t=1}^{T-1} A_{path_{t-1},path_t} * \prod_{t=0}^{T-1} E_{path_t,seq_t}$$

$$P(seq) = \sum_{path \in N^T} P(seq/path)$$

  - unpracticable, we have $N^T$ paths to sum on

## Forward

- greedy variation
- **alpha[i][t]**
  probability to observe seq **until t** and end up in **state i**
- initialization

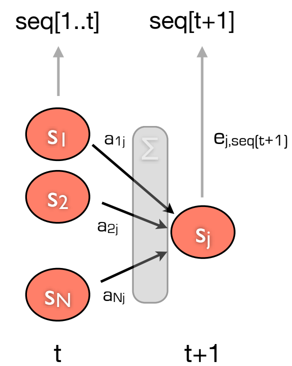
$$\alpha_{i,0} = \pi_i * E_{i,seq_0}$$

- alpha[i][0] = PI[i] * E[i][seq[0]]
- for i in range(N):
  alpha[i][0] = PI[i] * E[i][seq[0]]

## Forward - continued

- Recurrence

$$\alpha_{j,t+1} = \sum_{i=1}^{N} \alpha_{i,t} * A_{i,j} * E_{j,seq_{t+1}}$$




- alpha[j,t+1]=\
  reduce (add, [ alpha[i,t] * A[i,j] * E[j,seq[t+1]] for i in range(N))

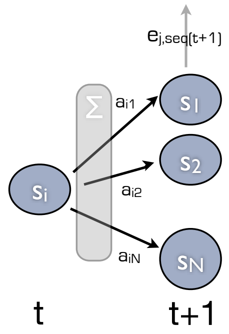
## Forward - finalization

- Original purpose : compute proba (seq)
  - obtained by summing all alpha[i][T]
- Note that the details of alpha are needed as well
  - as they will be re-used in the other algorithms
- Complexity of this algorithm
  - $\Theta$ ( $N^2$ * T ) in space and time

## Backward

- Same idea as *forward*, except .. the other way around
- **beta[i][t]**
  probability to observe seq **from t+1** and end up in **state i**
- beta[i][T-1] = 1
- beta[i][t] = <yours to say>
- final result

$$P(seq) = \sum_{i=0}^{N-1} \pi(seq_0) * beta(i, 0)$$




## Implementation

- Use your locally installed python (v2 recommended ?)
- To implement a function that can be used like this

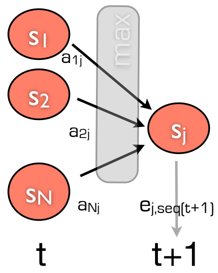
  (proba, alpha) = forward (PI, A, E, sequence)
- Using numpy or not - your choice

## Viterbi — decoding

- Given an observation *sequence*
  find out most likely path that has caused this sequence
  a.k.a. **Viterbi path**
- **delta[i][t]**
  best probability of path **until t** that **ends up at i**
  (and of course emits *sequence* until t)
- **psi[i][t]**
  state at t-1 for the path that corresponds to *delta*
  psi[i][0] undefined / does not matter

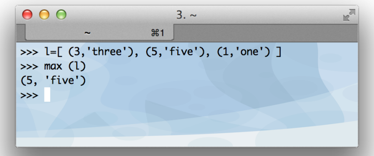
## Viterbi — details

- Initialization
  delta[i][0] = PI[i] * E[i][sequence[0]]
  psi[i][0] = None




- Recurrence
  delta[j][t+1] = \
    max ( [ delta [i][t] * A[i][j] for i in range(N) ]) * E[j][seq[t+1]
  psi[j][t] = argmax ( *same expression* )

## Viterbi — a useful trick



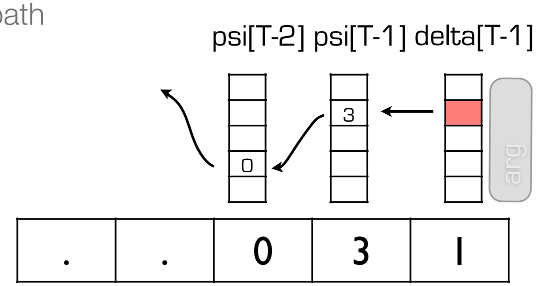

(proba, rank) = max ( [ ( delta[i][t-1] * A[i][j] , i ) for i in range(N) ] )

delta[j][t] = proba * E[j][seq[t]]

