

2 Euclidean Geometry

2.1 Euclid's Postulates and Book I of the Elements

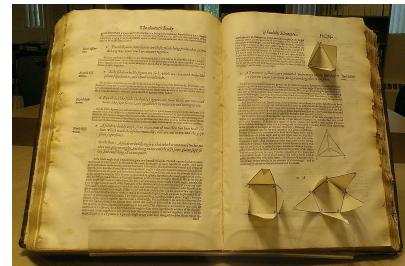
Euclid's *Elements* (c. 300 BC) formed a core part of European and Islamic curricula until the mid 20th century. Several examples are shown below.



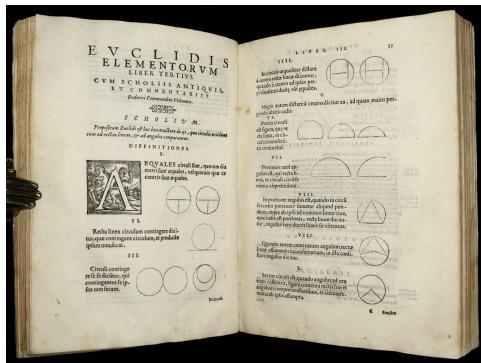
Earliest Fragment c. AD 100



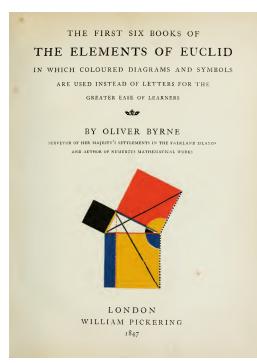
Full copy, Vatican, 9th C



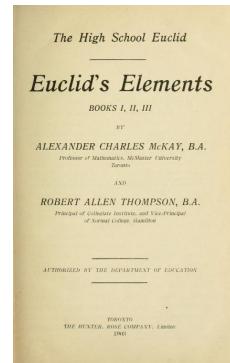
Pop-up edition, 1500s



Latin translation, 1572



Color edition, 1847



Textbook, 1903

Many of Euclid's arguments can be found online, and you can read Byrne's 1847 edition here: the cover is Euclid's proof of the Pythagorean Theorem. We present an overview of Book I.

Undefined Terms E.g., point, line, etc.⁴

*Axioms/Postulates*⁵ A1 If two objects equal a third, then the objects are equal (= is transitive)

A2 If equals are added to equals, the results are equal ($a = c \ \& \ b = d \implies a + b = c + d$)

A3 If equals are subtracted from equals, the results are equal

A4 Things that coincide are equal (in magnitude)

A5 The whole is greater than the part

P1 A pair of points may be joined to create a line

P2 A line may be extended

P3 Given a center and a radius, a circle may be drawn

P4 All right-angles are equal

P5 If a straight line crosses two others and the angles on one side sum to less than two right-angles then the two lines (when extended) meet on that side.

⁴In fact Euclid attempted to define these: 'A point is that which has no part,' and 'A line has length but no breadth.'

⁵In Euclid, an axiom is somewhat more general than a postulate. Here the postulates contain the *geometry*.

The first three postulates describe *ruler and compass constructions*. P4 allows Euclid to compare angles at different locations. P5 is usually called the *parallel postulate*.

Euclid's system doesn't quite fit the modern standard. Some axioms are vague (what are 'things'?) and we'll consider several more-serious shortcomings later. For now we clarify two issues and introduce some notation.

Segments To Euclid, a line had *finite* extent—we call such a (*line*) *segment*. The segment joining points A, B is denoted \overline{AB} . In modern geometry, a *line* extends as far as permitted, often infinitely.

Congruence Euclid uses *equal* where modern mathematicians say *congruent*. We'll express congruent segments and angles in the modern fashion, e.g., $\angle ABC \cong \angle DEF$. Equal angles/segments must be genuinely the same object (same location, etc.).

Basic Theorems à la Euclid

Theorems were typically presented as a *problem*. Euclid provides a constructive solution (P1–P3) before proving that his construction really does solve the problem.

Theorem 2.1 (I. 1). *Problem: to construct an equilateral triangle on a given segment.*

The labelling I. 1 indicates Book I, Theorem 1. A triangle is equilateral if its three sides are congruent.

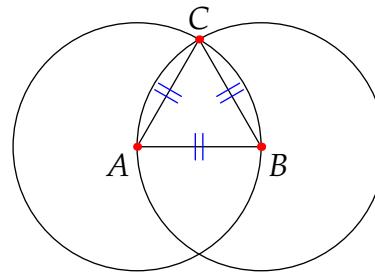
Proof. Given a line segment \overline{AB} :

By P3, construct circles centered at A and B with radius \overline{AB} .

Call one of the intersection points C . By P1, construct \overline{AC} and \overline{BC} .

We claim that $\triangle ABC$ is equilateral.

Observe that \overline{AB} and \overline{AC} are radii of the circle centered at A , while \overline{AB} and \overline{BC} are radii of the circle centered at B . By Axiom A1, the three sides of $\triangle ABC$ are congruent. ■



Euclid proceeds to develop several well-known constructions and properties of triangles.

- (I. 4) Side-angle-side (SAS) congruence: if two triangles have two pairs of congruent sides and the angles between these are congruent, then the remaining sides and angles are also congruent in pairs.

$$\begin{cases} \overline{AB} \cong \overline{DE} \\ \angle ABC \cong \angle DEF \\ \overline{BC} \cong \overline{EF} \end{cases} \implies \begin{cases} \overline{AC} \cong \overline{DF} \\ \angle BCA \cong \angle EFD \\ \angle CAB \cong \angle FDE \end{cases}$$

- (I. 5) An isosceles triangle has congruent base angles.
- (I. 9) To bisect an angle.
- (I. 10) To find the midpoint of a segment.
- (I. 15) If two lines/segments cut one another, opposite angles are congruent.

Have a look at some of Euclid's arguments, say in Byrne's edition. These are worth reading despite the logical issues in Euclid's presentation. We'll revisit these basic results in the Exercises and in the next two sections.

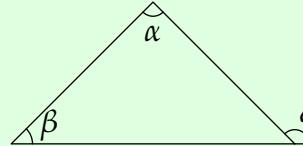
Parallel Lines: Construction & Existence

Definition 2.2. Lines are *parallel* if they do not intersect. Segments are parallel if no extensions of them intersect.

In Euclid, a line is not parallel to itself. The next result is one of the most important in Euclidean geometry, for it describes how to create a parallel line through a given point.

Theorem 2.3 (I. 16 Exterior Angle Theorem). If one side of a triangle is extended, then the exterior angle is larger than either of the opposite interior angles.

In the picture, we have $\delta > \alpha$ and $\delta > \beta$.



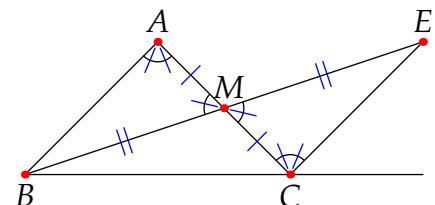
Euclid did not quantify angles numerically: $\delta > \alpha$ means that α is congruent to some angle *inside* δ .

Proof. Construct the bisector \overline{BM} of \overline{AC} (I. 10).

Extend \overline{BM} to E such that $\overline{BM} \cong \overline{ME}$ (I. 2) and connect \overline{CE} (P1).

The opposite angles at M are congruent (I. 15).

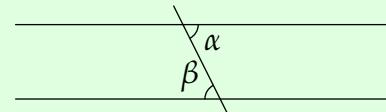
SAS (I. 4) applied to $\triangle AMB$ and $\triangle CME$ says $\angle BAM \cong \angle EMC$, which is clearly smaller than the exterior angle at C .



Bisect \overline{BC} and repeat the argument to see that $\beta < \delta$. ■

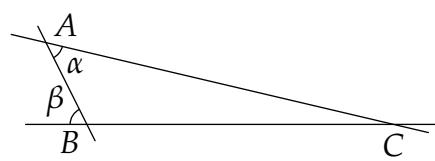
The proof in fact *constructs* a parallel (\overline{CE}) to \overline{AB} through C , as the next result shows.

Theorem 2.4 (I. 27). If a line falls on two other lines such that the alternate angles (α, β) are congruent, then the two lines are parallel.



The *alternate angles* in the exterior angle theorem are those at A and C : \overline{CE} really is parallel to \overline{AB} .

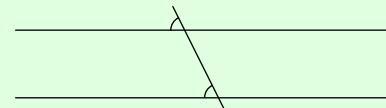
Proof. If the lines were not parallel, they would meet on one side. WLOG suppose they meet on the right side at C .



The angle β at B , being exterior to $\triangle ABC$, must be greater than the angle α at A (I. 16): contradiction. ■

Euclid combines this with the vertical angles theorem (I. 15) to finish the first half of Book I.

Corollary 2.5 (I. 28). If a line falling on two other lines makes congruent angles, then the two lines are parallel.



Thus far, Euclid uses only postulates P1–P4. In any model in which these hold:

Given a line ℓ and a point C not on ℓ , there exists a parallel to ℓ through C

Parallel Lines: Uniqueness, Angle-sums & Playfair's Postulate

Euclid finally invokes the parallel postulate to prove the converse of I. 27, showing that the congruent alternate angle approach is the *only* way to have parallel lines.

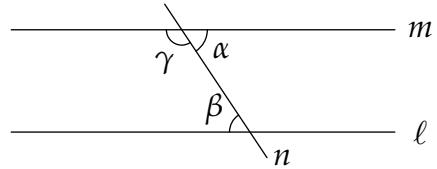
Theorem 2.6 (I. 29). *If a line falls on two parallel lines, then the alternate angles are congruent.*

Proof. Given the picture, we must prove that $\alpha \cong \beta$.

Suppose not and WLOG that $\alpha > \beta$.

But then $\beta + \gamma < \alpha + \gamma$, which is a straight edge.

By the parallel postulate, the lines ℓ, m meet on the left side of the picture, whence ℓ and m are not parallel. ■



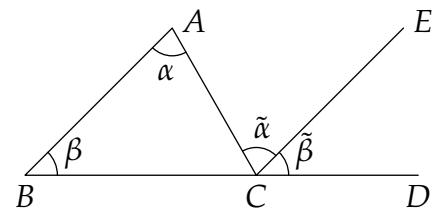
The most well-known result about triangles is now within our grasp: the interior angles sum to a straight-edge. Euclid words this slightly differently.

Theorem 2.7 (I. 32). *If one side of a triangle is extended, the exterior angle is congruent to the sum of the opposite interior angles.*

This is not a numerical sum, though for the sake of familiarity we'll often write 180° for a straight-edge and 90° for a right-angle.

In the picture we've labelled angles with Greek letters for clarity.

The result amounts to showing that $\tilde{\alpha} + \tilde{\beta} \cong \alpha + \beta$.



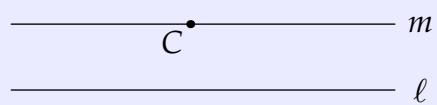
Proof. Construct \overline{CE} parallel to \overline{BA} as in I. 16, so that $\tilde{\alpha} \cong \alpha$.

\overline{BD} falls on parallel lines \overline{AB} and \overline{CE} , whence $\tilde{\beta} \cong \beta$ (Corollary of I. 29).

