5 Fractal Geometry

5.1 Natural Geometry, Self-similarity and Fractal Dimension

Classical geometry typically considers objects (lines, curves, spheres, etc.) which seem flatter and less interesting as one zooms in: a differentiable curve at small scales looks like a line segment!

By contrast, real-world objects tend to exhibit greater detail at smaller scales. A seemingly spherical orange is dimpled on closer inspection. Is its surface area that of a sphere, or is it greater due to the dimples? What if we zoom in further? Under a microscope, the dimples are seen to have minute cracks and fissures. With modern technology, we can see almost to the molecular level; what does *surface area* even mean at such a scale?

The Length of a Coastline In 1967 Benoit Mandelbrot asked a related question in a now-famous paper, *How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension.* His essential point was that the question has no simple answer:[30] Should one measure by walking along the mean high tide line? But where is this? Do we 'walk' round every pebble? Round every grain of sand? Every molecule? As one shrinks the scale, the measured length becomes absurdly large. We sketch Mandelbrot's approach.[31]

- Given a ruler of length R, measure how many N are required to trace round the coastline when laid end-to-end.
- Plotting $\log N$ against $\log(1/R)$ for several sizes of ruler seems to give a straight line!

  $$\log N \approx \log k + D\log(1/R) = \log(kR^{-D}) \implies N \approx kR^{-D}$$

  The number D is Mandelbrot's *fractal dimension* of the coastline.

Mandelbrot's fractal dimension is purely empirical, though it does seem to capture something about the 'bumpiness' of a coastline: the bumpier, the greater its fractal dimension. For mainland Britain with its smooth east and rugged west coasts, $D \approx 1.25$. Given its many fjords, Norway has a far rougher coastline and a higher fractal dimension $D \approx 1.52$.

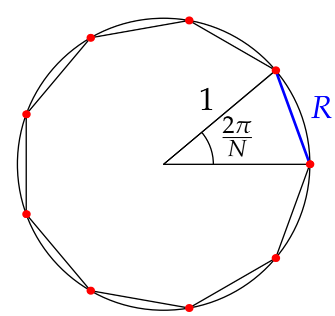
Example 5.1. As a sanity check, consider a smooth circular 'coastline.' Approximate the circumference using N rulers of length R: clearly

$$R = 2\sin\frac{\pi}{N}$$

As $N \to \infty$, the small angle approximation for sine applies,

$$R \approx \frac{2\pi}{N} \implies N \approx 2\pi R^{-1}$$

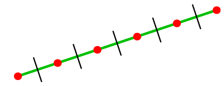
where the approximation improves as $N \to \infty$. The fractal dimension of a circle is therefore 1.



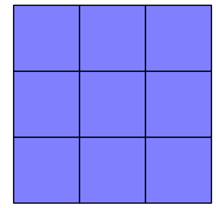
[30]The official answer from the Ordnance Survey (the UK government mapping office) is, 'It depends.' The all-knowing CIA states 7723 miles, though offers no evidence as to why.

[31]For more detail see the Fractal Foundation's website. Mandelbrot coined the word *fractal*, though he didn't invent the concept from nothing. Rather he applied earlier ideas of Hausdorff, Minkowski and others, and observed how the natural world contains many examples of fractal structures.

Our goal is to describe self-similar objects and thus create a new notion of dimension related to Mandelbrot's. To begin consideration of self-similarity, we first consider some of the standard objects of pre-fractal geometry.



Line A segment can be viewed as N copies of itself scaled by a factor $r = \frac{1}{N}$.



Square A square comprises N copies of itself scaled by a factor $r = \frac{1}{\sqrt{N}}$.

Cube A cube comprises N copies of itself scaled by a factor $r = \frac{1}{\sqrt[3]{N}}$.

