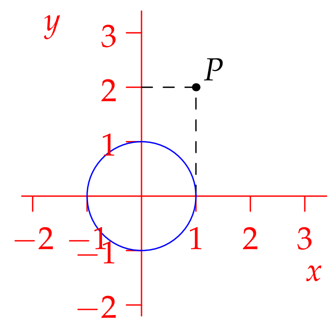
3 Analytic Geometry

Geometry in the style of Euclid and Hilbert is *synthetic*: axiomatic, without co-ordinates or explicit formulæ for length, area, volume, etc. By contrast, the practice of elementary geometry nowadays is typically *analytic*: reliant on co-ordinates & algebra, vectors. The critical invention was the *axis*, developed by René Descartes and Pierre de Fermat in the early 1600s; a fixed reference ruler against which objects can be measured using *co-ordinates.*

3.1 The Cartesian Co-ordinate System

Since Cartesian geometry (*Descartes'* geometry) should be familiar, we merely sketch the core ideas.



- Perpendicular *axes* meet at the *origin* O.
- The *co-ordinates* of a point are measured by projecting onto the axes; since these are real numbers we denote the set of these

  $$\mathbb{R}^2 = \{(x,y) : x,y \in \mathbb{R}\}$$

  E.g. P has co-ordinates $(1,2)$, we usually just write $P = (1,2)$.
- Algebra is introduced via *addition* and *scalar multiplication*

  $$P + Q = (p_1, p_2) + (q_1, q_2) = (p_1 + q_1, p_2 + q_2) \qquad \lambda P = (\lambda p_1, \lambda p_2)$$

- The *length* of a segment uses Pythagoras' Theorem[13]

  $$d(P,Q) = |PQ| = |Q - P| = \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}$$

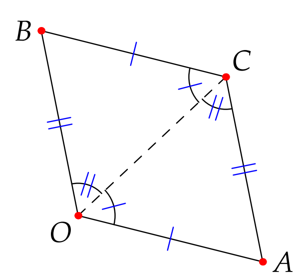
  In the picture $|OP| = \sqrt{1^2 + 2^2} = \sqrt{5}$. As in Section 2.5, segments are congruent if and only if they have the same length.
- *Curves* are defined using *equations.* E.g. $x^2 + y^2 = 1$ describes a circle.

Analytic geometry was conceived as a computational toolkit built on top of Euclid. At first, mathematicians felt the need to justify analytic arguments synthetically lest no-one believe their work.[14] Synthetic geometry is not without its benefits, but its study has increasingly become a fringe activity; co-ordinates are just too useful to ignore.

We may therefore assume anything from Euclid and mix strategies as appropriate. To see this at work, consider a simple result.

Lemma 3.1. *Non-collinear points $O = (0,0)$, $A = (x,y)$, $B = (v,w)$ and $C := (x+v, y+w)$, form a parallelogram $OACB$.*

Proof. Opposite sides have the same length ($|BC| = \sqrt{x^2 + y^2} = |OA|$, etc.) and are thus congruent. SAS shows $\triangle OAC \cong \triangle CBO$. Euclid's discussion of alternate angles (pages 10–12) forces opposite sides to be parallel. ∎



[13]It is useful to have another notation for distance, particular once we start applying functions to points (Section 3.3).

[14]This attitude persisted for some time; when Issac Newton published his groundbreaking *Principia* in 1687, his presentation was largely synthetic, even though he had used co-ordinates in his derivations.

Lemma 3.2. *The points X_t on the line $\overleftrightarrow{PQ}$ are in 1–1 correspondence with the real numbers via*

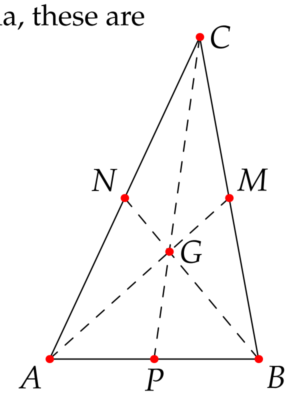
$$X_t = P + t(Q - P) = (1 - t)P + tQ$$

Moreover, $d(P, X_t) = |t|\,|PQ|$ so that t measures (signed) fractional distance along a segment.

The proof is an exercise. As an example of how easy it can be to work in analytic geometry, we repeatedly apply the Lemma to re-establish a famous result.

Theorem 3.3. *The medians of a triangle meet at a point $2/3$ of the way along each median.*

Proof. Given $\triangle ABC$, label the midpoints of each side as shown. By the Lemma, these are



$$M = \frac{1}{2}(B + C), \quad N = \frac{1}{2}(A + B), \quad P = \frac{1}{2}(A + C)$$

The point $\frac{2}{3}$ of the way along median $\overline{AM}$ is then

$$A + \frac{2}{3}(M - A) = A + \frac{2}{3}(B + C - 2A) = \frac{1}{3}(A + B + C)$$

By symmetry (check directly if you like!), this is also the point $\frac{2}{3}$ of the way along the other two medians.

The three points are therefore identical: the medians meet at the centroid $G = \frac{1}{3}(A + B + C)$. ■

Compare this to Exercise 2.5.8 where we used Ceva's Theorem!

Exercises 3.1. 1. By completing the square, identify the curve described by the equation

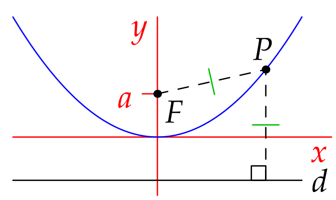
$$x^2 + y^2 - 4x + 2y = 10$$

2. (a) Perform a pure co-ordinate proof of Theorem 3.3. For simplicity, arrange the triangle so that $A = (0, 0)$ is the origin, and B points along the positive x-axis.

   (b) Descartes and Fermat did not have a fixed perpendicular second axis! Their approach was equivalent to choosing a second axis at an angle which made the problem as simple as possible.
   Given $\triangle ABC$, let A be the origin and choose axes which point along the edges $\overline{AB}$ and $\overline{BC}$. What are the co-ordinates of B and C with respect to these axes? Now give an even simpler proof of the centroid theorem.