Axiom A2 shows that $\angle ACD = \tilde{\alpha} + \tilde{\beta} \cong \alpha + \beta$. ■

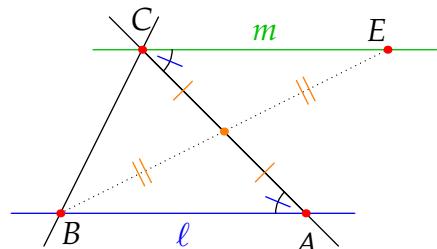
The parallel postulate is stated in the negative (angles don't sum to a straight-edge, therefore the lines are *not* parallel). Euclid possibly chose this formulation to facilitate proofs by contradiction, though the unfortunate effect is to obscure the meaning of the parallel postulate. Here is a more modern interpretation.

Axiom 2.8 (Playfair's Postulate). *Given a line ℓ and a point C not on ℓ , at most one parallel m to ℓ passes through C .*



Our discussion thus far shows that the parallel postulate implies Playfair.

- Let $A, B \in \ell$ and construct the triangle $\triangle ABC$.
- The exterior angle theorem constructs E and thus a parallel m to ℓ by I. 27.
- I. 29 invokes the parallel postulate to prove that this is the *only* such parallel.



In fact the postulates are equivalent.

Theorem 2.9. *In the presence of Euclid's first four postulates, Playfair's postulate and the parallel postulate (P5) are equivalent.*

Proof. ($P5 \Rightarrow \text{Playfair}$) We proved this above.

($\text{Playfair} \Rightarrow P5$) We prove the contrapositive. Assume postulates P1–P4 are true and that P5 is *false*. Using quantifiers, and with reference to the picture in I.29, we restate the parallel postulate:

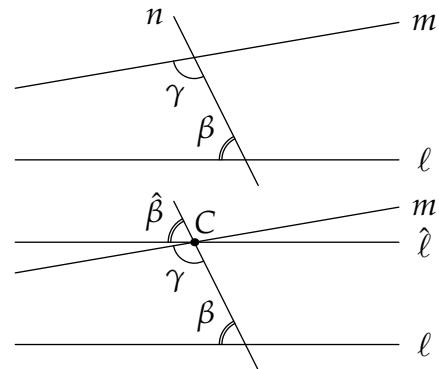
P5: \forall pairs of lines ℓ, m and \forall crossing lines n , $\beta + \gamma < 180^\circ \implies \ell, m \text{ not parallel.}$

Its *negation* (P5 *false*) is therefore:

\exists parallel lines ℓ, m and a crossing line n for which $\beta + \gamma < 180^\circ$

This is without loss of generality: if $\beta + \gamma > 180^\circ$, consider the angles on the other side of n .

By the exterior angle theorem/I.28, we may build a parallel line $\hat{\ell}$ to ℓ through the intersection C of m and n (in the picture, $\hat{\beta} \cong \beta$). Crucially, this only requires postulates P1–P4!



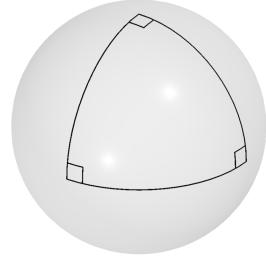
Observe that $\hat{\ell}$ and m are *distinct* since $\hat{\beta} + \gamma \cong \beta + \gamma < 180^\circ$. We therefore have a line ℓ and a point C not on ℓ , through which pass (at least) *two* parallels to ℓ : Playfair's postulate is *false*. ■

Non-Euclidean Geometry

That Euclid waited so long before invoking the uniqueness of parallels suggests he was trying to establish as much as he could about triangles and basic geometry in its absence. By contrast, everything from I.29 onwards relies on the parallel postulate, including the proof that the angle sum in a triangle is 180° . For centuries, many mathematicians believed, though none could prove it, that such a fundamental fact about triangles must be true independent of the parallel postulate.

Loosely speaking, a *non-Euclidean geometry* is a model for which a parallel through an off-line point either doesn't exist or is non-unique. It wasn't until the 17–1800s and the development of *hyperbolic geometry* (Chapter 4) that a model was found in which Euclid's first four postulates hold but for which the parallel postulate is *false*.⁶

We shall eventually see that every triangle in hyperbolic geometry has angle sum less than 180° , though this will require a lot of work! For a more easily visualized non-Euclidean geometry consider the sphere. A rubber band stretched between three points on its surface describes a *spherical triangle*: an example with angle sum 270° is drawn. A similar game can be played on a saddle-shaped surface: as in hyperbolic geometry, 'triangles' will have angle sum less than 180° .



⁶This shows that the parallel postulate is independent; in fact all Euclid's postulates are independent. They are also consistent (the 'usual' points and lines in the plane are a model), but incomplete: a sample undecidable is in Exercise 5.

Pythagoras' Theorem

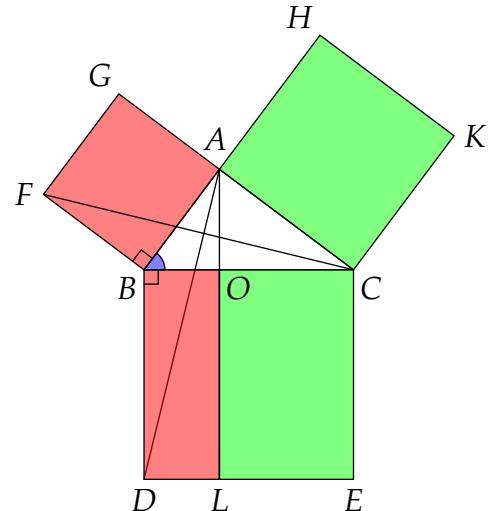
Following his discussion of parallels, Euclid shows that parallelograms with the same base and height are equal (in area) (I. 33–41), before providing constructions of parallelograms and squares (I. 42–46). Some of this is in Exercise 2. Immediately afterwards comes the capstone of Book I.

Theorem 2.10 (I. 47 Pythagoras' Theorem). *The square on the hypotenuse of a right triangle equals (has the same area as) the sum of the squares on the other sides.*

Proof. The given triangle $\triangle ABC$ is assumed to have a right-angle at A .

1. Construct squares on each side of $\triangle ABC$ (I. 46) and a parallel \overline{AL} to \overline{BD} (I. 16).
2. $\overline{AB} \cong \overline{FB}$ and $\overline{BD} \cong \overline{BC}$ since sides of squares are congruent. Moreover $\angle ABD \cong \angle FBC$ since both contain $\angle ABC$ and a right-angle.
3. Side-angle-side (I. 4) says that $\triangle ABD$ and $\triangle FBC$ are congruent (identical up to rotation by 90°).
4. I. 41 compares areas of parallelograms and triangles with the same base and height:

$$\begin{aligned}\text{Area}(\square ABFG) &= 2 \text{Area}(\triangle FBC) \\ &= 2 \text{Area}(\triangle ABD) \\ &= \text{Area}(\square BOLD)\end{aligned}$$



5. Similarly $\text{Area}(\square ACKH) = \text{Area}(\square OCEL)$.
6. Sum the rectangles to obtain $\square BCED$ and complete the proof. ■

Euclid finishes Book I with the converse, which we state without proof. Euclid's argument is very sneaky—look it up!

Theorem 2.11 (I. 48). *If the (areas of the) squares on two sides of a triangle equal the (area of the) square on the third side, then the triangle has a right-angle opposite the third side.*

The *Elements* contains thirteen books. Much of the remaining twelve discuss further geometric constructions, including in three dimensions. There is also a healthy dose of basic number theory including what is now known as the Euclidean algorithm.

While undoubtedly a masterpiece of logical reasoning, Euclid's presentation has several flaws. Most problematic is his reliance on pictorial reasoning: for instance, he 'proves' the SAS and SSS congruence theorems (I. 4 & 8) by laying one triangle on top of another, a process not justified by his axioms (look it up online or Byrne). In a modern sense, Euclid's approach is part axiomatic system and part model: his reasoning requires a visual/physical representation of lines, circles, etc. Because of these issues, we now turn to a more modern description of Euclidean geometry, courtesy of David Hilbert.

Exercises 2.1. 1. (a) Prove the *vertical angle theorem* (I.15): if two lines cut one another, opposite angles are congruent.

(Hint: This is one place where you will need to use postulate 4 regarding right-angles)

(b) Use part (a) to complete the proof of the exterior angle theorem: i.e., explain why $\beta < \delta$.

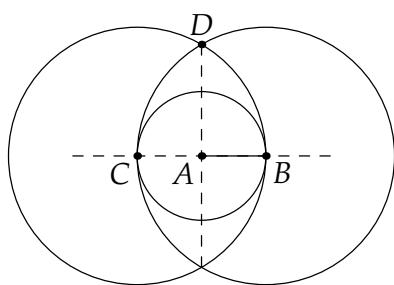
2. To help prove Pythagoras', Euclid makes use of the following results. Prove them as best as you can. Full rigor is tricky, but the pictures should help!

(a) (I.11) At a given point on a line, to construct a perpendicular.

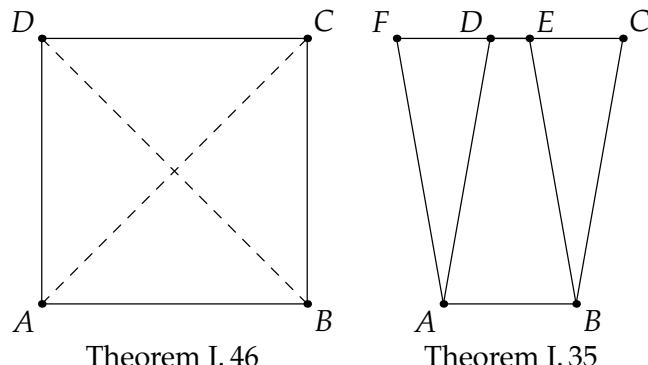
(b) (I.46) To construct a square on a given segment.

(c) (I.35) Parallelograms on the same base and with the same height have equal area.

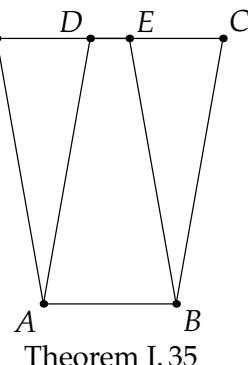
(d) (I.41) A parallelogram has twice the area of a triangle on the same base and with the same height.



Theorem I.11



Theorem I.46



Theorem I.35

3. Consider spherical geometry (page 12), where *lines* are paths of shortest distance (great circles).

(a) Which of Euclid's postulates P1–P5 are satisfied by this geometry?

(b) (Hard) Where does the proof of the exterior angle theorem fail in spherical geometry?

4. (a) State the negation of Playfair's postulate.

(b) Prove that Playfair's postulate is equivalent to the following statement:

Whenever a line is perpendicular to one of two parallel lines, it must be perpendicular to the other.

5. The *line-circle continuity* property states:

If point P lies inside and Q lies outside a circle α , then the segment \overline{PQ} intersects α .

By considering the set of rational points in the plane $\mathbb{Q}^2 = \{(x,y) : x, y \in \mathbb{Q}\}$, and making a sensible definition of line and circle, show that the line-circle continuity property is undecidable within Euclid's system.

6. The standard proof of the converse of Pythagoras' theorem (I.48) is, in fact, a *corollary* of the original! Look it up and explain the argument as best you can.

2.2 Hilbert's Axioms I: Incidence and Order

The long process of identifying and correcting the errors and omissions in Euclid's *Elements* culminated in the 1899 publication of David Hilbert's *Grundlagen der Geometrie* (*Foundations of Geometry*). In the next two sections we consider some of the details of Hilbert's approach, providing a modern and logically superior description of Euclidean geometry.

Hilbert's axioms for plane geometry⁷ are listed on the next page. The undefined terms consist of two types of object (*points* and *lines*), and three relations (*between* $*$, *on* \in and *congruence* \cong). For brevity we'll often use/abuse set notation, viewing a line as a set of points, though this is not necessary. At various places, definitions and notations are required.