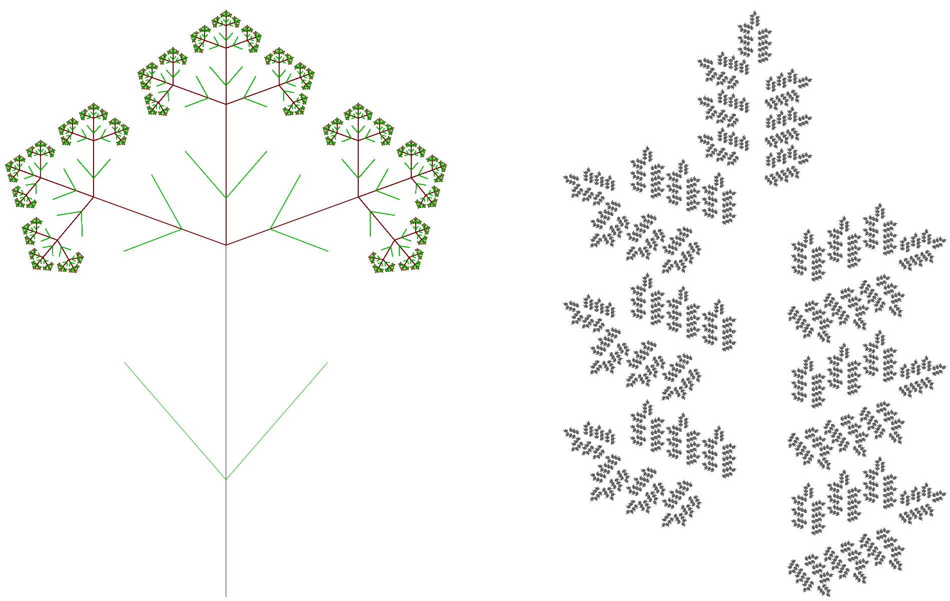
In each case observe that $N = \left(\frac{1}{r}\right)^D$ where D is the usual dimension of the object (1, 2 or 3). Inspired by this, we make a loose definition.

Definition 5.2. A geometric figure is *self-similar* if it may be subdivided into N similar copies of itself, each scaled by a magnification factor $r < 1$. The *fractal dimension* of such a figure is

$$D := \log_{1/r} N = \frac{\log N}{\log(1/r)} = -\frac{\log N}{\log r}$$

Example 5.3. The botanical pictures below offer some evidence for non-integer fractal dimension and that self-similarity is a natural phenomenon. The 'tree' comprises $N = 3$ copies of itself, each scaled by a factor of $r = 0.4$. Its fractal dimension is $D = -\frac{\log 3}{\log 0.4} \approx 1.199$.

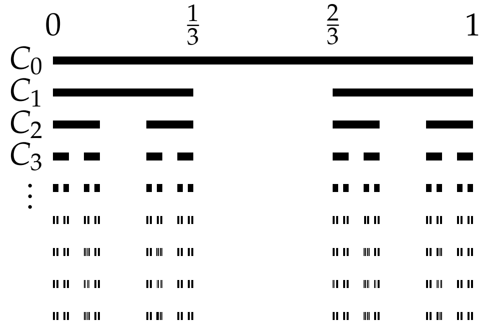
The fern has $N = 7$ and $r = 0.3$ for a fractal dimension $D = -\frac{\log 7}{\log 0.3} \approx 1.616$.



Tree fractal $D \approx 1.199$ Fern fractal $D \approx 1.616$

The pictures illustrate the interpretation of fractal dimension. Both objects seem to occupy more space than mere lines, but neither has positive area. Moreover, the fern seems to occupy more space than the tree. The 'trunk' and 'branches' in the first picture aren't part of the fractal; we've drawn them only to give the picture a skeleton. The fern, without stalks, is a more accurate approximation of a fractal.

Example 5.4 (Cantor's Middle-third Set). This famous example dates from the late 1800s.[32]



Starting with the unit interval $C_0 = [0,1]$, define a sequence of sets (C_n) where C_{n+1} is obtained by deleting the open 'middle-third' of each interval in C_n; for instance

$$C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

Cantor's set is essentially the limit of this sequence:

$$\mathcal{C} := \bigcap_{n=0}^{\infty} C_n$$

Cantor's set has several strange properties...

