GATE Exam

CSE, ECE, IN, ME

Short Notes

2025, GATE

Contents

General Aptitude

This document contains short notes of the most expected topics from the aptitude section in the GATE Examination. Weightage of General Aptitude in GATE Exam is 15 marks. It also contains some concepts, formulas and tricks to solve the problems.

1.1 Quantitative Aptitude

Face Value - The Face Value of a digit in a numeral is its own value, at whatever place it may be Ex: 6843 face value of 6 is 6
Place Value - The Place Value of a digit 'd' in position n of a numeral is $d * 10^{n-1}$.
In numeral 1856, the place value of 8 is 8*100 = 800
Perfect Number - Sum of Factors of a number (except the number itself) equals to the given number. Ex: $6 = 1 + 2 + 3$
Irrational Number - Numbers expressed in decimal would be in non-terminating and non-repeating form, are called irrational numbers. Ex: $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \pi, e$
Co-Prime or Relative Prime - HCF between two numbers is 1.
Ex: (2, 3), (8, 9)
Twin Prime - Two prime numbers whose difference is 2.
Ex: (3, 5), (5, 7)

Note: 1 is Unique Number, neither prime nor composite.

1.1.1 Unit Digit

The last digit of a numeral is called Unit Digit.
Ex: Unit digit of 56516 is 6.
If a number K is raised to power 'n' and the unit digit of K is U, then Unit digit of K^n will only depend on U^n

1.1.2 Cyclicity of a Unit Digit

Case 1 - If U = 0, 1, 5, 6, then U^n = 0, 1, 5, 6.
Case 2 - If U = 4, 9 then cyclicity is 2

$$4^{even} = 6, 4^{odd} = 4, 9^{even} = 1, 9^{odd} = 9$$

Case 3 - If U = 2, 3, 7, 8, then cyclicity is 4

$$U^p = U^{p/4+k} = U^k = U_v$$

Example: Unit digit of 7^{786}

$$7^{786} = 7^{784/4+2} = 7^2 = 9$$

1.1.3 Divisibility Rule

Divisible by 2, if the unit digit is 0, 2, 4, 6, 8.
Divisible by 3/9, if the sum of digits divisible by 3/9.
Divisible by 4, if the last two digits divisible by 4.
Divisible by 5, if the unit digit is 0 or 5.
Divisible by 6, if it is divisible by 2 and 3.
Divisible by 7/13, divide the number into group of 3 digits find the difference between odd and even place, if the result is 0 or divisible by 7/13 then it is divisible.
Divisible by 8, if the last three digits divisible by 8.
Divisible by 10, if the unit digit is 0.
Divisible by 11, if the difference between sum of digits in odd place & sum of digits in even place is 0 or 11 then it is divisible.

1.1.4 Test for a Number to be Prime

Let P be the given Number and let n be the smallest counting number such that $n^2 \geq P$
Now, test whether p is divisible by any of the prime numbers less than or equal to n. If the given number is divisible it is not prime else prime.

1.1.5 Eucliden or Division Algorithm

If we divide a given number with another number, then

$$Dividend = Divisor * Quotient + Remainder$$

1.1.6 Factors Terminology

Let N be a real positive number, to find the total number of factors the number should be expressed in terms of prime number as given below.

$$N = 2^a * p_1^b * p_2^c * p_3^d \dots$$

Total Number of factors (TNF) = $(a+1)(b+1)(c+1)(d+1)$...
Total Number of odd factors (TNOF) = $(b+1)(c+1)(d+1)$...
Total Number of even factors (TNEF) = TNF - TNOF
Total Number of prime factors (TNPF) = $a+b+c+d+$...
Total Number of distinct prime factors = no of primes

1.1.7 HCF and LCM

For two numbers $LCM * HCF = N1 * N2$
If a Numbers or ratio of numbers are given for the LCM (N), then use $LCM(2k, 3k, 4k, 5k) = N$
Note K can be taken out.

1.1.8 Highest power of 'p' in n!

$$= \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + \lfloor \frac{n}{p^3} \rfloor + \dots \ till\ 0$$

In simple terms Adding products of Division table. If the number is not prime then reduce it to prime and find products for each primes.

1.1.9 Number of Zeros in n!

Number of Zeros can be represented by highest power of 10. The highest power will be always less for large number $2^m 5^n; n < m$. Use 5 Divison table.

1.1.10 Remainder Theorem

1. $R\left(\frac{n1 * n2 * n3 \ldots}{d}\right) = R\left(\frac{r1 * r2 * r3 \ldots}{d}\right)$
2. $R\left(\frac{a^n}{d}\right) = R\left(\frac{a * a * a \ldots}{d}\right) = R\left(\frac{r^n}{d}\right)$
3. $R\left(\frac{(a+1)^n}{a}\right) = 1$
4. $R\left(\frac{a^n}{a+1}\right) = \begin{cases} a, & \text{if n is odd} \\ 1, & \text{if n is even} \end{cases}$

Example: Find the Remainder of 16*24*33 when divided by 5.

$R\left(\frac{16 * 24 * 33}{5}\right) = R\left(\frac{1 * 4 * 3}{5}\right) = R(3)$

1.1.11 Arithemetic Progression (A.P)

If a difference between two terms of a series is d then it is defined as given below.

$a,\ a+d,\ a+2d,\ \ldots$

n^{th}term of series $T_n = a + (n-1)d$

Sum of n terms $S_n = \frac{n}{2}[a + T_n] = \frac{n}{2}[2a + (n-1)d]$

1.1.12 Geometric Progression (G.P)

If a difference between two terms of a series is r then it is defined as given below.

$a,\ ar,\ ar^2,\ ar^3,\ \ldots$

n^{th}term of series $T_n = ar^{n-1}$

Sum of n terms $S_n = a\left(\frac{1-r^n}{1-r}\right)\ or\ a\left(\frac{r^n-1}{r-1}\right)$

Sum of $\infty\ terms = \begin{cases} \frac{a}{1-r}, & |r| < 1 \\ \infty, & |r| \geq 1 \end{cases}$

1.1.13 Ratio

The ratio of two quantities a and b in same units, is the fraction $\frac{a}{b}$ and we write it as $a : b$. Multiplication or Divison of each term of a ratio by the same non-zero number does not affect the ratio.
Note: a is antecedent and b is consequent

1.1.14 Proportions

The equality of two ratios is called proportion. If $a : b = c : d,$ then $a : b :: c : d$
Here a and d are extermes, while b and c are mean terms

$$\text{Product of means} = \text{Product of Extermes}$$

$$b * c = a * d$$

1.1.15 Percentage Results

1. If a certain value of p increases by x% , then increased value of p = (100 + x)% of p.

2. If a certain value of p decreases by x% , then decreased value of p = (100 - x)% of p.

3. If the price of a commodity increases by R% then the reduction in consumption so as not to increase the expenditure is
$$\left[\frac{R}{100+R} * 100\right] \%$$

4. If the price of a commodity decreases by R% then the increase in consumption so as not to decrease the expenditure is
$$\left[\frac{R}{100-R} * 100\right] \%$$

5. Let the population of a town be P now and suppose it increases at the rate of R% per annum, then:

   1. Population after n years = $P\left(1+\frac{R}{100}\right)^n$

   2. Population n years ago = $\dfrac{P}{(1+\frac{R}{100})^n}$

6. Let the present value of a machine be P. Suppose it depreciates at the rate of R% per annum, then:

   1. Value of the machine after n years = $P\left(1-\frac{R}{100}\right)^n$

   2. Value of the machine n years ago = $\dfrac{P}{(1-\frac{R}{100})^n}$

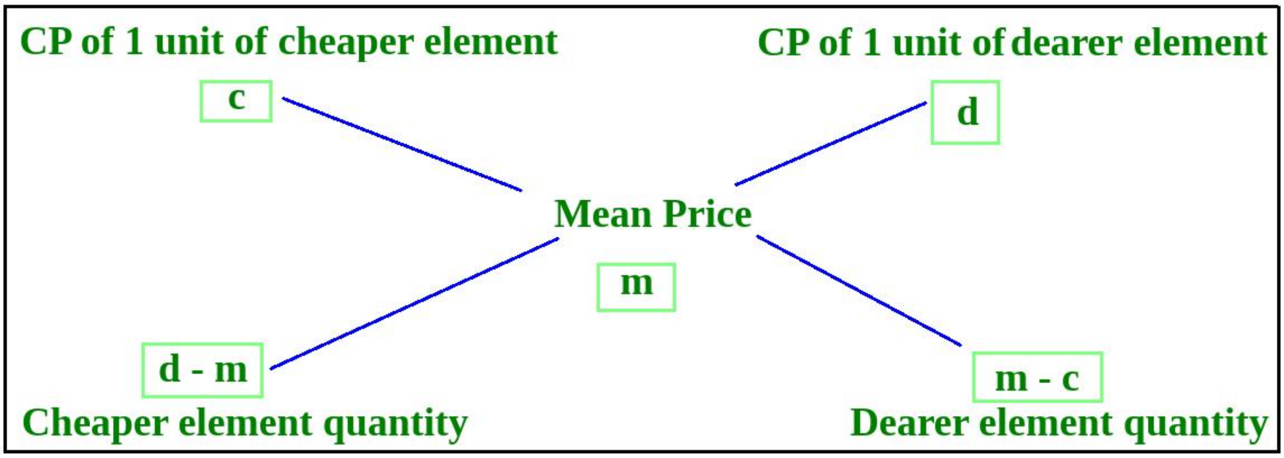
1.1.16 Mixtures and Allegations

Alligation - It is the rule that enables us to find the ratio in which two or more ingredients at the given price must be mixed to produce a mixture of a desired price.
Mean Price - The Cost Price(C.P) of a unit quantity of the mixture.
Rule of Alligation - If two ingredients are mixed, then

$$\left(\frac{\text{Quantity of cheaper}}{\text{Quantity of dearer}} = \frac{\text{(C.P of dearer) - (Mean price)}}{\text{(Mean price) - (C.P of cheaper)}}\right)$$

$$\text{(Cheaper Quantity)}:\text{(Dearer Quantity)} = (d - m) : (m - c)$$

Result - Suppose a container contains x units of liquid from which y units are taken out and replaced by water.
After n operations, the quantity of pure liquid = $\left[x\left(1 - \frac{y}{x}\right)^n\right]$ units.



1.1.17 Profit and Loss

If a product is sold, then the dealer obtains either gain or loss.

Gain = SP - CP	Loss = CP - SP
$Gain\% = \frac{SP-CP}{CP} * 100$	$Loss\% = \frac{CP-SP}{CP} * 100$
$S.P = \frac{100+Gain\%}{100} * CP$	$S.P = \frac{100-Loss\%}{100} * CP$
$C.P = \frac{100}{100+Gain\%} * SP$	$C.P = \frac{100}{100-Loss\%} * SP$

Selling a product with false value.

$$\%profit = \frac{True - False}{False} * 100$$

1.1.18 Partnership

When two or more than two persons run a business jointly, they are called partners and the deal is know as partnership.

Simple Partnership A simple parthership is the one in which the capitals of all the partners are invested for the same time. In this partnership, the gain or loss is distributed among the partners in the ratio of their investments.
Suppose A and B invest Rs.x and Rs.y respectively for a year in a business, then at the end of the year.

$$\text{(A's share of profit)}:\text{(B's share of profit)} = \text{x : y}$$

Compound Partnership A compound partnership is the one in which the capitals of the partners are invested for different time periods.
In this partnership the equivalent capitals are calculated for a unit of time by taking (capital x number of units of time). Now, gain or loss is divided in the ratio of these capitals.
Suppose A invests Rs.x for p months and B invest Rs.y for q months, then

(A's share of profit):(B's share of profit) = xp : yq

1.1.19 Simple and Compound Interest

$Simple\ Interest = \frac{Pnr}{100}$, p - Amount, n - time period, r - rate of interest

$Annual\ Compound\ Interest = P\left(1 + \frac{r}{100}\right)^n$

$Half - Yearly\ Compound\ Interest = P\left(1 + \frac{r/2}{100}\right)^{2n}$

$Quartely\ Compound\ Interest = P\left(1 + \frac{r/4}{100}\right)^{4n}$

Results

1. when interest is compounded annually but time is in fraction, $a\frac{b}{c}$

$$Compound\ Interest = P\left(1 + \frac{r}{100}\right)^a * \left(1 + \frac{r\frac{b}{c}}{100}\right)$$

2. when rates are different for different years, say $R_1\%, R_2\%, R_3\%$ for 1st, 2nd and 3rd year respectively.

$$Compound\ Interest = P\left(1 + \frac{R_1}{100}\right) * \left(1 + \frac{R_2}{100}\right) * \left(1 + \frac{R_3}{100}\right)$$

3. Present worth of Rs. x due in n years is given by:

$$Present\ Worth = \frac{x}{\left(1 + \frac{R}{100}\right)^n}$$

1.1.20 Time & Work

Efficiency - Work done in unit time $e = \frac{w}{t}$ If A takes N_1 days and B takes N_2 days to complete the same work, then their efficiency is $\frac{w}{N_1}$ and $\frac{w}{N_2}$.

$$Efficiency\ of\ (A + B) = \frac{w}{N_1} + \frac{w}{N_2} = w\frac{N_1 + N_2}{N_1 N_2}$$

$$Time\ taken\ by\ (A + B) = \frac{N_1 N_2}{N_1 + N_2}$$

$$Work = Men * Days * time * efficiency$$

Pipes and Cisterns - In pipes and cistern, if a pipe empty a tank then it is known as negative work.