Definition 2.12. Throughout, A, B, C denote points and ℓ, m lines.

Line: \overleftrightarrow{AB} denotes the line through distinct A, B . This exists and is unique by axioms I-1 and I-2.

Segment: $\overline{AB} := \{A, B\} \cup \{C : A * C * B\}$ consists of distinct *endpoints* A, B and all *interior* points C lying between them.

Ray: $\overrightarrow{AB} := \overline{AB} \cup \{C : A * B * C\}$ is a ray with *vertex* A . In essence we extend \overline{AB} beyond B .

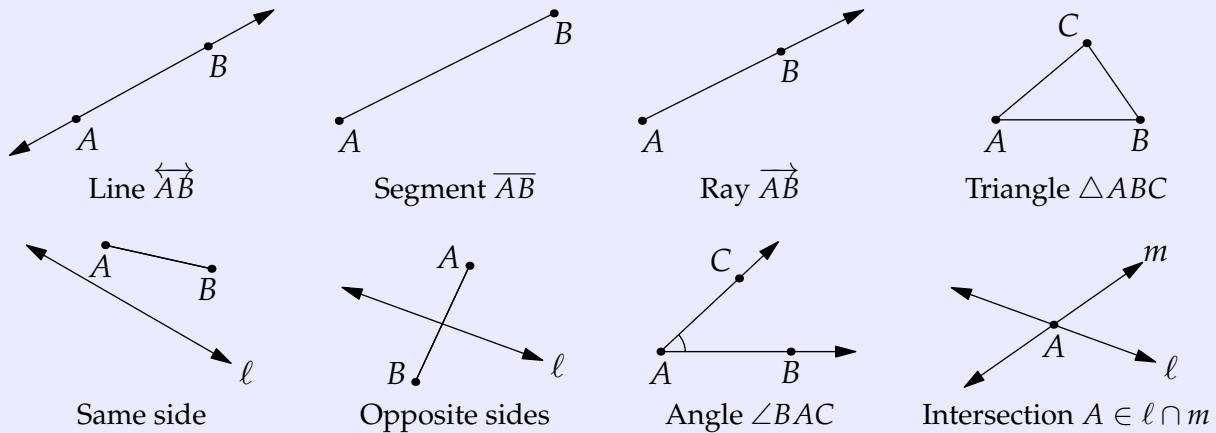
Triangle: $\triangle ABC := \overline{AB} \cup \overline{BC} \cup \overline{CA}$ where A, B, C are non-collinear. Triangles are *congruent* if their sides and angles are congruent in pairs.

Sidedness: Distinct A, B , not on ℓ , lie on the *same side* of ℓ if $\overline{AB} \cap \ell = \emptyset$. Otherwise A and B lie on *opposite sides* of ℓ .

Angle: $\angle BAC := \overrightarrow{AB} \cup \overrightarrow{AC}$ has *vertex* A and *sides* \overrightarrow{AB} and \overrightarrow{AC} .

Parallelism: Lines ℓ and m *intersect* if there exists a point lying on both: $\exists A \in \ell \cap m$. Lines are *parallel* if they do not intersect. Segments/rays are parallel when the corresponding lines are parallel.

The pictures represent these notions in the usual model of Cartesian geometry.



⁷Like Euclid, Hilbert also covered 3D geometry—we only give the axioms for plane geometry. With regard to our desired properties (Definition 1.6), his system is about as good as can be hoped. Essentially one only one model exists, which is almost the same thing as completeness. In the absence of the continuity axiom, the axioms are consistent; in line with Gödel's theorems (1.8), consistency cannot be proved once continuity is included. As stated, the axioms are not quite independent, though this can be remedied: O-3 does not require existence (follows from Pasch's axiom), C-1 does not require uniqueness (follows from uniqueness in C-4) and C-6 can be weakened slightly.

Hilbert's Axioms for Plane Geometry

Undefined terms

1. *Points*: use capital letters, $A, B, C \dots$
2. *Lines*: use lower case letters, ℓ, m, n, \dots
3. *On*: $A \in \ell$ is read ' A lies on ℓ '
4. *Between*: $A * B * C$ is read ' B lies between A and C '
5. *Congruence*: \cong is a binary relation on segments or angles

Axioms of Incidence

- I-1 For any distinct A, B there exists a line ℓ on which lie A, B .
- I-2 There is at most one line through distinct A, B (A and B both *on* the line).

Notation: *line* \overleftrightarrow{AB} through A and B

- I-3 On every line there exist at least two distinct points. There exist at least three points not all on the same line.

Axioms of Order

- O-1 If $A * B * C$, then A, B, C are distinct points on the same line and $C * B * A$.
- O-2 Given distinct A, B , there is at least one point C such that $A * B * C$.
- O-3 If A, B, C are distinct points on the same line, exactly one lies between the others.

Definitions: *segment* \overline{AB} and *triangle* $\triangle ABC$

- O-4 (Pasch's Axiom) Let $\triangle ABC$ be a triangle and ℓ a line not containing any of A, B, C . If ℓ contains a point of the segment \overline{AB} , then it also contains a point of either \overline{AC} or \overline{BC} .

Definitions: *sides* of line \overleftrightarrow{AB} and *ray* \overrightarrow{AB}

Axioms of Congruence

- C-1 (Segment transference) Let A, B be distinct and r a ray based at A' . Then there exists a unique point $B' \in r$ for which $\overline{AB} \cong \overline{A'B'}$. Moreover $\overline{AB} \cong \overline{BA}$.
- C-2 If $\overline{AB} \cong \overline{EF}$ and $\overline{CD} \cong \overline{EF}$, then $\overline{AB} \cong \overline{CD}$.
- C-3 If $A * B * C$, $A' * B' * C'$, $\overline{AB} \cong \overline{A'B'}$ and $\overline{BC} \cong \overline{B'C'}$, then $\overline{AC} \cong \overline{A'C'}$.

Definitions: *angle* $\angle ABC$

- C-4 (Angle transference) Given $\angle BAC$ and $\overleftrightarrow{A'B'}$, there exists a unique ray $\overrightarrow{A'C'}$ on a given side of $\overleftrightarrow{A'B'}$ for which $\angle BAC \cong \angle B'A'C'$.
- C-5 If $\angle ABC \cong \angle GHI$ and $\angle DEF \cong \angle GHI$, then $\angle ABC \cong \angle DEF$. Moreover, $\angle ABC \cong \angle CBA$.
- C-6 (Side-angle-side) Given triangles $\triangle ABC$ and $\triangle A'B'C'$, if $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, and $\angle BAC \cong \angle B'A'C'$, then the triangles are congruent.⁸

Axiom of Continuity

Suppose a line/segment is partitioned into non-empty subsets Σ_1, Σ_2 such that no point of Σ_1 lies between two points of Σ_2 and vice versa. Then there exists a unique point \mathcal{O} satisfying $P_1 * \mathcal{O} * P_2$, if and only if $\mathcal{O} \neq P_1, \mathcal{O} \neq P_2$ and one of P_1 or P_2 lies in Σ_1 and the other in Σ_2 .

Playfair's Axiom

Definition: *parallel lines*

Given a line ℓ and a point $P \notin \ell$, at most one line through P is parallel to ℓ .

⁸Its sides/angles are congruent in pairs. We extend congruence to other geometric objects similarly.

Axioms of Incidence: Finite Geometries

The axioms of incidence describe the relation *on*. An *incidence geometry* is any model satisfying axioms I-1, I-2, I-3. Perhaps surprisingly, there exist incidence geometries with *finitely many points*!

Examples 2.13. By I-3, an incidence geometry requires *at least three points*.

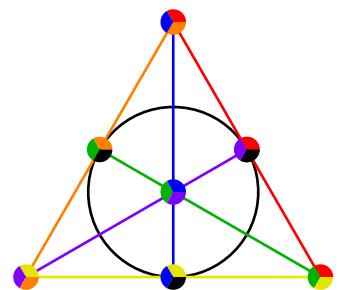
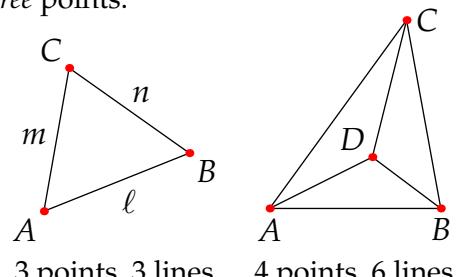
A 3-point geometry exists, and is unique up to relabelling:

I-3 says the points A, B, C must be non-collinear. By I-1 and I-2, each pair lies on a unique line, whence there are precisely three lines

$$\ell = \{A, B\}, \quad m = \{A, C\}, \quad n = \{B, C\}$$

Up to relabelling, there are two incidence geometries with four points: one is drawn; how many lines has the other?

The final picture is a seven-point incidence geometry called the *Fano plane*, which finds many applications particularly in combinatorics. Each point lies on precisely three lines and each line contains precisely three points—each dot is colored to indicate the lines to which it belongs. Don't be fooled by the black line looking 'curved' and seeming to cross the blue line near the top, for the line only contains three points!



We can even prove some simple theorems in incidence geometry. The second is an exercise.

Lemma 2.14. *If distinct lines intersect, then they do so in exactly one point.*

Proof. Suppose A, B are distinct points of intersection. By axiom I-2, there is at most one line through A and B . Contradiction. ■

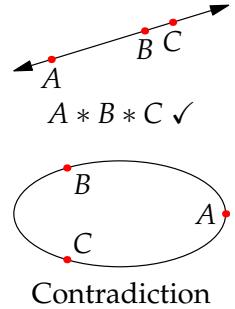
Lemma 2.15. *Given any point, there exist at least two lines on which it lies.*

Axioms of Order: Sides of a Line, Pasch's Axiom & the Crossbar Theorem

The order axioms describe the ternary relation *between*. The first three make explicit the idea that points on a line lie in a fixed order: travelling along a line, one encounters each point exactly once. In particular, these axioms prevent 'circular' lines (the pictured contradiction), and guarantee that a line contains infinitely many points.

Each of the above finite incidence examples fails to satisfy (some of) the order axioms.

The rest of this section is devoted to the consequences of Pasch's axiom (O-4)—named to honor the work of Moritz Pasch (c. 1882). Amongst other things, it permits us to define the *interiors* of several geometric objects and to see that these are non-empty. For instance:

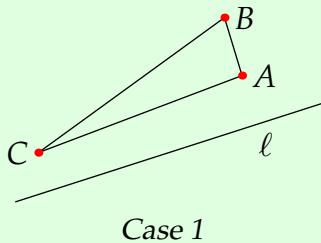


Lemma 2.16 (Exercise 5). *Every segment contains an interior point.*

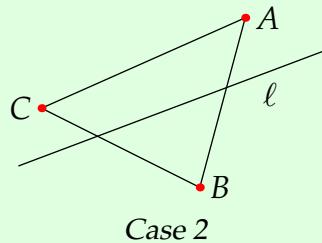
To get much further, it is necessary to establish that a line has precisely two *sides* (Definition 2.12). This concept lies behind several of Euclid's arguments, without being properly defined in the *Elements*.