Zero length If the *length* of a set is the sum of the lengths of its disjoint sub-intervals, then

$$\text{length}(C_n) = \left(\frac{2}{3}\right)^n$$

since we delete $\frac{1}{3}$ of the remaining set at each step. It follows that

$$\forall n \in \mathbb{N}_0,\ \text{length}(\mathcal{C}) \le \left(\frac{2}{3}\right)^n \implies \text{length}(\mathcal{C}) = 0$$

Otherwise said, $\mathcal{C}$ contains no subintervals.

Uncountable There exists a bijection between $\mathcal{C}$ and the original interval $[0,1]$!

Self-similarity Abusing notation somewhat,

$$C_{n+1} = \frac{1}{3}C_n \cup \left(\frac{1}{3}C_n + \frac{2}{3}\right)$$

where we mean that C_{n+1} consists of two copies of C_n, each shrunk by a factor of $\frac{1}{3}$ and one shifted $\frac{2}{3}$ to the right. The upshot is that the Cantor set itself satisfies

$$\mathcal{C} = \frac{1}{3}\mathcal{C} \cup \left(\frac{1}{3}\mathcal{C} + \frac{2}{3}\right)$$

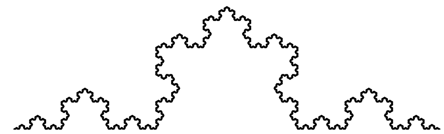
Being similar to two disjoint subsets of itself, its fractal dimension is $D = \frac{\log 2}{\log 3} \approx 0.631$. The above image links to an animation showing how the full set may be doubled to produce itself.

The Cantor set has many generalizations. Look up the Sierpiński triangle ($D = \frac{\log 3}{\log 2} \approx 1.585$) and carpet (Examples 5.8, $D = \frac{\log 8}{\log 3} \approx 1.893$), and the Menger sponge ($D = \frac{\log 20}{\log 3} \approx 2.727$).

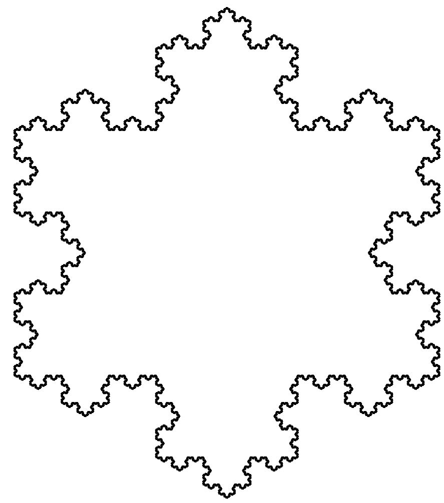
[32]Henry Smith discovered this set in 1874 while investigating integrability, in which context the 'length' of a set was later formalized using *measure theory*. Cantor's description in 1883 was more focused on topological properties. Self-similarity was less of a concern at the time.

Example 5.5 (The Koch Curve and Snowflake). The Koch curve is another generalization of the Cantor set, produced as the limit of a sequence of curves.

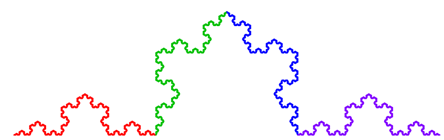
- Let K_0 be a segment of length 1.
- Replace the middle third of K_0 with the other two sides of an equilateral triangle to create K_1.
- Replacing the middle third of each segment in K_1 as before to create K_2.
- Repeat this process *ad infinitum.*



Koch Curve



Koch Snowflake



Self-similarity

The curve is drawn, along with the *Koch snowflake* obtained by arranging three copies around an equilateral triangle.

The relation to the Cantor set should be obvious in the construction. Indeed if $K_0 = [0,1]$, then the intersection of this with the Koch curve is the Cantor set!

The Koch curve is self-similar in that it comprises $N = 4$ copies of itself shrunk by a factor of $r = \frac{1}{3}$. Its fractal dimension is therefore $\frac{\log 4}{\log 3} \approx 1.2619$.

We may also consider the curve's length. Let s_n be the number of segments in K_n, each having length t_n. Also let $\ell_n = t_n s_n$ be the length of the curve K_n. We easily see that

$$s_n = 4^n, \quad t_n = \frac{1}{3^n} \implies \ell_n = \left(\frac{4}{3}\right)^n \to \infty$$

from which the Koch curve is *infinitely long*!