1.1.21 Speed, Time & Distance

$$speed = \frac{distance}{time}; \quad 1\ km/hr = \frac{5}{18} m/s; \quad 1\ m/s = \frac{18}{5} m/s$$

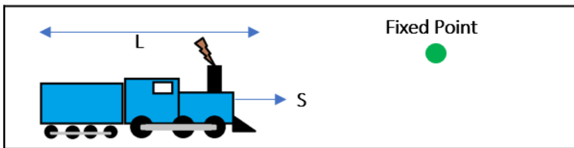
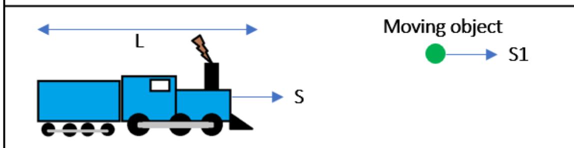
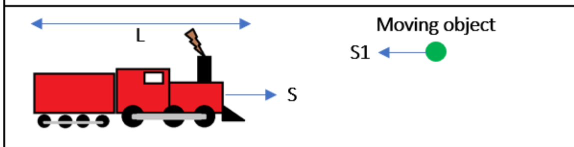
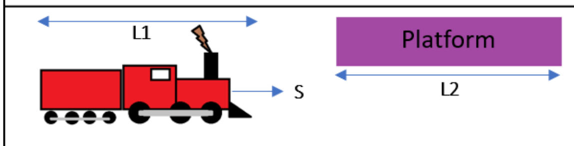
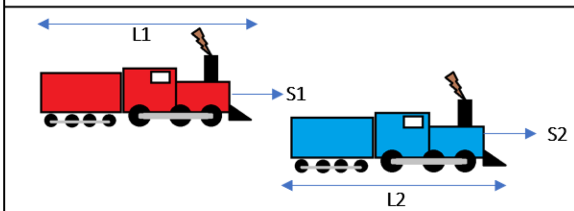
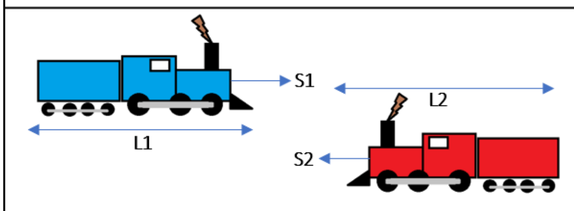
$$AverageSpeed = \frac{TotalDistance}{TotalTime} = \frac{2s_1s_2}{s_1+s_2}$$

Relative speed of A and B if they are travelling in same direction is
$R_s = S_a - S_b$
Relative speed of A and B if they are travelling in opposite direction is $R_s = S_a + S_b$
If a person is travelling is velocity s kmph for the time period of t. If he increases the speed by x to reduce the time taken as y, then the equation is given by

$$ys + xt = |x * y|$$

Problems on Trains

If two trains(or bodies) start at same time from point A and B towards each other and after crossing they take a and b hrs in reaching B and A respectively, then $A's\ Speed : B's\ Speed = \sqrt{b} : \sqrt{a}$

	Time = $\frac{L}{s}$
	Time = $\frac{L}{s-s1}$
	Time = $\frac{L}{s+s1}$
	Time = $\frac{L1+L2}{s}$
	Time = $\frac{L1+L2}{s1-s2}$
	Time = $\frac{L1+L2}{s1+s2}$

Problems on Boats

$$Upstream : S_u = S_b - S_w,\ Downstream : S_u = S_b + S_w,\ Stillwater : S_u = S_b$$

1.1.22 Permutations & Combinations

Permutation - The different arrangements of a given number of things by taking some or all at a time are called permutations.
Combination - Each of the different groups or selections which can be formed by taking some or all at a time, is called combination.

Results

1. Arrangement of 'k' items out of 'n' without repetition is given by $^{n}P_{k} = \frac{n!}{(n-k)!}$
2. Arrangement of 'n' items out of 'n' is given by $^{n}P_{n} = n!$
3. Arrangement of a word with length n and two repetitive letters of length a and b is $\frac{^{n}P_{n}}{^{a}P_{a} * {^{b}P_{b}}} = \frac{n!}{a! * b!}$
4. selection of 'k' things out of 'n' is given by $^{n}C_{k} = \frac{n!}{k!(n-k)!}$; $^{10}C_{3} = \frac{10*9*8}{1*2*3}$
5. Division of 'n' items among 'r' persons
   (a) If a person can receive zero (or more thing) $^{n+r-1}C_{r-1}$ $(0 \leq r \leq n)$
   (b) If a person can receive atleast one item $^{n-1}C_{r-1}$ $(1 \leq r \leq n)$

1.1.23 Probability

Random Experiment - An experiment in which all the possible outcomes (Sample Space) are known and the exact output cannot be predicted in advance, is called a random experiment.
Probability of Occurrence of an Event
Let S be the sample space and let E be an Event. Then,

$$P(E) = \frac{n(E)}{n(S)}$$

Results Binomial Probability Law
For a single event 'r' to happen in 'n' independent trials is given by

$$^{n}C_{r}(P(E))^{r}(P(\overline{E}))^{n-r}$$

Results

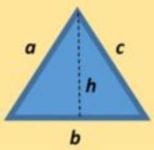
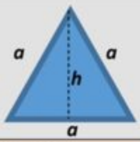
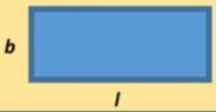
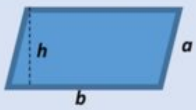
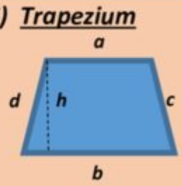
1. P(S) = 1
2. $0 \leq P(E) \leq 1$
3. For any events A and B, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
4. $P(\overline{A}) = 1 - P(A)$

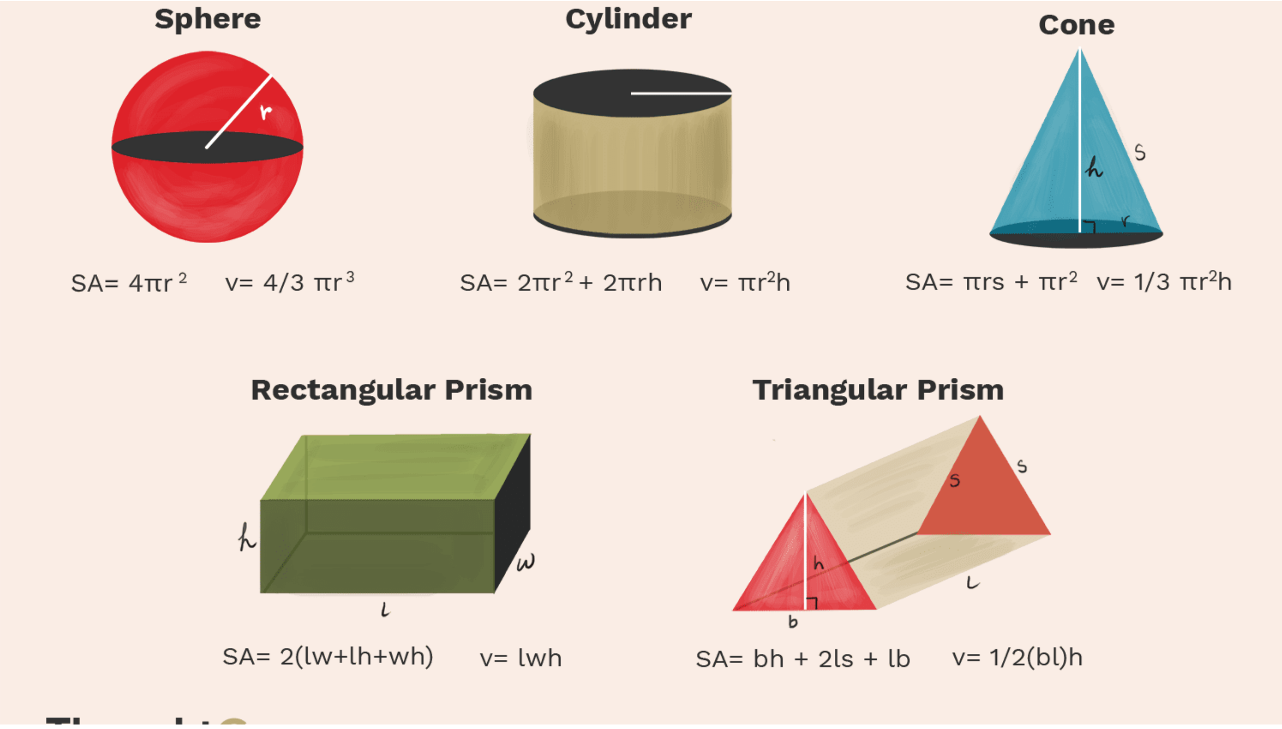
1.1.24 Mensuration & Geometry

Results

1. In cyclic quadrilateral, the sum of opposite angle is 180 degree.
2. the radial line to the tangent at point of contact is perpendicular to the tangent.
3. the perpendicular from the center of the circle to a chord bisects the chord.
4. Common pythagoras triplets are (3, 4, 5), (5, 12, 13), (8, 15, 17)

Area and Perimeter of 2-D Figures

Shape	Terms	Perimeter	Area
1) Triangle 	b = base of triangle h = height of triangle	Perimeter = a + b + c and (Semi perimeter)s = $\frac{a+b+c}{2}$	Area = $\frac{1}{2}$ x base x height Or Area = $\sqrt{s(s-a)(s-b)(s-c)}$
2) Equilateral Triangle 	a = length of sides of equilateral triangle	Perimeter = 3a	Area = $\frac{\sqrt{3}}{4}a^2$
3) Square 	a = length of side	Perimeter = 4a	Area = a^2
4) Rectangle 	l = length b = breadth	Perimeter = 2(l + b)	Area= l x b
5) Parallelogram 	b = base of parallelogram h = height of parallelogram	Perimeter = 2(a + b)	Area = b x h
6) Trapezium 	a , b = length of parallel sides h = distance between the parallel sides	Perimeter = a + b + c + d	Area = $\frac{1}{2}(a+b)$ x h
7) Circle 	r = radius d = diameter	Perimeter or circumference = $2\pi r$	Area = πr^2



1.1.25 Important Formulas

Lograthmic

$$log(mn) = log(m) + log(n) \qquad log\left(\frac{m}{n}\right) = log(m) - log(n) \qquad n\log(m) = log(m)^n$$

$$log_m(m) = 1 \qquad log(1) = 0 \qquad log_b a = \frac{log_{10}a}{log_{10}b}$$

$log(x)\ \forall\ x > 1;$ No of digits = characteristic + 1

$characteristic = \lfloor log(x) \rfloor$ mantissa = decimal

Example

$$log(0.36) = -0.44,\ \text{characteristic} = \lfloor -0.44 \rfloor = -1,\ mantissa = 1 - 0.44 = 0.56$$

$log(0.36) = \bar{1}.56$

$log(2.4) = 0.38,$ characteristic $= \lfloor 0.38 \rfloor = 0,$ $mantissa = 0.38$ No of digits = 1

Exponential $e^0 = 1;\ e^{-\infty} = 0;\ e^{\infty} = \infty$

Algebric

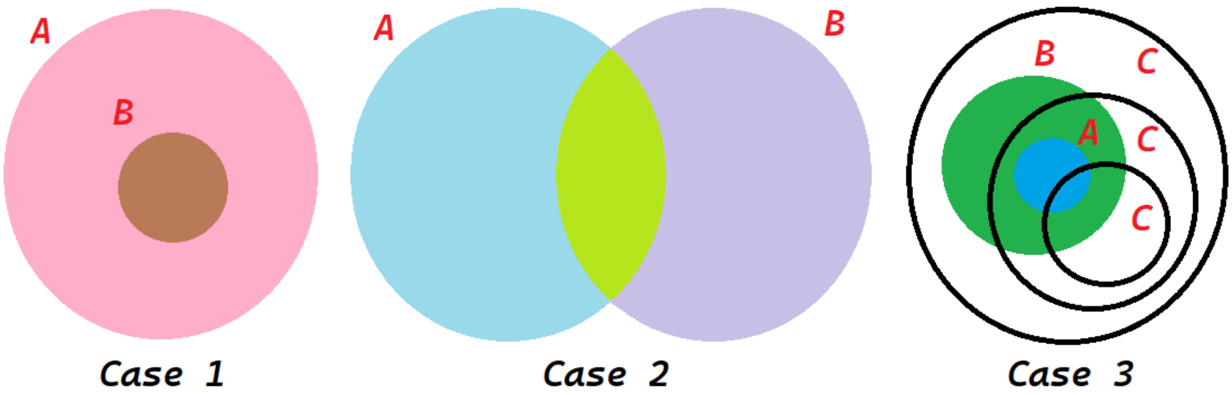
1.$(a+b)^3 = a^3 + b^3 + 3ab(a+b)$ 2.$a^3 + b^3 = (a+b)(a^2 + b^2 - ab)$
3.$(a-b)^3 = a^3 - b^3 - 3ab(a-b)$ 4.$a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$
5.$(a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$
6.$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca)$

1.2 Analytical Aptitude

1.2.1 Syllogism

If two or more statements and conclusions are given in the problem. Considering the given statements are true, we need to find whether the conclusion logically follows the statement.

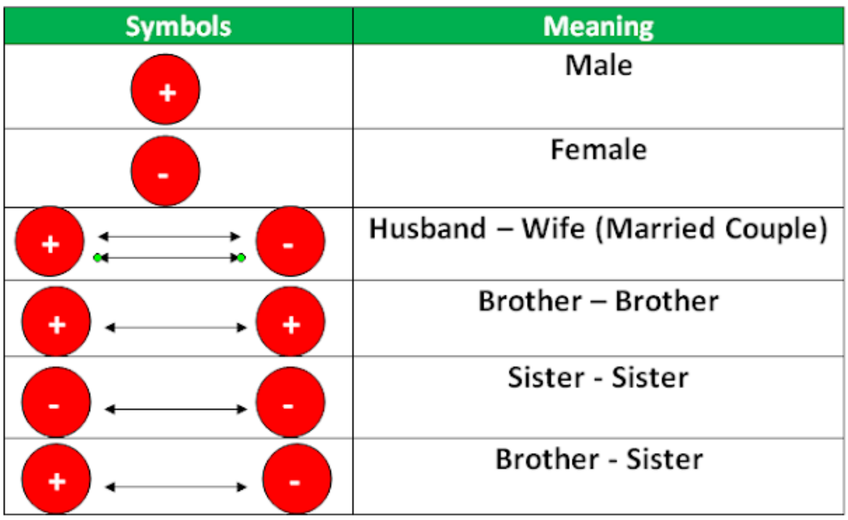
Venn Diagram Method
1. Statement: If All B's are A's. Conclusion: Some A's are B's.
2. Statement: Some B's are A's. Conclusion: Some A's are B's.
3. Statement: All A's are B's. Some C's are A's. Conclusion is true if it satisfy all the conditions.