3. Prove Lemma 3.2.

4. A *parabola* is a curve whose points are equidistant from a fixed point F, the *focus*, and a fixed line d (the *directrix*). Choose axes as shown in the picture so that $F = (0, a)$ and d has equation $y = -a$. Find the equation of the parabola.



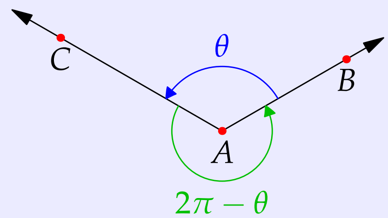
3.2 Angles and Trigonometry

We define angles a little differently to the method introduced in Section 2.5: this time we use radians, angles are *oriented*, and may be reflex (larger than a straight edge). This should all be as familiar as the previous section.

Definition 3.4. Suppose A, B, C are distinct points in the plane. Take any circular arc centered at A and define the *radian-measure*

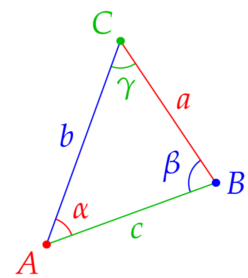
$$\measuredangle BAC := \frac{\text{arc-length}}{\text{radius}} \in [0, 2\pi)$$

where arc-length is measured *counter-clockwise*[15] from $\overrightarrow{AB}$ to $\overrightarrow{AC}$.



Being ratios of lengths, radians are naturally *unitless*; since arc-length scales with radius, the definition is independent of the radius of the circular arc. Indeed

> Angles are congruent if and only if their radian measures are equal.



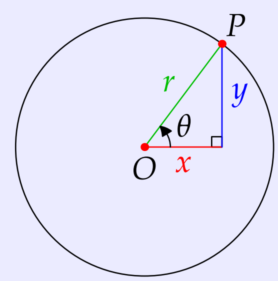
For this reason, it is common to label angles with their radian measure. Standard convention for triangles is to label the edge-length opposite a vertex with the corresponding lower-case letter, and the radian-measure with a Greek letter.

Trigonometric functions may now be defined for all angles simultaneously.

Definition 3.5. Let $P = (x, y)$ lie on a circle of radius r such that the $\overrightarrow{OP}$ makes angle θ radians measured counter-clockwise from the positive x-axis. The *cosine*, *sine* and *tangent* of θ are defined by

$$\cos\theta := \frac{x}{r} \qquad \sin\theta := \frac{y}{r} \qquad \tan\theta := \frac{y}{x}\ (x \neq 0)$$

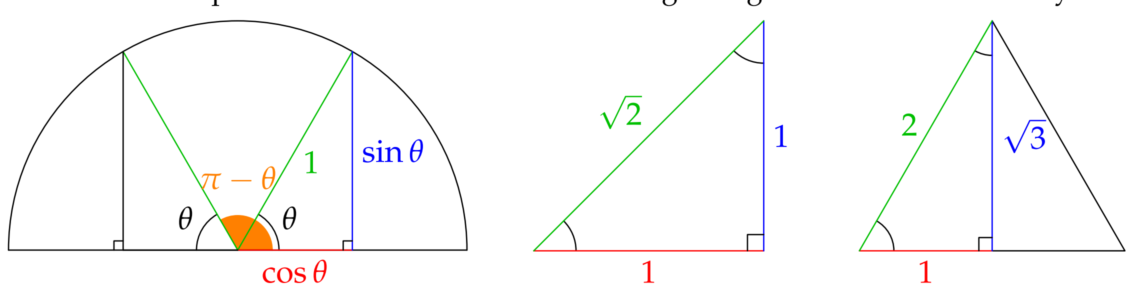
The AAA theorem for similar triangles says these are well-defined.



Example 3.6. Basic relationships should be obvious from the picture: e.g.

$$\cos^2\theta + \sin^2\theta = 1 \quad \text{and} \quad \sin\theta = \cos\left(\tfrac{\pi}{2} - \theta\right)$$

Here are three further pictures. What well-known facts regarding sine and cosine do they illustrate?



[15]Some versions of analytic geometry permit negative angles or measures $\geq 2\pi$ radians. Note the distinction from synthetic geometry where all angles are smaller than a straight edge and $\angle BAC \cong \angle CAB$. In the picture, $\measuredangle CAB$ is *not* the radian-measure of $\angle CAB$!

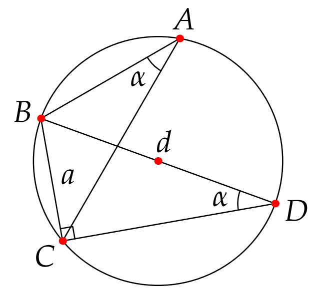
Solving Triangles A triangle is described by six values: three side lengths and three angle measures. The triangle congruence theorems (SAS, ASA, SSS, SAA) say that three of these in a suitable combination is enough to recover the remainder. This is done using the cosine and sine rules.

Theorem 3.7. *For any triangle $\triangle ABC$:*

Sine Rule *If d is the diameter of the circumcircle (Definition 2.31), then* $\dfrac{\sin\alpha}{a} = \dfrac{\sin\beta}{b} = \dfrac{\sin\gamma}{c} = \dfrac{1}{d}$

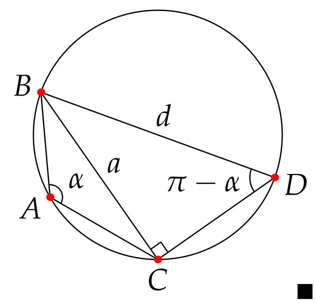
Cosine Rule $c^2 = a^2 + b^2 - 2ab\cos\gamma$

Proof. We prove the sine rule and leave the cosine rule as an exercise. Everything relies on Corollary 2.33. Draw the circumcircle of $\triangle ABC$. Construct $\triangle BCD$ with diameter $\overline{BD}$; this is right-angled at C by Thales' Theorem. There are two cases:



1. If A lies on the same side of $\overleftrightarrow{BC}$ as D, then A and D share the same arc, whence $\measuredangle BDC = \alpha$ and

$$a = d\sin\measuredangle BDC = d\sin\alpha$$

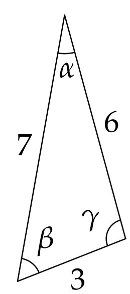


2. If A lies on the opposite side, then the quadrilateral $ABDC$ lies on a circle. Opposite angles at A, D are supplementary, whence

$$\sin\alpha = \sin(\pi - \alpha) = \sin\measuredangle BDC = \frac{a}{d}$$

The two other angle-side combinations follow by permutation. ∎

Examples 3.8. 1. The SSS congruence corresponds to solving a triangle using the cosine rule. For instance, the given triangle has angles

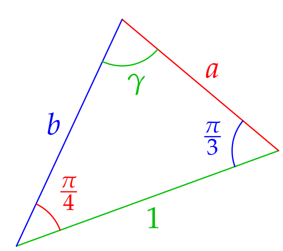


$$\alpha = \frac{6^2+7^2-3^2}{2\cdot 6\cdot 7} = \cos^{-1}\frac{19}{21} \approx 25^\circ \qquad \beta = \frac{3^2+7^2-6^2}{2\cdot 3\cdot 7} = \cos^{-1}\frac{11}{21} \approx 58^\circ$$

$$\gamma = \frac{3^2+6^2-7^2}{2\cdot 3\cdot 6} = \cos^{-1}\frac{-1}{9} \approx 96^\circ$$

Once you have one angle-side pair, you could switch to the sine rule, and the last angle could instead be found by subtracting from π radians.

2. To solve a triangle with data corresponding to the ASA congruence, find the remaining angle $\gamma = \pi - \frac{\pi}{4} - \frac{\pi}{3} = \frac{5\pi}{12}$ then apply the sine rule



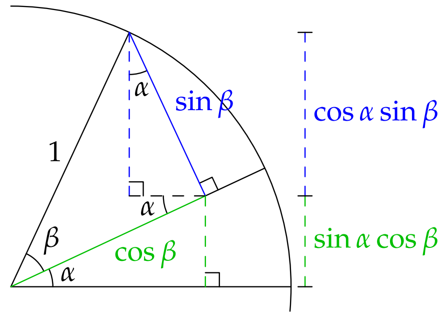
$$\frac{\sin\frac{\pi}{4}}{a} = \frac{\sin\frac{\pi}{3}}{b} = \sin\frac{5\pi}{12} = \cos\frac{\pi}{12} \implies a = \frac{1}{\sqrt{2}\cos\frac{\pi}{12}} \approx 0.732$$

$$b = \frac{\sqrt{3}}{2\cos\frac{\pi}{12}} \approx 0.897$$

Multiple-angle formulæ The picture provides a very simple proof of the expressions

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

at least when $\alpha+\beta < \frac{\pi}{2}$. A little algebraic manipulation produces the double-angle and difference formulæ; and verifies that these hold for all possible angle inputs.



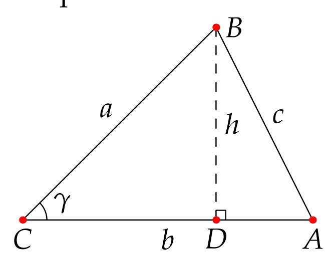
$$\sin 2\alpha = 2\sin\alpha\cos\alpha \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

Exercises 3.2.

1. A triangle has angle of $\frac{2\pi}{3}$ radians between sides of lengths 2 and $\sqrt{3}-1$. Find the length of the remaining side, and the remaining angles.
2. Describe how to solve a triangle given data in line with the SAA congruence theorem.
3. Two measurements for the height of a mountain are taken at sea level 5000 ft apart in a line pointing away from the mountain. The angles of elevation to the mountain top from the horizontal are 15° and 13° respectively. What is the height of the mountain?
4. Use a multiple angle formula to find an exact value for $\cos\frac{\pi}{12}$ and thus exact values for the side lengths of the triangle in Exercise 3.8.2.
5. The area of a triangle is $\frac{1}{2}$(base)·(height). By using each side of the triangle alternately as the 'base,' provide an alternative proof of the sine rule without the relationship to the circumcircle.
6. (a) By dropping a perpendicular from B to $\overleftrightarrow{AC}$ at D, construct a proof of the cosine rule.

   (*Hint: apply Pythagoras' to the two right-triangles*)

   (b) Is your argument valid if D is not interior to $\overline{AC}$?



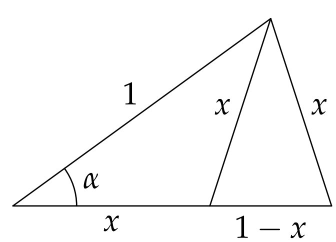
7. The *dot product* of $A = (a_1, a_2)$ and $B = (b_1, b_2)$ is $A \cdot B := a_1b_1 + a_2b_2$. Apply the cosine rule to $\triangle OAB$ to prove that

   $$A \cdot B = |OA|\,|OB|\cos\measuredangle AOB$$

8. Derive the multiple-angle formula for $\sin(\alpha-\beta)$.

   (*Remember that* $0 \le \alpha, \beta, \alpha-\beta < 2\pi$ *so you can't simply switch the sign of* β*!*)

9. Given the arrangement in the picture, find x, the radian-measure α and the exact value of $\cos\alpha$.
   (*Hint: first show that you have similar isosceles triangles*)



3.3 Isometries

At the heart of elementary geometry is the notion of *congruence*: the idea that geometric figures can be essentially the same without necessarily being equal.

Euclid Uses *equal* instead of *congruent*. Assumes without justification that congruent angles or segments can *superimposed* on one another.

Hilbert Congruence is an *undefined term* described by axioms C1–5. These formalize Euclid's superposition ideas and ruler-and-compass constructions.

In analytic geometry, congruence may be described *algebraically*. Recall that two segments have the same length if and only if they are congruent; this motivates a definition.

Definition 3.9. A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a *(Euclidean) isometry*[16] if it preserves lengths:

$$\forall P, Q \in \mathbb{R}^2, \text{ we have } d\big(f(P), f(Q)\big) = |PQ|$$

Two figures (segments, angles, triangles, etc.) are *congruent* precisely when there is an isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ mapping one to the other.