Exercises 5.1. 1. By removing a constant middle fraction of each interval, construct a fractal analogous to the Cantor set but with dimension $\frac{1}{2}$.

2. Prove that the area inside the n^{th} iteration of the construction of the Koch snowflake is

$$A_n = \left(1 + \frac{3}{5}\left[1 - \left(\frac{4}{9}\right)^n\right]\right)\frac{\sqrt{3}}{4}$$

The area inside the complete snowflake is therefore $\frac{8}{5}$ *that of the original triangle.*

3. Suppose $\mathbf{r}(t)$, $t \in [0,1]$ is a regular (smooth) curve in the plane.

   (a) Use the arc-length formula $L = \int_0^1 |\mathbf{r}'(t)|\,\mathrm{d}t$ together with Riemann sums and the linear approximation $\mathbf{r}(t+\epsilon) \approx \mathbf{r}(t) + \epsilon\mathbf{r}'(t)$ with $\epsilon = \frac{1}{N}$ to argue that

   $$L \approx \sum_{k=0}^{N-1} \left|\mathbf{r}\left(\frac{k+1}{N}\right) - \mathbf{r}\left(\frac{k}{N}\right)\right| \tag{$*$}$$

   (b) Suppose the curve is parametrized such that each segment on the right side of $(*)$ has the same length R. Prove that $L \approx NR$.
   Any regular curve thus has fractal dimension 1 in the sense stated by Mandelbrot (pg. 78).

5.2 Contraction Mappings & Iterated Function Systems

Thus far we have only dealt with fractals where the whole consists of pieces scaled by the same factor. In general we can mix up scaling factors. To do this it is helpful to borrow some language from topology.

Definition 5.6. A *contraction mapping* is a function S on a subset of $\mathbb{R}^n$ such that $\exists c \in [0,1)$ with

$$|S(x) - S(y)| \leq c\,|x - y|$$

A contraction mapping therefore moves points closer together. It should be clear that every contraction mapping is continuous. The main idea of this section is that fractals may be generated by repeatedly applying contraction mappings to an initial shape. We have already seen a example:

Example (5.4, mk. II). Consider the following functions $S_1, S_2 : \mathbb{R} \to \mathbb{R}$

$$S_1(x) = \frac{x}{3} \qquad S_2(x) = \frac{x}{3} + \frac{2}{3}$$

These are certainly contraction mappings

$$\forall x, y \in \mathbb{R},\ |S_1(x) - S_1(y)| = |S_2(x) - S_2(y)| = \frac{1}{3}\,|x - y|$$

with scale factor $c = \frac{1}{3}$. More importantly, these functions *define* the Cantor set: at each stage of its construction, we have

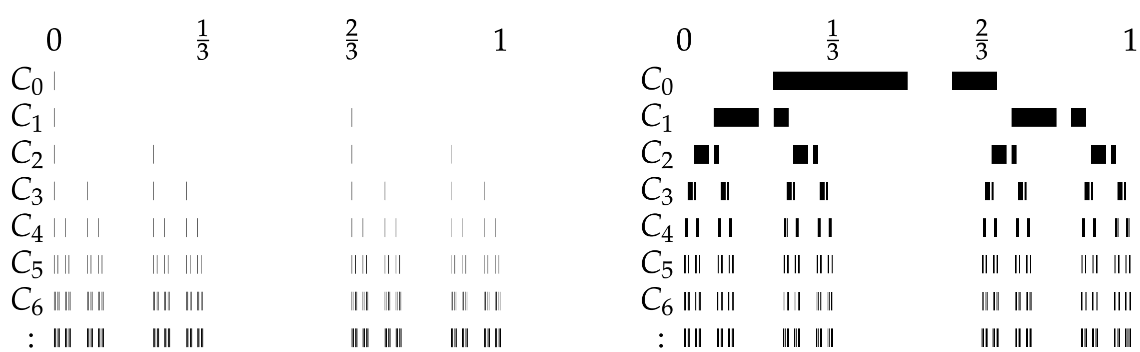
$$C_{n+1} := S_1(C_n) \cup S_2(C_n)$$

As the limit of this process, the self-similarity of the Cantor set can be expressed in the same manner: $\mathcal{C} = S_1(\mathcal{C}) \cup S_2(\mathcal{C})$.

