# Understanding and Solving the 1D Heat PDE

Hasif Ahmed*

20th November, 2022

**Abstract**

This paper presents a personal exploration of the 1D heat equation which is a Partial Differential Equation (PDE), driven by a fascination with its structure and a desire to understand heat diffusion in a mathematical framework. This was done as a coursework project of *PHYS425: Advanced Mechanics and Computational Physics*. The study involves a detailed analysis of the equation and its numerical solution using the finite difference method.

## 1 Introduction

Heat transfer is a fundamental phenomenon in physics, and understanding its dynamics is critical for various applications. The 1D heat equation offers a simplified yet insightful model for studying this process. This paper delves into the mathematical formulation of the equation and presents a method for its numerical solution.

## 2 The 1D Heat Equation

The 1D heat equation is expressed as:

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} \tag{1}$$

where $u$ represents temperature, $\alpha$ is a constant, $x$ is the spatial dimension, and $t$ is time.

### 2.1 Understanding the Equation

The equation is built upon the idea that the rate of temperature change in an object is proportional to the spatial derivative of temperature. Considering a one-dimensional object with particles at positions $x_1, x_2, x_3$, and respective temperatures $u_1, u_2, u_3$, we focus on the temperature change of the middle particle.

*Physics Department, Lawrence University, Appleton, WI

### 2.2 Intuitive Explanation

For a particle, if its neighbors are hotter, it heats up; if cooler, it cools down. The rate of change in temperature is proportional to the difference in temperature with its neighbors:

$$\frac{\partial u_2}{\partial t} = \alpha \left( \frac{u_1 + u_3}{2} - u_2 \right) \tag{2}$$

This can be reformulated as:

$$\frac{\partial u_2}{\partial t} = \frac{\alpha}{2} \left( (u_3 - u_2) - (u_2 - u_1) \right) \tag{3}$$

We can simplify: $\Delta u_1 = (u_2 - u_1)$ and $\Delta u_2 = u_3 - u_2$.

So we have: $\frac{\partial u_2}{\partial t} = \frac{\alpha}{2}(\Delta u_2 - \Delta u_1)$

Notice that $(\Delta u_2 - \Delta u_1)$ can be viewed as a "difference of differences" which Numerical Analysis allows us to reformulate as the second derivative with respect to position $x$. We can also absorb the constant $\frac{\alpha}{2}$ to be a new constant $\alpha$. This leads to the derived 1D heat PDE:

$$\frac{\partial u(x,t)}{\partial t} = \alpha \frac{\partial^2 u(x,t)}{\partial x^2} \tag{4}$$

## 3 Numerical Solution: Finite Difference Method

The finite difference method (FDM) is a numerical approach used to approximate solutions to differential equations. [1] This method is particularly useful when the exact analytical solutions are not feasible. We apply FDM to the heat equation, which is a partial differential equation (PDE) modeling heat distribution over time.

We start by discretizing the continuous domain into a grid by choosing step sizes for time ($\Delta t$) and space ($\Delta x$). The domain is divided into a set of discrete points, laying the groundwork for the application of difference formulas.

The continuous partial derivatives are approximated by the differences between the function values at these discrete points. For the first derivative with respect to time, we use the forward difference formula:

$$\frac{\partial u}{\partial t} \approx \frac{u(x, t + \Delta t) - u(x,t)}{\Delta t}$$

For the second derivative with respect to space, we apply the central difference formula:

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x + \Delta x, t) - 2u(x,t) + u(x - \Delta x, t)}{(\Delta x)^2}$$

These approximations are substituted into the differential equation to form difference equations that relate the function values at different grid points.

The resulting system of algebraic equations can be solved for the unknown function values at the grid points using numerical methods, such as matrix inversion or iterative solvers.

For time-dependent problems, the initial condition is specified, and the solution is iteratively advanced in time. The stability and accuracy of the method are dependent on the chosen step sizes, with conditions such as the Courant–Friedrichs–Lewy (CFL) condition often dictating the maximum allowable time step for stability.

Applying these principles to the heat equation, we obtain an explicit scheme that allows us to compute the temperature at each grid point based on the current and neighboring values:

$$u(x, t + \Delta t) = u(x, t) + k\Delta t[u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)]$$

Here we made the simplification $k = \frac{\alpha}{(\Delta x)^2}$. This equation represents the numerical solution to the heat equation using the finite difference method.

