Symmetry and Quantum Mechanics- Problem Solutions

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Chapter 1

5.8

Show that the only invariant subspaces of the Heisenberg group are $\{0\}$, $\mathrm{span}\{e_1\}$, $\mathrm{span}\{e_1, e_2\}$, and $\mathbb{C}^3$, and conclude that this representation cannot be written as a direct sum of positive-dimensional representations.

Take $H = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \in H_1$. We want to consider $H_1 \subset GL(3, \mathbb{C})$ and its representation on $\mathbb{C}^3$ by left- multiplication. In the case of checking which are the invariant subspaces , we have to check all the subspaces of $\mathbb{C}^3$.

$\{0\}$: The zero vector is invariant under any linear transformation, including H. Given $\{0\}$ is the only zero-dimensional subspace of $\mathbb{C}^3$, we are confirmed that the only zero-dimensional invariant subspace is $\{0\}$.

$\langle e_1 \rangle$: Consider a vector $\begin{bmatrix} z_1 \\ 0 \\ 0 \end{bmatrix} \in \langle e_1 \rangle$. Applying H to this vector gives:

$$H \begin{bmatrix} z_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} z_1 \\ 0 \\ 0 \end{bmatrix}$$

The resulting vector lies entirely in $\langle e_1 \rangle$. Therefore, $\langle e_1 \rangle$ is invariant.

To show that $\mathrm{span}\{e_1\}$ is the only one-dimensional invariant subspace, we just need to use the idea of eigenvectors. I claim that the eigenspace would be the only invariant one-dimenstional invariant subspace for our consideration. Let's compute the eigenvector of a heisenberg matrix using computer. We get $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ to be the only eigenvector of the matrix. This implies that the eigenspace would just be $\langle e_1 \rangle$. So we are done proving that $\mathrm{span}\{e_1\}$ is the only one-dimensional invariant subspace for heisenberg group representations on $\mathbb{C}^3$.

2-dimensional subspaces:

$\langle e_1, e_2\rangle$: Consider a vector $\begin{bmatrix} z_1 \\ z_2 \\ 0 \end{bmatrix} \in \langle e_1, e_2\rangle$. Applying H to this vector gives:

$$H\begin{bmatrix} z_1 \\ z_2 \\ 0 \end{bmatrix} = \begin{bmatrix} z_1 + az_2 \\ z_2 + bz_2 \\ 0 \end{bmatrix}$$

The resulting vector lies entirely in $\langle e_1, e_2\rangle$. Thus, $\langle e_1, e_2\rangle$ is invariant.

To determine all the two-dimensional invariant subspaces of the matrix $H = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$, consider the geometric interpretation and observe that any two-dimensional invariant subspace (plane) must contain the one-dimensional invariant subspace $\langle e_1\rangle$ (a line).

Consider a generic two-dimensional subspace spanned by v_1, v_2. To determine the conditions for this subspace to be invariant under the representation of H, we need to verify that Hv_1 and Hv_2 remain within the span of v_1, v_2.

Let's calculate Hv_1 and Hv_2:
$\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_{1,1} \\ v_{1,2} \\ v_{1,3} \end{bmatrix} = \begin{bmatrix} v_{1,1} + av_{1,2} + cv_{1,3} \\ v_{1,2} + bv_{1,3} \\ v_{1,3} \end{bmatrix}$, $Hv_2 = \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} v_{2,1} \\ v_{2,2} \\ v_{2,3} \end{bmatrix} = \begin{bmatrix} v_{2,1} + av_{2,2} + cv_{2,3} \\ v_{2,2} + bv_{2,3} \\ v_{2,3} \end{bmatrix}$.

For Hv_1 and Hv_2 to be in $\text{span}\{v_1, v_2\}$, they must be linear combinations of v_1 and v_2. Therefore, we can write:

$$Hv_1 = x_1v_1 + x_2v_2, Hv_2 = y_1v_1 + y_2v_2,$$

where x_1, x_2, y_1, y_2 are scalars. Equating the components, we have the following system of equations:

$$\begin{aligned} v_{1,1} + av_{1,2} + cv_{1,3} &= x_1v_{1,1} + x_2v_{2,1}, && (1) \\ v_{1,2} + bv_{1,3} &= x_1v_{1,2} + x_2v_{2,2}, && (2) \\ v_{1,3} &= x_1v_{1,3} + x_2v_{2,3}, && (3) \end{aligned}$$

and

$$\begin{aligned} v_{2,1} + av_{2,2} + cv_{2,3} &= y_1v_{1,1} + y_2v_{2,1}, && (4) \\ v_{2,2} + bv_{2,3} &= y_1v_{1,2} + y_2v_{2,2}, && (5) \\ v_{2,3} &= y_1v_{1,3} + y_2v_{2,3}. && (6) \end{aligned}$$

