MTH 3120 Assignment 1
Due Jan 24, 2017

1. Consider the heat equation $u_t = u_{xx}$ with initial condition $u(0, x) = \sin(x) + 5\sin(5x) + 10\sin(10x)$.
   (a) Find the solution. (Use superposition!)
   (b) Plot this solution at $t = 0, 0.01, 0.1$ and 1.

   Your plots should exhibit the heat equation's tendency to filter out high frequencies.

2. In class we defined the function $S(t, x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$. Show that S solves the heat equation, i.e. that $S_t = S_{xx}$.

3. Which of the following functions are solutions of the heat equation $u_t = u_{xx}$? Justify your answer (a few words will suffice)
   (a) $3e^{-9t}\cos(3x) + 19e^{-25t}\cos(5x)$
   (b) $3e^{-9t}\cos(3x + 5) + 2e^{-4t}\sin(2x - 4)$
   (c) $e^{-9t}\sin(2x) + e^{-4t}\cos(3x)$.
   (d) $S(t + 2, x - 3)$
   (e) $S(t + .1, x - 2) + S(t + .2, x + 3)$
   (f) $S(t, x)^2$
   (g) $S(t, x) + e^{-t}\sin(x)$
   (h) $\frac{\partial S}{\partial x}(t, x)$
   (i) $\int_{-\infty}^{x} S(t, y)dy$

4. Let $f(x)$ be any function and define $u(t, x) = \int_{-\infty}^{\infty} S(t, x-y)f(y)dy$. Show that u solves the heat equation $u_t = u_{xx}$.

5. The formula $\int_{-\infty}^{\infty} S(t, x-y) f(y) dy$ which is also called **convolution** and written $S * f$ can be intimidating. The purpose of this problem is to build up some intuition.

   (a) Generate plots of $\frac{1}{2}\left(S(t, x-\frac{1}{2}) - S(t, x+\frac{1}{2})\right)$ for $t = .1, .2, 1, 2$ on the same set of axes. This is the solution to the heat equation with a hot spot at $x = \frac{1}{2}$ and a cold spot at $x = -\frac{1}{2}$.

   (b) On a new set of axes, generate plots of

   $$\frac{1}{6}\left(S(t, x-\frac{1}{4}) + S(t, x-\frac{1}{2}) + S(t, x-\frac{3}{4}) - S(t, x+\frac{1}{4}) - S(t, x+\frac{1}{2}) - S(t, x+\frac{3}{4})\right)$$

   again for $t = .1, .2, 1, 2$.

   (c) What should $\frac{1}{n}\sum_{k=1}^{n}(S(t, x-\frac{k}{n+1}) - S(t, x+\frac{k}{n+1}))$ look like for large n? Make a conjecture and sketch it.

   (d) I claim that the function

   $$\frac{1}{n}\sum_{k=1}^{n}(S(t, x-\frac{k}{n+1}) - S(t, x+\frac{k}{n+1}))$$

   approximates the convolution $S * f$ with

   $$f(x) = \begin{cases} 1 & 0 < x < 1 \\ -1 & -1 < x \leq 0 \\ 0 & |x| \geq 1 \end{cases}$$

   and hence the solution to the heat equation with initial condition f. Explain why my claim is valid.

6. Solve the heat equation with initial condition $u(0, x) = \sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \frac{1}{7}\sin(7x)$. Produce plots of the solution for different values of t. How is this solution similar to

   $$\frac{1}{6}\left(S(t, x-\frac{1}{4}) + S(t, x-\frac{1}{2}) + S(t, x-\frac{3}{4}) - S(t, x+\frac{1}{4}) - S(t, x+\frac{1}{2}) - S(t, x+\frac{3}{4})\right)$$

   which you analyzed above? How is it different?

7. In class we derived the heat equation as follows. We defined the total heat energy $E(t) = \int_a^b \rho c T dx$. The conservation of energy implies the existence of a heat flux function $q(t, x)$ so that $\frac{dE}{dt} = -q|_a^b$. We manipulated this expression to obtain

$$\rho c \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x}$$

Together with Fourier's Law :

$$q = -k \frac{\partial T}{\partial x}$$

we obtained the heat equation

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c} \frac{\partial^2 T}{\partial x^2}$$

Repeat this process (except show all the steps that are missing above) to derive an equation for the concentration of dye in a long narrow tube of water. Temperature is replaced by concentration. The conservation of energy is replaced by conservation of mass. Finally, Fourier's Law is replaced by Fick's law which says that mass diffuses down its concentration gradient.