Trigonometric Formulas and the Congruence Theorems

Euclidean Geometry Label triangle in the usual way: sides a, b, c, angles α, β, γ.

SAS Given a, γ, b:

1. Cosine rule $c^2 = a^2 + b^2 - 2ab\cos\gamma$ gives c.
2. Sine rule $\sin\alpha = \frac{a}{c}\sin\gamma$ gives α.
3. $\Sigma_\triangle = 180°$ gives β (or sine rule again).

SSS Given a, b, c:

1. Cosine rule $\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$ gives α.
2. Cos rule or sine rule for β.
3. $\Sigma_\triangle = 180°$ gives γ.

SAA Given b, α, β:

1. $\Sigma_\triangle = 180°$ gives γ.
2. Sine rule $a = b\frac{\sin\alpha}{\sin\beta}$ and $c = b\frac{\sin\gamma}{\sin\beta}$.

ASA Given α, c, β:

1. $\Sigma_\triangle = 180°$ gives γ.
2. Sine rule $a = c\frac{\sin\alpha}{\sin\gamma}$ and $b = c\frac{\sin\beta}{\sin\gamma}$.

SSA One/two solutions (not a congruence theorem!) Given a, b, α:

1. Sine rule $\sin\beta = \frac{b}{a}\sin\alpha$ gives two values for β (one acute, one obtuse). If $\alpha \geq 90°$, then only acute angle is valid.
2. Could instead use cosine rule $a^2 = c^2 + b^2 - 2bc\cos\alpha \implies c = b\cos\alpha \pm \sqrt{a^2 - b^2\sin^2\alpha}$ gives one/two values for c.

Hyperbolic Geometry Same notation where sides are *hyperbolic* lengths.

Pythagoras/trig for right-triangles

$$\cosh h = \cosh x \cosh y \qquad \sin\theta = \frac{\sinh y}{\sinh h} \qquad \cos\theta = \frac{\tanh x}{\tanh h} \qquad \tan\theta = \frac{\tanh y}{\sinh x}$$

Cos rule $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma$

Sine rule $\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c}$

SAS Given a, γ, b:

1. Cosine rule gives (cosh) c.
2. Sine rule $\sin\alpha = \frac{\sinh a}{\sinh c}\sin\gamma$ gives α.
3. Sine rule again $\sin\beta = \frac{\sinh b}{\sinh c}\sin\gamma$ gives β (can't use $\Sigma_\triangle$!).

SSS Given a, b, c:

1. Cosine rule $\cos\alpha = \frac{\cosh b \cosh c - \cosh a}{\sinh a \sinh b}$ gives α.
2. Repeat twice more, or use sine rule for β, γ.

SAA Given b, α, β:

1. Sine rule $\sinh a = \frac{\sin\alpha}{\sin\beta}\sinh b$ gives a.
2. Cosine formula by dropping perpendicular to compute

$$c = \tanh^{-1}\big(\cos\alpha \tanh b\big) + \tanh^{-1}\big(\cos\beta \tanh a\big)$$

3. Sine rule $\sin\gamma = \sin\alpha \frac{\sinh c}{\sinh a}$.

ASA (Two methods—hard!) Given α, c, β:

1. Drop perp from angle at $\beta = \beta_1 + \beta_2$ to get two right triangles: compute $\beta_1, \beta_2 = \beta - \beta_1$, basic trig and Pythag (mess).
2. Compute a. Cosine rule twice plus sine rule:

$$\begin{aligned}\cosh a &= \cosh b \cosh c - \sinh b \sinh c \cos\alpha \\ &= \big(\cosh a \cosh c - \sinh a \sinh c \cos\beta\big)\cosh c - \sinh a \sinh c \frac{\cos\alpha \sin\beta}{\sin\alpha} \\ &= \cosh a \cosh^2 c - \sinh a \sinh c\big(\cos\beta \cosh c + \cot\alpha \sin\beta\big)\end{aligned}$$

   Use $\cosh^2 c = 1 + \sinh^2 c$ to rearrange and divide through by $\sinh c$:

$$\tanh a = \frac{\sinh c}{\cos\beta \cosh c + \cot\alpha \sin\beta} = \frac{\sin\alpha \sinh c}{\sin\alpha \cos\beta \cosh c + \cos\alpha \sin\beta}$$

   Compare with $a = c\frac{\sin\alpha}{\sin(\alpha+\beta)} = c\frac{\sin\alpha}{\sin\gamma}$ which is the sine rule in Euclidean geometry.

3. Sine rule $\sinh b = \sinh a \frac{\sin\beta}{\sin\alpha}$ and $\sin\gamma = \sin\alpha \frac{\sinh c}{\sinh a}$.

AAA (Hard!) Given angles α, β, γ.

1. Know ASA formula (part 1); now apply sine rule to numerator

$$\frac{\sinh a}{\cosh a} = \frac{\sin\gamma \sinh a}{\sin\alpha \cos_\beta \cosh c + \cos\alpha \sin\beta} \implies s_\gamma \cosh a = s_\alpha c_\beta \cosh c + c_\alpha s_\beta \qquad (*)$$

   Square, use identities and sine rule:

$$\begin{aligned} s_\alpha^2 c_\beta^2 \cosh^2 c + 2 s_\alpha s_\beta c_\alpha c_\beta \cosh c + c_\alpha^2 s_\beta^2 &= s_\gamma^2 (1 + \sinh^2 a) = s_\gamma^2 + s_\alpha^2 \sinh^2 c \\ &= s_\gamma^2 + s_\alpha^2 (\cosh^2 c - 1) \end{aligned}$$

   Rearrange to obtain quadratic in $\cosh c$:

$$\begin{aligned} & s_\alpha^2 (1 - c_\beta^2) \cosh^2 c - 2 s_\alpha s_\beta c_\alpha c_\beta \cosh c - c_\alpha^2 (1 - c_\beta^2) + s_\gamma^2 - s_\alpha^2 = 0 \\ \implies & s_\alpha^2 s_\beta^2 \cosh^2 c - 2 s_\alpha s_\beta c_\alpha c_\beta \cosh c + c_\alpha^2 c_\beta^2 - c_\gamma^2 = 0 \\ \implies & \big(s_\alpha s_\beta \cosh c - c_\alpha c_\beta \big)^2 = \cos^2\gamma \\ \implies & \cosh c = \frac{\cos\alpha \cos\beta + \cos\gamma}{\sin\alpha \sin\beta} \end{aligned}$$

   By considering $(*)$, it may be checked that only the positive square root is legitimate. This is essentially an alternative version of the cosine rule that comes by evaluating on imaginary angles and sides.

   Compare: in Euclidean limit, $\gamma = \pi - \alpha - \beta$, whence RHS becomes 1.

2. Same formula can be used three times to compute side length, or use previous methods.

SSA Two solutions (not a congruence theorem!) Given a, b, α:

1. Sine rule $\sin\beta = \frac{\sinh b}{\sinh a} \sin\alpha$ gives two values for β (one acute, one obtuse). Can use cosine rule as in Euclidean geometry, but result ugly since not a quadratic for c…
2. Now in one of above situations.