Now, let's solve these equations to find the conditions on v_1 and v_2 for the subspace to be invariant. From equation (3), we see that the third component of both v_1 and v_2 must be zero: $v_{1,3} = v_{2,3} = 0$. Now, let's consider equations (1) and (2). By comparing the coefficients of $v_{1,1}$ and $v_{1,2}$ on both sides, we get:

$$1 + av_{1,2} = x_1, \tag{7}$$
$$0 + bv_{1,3} = x_2. \tag{8}$$

Similarly, from equations (4) and (5), we have:

$$1 + av_{2,2} = y_1, \tag{9}$$
$$0 + bv_{2,3} = y_2. \tag{10}$$

Equations (7)-(10) give us the conditions on v_1 and v_2 for the subspace spanned by $\{v_1, v_2\}$ to be invariant under the action of H. These conditions are: 1. $v_{1,3} = v_{2,3} = 0$. 2. $x_1 = 1 + av_{1,2}$ and $x_2 = bv_{1,3}$. 3. $y_1 = 1 + av_{2,2}$ and $y_2 = bv_{2,3}$. By satisfying these conditions, the vectors Hv_1 and Hv_2 will remain in the span of $\{v_1, v_2\}$. Now, let's prove that span$\{e_1, e_2\}$ is indeed invariant under the given representation. The vector $e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ satisfies the conditions mentioned above since $e_{1,3} = 0$, and we can choose a and c such that $1 + ae_{1,2} = 1$ (i.e., $a = 0$) and $b = 0$ in this case. Similarly, the vector $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ satisfies the conditions by setting $a = 0$ and $b = 1$. Therefore, span$\{e_1, e_2\}$ is invariant under the action of H and it's the **only** 2-dimensional subspace with the invariant condition fulfilled from our checking process.

3-dimensional subspaces:

Take $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{C}^3$. Check for invariance:

$Hv = \begin{bmatrix} v_1 + av_2 + cv_3 \\ v_2 + bv_3 \\ v_3 \end{bmatrix} \in \mathbb{C}^3$. Hence $\mathbb{C}^3$ is an invariant subspace of $\mathbb{C}^3$ for our case.

But $\mathbb{C}^3$ is the only 3 dimensional subspace of $\mathbb{C}^3$. If you give me any other 3-dimensional subspace, I will just show you that your subspace will be isomorphic to $\mathbb{C}^3$. In that case, we are done as we proved that $\mathbb{C}^3$ is the only 3 dimensional invariant subspace.

1.21

Show that left- multiplication $(A, x) \mapsto A \star x := Ax$ **defines an action of** $SO(3)$ **on** R^3 Closure: Let A be an arbitrary matrix in $SO(3)$:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Take a vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$. The product $A \star x$ is given by:

$$A \star x = Ax = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \text{expression1} \\ \text{expression2} \\ \text{expression3} \end{bmatrix}.$$

Each expression in the resulting vector is a linear combination of x_1, x_2, x_3, which demonstrates closure.

Identity: The identity matrix in $SO(3)$ is given by: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Take an arbitrary vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$. Applying left-multiplication, we have:

$$I \star x = Ix = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x.$$

Hence, the identity property is satisfied.
2. Take two matrices A and B in $SO(3)$:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}.$$

Consider a vector $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ in $\mathbb{R}^3$. We have $(AB) \star x$:

$$(AB) \star x = (AB)x = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & \cdots \\ \vdots & \cdots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Next, we consider $A \star (B \star x)$:

$$A \star (B \star x) = A \star (Bx) = A(Bx) = A \begin{bmatrix} b_{11}x_1 + b_{12}x_2 + b_{13}x_3 \\ \vdots \end{bmatrix}.$$

By comparing the expressions, we can see that $(AB) \star x = A \star (B \star x)$.