1.2.2 Blood Relations

If the speaker points out a person and then explains the relation between the speaker and the person, then try to minimize the given statement to solve the problem else use family tree method. Refer the attached Images.

Family Tree Method



Symbols	Meaning
+	Male
-	Female
+ ⟷ -	Husband – Wife (Married Couple)
+ ⟷ +	Brother – Brother
- ⟷ -	Sister - Sister
+ ⟷ -	Brother - Sister

Type of Relationship	Relationship Name
Mother's or Father's son	Myself/Brother
Mother's or Father's daughter	Myself/Sister
Mother's or Father's brother	Uncle
Mother's or Father's sister	Aunt
Mother's or Father's father	Grandfather
Mother's or Father's mother	Grandmother
Son's Wife	Daughter-in-law
Daughter's husband	Son-in-law
Husband's or wife's sister	Sister-in-law
Husband's or wife's brother	Brother-in-law
Brother's son	Nephew
Brother's daughter	Niece
Uncle or Aunt's son or daughter	Cousin
Sister's Husband	Brother-in-law
Brother's Wife	Sister-in-law

1.2.3 Clocks and Calendars

Clock Time if the image of the clock is mirrored or water.

$Mirror\ time = 11:60 - hh:mm$

$$Water\ time = \begin{cases} |18:30 - hh:mm|, & \text{if minutes} \leq 30 \\ |18:90 - hh:mm|, & \text{if minutes} > 30 \end{cases}$$

Angle between the hour hand and minute hand is given by the formula.

$$\theta = |30H - \frac{11}{2}M|$$

If a clock is showing time behind the actual time at t1 and the clock is showing time ahead of the actual time at t2. Clock showed the correct time between t2 and t1. To find that sum up the total hours and find the rate at which the time changes.
Leap year - If the current year is century year, then it should be divisible by 400 else divisible by 4.

Find the day - If a day is given and we need to find the n'th day we use odd days method.

Odd days is number of days other than complete week.

$$Odd\ days = R\left[\frac{\text{Total No of days}}{7}\right]$$

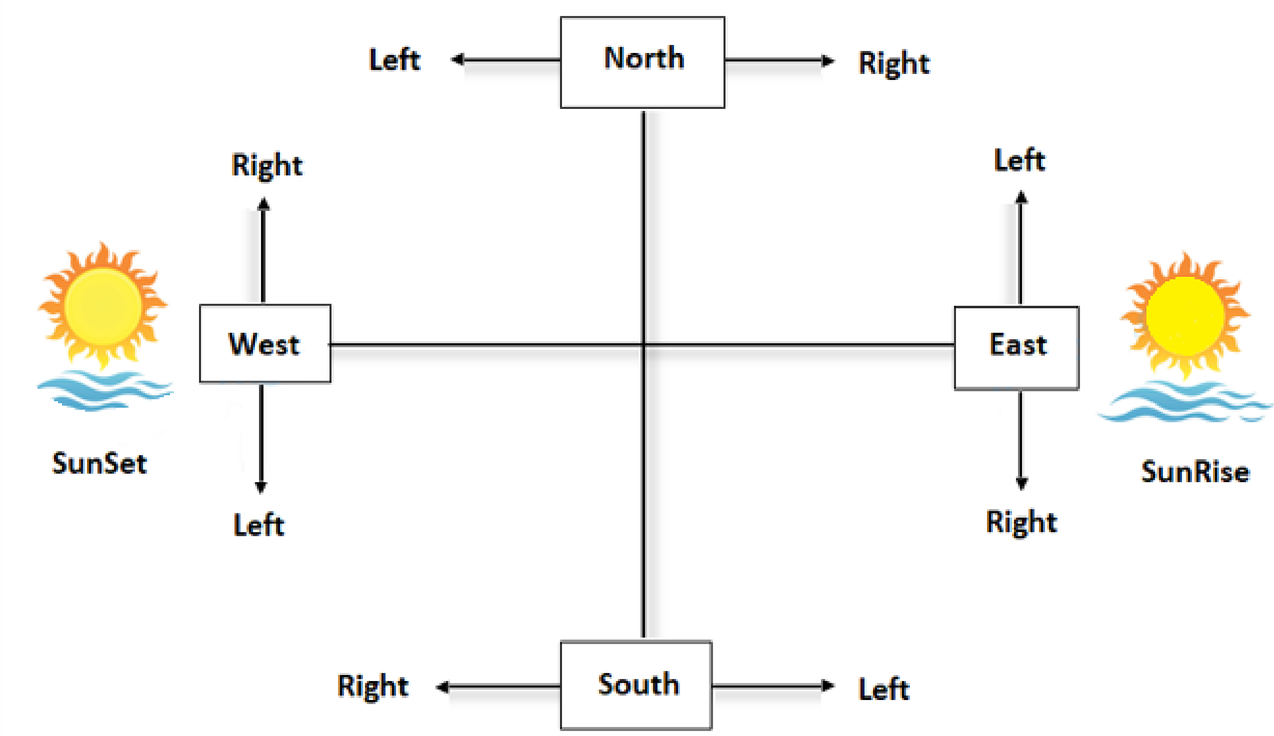
If Odd days is 0, then the day is the given day else add the odd days after the given day.

Calendar Matching

1. Add Odd days of the year from the given year till 7.
2. if the year next to that is leap and the actual year is not leap or viceversa, Repeat Step 1, else the next year is matching year.

1.2.4 Directions

Draw follow-up diagram to the question to solve it. Apply pythagoras theorem if neccessary.



1.3 Spatial Aptitude

Mirror & Water Image, Paper Cutting & Folding, Pattern Finding, Figure Completion, Rotation objects and Seating Arrangement are not included as it does not have concepts, formulas and tricks as they only needs practice to solve it.

1.3.1 Figure Counting

1. Number of squares in a square of "n" rows x "n" columns.

$$1^2 + 2^2 + 3^2 + \ldots\ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

2. Number of rectangles in a square of ”n” rows x ”n” columns.

$$1^3 + 2^3 + 3^3 + \ldots\ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

3. Number of squares in a rectangle of ”m” rows x ”n” columns

$$= mn + (m-1)(n-1) + (m-2)(n-2) + \ldots . till\ 0$$

4. Number of rectangles in a rectangle of ”m” rows x ”n” columns

$$= (1 + 2 + 3 + \ldots . + m)(1 + 2 + 3 + \ldots . + n)$$

1.3.2 Coding-Decoding

If a certain word is coded using a logic, then it can be used to code (or) decode another word. To remember a position of a letter quickly, Remember the word **EJOTY**.
Position of E,J,O,T,Y = **5,10,15,20,25**

1.4 Verbal Aptitude

Verbal Aptitude is all about standard written english, at first make sure you have understood parts of speech and the rules of it. This section is the vast area and requires constant practice to get marks. Choose the Appropriate word, Complete the sentence and Error Spotting requires us to understand the common rules of english grammer like preposition, verb and article usage.

1.4.1 Important Points to Note

- Conditionals
  1. If + Present,...........future
  2. If + Past,................would + verb
  3. If + Past perfect,....would + have + past participle
  4. If + were,................would + verb // Irrespective of previous rules
- superflucity - excessive usage of pronoun/noun
- Reading Comprehension - Do not answer from out of context
- Courses of Action - No Assumption/opinions should be considered

Engineering Mathematics

Engineering Mathematics is one of the key subject related to all other subjects. In GATE, this subject has a good weightage of around 13-14 marks. This document contains all the concepts, formulas and key notes for all the chapters in the syllabus of GATE 2023 Mechanical Engineering.
The most common type of questions from each unit is given in the below mentioned figure.

Topic	Important Topics	Common Model Questions
Linear Algebra	• Eigen Values and Vectors • Rank and Determinant of matrices • Systems of Linear Equations	• Find the Matrix for the given Eigenvalues and Eigenvectors • Properties of Eigenvalues for symmetric Matrices • Find the rank and determinant of the given matrix • Properties of Matrices based on Determinants, Rank, etc. • Find the solution for the given system of Linear Equations
Calculus	• Limit • Maxima and Minima • Gradient, Divergence and Curl	• Find the maximum and minimum values for the given functions • Simple questions on limit and continuity • Find the Divergence of the given vector field
Differential Calculus	• First-order equations(Linear and Nonlinear) • Cauchy's and Euler's equations	Find the solution of the given Differential Equation
Complex Analysis	• Analytic Functions • Cauchy-Riemann Equations • Taylor's Series	• Find the expression for one of u(x, y) and v(x, y) for the given analytical function f(z) = u(x, y) + iv(x, y) and also provided value of j with other expression • Integration of given complex function in either counterclockwise or anticlockwise direction
Numerical Methods	• Newton-Raphson Method • Integration by trapezoidal and Simpson's Rules	• Find the iteration value of the equation using Newton-Raphson method • Find the value of the given integral using Trapezoidal and Simpson's rules
Probability and Statistics	• Joint and Conditional Probability • Uniform, Normal, Exponential Distributions	• Finding the probability for coin-based problems, dice-based problems, etc. • Finding the probability using distributions
Transform Theory	Laplace Transformation	Find the Laplace Transform and Inverse Laplace transform of the given function.

Linear Algebra

2.1 Linear Algebra

2.1.1 Basic Matrix Operations

Let A be a matrix of size 3x3, A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

- Inverse of a Matrix: $A^{-1} = \frac{1}{|A|}adj(A)$
- $adj(A) = C^T$, C = $\begin{bmatrix} +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{33} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ +\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & +\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$
- LU Decomposition - given matrix A is reduced to Lower L and Upper U triangular matrix [A=LU]
  - Crout's method (Strict last column operation)
    Column-wise Elementary transformation (one element at a time) can be used to obtain lower triangular matrix L. The Constants used for deducing matrix L is used to create upper triangular matrix U by replacing the specific constant (K).
    A = LU = $\begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$ $C_i \rightarrow C_i + KC_j$
  - DooLittle's method (Strict last row operation)
    Row-wise Elementary transformation (one element at a time) can be used to obtain upper triangular matrix U. The Constants used for deducing matrix U is used to create lower triangular matrix L by replacing the specific constant (K).
    A = LU = $\begin{bmatrix} 1 & 0 \\ l_{21} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} \\ 0 & u_{22} \end{bmatrix}$ $R_i \rightarrow R_i + KR_j$

2.1.2 Rank of a Matrix

It is defined as the number of non-zero rows present after obtaining eucleon form of the given matrix. Rank of the matrix is evaluated using Elementary Transform method(Row or Column) or Minor method.

Properties

- For a null matrix '0' of any order rank(0) = 0
- For a non-zero matrix of order n*n; $1 \leq rank(A) \leq n$
- For a non-zero & singular matrix $1 \leq rank(A) < n$
- For a non-zero & non singular matrix $A_{n*n} rank(A) = n$
- For a non-zero rectangular matrix A of order m*n $1 \leq rank(A) \leq min(m, n)$
- Rank of a matrix A is equal to Rank of transpose of A $\rho(A) = \rho(A^T)$

$\rho(A_{n*n})$	$\rho(adj(A))$
n	n
n-1	1
<n-1	0

2.1.3 System of Linear Equation

Consider the following system of linear equation with three variables.

$$a_{11}x + a_{12}y + a_{13}z = b1; a_{21}x + a_{22}y + a_{23}z = b2; a_{31}x + a_{32}y + a_{33}z = b3;$$

The system of linear equation can be represented as matrices **AX = B** as mentioned below.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b1 \\ b2 \\ b3 \end{bmatrix}$$

In this matrix A is co-efficient matrix, X is variable co-efficient matrix and B is constant column matrix. If B = 0, then it is known as homogeneous system and If $B \neq 0$, then the given equations form non-homogeneous system.

Homogeneous System [AX=0]

- Trivial Solution(Zero Solution) if $|A| \neq 0$
- Non-Trivial Solution(many Solution) if $|A| = 0$
- Number of Independent Solution = no of variables(n) - rank of matrix(p)

Non-homogeneous System [AX=B]

Consider a **2x2 Matrix** with the system of linear equations.

$$a_1x + b_1y = c_1;\ \ a_2x + b_2y = c_2$$

The solution for these equation are given by

- if $\frac{a1}{a2} \neq \frac{b1}{b2}$ system is consistent and have a unique solution.
- if $\frac{a1}{a2} = \frac{b1}{b2} = \frac{c1}{c2}$ system is consistent and have infinitely many solution.
- if $\frac{a1}{a2} = \frac{b1}{b2} \neq \frac{c1}{c2}$ system is inconsistent and have no solution.