Theorem 2.17 (Plane Separation). A line ℓ separates all points not on ℓ into two half-planes: the two sides of ℓ . To be explicit, suppose none of the points A, B, C lie on ℓ , then:

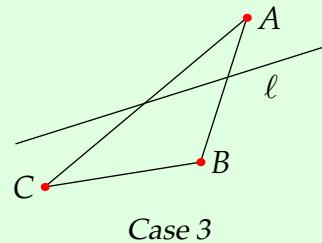
1. If A, B lie on the same side of ℓ and B, C lie on the same side, then A, C lie on the same side.
2. If A, B lie on opposite sides and B, C lie on opposite sides, then A, C lie on the same side.
3. If A, B lie on opposite sides and B, C lie on the same side, then A, C lie on opposite sides.



Case 1



Case 2



Case 3

Proof. We prove the contrapositive of case 1. Suppose A, B, C are non-collinear. If \overline{AC} intersects ℓ , then ℓ intersects one side of $\triangle ABC$. By Pasch's axiom, it also intersects either \overline{AB} or \overline{BC} .

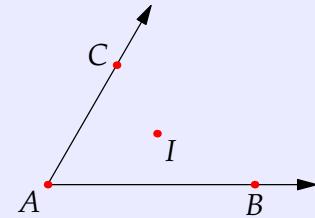
The other cases are exercises, and we omit the tedious collinear possibilities. ■

Plane separation/sidedness allows us to properly define interiors of angles and triangles.

Definition 2.18. A point I is *interior* to angle $\angle BAC$ if:

- I lies on the same side of \overleftrightarrow{AB} as C , and,
- I lies on the same side of \overleftrightarrow{AC} as B .

Otherwise said, I lies in the intersection of two half-planes.



A point I is *interior* to triangle $\triangle ABC$ if it is interior to all three of its angles $\angle ABC$, $\angle BAC$ and $\angle ACB$. Otherwise said, I lies in the triple intersection of three of the half-planes defined by the triangle's sides.

Interior points permit us to compare angles which share a vertex: if I is interior to $\angle BAC$, then $\angle BAI < \angle BAC$ has obvious meaning *without resorting to numerical angle measure*.

Corollary 2.19. Every angle has an interior point.

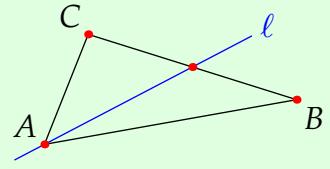
Proof. Given $\angle BAC$, consider any interior point I of the segment \overline{BC} . This plainly lies on the same side of \overleftrightarrow{AB} as C and on the same side of \overleftrightarrow{AC} as B . ■

In Exercise 8, we check that the interior of a triangle is non-empty.

Pasch's axiom could be paraphrased: *If a line enters a triangle, it must come out*. We haven't quite established this crucial fact, however. What if the line passes through a vertex?

Theorem 2.20 (Crossbar Theorem). Suppose I is interior to $\angle BAC$. Then \overleftrightarrow{AI} intersects \overline{BC} .

In particular, if a line passes through a vertex and an interior point of a triangle, then it intersects the side opposite the vertex.

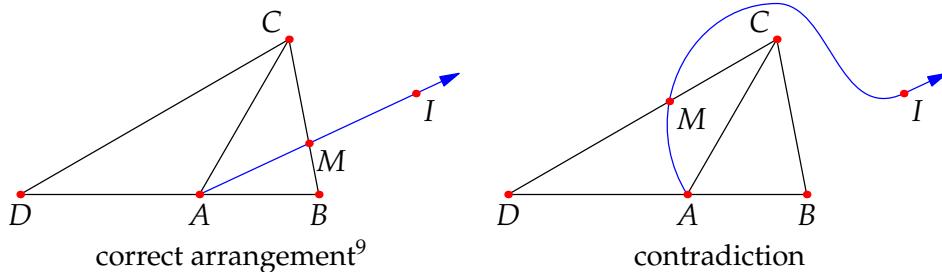


Proof. Extend \overline{AB} to a point D such that A lies between B and D (O-2). Since C is not on $\overleftrightarrow{BD} = \overleftrightarrow{AB}$ we have a triangle $\triangle BCD$. Since \overleftrightarrow{AI} intersects one edge of $\triangle BCD$ at A and does not cross any vertices (think about why...), Pasch says it intersects one of the other edges (\overline{BC} or \overline{CD}) at some point M .

The result follows from applying plane separation to the lines $\overleftrightarrow{AB} = \overleftrightarrow{BD}$ and \overleftrightarrow{AC} . First observe:

Since I, M lie on the same side of $\overleftrightarrow{AB} = \overleftrightarrow{BD}$ as C , it follows that \overline{IM} does not intersect \overleftrightarrow{AB} .
Since A, I, M are collinear and $A \in \overleftrightarrow{AB}$, it follows that $A \notin \overline{IM}$.

If $M \in \overline{BC}$, we are done. Our goal is to show that $M \in \overline{CD}$ is a contradiction.



Suppose, for contradiction, that $M \in \overline{CD}$. Relative to \overleftrightarrow{AC} :

- I and B lie on the same side since I is interior to $\angle BAC$;
- B and D lie on opposite sides, since $B * A * D$ and $\overleftrightarrow{AC} \neq \overleftrightarrow{BD} = \overleftrightarrow{AB}$;
- D and M lie on the same side since $M \in \overline{CD}$ and $\overleftrightarrow{CD} \neq \overleftrightarrow{AC}$.

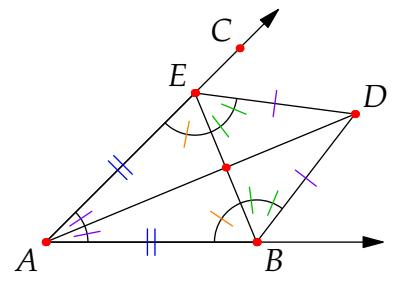
By plane separation, I, M lie on opposite sides of \overleftrightarrow{AC} . The collinearity of A, I, M then forces the contradiction $A \in \overline{IM}$. ■

Euclid repeatedly uses the crossbar theorem without justification, including in his construction of perpendiculars and angle/segment bisectors (Theorems I. 9+10). We sketch the latter here.

Given $\angle BAC$, construct E such that $\overline{AB} \cong \overline{AE}$. Construct D using an equilateral triangle (I. 1). SSS (I. 8) shows that $\angle BAC$ is bisected, and SAS (I. 4) that \overleftrightarrow{AD} bisects \overline{BE} .

Quite apart from Euclid's arguments for SAS and SSS being suspect (we'll deal with these in the next section), he gives no argument for why D is interior to $\angle BAC$ or why \overleftrightarrow{AD} should intersect \overline{BE} !

Even with Pasch's axiom and the crossbar theorem, it requires some effort to repair Euclid's proof. No matter, we'll provide an alternative construction of the bisector once we've considered congruence.



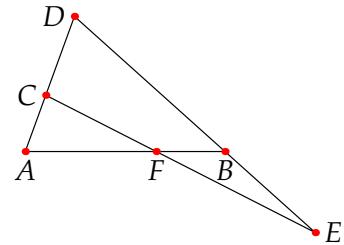
⁹The pictures could be modified: e.g., $I = M$ and $A * I * M$ are also correct arrangements ($M \in \overline{BC}$).

Exercises 2.2. 1. Label the vertices in the Fano plane 1 through 7 (any way you like). As we did in Example 2.13 for the 3-point geometry, describe each line in terms of its points.

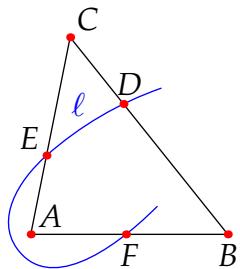
2. Prove Lemma 2.15.
3. (a) Give a model for each of the 5-point incidence geometries. How many are there?
(Hint: remember that order doesn't matter, so the only issue is how many points lie on each line)
- (b) Is it possible for there to be a 6-point incidence geometry so that each line contains precisely three points? Why/why not?
4. Consider the proof of the crossbar theorem. How can we be certain that \overleftrightarrow{AI} does not contain any of the vertices of $\triangle BCD$.
5. You are given distinct points A, B . Using the axioms of incidence and order and Lemma 2.14 (follows from I-2), show the existence of each of the points C, D, E, F in the picture *in alphabetical order*. Hence conclude the existence of a point F lying *between* A and B (Lemma 2.16).

During your construction, address the following issues:

- (a) Explain why D does not lie on \overleftrightarrow{AB} .
- (b) Explain why E does not lie on $\triangle ABD$.
- (c) Explain why $E \neq C$ (whence \overleftrightarrow{CE} exists).
- (d) Explain why F lies on \overline{AB} and *not* on \overline{BD} .
6. We complete the proof of the plane separation theorem (2.17).
 - (a) Prove part 3 (it is almost a verbatim application of Pasch's axiom).
 - (b) Suppose a line ℓ intersects all three sides of $\triangle ABC$ but no vertices. This results in a very strange picture (we've labelled the intersections D, E, F and WLOG chosen $D * E * F$). Apply Pasch's axiom to $\triangle DBF$ and \overleftrightarrow{AC} to obtain a contradiction. Hence establish part 2 of the plane separation theorem.



7. Suppose A, B, C are distinct points on a line ℓ .
 - (a) Explain why there exists a line $m \neq \ell$ such that $B \in m$.
 - (b) Prove that $A * B * C \iff A$ and C lie on opposite sides of m .
 - (c) Suppose $A * B * C$. Use part (b) to prove the following:
 - i. B is the only point common to the rays \overrightarrow{BA} and \overrightarrow{BC} .
 - ii. If $D \in \ell$ is any point other than B , prove that D lies in precisely one of \overrightarrow{BA} or \overrightarrow{BC} .
8. Prove that the interior of a triangle is non-empty.
(Hint: use Exercise 5 to construct a suitable I , then prove that it lies on the correct side of each edge)
9. The existence of infinitely many points on a line follows easily from the fact that every segment has an interior point. Find an alternative proof that does not depend on Pasch's axiom.



2.3 Hilbert's Axioms II: Congruence

Hilbert's congruence axioms address two primary issues in Euclid.

1. Euclid's use of *equal* is confusing. In Hilbert, segments/angles are now equal only when they are precisely the same (this amounts to the *reflexivity* part of the next result).
2. Euclid's frequent and unjustified use of pictorial reasoning. We previously discussed Euclid's erroneous approach to the SAS and SSS triangle congruence theorems. It was eventually realized that one of the triangle congruences has to be an axiom: SAS is Hilbert's C-6.

We start with a small piece of bookkeeping.

Lemma 2.21. *Congruence of segments/angles is an equivalence relation.*

Proof. (Reflexivity) Let \overline{AB} be given. Apply C-1 to obtain $\overline{A'B'}$ such that $\overline{AB} \cong \overline{A'B'}$. We sneakily use this *twice* and apply C-2 to obtain

$$\overline{AB} \cong \overline{A'B'} \text{ and } \overline{AB} \cong \overline{A'B'} \implies \overline{AB} \cong \overline{AB}$$

(Symmetry) Assume $\overline{AB} \cong \overline{CD}$. By reflexivity, $\overline{CD} \cong \overline{CD}$. By C-2 we have $\overline{CD} \cong \overline{AB}$.

(Transitivity) Suppose $\overline{AB} \cong \overline{CD}$ and $\overline{CD} \cong \overline{EF}$. By symmetry, $\overline{EF} \cong \overline{CD}$. Axiom C-2 now shows that $\overline{AB} \cong \overline{EF}$.

Axioms C-4 and C-5 say essentially the same thing for angles (see Exercise 2). ■

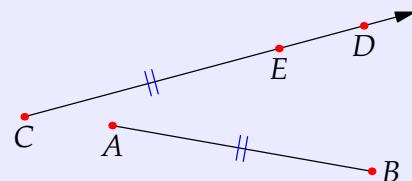
Segment/Angle Transfer and Comparison

Hilbert's axioms of segment and angle transference are crucial for comparing non-collinear segments and angles with distinct vertices.