By explicitly taking elements and showing the action, we have established that left-multiplication defines an action of $SO(3)$ on $\mathbb{R}^3$.

Exercise 1.2

An *inner product* on V is a function $\langle , \rangle : V \times V \mapsto \mathbb{R}$ such that if c $\in \mathbb{R}$ and **v,v',w**$\in$ V, then:

- $\langle$cv+v',w$\rangle$ = c$\langle$v,w$\rangle$ + $\langle$v',w$\rangle$
- $\langle$v,w$\rangle$ = $\langle$w,v$\rangle$
- $\langle$v,w$\rangle \geq 0$ with equality iff $\mathbf{v} = \mathbf{0}$

Show that the the first two properties- Linearity in first component and symmetry, imply linearity in the second component.

$$\begin{aligned}\langle cv + v', w\rangle &= c\langle v, w\rangle + \langle v', w\rangle \\ &= c\langle w, v\rangle + \langle w, v'\rangle \text{[Using symmetry]}\end{aligned}$$

But, symmetry also gives us that $\langle cv + v', w\rangle = \langle w, cv + v'\rangle$
This gives us: $\langle w, cv + v'\rangle = c\langle w, v\rangle + \langle w, v'\rangle$, which is what we wanted to show.

Problem 2.15

Show that the unitary group $U(1)$ is isomorphic to the special orthogonal group $SO(2)$

$SO(2) = \{A \in \mathbb{R}^{2\times 2} \mid A = \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}, \quad \theta \in [0, 2\pi)\}$

We may now construct the isomorphism from $U(1)$ to $SO(2)$ as follows: If $z \in U(1)$, take a map:

$$\begin{aligned} f : U(1) &\longrightarrow SO(2) \\ z &\longmapsto \begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}, \end{aligned}$$

where θ is the argument of z.

f is clearly 1–1 because one can retrieve z directly from the entries of the matrix. It is onto because all elements $\in$ SO(2) correspond to some argument θ for z $\in$ U(1).

The map f takes complex products to matrix products- so verifying homomorphism is trivial.

Hence, f is an isomorphism. That completes the exercise.

Problem 2.32

Show that, taking $\frac{1}{2i}\sigma_j$ as an orthonormal basis, the squared length of an element $iY \in iH_0(2)$ is given by $< iY, iY >= 4\mathbf{det}(iY)$

Let's start by expressing the element iY in terms of the given orthonormal basis, which consists of $\frac{1}{2i}\sigma_j$, where σ_j represents the Pauli matrices. We can write iY as:
$iY = x_1\frac{1}{2i}\sigma_1 + x_2\frac{1}{2i}\sigma_2 + x_2\frac{1}{2i}\sigma_3$
where x_1, x_2, x_3 are real coefficients representing the coordinates of iY in the basis.

Now, we can calculate the squared length of iY, denoted as $< iY, iY >$, using the inner product.

$$< iY, iY > = x_1^2 + x_2^2 + x_3^2$$

Now let's compute the determinant of the iY in its matrix representation.
$iY = \frac{1}{2i}\begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix}$
Using determinant formula, we get $det(iY) = \frac{(x_1^2+x_2^2+x_3^2)}{4}$
Hence, we see that $< iY, iY >= 4det(iY)$

Problem 2.33

Suppose that $B \in SU(2)$ and $iY \in iH_0(2)$. Then define $B \star iY := B(iY)B^{-1}$. Show that $\star$ defines an SU(2) action on $iH_0(2)$ in the sense of definition 1.20 from Chapter 1.

To show that the operation $\star$ defined as $B * iY := B(iY)B^{-1}$ defines an SU(2)-action on $iH_0(2)$, we need to verify the properties of a group action as mentioned in Definition 1.20.
Closure: We need to show that for any $B \in$ SU(2) and $iY \in iH_0(2)$, $B * iY$ is also in $iH_0(2)$. $iH_0(2)$ is the set of 2×2 Hermitian matrices with trace zero, and SU(2) is the special unitary group of 2×2 unitary matrices with determinant 1. Since B is in SU(2), it is a unitary matrix with determinant 1, and since iY is in $iH_0(2)$, it is a Hermitian matrix with trace zero. Thus, $B * iY$ is a product of unitary matrices and Hermitian matrices, and its trace is also zero, making it an element of $iH_0(2)$.

