

# Principles

*Notes on Statistical Mechanics for Molecular Simulation*

*Xufan Gao*

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西安交通大学

XI'AN JIAOTONG UNIVERSITY

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- 作者: Xufan Gao
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# Preface

Oxford Graduate Texts

Tuckerman, M. E. (2010). Statistical mechanics: Theory and molecular simulation.

Oxford: Oxford University Press.

As a freshman in college physics, there must be errors.

For non-essential examples, I won't number the equations.

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# Chapter 1 Classical Mechanics

## § 1.1 Math recap

Take the derivative with respect to vectors

## § 1.2 Newton's laws of motion

Review basic laws.

Newton's second law:

$$\mathbf{F} = m \frac{d^2 \mathbf{r}}{dt^2} = m \ddot{\mathbf{r}} \quad (1.1)$$

And Newton's third law:

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA} \quad (1.2)$$

The definition of work:

$$W_{AB} = \int_A^B \mathbf{F} \cdot d\mathbf{l} \quad (1.3)$$

In statistical mechanics, a particle  $i$  will experience a force  $\mathbf{F}_i(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N, \dot{\mathbf{r}}_i)$ , which is determined by the positions of all other particles and its velocity.

## § 1.3 Phase space: visualization

We use position and momentum in the phase space because

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \quad (1.4)$$

Phase space:  $3N$  position variables and  $3N$  momentum variables constitute the microscopic state of the system at time  $t$ , if the dimensionality is 3. These variables forms a so-called *phase space vector*  $x$ . And by solving the Newton's second law, we obtain a *trajectory*:

$$x_t = (\mathbf{r}_1(t), \dots, \mathbf{r}_N(t), \mathbf{p}_1(t), \dots, \mathbf{p}_N(t)) \quad (1.5)$$

We can visualize the trajectory for a single one-dimensional particle. Some examples are (not explained here):

- a free particle with momentum  $p$ .
- a harmonic oscillator, with an elliptical phase space.
- a particle crossing a "hill" potential

Visualizing a many-particle system is hard. Maybe we can consider a particular cut or surface representing a set of variables of interest, known as a *Poincaré section*.

## § 1.4 Lagrangian formulation of classical mechanics

### 1.4.1 Conservative forces

Conservative forces, means **energy conservation**. Defined as the negative gradient of  $U$ , the *potential energy function*, Or for each particle  $i$

$$\mathbf{F}_i = -\nabla_i U(\mathbf{r}_1, \dots, \mathbf{r}_N) \quad (1.6)$$

where  $\nabla = \frac{\partial}{\partial \mathbf{r}}$ .

An important property is that, the work done by conservative forces only depends on the difference of  $U$  of the start and end state, i.e. independent of the path taken.

$$W_{AB} = U_B - U_A$$

The kinetic energy of the system is given by

$$K(\dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2, \dots, \dot{\mathbf{r}}_N) = \frac{1}{2} \sum_{i=1}^N m_i \dot{\mathbf{r}}_i^2 \quad (1.7)$$

### 1.4.2 Formulation

Here we introduce the *Lagrangian* of a system

$$\mathcal{L} = K(\dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N) - U(\mathbf{r}_1, \dots, \mathbf{r}_N) \quad (1.8)$$

which is the difference between the kinetic energy and potential energy. In contrast, the total energy is the sum of the two:

$$E = K(\dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N) + U(\mathbf{r}_1, \dots, \mathbf{r}_N) \quad (1.9)$$

#### The Euler-Lagrangian equation

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_i} = \frac{\partial \mathcal{L}}{\partial \mathbf{r}_i} \quad (1.10)$$

For any one of position coordinates, taking the partial derivative with respect to the velocity and then time, is equal to that with respect to the position directly.

which can be verified with the substitution of eqn. 1.6 and 1.7 (poco).

This equation helps generate the equations of motion. Take the example of the one-dimensional harmonic oscillator, where

$$\begin{aligned} U(x) &= \frac{1}{2}kx^2 \\ K(\dot{x}) &= \frac{1}{2}m\dot{x}^2 \\ \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} &= \frac{d(m\dot{x})}{dt} = m\ddot{x} \\ \frac{\partial \mathcal{L}}{\partial x} &= -kx \end{aligned}$$

It also helps to verify conservation of the energy.