Surprisingly, it barely seems to matter what initial set C_0 we choose. For example, we could start with the singleton set $C_0 = \{0\}$, from which

$$C_1 = \{0, \tfrac{2}{3}\}, \qquad C_2 = \{0, \tfrac{2}{9}, \tfrac{2}{3}, \tfrac{8}{9}\}, \qquad C_3 = \{0, \tfrac{2}{27}, \tfrac{2}{9}, \tfrac{8}{27}, \tfrac{2}{3}, \tfrac{20}{27}, \tfrac{8}{9}, \tfrac{26}{27}\}, \ \ldots$$

We draw the first few iterations below. In the second picture, we start with a very different initial set $C_0 = [0.2, 0.5] \cup [0.6, 0.7]$. Iterating this also appears to produce the Cantor set!



Iterated Function Systems

It certainly seems as if the Cantor set might be generated by the contraction maps S_1, S_2 independently of the initial data C_0. The following result shows in what sense this is the case, though it relies on some heavy lifting from topology. If you've done some analysis, then several of the concepts will be familiar. We summarize the discussion without proof.

- A subset of $\mathbb{R}^m$ is *compact* if it is *closed* (contains its boundary points) and *bounded* (all points lie within some ball centered at the origin).
- The set of all compact subsets of $\mathbb{R}^m$ is a metric space $\mathcal{H}$. This means that the *distance* $d(X,Y)$ between two compact sets $X, Y \in \mathcal{H}$ may sensibly be defined, though it is a little tricky…[33]
- Since $\mathcal{H}$ is a metric space, we can discuss convergent sequences (K_n) of compact sets

  $$\lim_{n\to\infty} K_n = K \iff \lim_{n\to\infty} d(K_n, K) = 0$$

  It also makes sense to speak of Cauchy sequences in $\mathcal{H}$. Moreover, $\mathcal{H}$ is *complete* in that every Cauchy sequence $(K_n) \subseteq \mathcal{H}$ converges to some $K \in \mathcal{H}$.
- The *Banach Fixed Point Theorem* now applies.

  > If $S : \mathcal{H} \to \mathcal{H}$ is a contraction mapping on a complete metric space $\mathcal{H}$, then S has a unique fixed point (some $F \in \mathcal{H}$ such that $S(F) = F$). Moreover, if $F_0 \in \mathcal{H}$ is any initial value, then the sequence defined iteratively by $F_{k+1} := S(F_k)$ converges to F.

  This powerful result has applications throughout mathematics.

Theorem 5.7. *Let $S_1, \ldots, S_n$ be contraction mappings on $\mathbb{R}^m$ with ratios $c_1, \ldots, c_n$. Define*

$$S : \mathcal{H} \to \mathcal{H} \quad by \quad S(D) = \bigcup_{i=1}^{n} S_i(D)$$

1. *S is a contraction mapping on $\mathcal{H}$, with contraction ratio $c = \max\{c_i\}$.*
2. *S has a unique fixed set $F \in \mathcal{H}$ given by $F = \lim_{k\to\infty} S^k(E)$ for any non-empty $E \in \mathcal{H}$.*

Part 1 is not difficult to prove if you're willing to work with the definition of the Hausdorff metric (try it if you're comfortable with analysis!). Part 2 is Banach's theorem.

The upshot is this: if we take any non-empty compact set E and repeatedly apply contraction mappings, the process will converge to a limit which is *independent of E!* We call the limit set F for *fractal.* Such fractals are often called *attractors*: being limit-sets, they 'attract' data towards themselves.

[33] This is the *Hausdorff metric.* Given $Y \in \mathcal{H}$, and $x \in \mathbb{R}^n$, define $d_Y(x) = \inf_{y\in Y} ||x - y||$ to be the distance from x to the 'nearest' point of Y. Define $d_X(y)$ similarly. The Hausdorff distance between X and Y is then

$$d(X,Y) := \max\left\{\sup_{x\in X} d_Y(x), \sup_{y\in Y} d_X(y)\right\}$$

Roughly speaking, find $x \in X$ which is as far away $d_Y(x)$ as possible from anything in Y, and find $y \in Y$ similarly; $d(X,Y)$ is the larger of these distances.