Definition 2.22. Let segments \overline{AB} and \overline{CD} be given.

By axiom C-1, let E be the unique point on \overrightarrow{CD} such that $\overline{CE} \cong \overline{AB}$: we have *transferred* \overline{AB} onto \overline{CD} .

We write $\overline{AB} < \overline{CD}$ if E lies between C and D , etc.



By O-3, any two segments are comparable: given \overline{AB} & \overline{CD} , precisely one of the following holds,

$$\overline{AB} < \overline{CD}, \quad \overline{CD} < \overline{AB}, \quad \overline{AB} \cong \overline{CD}$$

C-3 says that congruence respects the 'addition' of adjacent congruent segments. Unique angle transfer, comparison and addition follow similarly from axiom C-4 and Definition 2.18 (interior points).

Neither Hilbert nor Euclid use or require an *absolute* notion of length/angle-measure: the comparison $\overline{AB} < \overline{CD}$ does *not* indicate a relationship between numerical quantities (lengths). Introducing numerical length requires the inclusion of the real numbers (and thus far more axioms)—for purity reasons, we postpone this until Section 2.5.

The Triangle Congruence Theorems: SAS, ASA, SSS & SAA

Hilbert assumes side-angle-side (SAS) and proceeds to prove the remainder. Here is the first of these; we'll cover SSS momentarily and SAA in Exercise 6.

Theorem 2.23 (Angle-Side-Angle/ASA, Euclid I. 26, case I). Suppose $\triangle ABC$ and $\triangle DEF$ satisfy

$$\angle ABC \cong \angle DEF, \quad \overline{AB} \cong \overline{DE}, \quad \angle BAC \cong \angle EDF$$

Then the triangles are congruent ($\angle ACB \cong \angle DFE$, $\overline{AC} \cong \overline{DF}$ and $\overline{BC} \cong \overline{EF}$).

Hilbert's approach modifies Euclid's: instead of laying $\triangle ABC$ on top of $\triangle DEF$, he creates a new triangle $\triangle DEG \cong \triangle ABC$ and proves that $G = F$.

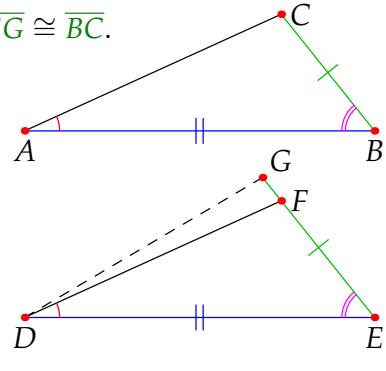
Proof. Segment transfer provides the unique point $G \in \overrightarrow{EF}$ such that $\overline{EG} \cong \overline{BC}$.

SAS applied to $\overline{AB} \cong \overline{DE}$, $\angle ABC \cong \angle DEG$ ($= \angle DEF$), $\overline{BC} \cong \overline{EG}$, says $\angle BAC \cong \angle EDG$ ($\cong \angle EDF$) (this last is by assumption).

Since F and G lie on the same side of \overleftrightarrow{DE} , angle transfer (C-4) says they lie on the *same ray* through D .

But then F and G both lie on two distinct lines ($\overleftrightarrow{EF} = \overleftrightarrow{EG}$ and $\overleftrightarrow{DF} = \overleftrightarrow{DG}$). We conclude that $F = G$.

By SAS we conclude that $\triangle ABC \cong \triangle DEF$. ■



Geometry Without Circles

Circles are at the heart of Euclid's constructions. Yet, for reason we'll address in Section 2.4, Hilbert essentially ignores them. We sketch a few of his alternative approaches to Euclid's basic results.

Theorem 2.24 (Euclid I. 5). An isosceles triangle has congruent base angles.

Isosceles means *equal legs*: two sides of the triangle are congruent. The remaining side is the *base*. Euclid's argument relies on a famously complicated construction (look it up!). Hilbert does things more speedily and sneakily, by relabelling the original triangle and applying SAS.

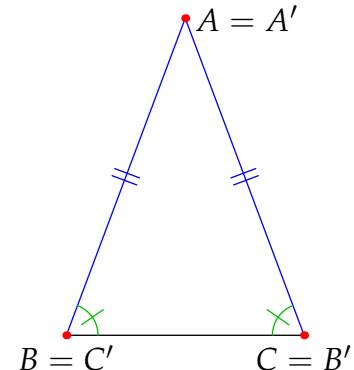
Proof. Suppose $\triangle ABC$ is isosceles where $\overline{AB} \cong \overline{AC}$. Consider a 'new' triangle $\triangle A'B'C' = \triangle ACB$ where the base points are switched:

$$A' := A, \quad B' := C, \quad C' := B$$

Observe:

- $\angle BAC \cong \angle CAB$ (axiom C-5) $\implies \angle BAC \cong \angle B'A'C'$.
- $\overline{AB} \cong \overline{AC} \implies \overline{AB} \cong \overline{A'B'} \text{ and } \overline{AC} \cong \overline{A'C'}$.

SAS says that $\angle ABC \cong \angle A'B'C' \cong \angle ACB$. ■



Dropping a Perpendicular As with the majority of Book I, Euclid accomplishes this using circle intersections.¹⁰ Hilbert instead uses segment/angle transference and the concept of sidedness.

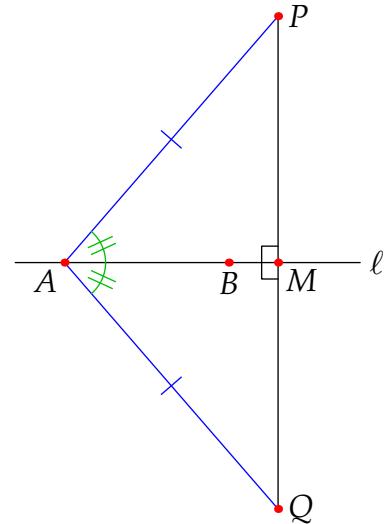
Suppose we are given a line ℓ and a point P not on ℓ . Our goal is to construct a point $M \in \ell$ such that \overline{PM} intersects ℓ in a right-angle.

Let A, B be distinct points on ℓ (axiom I-3) so that $\ell = \overleftrightarrow{AB}$.

By axioms C-4 and C-1, we may transfer \overline{AP} to the other side of ℓ at A , creating a new point Q .

Since P and Q lie on opposite sides of ℓ , the line intersects \overline{PQ} at some point M . There are two cases to consider.

- In the generic case $M \neq A$ (pictured), SAS applied to $\triangle MAP$ and $\triangle MAQ$ shows that $\angle AMP \cong \angle AMQ$. Since these angles sum to a straight edge (\overline{PQ}), they are both right-angles.
- In the extreme case $M = A$, there are no triangles and SAS cannot be applied. Instead, observe that B does not lie on \overline{PQ} (which axioms/results make this clear?!?) and apply the above argument with B instead of A .



A generalization of this construction facilitates a corrected argument for the SSS triangle congruence.

Theorem 2.25 (Side-Side-Side/SSS, Euclid I.8). Suppose $\triangle ABC$ and $\triangle DEF$ have sides congruent in pairs:

$$\overline{AB} \cong \overline{DE}, \quad \overline{BC} \cong \overline{EF}, \quad \overline{AC} \cong \overline{DF}$$

Then the triangles are congruent ($\angle ABC \cong \angle DEF, \angle BCA \cong \angle EFD, \angle CAB \cong \angle FDE$).

The strategy is similar to the proof of ASA. Hilbert creates a new triangle $\triangle DEG \cong \triangle ABC$, though this time with G on the opposite side of \overleftrightarrow{DE} to F .

Proof. Transfer $\angle BAC$ to D on the other side of \overleftrightarrow{DE} from F to obtain G (axioms C-4 and C-1).

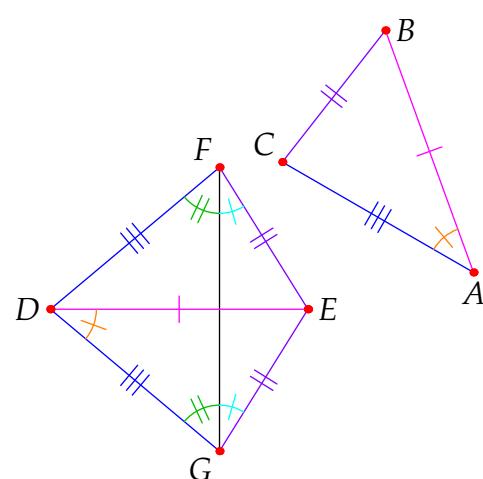
SAS ($\overline{AB} \cong \overline{DE}, \angle BAC \cong \angle EDG, \overline{AC} \cong \overline{DG}$) shows that $\overline{EG} \cong \overline{BC} \cong \overline{EF}$. Otherwise said, $\triangle DEG \cong \triangle ABC$.

Join \overline{FG} to produce isosceles triangles $\triangle FDG$ and $\triangle FEG$ with base \overline{FG} , both with congruent angles at F and G .

Sum angles at F and G and apply SAS ($\overline{DF} \cong \overline{DG}, \angle DFE \cong \angle DGE, \overline{EF} \cong \overline{EG}$) to see that $\triangle DEF \cong \triangle DEG$.

We conclude that $\triangle ABC \cong \triangle DEG \cong \triangle DEF$, as required.

To be completely formal, we should also carefully deal with the situations where the sum is a subtraction or the triangle is right-angled at A or B .

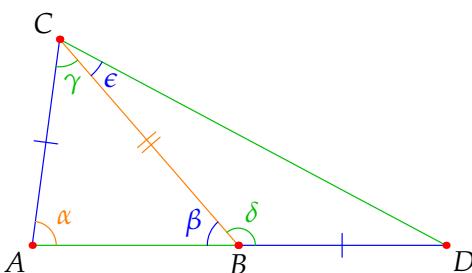


¹⁰Consider the picture for Thm. I. 11 in Exercise 2.2.2.

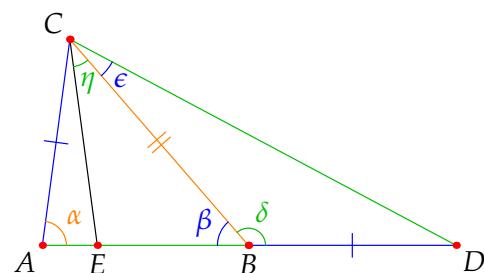
Exterior Angle Theorem (Thm. 2.3, I. 16) Euclid's approach uses a bisector which he obtains from circles. Hilbert does things a little differently.

Proof. Given $\triangle ABC$, extend \overline{AB} to D such that $\overline{AC} \cong \overline{BD}$. For clarity, we label angles with Greek letters as in the first picture below. We show that $\gamma < \delta$ by proving that the alternatives are impossible.

1. ($\delta \not\cong \gamma$) Assume $\delta \cong \gamma$. By SAS, $\triangle ACB \cong \triangle DBC$; in particular $\epsilon \cong \beta$. Since A and D lie on opposite sides of \overleftrightarrow{BC} , we see that $\epsilon + \gamma \cong \beta + \delta$ is a straight edge. But then A, D are distinct points lying on *two* lines! Contradiction.
2. ($\delta \not< \gamma$) Assume $\delta < \gamma$. Transfer δ to C as shown to obtain $\eta \cong \delta$. By the crossbar theorem, we obtain an intersection point E . But now δ is an exterior angle of $\triangle EBC$ congruent to an interior angle η of the same triangle, contradicting part 1.