Consider a **3x3 Matrix** with the system of linear equations.

$$a_{11}x + a_{12}y + a_{13}z = b1; a_{21}x + a_{22}y + a_{23}z = b2; a_{31}x + a_{32}y + a_{33}z = b3;$$

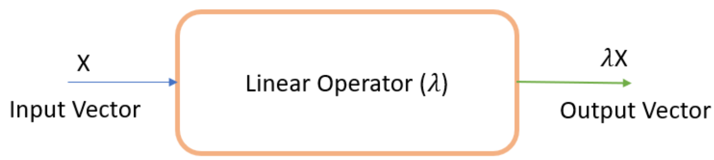
Then Matrix C is given by C = [A:B] C = $\begin{bmatrix} a_{11} & a_{12} & a_{13} & b1 \\ a_{21} & a_{22} & a_{23} & b2 \\ a_{31} & a_{32} & a_{33} & b3 \end{bmatrix}$

$\rho(A) = p;\ \ \rho(c) = r;$ n - no of variables

The solution for these equations are given by

- if $r = p = N$, then the system is consistent and has unique solution.
- if $r = p < N$, then the system is consistent and has infinetly many solution.
- if $r \neq p$, then the system is inconsistent and has no solution.

2.1.4 Eigen Values & Eigen Vectors



$$AX = \lambda X \Rightarrow AX - \lambda X = 0 \qquad (A - \lambda)X = 0 \Rightarrow BX = 0 \text{ [Homogenous]}$$

Characteristic Equation for Matrix A is given as
For 2x2 Matrix, $\lambda^2 - \beta_1\lambda + |A| = 0$
For 3x3 Matrix, $\lambda^3 - \beta_1\lambda^2 + \beta_2\lambda - |A| = 0$
$\beta_1 \rightarrow$ sum of the diagnol elements(trace); $\beta_2 \rightarrow$ Sum of minors of diagnol elements

Algebric Multiplicity (AM) of an eigen value
If "λ" is repeated "m" times, then AM = m

Geometric Multiplicity (GM) of an eigen value
If "λ" has "p" linear independent eigen vectors then GM=p
$GM\ of\ \lambda = n - p(A - \lambda I) = n - r$

Properties of Eigen Values and Eigen Vectors

1. sum of Eigen Values = trace of the matrix
2. product of Eigen Values = $|A|$
3. A and A^T will have same eigen values
4. The rank of the matrix is equal to the number of non-zero eigen values of the matrix(except triangular matrix)
5. For a triangular matrix, the eigen values are equal to diagnol elements.
6. Relation between Eigen Value and Eigen Vector for different form of the given matrix.

Matrix	**Eigen Value**	**Eigen Vector**
A	λ	X
A^n	λ^n	X
A+I	$\lambda + I$	X
KA	$K\lambda$	X

7. If the sum of elements of every row/column is same then one of the eigen value of the matrix is equal to the sum.
8. For a real matrix, if $\alpha+i\beta$ is an eigen value then $\alpha-i\beta$ will also be an eigen value[conjugate pair].
9. If a eigen value is $\alpha + i\beta$ for a complex matrix, then there is no conjugate pair.
10. The eigen values of a real symmetric matrix $[A = A^T]$ are real.
11. The eigen values of a real skew-symmetric matrix $[A = -A^T]$ are either zero (or) purely imaginary.
12. The eigen vector corresponding to the distinct eigen values of a real symmteric matrix will be orthogonal to each other.
13. The eigen values of a hermition matrix are always real.
14. The eigen values of a unitary matrix (or) real orthogonal matrix satisfies $|\lambda| = 1$
15. The number of linearly independent eigen vectors of a matrix are equal to the number of distinct eigen values of the matrix.

Cayley Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

$$\lambda^2 - (trace)\lambda + |A| = 0 \;\Rightarrow A^2 - (trace)A + |A| = 0$$

2.2 Trignometric Relations

All the basic and important trignometric relations are listed in this section.

2.2.1 Basics

Comman Angles of sin, cos and tan Reciprocal Functions

Degrees	$0°$	$30°$	$45°$	$60°$	$90°$
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

$$csc\ x = \frac{1}{sin\ x} \qquad sec\ x = \frac{1}{cos\ x} \qquad cot\ x = \frac{1}{tan\ x}$$

Even/Odd Relation

$$sin\ (-x) = -sin\ x \qquad cos\ (-x) = cos\ x \qquad tan\ (-x) = -tan\ x$$

2.2.2 Identities

Pythagorous Identities

$$sin^2x + cos^2x = 1 \qquad 1 + cot^2x = csc^2x \qquad 1 + tan^2x = sec^2x$$

Cofunction Identities

$$\sin(\frac{\pi}{2} - x) = \cos x \qquad \cos(\frac{\pi}{2} - x) = \sin x \qquad \tan(\frac{\pi}{2} - x) = \cot x$$
$$\cot(\frac{\pi}{2} - x) = \tan x \qquad \sec(\frac{\pi}{2} - x) = \csc x \qquad \csc(\frac{\pi}{2} - x) = \sec x$$

2.2.3 Angular Relations

Sum and Difference of Angles

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \qquad \sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y \qquad \cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \qquad \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Dobule Angles

$$\sin(2x) = 2\sin x\cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$
$$\tan(2x) = \frac{2\tan x}{1-\tan^2 x}$$

Half Angles

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} \qquad \cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}$$
$$\tan\frac{x}{2} = \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$$

2.2.4 sum, Power and Products Relations

Power Reducing Formulas

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2} \qquad \tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$$

Product to sum

$$\sin x\sin y = \frac{1}{2}\big[\cos(x-y) - \cos(x+y)\big]$$
$$\cos x\cos y = \frac{1}{2}\big[\cos(x-y) + \cos(x+y)\big]$$
$$\sin x\cos y = \frac{1}{2}\big[\sin(x+y) + \sin(x-y)\big]$$
$$\tan x\tan y = \frac{\tan x + \tan y}{\cot x + \cot y}$$
$$\tan x\cot y = \frac{\tan x + \cot y}{\cot x + \tan y}$$

Sum to Product

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\tan x + \tan y = \frac{\sin(x+y)}{\cos x\cos y}$$
$$\tan x - \tan y = \frac{\sin(x-y)}{\cos x\cos y}$$

2.3 Differentiation Formulas

1. $\frac{d}{dx}k = 0$
2. $\frac{d}{dx}f(g(x)) = f'(g(x)).g'(x)$
3. $\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g'(x)f(x)$
4. $\frac{d}{dx}x^n = nx^{n-1}$
5. $\frac{d}{dx}e^x = e^x$
6. $\frac{d}{dx}sin\ x = cos\ x$
7. $\frac{d}{dx}tan\ x = sec^2\ x$
8. $\frac{d}{dx}sec\ x = sec\ x\ tan\ x$
9. $\frac{d}{dx}\ sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$
10. $\frac{d}{dx}\ tan^{-1}x = \frac{1}{\sqrt{x^2+1}}$
11. $\frac{d}{dx}\ sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$
12. $\frac{d}{dx}[k(f(x)] = k.f'(x)$
13. $\frac{d}{dx} = [f(x) \pm g(x)] = f'(x) \pm g'(x)$
14. $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$
15. $\frac{d}{dx}\ ln|x| = \frac{1}{x};\ \frac{d}{dx}\ log_a|x| = \frac{1}{xlna}$
16. $\frac{d}{dx}a^x = a^x\ ln\ a$
17. $\frac{d}{dx}cos\ x = -sin\ x$
18. $\frac{d}{dx}cot\ x = -csc^2\ x$
19. $\frac{d}{dx}csc\ x = -csc\ x\ cot\ x$
20. $\frac{d}{dx}\ cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\ cot^{-1}x = \frac{-1}{\sqrt{x^2+1}}$
22. $\frac{d}{dx}\ csc^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}$

2.4 Integration Formulas

1. $\int\ dx = x + C$
2. $\int \frac{dx}{x} = ln|x| + C$
3. $\int e^x\ dx = e^x + C$
4. $\int \frac{f'(x)}{f(x)}\ dx = ln(f(x)) + C$
5. $\int sin\ x\ dx = -cos\ x + C$
6. $\int tan\ x\ dx = -ln|cos\ x| = ln|sec\ x|\ + C$
7. $\int sec\ x\ dx = ln|sec\ x + tan\ x| + C$
8. $\int sec\ x\ tan\ x\ dx = sec\ x + C$
9. $\int \frac{dx}{\sqrt{a^2-x^2}} = sin^{-1}\frac{x}{a} + C$
10. $\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}sec^{-1}\frac{|x|}{a} + C$
11. $\int x^n\ dx = \frac{x^{n+1}}{n+1} + C$
12. $\int a^x\ dx = \frac{1}{ln\ a}a^x + C$
13. $\int ln\ x\ dx = x\ ln\ x - x + C$
14. $\int \frac{f'(x)}{\sqrt{f(x)}}\ dx = 2\sqrt{f(x)} + C$
15. $\int cos\ x\ dx = sin\ x + C$
16. $\int cot\ x\ dx = ln|sin\ x| = -ln|csc\ x|\ + C$
17. $\int csc\ x\ dx = -ln|csc\ x + cot\ x| + C$
18. $\int csc\ x\ cot\ x\ dx = -csc\ x + C$
19. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}tan^{-1}\frac{x}{a} + C$
20. $\int_0^\infty cos^2x\ dx = \int_0^\infty sin^2x\ dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

Integration by Parts $\int u\ dv = uv_1 - u' \int v_2$ where v^n stands for integrating the given function n times. [priority for u - ILATE]

Calculus

2.5 Calculus

2.5.1 Limit, Continuity & Differentiability

Conditions for Continuvity

f(x) is continuous at point x = a, if

1. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$ (LHL=RHL)
2. f(a) is finite
3. $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = f(a)$

Conditions for Differentiability

f(x) is differentiable at point x=a, if

1. f(x) should be continuous
2. $\lim_{h\to 0^-} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0^+} \frac{(x+h)-f(x)}{h}$ (LHD=RHD)
3. f(x) should not have vertical tangent at x=0

Key Points

1. The derivative at any point on the curve given the value of the slope of the tangent to the curve at that point.
2. $sinx$, $cosx$, e^x and polynomial functions are always continuous & differentiable.
3. lnx is continuous and differentiable $\forall x > 0$
4. If f(x) and g(x) are continuous and differentiable functions then the following function will also be continuous and differentiable. $i) f(x) \pm g(x)$ $ii) f(x)g(x)$ $iii) \frac{f(x)}{g(x)}, g(x) \neq 0$
5. sharp (or) edge (or) break point is the point in which the nature of the curve changes(sudden change in slope). If a function is continuous at a point but RHD $\neq$ LHD, then we get a sharp point.
6. $|x - a|$ is continuous but not differentiable at x = a.

Applying Limit to f(x)

$$\lim_{x \to a} f(x) = k$$

L'Hospital Rule if Applying a limit in f(x) results $\frac{0}{0}$ *or* $\frac{\infty}{\infty}$, then differentiate numerator and denominator, continue differntiating and applying limits till the results not infinite.
If Applying Limits in f(x) results in $\infty - \infty$, then substitution method is used. Choose a substitution value in such a way a limit is not infinity.

2.5.2 Mean Value Theorems

Lagrange's Mean Value Theorem

If f(x) be a real valued function such that
1. It is continuous in [a, b] 2. It is differentiable in (a, b)
Then there exists a point $x = C \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Rolle's Theorem

If f(x) be a real valued function such that
1. It is continuous in [a, b] 2. It is differentiable in (a, b) 3. f(a) = f(b)
Then there exists a point $x = C \in (a, b)$ such that

$$f'(c) = 0$$

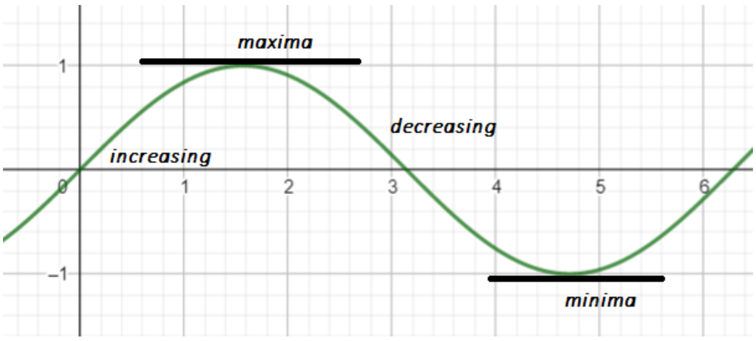
Cauchy's Mean Value Theorem

If f(x) and g(x) be two real valued functions such that
1. It is continuous in [a, b] 2. differntiable in (a, b)
Then there exists a point $x = C \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Increasing and Decreasing Functions

A function is said to be increasing if $f'(x) > 0$.
A function is said to be decreasing if $f'(x) < 0$.



2.5.3 Maxima and Minima

For any function f(x) the tangent to local maxima and local minima is always parallel to x-axis, then f'(x) = 0 gives the stationary points.
Determine the maxima and minima for the given function using the below methods.
1st Derivative Test

1. Calculate the stationary point using the equation f'(x) = 0
2. If the point around the 1st derivative changes its sign from +ve to -ve then that stationary point is a point of local maxima.
3. If the point around the 1st derivative changes its sign from -ve to +ve then that stationary point is a point of local minima.