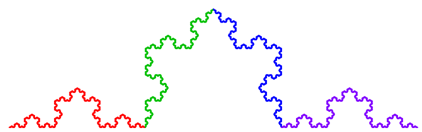
Examples 5.8. 1. (Cantor set Ex. 5.4) Theorem 5.7 shows that we may let C_0 be *any* closed bounded subset of $\mathbb{R}$. Repeatedly applying the contraction mappings S_1 and S_2 will always result in the same set $\mathcal{C}$.

A nice application is that one can easily find all sorts of interesting points in the Cantor set. For instance, suppose $x, y \in \mathbb{R}$ are a pair such that $y = S_1(x)$ and $x = S_2(y)$: otherwise said

$$y = \frac{1}{3}x \quad \text{and} \quad x = \frac{1}{3}(y+2)$$

Since $E = \{x, y\}$ is a compact set satisfying $E \subseteq S(E)$, it follows that $E \subseteq \lim S^k(E) = \mathcal{C}$, from which x, y both lie in the Cantor set! However, we can easily solve to see that $(x, y) = (\frac{3}{4}, \frac{1}{4})$. This seems paradoxical: $\frac{1}{4}$ does not lie at the end of any deleted interval (denominators of the form 3^n) but yet the Cantor set contains no intervals. How does $\frac{1}{4}$ end up in there?!

2. (Koch curve, Ex. 5.5) Define four mappings $S_i : \mathbb{R}^2 \to \mathbb{R}^2$, each with scale factor $c = \frac{1}{3}$.

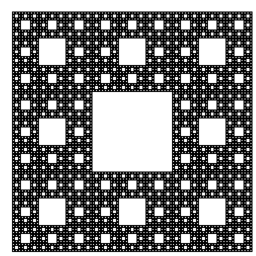


Mapping	Effect
$S_1(x,y) = (\frac{x}{3}, \frac{y}{3})$	Scale $\frac{1}{3}$
$S_2(x,y) = \left(\frac{1}{6}x - \frac{\sqrt{3}}{6}y + \frac{1}{3}, \frac{\sqrt{3}}{6}x + \frac{1}{6}y\right)$	Scale $\frac{1}{3}$, rotate 60°, translate
$S_3(x,y) = \left(\frac{1}{6}x + \frac{\sqrt{3}}{6}y + \frac{1}{2}, \frac{\sqrt{3}}{6}x - \frac{1}{6}y + \frac{\sqrt{3}}{6}\right)$	Scale $\frac{1}{3}$, rotate $-60°$, translate
$S_4(x,y) = (\frac{x}{3} + \frac{2}{3}, \frac{y}{3})$	Scale $\frac{1}{3}$, translate

Applied to the Koch curve, the image of each map corresponds by color. The picture links to a series of animated constructions of the curve starting with different initial sets E.

3. (Sierpiński carpet) Eight contraction mappings produce this fractal, each reducing the whole by a (length-scale) factor of $\frac{1}{3}$.

As with the Koch curve, the image links to several alternative constructions using different initial starting sets.



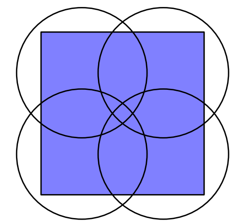
4. (A Fractal Fern) This is built from three contraction mappings:

   S_1: Scale by $\frac{3}{4}$, rotate 5° clockwise, and translate by $(0, \frac{1}{4})$

   S_2: Scale by $\frac{1}{4}$, rotate 60° counter-clockwise, and translate by $(0, \frac{1}{4})$

   S_3: Scale by $\frac{1}{4}$, rotate 60° clockwise, and translate by $(0, \frac{1}{4})$

Fractal Dimension Revisited



Since Theorem 5.7 permits several different contraction factors, we need a new approach to computing fractal dimension. We ask how many disks of a given radius ϵ are required to cover a set. In the picture, the unit square requires four disks of radius $\varepsilon = 0.4$. For smaller ε, we will plainly need more disks...