Associativity: We need to show that for any $B_1, B_2 \in$ SU(2) and $iY \in iH_0(2)$, $(B_1B_2) * iY = B_1 * (B_2 * iY)$. Using the definition of $*$, we have:
$(B_1B_2) * iY = (B_1B_2)(iY)(B_1B_2)^{-1}$ [Definition of *]
$= B_1(B_2(iY)B_2^{-1})B_1^{-1}$ [Associativity of matrix multiplication]
$= B_1 * (B_2 * iY)$

Identity element: We need to show that for any $iY \in iH_0(2)$, $e * iY = iY$, where e is the identity element of SU(2). Using the definition of $*$ and the fact that e is the identity element of SU(2), we have:
$e * iY = e(iY)e^{-1}$ [Definition of *]
$= iY$

In conclusion, we have shown that the operation $*$ defined as $B * iY := B(iY)B^{-1}$ defines an SU(2)-action on $iH_0(2)$, satisfying all the properties of Definition 1.20 from Chapter 1.

Problem 2.34

Show that the function $SU(2) \times \mathbb{R}^3 \mapsto \mathbb{R}^3$ defined by :
$(B, x) \mapsto F(\frac{1}{2i}B \begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix} B^{-1})$**defines an action of $SU(2)$ on $\mathbb{R}^3$.**

We need to check that the result $F(\frac{1}{2i}B \begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix} B^{-1})$ is an element of $\mathbb{R}^3$. Since $B \in$ SU(2) and $B^{-1} \in$ SU(2), we know that $B * iY = B(iY)B^{-1}$ where * defines an SU(2) action on $iH_0(2)$ from previous exercise. So this gives us: $F(\frac{1}{2i}*\begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix})$. Given that $\frac{1}{2i}B*\begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix} \in iH_0(2)$ and $F : (iH_0(2), <,>) \mapsto (R^3, .)$ is an isomorphism, it follows that $F(\frac{1}{2i}B \begin{bmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{bmatrix} B^{-1}) \in R^3$.

We need to show that for any $B_1, B_2 \in$ SU(2) and $x \in R^3$, $(B_1B_2) * x = B_1 * (B_2 * x)$. Using the definition of $*$, we have:
$(B_1B_2) * x = (B_1B_2)(x)(B_1B_2)^{-1}$ [Definition of *]
$= B_1(B_2(x)B_2^{-1})B_1^{-1}$ [Associativity of matrix multiplication]
$= B_1 * (B_2 * x)$

We need to show that for any $x \in R^3$, $e * x = x$, where e is the identity element of SU(2). Using the definition of $*$ and the fact that e is the identity element of SU(2), we have:
$e * x = e(x)e^{-1}$ [Definition of *]
$= x$

So we are done defining an action of SU(2) on R^3

Commutator properties

The Commutator $[iX, iY] = (iX)(iY) - (iY)(iX)$ satisfies the following properties for all $iX, iY, iZ \in iH_0(2)$ and all $a \in \mathbb{R}$:

- $[iX, iY] \in iH_0(2)$
- $[aiX + iY, iZ] = a[iX, iZ] + [iY, iZ]$
- $[iX, iY] = -[iY, iX]$
- $[iX, [iY, iZ]] + [iY, [iZ, iX]] + [iZ, [iX, iY]] = 0$

Problem 3.7

Check that the real vector space $i\mathbb{R}$ with the trivial bracket $[ix, iy] = 0$ for all $x, y \in R$ is a Lie Algebra. We are given that $i\mathbb{R}$ is a real vector space, and the Lie bracket operation $[,]$ is defined as the trivial bracket, which means that $[ix, iy] = 0$ for all $ix, iy \in i\mathbb{R}$. We need to show that the four properties listed in the problem statement hold for this vector space and bracket.

Closure: $[ix, iy] \in i\mathbb{R}$ for all $ix, iy \in i\mathbb{R}$.

$[ix, iy] = (ix)(iy) - (iy)(ix) = 0$, Since $x, y \in \mathbb{R}$ commute.

Linearity: $[aix + iy, iz] = a[ix, iz] + [iy, iz]$ for all $a \in \mathbb{R}$ and $ix, iy, iz \in i\mathbb{R}$.