Example 3.10. The map $f(x, y) = \frac{1}{5}\big(3x + 4y, 4x - 3y\big) + (2, 3)$ is an isometry, as is readily checked: if $P = (x, y)$ and $Q = (v, w)$, then

$$\begin{aligned} d\big(f(P), f(Q)\big)^2 &= \left(\frac{3v + 4w - 3x - 4y}{5}\right)^2 + \left(\frac{4v - 3w - 4x + 3y}{5}\right)^2 \\ &= \frac{3^2 + 4^2}{5^2}\big((v - x)^2 + (w - y)^2\big) = |PQ|^2 \end{aligned}$$

Isometric segments are certainly congruent; we should make sure the same holds for angles.

Lemma 3.11. *Isometries preserve (non-oriented) angles: if $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry, then*

$$\angle PQR \cong \angle f(P) f(Q) f(R)$$

Proof. Since f is an isometry, the sides of $\triangle PQR$ and $\triangle f(P) f(Q) f(R)$ are mutually congruent in pairs. The SSS triangle congruence theorem says that the angles are also mutually congruent. ■

Our next task is to verify our intuition that isometries are rotations, reflections and translations. Given an isometry f, define g via $g(X) = f(X) - f(O)$, where O is the origin. Then g is an isometry

$$g(P) - g(Q) = f(P) - f(Q) \implies d\big(g(P), g(Q)\big) = d\big(f(P), f(Q)\big) = |PQ|$$

which moreover *fixes the origin*: $g(O) = O$. We conclude that every isometry f is the composition of an *origin-preserving* isometry g followed by a *translation* "$+C$:"

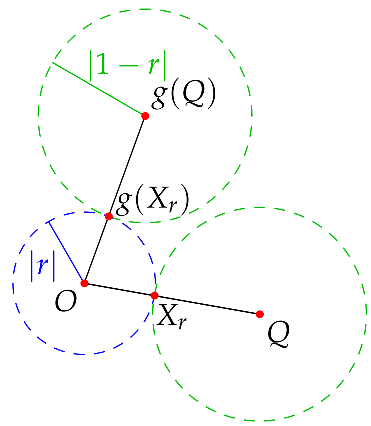
$$f(X) = g(X) + C$$

[16] In the Greek, the prefix *iso* means 'same' and *metros* 'measure,' i.e. *same length*.

It thus suffices to describe the origin-preserving isometries g. For these, we make two observations.

1. Suppose $|OQ| = 1$ and let $X_r = rQ$ for some $r \in \mathbb{R}$. Then

   - $g(X_r)$ is a distance $|r| = |OX_r|$ from the origin.
   - $g(X_r)$ is a distance $|1-r| = |QX_r|$ from Q.

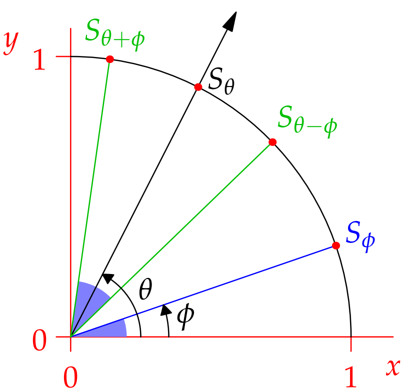
   

   $g(X_r)$ therefore lies on the intersection of two circles, which intersect at a single point: we conclude that

   $$g(rQ) = rg(Q)$$

   The picture shows the case $r \in (0,1)$, where the uniqueness of intersection follows from $1 = |r| + |1-r|$.

2. $g(1,0)$ lies on the unit circle and therefore has the form

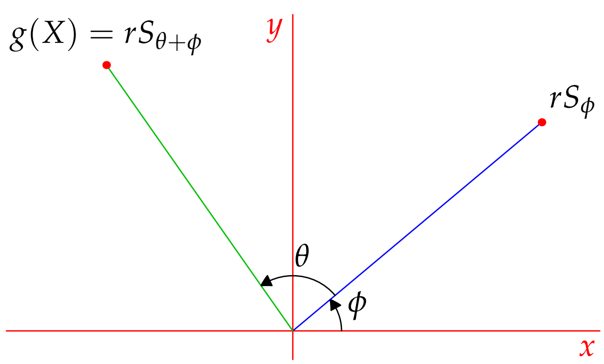
   $$g(1,0) = S_\theta := (\cos\theta, \sin\theta)$$

   for some $\theta \in [0, 2\pi)$. By preservation of length and angle (Lemma 3.11), any other point $S_\phi = (\cos\phi, \sin\phi)$ on the unit circle must therefore be mapped to one of two points

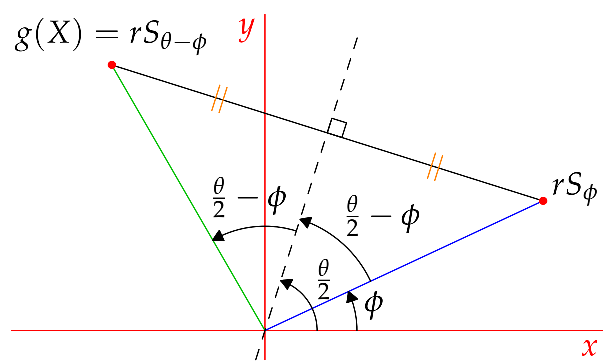
   $$g(S_\phi) = S_{\theta\pm\phi} = \big(\cos(\theta\pm\phi), \sin(\theta\pm\phi)\big)$$

   That is, we transfer the angle ϕ to either side of the ray $\overrightarrow{OR_\theta}$.

Putting these together by writing $X = rS_\phi = (r\cos\phi, r\sin\phi)$ in polar co-ordinates, we conclude that g has one of two forms:



Rotation counter-clockwise by θ



Reflection across the line making angle $\frac{\theta}{2}$ with positive x-axis

Theorem 3.12. *Every isometry of $\mathbb{R}^2$ has the form*

$$f(X) = g(X) + C$$

where g is either a rotation about the origin, or a reflection across a line through the origin.

