Math 161 - Notes

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1 Geometry and the Axiomatic Method

1.1 The Early Origins of Geometry: Thales and Pythagoras

We start with a very brief overview of geometric history. The term *geometry* is of ancient Greek origin: *geo* (Earth) + *metros* (measure). Measurement (distance, area, height, angle) has obvious practical application: construction, taxation, commerce, navigation, etc. Astronomy/astrology provided a related religious/cultural imperative for the development of ancient geometry.

Ancient Times (pre-500 BC) Basic rules for measuring lengths, areas and volumes of simple shapes were developed in Egypt, Mesopotamia, China & India. Applications: surveying, tax collection, construction, religious practice, astronomy, navigation. Problems were typically worked examples without general formulæ/abstraction.

Ancient Greece (from c. 600 BC) Philosophers such as Thales and Pythagoras began the process of *abstraction.* General statements (theorems) formulated and proofs attempted. Concurrent development of early scientific reasoning.

Euclid of Alexandria (c. 300 BC) Collected and expanded earlier work, especially that of the Pythagoreans. His compendium the *Elements* motivated mathematical development in Eurasia and North Africa, remaining a standard school textbook well into the 1900's. The *Elements* is an early exemplar of the axiomatic method at the heart of modern mathematics.

Later Greek Geometry Archimedes' (c. 270–212 BC) work on area and volume included techniques similar to those of modern calculus. Apollonius studies conics (ellipses, parabolæ and hyperbolæ). Ptolemy's (c. AD 100–170) *Almagest* applies basic geometry to astronomy and includes the foundations of trigonometry.

Post-Greek Geometry The work of Euclid and Ptolemy was expanded and enhanced by Indian and Islamic mathematicians, who particularly developed trigonometry (as well as algebra and our modern system of decimal enumeration).

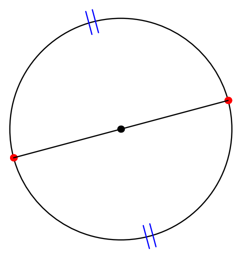
Analytic Geometry (early 1600's France) Descartes and Fermat begin using axes and co-ordinates, melding geometry and algebra.

Modern Development Non-Euclidean geometries help provide the mathematical foundation for Einstein's relativity and the study of curvature. Following Klein (1872), modern geometry is highly dependent on group theory.

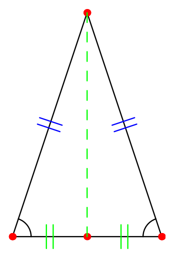
Thales of Miletus (c. 624–546 BC) Thales was an olive trader from Miletus, a city-state on the west coast of modern Turkey. He absorbed mathematical ideas from nearby cultures including Egypt and Mesopotamia. Here are five results partly attributable to Thales.

1. A circle is bisected by a diameter.
2. The base angles of an isosceles triangle are equal.
3. The pairs of angles formed by two intersecting lines are equal.
4. Triangles are congruent if they have two angles and the included side equal (ASA congruence).
5. An angle inscribed in a semicircle is a right angle.

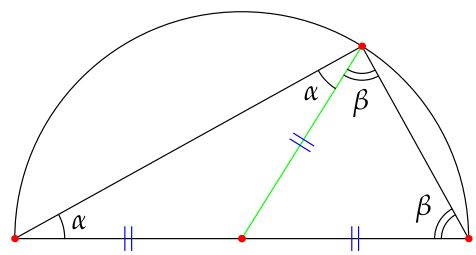
This last is still known as *Thales' Theorem.* Thales' innovation was to state *universal, abstract principles*: e.g., *any* circle is bisected by *any* of its diameters. The Greek word *θεωρεω* (*theoreo*), from which we get *theorem,* has several meanings: 'to look at,' 'speculate,' or 'consider.' His arguments were not rigorous by modern standards, but were supposed to be clear just by looking at a picture.



Theorem 1



Theorem 2



Theorem 5

Arguments for Theorems 1 and 2 might be as simple as 'fold.' Theorem 5 follows from the observation that the radius of the circle splits the large triangle into two isosceles triangles: Theorem 2 says that these have equal base angles (α, α and β, β), whence $\alpha + \beta$ is half the angles in a triangle, namely a right-angle.