2nd Derivative Test

1. Calculate the stationary point by the equation f'(x) = 0
2. let x=x0 be the stationary point.

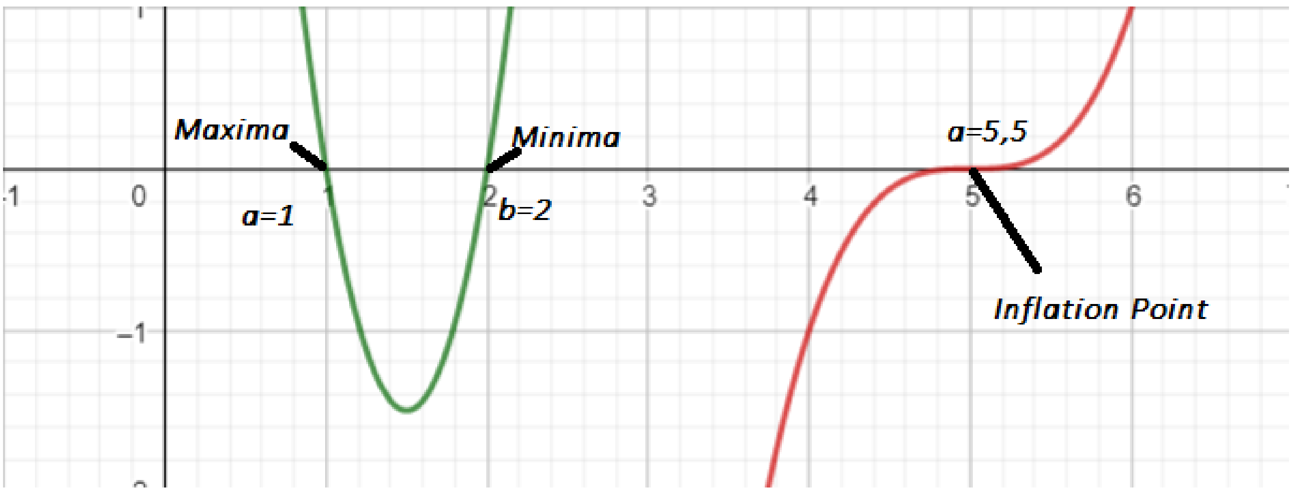
$$f''(x)_{x=x_0} = \begin{cases} +ve, & minima \\ 0, & gofurther \\ -ve, & maxima \end{cases} \qquad f^{iv}(x)_{x=x_0} = \begin{cases} +ve, & minima \\ 0, & gofurther \\ -ve, & maxima \end{cases}$$

$$f'''(x)_{x=x_0} = \begin{cases} +ve, & \text{neither maximum or minimum} \\ 0, & \text{go further} \\ -ve, & \text{neither maximum or minimum} \end{cases}$$

Inflation Point - stationary point at which the curve will be increasing/decreasing before and after stationary points then it known as point of inflation.

Wavy Curve Method

Easy method to find maxima and minima but only applicable to f'(x) having polynomial inequalities.
Wavy Curve always start from right at positive and then change sign for every different roots.



Global Maxima and Minima

For a function f(x) in [a, b] global maxima and minima can be determined by

1. Find stationary points f'(x) = 0
2. find the values of $f(a), f(b), f(x_n)$ and then compare the values to get maxima and minima.

Important Note:

1. If interval not mentioned then find local maxima and local minima else find global maxima and global minima.
2. No of Stationary Point α No of Maxima and minima.
3. If No of stationary point is 1 then local maxima/minima is global maxima/minima.

2.5.4 Partial Differentiation

1. For a function of single independent variable, simple or ordinary derivative will exist.
$$y = f(x) = \frac{dy}{dx}$$
2. For a function of more than one independent variable partial derivative will exist.
$$u = f(x, y),\ \frac{\partial u}{\partial x},\ \frac{\partial u}{\partial y}$$
3. The number of partial derivatives existing for a functional relation is exactly equal to the number of independent variables.
4. The procedure to evaluate partial derivatives is same as that of the simple derivatives but the only difference is that all other independent variable will remain constant.
5. Dependent variable should never be assumed constant.

Conditions for Maxima and Minima for Paritial Derivaties

Let f(x, y) be any two functions of variables x and y, then

$$r = \frac{\partial^2 f}{\partial x^2},\quad s = \frac{\partial^2 f}{\partial x \partial y},\quad t = \frac{\partial^2 f}{\partial y^2}$$

1. if $rt - s^2 > 0;\ r < 0,\ t < 0$, then the point is local maxima
2. if $rt - s^2 > 0;\ r > 0,\ t > 0$, then the point is local minima
3. if $rt - s^2 < 0$; Neither Maxima or Minima
4. if $rt - s^2 = 0;\ r < 0,\ t < 0$, No conclusion further investigation required.

Homogeneous Function
A function having same degree in all its terms is called Homogeneous function. A homogeneous function can be expressed either as a function of 'y/x' or as a function of "x/y".

$$f(x, y) = x^n F(x/y)\ (or)\ x^n F(y/x)$$

Euler's Theorem for Homogeneous function

Corallory 0: If u=f(x,y) be a homogeneous function of degree n.

1. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$ 2. $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = n(n-1)u$

Corallory 1: If u is not a homogeneous function but f(u) is a homogeneous function of degree n.

1. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{nf(u)}{f'(u)} = g(u)$ 2. $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = g(u)[g'(u) - 1]$

Corallory 2: If u=f(x,y) + g(x,y) + h(x,y) where f, g, h are homogeneous function of degree m, n, p respectively, then.

1. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = mf + ng + ph$ 2. $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = m(m-1)f + n(n-1)g + p(p-1)h$

2.5.5 Total Derivatives

1. In a composite relation, the derivative of an initial variable w.r.t final variable is called as Total derivative.
2. The number of total derivatives existing for a composite relation is exactly equal to the number of final variables.
3. The number of terms present in the expression of a total derivative is exactly equal to the number of intermediate variables.

$$w \to x, y \to t \quad \frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Taylor Series

If f(X) has all its derivative as finite and contiuous $\forall x$

$$f(x) = f(a) + hf'(a) + h^2 f''(a) + h^3 f'''(a) + \ldots\ldots\infty$$

If f(x,y) has all its derivative as finite and continuous at all point(x,y)

$$f(x+h, y+k) = f(x,y) + [hf_x + kf_y] + \frac{1}{2!}[h^2 f_{xx} + k^2 f_{yy} + 2hkf_{xy}] + \frac{1}{3!}[h^3 f_{xxx} + k^3 f_{yyy} + 3hk^2 f_{xyy} + 3h^2 k f_{xxy}] + \ldots$$

2.5.6 Gamma Function

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx \qquad \frac{\Gamma n}{k^n} = \int_0^\infty e^{-kx} x^{n-1}\, dx$$

$$\Gamma\frac{1}{2} = \sqrt{\pi} \qquad \Gamma n\ \Gamma(1-n) = \frac{\pi}{sin\ n\pi} \qquad \Gamma(n+1) = \begin{cases} n\Gamma n & \text{'n' is fraction} \\ n! & \text{'n' is integer} \end{cases}$$

2.5.7 Beta Function

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad \beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\beta(m,n) = \beta(n,m) \qquad \beta(m,n) = \frac{\Gamma m\ \Gamma n}{\Gamma(m+n)} \qquad \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right) = \int_0^{\pi/2} sin^p\theta\ cos^q\theta\,d\theta$$

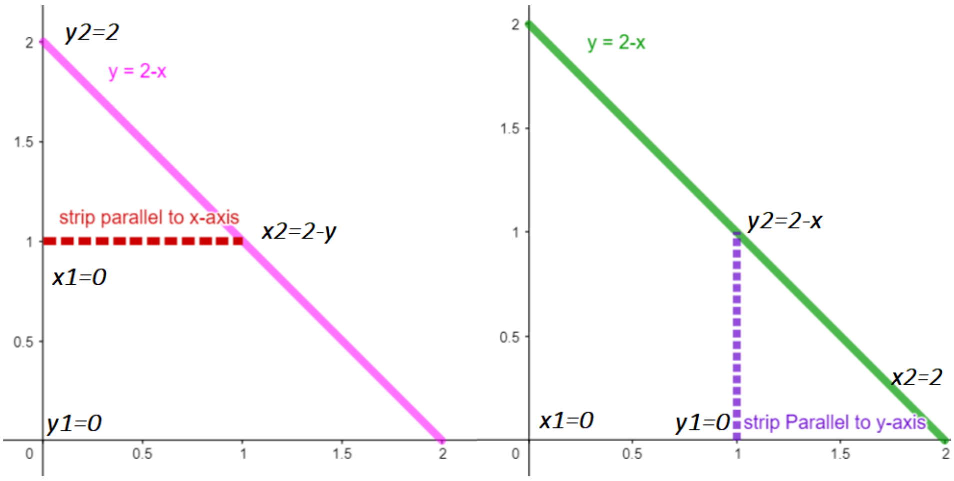
Points to Remember

1. $\int_0^a f(x)\,dx = \begin{cases} 0 & \text{if f(a-x)=-f(x)} \\ 2\int_0^{a/2} f(x)\,dx & \text{if f(a-x)=f(x)} \end{cases}$ 2. $\int_{-a}^a f(x)\,dx = \begin{cases} 0 & \text{if f(x) is odd} \\ 2\int_0^a f(x)\,dx & \text{if f(x) is even} \end{cases}$

2.5.8 Double and Triple Integrals

In Double and Triple Integrals, we need to find the limits of the function with respect to x, y and z-axis based on the given equation. To do that use the given equations in the problem.
1. Find the intersection point and plot the graph.
2. Draw a strip parallel to x-axis or y-axis based on the integration order.
3. Now, use sliding approach on the parallel axis to obtain limits for the constant set and find the limits for variable set using the strip and the equation of the given curve.

0	a	Constant Set	$\|\|el\ to\ y-axis$
0	x	Variable Set	First Priority



2.5.9 Fourier Series

Periodic Function - A function f(x) is periodic function of period $T(T > 0)\ if\ f(x) = f(x+T)\forall x$

Orthogonal Function - Two non-zero function f(x) and g(x) are said to be orthogonal on $a \leq x \leq b$ if,

$$\int_a^b f(x)g(x)dx = 0$$

Dirichlet's Condition

The fourier series f(x) converges if

1. f(x) is periodic, single values and finite.
2. The function f(x) has finite number of finite discontinuties in any one period.
3. The function f(x) must have finite number of maxima and minima.

Fourier Series

The fourier series of a periodic function f(x) of period 2L on the interval (-L, L) is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n cos\frac{n\pi x}{L} + b_n sin\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)cos\frac{n\pi x}{L}\,dx \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)sin\frac{n\pi x}{L}\,dx$$

The fourier series of periodic function f(x) of period 2π on the interval $(-\pi, \pi)$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n cosnx + b_n sinnx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)cosnx\,dx \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)sinnx\,dx$$

Note: Fourier Series expressing given function f(x) in terms of cos & sin functions.

Example: Fourier series of $sinx - sin3x + cos2x\ in\ [-\pi, \pi]\ is$

The given function is itself a fourier series.

The values of $a_0 = 0, a_1 = 0, a_2 = 1$ and $b_1 = 1, b_2 = 0, b_3 = -1$

Convergence of Fourier Series

- If f(x) is continuous at x=c; $[Fourier\ Series\ of\ f(X)]_{x=c} = f(c)$
- If f(x) is not continuous at x=c; $[Fourier\ Series\ of\ f(X)]_{x=c} = \frac{f(c^-)+f(c^+)}{2}$

Parseval Identity

The parseval identity for fourier series in the interval (c,c+2l) is defined as

$$\frac{1}{2l}\int_{c}^{c+2l}[f(x)]^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left[a_n^2 + b_n^2\right]$$

Even Function - f(-x) = f(x) then the graph of y=f(x) is symmetric about y-axis.

Odd Function - f(-x) = -f(x) then the graph of y=f(x) is symmetric about origin.

Parseval identity is used to derive standard series formula from fourier series.

Important Results

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots\ldots=\frac{\pi^2}{6}$$

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\ldots\ldots\ldots=\frac{\pi^2}{90}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\ldots\ldots=\frac{\pi^2}{8}$$

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\ldots\ldots\ldots=\frac{\pi}{4}$$

$$\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\ldots\ldots\ldots=\frac{\pi^2}{24}$$

Fourier Cosine Series

The Fourier series of Even periodic function f(x) of period 2π on the interval $-\pi, \pi$ is fourier cosine series.

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n cosnx \qquad a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\ dx \qquad a_n=\frac{2}{\pi}\int_0^{\pi}f(x)cosnx\ dx$$

Fourier Sine Series

The Fourier series of Odd periodic function f(x) of period 2π on the interval $-\pi, \pi$ is fourier sine series.

$$f(x)=\sum_{n=1}^{\infty}b_n sinnx \qquad b_n=\frac{2}{\pi}\int_0^{\pi}f(x)sinnx\ dx$$

Half Range Fourier Series

Half Range Cosine Series - It is defined in the interval $(0, \pi)$

$$f(x)=a_0+\sum_{n=1}^{\infty}a_n cosnx \qquad a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\ dx \qquad a_n=\frac{2}{\pi}\int_0^{\pi}f(x)cosnx\ dx$$

Half Range Sine Series - It is defined in the interval $(0, \pi)$

$$f(x)=\sum_{n=1}^{\infty}b_n sinnx \qquad b_n=\frac{2}{\pi}\int_0^{\pi}f(x)sinnx\ dx$$

2.5.10 Gradient, Divergence and Curl

Vector Differential Operator also known as Del Operator is given by

$$\overrightarrow{\nabla}=\overrightarrow{\mathrm{i}}\frac{\partial}{\partial x}+\overrightarrow{\mathrm{j}}\frac{\partial}{\partial y}+\overrightarrow{\mathrm{k}}\frac{\partial}{\partial z}$$

Gradient is a vector quantity and it is defined as

$$Gradient(\emptyset)=\overrightarrow{\nabla}\emptyset=\overrightarrow{\mathrm{i}}\frac{\partial\emptyset}{\partial x}+\overrightarrow{\mathrm{j}}\frac{\partial\emptyset}{\partial y}+\overrightarrow{\mathrm{k}}\frac{\partial\emptyset}{\partial z}$$

Let $\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{F}$

Divergence is a scalar quantity and it is defined as

$$Divergence\,\vec{F} = \vec{\nabla}\ \vec{F} = \frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y + \frac{\partial}{\partial z}F_z$$

Curl is a vector quantity and it is defined as

$$Curl\,\vec{F} = |\vec{\nabla} \times \vec{F}| = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

Properties

1. $\vec{\nabla}^2$ is known as Laplacian Operator and it is defined as below

$$\vec{\nabla}.(\vec{\nabla}\emptyset) = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} + \frac{\partial^2\emptyset}{\partial z^2}$$