Calculating with isometries Computation benefits from column-vector notation and matrix multiplication. If we write $\mathbf{x} = \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} r\cos\theta \\ r\sin\theta \end{smallmatrix}\right)$ for the position vector of X and apply the multiple-angle formulæ (page 39), then the expression for rotation becomes

$$g(\mathbf{x}) = r\begin{pmatrix} \cos(\theta+\phi) \\ \sin(\theta+\phi) \end{pmatrix} = r\begin{pmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\mathbf{x}$$

For reflections, the sign of the second column is reversed: $\left(\begin{smallmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{smallmatrix}\right)$. Every isometry may therefore be written $f(\mathbf{x}) = A\mathbf{x} + \mathbf{c}$ where A is an *orthogonal matrix*.[17]

Examples 3.13. 1. We revisit Example 3.10 in matrix format:

$$f(\mathbf{x}) = \frac{1}{5}\begin{pmatrix} 3x+4y \\ 4x-3y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

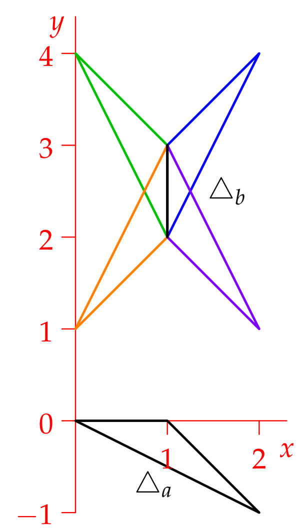
Since $\frac{\sin\theta}{\cos\theta} = \frac{4/5}{3/5} = \frac{4}{3}$, we see that its effect is to *reflect* across the line through the origin making angle $\frac{1}{2}\tan^{-1}\frac{4}{3} \approx 26.6°$ with the positive x-axis, before *translating* by $(2,3)$.

2. $\triangle_a$ has vertices $(0,0)$, $(1,0)$, $(2,-1)$ and is congruent to $\triangle_b$, two of whose vertices are $(1,2)$ and $(1,3)$. Find all isometries transforming $\triangle_a$ to $\triangle_b$ and the location(s) of the third vertex of $\triangle_b$.

Let $f = A\mathbf{x} + \mathbf{c}$ be the isometry. Since $d\big((1,2),(1,3)\big) = 1$ these points must be the images under f of $(0,0)$ and $(1,0)$. There are *four* distinct isometries:

Cases 1, 2: If $f(0,0) = (1,2)$ and $f(1,0) = (1,3)$, then $\mathbf{c} = f\left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ and

$$A\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mathbf{c} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \implies A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies A = \begin{pmatrix} 0 & a_{12} \\ 1 & a_{22} \end{pmatrix}$$



for some a_{12}, a_{22}. Since A is orthogonal, the options are $A = \left(\begin{smallmatrix} 0 & \mp 1 \\ 1 & 0 \end{smallmatrix}\right)$ and we obtain two possible isometries:

- $f_1(\mathbf{x}) = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\mathbf{x} + \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ rotates by 90°, then translates by $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$.
- $f_2(\mathbf{x}) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\mathbf{x} + \left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$ reflects across $y = x$, then translates by $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$.

The third point of $\triangle_b$ is $f_1(2,-1) = (2,4)$ or $f_2(2,-1) = (0,4)$.

Cases 3, 4: $f(0,0) = (1,3)$ and $f(1,0) = (1,2)$ results in two further isometries f_3 and f_4. The details are an exercise.

All four possible triangles $\triangle_b$ are drawn in the picture.

In 1872, Felix Klein suggested that the geometry of a set is the study of its *invariants*: properties preserved by its *group* of structure-preserving transformations. In Euclidean geometry, this is the group of *Euclidean isometries* (Exercise 9). Klein's approach provided a method for analyzing and comparing the non-Euclidean geometries appearing in the late 1800s. By the mid 1900s, the resulting theory of *Lie groups* had largely classified classical geometries. Klein's approach, and the application of abstract algebra, remains dominant in modern research, both in mathematics and physics.

[17] An orthogonal matrix satisfies $A^TA = I$: all such have the form $\left(\begin{smallmatrix} \cos\theta & \mp\sin\theta \\ \sin\theta & \pm\cos\theta \end{smallmatrix}\right) = \left(\begin{smallmatrix} a & \mp b \\ b & \pm a \end{smallmatrix}\right)$ where $a^2 + b^2 = 1$.

Exercises 3.3. 1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the isometry, "reflect across the line through the origin making angle $\frac{\pi}{3}$ with the positive x-axis." Find a 2×2 matrix A such that $f(\mathbf{x}) = A\mathbf{x}$.

2. Describe the geometric effect of the isometry $f(\mathbf{x}) = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}\mathbf{x} + \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

3. Find the remaining isometries f_3, f_4 and the third points of $\triangle_b$ in Exercise 3.13.2.

4. Find the reflection of the point $(4,1)$ across the line making angle $\frac{1}{2}\tan^{-1}\frac{12}{5} \approx 33.7°$ with the positive x-axis.

   (*Hint: if* $\tan\theta = \frac{12}{5}$, *what are* $\cos\theta$ *and* $\sin\theta$*?*)

5. An origin-preserving isometry $f(\mathbf{v}) = A\mathbf{v}$ moves the point $(7,4)$ to $(-1,8)$.

   (a) If f is a rotation, find the matrix A. Through what angle does it rotate?

   (b) If f is a reflection, find the matrix A. Across which line does it reflect?

6. Let $ABCD$ be the rectangle with vertices $A = (0,0)$, $B = (4,0)$, $C = (4,3)$, $D = (0,3)$. Suppose an isometry $f : \mathbb{R}^2 \to \mathbb{R}^2$ maps $ABCD$ to a new rectangle $PQRS$ where

   $$P = f(A) := (2,4) \quad \text{and} \quad R = f(C) := (2,9)$$

   Find all possible isometries f and the remaining points $Q = f(B)$ and $S = f(D)$.

7. (a) If $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $\mathbf{p}$ is constant, explain why $f(\mathbf{x}) = A(\mathbf{x} - \mathbf{p}) + \mathbf{p} = A\mathbf{x} + (I - A)\mathbf{p}$ rotates by θ around the point with position vector $\mathbf{p}$.