Pythagoras of Samos (570–495 BC) Hailing from Samos, an island in the Aegean Sea not far from Miletus, Pythagoras travelled widely, eventually settling in Croton, southern Italy, around 530 BC, where he founded a philosophical school devoted to the study of number, music and geometry. As a mysterious, cult-like group, the Pythagoreans' output is not fully understood, though they are typically credited with the classification of the regular (Platonic) solids and with the development of the relationship between the length of a vibrating string and its (musical) pitch. The Pythagorean obsession with number and the 'music of the universe' inspired later Greek mathematicians who believed they were refining and clarifying this earlier work.

Of course, Pythagoras is best known for the result that still bears his name.

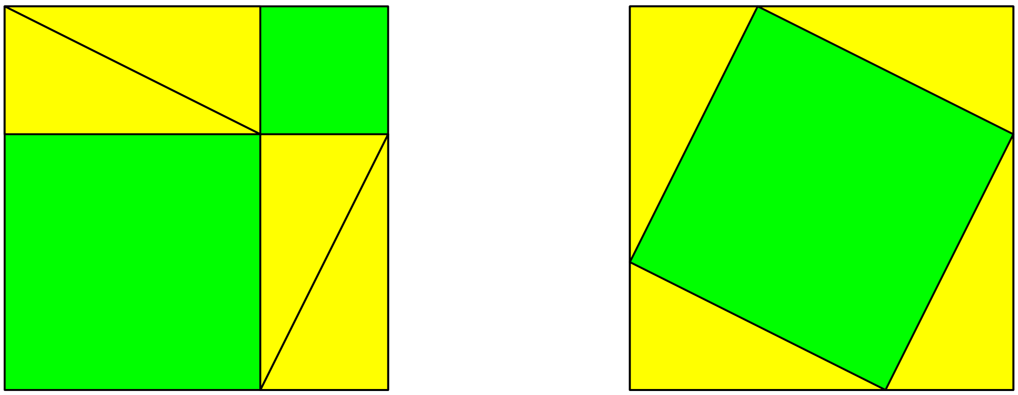
Theorem 1.1 (Pythagoras). *The square on the hypotenuse (longest side) of a right-triangle equals the sum of the squares on the remaining sides.*

Two important clarifications are needed for modern readers.

1. By *square,* the Greeks meant an honest square (shape)! To the Greeks, Pythagoras is not a statement about multiplication ('squaring'): there is no algebra, no numerical length, and the equation $a^2 + b^2 = c^2$ won't be seen for another 2000 years.

2. The word *equals* means *equal area,* though without any numerical concept of such. The Greeks meant that the square on the hypotenuse can be subdivided into pieces which may be rearranged to produce the two squares on the remaining sides.

The result suddenly seems less easy! The pictures below provide a simple visualization.



A simple proof of Pythagoras' Theorem

Book I of Euclid's *Elements* seems to have been structured precisely to provide a rigorous constructive proof in line with the Greek *additive* notion of area. By contrast, the above visualization relies on *subtracting* the areas of four congruent triangles from a single large square. It is possible, though ugly, to apply the 47 results up to and including Euclid's proof so as to explicitly subdivide the hypotenuse square and rearrange the pieces into the two smaller squares.

Much has been written about the Pythagorean Theorem, and many, many proofs have been given.[1] While it is sometimes believed that Pythagoras himself first proved the result, this is generally considered incorrect: the 'proof' most often attributed to the Pythagoreans is based on contradictory ideas number which were debunked by the time of Aristotle. Moreover, the result was in use in example form (e.g., in ancient China and Mesopotamia[2]) several hundred years before Pythagoras. Regardless, any argument over attribution is pointless without an agreement on what constitutes a *proof*. To discuss the modern meaning, we need to spend some time considering Axiomatic Systems...

Exercises 1.1. 1. In the above pictorial argument, let the side-lengths of the triangles be a, b, c. Can you rephrase the proof algebraically?

2. A theorem of Euclid states:

> The square on the parts equals the sum of the squares on each part plus twice the rectangle on the parts

By referencing the above picture, state Euclid's result using modern algebra.

(*Hint: let a and b be the 'parts'...*)

[1] Including a proof by former US President James Garfield. Would that current presidents were so learned...

[2] In China, the Pythagorean Theorem is known as the *gou gu,* in reference to the two non-hypotenuse sides of the triangle.

1.2 Axiomatic Systems

Arguably the most revolutionary aspect of Euclid's *Elements* was its axiomatic presentation.

Definition 1.2. An *axiomatic system* comprises four types of object.

1. *Undefined terms*: Concepts accepted without definition/explanation.
2. *Axioms*: Logical statements regarding the undefined terms which are accepted without proof.
3. *Defined terms*: Concepts defined in terms of 1 & 2.
4. *Theorems*: Logical statements deduced from 1–3.