2. $\vec{\nabla} \times (\vec{\nabla}\emptyset) = 0, \qquad \vec{\nabla} \times \vec{\nabla} = 0$
3. $\vec{\nabla}.(\vec{\nabla} \times \vec{F} = 0)$
4. $\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla}.(\vec{\nabla}.\vec{F}) - \vec{\nabla}^2\vec{F}$

Important Results

- Directional Derivative $= \vec{\nabla}.\emptyset\ \hat{a} \qquad \hat{a} = \left[\frac{\vec{a}}{|\vec{a}|}\right] \qquad |\vec{a}| = \sqrt{a^2 + b^2 + c^2}$
- Curl $\vec{F} = 0$ means it is irrotational.
- Divergence $\vec{F} = 0$ means it is solenoidal
- Angle between two planes is given by $cos\theta = \frac{\vec{\nabla}f1.\vec{\nabla}f2}{|\vec{\nabla}f1|.|\vec{\nabla}f2|}$
- Two planes are said to be orthogonal if, $\vec{\nabla}f1.\vec{\nabla}f2 = 0$

Vector Identities

1. $\vec{\nabla}(f+g) = \vec{\nabla}f+\vec{\nabla}g, \qquad \vec{\nabla}(\vec{A}+\vec{B}) = \vec{\nabla}\vec{A}+\vec{\nabla}\vec{B}, \qquad \vec{\nabla}\times(\vec{A}+\vec{B}) = \vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$
2. $\vec{\nabla}(f.g) = f(\vec{\nabla}g) + g(\vec{\nabla}f), \qquad \vec{\nabla}(f\vec{A}) = f(\vec{\nabla}\vec{A}) + \vec{A}(\vec{\nabla}.f)$
3. $\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) + \vec{A} \times (\vec{\nabla}.f), \qquad \vec{\nabla}(\vec{A} \times \vec{B}) = \vec{B}(\vec{\nabla} \times \vec{A}) - \vec{A}(\vec{\nabla} \times \vec{B})$
4. $\vec{A}.(\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

Line Integral An integration evaluated along the given curve/line.

$$\overrightarrow{\mathrm{F}} = F_1 \overrightarrow{\mathrm{i}} + F_2 \overrightarrow{\mathrm{j}} + F_3 \overrightarrow{\mathrm{k}} \quad \overrightarrow{\mathrm{dr}} = dx \overrightarrow{\mathrm{i}} + dy \overrightarrow{\mathrm{j}} + dz \overrightarrow{\mathrm{k}}$$

$$\oint_C \overrightarrow{\mathrm{F}} \overrightarrow{\mathrm{dr}} = \oint_C F_1 dx + F_2 dy + F_3 dz$$

If a Multi Variable function is given, then equate it to single variable function using the given equation/curve/points/parameters. Then the Limit will be fixed to 0 to 1

$$Ex: \; x = y = z = t \quad dx = dy = dz = t \quad \oint_C \overrightarrow{\mathrm{F}} \overrightarrow{\mathrm{dr}} = \int_C f(x) t \, dt$$

Surface Integral An integration evaluated along the given surface by projecting the give surface to xy, yz, zx plane. Consider $\emptyset$ as given equation of the surface plane. Then the surface integral is defined as

$$\iint_S \overrightarrow{\mathrm{F}} . \hat{n} \, ds \; where \; \hat{n} = \frac{\overrightarrow{\nabla} \emptyset}{|\overrightarrow{\nabla} \emptyset|}$$

Consider if the surface is projected to x-y plane, then the surface integral is given as

$$\iint_S \overrightarrow{\mathrm{F}} . \hat{n} \, ds = \iint_R \overrightarrow{\mathrm{F}} . \hat{n} \frac{dxdy}{|\overrightarrow{\mathrm{n}} . \overrightarrow{\mathrm{k}}|}$$

Volume Integral An integral evaluated over a volume bounded by the surface.

$$\iiint_v F(x, y, z) \, dx \, dy \, dz$$

2.5.11 Vector Calculus Theorems

Green's Theorem

If R is a closed Region of the XY plane bounded by a simple curve C is given as

$$\int_C (M dx + N dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

Gauss Divergence Theorem

If a closed surface S bounding a Volume is given by the Gauss Divergence Theorem as below.

$$\oint \oint_S \overrightarrow{\mathrm{F}} . \hat{n} \, ds = \iiint_V \overrightarrow{\nabla} . \overrightarrow{\mathrm{F}} \, dv \qquad where \iiint_V dv \text{ volume of the given equation}$$

Stokes Theorem

Stokes Theorem is defined as

$$\int_C \overrightarrow{\mathrm{F}} \overrightarrow{\mathrm{dr}} = \iint_S (\overrightarrow{\nabla} \times \overrightarrow{\mathrm{F}}) . \hat{n} \, ds$$

Differential Equations

2.6 Differential Equations

Order of the highest order derivative occurring in Differential Equation.

Degree the power of highest order derivative occurring in the Differential Equation after the dependent and it's derivative are made free of radicals and fraction.

Example

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} = C\left(\frac{d^2y}{dx^2}\right); \qquad \text{Order} = 2 \text{ degree} = 2$$

Linear Differential Equation - A differential equation is said to be linear if the dependent variable and its differential co-efficient occurs only in the first degree and not multiplied together.

$$y(y'+1) = sin(x)\ and\ sec^2(y)\ y' + x\ tan(y) = x^2 are \text{ non-linear.}$$

$$y'' + 3y + 4y = x^3\ and\ x^2y'' + 2xy' + 3y = cos(x)\ are\ linear.$$

Solution of Differential Equation

The relation between dependent and independent variable which satisfies the given differential equations.

General Solution - A solution of differential equation containing n arbitary constants which is same as the order of the equation.

Particular Solution - A solution which can be obtained from general solution and has no arbitary constant.

Initial Value Problem - Differential Equation + Initial Condition (conditions specified at same point).

Boundary Value Problem - Differential Equation + Boundary Condition (conditions specified at different Point).

Law of Natural Growth

The rate at which populatuib grows is proportional to instantaneous population present.

$$\frac{dx}{dt}\alpha x \qquad \frac{dx}{dt} = Kx \qquad x = Ce^{kt}$$

Newton's Law of Cooling

The rate at change of temperature of a body is proportional to the difference of temperature of body and temperature of surrounding medium. $\theta-$ Temperature of body. θ_o- Temperature of surrounding medium.

$$\frac{d\theta}{dt}\alpha(\theta-\theta_o) \qquad \frac{d\theta}{dt}=K(\theta-\theta_o) \qquad \theta=\theta_o+Ce^{Kt}$$

2.6.1 First Order Differential Equation

Variable Seperable Method

Seperating x and y terms and then integrating them to get the solution.

$$f(x,y,y')=0 \qquad g(x)dx=h(y)dy \qquad \int g(x)\ dx=\int h(y)\ dy+C$$

Reducible to Variable Seperable Method using substitution.

$$\frac{dy}{dx}=f(x,y) \qquad t=f(x,y)$$

Leibnitz Differential Equation

$$\frac{dy}{dx}+Py=Q \qquad \text{P, Q - functions of x/ Linear in y}$$

$$\text{Integrating Factor (IF)}=e^{\int P\ dx} \qquad \text{General Solution (GS)}\rightarrow y(IF)=\int Q(IF)\ dx+C$$

$$\frac{dx}{dy}+Px=Q \qquad \text{P, Q - functions of y/ Linear in x}$$

$$\text{Integrating Factor (IF)}=e^{\int P\ dy} \qquad \text{General Solution (GS)}\rightarrow x(IF)=\int Q(IF)\ dy+C$$

Reducible to Lebinitz Linear form

$$f'(y)\frac{dy}{dx}+Pf(y)=Q \qquad \text{P, Q - functions of x/ Linear in y}$$

$$\text{Put } f(y)=t, \quad f'(y)\frac{dy}{dx}=\frac{dt}{dx}, \qquad \frac{dt}{dx}+Pt=Q \quad \text{Linear in t}$$

Bernoulli's Differential Equation

$$\left(\frac{dy}{dx}\right)+Py=Qy^n \qquad y^{-n}\left(\frac{dy}{dx}\right)+Py^{1-n}=Q$$

$$\text{Let } v=y^{1-n} \qquad \frac{dv}{dx}=(1-n)y^{-n}\frac{dy}{dx} \qquad y^{-n}\frac{dy}{dx}=\frac{1}{1-n}\frac{dv}{dx}$$

$$\frac{1}{1-n}\frac{dv}{dx}+Pv=Q \qquad \frac{dv}{dx}+(1-n)Pv=(1-n)Q \ \ \text{linear in v}$$

2.6.2 Exact Differential Equation

A differential Equation of form M(x,y)dx + N(x,y)dy=0 is said to be exact if Mdx + Ndy=du for some function u(x,y), then the general solution for u(x, y)=C.

A differential Equation of the form M(x,y)dx + N(x,y)dy=0 is Exact Differential Equation if and only if

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

General Solution for the Exact Differential Equation is given by

$$\int_{y \ as \ constant} M(x, y)dx + \int_{independent \ of \ x} N(x, y)dy = C$$

If a Non-Exact differential equation is converted to exact Differential equation (Reducible to Exact diferential equation) by a multiplying a factor then it is called integrating factor.

Methods to Find Integrating Factor(IF)

1. If Mdx + Ndy = 0 is non-exact differential equation and M and N are homogeneous functions of same degree, then $IF = \frac{1}{M_x+N_y}$
2. If Mdx + Ndy = 0 is non-exact differential equation and M=yf(x,y) and N=xg(x,y), then $IF = \frac{1}{M_x-N_y}$
3. If Mdx + Ndy = 0 is non-exact differential equation and $\frac{\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}}{N} = f(x)$ then $IF = e^{\int f(X)\ dx}$
4. If Mdx + Ndy = 0 is non-exact differential equation and $\frac{\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}}{M} = f(y)$ then $IF = e^{\int f(y)\ dy}$

2.6.3 Orthogonal Trajectories

Two families of curves $F_1\ andF_2$ are said to be orthogonal trajectories if every number of either family cuts every member of the other family at right angles.

Procedure to find Orthogonal Trajectories(cartesian form)

1. Given Family of Curves $F(x, y, C) = 0$
2. Form Differential equation of the given family of curves $f(x, y, y') = 0$
3. Replace $\frac{dy}{dx}\ with\ \frac{-dx}{dy}\ i.e\ f\left(x, y, \frac{-1}{y'}\right) = 0$ which will give the required differential equation of the Orthogonal Trajectories.
4. Solve the differential equation $G(x, y, c)$ to get equation of Orthogonal Trajectory.

Procedure to find Orthogonal Trajectories(Polar form)

1. Given Family of Curves $f(r, \theta, C) = 0$
2. Form Differential equation of the given family of curves $f\left(r, \theta, \frac{dr}{d\theta}\right) = 0$
3. Replace $\frac{dr}{d\theta}\ with\ -r^2\frac{d\theta}{dr}\ i.e\ f\left(x, y, -r^2\frac{d\theta}{dr}\right) = 0$ which will give the required differential equation of the Orthogonal Trajectories.
4. Solve the differential equation $G(r, \theta, c)$ to get equation of Orthogonal Trajectory.

2.6.4 Higher Order Differential Equation

$$K_0\frac{d^n y}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + K_n y = f(x)$$

$$K_0 D^n y + k_1 D^{n-1} y + \ldots\ldots + K_n y = f(x) \qquad F(D)y = f(x)$$

Homogeneous Linear Differential Equation

If f(x) = 0, then the given differential equation is homogeneous equation F(D)y=0. Then, the solution to the equation depends on the roots of the auxilary equation i.e

$$F(m) = 0 \quad K_0 m^2 + k_1 m + k_2 = 0,\ gives\ roots$$

General Solution

Roots	Linearly Independent Solution	General Solution
Real and distinct (b1 & b2)	$e^{b_1 x}$, $e^{b_2 x}$	$y = C_1 e^{b_1 x} + C_2 e^{b_2 x}$
Real and Equal (b)	e^{bx}, xe^{bx} $x^2 e^{bx}$	$y = C_1 e^{bx} + xC_2 e^{bx} + x^2 C_3 e^{bx}$ $y = e^{bx}(C_1 + xC_2 + x^2 C_3)$
Complex Conjugates (a+ib)	$e^{ax}cosbx,\ \ e^{ax}sinbx$	$y = e^{ax}(C_1 cosbx + C_2 sinbx)$
Real and distinct (For roots) $a \pm \sqrt{b}$	$e^{ax}cosh\sqrt{b}x$ $e^{ax}sinh\sqrt{b}x$ $e^{a+\sqrt{b}x}$, $e^{a-\sqrt{b}x}$	$y = e^{ax}[C_1 cosh\sqrt{b}x + C_2 sinh\sqrt{b}x]$ $y = C_1 e^{a+\sqrt{b}x} + C_2 e^{a-\sqrt{b}x}$

Non-Homogeneous Linear Differential Equation

If the given linear differential equation is non-homogeneous i.e F(D)y = f(x), then the differential equation has two solution namely Complementary function y_c and particular Integraly_p

$$y_c\ - \text{solution of } F(D)y = 0 \qquad y_p - \text{solution of } F(D)y = f(x) \quad Solution\ y = y_c + y_p$$

Results

$$1.\frac{1}{D}f(x) = \int f(x)\ dx \qquad 2.\frac{1}{D-a}f(x) = e^{ax}\int f(x)e^{-ax}\ dx$$

Methods for finding Particular Integral

Refer Table Below to find Particular Integral for different Combinations.