$$\begin{aligned} \frac{dE}{dt} &= \sum_{i=1}^N m\dot{\mathbf{r}}_i \ddot{\mathbf{r}}_i + \sum_{i=1}^N \frac{\partial U}{\partial \mathbf{r}_i} \frac{d\mathbf{r}_i}{dt} \\ &= \sum_{i=1}^N m\dot{\mathbf{r}}_i \ddot{\mathbf{r}}_i - \sum_{i=1}^N \mathbf{F}_i \dot{\mathbf{r}}_i \\ &= \sum_{i=1}^N m\dot{\mathbf{r}}_i \ddot{\mathbf{r}}_i - m\ddot{\mathbf{r}}_i \dot{\mathbf{r}}_i \\ &= 0 \end{aligned}$$

### 1.4.3 Generalized coordinates

The power of Lagrangian formulation lies in the fact that the equations in an arbitrary coordinate system can be derived easily in order to address a particular problem. A set of  $3N$  *generalized coordinates* are related to the original Cartesian coordinate via

$$q_\alpha = f_\alpha(\mathbf{r}_1, \dots, \mathbf{r}_N) \quad \alpha = 1, 2, \dots, 3N \quad (1.11)$$

It is assumed that the transformation has a unique inverse

$$\mathbf{r}_i = \mathbf{g}_i(q_1, \dots, q_{3N}) \quad i = 1, 2, \dots, N \quad (1.12)$$

Thus, through the chain rule, we have

$$\dot{\mathbf{r}}_i = \sum_{\alpha=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \dot{q}_\alpha \quad (1.13)$$

The kinetic energy can then be written as<sup>(1)</sup>

$$\begin{aligned}
 K(\mathbf{q}, \dot{\mathbf{q}}) &= \frac{1}{2} \sum_{i=1}^N m_i \sum_{\alpha=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \dot{q}_\alpha \sum_{\beta=1}^{3N} \frac{\partial \mathbf{r}_i}{\partial q_\beta} \dot{q}_\beta \\
 &= \frac{1}{2} \sum_{\alpha=1}^{3N} \sum_{\beta=1}^{3N} \sum_{i=1}^N m_i \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \frac{\partial \mathbf{r}_i}{\partial q_\beta} \dot{q}_\alpha \dot{q}_\beta \\
 &= \frac{1}{2} \sum_{\alpha=1}^{3N} \sum_{\beta=1}^{3N} G_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta
 \end{aligned} \tag{1.14}$$

where

$$G_{\alpha\beta}(\mathbf{q}) = \sum_{i=1}^N m_i \frac{\partial \mathbf{r}_i}{\partial q_\alpha} \cdot \frac{\partial \mathbf{r}_i}{\partial q_\beta} \tag{1.15}$$

is the function of  $\mathbf{q}$  (so is  $U(\mathbf{q})$ ).  $\alpha$  and  $\beta$  as indices, these elements forms a matrix  $G$ , the *mass metric matrix*.

Then the Lagrangian is expressed as a function of  $\mathbf{q}$  and  $\dot{\mathbf{q}}$ . Adopting the Euler-Lagrangian equation, considering  $q_\gamma$  as  $q_\alpha$ :

$$\frac{d}{dt} \left( \sum_{\beta=1}^{3N} G_{\gamma\beta} \dot{q}_\beta \right) = \sum_{\alpha=1}^{3N} \sum_{\beta=1}^{3N} \frac{\partial G_{\alpha\beta}}{\partial q_\gamma} \dot{q}_\alpha \dot{q}_\beta - \frac{\partial U}{\partial q_\gamma} \tag{1.16}$$

where  $\gamma = 1, \dots, 3N$ . These are the  $3N$  equations of motion of the system.

## 1.4.4 Example: two-particle system

A two particle system subject to a potential  $U$  which only depends on the distance between them. We can write:

$$\mathcal{L} = \frac{1}{2} m_1 \dot{\mathbf{r}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{r}}_2^2 - U(|\mathbf{r}_1 - \mathbf{r}_2|) \tag{1.17}$$

And knowing that (not important)

$$\begin{aligned}
 |\mathbf{r}_1 - \mathbf{r}_2| &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \\
 \frac{\partial |\mathbf{r}_1 - \mathbf{r}_2|}{\partial x_1} &= \frac{2x_1}{2\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}} = \frac{x_1}{|\mathbf{r}_1 - \mathbf{r}_2|} \\
 \frac{\partial U(|\mathbf{r}_1 - \mathbf{r}_2|)}{\partial \mathbf{r}_1} &= \frac{dU(|\mathbf{r}_1 - \mathbf{r}_2|)}{d(|\mathbf{r}_1 - \mathbf{r}_2|)} \frac{\partial |\mathbf{r}_1 - \mathbf{r}_2|}{\partial \mathbf{r}_1} = \frac{dU(|\mathbf{r}_1 - \mathbf{r}_2|)}{d(|\mathbf{r}_1 - \mathbf{r}_2|)} \frac{\mathbf{r}_1}{|\mathbf{r}_1 - \mathbf{r}_2|} \\
 \frac{\partial U(|\mathbf{r}_1 - \mathbf{r}_2|)}{\partial \mathbf{r}_2} &= - \frac{dU(|\mathbf{r}_1 - \mathbf{r}_2|)}{d(|\mathbf{r}_1 - \mathbf{r}_2|)} \frac{\mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}
 \end{aligned}$$