   (b) Suppose $f(\mathbf{x}) = A\mathbf{x} + \mathbf{c}$ rotates the plane around the point $P = (-2,1)$ by an angle $\theta = \tan^{-1}\frac{3}{4}$. Find A and $\mathbf{c}$.

   (c) Suppose f rotates by θ around $\mathbf{p}$ and g rotates by ϕ around $\mathbf{q}$ where θ, ϕ are non-zero.

      i. If $\theta + \phi \neq 2\pi$, show that $f \circ g$ is a rotation: by what angle and about which point?

      ii. What happens instead if $\theta + \phi = 2\pi$?

8. Make an argument involving circle intersections (see page 41) to prove that for any isometry f,

   $$f\big((1-t)P + tQ\big) = (1-t)f(P) + tf(Q)$$

9. Throughout this question, we use the notation $f_{A,\mathbf{c}} : \mathbf{x} \mapsto A\mathbf{x} + \mathbf{c}$.

   (a) Prove that isometries obey the composition law $f_{A,\mathbf{c}} \circ f_{B,\mathbf{d}} = f_{AB,\mathbf{c}+A\mathbf{d}}$.

   (b) Find the inverse function of the isometry $f_{A,\mathbf{c}}$. Otherwise said, if $f_{A,\mathbf{c}} \circ f_{C,\mathbf{d}} = f_{I,\mathbf{0}}$, where I is the identity matrix, how do $B, \mathbf{d}$ depend on $A, \mathbf{c}$?

   (c) Verify that the following composition $f_{A,\mathbf{c}} \circ f_{I,\mathbf{d}} \circ f_{A,\mathbf{c}}^{-1}$ is a translation.

   The composition law in part (a) is often written in matrix format using augmented matrices:

   $$(A|\mathbf{c})(B|\mathbf{d}) := (AB|\mathbf{c} + A\mathbf{d})$$

   If you've done group theory, parts (a) and (b) are the closure and inverse properties of the group of Euclidean isometries E. By part (c), the translations T form a normal subgroup. We may therefore write E as a semi-direct product of T and the orthogonal group of origin-preserving isometries

   $$E = T \rtimes O_2(\mathbb{R})$$

3.4 The Complex Plane

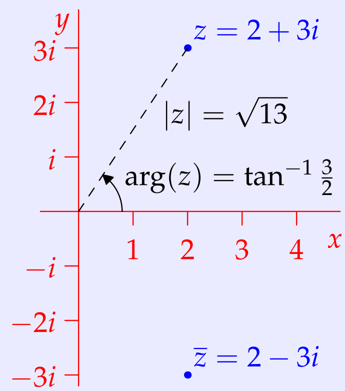
Complex numbers date to 16th century Italy. Their application to geometry really begins with Leonhard Euler (1707–1783) who identified the set of complex numbers $\mathbb{C}$ with the plane (what is now known as the *Argand diagram*).

Definition 3.14. Let i be an abstract symbol satisfying the property $i^2 = -1$.

Given real numbers x, y, the *complex number* $z = x + iy$ is simply the point (x, y) in the standard Cartesian plane.[18]

Given $z = x + iy$, its:

- *Complex conjugate* $\overline{z} = x - iy$ is its *reflection* across the real axis.
- *Modulus* $|z| = \sqrt{z\overline{z}} = \sqrt{x^2 + y^2}$ is its distance from the origin.
- *Argument* $\arg(z)$ is the angle $\measuredangle 10z$ between the segment $\overline{0z}$ and the positive real axis.



Addition, *scalar multiplication* (by real numbers) and *complex multiplication* follow the usual algebraic rules while using $i^2 = -1$ to simplify.

Example 3.15. A simple example of multiplication of complex numbers:

$$\begin{aligned}(2+3i)(4+5i) &= 2\cdot 4 + 2\cdot 5i + 3i\cdot 4 + 3i\cdot 5i && \text{(multiply out)}\\ &= 8 + 10i + 12i - 15 && \text{(use } i^2 = -1 \text{ to simplify)}\\ &= -7 + 22i\end{aligned}$$

The algebra screams *geometry*! Definition 3.14 already length, angle and reflection in the real axis. Two other aspects of basic geometry are immediate:

- Addition by z *translates* all points by z.
- Scalar multiplication *scales* distances from the origin (similarity).

The algebraic property distinguishing the complex numbers from the standard Cartesian plane is *complex multiplication*. To start visualizing this, consider multiplication by i,

$$iz = i(x + iy) = -y + ix$$

This is the result of *rotating* z counter-clockwise $\frac{\pi}{2}$ radians about the origin. To obtain all rotations and reflections, we need an alternative description of a complex number.

Lemma 3.16. 1. *(Euler's Formula) For any $\theta \in \mathbb{R}$, $e^{i\theta} = \cos\theta + i\sin\theta$.*

2. *(Exponential laws) $e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$ and $(e^{i\theta})^n = e^{in\theta}$ for any $n \in \mathbb{Z}$.*

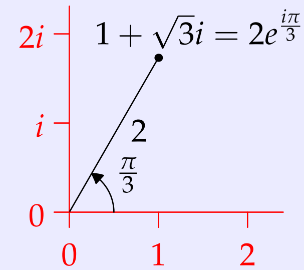
Evaluating at $\theta = \pi$ yields the famous *Euler identity* $e^{i\pi} = -1$. Part 1 can be taken as a definition. To see that it is a reasonable definition requires either power series or elementary differential equations, topics best described elsewhere. Part 2 is an exercise.

[18] In the language of linear algebra, $\mathbb{C}$ is a vector space over $\mathbb{R}$ with basis $\{1, i\}$.

Definition 3.17. Let $z = x + iy$ be a non-zero complex number. Writing $x = r\cos\theta$ and $y = r\sin\theta$, we obtain the *polar form*

$$z = re^{i\theta} = r(\cos\theta + i\sin\theta)$$

where $r = |z|$ is the modulus and $\theta = \arg(z)$ the argument of z.



Now consider the effect of multiplying a complex number $z = re^{i\phi}$ by $e^{i\theta} = \cos\theta + i\sin\theta$: according to the Lemma

$$e^{i\theta}z = re^{i\theta}e^{i\phi} = re^{i(\theta+\phi)}$$

which has the same modulus (r) as z but a new argument.