Step 1: $\delta \cong \gamma$ is a contradiction



Step 2: $\delta < \gamma$ is a contradiction

Take the vertical angle to δ at B and repeat the argument to see that $\alpha < \delta$. ■

The proof also shows that the sum of any two angles in a triangle is strictly less than a straight edge: $\alpha + \beta < \delta + \beta = 180^\circ$.

Is Euclid now fixed? Almost! In the exercises we show how the following may be achieved:

- Construction of an isosceles triangle on a segment \overline{AB} . With this one can construct segment and angle bisectors (Euclid I. 9+10).
- SAA congruence (Euclid I. 26, case II), the last remaining triangle congruence theorem.

We've now recovered almost all of Book I prior to the application of the parallel postulate. Including Playfair's axiom completes the remainder, including Pythagoras', all *without circles!*

Exercises 2.3. Except for question 8, answer everything without reference to the continuity axiom, circles, or the uniqueness of parallels (e.g., Playfair's axiom, (tri)angle sum = 180°).

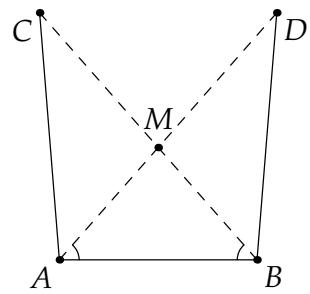
1. Draw pictures to suggest why you don't expect Angle-Angle-Angle (AAA) and Side-Side-Angle (SSA) to be triangle congruence theorems.
2. Use Hilbert's axioms C-4 and C-5 to prove that congruence of angles is an equivalence relation.
3. (a) Use ASA to prove that if the base angles are congruent then a triangle is isosceles.
 (b) Find an alternative argument that relies the exterior angle theorem.
(Hint: this is essentially the same as the proof of Exercise 5 (a))
 (c) Explain why the base angles of an isosceles triangle are *acute* (less than a right-angle).

4. Given \overline{AB} , axiom I-3 says $\exists C \notin \overleftrightarrow{AB}$.

If $\triangle ABC$ is not isosceles, then WLOG assume $\angle ABC < \angle BAC$.

Transfer $\angle ABC$ to A to produce D on the same side of \overleftrightarrow{AB} as C with

$$\angle ABC \cong \angle BAD, \quad \overline{BC} \cong \overline{AD}$$



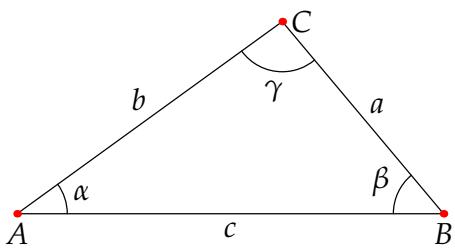
(a) Explain why rays \overrightarrow{AD} and \overrightarrow{BC} intersect (at some point M).

(b) Why is $\triangle MAB$ isosceles?

(c) Describe how to produce the perpendicular bisector of \overline{AB} .

(d) Explain how to construct an angle bisector using the above discussion.

5. We prove Theorems I. 18, 19 and 20 on comparisons of angles and sides in a triangle. For clarity, suppose $\triangle ABC$ has sides and angles labelled as in the picture.



(a) (I. 18) Assume $a < c$. Prove that $\alpha < \gamma$.

(Hint: let D on \overline{AB} satisfy $\overline{BD} \cong a$)

(b) (I. 19: converse to I. 18) $\alpha < \gamma \implies a < c$.

(Hint: Prove the contrapositive)

(c) (I. 20: triangle inequality) $a + b > c$.

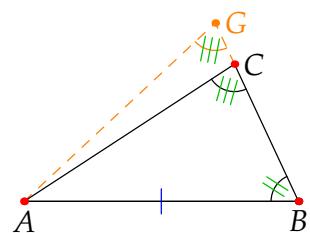
(Hint: Let E lie on \overrightarrow{BC} such that $\overline{CE} \cong b$ and apply I. 19)

6. Prove the SAA congruence. If $\triangle ABC$ and $\triangle DEF$ satisfy

$$\overline{AB} \cong \overline{DE}, \quad \angle ABC \cong \angle DEF \quad \text{and} \quad \angle BCA \cong \angle EFD$$

then the triangles are congruent: $\triangle ABC \cong \triangle DEF$.

(Hint: Let $G \in \overrightarrow{BC}$ be such that $\overline{BG} \cong \overline{EF}$, apply SAS and the exterior angle theorem)



7. Hilbert's published SAS axiom is weaker than we've stated.

Given triangles $\triangle ABC$ and $\triangle A'B'C'$, if $\overline{AB} \cong \overline{A'B'}$, $\overline{AC} \cong \overline{A'C'}$, and $\angle BAC \cong \angle B'A'C'$, then $\angle ABC \cong \angle A'B'C'$.

Use this to prove the full SAS congruence theorem (axiom C-6 as we've stated it).

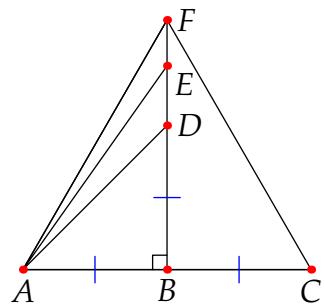
(Hint: try a trick similar to the proof of ASA)

8. Construct the picture on the right, where \overline{BF} is the perpendicular bisector of \overline{AC} , and

$$\overline{BD} \cong \overline{AB}, \quad \overline{BE} \cong \overline{AD}, \quad \overline{BF} \cong \overline{AE}$$

Use Pythagoras' Theorem to prove that $\triangle ACF$ is equilateral.

This construction requires Playfair's axiom and thus unique parallels. It does not require circle intersections (continuity) like Theorem 2.1 (I. 1).



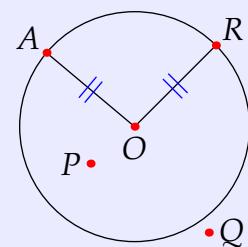
2.4 Circles and Continuity

Definition 2.26. Let O and R be distinct points. The *circle* \mathcal{C} with center O and radius \overline{OR} is the collection of points A such that $\overline{OA} \cong \overline{OR}$.

A point P lies *inside* the circle \mathcal{C} if $P = O$ or $\overline{OP} < \overline{OR}$.

A point Q lies *outside* if $\overline{OR} < \overline{OQ}$.

Since all segments are comparable, any point lies *inside*, *outside* or *on* a given circle.



A major weakness of Euclid is that many of his proofs rely on circle intersections, rather than lines. To use circles in this manner requires the *Axiom of Continuity*. This much more technical than the other axioms. It is likely for this reason that Hilbert barely mentions circles, instead wanting to build as much geometry as possible using only the simplest axioms.

Here are the facts necessary before Euclid's approach can be followed.

Theorem 2.27. Suppose \mathcal{C} and \mathcal{D} are circles.

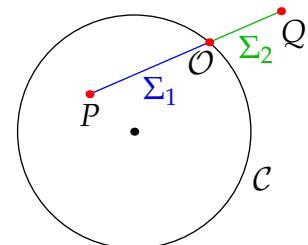
1. (Elementary/line-circle Continuity Principle) If P lies inside and Q outside \mathcal{C} , then \overline{PQ} intersects \mathcal{C} in exactly one point.
2. (Circular Continuity Principle) If \mathcal{D} contains a point inside and another outside \mathcal{C} , then they intersect in exactly two points. These lie on opposite sides of the line joining the circle centers.

The idea of the first principle is to partition \overline{PQ} into two pieces:

Σ_1 consists of the points lying on or inside \mathcal{C}

Σ_2 consists of the points lying outside \mathcal{C}

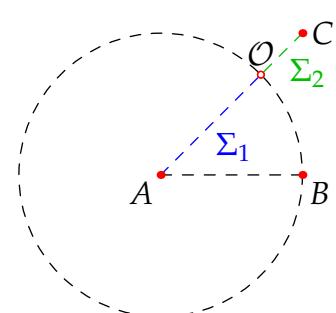
One shows that Σ_1 and Σ_2 satisfy the assumptions of the axiom. The unique point O then exists and is shown to lie on \mathcal{C} itself. Some of this are in Exercise 6. The circular continuity principle is harder.



What is perhaps more interesting is to consider a geometry in which the axiom of continuity is false.

Example 2.28. The geometry $\mathbb{Q}^2 = \{(x, y) \in \mathbb{R}^2 : x, y \in \mathbb{Q}\}$ of points in the plane with *rational* co-ordinates satisfies *almost all* of Hilbert's axioms, however C-1 and continuity are false.

Axiom C-1 Given points $A = (0, 0)$, $B = (1, 0)$ and $C = (1, 1)$, we see that $O = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ is the unique point (in \mathbb{R}^2) on the ray $r = \overrightarrow{AC}$ such that $\overline{AC} \cong \overline{AB}$. Clearly O is an irrational point and therefore not in the geometry.



Continuity The circle centered at $A = (0, 0)$ with radius 1 does not intersect the segment \overline{AC} . More properly, $\overline{AC} = \Sigma_1 \cup \Sigma_2$ may be partitioned as shown and yet no point O in the geometry separates Σ_1, Σ_2 .

Equilateral triangles We can finally correct Euclid's proof of the first proposition of the *Elements*!

Theorem 2.29 (Euclid I.1). An equilateral triangle may be constructed on a given segment \overline{AB} .

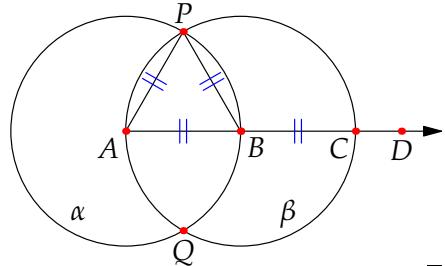
Proof. Following Euclid, take the circles α and β centered at A and B , with radii \overline{AB} .

Axiom O-2: $\exists D$ such that $A * B * D$.

Axiom C-1: let $C \in \overrightarrow{BD}$ be such that $\overline{BC} \cong \overline{AB}$.

Circular continuity principle: β contains A (inside α) and C (outside α) so the circles intersect in two points P, Q .

Since P lies on both circles (and is therefore distinct from A and B), we have $\overline{AB} \cong \overline{AP} \cong \overline{BP}$ whence $\triangle ABC$ is equilateral.



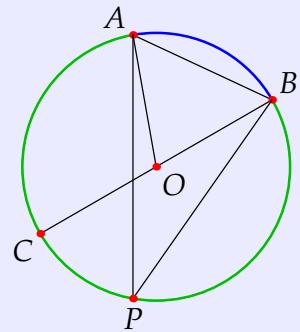
If one allows Playfair's axiom on unique parallels, Euclid's result can be proved without using circles or the continuity axiom (see Exercise 2.3.8). Nevertheless, we are finally able to say that every result in Book I of Euclid is correct, even if his original axioms and arguments are insufficient!

Basic Circle Geometry

We continue our survey of Euclidean geometry with a few results about circles, many of which are found in Book III of the *Elements*. From this point on, we assume all Hilbert's axioms including Playfair and continuity. Indeed what follows often relies on their consequences, particularly angle-sums in triangles and the circular continuity principle.

Definition 2.30. With reference to the picture:

- A *chord* \overline{AB} is a segment joining two points on a circle.
- A *diameter* \overline{BC} is a chord passing through the center O .
- An *arc* \widehat{AB} is part of the circular edge between chord points (major or minor by length).
- $\angle AOB$ is a *central angle* and $\angle APB$ an *inscribed angle*.
- $\triangle ABP$ is *inscribed* in its *circumcircle*.



