Trigonometric Formulas and the Congruence Theorems

Euclidean Geometry Label triangle in the usual way: sides a, b, c, angles α, β, γ.

SAS Given a, γ, b:

1. Cosine rule $c^2 = a^2 + b^2 - 2ab\cos\gamma$ gives c.
2. Sine rule $\sin\alpha = \frac{a}{c}\sin\gamma$ gives α.
3. $\Sigma_\triangle = 180°$ gives β (or sine rule again).

SSS Given a, b, c:

1. Cosine rule $\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$ gives α.
2. Cos rule or sine rule for β.
3. $\Sigma_\triangle = 180°$ gives γ.

SAA Given b, α, β:

1. $\Sigma_\triangle = 180°$ gives γ.
2. Sine rule $a = b\frac{\sin\alpha}{\sin\beta}$ and $c = b\frac{\sin\gamma}{\sin\beta}$.

ASA Given α, c, β:

1. $\Sigma_\triangle = 180°$ gives γ.
2. Sine rule $a = c\frac{\sin\alpha}{\sin\gamma}$ and $b = c\frac{\sin\beta}{\sin\gamma}$.

SSA One/two solutions (not a congruence theorem!) Given a, b, α:

1. Sine rule $\sin\beta = \frac{b}{a}\sin\alpha$ gives two values for β (one acute, one obtuse). If $\alpha \geq 90°$, then only acute angle is valid.
2. Could instead use cosine rule $a^2 = c^2 + b^2 - 2bc\cos\alpha \implies c = b\cos\alpha \pm \sqrt{a^2 - b^2\sin^2\alpha}$ gives one/two values for c.

Hyperbolic Geometry Same notation where sides are *hyperbolic* lengths.

Pythagoras/trig for right-triangles

$$\cosh h = \cosh x \cosh y \qquad \sin\theta = \frac{\sinh y}{\sinh h} \qquad \cos\theta = \frac{\tanh x}{\tanh h} \qquad \tan\theta = \frac{\tanh y}{\sinh x}$$

Cos rule $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma$

Sine rule $\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c}$

SAS Given a, γ, b:

1. Cosine rule gives (cosh) c.
2. Sine rule $\sin\alpha = \frac{\sinh a}{\sinh c}\sin\gamma$ gives α.
3. Sine rule again $\sin\beta = \frac{\sinh b}{\sinh c}\sin\gamma$ gives β (can't use $\Sigma_\triangle$!).

SSS Given a, b, c:

1. Cosine rule $\cos\alpha = \frac{\cosh b \cosh c - \cosh a}{\sinh a \sinh b}$ gives α.
2. Repeat twice more, or use sine rule for β, γ.

SAA Given b, α, β:

1. Sine rule $\sinh a = \frac{\sin\alpha}{\sin\beta}\sinh b$ gives a.
2. Cosine formula by dropping perpendicular to compute
$$c = \tanh^{-1}\big(\cos\alpha \tanh b\big) + \tanh^{-1}\big(\cos\beta \tanh a\big)$$
3. Sine rule $\sin\gamma = \sin\alpha \frac{\sinh c}{\sinh a}$.

ASA (Two methods—hard!) Given α, c, β:

1. Drop perp from angle at $\beta = \beta_1 + \beta_2$ to get two right triangles: compute $\beta_1, \beta_2 = \beta - \beta_1$, basic trig and Pythag (mess).
2. Compute a. Cosine rule twice plus sine rule:
$$\begin{aligned}\cosh a &= \cosh b \cosh c - \sinh b \sinh c \cos\alpha \\ &= \big(\cosh a \cosh c - \sinh a \sinh c \cos\beta\big)\cosh c - \sinh a \sinh c \frac{\cos\alpha \sin\beta}{\sin\alpha} \\ &= \cosh a \cosh^2 c - \sinh a \sinh c\big(\cos\beta \cosh c + \cot\alpha \sin\beta\big)\end{aligned}$$
Use $\cosh^2 c = 1 + \sinh^2 c$ to rearrange and divide through by $\sinh c$:
$$\tanh a = \frac{\sinh c}{\cos\beta \cosh c + \cot\alpha \sin\beta} = \frac{\sin\alpha \sinh c}{\sin\alpha \cos\beta \cosh c + \cos\alpha \sin\beta}$$
Compare with $a = c\frac{\sin\alpha}{\sin(\alpha+\beta)} = c\frac{\sin\alpha}{\sin\gamma}$ which is the sine rule in Euclidean geometry.
3. Sine rule $\sinh b = \sinh a \frac{\sin\beta}{\sin\alpha}$ and $\sin\gamma = \sin\alpha \frac{\sinh c}{\sinh a}$.

AAA (Hard!) Given angles α, β, γ.

1. Know ASA formula (part 1); now apply sine rule to numerator

$$\frac{\sinh a}{\cosh a} = \frac{\sin\gamma \sinh a}{\sin\alpha \cos_\beta \cosh c + \cos\alpha \sin\beta} \implies s_\gamma \cosh a = s_\alpha c_\beta \cosh c + c_\alpha s_\beta \qquad (*)$$

   Square, use identities and sine rule:

$$\begin{aligned} s_\alpha^2 c_\beta^2 \cosh^2 c + 2 s_\alpha s_\beta c_\alpha c_\beta \cosh c + c_\alpha^2 s_\beta^2 &= s_\gamma^2 (1 + \sinh^2 a) = s_\gamma^2 + s_\alpha^2 \sinh^2 c \\ &= s_\gamma^2 + s_\alpha^2 (\cosh^2 c - 1) \end{aligned}$$

   Rearrange to obtain quadratic in $\cosh c$:

$$\begin{aligned} & s_\alpha^2 (1 - c_\beta^2) \cosh^2 c - 2 s_\alpha s_\beta c_\alpha c_\beta \cosh c - c_\alpha^2 (1 - c_\beta^2) + s_\gamma^2 - s_\alpha^2 = 0 \\ \implies & s_\alpha^2 s_\beta^2 \cosh^2 c - 2 s_\alpha s_\beta c_\alpha c_\beta \cosh c + c_\alpha^2 c_\beta^2 - c_\gamma^2 = 0 \\ \implies & \big(s_\alpha s_\beta \cosh c - c_\alpha c_\beta \big)^2 = \cos^2 \gamma \\ \implies & \cosh c = \frac{\cos\alpha \cos\beta + \cos\gamma}{\sin\alpha \sin\beta} \end{aligned}$$

   By considering $(*)$, it may be checked that only the positive square root is legitimate. Compare: in Euclidean limit, $\gamma = \pi - \alpha - \beta$, whence RHS becomes 1.

2. Same formula can be used three times to compute side length, or use previous methods.

SSA Two solutions (not a congruence theorem!) Given a, b, α:

1. Sine rule $\sin\beta = \frac{\sinh b}{\sinh a} \sin\alpha$ gives two values for β (one acute, one obtuse). Can use cosine rule as in Euclidean geometry, but result ugly since not a quadratic for c...
2. Now in one of above situations.