Trigonometric Formulas and the Congruence Theorems

Euclidean Geometry Label triangle in the usual way: sides a, b, c, angles α, β, γ.

SAS Given a, γ, b:

1. Cosine rule $c^2 = a^2 + b^2 - 2ab\cos\gamma$ gives c.
2. Sine rule $\sin\alpha = \frac{a}{c}\sin\gamma$ gives α.
3. $\Sigma_\triangle = 180°$ gives β (or sine rule again).

SSS Given a, b, c:

1. Cosine rule $\cos\alpha = \frac{b^2+c^2-a^2}{2bc}$ gives α.
2. Cos rule or sine rule for β.
3. $\Sigma_\triangle = 180°$ gives γ.

SAA Given b, α, β:

1. $\Sigma_\triangle = 180°$ gives γ.
2. Sine rule $a = b\frac{\sin\alpha}{\sin\beta}$ and $c = b\frac{\sin\gamma}{\sin\beta}$.

ASA Given α, c, β:

1. $\Sigma_\triangle = 180°$ gives γ.
2. Sine rule $a = c\frac{\sin\alpha}{\sin\gamma}$ and $b = c\frac{\sin\beta}{\sin\gamma}$.

SSA One/two solutions (not a congruence theorem!) Given a, b, α:

1. Sine rule $\sin\beta = \frac{b}{a}\sin\alpha$ gives two values for β (one acute, one obtuse). If $\alpha \geq 90°$, then only acute angle is valid.
2. Could instead use cosine rule $a^2 = c^2 + b^2 - 2bc\cos\alpha \implies c = b\cos\alpha \pm \sqrt{a^2 - b^2\sin^2\alpha}$ gives one/two values for c.

Hyperbolic Geometry Same notation where sides are *hyperbolic* lengths.

Pythagoras/trig for right-triangles

$$\cosh h = \cosh x \cosh y \qquad \sin\theta = \frac{\sinh y}{\sinh h} \qquad \cos\theta = \frac{\tanh x}{\tanh h} \qquad \tan\theta = \frac{\tanh y}{\sinh x}$$

Cos rules $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos\gamma$ and[34] $\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\cosh c$

Sine rule $\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c}$

SAS Given a, γ, b:

1. Cosine rule gives (cosh) c.
2. Sine rule $\sin\alpha = \frac{\sinh a}{\sinh c}\sin\gamma$ gives α.
3. Sine rule again $\sin\beta = \frac{\sinh b}{\sinh c}\sin\gamma$ gives β (can't use $\Sigma_\triangle$!).

SSS Given a, b, c:

1. Cosine rule $\cos\alpha = \frac{\cosh b\cosh c - \cosh a}{\sinh a \sinh b}$ gives α.
2. Repeat twice more, or use sine rule for β, γ.

SAA Given b, α, β:

1. Sine rule $\sinh a = \frac{\sin\alpha}{\sin\beta}\sinh b$ gives a.
2. Cosine formula by dropping perpendicular to compute
$$c = \tanh^{-1}\big(\cos\alpha\tanh b\big) + \tanh^{-1}\big(\cos\beta\tanh a\big)$$
3. Sine rule $\sin\gamma = \sin\alpha\frac{\sinh c}{\sinh a}$.

ASA (Two methods) Given α, c, β:

1. Drop perp from angle at $\beta = \beta_1 + \beta_2$ to get two right triangles: compute $\beta_1, \beta_2 = \beta - \beta_1$, basic trig and Pythag (mess).
2. Use second cosine rule to compute γ, now have AAA data...

AAA Given angles α, β, γ.

1. Use second cosine rule three times to find sides.

SSA Two solutions (not a congruence theorem!) Given a, b, α:

1. Sine rule $\sin\beta = \frac{\sinh b}{\sinh a}\sin\alpha$ gives two values for β (one acute, one obtuse). Can use cosine rule as in Euclidean geometry, but result ugly since not a quadratic for c...
2. Now in one of above situations.

[34] In Euclidean limit, second version has $\gamma = \pi - \alpha - \beta$, which reduces to $\cosh c = 1$ ($c = 0$).