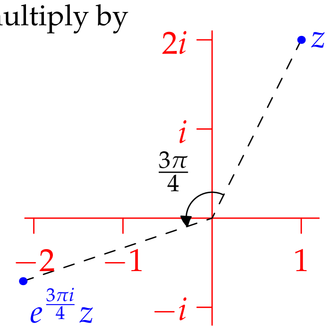
Theorem 3.18. *The complex number $e^{i\theta}z$ is the result of rotating z counter-clockwise about the origin through an angle θ.*

Example 3.19. To rotate $z = 1 + 2i$ counter-clockwise by $\frac{3\pi}{4}$ radians, we multiply by

$$e^{\frac{3\pi i}{4}} = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}(-1+i)$$

to obtain

$$e^{\frac{3\pi i}{4}}z = \frac{1}{\sqrt{2}}(-1+i)(1+2i) = -\frac{1}{\sqrt{2}}(3+i)$$



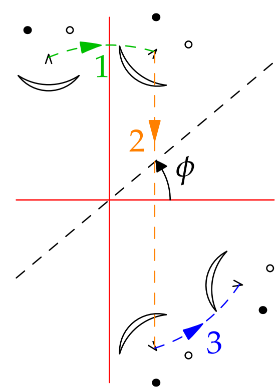
You could try to keep things in polar form, though it doesn't result in a nice answer:

$$z = \sqrt{5}e^{i\tan^{-1}2} \implies e^{\frac{3\pi i}{4}}z = \sqrt{5}e^{\frac{3\pi i}{4}+i\tan^{-1}2}$$

Reflections may be described by combining rotations with complex conjugation. To reflect across the line making angle θ with the positive real axis, we rotate the plane so that the reflection appears to be vertical:

1. Rotate the plane *clockwise* by θ, that is $z \mapsto e^{-i\theta}z$.
2. Reflect across the real axis by complex conjugation.
3. Rotate counter-clockwise by θ.

Combining these steps gives the formula.



Theorem 3.20. *To reflect z across the line making angle θ with the positive real axis, we compute*

$$z \mapsto e^{i\theta}\left(\overline{e^{-i\theta}z}\right) = e^{2i\theta}\overline{z}$$

Example 3.21. Reflect $z = -2+3i$ across the line through the origin and $w = \sqrt{3}+i$.
First compute $\theta = \arg(w) = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$. The desired point is therefore

$$e^{\frac{i\pi}{3}}(-2-3i) = \left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)(-2-3i) = \left(\frac{3\sqrt{3}}{2}-1\right) - \left(\sqrt{3}+\frac{3}{2}\right)i$$

To describe general rotations and reflections about arbitrary points/lines, we combine our approach with *translations* (compare Exercise 3.3.7).

Corollary 3.22. 1. *To rotate z by θ about a point w, compute $z \mapsto e^{i\theta}(z-w)+w$.*

2. *To reflect z across the line with slope θ through a point w, compute $z \mapsto e^{2i\theta}(\overline{z}-\overline{w})+w$.*

Example 3.23. The combination of translation by $-i$, rotation by $\frac{\pi}{3}$ around the origin, then translation by 1, may be expressed

$$z \mapsto e^{\frac{\pi}{3}}(z-i)+1 = i + e^{\frac{\pi}{3}}(z-i) + 1 - i$$

Alternatively, this is rotation by $\frac{\pi}{3}$ around i followed by translation by $1-i$.

We have now described all the Euclidean isometries of the previous section in the language of complex numbers. Here is the full dictionary.[19]

Isometry/Transformation	Complex numbers	Matrices/vectors
Addition/Translation	$z+w = (x+iy)+(u+iv)$	$\mathbf{z}+\mathbf{w} = \begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}u\\v\end{pmatrix}$
Scaling	$\lambda z = (\lambda x)+i(\lambda y)$	$\lambda\mathbf{z} = \begin{pmatrix}\lambda x\\\lambda y\end{pmatrix}$
Rotation CCW by $\frac{\pi}{2}$	$z \mapsto iz$	$\mathbf{z} \mapsto \begin{pmatrix}0&-1\\1&0\end{pmatrix}\mathbf{z}$
Rotation CCW by θ	$z \mapsto e^{i\theta}z$	$\mathbf{z} \mapsto \begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}\mathbf{z}$
Vertical reflection	$z \mapsto \overline{z}$	$\mathbf{z} \mapsto \begin{pmatrix}1&0\\0&-1\end{pmatrix}\mathbf{z}$
Reflection across line with slope $\frac{\theta}{2}$	$z \mapsto e^{i\theta}\overline{z}$	$\mathbf{z} \mapsto \begin{pmatrix}\cos\theta&\sin\theta\\\sin\theta&-\cos\theta\end{pmatrix}\mathbf{z}$

It is perhaps surprising to modern readers, but complex numbers came before vectors and matrix-geometry! During the 1800s mathematicians tried unsuccessfully to replicate the complex number approach in higher dimensions. This ultimately led (via Hamilton's quaternions) to the adoption of vectors and linear algebra/matrix calculations.

One reason for the desire to keep the complex number description is that it may be used to describe further (non-isometric) transformations of the plane: for instance $z \mapsto \overline{z}^{-1}$ is *reflection in a circle*! We'll discuss some of this at the end of Chapter 4.

[19]Scaling isn't an isometry, but it is worth including nonetheless!

Exercises 3.4. 1. Use complex numbers to compute the result of the following transformations: you can answer in either standard or polar form.

(a) Rotate $3-5i$ counter-clockwise around the origin by $\frac{3\pi}{4}$ radians.

(b) Reflect $2-i$ across the line joining $1+i\sqrt{3}$ and the origin.

(c) Reflect $1+i$ across the line through the origin making angle $\frac{\pi}{5}$ radians with the positive real axis.

2. Find the reflection of the point $(2,3)$ across the line making angle $\frac{3\pi}{8}$ with the positive x-axis. Give your answer using both complex numbers and matrices/vectors.