We implement this numerical scheme in python to understand the behavior of the solution.

# 4 Numerical Results

We present the numerical solution to the one-dimensional heat equation as obtained by the finite difference method. The initial condition for the temperature distribution is a Gaussian function centered at the middle of the domain, which is discretized into 120 spatial points.

## 4.1 Initial Temperature Distribution

The initial temperature distribution is depicted in Figure 1, showing a Gaussian profile due to the exponential function $e^{-x^2}$.

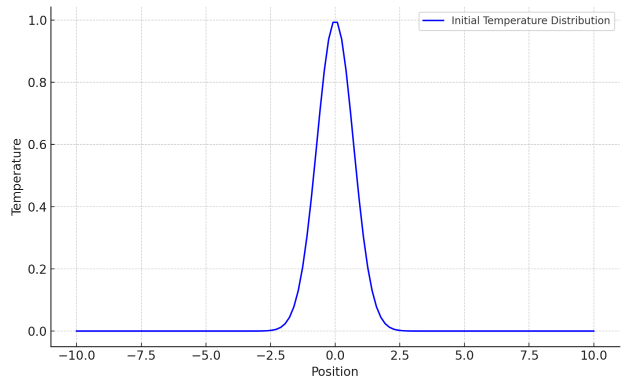


Figure 1: Initial temperature distribution across the object.

### 4.2 Temperature Distribution after 100 Time Steps

After applying the finite difference method for 100 time steps, we observe the diffusion of heat across the object. The boundaries are maintained at zero temperature due to the Dirichlet boundary conditions. The resulting temperature distribution is shown in Figure 2.

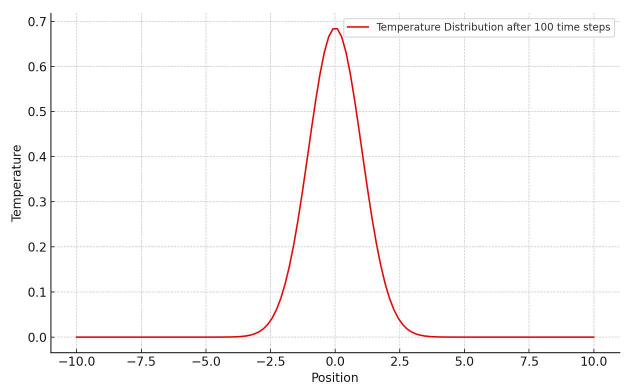


Figure 2: Temperature distribution after 100 time steps.

## 5 Conclusion

The journey through the mathematical landscape of the 1D heat equation has been a bridge between the abstract world of differential equations and the tangible realm of physical heat transfer phenomena. The derivation of the heat equation itself is a testament to the elegance with which mathematics can encapsulate natural laws. By employing the finite difference method, a numerical technique steeped in the rigor of approximation theory, we have transcended mere analytical exercises and ventured into the domain of computational physics. This approach has enabled us to visualize the subtle dance of heat energy as it diffuses through a medium over time, revealing the interplay between thermal gradients and the flow of heat.

Moreover, the practical implications of this exploration cannot be overstated. In engineering and the physical sciences, the heat equation serves as a fundamental model for a myriad of applications, from the design of thermal insulation and heat exchangers to the analysis of planetary temperature distributions. The numerical solutions we have obtained offer not just a confirmation of theoretical predictions, but also serve as a guide for experimental and industrial endeavors where precise control of heat is paramount.

In conclusion, this work has not only quenched a personal intellectual thirst but has also laid down a foundation for further inquiry and application. The numerical methods and the associated computational algorithms developed herein are not confined to the study of the heat equation alone but can be adapted to a wide spectrum of differential equations that arise across various fields of science and engineering. Thus, the significance of this study is anchored not just in its immediate findings, but in its broader contribution to the toolkit of numerical analysis and its potential to inform future innovations in the thermal sciences.

# References

[1] R.L. Herman, *Numerical Solution of 1D Heat Equation*, Available at: `http://people.uncw.edu/hermanr/pde1/NumHeatEqn.pdf`