<sup>(1)</sup>separate everything containing  $q$  ( $i$ ) and the  $\dot{q}$  velocities, by summing over  $i$  first. Imagina a 3D cube, the z-axis is  $m_i$ , while on the x-y plane sits pairs of  $q$  terms...



,using eqn 1.10, we can get

$$\begin{aligned} m_1 \ddot{\mathbf{r}}_1 &= -\frac{dU(|\mathbf{r}_1 - \mathbf{r}_2|)}{d(|\mathbf{r}_1 - \mathbf{r}_2|)} \frac{\mathbf{r}_1}{|\mathbf{r}_1 - \mathbf{r}_2|} \\ m_2 \ddot{\mathbf{r}}_2 &= \frac{dU(|\mathbf{r}_1 - \mathbf{r}_2|)}{d(|\mathbf{r}_1 - \mathbf{r}_2|)} \frac{\mathbf{r}_1}{|\mathbf{r}_1 - \mathbf{r}_2|} \end{aligned}$$

We now want to introduce a more natural set of general coordinates: *center of mass* and *relative position*:

$$\begin{aligned} \mathbf{R} &= \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \\ \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 \end{aligned}$$

Let  $M = m_1 + m_2$ . The inverse of this transformation is

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{R} + \frac{m_2}{M} \mathbf{r} \\ \mathbf{r}_2 &= \mathbf{R} - \frac{m_1}{M} \mathbf{r} \end{aligned}$$

Substituted into eqn 1.17, the Lagrangian becomes

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} m_1 \left( \dot{\mathbf{R}} + \frac{m_2}{M} \dot{\mathbf{r}} \right)^2 + \frac{1}{2} m_2 \left( \dot{\mathbf{R}} - \frac{m_1}{M} \dot{\mathbf{r}} \right)^2 - U(|\mathbf{r}|) \\ &= \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \mu \dot{\mathbf{r}}^2 - U(|\mathbf{r}|) \end{aligned}$$

where *reduced mass*  $\mu = \frac{m_1 m_2}{M}$ .

Since the energy does not change as the center of mass moves,  $\frac{\partial \mathcal{L}}{\partial \mathbf{R}} = 0$ . Then the equation of motion will be

$$\begin{aligned} M \ddot{\mathbf{R}} &= 0 \\ \mu \ddot{\mathbf{r}} &= -\frac{dU}{d|\mathbf{r}|} \frac{\mathbf{r}}{|\mathbf{r}|} \end{aligned}$$

We can also transform  $\mathbf{r}$  further into spherical coordinates to obtain the one-dimensional equation of motion.

## § 1.5 Legendre transforms

Consider a derivable function  $f(x)$ . Define

$$s = f'(x) = g(x) \tag{1.18}$$

and assume  $s = g(x)$  is a one-to-one mapping (i.e.  $x = g^{-1}(s)$  exists).

For all  $x = x_0$ ,

$$f(x_0) = f'(x_0)x_0 + b(x_0)$$

holds, where  $b(x_0)$  is the  $y$ -intercept. Thus, for all  $x$

$$f(x) = f'(x)x + b(x) \quad (1.19)$$

We now express the relation in the function of  $s$ , i.e. find the equivalent expression. Let  $\tilde{f} = b$ , then

$$\tilde{f} = b(g^{-1}(s)) = f(x(s)) - sx(s) \quad (1.20)$$

This is called a *Legendre transform*.

For multiple variables  $x_1, \dots, x_n$ , there will be corresponding  $s_1, \dots, s_n$ , and the generalized form is

$$\tilde{f}(s_1, \dots, s_n) = f(x_1(s_1, \dots, s_n), \dots, x_n(s_1, \dots, s_n)) - \sum_{i=1}^n s_i x_i(s_1, \dots, s_n) \quad (1.21)$$

$f$  is constituted of  $n$  pairs of variable  $\times$  its derivative, plus a bias term  $\tilde{f}$ .

## § 1.6 Hamiltonian formulation

### 1.6.1 Notes

Before the derivation, something is worth noting.