Definition 5.9. Let A be a compact subset of $\mathbb{R}^m$.

1. If $\varepsilon > 0$, the *closed ε-ball centered at $x \in A$* consists of the points at most a distance ε from x:
$$B_\epsilon(x) = \{y \in \mathbb{R}^m : d(x,y) \le \varepsilon\}$$
2. The *minimal ε-covering number* for A is
$$\mathcal{N}(A,\varepsilon) = \min\left\{ M : \exists x_1,\ldots,x_M \in A \text{ with } A \subseteq \bigcup_{n=1}^{M} B_\epsilon(x_n)\right\}$$
3. Given a compact set $A \subseteq \mathbb{R}^m$, its *fractal dimension* is the limit
$$D = \lim_{\varepsilon\to 0} \frac{\log \mathcal{N}(A,\varepsilon)}{\log(1/\varepsilon)}$$

We don't claim to prove that D must exist, though a simple example should at least convince you that the definition is reasonable!

Example 5.10. Let $A = [0,1]$ be the interval of length 1. It is not hard to see that
$$\varepsilon \ge \frac{1}{2} \iff \mathcal{N}(\varepsilon) = 1, \quad \text{and} \quad \frac{1}{4} \le \varepsilon < \frac{1}{2} \iff \mathcal{N}(\varepsilon) = 2$$
etc. More generally, $\mathcal{N}$ and ϵ are related via
$$\frac{1}{2\mathcal{N}} \le \epsilon < \frac{1}{2(\mathcal{N}-1)}$$
The dimension of the line (1) may therefore be recovered via the squeeze theorem
$$D = \lim_{\epsilon\to 0} \frac{\log \mathcal{N}}{\log(1/\varepsilon)} = 1$$

Thankfully an easier-to-use modification is available using boxes.

Theorem 5.11 (Box-counting). *Let A be compact and cover $\mathbb{R}^m$ by boxes of side length $\frac{1}{2^n}$. Let $\mathcal{N}_n(A)$ be the number of boxes intersecting A. Then*
$$D = \lim_{n\to\infty} \frac{\log \mathcal{N}_n(A)}{\log 2^n}$$

We finish with a formula satisfied by the dimension of an iterated function system (Theorem 5.7).

Theorem 5.12. *Let $\{S_n\}_{n=1}^{M}$ be an iterated function system with attractor (limiting fractal) F and where each contraction S_n has scale factor $c_n \in (0,1)$. At each stage of the construction, suppose portions of the fractal generated by each contraction map meet only at boundary points. Then the fractal dimension is the unique D satisfying*

$$\sum_{n=1}^{M} c_n^D = 1$$

Examples 5.13. 1. If all scale-factors are identical $c_n = r$, we recover Definition 5.2,

$$Mr^D = 1 \implies D = \frac{-\log M}{\log r} = \frac{\log M}{\log(1/r)}$$

2. The fractal fern (Examples 5.8) is generated by three contraction maps with scale factors $\frac{3}{4}, \frac{1}{4}, \frac{1}{4}$. Its dimension is the solution to the equation

$$\left(\frac{3}{4}\right)^D + \left(\frac{1}{4}\right)^D + \left(\frac{1}{4}\right)^D = 1 \implies D \approx 1.3267$$

3. Numerical approximation is usually required to solve for D, though sometimes an exact solution is possible. For instance, if $c_1 = c_2 = \frac{1}{2}$ and $c_3 = c_4 = c_5 = \frac{1}{4}$, then