psi[j][t] = rank

## Viterbi - finalization

- retrieve full path



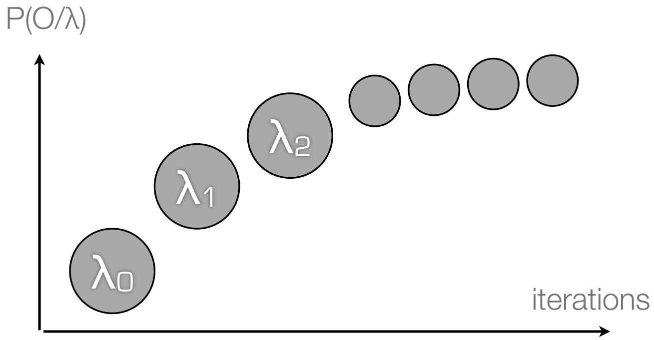

- Viterbi is a greedy algorithm too
  - $\Theta$ ( $N^2$ * T ) in space and time

## Baum-Welch — learning

- Problem: given a set of output sequences
  Find out the “best” HMM that would give this out
- Improve model λ = (A, E, PI) by successive iterations
- maximize likelihood over λ
  likelihood defined as the **product** of probabilities of all sequences in a sample
- in our case, multiply P(O/λ) over sequences

## Expectation Maximization (EM)

- Baum-Welch is a special case of "Expectation - Maximization" class of algorithms



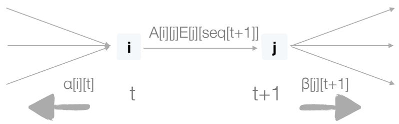

## Baum Welch basics

- Given sequence *seq,* and model λ
- **xi[i][j][t]** $\xi_t[i,j]$
  probability to be in state **i** at time **t**, and in state **j** at **t+1**
- **gamma[i][t]** $\gamma_t[i]$
  probability to be in state **i** at time **t**

## computing xi and gamma

- xi ( $\xi$ ) probability to be in **i** at **t**, **j** at **t+1**(knowing seq)

$$\xi_t(i,j) = \frac{\alpha_t(i) A_{i,j} E_{j,seq_{t+1}} \beta_{t+1}(j)}{P(seq/\lambda)}$$




- gamma ( $\gamma$ ) probability to be in **i** at **t** (knowing seq)

$$\gamma_t(i) = \sum_{j=0}^{N-1} \xi_t(i)(j)$$

## Rationale for re-estimation

$$\sum_{t=0}^{T-2} \gamma_t(i) = \text{expected number of transitions from state } \mathbf{i}$$

$$\sum_{t=0}^{T-2} \xi_t(i,j) = \text{expected number of transitions from state } \mathbf{i} \text{ to } \mathbf{j}$$

- note that one instant is not taken into account
  we are counting transitions, so there are T-1
  will be used for re-estimating PI instead

## Re-estimation

$$\overline{\pi_i} = \gamma_0(i)$$

$$\overline{A_{ij}} = \frac{expected\ number\ of\ transitions\ from\ \mathbf{i}\ to\ \mathbf{j}}{expected\ number\ of\ transitions\ from\ \mathbf{i}}$$

$$= \frac{\sum_{t=0}^{T-2} \xi_t(i,j)}{\sum_{t=0}^{T-2} \gamma_t(i)}$$

$$\overline{E_{jo}} = \frac{expected\ number\ of\ times\ in\ \mathbf{j}\ and\ observing\ \mathbf{o}}{expected\ number\ of\ times\ in\ \mathbf{j}}$$

$$= \frac{\sum_{\substack{t=0 \\ so\ that\ \mathbf{o}=seq(t)}}^{T-1} \gamma_t(j)}{\sum_{t=0}^{T-1} \gamma_t(j)}$$

reminder from previous slides

* in **i** at **t** and in **j** at **t+1**

$$\xi_t(i,j) = \frac{\alpha_t(i) A_{i,j} E_{j,seq_{t+1}} \beta_{t+1}(j)}{P(seq/\lambda)}$$

* in **i** at **t**

$$\gamma_t(i) = \sum_{j=0}^{N-1} \xi_t(i)(j)$$

## Convergence criteria

- monitor overall P(seq) at each iteration
- should grow asymptotically
- stop when (P'/P) < (1+ε)
- ε passed as argument

## Deal with multiple sample sequences

$$\overline{\pi_i} = \gamma_0(i)$$ Average on sequences

$$\overline{A_{ij}} = \frac{expected\ number\ of\ transitions\ from\ \mathbf{i}\ to\ \mathbf{j}}{expected\ number\ of\ transitions\ from\ \mathbf{i}}$$