Type	f(x)	Particular Integral	Substitution
1	e^{ax}	$\frac{1}{F(D)}e^{ax}$	$D = a$

Note: $if\ D_r = 0\ y_p = x\frac{1}{F'(D)}e^{ax}$

Add x for each time D_r gets 0

Type	f(x)	Particular Integral	Substitution
2	$sinax/cosax$	$\frac{1}{F(D)}sinax/cosax$	$D^2 = -a^2$

Note: $if\ D_r = 0\ y_p = x\frac{1}{F'(D)}sinax/cosax$

Add x for each time D_r gets 0

Results: $\frac{1}{D^2+a^2}sinax = \frac{-x}{2a}cosax \quad \frac{1}{D^2+a^2}cosax = \frac{x}{2a}sinax$

Type	f(x)	Particular Integral	Substitution
3	x^m	$\frac{1}{F(D)}x^m$	Nil

Results: $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + ...$

1. $(1-x)^-1 = 1 + x + x^2 + x^3 + x^4 + ...$
2. $(1-x)^-2 = 1 + 2x + 3x^2 + 4x^3 + ...$
3. $(1+x)^-1 = 1 - x + x^2 - x^3 + x^4 - ....$
4. $(1+x)^-2 = 1 - 2x + 3x^2 - 4x^3 + ...$

Type	f(x)	Particular Integral
4	$e^{ax}V$	$\frac{1}{F(D)}e^{ax}V$ $e^{ax}\frac{1}{F(D+a)}V$ Now it will be in Type 2 or 3
5	xV $V = sinax/cosax$	$\frac{1}{F(D)}xV$ $x\frac{1}{F(D)}V - \frac{F'(D)}{[F(D)]^2}V$

Euler-Cauchy's Homogeneous Equation

$$a_0x^n\frac{d^ny}{dx^n}+a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\ldots\ldots+a_{n-1}x\frac{dy}{dx}+a_ny=X$$

$$\left(a_0x^nD^n+a_1x^{n-1}D^{n-1}+\ldots\ldots+a_n\right)y=X$$

$$f(XD)y=X \qquad Put\ x=e^z \quad log(x)=z, \qquad XD=D_1 \quad X^2D^2=D_1(D_1-1)$$

Note: Cauchy's Homogeneous form is a variable co-efficient linear differential equation in order to solve cauchy's form, it must be converted into constant co-efficient linear differential equation.

2.6.5 Transform Theory

Laplace Transform - It is used to transform one domain of variables to time domain.

$$F(S)=L\{f(t)\}=\int_0^{\infty}e^{-st}f(t)\ dt$$

Inverse Laplace Transform - It is used to transform a function f(S) to F(t).

$$L^{-1}\{F(S)\}=f(t)$$

Standard Function

Y(t)	**Y(s)**	**Y(t)**	**Y(s)**
$\theta(t)\ or\ 1$	$\frac{1}{s}$	t^n	$\frac{n!}{s^{n+1}}$
$t^{1/2}$	$\frac{1}{2}\sqrt{\frac{\pi}{s^3}}$	$t^{-1/2}$	$\sqrt{\frac{\pi}{s}}$
e^{-at}	$\frac{1}{s+a}$	e^{at}	$\frac{1}{s-a}$
$sinwt$	$\frac{w}{s^2+w^2}$	$sinhwt$	$\frac{w}{s^2-w^2}$
$coswt$	$\frac{s}{s^2+w^2}$	$coshwt$	$\frac{s}{s^2-w^2}$
$L^{-1}\left(\frac{1}{s^n}\right)$	$\frac{1}{\Gamma n}t^{n-1}$	$L^{-1}\left(\frac{1}{s^n}\right)$	$\frac{1}{(n-1)!}t^{n-1}$

Functions

Unit Step Function - If a Unit Step Functions u(t) is defined as below, then it can be expressed in Laplace Transform.

$$u(t-a)=\begin{cases}0, & t<a\\ 1, & t\geq a\end{cases}$$

$$L\{u(t-a)\}=\frac{e^{-as}}{s} \qquad L^{-1}\left(\frac{e^{-as}}{s}\right)=u(t-a)$$

Impulse Unit Function - If there is sudden peak raise in peak at some point of the function then it is called Impulse Unit function.

$$\delta(t-a) = \begin{cases} \frac{1}{\in}, & a < t < a+\in \\ 0, & otherwise \end{cases}$$

$$L\{\delta(t-a)\} = e^{-as} \qquad L^{-1}\left(e^{-as}\right) = \delta(t-a)$$

Shifted Function

$$f^{\sim}(t) = \begin{cases} 0, & t < 0 \\ f(t-a), & t \geq a, \quad a \geq 0 \end{cases}$$

Result

$$L\left(\frac{e^{at}-e^{bt}}{t}\right) = log\left(\frac{s-b}{s-a}\right)$$

Theorems and Properties

Initial Value Theorem

If $L^{-1}\{F(S)\} = f(t)$, then

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty}[SF(S)]$$

Final Value Theorem

If $L^{-1}\{F(S)\} = f(t)$, then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0}[SF(S)]$$

Note - Take co-efficient of highest degree polynomial(same) on numerator and denominator.

First Shifting Theorem

If $F(S) = L\{f(t)\}$, then

$$L\left(e^{at}f(t)\right) = F(s-a)$$

Second Shifting Theorem

If $F(S) = L\{f(t)\}$, then

$$L\left(f^{\sim}(t)\right) = L\left(f(t-a)u(t-a)\right) = e^{-as}F(s)$$

First Translational Property

$$L^{-1}\left(F(s-a)\right) = e^{at}L^{-1}\left(F(s)\right)$$

Change of Scale Property

If $F(S) = L\{f(t)\}$, then

$$L\left(f(at)\right) = \frac{1}{a}F(s/a)$$

Multiplication by S(Laplace Transform of derivative)

If $F(S) = L\{f(t)\}$, then

$$L\left(f^n(t)\right) = s^n F(s) - s^{n-1} f(0) - s^{n-2} f''(0) - \ldots\ldots. f^{n-1}(0)$$

Division by S(Laplace Transform of Integration)

$$L\left(\int_0^t f(\tau)\ d\tau\right) = \frac{F(s)}{s} \qquad L\left(\int_0^t \int_0^t f(\tau)\ d\tau dt\right) = \frac{F(s)}{s}$$

Multiplication by t(Derivative of Laplace Transform)

$$L\left(tf(t)\right) = -\frac{dF(s)}{dn} \qquad L\left(t^n f(t)\right) = -\frac{d^n}{ds^n} F(s)(-1)^n$$

Division by t(Integral of Laplace Transform)

$$L\left(\frac{f(t)}{t}\right) = \int_s^\infty F(s)\ ds \qquad L\left(\frac{f(t)}{t^n} = \int_s^\infty \int_s^\infty \ldots. \int_s^\infty F(s)\ ds.ds.\ldots.ds\right)$$

Convolution Theorem

$$f(t) * g(t) = \int_0^t f(u)g(t-u)\ du \qquad L^{-1}\left(F(s)\right) = f(t) \qquad L^{-1}\left(G(s)\right) = g(t)$$

$$L^{-1}\left(F(s) * G(s)\right) = f(t) * g(t) = \int_0^t f(\tau)g(\tau - t)\ d\tau$$

Laplace Transform of a Periodic Function of period T

If a function is said to be periodic $f(t+T) = f(t)$,then T is the period of function f(t).

$$L\left(f(t)\right) = \frac{1}{1-e^{-st}} \int_0^T e^{-st} f(t)\ dt$$

Complex Variables

2.7 Complex Variables

2.7.1 Basics of Complex Variables

- $|Z| = \sqrt{x^2 + y^2}$ $\quad arg(z) = \theta = tan^{-}1(y/x)$ $\quad e^{iz} = cosz + isinz$
- $cosiz = coshz$ $\quad coshiz = cosz$ $\quad siniz = isinhz$ $\quad sinhiz = isinz$ $\quad taniz = isinhz$
- $sinz = \frac{1}{2i}(e^{iz} - e^{-iz})$ $\quad cosz = \frac{1}{2}(e^{iz} + e^{-iz})$ $\quad sinhz = \frac{e^z - e^{-z}}{2}$ $\quad coshz = \frac{e^z + e^{-z}}{2}$

Analytic Function

A function f(z) is said to be analytical function in a region R, if it is analytic at every point of R. It is also called Regular Function (or) holomorphic function.

Singular Point If there are some points C for an analytic function in which the function fails to be analytic.

Cauchy-Reimann's Equation

Let $w = f(z) = u + iv$ is an analytic function.

$$\text{For Cartesian System } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}; \text{ For Polor System } \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Milne Thomson's Method

To find f(z) when real (or) imaginary part is given by

$$f(z) = u + iv \qquad f'(z) = u_x + iv_x \ = u_x - iu_y \qquad f'(x) = u_x(z,0) - iu_y(z,0)$$

$$\int f'(z)\ dz = \int u_x(z,0)\ dz - i\int v_y(z,0)\ dz + C$$

C = k to find real part and C = ik to find imaginary part

Harmonic Function

A real function of two variables x and y posses continuous second order partial differential equations and satisfies laplace equation is called harmonic function.

$$\vec{\nabla}\emptyset = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} = 0$$

2.7.2 Cauchy's Integral Theorem

If f(x) is analytic and f'(z) is continuous at all points inside and on simple closed curve C.

$$\int_C f(z)\ dz = 0$$

For Multiple Connected Regions

$$\int_C f(z)\ dz = \int_{C1} f(z)\ dz + \int_{C2} f(z)\ dz + \ldots\ldots + \int_{Cn} f(z)\ dz$$

2.7.3 Cauchy's Integral Formula

If f(z) is analytic function which has a singular point inside the closed surface C.

$$\oint_C \frac{f(z)}{z-a}\ dz = 2\pi i\ f(a) \qquad \oint_C \frac{f(z)}{(z-a)^2}\ dz = \frac{2\pi i}{2!} f'(a) \oint_C \frac{f(z)}{(z-a)^{n+1}}\ dz = \frac{2\pi i}{n!} f^n(a)$$

If two function are inside curve partial fractions needs to be applied.

2.7.4 Evaluation of Residues/ Contour Integration

1. Residue at a simple pole - If z=a is a simple pole, then

$$[Res\ f(z)]_{z=a} = \lim_{z\to a}(z-a)f(z)$$

2. Residue at a pole of order n+1, If z=a is a pole of order n+1, then

$$[Res\ f(z)]_{z=a} = \frac{1}{n!}\lim_{z\to a}\frac{d^n}{dz^n}(z-a)^{n+1}f(z)$$

Cauchy's Residue Theorem

If f(z) is analytic at all points inside and on a simple closed curve C expect for a finite number of isolated singularities $z_1, z_2, z_3, \ldots . z_n$

$$\int_C f(z)\ dz = 2\pi i[\text{sum of residues at } z_1, z_2, z_3, \ldots . z_n]$$

2.7.5 Taylor Series

If f(z) is analytic function inside a circle C with center at 'a' and radius R, then at each point Z inside C.

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \frac{f'''(a)}{3!}(z-a)^3 + \ldots\ldots\infty$$

2.7.6 Laurent Series

If C_1 *and* C_2 are two concentric circles with center at 'a' and radii r1 and r2 $(r1 > r2)$ and if f(z) is analytic on C_1 *and* C_2 and throughput the annular region R between them, then at each point Z in R.

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=0}^{\infty} b_n(z-a)^{-n}, \qquad n = 0, 1, 2$$

$$a_n = \frac{1}{2\pi i}\oint_{C1}\frac{f(w)}{(w-a)^{n+1}}dw \qquad b_n = \frac{1}{2\pi i}\oint_{C2}\frac{f(w)}{(w-a)^{-(n+1)}}dw$$

Probability and Statistics

2.8 Probability and Statistics

Events

1. Mariginal Probability - The occurrence of event has no effect on the probability of the occurrence of another event. This is alson known as independent Event.
2. Conditional Probability is the probability of occurrence of event A given that the ecent B has already occurred. [P(A/B)]
3. If A and B are independent to each other.$P(A \cap B) = P(A).P(B)$
4. If A and B are mutually dependent to each other. $P(A \cap B) = P(A).P(B/A)$
5. If A and B are mutually exclusive $P(A \cap B) = \frac{P(A \cup B)}{n(S)} = 0$
6. If two events occurs and their sum is 1, then they are collectively exhaustive $P(A) + P(B) = 1$

2.8.1 General Discrete Probability Distribution

Probability Mass Function

Let f(x) is probability at given point f(X) = P(X=x)

Properties of Probability Mass Function

$$1.P(X) \geq 0;\ f(x) \geq 0 \qquad 2.\sum_x P(X) = 1;\ \ \sum_x f(x) = 1;$$

Probability Distribution Function

It is also Known as cummulative distribution function. Let F(x) is probability upto given point then $F(x) = P(X \leq x)$

Properties of Probability Distribution Function 1. F(x)=1; Highest break point $\geq X$ 2. F(x)=0; Lowest break point $\geq X$

Mathematical Expectation (or) Mean Value

$$E(x) = \mu = \sum_x xP(x)$$

Variance & Standard Deviation

$$Var(X) = E(X^2) - [E(X)]^2 \qquad \sigma = \sqrt{Var(X)}$$

2.8.2 Bernoulli's Trial and Non-Bernoulli's Trial

Bernoulli's Trial - The experiment in which the probability of success and failure remains constant throughout the trials

Non-Bernoull's Trial - The experiment in which the probability of success and failure does not remain constant throughout the trials.

Bayes Theorem

If A and B are collectively exhaustive and mutually exclusive and E is dependent on both A and B.

$$A \to E \to P(A \cap E) = P(A) - P(E/A) \qquad P(A/E) = \frac{P(A \cap E)}{P(E)}$$

$$B \to E \to P(B \cap E) = P(B) - P(E/B) \qquad P(B/E) = \frac{P(B \cap E)}{P(E)}$$

2.8.3 Binomial Distribution

The experiment Should follow bernoulli's trial.
$P(X) = {}^nC_x \; p^x q^{n-x}$
n - no of trials, p - probability of success, q - probability of failure x - Count/draw

$$\mu = E(X) = n.p \qquad Var(X) = n.p.q \qquad \sigma = \sqrt{n.p.q}$$

Mean of Geometric Distribution - If the binomial trial is repeated for the first trial. Then, the mean is 1/p.