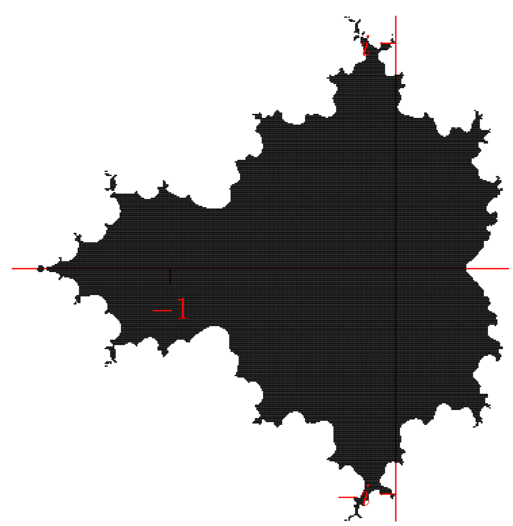
$$2\left(\frac{1}{2}\right)^D + 3\left(\frac{1}{4}\right)^D = 1$$

Writing $\alpha = \left(\frac{1}{2}\right)^D$ yields the quadratic equation

$$2\alpha + 3\alpha^2 = 1 \implies \alpha = \frac{1}{3} \implies D = \log_2 3 \approx 1.584$$

Other methods of creating fractals

The contraction mapping approach is one of many ways to create fractals. Two other famous examples are the *logistic map* (related to numerical approximations to non-linear differential equations) and the *Mandelbrot set* (pictured).



The Mandelbrot set arises from a construction in the complex plane. For a given $c \in \mathbb{C}$, we iterate the function

$$f_c(z) = z^2 + c$$

If $f(f(f(\cdots f(c)\cdots)))$ remains bounded, no matter how many times f is applied, then c lies in the Mandelbrot set.

Much better pictures and some trippy videos can be found online...

Exercises 5.2. 1. Let $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$ be the contraction mappings defining the Cantor set and suppose $x, y, z \in \mathbb{R}$ satisfy

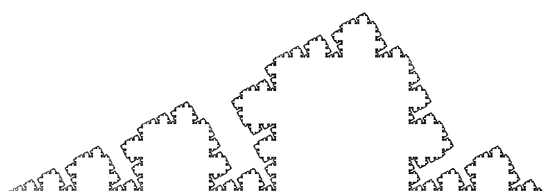
$$y = S_1(x), \qquad z = S_2(y), \qquad x = S_2(z)$$

Show that x, y, z lie in the Cantor set, and find their values.

2. The construction of a Cantor-type set starts by removing the open intervals $(0.1, 0.2)$ and $(0.6, 0.8)$ from the unit interval.
   (a) Sketch the first three iterations of this fractal.
   (b) This construction may be described using three contraction mappings; what are they?
   (c) State an equation satisfied by the dimension D of the set and use a computer algebra package to estimate its value.

3. A variation on the Koch curve is constructed using the following contraction mappings. Each is built by first scaling the whole picture by a factor c, rotating the picture through an angle counter-clockwise, and then translating the picture by adding a constant. The resulting fractal is drawn.

map	scale	rotate	translate (add (x, y))
S_1	$\frac{1}{2}$	0	0
S_2	$\frac{1}{4}$	90°	$(\frac{1}{2}, 0)$
S_3	$\frac{1}{4}$	0	$(\frac{1}{2}, \frac{1}{4})$
S_4	$\frac{1}{4}$	−90°	$(\frac{3}{4}, \frac{1}{4})$
S_5	$\frac{1}{4}$	0	$(\frac{3}{4}, 0)$



   (a) Suppose you start with the straight line segment from $(0, 0)$ to $(1, 0)$. Draw the first two iterations of the fractal's construction.
   (b) The dimension of the fractal is the unique solution D to the equation

$$\left(\frac{1}{2}\right)^D + \left(\frac{1}{4}\right)^D + \left(\frac{1}{4}\right)^D + \left(\frac{1}{4}\right)^D + \left(\frac{1}{4}\right)^D = 1$$

   By observing that $\frac{1}{4} = \left(\frac{1}{2}\right)^2$, convert this to a quadratic equation in the variable $\alpha := \left(\frac{1}{2}\right)^D$. Hence compute the dimension of the fractal.
   (c) The dimension computed in part (b) is *larger* than the dimension $\frac{\log 4}{\log 3}$ of the Koch curve. Explain what this means.

4. Verify the details of Example 5.10, including the computation of the limit.

5. In Theorem 5.12, prove that D exists and is unique.
   (*Hint: You'll need the intermediate value theorem from calculus*)