3. Repeat the previous question for the point $(3,4)$ and the angle $\frac{5\pi}{12}=75°$.

4. Describe the geometric effect of the map $z\mapsto\frac{1}{\sqrt{2}}(-1-i)\big(\overline{z}-3+4i\big)$.

(*Hint: compare Example 3.23*)

5. (Hard) Consider the line ℓ through the origin and $\left(\sqrt{2+\sqrt{2}},\sqrt{2-\sqrt{2}}\right)$. Compute the result of reflecting $-2+3i$ across ℓ.

6. By letting $n=3$ in Lemma 3.16, prove that

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

Find a corresponding trigonometric identity for $\sin 3\theta$.

7. Prove part 2 of Lemma 3.16.

(*Hint: use the multiple-angle formulae (page 39) to expand* $e^{i(\theta+\phi)}$)

3.5 Birkhoff's Axiomatic System for Analytic Geometry (non-examinable)

Analytic geometry was originally conceived as an addition to Euclidean geometry. In 1932, courtesy of George David Birkhoff, it was axiomatized in its own right.

Background Assume the usual properties/axioms of the real numbers as a complete ordered field. Birkhoff's system is typical of modern axiomatic systems in that it is built on top of pre-existing systems (set theory, complete ordered fields, etc.).

Undefined terms Two objects: *Point, line.* Two function: *distance d, angle measure $\measuredangle$.* If the set of points is $\mathcal{S}$, then,

$$d : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_0^+, \qquad \measuredangle : \mathcal{S} \times \mathcal{S} \times \mathcal{S} \to [0, 2\pi)$$

Axioms *Euclidean* Given two distinct points, there exists a unique line containing them.

Ruler Points on a line ℓ are in bijective correspondence with the real numbers in such a way that if t_A, t_B correspond to $A, B \in \ell$, then $|t_A - t_B| = d(A, B)$.

Protractor The rays emanating from a point O are in bijective correspondence with the set $[0, 2\pi)$ so that if α, β correspond to rays $\overrightarrow{OA}, \overrightarrow{OB}$, then $\measuredangle AOB \equiv \beta - \alpha \pmod{2\pi}$. This correspondence is continuous in A, B.

SAS similarity[20] If triangles have a pair of angles with equal measure, and the sides adjacent to said angles are in the same ratio, then the remaining angles have equal measure and the final sides are in the same ratio.

Definitions As with Hilbert, some of these are required before later axioms make sense. In particular, the definition of *ray* is required before the *protractor* axiom.

Betweenness B lies between A and C if $d(A, B) + d(B, C) = d(A, C)$

Segment $\overline{AB}$ consists of the points A, B and all those between

Ray $\overrightarrow{AB}$ consists of the segment $\overline{AB}$ and all points C such that B lies between A and C.

Basic shapes Triangles, circles, etc.

Analytic Geometry as a *Model*

The axioms should feel familiar. Being shorter than Hilbert's list, and being built on familiar notions such as the real line, it is somewhat easier for us to understand what the axioms are saying and to visualize them. There is something to *prove* however; indeed the major point of Birkhoff's system!

Theorem 3.24. *Cartesian analytic geometry is a model of Birkhoff's axioms.*

Recall what this requires: we must provide a *definition* of each of the undefined terms and prove that these satisfy each of Birkhoff's axioms. Here are suitable definitions for Cartesian analytic geometry:

[20] As with Hilbert, Birkhoff makes SAS an *axiom*: Birkhoff's version is stronger, for it also applies to similar triangles

Point An ordered pair (x, y) of real numbers.

Distance $d(A, B) = \sqrt{(A_x - B_x)^2 + (A_y - B_y)^2}$

Line All points satisfying a linear equation $ax + by + c = 0$.

Angle Define column vectors as differences ($\mathbf{v} = P - O$ and $\mathbf{w} = Q - O$) and consider the matrix $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Now define angle via

$$\cos \measuredangle POQ = \frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|} \quad \text{where} \quad \measuredangle POQ \in \begin{cases} [0, \pi] & \iff \mathbf{w} \cdot J\mathbf{v} \geq 0 \\ (\pi, 2\pi) & \iff \mathbf{w} \cdot J\mathbf{v} < 0 \end{cases} \tag{$*$}$$

In essence, J is 'rotate counter-clockwise by $\frac{\pi}{2}$.' Cosine may be defined using power series, so no pre-existing geometric meaning is required.

Proof. (Euclidean axiom) If (x_1, y_1) and (x_2, y_2) satisfy $ax + by + c = 0$ then

$$a(x_1 - x_2) + b(y_1 - y_2) = 0,$$

whence $a = y_1 - y_2$, $b = x_2 - x_1$ up to scaling. It follows that the line has equation

$$(y_1 - y_2)x + (x_2 - x_1)y + x_1 y_2 - x_2 y_1 = 0$$

unique up to multiplication of all three of a, b, c by a non-zero constant.

The remaining axioms are exercises. ∎

Exercises 3.5. 1. Prove that the ruler axiom is satisfied:

(a) First show that if $P \neq Q$ lie on ℓ, then any point A on the line has the form

$$A = P + \frac{t_A}{d(P, Q)}(Q - P) \qquad \text{where } t_A \in \mathbb{R}$$

(b) Use this formula to verify that $d(A, B)^2 = (t_A - t_B)^2$.

2. Let $\mathbf{i} = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Given any non-zero point B, define $\mathbf{b} = B - O$ and let $\beta = \cos^{-1} \frac{\mathbf{i} \cdot \mathbf{b}}{|\mathbf{b}|}$ in accordance with $(*)$. This is a continuous function of $\mathbf{b}$.

   (a) If $\hat{B}$ is any other point on the same ray $\overrightarrow{OB}$, explain why we get the same value β. (β *is thus a continuous function of* B)

   (b) If $B = (x, y)$, what is are values of $\cos \beta$ and $\sin \beta$?

   (c) Suppose A corresponds to α under this identification. Evaluate $\cos(\beta - \alpha)$ and therefore prove that the protractor axiom is satisfied.

3. Use the cosine rule (Theorem 3.7) to prove that the SAS similarity axiom is satisfied.