Using the definition of the Lie bracket, we have:

$$\begin{aligned}
[aix + iy, iz] &= ((aix + iy)(iz) - (iz)(aix + iy)) \\
&= aix(iz) + iy(iz) - iz(aix) - iz(iy) \\
&= a\{(ix)(iz) - (iz)(ix)\} + \{(iy)(iz) - (iz)(iy)\} \\
&= a[ix, iz] + [iy, iz] \quad \text{(using the definition of } ix, iy, iz).
\end{aligned}$$

Therefore, the linearity property holds.

Antisymmetry: $[ix, iy] = -[iy, ix]$ for all $ix, iy \in i\mathbb{R}$.

Since $[ix, iy] = 0$ and $[iy, ix] = 0$ by the definition of the trivial bracket, we have $[ix, iy] = -[iy, ix] = 0$. Therefore, the antisymmetry property holds.

Jacobi identity: $[ix, [iy, iz]] + [iy, [iz, ix]] + [iz, [ix, iy]] = 0$ for all $ix, iy, iz \in i\mathbb{R}$.

Since the bracket is trivial, we have $[ix, [iy, iz]] = 0$, $[iy, [iz, ix]] = 0$, and $[iz, [ix, iy]] = 0$ for all $ix, iy, iz \in i\mathbb{R}$ as $[ix, 0] = 0$... and so on. Therefore, the left-hand side of the Jacobi identity is zero, and the identity holds.

Therefore, we have shown that $i\mathbb{R}$ with the trivial bracket is a Lie algebra.

Problem 3.8

Verify the commutation relations for $\mathfrak{su}(2)$.

$$\begin{aligned}
\left[\frac{1}{2i}\sigma_1, \frac{1}{2i}\sigma_2\right] &= \frac{1}{4i^2}(\sigma_1\sigma_2 - \sigma_2\sigma_1) \\
&= \frac{1}{4i^2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} - \frac{1}{4i^2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \\
&= \frac{1}{4i^2}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} - \frac{1}{4i^2}\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix} \\
&= \frac{1}{4i^2}\begin{pmatrix}2i & 0\\ 0 & -2i\end{pmatrix} \\
&= \frac{1}{2i}\sigma_3
\end{aligned}$$

Similarly, we have:

$$\begin{aligned}
\left[\frac{1}{2i}\sigma_2, \frac{1}{2i}\sigma_3\right] &= \frac{1}{4i^2}(\sigma_2\sigma_3 - \sigma_3\sigma_2) \\
&= \frac{1}{4i^2}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} - \frac{1}{4i^2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix} \\
&= \frac{1}{4i^2}\begin{pmatrix}0 & i \\ i & 0\end{pmatrix} - \frac{1}{4i^2}\begin{pmatrix}0 & -i \\ -i & 0\end{pmatrix} \\
&= \frac{1}{4i^2}\begin{pmatrix}0 & 2i \\ 2i & 0\end{pmatrix} \\
&= \frac{1}{2i}\sigma_1
\end{aligned}$$

Lastly:

$$\begin{aligned}
[\frac{1}{2i}\sigma_3, \frac{1}{2i}\sigma_1] &= \frac{1}{4i^2}[\sigma_3, \sigma_1] \\
&= \frac{1}{4i^2}(\sigma_3\sigma_1 - \sigma_1\sigma_3) \\
&= \frac{1}{4i^2}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} - \frac{1}{4i^2}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix} \\
&= \frac{1}{4i^2}\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix} - \frac{1}{4i^2}\begin{pmatrix}0 & -1 \\ 1 & 0\end{pmatrix} \\
&= \frac{1}{4i^2}\begin{pmatrix}0 & 2 \\ -2 & 0\end{pmatrix} \\
&= \frac{1}{2i^2}\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix} \\
&= \frac{1}{2i}(-i)\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix} \\
&= \frac{1}{2i}\sigma_2
\end{aligned}$$

where we used the fact $[\frac{1}{i} = -i]$.