An interesting fact is

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_i} = \frac{\partial}{\partial \dot{\mathbf{r}}_i} \left[ \sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 - U(\mathbf{r}_1, \dots, \mathbf{r}_N) \right] = m_i \dot{\mathbf{r}}_i = \mathbf{p}_i \quad (1.22)$$

$$K = \sum_{i=1}^N \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2} \sum_{i=1}^N \mathbf{p}_i \dot{\mathbf{r}}_i \quad (1.23)$$

$\dot{\mathbf{r}}_i$  is a function of  $\mathbf{p}_i$  ( $\dot{\mathbf{r}}_i(\mathbf{p}_i) = \mathbf{p}_i/m_i$ ;  $s_i$  is  $\mathbf{p}_i$ ;  $x_i$  is  $\dot{\mathbf{r}}_i$ ).

So we can define the *generalized momentum*  $p_\alpha$  as

$$p_\alpha = \frac{\partial \mathcal{L}}{\partial \dot{q}_\alpha} \quad (1.24)$$

$$K = \frac{1}{2} \sum_{\alpha=1}^{3N} p_\alpha \dot{q}_\alpha \quad (1.25)$$

for every set of generalized coordinates. These momenta are referred to as *conjugated* to the coordinates.  $\dot{q}_\alpha$  is a function of  $p_\alpha$ .

## 1.6.2 Formulation

The Hamiltonian is just the negative Legendre transformation of the Lagrangian.

The reason we introduce the Hamiltonian is to replace  $\dot{\mathbf{r}}_i$  in  $\mathcal{L}$  into  $\mathbf{p}_i$ . Do the Legendre transform with  $\mathbf{r}_i$  fixed:

$$\begin{aligned}\tilde{\mathcal{L}} &= \mathcal{L}(\mathbf{r}_1, \dots, \mathbf{r}_N, \dot{\mathbf{r}}_1(\mathbf{p}_1), \dots, \dot{\mathbf{r}}_N(\mathbf{p}_N)) - \sum_{i=1}^N \mathbf{p}_i \dot{\mathbf{r}}_i(\mathbf{p}_i) \\ &= \sum_{i=1}^N \frac{1}{2} \frac{\mathbf{p}_i^2}{m_i} - U(\mathbf{r}_1, \dots, \mathbf{r}_N) - \sum_{i=1}^N \frac{\mathbf{p}_i^2}{m_i} \\ &= - \left( \sum_{i=1}^N \frac{1}{2} \frac{\mathbf{p}_i^2}{m_i} + U(\mathbf{r}_1, \dots, \mathbf{r}_N) \right)\end{aligned}\quad (1.26)$$

Thus, the Hamiltonian is defined as

$$\mathcal{H}(\mathbf{r}_1, \dots, \mathbf{r}_N, \mathbf{p}_1, \dots, \mathbf{p}_N) = \sum_{i=1}^N \frac{1}{2} \frac{\mathbf{p}_i^2}{m_i} + U(\mathbf{r}_1, \dots, \mathbf{r}_N) \quad (1.27)$$

which is the **total energy** of the system.

## 1.6.3 Equations of motion

Recall eqn 1.14, our mass metric matrix  $G$  where

$$K(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \sum_{\alpha=1}^{3N} \sum_{\beta=1}^{3N} G_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta \quad (1.28)$$

Comparing with eqn. 1.25, we can infer that

$$\mathbf{p}_\alpha = \sum_{\beta=1}^{3N} G_{\alpha\beta} \dot{q}_\beta \quad (1.29)$$

In the form of matrix multiplication, it's  $\mathbf{p} = G\dot{\mathbf{q}}$ , or reversely

$$\dot{\mathbf{q}} = G^{-1}\mathbf{p} \quad (1.30)$$

where  $\mathbf{p} = (p_1, \dots, p_{3N})$  and  $\dot{\mathbf{q}} = (\dot{q}_1, \dots, \dot{q}_{3N})$ .

For any set of generalized coordinates, the Hamiltonian is given similarly:

$$\mathcal{H}(\mathbf{q}, \mathbf{p}) = K(\mathbf{p}, \dot{\mathbf{q}}(\mathbf{p})) + U(\mathbf{q}) \quad (1.31)$$

where  $K$  is only the function of  $\mathbf{p}$  ( $m_i$  as constants) and  $U$  is only the function of  $\mathbf{q}$ .

From eqn. 1.31, we can obtain by referring to eqn. 1.25 and 1.6:

$$\frac{\partial \mathcal{H}}{\partial p_\alpha} = \dot{q}_\alpha \quad \frac{\partial \mathcal{H}}{\partial q_\alpha} = -\dot{p}_\alpha \quad (1.32)$$

for  $\alpha = 1, \dots, 3N$ , which is known as *Hamiltonian's equations of motion*. You may verify it in the example of harmonic oscillation:

$$\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

$$m\ddot{x} = \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -kx$$

Notes on the Hamiltonian equations of motion:

- It's equivalent to Newton's second law of motion (eqn. 1.1)
- It's equivalent to the Lagrangian equations of motion (eqn. 1.16), which is a set of  $3N$  second-order DEs, while eqn. 1.32 is a set of  $6N$  first-order DEs.