Sum over sequences

Sum over sequences

$$\cancel{= \frac{\sum_{t=0}^{T-2} \xi_t(i,j)}{\sum_{t=0}^{T-2} \gamma_t(i)}}$$

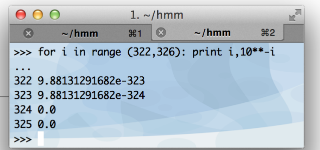
$$\overline{E_{jo}} = \frac{expected\ number\ of\ times\ in\ \mathbf{j}\ and\ observing\ \mathbf{o}}{expected\ number\ of\ times\ in\ \mathbf{j}}$$

Sum over sequences

Sum over sequences

$$\cancel{= \frac{\sum_{\substack{t=0 \\ so\ that\ \mathbf{o}=seq(t)}}^{T-1} \gamma_t(j)}{\sum_{t=0}^{T-1} \gamma_t(j)}}$$

## Scaling




- If $P_{max}$ is the maximum of the terms in A and E
- Order of magnitude for e.g. alpha[i][t] : $P_{max}^{2T}$
- Even if $P_{max}$ is, say, 0.9, T=300 gives you fairly small values
- Quickly reach limits of hardware implementation
- Solution: for each time t
  select some relevant factor (e.g. based on sum of values)
  store in memory real value multiplied by factor

## Continuous Hidden Markov Models

## Continuous Hidden Markov Model

- Replace finite set of O signals (outputs)
- With a finite set of O gaussian distributions in $R^d$
- Each of these being defined by
  - an average $\mu = \{\mu_1, \mu_2, .. \mu_d\}$
  - a covariance matrix $\sigma_{ij}$ in dimension d
- Then each state is attached a linear stochastic combination of these gaussians (a **mixture**)

## Continuous HMM — example

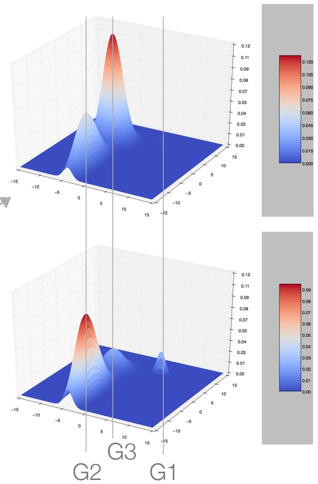
- 3 Outputs:

  G1 : μ=(4., 8.), σ=(1., 0.25)
  G2 : μ=(-4.,-8.), σ=(1., 4.)
  G3 : μ=(-6., 6.), σ=(2., 2)

  say G1~tired,
  G2~feverish, G3 ~ feels good

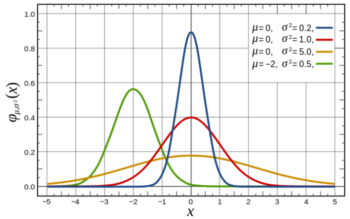
- 2 states:

  E[1]: (.1, .1, .8)
  E[2]: (.7, .15, .15)




## Gaussian distribution - single dimension

- a.k.a. Normal Distribution (wiki)
- basic shape $f(x) = e^{-x^2}$
- normalize $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$
- translate $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2}$
- scale $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$



## General form of d-dimension gaussian

- a.k.a. multivariate normal distribution (wiki)
- μ average; Σ covariance
  Σ is positive semidefinite and symmetric

$\mu \in \mathbb{R}^d, \Sigma \in \mathbb{R}^{d.d}, G_{\mu,\Sigma} : \mathbb{R}^d \longmapsto \mathbb{R}$

$$G_{\mu,\Sigma}(X) = \frac{1}{\sqrt{(2\pi)^d . det(\Sigma)}} e^{(-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu))}$$

- check out “covariance matrix” in wiki

## Degenerate case

- Diagonal covariance matrix (unrelated data)

$$\Sigma = \begin{pmatrix} {\sigma_1}^2 & 0 & . & . & 0 \\ 0 & {\sigma_2}^2 & . & . & 0 \\ . & . & . & . & . \\ 0 & . & . & {\sigma_{d-1}}^2 & 0 \\ 0 & . & . & 0 & {\sigma_d}^2 \end{pmatrix}$$

$$G_{\mu,\Sigma}(X) = \frac{1}{\prod_{i=1}^{d} \sigma_i \sqrt{(2\pi)^d}} e^{(-\frac{1}{2}\sum_{i=1}^{d}(\frac{xi-\mu_i}{\sigma_i})^2)}$$