3.19

Show that if $\bar{H}\bar{K} = (H- < H >_\psi I)(K- < K >_\psi I) = X + iY$ **is the cartesian decomposition of** $\bar{H}\bar{K}$**, then** $Y = \frac{1}{2i}[H, K]$

So we want to be clever and use what has already been established at 3.12 by Scott. By that problem's proof, we can claim that $\bar{H}\bar{K} = M$ can be cartesianically decomposed. So we can claim that there exists hermitian matrices X and Y such that $M = \bar{H}\bar{K} = X + iY$, $X = \frac{1}{2}(M + M^\dagger)$ and $Y = \frac{1}{2i}(M - M^\dagger)$. You can check that pretty easily. What the question then asks us to show is that $(M - M^\dagger) = [\bar{H}, \bar{K}]$ because that would prove that $Y = \frac{1}{2i}[\bar{H}, \bar{K}]$

Now by construction, we have:

$$\begin{aligned} M - M^\dagger &= \bar{H}\bar{K} - (\bar{H}\bar{K})^\dagger \\ &= \bar{H}\bar{K} - \bar{K}\bar{H} \\ &= [\bar{H}, \bar{K}] \end{aligned}$$

Where we used the hermiticity of $\bar{K}$ and $\bar{H}$

So we have proved that

$$Y = \frac{1}{2i}(M - M^\dagger) = \frac{1}{2i}[\bar{H}, \bar{K}]$$

Because $\bar{H} = H- < H > I$ and $\bar{K} = K- < K > I$ it just follows that $Y = \frac{1}{2i}[\bar{H}, \bar{K}] = \frac{1}{2i}[H, K])$

4.2

Verify the expressions for the expectation values in example 4.1

$$
\begin{aligned}
< \psi|S_z|\psi > &= \frac{\hbar}{2}\begin{bmatrix} cos(\alpha) & sin(\alpha)e^{-i\omega t} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} cos(\alpha) \\ sin(\alpha)e^{i\omega t} \end{bmatrix} \\
&= \frac{\hbar}{2}\begin{bmatrix} cos(\alpha) & sin(\alpha)e^{-i\omega t} \end{bmatrix} \begin{bmatrix} cos(\alpha) \\ -sin(\alpha)e^{i\omega t} \end{bmatrix} \\
&= \frac{\hbar}{2}\begin{bmatrix} cos^2(\alpha) - sin^2(\alpha) \end{bmatrix} \\
&= \frac{\hbar}{2}cos(2\alpha)
\end{aligned}
$$

Hence it shows that the expectation value is S_z is time-independent.

$$
\begin{aligned}
< \psi|S_y|\psi > &= \frac{\hbar}{2}\begin{bmatrix} cos(\alpha) & sin(\alpha)e^{-i\omega t} \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} cos(\alpha) \\ sin(\alpha)e^{i\omega t} \end{bmatrix} \\
&= \frac{\hbar}{2}\begin{bmatrix} cos(\alpha) & sin(\alpha)e^{-i\omega t} \end{bmatrix} \begin{bmatrix} -isin(\alpha)e^{i\omega t} \\ icos(\alpha) \end{bmatrix} \\
&= \frac{\hbar}{2}[-isin(\alpha)cos(\alpha)e^{i\omega t} + icos(\alpha)sin(\alpha)e^{-i\omega t}] \\
&= \frac{\hbar isin(\alpha)cos(\alpha)}{2}[e^{-i\omega t} - e^{i\omega t}] \\
&= \frac{\hbar isin(\alpha)cos(\alpha)}{2}[-2isin(\omega t)] \\
&= \hbar sin(\alpha)cos(\alpha)sin(\omega t) \\
&= \frac{\hbar}{2}sin(2\alpha)sin(\omega t)
\end{aligned}
$$

The expectation value of S_x has already been done in the text, so we are done.

3.23

Show that the polar decomposition of $M = UP$ of an inveritble matrix M is unique.

Claim that there exists polar decompositions $M = UP = U'P'$ where U and V are unitary matrices and P and P' are positive definite hermitian matrices. We want to show that the decomposition is unique as in $U = U'$ and $P = P'$

Now, $M^{\dagger}M = P^{\dagger}U^{\dagger}UP = P^{\dagger}IP = P^{\dagger}P = P^2$

Similarly $M^{\dagger}M = P'^{\dagger}U'^{\dagger}U'P' = P'^{\dagger}IP' = P'^{\dagger}P' = P'^2$

So, it follows that $P^2 = P'^2$. But we know that P and P' are positive definite hermitian matrices so, it follows that $P = P'$

Now we have,

$$\begin{aligned} M = U'P &= UP \\ U'PP^{-1} &= UPP^{-1} \\ U' &= U \end{aligned}$$

So the decomposition is unique!