Since these ideas shouldn't be new, most of the details are left as exercises.

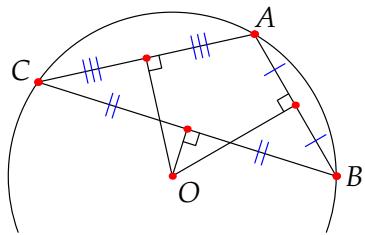
Theorem 2.31 (III. 20). The central angle is twice the inscribed angle: $\angle AOB = 2\angle APB$.

For a sketch proof, join \overline{OP} , breaking $\triangle ABP$ into three isosceles triangles and count angle sums.

- Corollary 2.32.**
1. (III. 21) If inscribed triangles share a side, the opposite angles are congruent.
 2. (III. 22) An inscribed quadrilateral has opposite angles supplementary (summing to 180°).
 3. (Thales' Theorem III. 31) A triangle in a semi-circle is right-angled.

Theorem 2.33. Any triangle has a unique circumcircle.

This is similar to III. 1: construct the perpendicular bisectors of two sides as in the picture.



Definition 2.34. A line is *tangent* to a circle if it intersects the circle exactly once.

Theorem 2.35 (III. 18, 19 (part)). A line is tangent to a circle if and only if it is perpendicular to the radius at an intersection point.

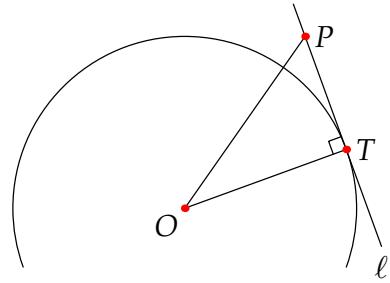
Proof. (\Leftarrow) Suppose ℓ through T is perpendicular to the radius \overline{OT} .

Let P be any another point on ℓ . But then (Exercise 2.3.5),

$$\angle OPT < 90^\circ = \angle OTP \implies \overline{OT} < \overline{OP}$$

thus P lies *outside* the circle. Every point on ℓ except T lies outside the circle, so T is the unique intersection, and ℓ is therefore tangent.

The converse is an exercise. ■



Theorem 2.36. Through a point outside a circle, exactly two lines are tangent to the circle.

Exercises 2.4. 1. Give formal proofs of all parts of Corollary 2.32.

2. Prove Theorem 2.33.

3. Complete the proof of Theorem 2.35 by showing the (\Rightarrow) direction.

(Hint: suppose T is an intersection and that the angle isn't 90° , and drop a perpendicular to ℓ ...)

4. Given a circle centered at O and a point P outside the circle, draw the circle centered at the midpoint of \overline{OP} passing through O and P . Explain why the intersections of these circles are the points of tangency required in Theorem 2.36, and hence complete its proof.

5. (a) Prove Theorem 2.31 when O is an *interior* point to $\triangle ABP$.

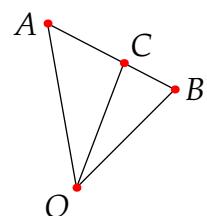
(b) Prove Theorem 2.31 when O is an *exterior* point to $\triangle ABP$.

6. Suppose $A * C * B$ and that $O \notin \overleftrightarrow{AB}$. Use Exercise 2.3.5 to show that

$$\overline{OC} < \max(\overline{OA}, \overline{OB})$$

If A, B are interior to a circle centered at O , conclude that C is also.

(This is part of what's needed to demonstrate the elementary continuity principle: no point of Σ_2 lies between two points of Σ_1 . Can you prove the other condition?)



7. If a line contains a point inside a circle, show that it intersects the circle in two points.

(Hint: first construct two points on the line lying outside the circle)

2.5 Similar Triangles, Length and Trigonometry

In the geometry of Euclid & Hilbert, there are no numerical measures of length or angle. *Relative* measure is built in (Definition 2.22), and we've denoted right-angles and straight edges by 90° & 180° purely for convenience. To avoid continued frustration it is time we introduced explicit *numerical* measure, though to do so properly requires more axioms!

Axioms 2.37 (Length and Angle Measure).

L1 To each segment \overline{AB} corresponds a unique *length* $|AB|$, a positive real number

L2 $|AB| = |CD| \iff \overline{AB} \cong \overline{CD}$

L3 $|AB| < |CD| \iff \overline{AB} < \overline{CD}$ (Definition 2.22)

L4 If $A * B * C$, then $|AB| + |BC| = |AC|$

A1 To each $\angle ABC$ corresponds a unique *degree measure* $m\angle ABC$, a real number between 0 and 180

A2 $m\angle ABC = m\angle DEF \iff \angle ABC \cong \angle DEF$

A3 $m\angle ABC < m\angle DEF \iff \angle ABC < \angle DEF$ (Definition 2.18)

A4 If P is interior to $\angle ABC$, then $m\angle ABP + m\angle PBC = m\angle ABC$

A5 Right-angles measure 90°

Don't memorize these axioms, just observe how they fit your intuition. Angle measure in Euclidean geometry has two notable differences from what you might expect:

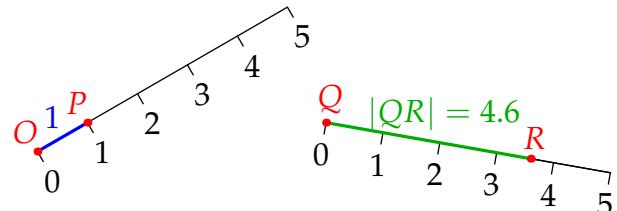
- (A1) All angles measure strictly between 0° and 180° . A straight edge isn't an angle, though such is commonly denoted 180° , and there are no *reflex angles* ($> 180^\circ$).
- (A2) Angles are *non-oriented*, measuring the same in reverse ($m\angle ABC = m\angle CBA$).

The axioms for length and angle follow the same pattern except for us explicitly fixing the scale of angle measure (A5). To do the same for length requires only a *choice* of a reference segment of length 1. The following is a consequence of the continuity axiom.

Theorem 2.38 (Uniqueness of measure).

1. Given \overline{OP} , there is a unique way to assign a length to every segment such that $|OP| = 1$.
2. There is a unique way to assign a degree measure to every angle.

The segment \overline{OP} in part 1 provides a length-scale for a *ruler*. We measure the length of any segment by moving this rule on top of the desired segment (congruence!).



Area Measure If we also include Playfair's axiom, then the discussion at the end of Book I of Euclid becomes valid, and *rectangles* can be defined (see e.g., Exercise 2.1.2).

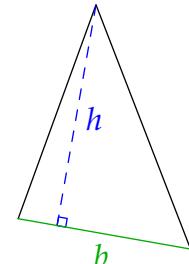
Definition 2.39. The *area (measure)* of a rectangle is the product of its base and height (measures).

Given a length measure, a square with side length 1 necessarily has area 1. Relative to a base segment, the *height* of a triangle is the length of the perpendicular dropped from the vertex.

Since every rectangle is a parallelogram and a triangle half a parallelogram, Euclid's discussion (Thm. I. 35) amounts to the familiar area formulæ:

$$\text{area(parallelogram)} = bh, \quad \text{area(triangle)} = \frac{1}{2}bh$$

While these expressions are nice to have, they are not necessary. Indeed everything that follows depends only on *area ratios*: e.g., $\frac{\frac{1}{2}bh_1}{\frac{1}{2}bh_2} = \frac{h_1}{h_2}$.

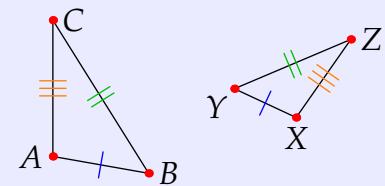


Lemma 2.40. If triangles have congruent bases, then their areas are in the same ratio as their heights. The same holds with the roles of heights and bases reversed.

Similarity and the AAA Theorem Similar triangles are the concern of Book VI of the *Elements*.

Definition 2.41. Triangles are *similar*, written $\triangle ABC \sim \triangle XYZ$, if their sides are in the same length ratio

$$\frac{|AB|}{|XY|} = \frac{|BC|}{|YZ|} = \frac{|CA|}{|ZX|}$$

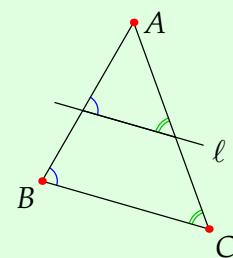


Euclid discusses these using non-numerical ratios of segments (e.g., $\overline{AB} : \overline{XY} = \overline{BC} : \overline{YZ}$). This is unnecessarily confusing for modern readers, indeed some of the most difficult parts of the *Elements* are where he describes what this should mean, particularly for irrational ratios (Books V & X).

Our primary result comes in two versions, where the second (which we'll prove) is a special case of the first.

Theorem 2.42 (Angle-Angle-Angle/AAA, Euclid VI. 2–5).

1. Triangles are similar if and only if their angles come in mutually congruent pairs.
2. Suppose a line intersects two sides of a triangle. The smaller triangle so created is similar to the original if and only if the line is parallel to the third side of the triangle.



The picture should convince you that $1 \Rightarrow 2$ follows from the uniqueness of parallels (Playfair's axiom, Corollary 2.5 & Theorem 2.6). This reliance is crucial! We should not expect AAA similarity in non-Euclidean geometry, and indeed shall see later that it is *false* in hyperbolic geometry (Chapter 4), where AAA is a theorem for *congruent* triangles! The converse ($2 \Rightarrow 1$) is left to Exercise 10.

Proof (AAA similarity, part 2). Suppose ℓ intersects $\triangle ABC$ at points D, E as shown. Drop perpendiculars to create distances h, k, d_1, d_2 as indicated. We prove:

$$\ell \text{ is parallel to } \overline{BC} \iff \triangle ABC \sim \triangle ADE$$

- (\Rightarrow) Suppose ℓ is parallel to \overline{BC} . Playfair¹¹ tells us that $d_1 = d_2$. By Lemma 2.40, triangles with the same height have areas proportional to their bases:

$$\frac{|BD|}{|AD|} = \frac{\text{area}(\triangle BDE)}{\text{area}(\triangle ADE)} \quad (\triangle BDE, \triangle ADE \text{ have same height } h)$$

$$\frac{|CE|}{|AE|} = \frac{\text{area}(\triangle CDE)}{\text{area}(\triangle ADE)} \quad (\triangle CDE, \triangle ADE \text{ have same height } k)$$

Since $\triangle BDE$ and $\triangle CDE$ share base \overline{DE} , we see that

$$\begin{aligned} \ell \text{ is parallel to } \overline{BC} &\iff d_1 = d_2 \iff \text{area}(\triangle BDE) = \text{area}(\triangle CDE) \\ &\iff \frac{|BD|}{|AD|} = \frac{|CE|}{|AE|} \end{aligned} \tag{*}$$

Add $1 = \frac{|AD|}{|AD|} = \frac{|AE|}{|AE|}$ to both sides to obtain one part of the similarity ratio

$$\frac{|AB|}{|AD|} = \frac{|AD| + |BD|}{|AD|} = \frac{|AE| + |CE|}{|AE|} = \frac{|AC|}{|AE|}$$

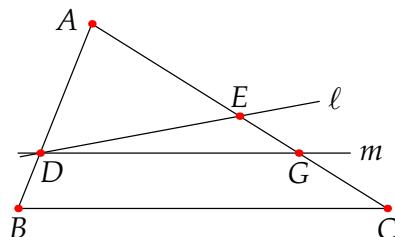
It remains to see that this ratio equals $\frac{|BC|}{|DE|}$. Again using common heights (h, k) of triangles,

$$\begin{aligned} \frac{|AB|}{|BD|} &= \frac{\text{area}(ABE)}{\text{area}(\triangle BDE)} \quad \frac{|CE|}{|AE|} = \frac{\text{area}(BCE)}{\text{area}(ABE)} \\ \implies \frac{|AB|}{|AD|} &\stackrel{(*)}{=} \frac{|AB|}{|BD|} \frac{|CE|}{|AE|} \stackrel{(\dagger)}{=} \frac{\text{area}(BCE)}{\text{area}(\triangle BDE)} = \frac{|BC|}{|DE|} \end{aligned} \tag{\dagger}$$

where the last equality follows since $\triangle BCE$ and $\triangle BDE$ have common height $d_1 = d_2$.