## Algorithms adaptation for continuous HMMs

- Replace “E[i][seq[t]]”
- With the evaluation of the gaussian mixture defined for i, on that point in space seq[t]

## Source Code Management systems

## Source Code Management systems

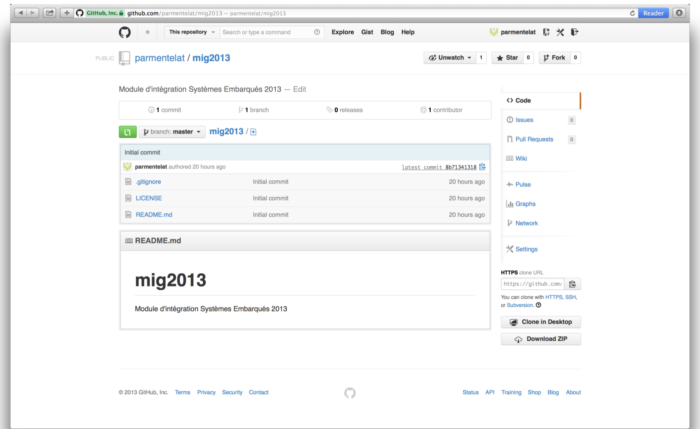
- purpose
  collaborative work
  keep track of changes
  changes are done concurrently
  teams do not necessarily know of each other
- git
  http://git-scm.com
  decentralized
  used by many open source projects
- svn
  http://subversion.tigris.org
  deprecated / previous generation
  simpler mental model, but centralized
- mercurial
  http://mercurial.selenic.com is nice too

## git basics

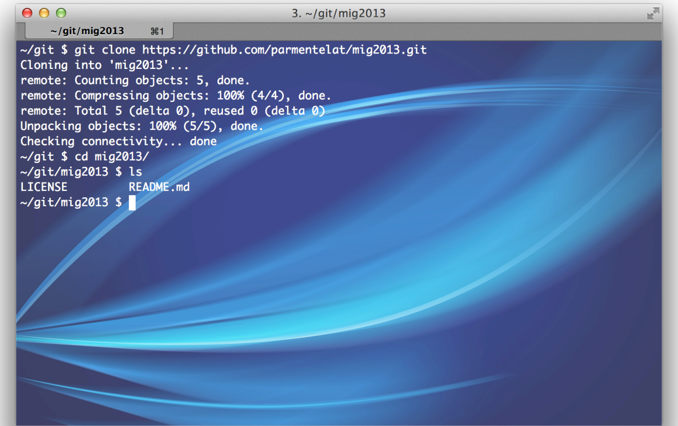
- git init — initialize git from a working directory
- git clone — create a local working dir from remote
- git add — put changes aside for next commit
- git commit — create a commit
- git pull — get changes from remote
- git push — push your commits on remote
- git stash — hide local changes under carpet