5.12

Show that $SU(2) - action$ on $C[w_1, w_2]$ preserves degrees, in the sense that if $p(w_1, w_2)$ is homogeneous of degree $m > 0$, then so is $B \star p$ for every $B \in SU(2)$

We are given that $p(w_1, w_2)$ is homogeneous of degree $m \geq 0$. So we have $p(w_1, w_2) = \Sigma_i{}^m c_i w_1^i w_2^{m-i}$

We also have $B \in SU(2)$. We want to show that $B \star p$ is also a homogeneous polynomial of degree $m \geq 0$

$$\begin{aligned} B \star p &= p \circ B^{-1} \\ &= p(w_1, w_2) \circ \begin{bmatrix} \bar{\alpha} & \bar{\beta} \\ -\beta & \alpha \end{bmatrix} \\ &= \Sigma_i{}^m c_i (w_1\bar{\alpha} + w_2\bar{\beta})^i (w_2\alpha - z_1\beta)^{m-i} \\ &= \Sigma_i{}^m c_i \Sigma_k{}^i \binom{i}{k} (w_1\bar{\alpha})^k (w_2\bar{\beta})^{i-k} \Sigma_j{}^{m-i} \binom{(m-i)}{j} (w_2\alpha)^j (-w_1\beta)^{m-i-j} \end{aligned}$$

At this point for the sake of simplicity in computation we can absorb all the coefficients $\alpha, \beta \cdots$ into L. So we get:

$$B \star p = L\Sigma_i{}^m c_i \Sigma_k{}^i \binom{i}{k} \Sigma_j{}^{m-i} \binom{(m-i)}{j} w_1^{k+m-i-j} w_2^{i-k+j}$$

It is easy to see that it is a homogeneuous polynomial of degree $m \geq 0$ (The powers of w_1, w_2 will always add up to m.) So we are done.

5.13

Finish the argument by showing that $\phi(B \star \phi^{-1}(\sigma_1 \pm i\sigma_2)) = B(\sigma_1 \pm i\sigma_2)B^{-1}$

Take the pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We get $\sigma_1 - i\sigma_2 = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$ and $\sigma_1 + i\sigma_2 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ by matrix algebra.

We wish to compute $\phi(B \star \phi^{-1}(\sigma_1 + i\sigma_2))$. Now:

$$\begin{aligned}
\phi(B \star \phi^{-1}(\sigma_1 - i\sigma_2)) &= \phi(B \star (-w_1^2)) \\
&= \phi(-w_1^2 \circ B^{-1}) \\
&= \phi(-w_1^2 \circ \begin{bmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{bmatrix}) \\
&= \phi(-(\alpha^* w_1 + \beta^* w_2)^2) \\
&= \phi(-{\alpha^*}^2 w_1^2 - 2\alpha^*\beta^* w_1 w_2 - {\beta^*}^2) \\
&= {\alpha^*}^2(\sigma_1 - i\sigma_2) - 2\alpha^*\beta^*\sigma_3 - {\beta^*}^2(\sigma_1 + i\sigma_2)
\end{aligned}$$

Now compute:

$$\begin{aligned}
B(\sigma_1 - i\sigma_2)B^{-1} &= \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{bmatrix} \\
&= \begin{bmatrix} -2\alpha^*\beta^* & -2{\beta^*}^2 \\ 2{\alpha^*}^2 & 2\alpha^*\beta^* \end{bmatrix} \\
&= {\alpha^*}^2(\sigma_1 - i\sigma_2) - 2\alpha^*\beta^*\sigma_3 - {\beta^*}^2(\sigma_1 + i\sigma_2)
\end{aligned}$$

So, $B(\sigma_1 - i\sigma_2)B^{-1} = \phi(B \star \phi^{-1}(\sigma_1 - i\sigma_2))$

Similarly, compute:

$$\begin{aligned}
\phi(B \star \phi^{-1}(\sigma_1 + i\sigma_2)) &= \phi(B \star w_2^2) \\
&= \phi(w_2^2 \circ B^{-1}) \\
&= \phi(w_2^2 \circ \begin{bmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{bmatrix}) \\
&= \phi((\alpha w_2 - \beta w_1)^2) \\
&= \phi(\beta^2 w_1^2 + \alpha^2 w_2^2 - 2\alpha\beta w_1 w_2) \\
&= -\beta^2(\sigma_1 - i\sigma_2) + \alpha^2(\sigma_1 + i\sigma_2) - 2\alpha\beta\sigma_3
\end{aligned}$$