## § 1.7 Summary of chapter

- 
- the Hamiltonian is a Legendre transform of the Lagrangian.

# Chapter 2 内容示例

## § 2.1 用户环境示例

### GRE 备考指南

<https://qyxf.site/latest/GRE备考指南-v2.0.pdf>

### 军事理论教程

<https://qyxf.site/latest/军事理论教程.pdf>

### 分析力学笔记

<https://qyxf.site/latest/分析力学笔记-v1.0.pdf>

### 大学物理题解

<https://qyxf.site/latest/大物题解（上）.pdf>

### 实变函数习题解答

<https://qyxf.site/latest/实变函数习题解答.pdf>

### 计算方法撷英

<https://qyxf.site/latest/计算方法撷英-v1.1.pdf>

### 计算机程序设计指南

<https://qyxf.site/latest/计算机设计程序指南.pdf>

### Equation 1 *Euler-Lagrangian equation*

$$\frac{d^2}{dt^2} \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_i} =$$

**定义 1** 极限就是超越自我。

**定理 1** 任何极限都可以直接观察得出。

**引理 2** 以上内容，纯属扯淡。

**推论 3** 这是一个推论。

注记

好好学习，天天向上。

警告

今天你学习了吗？

§ 2.2 列表样式

- 这是第一层
- 这也是第一层
  - 这是第二层
    - 这是第三层
- 1. 这是第一层
- 2. 这也是第一层
  - 1. 这是第二层
    - (1) 这是第三层

§ 2.3 正文示例

**微分学**（*differential calculus*）是微积分的一部分，是通过导数和微分来研究曲线斜率、加速度、最大值和最小值的一门学科，也是探讨特定数量变化速率的学科。微分学是微积分的两个主要分支之一，另一个分支则是**积分学**，探讨曲线下的面积。

Tab. 2.1 常用导数

原函数	导函数	原函数	导函数
$C$	0	$\ln x$	$\frac{1}{x}$
$x^\mu$	$\mu x^{\mu-1}$	$\sin x$	$\cos x$
$e^x$	$e^x$	$\cos x$	$-\sin x$

……几乎所有量化的学科中都有微分的应用。例如在物理学中，运动物体其位移对时间的导数即为其速度，速度对时间的导数就是加速度、物体动量对时间的导数即为物体所受的力，重新整理后可以得到牛顿第二运动定律  $F = ma$ 。化学反

应的化学反应速率也是导数。在运筹学中，会透过导数决定在运输或是设计上最有效率的做法。

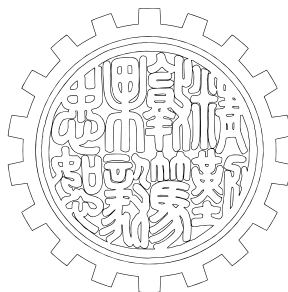


Fig. 2.1 V2 版本的封面图片

导数常用来找函数的极值。含有微分项的方程式称为**微分方程**，是自然现象描述的基础。微分以及其广义概念出现在许多数学领域中，例如复分析、泛函分析、微分几何、测度及抽象代数<sup>(1)</sup>。

## § 2.4 引导命令示例

**练习 1** 试用配方法求解方程：

$$ax^2 + bx + c = 0 \quad (2.1)$$

**解** 首先，方程左右两侧同除以  $a$ ，得到

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

根据一次项来配方，按公式  $(x + A)^2 = x^2 + 2Ax + A^2$  配出常数项：

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2 = 0$$

配方并移项得到

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

方程左右开方，得

$$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

<sup>(1)</sup>以上内容摘自维基百科中文词条 — 微分学：<https://zh.wikipedia.org/wiki/微分学>。

从而得到方程 (2.1) 之解为

$$x = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \quad (2.2)$$

该式即为一元二次方程的通用求根公式。

分析 在这一问题中，需要注意以下几点 [1, 2]:

- .....
- .....
- .....



# References

- [1] KNUTH D E. The  $\text{T}_{\text{E}}\text{X}$ book [M]. Addison-Wesley: Reading, 1986.
- [2] 刘海洋.  $\text{L}^{\text{A}}\text{T}_{\text{E}}\text{X}$  入门 [M]. 人民邮电出版社: 北京, 2013.