- (\Leftarrow) Suppose $\triangle ABC \sim \triangle ADE$. By Playfair, let m be the unique parallel to \overline{BC} through D . This intersects \overline{AC} at a point G . We must prove that $G = E$ (consequently $m = \ell$). By the (\Rightarrow) direction above,

$$\triangle ABC \sim \triangle ADG$$



However, \sim is plainly transitive (it is an equivalence relation), whence $\triangle ADE \sim \triangle ADG$. The similarity ratio is $1 = \frac{|AD|}{|AD|}$, whence

$$\frac{|AE|}{|AG|} = 1 \implies |AE| = |AG| \implies E = G$$

¹¹ $d_1 = d_2 \iff \ell$ parallel to \overline{BC} is Playfair! Compare Exercise 2.1.2 (Thm I.46) on the construction of a square...

Applications of Similarity: Trigonometric Functions, Cevians and the Butterfly Theorem

We finish with several applications of similarity which hopefully give an idea of what can be done *without* co-ordinates. None of these ideas were known to Euclid.

Definition 2.43. Given an acute angle $\angle ABC$ ($m\angle ABC < 90^\circ$), drop a perpendicular from A to \overrightarrow{BC} at D so that $\angle ADB$ is a right-angle. Define

$$\sin \angle ABC := \frac{|AD|}{|AB|} \quad \cos \angle ABC := \frac{|BD|}{|AB|}$$

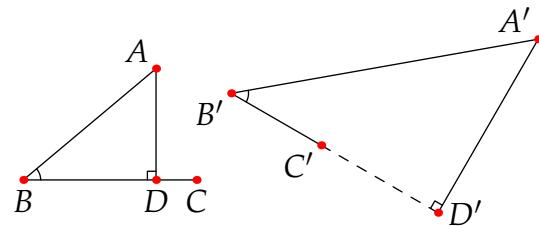
Early trigonometry dates to a few hundred years after Euclid, though the approach was different.¹²

Theorem 2.44. Angles have the same sine (cosine) if and only if they are congruent.

Proof. Assume $\angle ABC \cong \angle A'B'C'$ as in the picture, and drop perpendiculars to D, D' .

Since $\triangle ABD$ and $\triangle A'B'D'$ have two pairs of mutually congruent angles, the third pair is congruent also. AAA applies, the triangles are similar and

$$\frac{|AD|}{|AB|} = \frac{|A'D'|}{|A'B'|} \quad \frac{|BD|}{|AB|} = \frac{|B'D'|}{|A'B'|}$$



In particular, $\sin \angle ABC = \sin \angle A'B'C'$ and $\cos \angle ABC = \cos \angle A'B'C'$.

The converse is an exercise. ■

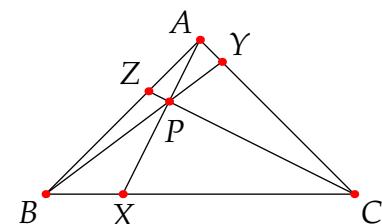
After Giovanni Ceva (1647–1734), a *cevian* is a segment joining a vertex to the opposite side of a triangle. Here is a beautiful result from the height of Euclidean geometry—good luck trying to prove it using co-ordinates!

Theorem 2.45 (Ceva's Theorem). Given $\triangle ABC$ and cevians $\overline{AX}, \overline{BY}, \overline{CZ}$,

$$\frac{|BX|}{|XC|} \frac{|CY|}{|YA|} \frac{|AZ|}{|ZB|} = 1 \iff \text{the cevians meet at a common point } P$$

Proof. (\Leftarrow) This is simply a repeated application of Lemma 2.40.

$$\begin{aligned} \frac{\text{area}(ABX)}{\text{area}(AXC)} &= \frac{|BX|}{|XC|} = \frac{\text{area}(PBX)}{\text{area}(PXC)} \\ \Rightarrow \frac{|BX|}{|XC|} &\stackrel{(*)}{=} \frac{\text{area}(ABX) - \text{area}(PBX)}{\text{area}(AXC) - \text{area}(PXC)} = \frac{\text{area}(ABP)}{\text{area}(APC)} \end{aligned}$$



Repeat for the other ratios and multiply to get 1.

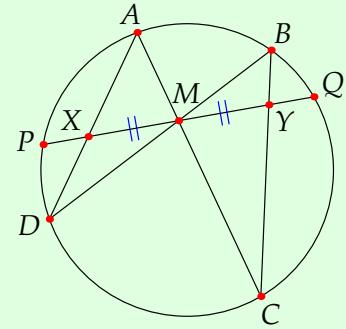
A simple justification of (*) and the converse are an exercise. ■

¹²The ancient forerunners of sine and cosine were defined using chords of circles rather than triangles. The word *trigonometry* (literally *triangle measure*) wasn't coined until 1595.

Theorem 2.46 (Butterfly Theorem). We are given the following data as in the picture:

- \overline{PQ} is a chord of a circle with midpoint M .
- \overline{AC} and \overline{BD} are chords meeting at M .
- X, Y are the chord-intersections as shown.

Then M is the midpoint of \overline{XY} .



This beautiful result dates to 1803-5 and has several proofs. We present an argument relying on similar triangles.

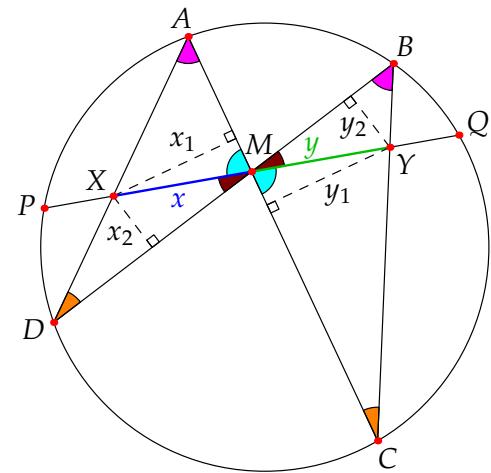
Proof. For convenience we introduce several numerical lengths:

- $z = |PM| = |MQ|$, $x = |XM|$ and $y = |YM|$.
- Drop perpendiculars from X, Y to chords $\overline{AD}, \overline{BC}$, and label the lengths x_1, x_2, y_1, y_2 as shown.

The four colored pairs of angles are congruent: vertical angles at M , and inscribed angles at A, B, C, D .

We compare sides of several similar triangles:

- $\frac{x}{x_1} = \frac{y}{y_1}$ and $\frac{x}{x_2} = \frac{y}{y_2} \implies \frac{x}{y} = \frac{x_1}{y_1} = \frac{x_2}{y_2}$
- $\frac{|AX|}{|BY|} = \frac{x_1}{y_2}$ and $\frac{|XD|}{|YC|} = \frac{x_2}{y_1}$



The result follows by putting these observations together and applying Exercise 1 twice:¹³

$$\begin{aligned} \frac{x^2}{y^2} &= \frac{x_1 x_2}{y_1 y_2} = \frac{|AX| |XD|}{|BY| |YC|} = \frac{|PX| |XQ|}{|PY| |YQ|} = \frac{(z-x)(z+x)}{(z+y)(z-y)} = \frac{z^2 - x^2}{z^2 - y^2} \\ &\implies x^2(z^2 - y^2) = y^2(z^2 - x^2) \\ &\implies x^2 = y^2 \implies x = y \end{aligned}$$

as required. ■

Exercises 2.5. 1. Let \overline{AD} and \overline{PQ} be chords of a circle which intersect at X . Use similar triangles to prove that

$$|AX| |XD| = |PX| |XQ|$$

¹³The denominators are equal by applying the Exercise to the chords \overline{BC} and \overline{PQ} .

2. Let $\triangle ABC$ have a right-angle at C . Drop a perpendicular from C to \overleftrightarrow{AB} at D .

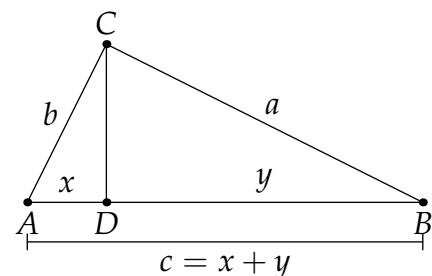
(a) Prove that D lies between A and B .

(b) Prove that you have *three* similar triangles

$$\triangle ACB \sim \triangle ADC \sim \triangle CDB$$

(c) Use these facts to prove Pythagoras' Theorem.

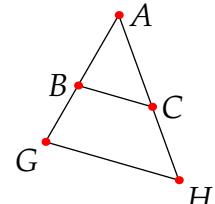
(Use the picture, where a, b, c, x, y are lengths)



3. Prove a simplified version of the SAS similarity theorem:

$$\frac{|AB|}{|AG|} = \frac{|AC|}{|AH|} \iff \triangle ABC \sim \triangle AGH$$

(Hint: construct \overline{BJ} parallel to \overline{GH} and appeal to AAA)



4. By excluding the other possibilities, prove the converse of length axiom L4:

If A, B, C are distinct and $|AB| + |BC| = |AC|$, then B lies between A and C .

5. Use Pythagoras' to prove that $\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}$, that $\sin 60^\circ = \frac{\sqrt{3}}{2}$ and $\cos 60^\circ = \frac{1}{2}$.

6. Prove the converse of Theorem 2.44: if $\sin \angle ABC = \sin \angle A'B'C'$, then $\angle ABC \cong \angle A'B'C'$.

(Hint: create right-triangles and prove they are similar. Label the side lengths o, a, h , etc.)

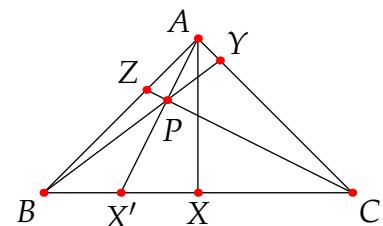
7. We complete the proof of Ceva's theorem.

(a) If p, q, r, s are non-zero real numbers, verify that $\alpha = \frac{p}{q} = \frac{r}{s} \implies \alpha = \frac{p-r}{q-s}$.

(b) Assume X, Y, Z satisfy Ceva's formula.

Define P as the intersection of \overline{BY} and \overline{CZ} and let \overrightarrow{AP} meet \overline{BC} at X' .

Prove the (\Rightarrow) direction of Ceva's theorem by using the (\Leftarrow) direction to show that $X' = X$.



8. (a) A *median* of a triangle is a segment from a vertex to the midpoint of the opposite side. Use Ceva's Theorem to prove that the medians of a triangle meet at a point (the *centroid*).

(b) (Hard) Medians split a triangle into six sub-triangles. Prove that all have the same area.

(c) Prove that the centroid is exactly $2/3$ of the distance along each median.

9. Prove that similarity of triangles is an equivalence relation.

(Don't use AAA since its proof requires this fact!)

10. (Hard) Explain how to prove ($2 \Rightarrow 1$) in the AAA Theorem (2.42).