Also compute :

$$\begin{aligned} B(\sigma_1 + i\sigma_2)B^{-1} &= \begin{bmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha^* & \beta^* \\ -\beta & \alpha \end{bmatrix} \\ &= \begin{bmatrix} -2\beta\alpha & 2\alpha^2 \\ -2\beta^2 & 2\alpha\beta \end{bmatrix} \\ &= -\beta^2(\sigma_1 - i\sigma_2) + \alpha^2(\sigma_1 + i\sigma_2) - 2\alpha\beta\sigma_3 \end{aligned}$$

$B(\sigma_1 + i\sigma_2)B^{-1} = \phi(B \star \phi^{-1}(\sigma_1 + i\sigma_2))$
So we are done.

5.38

Show that if f and g are smooth functions on $\mathbb{R}^3$, then $\Delta(fg) = \Delta(f) \cdot g + 2\nabla f \cdot \nabla g + f\Delta(g)$.

$$\begin{aligned} \Delta(fg) &= \frac{\partial^2}{\partial x^2}(fg) + \frac{\partial^2}{\partial y^2}(fg) + \frac{\partial^2}{\partial z^2}(fg) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(fg)\right) + \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}(fg)\right) + \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}(fg)\right) \\ &= \frac{\partial}{\partial x}\left(g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}\right) + \frac{\partial}{\partial y}\left(g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial z}\left(g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z}\right) \\ &= g\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial g}{\partial x}\frac{\partial f}{\partial x} + f\frac{\partial^2 g}{\partial x^2} + g\frac{\partial^2 f}{\partial y^2} + 2\frac{\partial g}{\partial y}\frac{\partial f}{\partial y} + f\frac{\partial^2 g}{\partial y^2} + g\frac{\partial^2 f}{\partial z^2} + 2\frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + f\frac{\partial^2 g}{\partial z^2} \\ &= g\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + 2\left(\frac{\partial g}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial f}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial f}{\partial z}\right) + f\left(\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} + \frac{\partial^2 g}{\partial z^2}\right) \\ &= \Delta(f) \cdot g + 2\left(\frac{\partial g}{\partial x}\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial f}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial f}{\partial z}\right) + f \cdot \Delta(g). \\ &= \Delta(f) \cdot g + 2\nabla f \cdot \nabla g + f\Delta(g) \end{aligned}$$

5.34

Compute $P_1(-2, -2, \alpha), P_1(-2, 2, \alpha), P_1(2, 2, \alpha)$.

$$\begin{aligned} P_1(-2, -2, \alpha) &= |\langle -2' | -2 \rangle|^2 \\ &= |\langle -2 | -2' \rangle|^2 \\ &= |\langle -2 | (\cos^2(\frac{\alpha}{2}) | -2\rangle - \frac{1}{\sqrt{2}}\sin(\alpha)|0\rangle + \sin^2(\frac{\alpha}{2})| + 2\rangle)|^2 \\ &= \cos^4(\frac{\alpha}{2}) \end{aligned}$$

$$
\begin{aligned}
P_1(-2, 2, \alpha) &= |\langle -2|2'\rangle|^2 \\
&= |\langle -2|(\sin^2(\frac{\alpha}{2})| - 2\rangle + \frac{1}{\sqrt{2}}\sin\alpha|0\rangle + \cos^2\frac{\alpha}{2}| + 2\rangle)|^2 \\
&= \sin^4(\frac{\alpha}{2})
\end{aligned}
$$

$$
\begin{aligned}
P_1(2, 2, \alpha) &= |\langle 2|2'\rangle|^2 \\
&= |\langle 2|(\sin^2(\frac{\alpha}{2})| - 2\rangle + \frac{1}{\sqrt{2}}\sin\alpha|0\rangle + \cos^2\frac{\alpha}{2}| + 2\rangle)|^2 \\
&= \cos^4(\frac{\alpha}{2})
\end{aligned}
$$

We are done!