Check out UI tools : http://www.sourcetreeapp.com

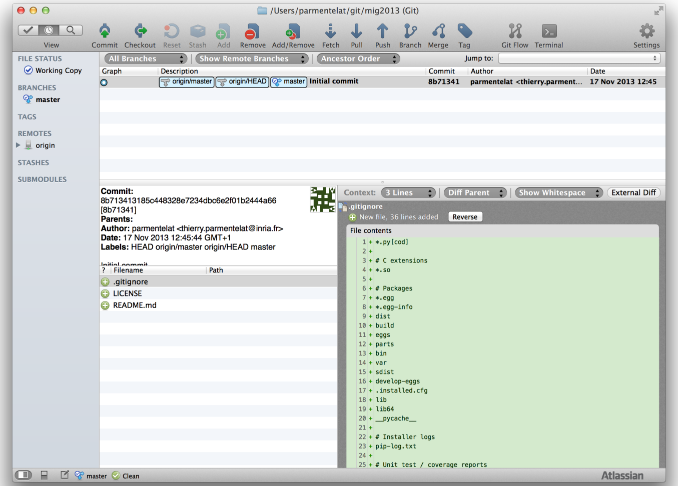
## example (1) - create repo in github




## example (2) - clone repo locally



## example (3) - visualize local repo with sourcetree

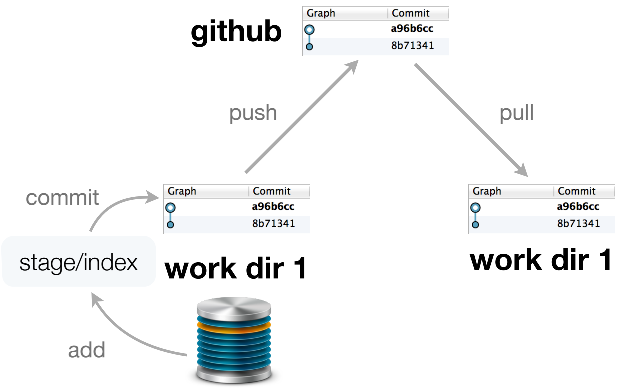


## example (4) - two working directories

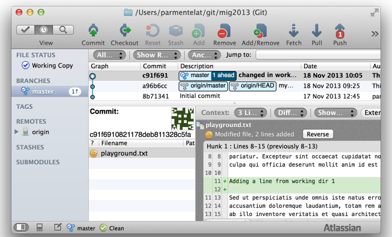
$ git ls-files playground.txt
$ git add playground.txt
$ git commit -m “my message”
$ git ls-files playground.txt
playground.txt
$ git push

$ git ls-files playground.txt
$
$
$ git ls-files playground.txt
$
$
$ git pull
$ git ls-files playground.txt
$ playground.txt

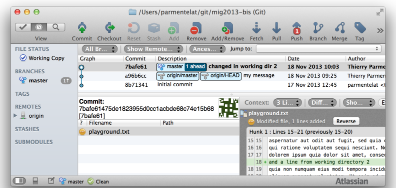
## example (5) - the repositories at work



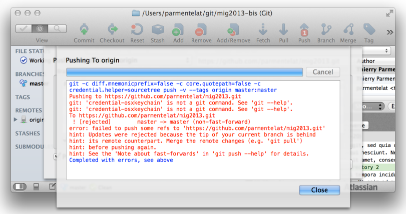

## example (6) - merging two concurrent changes



user 1 can push fine

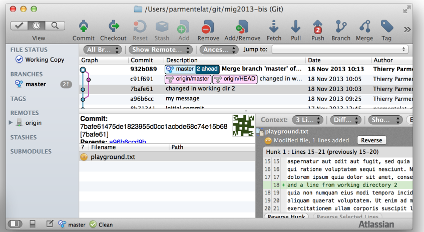


user 2 gets conflicts when pushing

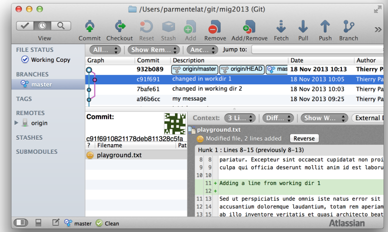


## example (7) - merging - continued

user 2 needs to pull first



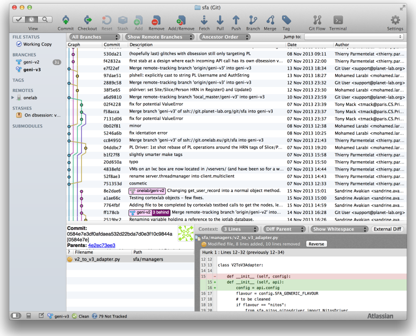

user 1 sees once he pulls again




user 2 can then push fine

## Good practices

- Branches are easy to create, merge, delete..
- Use as many branches as needed
- Typically one per “feature”



## ssh basics

## ssh history and purpose

- successor of (very unsecure) *rsh* (remote shell)
- ssh = secure shell
- basic purpose : remote terminal
- advanced purpose : all sorts of secure tunnels / bouncing
- esp. e.g. rsync (to keep files in sync), git, …

## ssh authentication mechanism(s)

- **password** authentication is **evil**, don't ever enable
- use **public key** authentication only
- ssh-keygen : create a key pair
- keep private key (id_rsa), well… private
  - never out of your computer
  - watch out access rights
  - private keys are password-protected
- expose public key (id_rsa.pub) to your peers

## ssh public key authentication - basics

- how to enable access
  - add public key in ~/.ssh/authorized_keys
  - again watch out for access rights
- how it works
  - Public o $E_{private}$ = $E_{private}$ o $E_{public}$ = Identity
  - challenge remote to "be" X (to have private key X)
  - send $E_{public}$(message) over the wire
  - $E_{private}$ is required to decode message