A *proof* is a logical argument demonstrating the truth of a theorem within an axiomatic system.

Examples 1.3. Here are two systems. In each case we provide only *examples* of each type of object.

Basic Geometry
1. *Line* and *point*.
2. Given two points, there exists a line joining them.
3. A *triangle* consists of three non-collinear points and the segments between them.
4. The theorems of Thales and Pythagoras.

Chess
1. Pieces (as black/white objects) and the board.
2. Rules for how each piece moves.
3. Concepts such as *check*, *stalemate*, *en-passant*.
4. Given a particular position, Black can win in five moves.

Definition 1.4. A *model* is a choice/definition of the undefined terms such that all axioms are true.

Models are often *abstract* in that they depend on another axiomatic system. In a *concrete* model, the undefined terms are real-world objects (where contradictions are impossible(!)). The big idea is this:

> Any theorem proved within an axiomatic system is true in any model of that system

Mathematical discoveries often hinge on the realization that seemingly separate discussions can be described in terms of models of a common axiomatic system.

Example 1.5. Here is the axiomatic system for a *monoid*, built using the language of standard set theory (itself an axiomatic system).

1. A set G and a binary operation $*$.
2. (A1) Closure: $\forall a, b \in G, a * b \in G$
   (A2) Associativity: $\forall a, b, c \in G,\ a * (b * c) = (a * b) * c$
   (A3) Identity: $\exists e \in G$ such that $\forall a \in G,\ a * e = e * a = a$
3. Concepts such as *square* $a^2 = a * a$, or *commutativity* $a * b = b * a$.
4. For example, *The identity is unique.*

$(G, *) = (\mathbb{Z}, +)$ is an abstract model, where $e = 0$. If you really want a concrete model, consider a single dot $\bullet$ on the page, equipped with the operation $\bullet * \bullet = \bullet$!

Definition 1.6. Certain properties are desirable in an axiomatic system.

Consistency The system is free of contradictions.

Independence An axiom is independent if it is not a theorem of the others. An axiomatic system is independent if all its axioms are.

Completeness Every valid proposition within the theory is *decidable*: can be proved or disproved.

We unpack these ideas slightly, though our descriptions are vague by necessity: some notions must be clarified (e.g., *valid proposition*) before these ideas can be made rigorous.

Consistency May be demonstrated by exhibiting a *concrete model*. An *abstract model* demonstrates *relative consistency*, where consistency depends on that of the underlying system. An inconsistent system is essentially useless to practicing mathematicians.

Independence To demonstrate the independence of an axiom, exhibit two models: one in which all axioms are true, the other in which only the considered axiom is false.

Completeness This is very unlikely to hold for a useful axiomatic system in mathematics, though examples do exist. To show incompleteness, an *undecidable*[3] statement is required, which can be viewed as a new independent axiom in an enlarged axiomatic system.

Example (1.5, cont). The axiomatic system for a monoid is:

Consistent We have a (concrete) model.

Independent Consider three models:

- $(\mathbb{N}, +)$ satisfies axioms A1 and A2, but not A3.
- $(\{e, a, b\}, *)$ defined by the following table satisfies A1 and A3, but not A2

$*$	e	a	b
e	e	a	b
a	a	e	a
b	b	b	a

e.g. $\quad a * (b * b) = a * a = e \neq a = a * b = (a * b) * b$

- $(\mathbb{Z} \setminus \{1\}, +)$ satisfies axioms A2 and A3, but not A1.

Incomplete The proposition '*A monoid contains at least two elements*' is undecidable *just from the axioms.* For instance, $(\{0\}, +)$ and $(\mathbb{Z}, +)$ are models with one/infinitely many elements.

We could also ask if all elements have an inverse. That this is undecidable is the same as saying that a new axiom is independent of A1, A2, A3.

(A4) Inverse: $\quad \forall g \in G,\ \exists g^{-1} \in G$ such that $g * g^{-1} = g^{-1} * g = e$.

The new system defined by the four axioms is also consistent and independent—this is the structure of a *group*. Even this new system is incomplete; for instance, consider a new axiom of commutativity...

[3] A famous example of an undecidable statement from standard set theory is the *Continuum Hypothesis*, which states that there is no uncountable set with cardinality strictly smaller than that of the real numbers.

Example 1.7 (Bus Routes). Here is a loosely defined axiomatic system.