2.8.4 Poisson's Distribution

n - no of trials, λ - poisson's factor, α - no of occurrence per unit time. Δt - no of unit time.

$$P(X) = \frac{\lambda^x \; e^{-\lambda}}{x!}$$

$$E(X) = Var(X) = \lambda \qquad E(X^2) = \lambda^2 + \lambda \qquad \lambda = n.p = \alpha \Delta t$$

2.8.5 Probability Density Function

The probability of a single value must be zero, P(x=a)=0. The probability if certain range of X is defined by

$$P(a \leq x \leq b) = \int_a^b f(x) \; dx$$

Conditions

$$a) f(x) \geq 0 \; \forall \; x \qquad b) \int_{-\infty}^{\infty} f(x) \; dx = P(-\infty < x < \infty) = 1$$

Properties

1. $E(X) = \int_{-\infty}^{\infty} xf(x)\ dx$
2. $E(X^n) = \int_{-\infty}^{\infty} x^n f(x) dx$
3. $Var(X) = E(X^2) - [E(X)]^2$
4. $\int_{-\infty}^{\infty} [X - E(X)]^2 f(x) dx$
5. If $Y = aX + b$, then $E(Y) = aE(x) + b\ Var(Y) = a^2 Var(X)$
6. If the PDF of a random variable is even symmetric, then the mean is zero.

Covariance and Correlation

The Co-variance of two random variable's X and Y is defined by

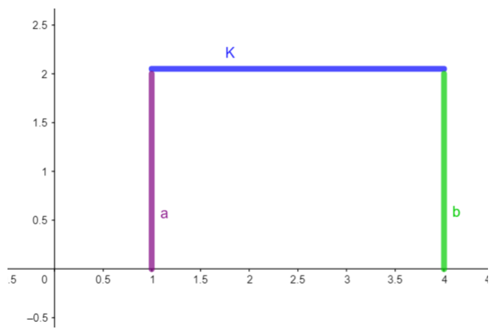
$$Cov(X, Y) = E[(X - E(X))(Y - E(X))] = E(XY) - E(X)E(Y)$$

Properties

1. $Cov(X, Y) = Var(X)$
2. $Cov(X, aY + b) = aCov(X, Y)$
3. $Cov(X, Y + Z) = Cov(X, Y) + Cov(X, Z)$
4. If X and Y are independent, $E[X, Y] = E(X)E(Y);\ Cov(X, Y) = 0$
5. Variance of sum of Random Variable $Var(X_1 + X_2) = Var(X_1) + Var(X_2) + 2Cov(X_1, X_2)$
6. Variance of sum of independent random variable $Var(X_1 + X_2) = Var(X_1) + Var(X_2)$
7. The correlation Co-efficient P(x,y) of two random variables X and Y that have non-zero Variance
$$P(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

2.8.6 Uniform Random Variable

A random variable is said to be uniform if its probability density function is consistent over a given range $X \in [a, b]$.



$$f(x) = k,\ a \leq x \leq b \qquad Area = (b-a)k = 1 \qquad k = \frac{1}{b-a}$$

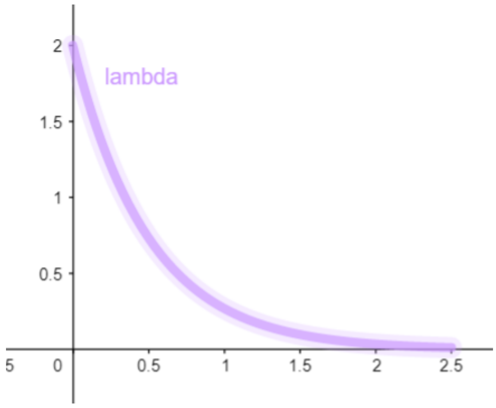
$$f(x) = \frac{1}{b-a} a \leq x \leq b \qquad \mu = E(x) = \frac{a+b}{2} \qquad Var(X) = \frac{(b-a)^2}{12}$$

$$Let\ a < b < c < d P(c \leq x \leq d) = \frac{d-c}{b-a}; \qquad Area(c \leq x \leq d) = (d-c)k$$

$$P(c < x < d) = \frac{d-c}{b-a} = \frac{favourable\ length}{total\ length}$$

2.8.7 Exponential Random Variable

Exponential random variable function f(x) can be understood from the given figure.

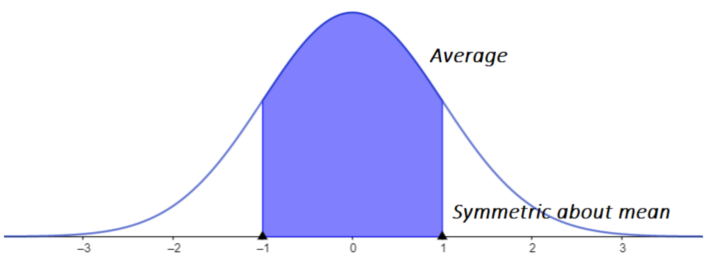


$$f(x) = \lambda e^{-\lambda x},\ x \geq 0 \qquad E(x) = \frac{1}{\lambda}; \qquad Var(x) = \frac{1}{\lambda^2}$$

For all Values of $P(X \geq a) = e^{-\lambda a}$

2.8.8 Gauss or Normal Random Variable

A Gauss Random Variable Function f(x) can be visualized as below.



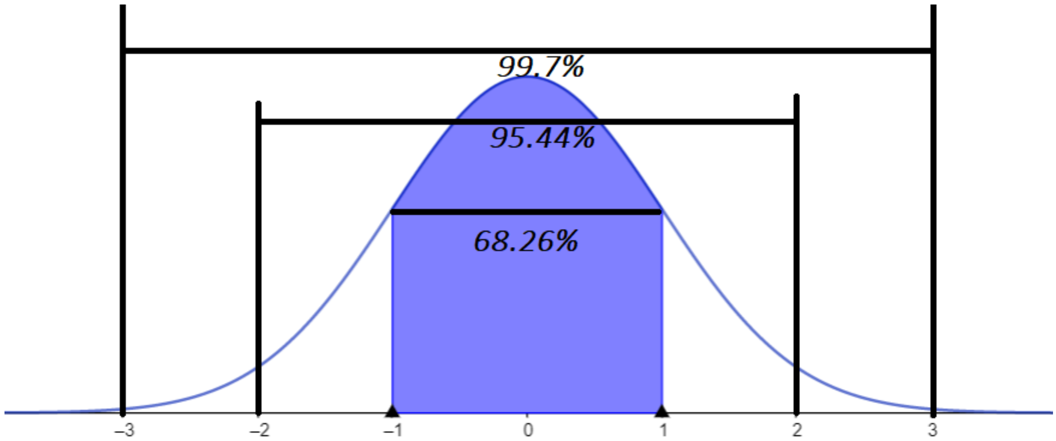
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(X-\mu)^2}{2\sigma^2}}, \forall\ x$$

2.8.9 Standard Normal Random Variable

A normal random variable "z" is standard normal random variable if $\mu = 0,\ \sigma^2 = 1$.

$$Let\ \emptyset(z) = P(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\ dx$$

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \qquad Y = ax + b \quad E(Y) = aE(X) + b \quad Var(X) = a^2 Var(z)$$



Key Points

1. The Normal distribution is symmetric about the mean.
2. $Area(X \leq \mu) = P(X \geq \mu) = \frac{1}{2}$
3. $\int_{-\infty}^{\mu} f(x)\ dx = \int_{\mu}^{\infty} f(x)\ dx = \frac{1}{2}$
4. $P(\mu - \sigma \leq X \leq \mu + \sigma) = 68.26\%$
5. $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = 95.44\%$
6. $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = 99.7\%$

Numerical Methods

2.9 Numerical Methods

2.9.1 Linear Equations

Consider the linear equations $a_{11}x+a_{12}y+a_{13}z = b1; a_{21}x+a_{22}y+a_{23}z = b2; a_{31}x+a_{32}y+a_{33}z = b3;$

A = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Diagnolly Dominant Matrix is defined as

$$|a_{11}| \geq |a_{12}| + |a_{13}| \qquad |a_{22}| \geq |a_{21}| + |a_{23}| \qquad |a_{33}| \geq |a_{31}| + |a_{32}|$$

If the given matrix or equations are not in diagnol form then try re-arranging to obtain diagnolly dominant matrix.

Jacobi's Method

Let initial guess be x_o, y_o, z_o

$$X_1 = \frac{b_1 - a_{12}y_o - a_{13}z_o}{a_{11}} \quad Y_1 = \frac{b_2 - a_{21}x_o - a_{23}z_o}{a_{22}} \quad Z_1 = \frac{b_3 - a_{31}x_o - a_{32}y_o}{a_{33}}$$

Gauss Seidel Method

This method is twice faster than Jacobi's method because simultaneous substitutions. Let initial guess be x_o, y_o, z_o

$$X_1 = \frac{b_1 - a_{12}y_o - a_{13}z_o}{a_{11}} \quad Y_1 = \frac{b_2 - a_{21}x_1 - a_{23}z_o}{a_{22}} \quad Z_1 = \frac{b_3 - a_{31}x_1 - a_{32}y_1}{a_{33}}$$

2.9.2 Non-Linear Equations

If f(x)=0 is a transcedental equation, then y=f(x) is its corresponding curve.
If $x = \alpha$ is the exact root of equation f(x)=0, then $x = \alpha$ is the intersection of curve y=f(x) with x-axis.
If α is the only root of the equation f(x)=0, between a and b then f(a) and f(b) must be of opposite sign else same sign.

Bisection/Bolzona Method

In this method two initial approximation and has rate of convergence is One [linear].

$$C = \frac{a+b}{2}$$

After obtaining C value, then substitute C in f(x) to get the next root and repeat it. No of Iterations required to get the appropriate value is given by the formula

$$\frac{b-a}{2^n}$$

Regula Falsi Method

This method is also known as method of false position. This method requires two initial approximation and the rate of convergence is One [Linear].

$$C = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Secant Method

$$C = \frac{af(b) - bf(a)}{f(b) - f(a)} \qquad f(a) \& f(b) \text{ can have same sign}$$

Difference between Secant and Regula Falsi Method
After 1st Iteration the highest value of f(a) is eliminated in secant while the sign of f(a) is considered for elimination in regula falsi method.

Iterative Method

In this method, Initial approximation is assumed (real value) and newton raphson's method is applied.

Newton-Raphson's Method

This method is also known as method of tangents. This method requires one initial approximation and rate of convergence is two [Quadratic]. $f'(x_n) - slope\ of\ tangent$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Condition of Convergence of Newton-Raphson's Method

$$\left| \frac{f(x_o).f''(x_0)}{[f'(x_o)]^2} \right| < 1$$

2.9.3 Numerical Integration

Trapezoidal Rule

This method is also known as Composite or Complex Trapezoidal Rule.

$$\int_a^b f(x)\ dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + y_3 + \dots\dots + y_{n-1})]$$

Simple Trapezoidal Rule

$$\int_a^b f(x)\ dx = \frac{h}{2}[(y_o + y_n)]$$

Simpson's Rule

Order of Error $= h^4$ and Accuracy $O(h^4)$
1/3rd Rule: if N is even (14/2)

$$\int_a^b f(x)\ dx = \frac{h}{3}[1(y_o + y_n) + 4(y_1 + y_3 + y_5 + \dots\dots + y_{n-1}) + 2(y_2 + y_4 + \dots\dots y_{n-2})]$$

3/8th Rule: if N is odd (13/2)

$$\int_a^b f(x)\ dx = \frac{3h}{8}[1(y_o + y_n) + 3(y_1 + y_2 + y_4 + \dots\dots + y_{n-1}) + 2(y_3 + y_6 + \dots\dots y_{n-3})]$$

For Both Trapezoidal Rule and Simpsons Rule, $h = \frac{b-a}{n}$

Order of fitting Polynomial	**Rule**
1 (Linear for single Variable and hyperbolic for two variables)	Trapezoidal Rule
2 (Parabolic)	1/3rd Simpson's Rule
3	3/8th Simpson's Rule

2.9.4 Numerical Solutions of Oridinary Differential Equation

Single Step Method

Taylor Series - Consider a differential equation $\frac{dy}{dx} = f(x, y)$ with initial condition $y(x_o) = y_0$, $\quad h = x - x_o$

$$y_{n+1} = y_n + hy'_n + \frac{h^2}{2!}y''_n + \frac{h^3}{3!}y'''_n + \dots\dots$$

Picard's Method of successive Approximation - $\frac{dy}{dx} = f(x, y) \quad y_n = y_o + \int_{x_0}^{x} f(x, y_{n-1})\ dx$

Multi step Method

Euler's Method - For the differential Equation $\frac{dy}{dx} = f(x, y)$ with initial conditions $y(x_o) = y_o$, the euler's Iteration formula is

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1}), \qquad n = 1, 2, 3 \ldots\ldots.$$

Modified Euler's Method

$$y_n = y_{r-1} + \frac{h}{2}[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{n-1})]$$

Iterations are repeated unless two successive approximation are appropriately equal.

Runge Kutta's Method

Order	**Equation**	
1	$y_0 + hy_0'$	
2	$y_0 + \frac{1}{2}(k_1 + k_2)$	$k_1 = hf(x_0, y_0)$ $k_2 = hf(x_0 + h, y_0 + h)$
3	$y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$	$k_1 = hf(x_0, y_0)$ $k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$ $k_3 = hf(x_0 + h, y_0 + k_2)$
4	$y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$	$k_1 = hf(x_0, y_0)$ $k_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$ $k_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$ $k_4 = hf(x_0 + h, y_0 + k_3)$