Undefined Terms: Route, Stop

Axioms: (A1) Each route is a list of stops in some order. These are the stops visited by the route.

(A2) Each route visits at least four distinct stops.

(A3) No route visits the same stop twice, except the first stop which is also the last stop.

(A4) There is a stop called downtown that is visited by each route.

(A5) Every stop other than downtown is visited by at most two routes.

Discuss the following questions:

1. Construct a model of the Bus Routes system with exactly three routes. What is the fewest number of stops you can use?

2. Your answer to 1 shows that this system is: complete, consistent, inconsistent, independent?

3. Does the following describe a model for the Bus Routes system? If not, determine which axioms are satisfied and which are not?

Stops:	Downtown, Walmart, Albertsons, Main St., CVS, Trader Joes, Zoo
Route 1:	Downtown, Walmart, Main St., CVS, Zoo, Downtown
Route 2:	Main St., CVS, Zoo, Albertsons, Downtown, Main St.
Route 3:	Walmart, Main St., Downtown, Albertsons, Main St., Walmart

4. Show that A3 is independent of the other axioms.

5. Demonstrate that '*There are exactly three routes*' is not a theorem in this system by finding a model in which it is false.

We are only scratching the surface of axiomatics. If you really want to dive down this rabbit hole, consider taking a class in formal logic or model theory. As an example of the ideas involved, we finish with two results proved in 1931 by the German logician Kurt Gödel.

Theorem 1.8 (Gödel's incompleteness theorems).

1. *Any consistent system containing the natural numbers is incomplete.*

2. *The consistency of such a system cannot be proved within the system itself.*

Gödel's first theorem tells us that there is no *ultimate* consistent complete axiomatic system. Perhaps this is reassuring: there will always be undecidable statements, so mathematics will never be finished! However, the undecidable statements cooked up by Gödel are analogues of the famous *liar paradox* ('This sentence is false'), so the profundity of this is a matter of debate.

Gödel's second theorem fleshes out the difficulty in proving the consistency of an axiomatic system. If a system is sufficiently detailed so as to describe the natural numbers, its consistency can at best be proved relative to some other axiomatic system. In practice, demonstrating that a useful axiomatic system really is consistent is essentially impossible!

Exercises 1.2. 1. Between two players are placed several piles of coins. On each turn a player takes as many coins as they want from *one pile*, though they must take at least one coin. The player who takes the last coin wins.

If there are two piles where one pile has more coins than the other, prove that the first player can always win the game.

2. Consider an axiomatic system where children in a classroom choose different flavors of ice cream. Suppose we have the following axioms:

   (A1) There are exactly five flavors of ice cream: vanilla, chocolate, strawberry, cookie dough, and bubble gum.
   (A2) Given any two distinct flavors, exactly one child likes both.
   (A3) Every child likes exactly two flavors of ice cream.

   (a) How many children are in the classroom? Prove your assertion.
   (b) Prove that any pair of children likes at most one common flavor.
   (c) Prove that for each flavor, there are exactly four children who like that flavor.

3. Suppose S is a set and $P \subseteq S \times S$ is a set of ordered pairs of elements (a, b) that satisfy the following axioms:

   (A1) If (a, b) is in P, then (b, a) is not in P.
   (A2) If (a, b) is in P and (b, c) is in P, then (a, c) is in P.

   (a) Let $S = \{1, 2, 3, 4\}$ and $P = \{(1, 2), (2, 3), (1, 3)\}$. Is this a model for the axiomatic system? Why/why not?
   (b) Let S be the set of real numbers and P consist of all pairs (x, y) where $x < y$. Is this a model for the system? Explain.
   (c) Use the results of (a) and (b) to argue that the axiomatic system is incomplete. Otherwise said, think of an independent axiom that could be added to the system for which part (a) is a model, but for which part (b) is not.

4. The undefined terms of an axiomatic system are 'brewery' and 'beer'. Here are some axioms.

   (A1) Every brewery is a non-empty collection of *at least* two beers (each brewery brews at least two beers).
   (A2) Any two distinct breweries have at most one beer in common.
   (A3) Every beer belongs to exactly three breweries.
   (A4) There exist exactly six breweries.

   (a) Prove the following theorems.
      i. There are exactly four beers.
      ii. There are exactly two beers in each brewery.
      iii. For each brewery, there is exactly one other brewery which has no beers in common.
   (b) Prove that the axioms are independent.
   (*When negating A1, you should assume that a brewery is still a collection of beers, but that any such could contain none or one beer*)