Date
Page

Lec 20 : 30th March

FRACTIONAL INDEPENDENT SET

- → $G = (V, E)$ be an undirected graph
- → A set is ind if no two ~~neighbo.~~ ~~are~~ vertices of S are ~~mutu~~ mutual neighbours in G.

Several Variants :

- Maximal IS : S cannot be expanded further
- Maximum IS : $|S|$ is largest possible

→ Fractional IS (FIS)

A set $S \subseteq V$ is called a (c, d) - FIS of it if it satisfies:

a) S is IS
b) for every vertex v in S degree(v) is at most d.
c) $|S|$ is at least $\frac{|V|}{c}$

$\Rightarrow c > 1$

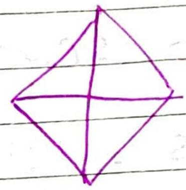
→ Not every graph may have FIS
→ Planar graphs have an FIS

Why Planar graphs have FIS

→ **Theorem (Euler):**

if $G(V, E)$ is planar with $|V|$ at least 3, then $|E|$ is at most $3|V| - 6$

eg.



$|E| = 6$
$|V| = 4$

$6 = 3 \times 4 - 6$

→ Using this theorem we develop another theorem to show that in a planar graph G, there are lots of vertices of a degree at most 'd':

Theorem: Let $G = (V, E)$ be a planar & d be an integer at least 6. $(d \geq 6)$

Let V_d be set of vertices of degree at most d. Then,

$|V_d|$ is at least $\frac{|V|}{c}$ for some constant c.

Proof : V_n be a complement of V_d

We know, $\sum \deg(V) = 2|E|$ (handshake theorem)

~~⇒ Σ deg(V) ≥~~

Now,

$$\sum_{V} \text{degree}(V) \geq \sum_{V \in V_n} \text{degree}(V)$$

$$\geq (d+1)|V_n|$$

in V_n degree is $d+1$, if less it would be in V_d

{ (if every node has degree $d+1$ then summation is $(d+1)|V_n|$) }

Using Eulers and handshake —

$$(d+1)|V_n| \leq 2(3|V|-6)$$

So, $|V_d| \geq |V| - |V_n| \geq |V| \cdot \frac{(d-5)}{d+1}$

→ $d = 6$

$|V_6|$ is at least $\frac{|V|}{7}$

finding FIS

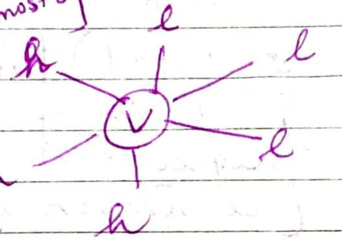
→ Start with any vertex v
→ if v has degree at most 6, add v to set S
→ Remove all neighbours of v
→ Continue till more vertices that are not removed

So, foro every vertex v, which has degree at most 6 we, remove its neighbours

So, we may remove vertices (neighbours) which have degree at most 6.

l - low degree vertices [i.e. deg at most 6]

h - high deg (we dont care about these)



when $v \in S$, we remove at most 6 other vertices are eliminated

So, the size of IS with vertices c degree at most 6 will be less than $|V_6|$ and will be $\geq \frac{|V_6|}{7}$

as for every $v \in S$, we delete at most 6 others, so 1 out of 7 is included in worst case

So, $|S|$ is at least $\frac{|V_6|}{7}$

and $|S|$ is a $(49, 6)$ – FIS

How?

We know $|V_d| \geq |V|/c$

$\Rightarrow |V_6| \geq |V|/7$

$\Rightarrow |S| \geq \frac{|V_6|}{7} = \frac{|V|}{49}$

$c = 49$

→ This is a sequential ~~alg~~. algorithm not efficient in parallel.

So, we need better approach where multiple nodes decide to join the FIS or not on their own

Parallel FIS

Consider each vertex of degree at most 6.
For each such vertex:

$$\text{Label}(V) = \begin{cases} 1 & 1/2 \\ 0 & 1/2 \end{cases}$$

$\text{Label}(V) = 1 \Rightarrow$ wants to join F/S

So, several vertices ~~will~~ & their neighbour may choose label 1 but both cannot join IS

So, set of vertices c label 1 is not ~~id~~ independent

So, we will make this set independent.

If a node v of degree at most 6, has label $(v) = 1$ and all its neighbours have label 0, then v enters the set S.

Otherwise v drops out.

$V \mid degree(v) \leq 6$

$Pr(v \in S) \geq \frac{1}{2^7}$

Now, we want to claim that S is a $(c, 6)$-FIS for some constant c. (only for planar graphs)

For this, we need to find a suitable value for c. So, that

$$|S| \geq |V| \cdot \frac{1}{c}$$

$$V_6 = \{u_1, u_2, u_3 \dots u_\ell\}$$

$$X_r = \begin{cases} 1 & \text{if } u_r \in S \\ 0 & o/w \end{cases}$$

$$E[X_r] = Pr(u_r = 1)$$
$$= \frac{1}{2^7}$$

$$X = \sum_{r=1}^{l} X_r$$

$$E[X] = \frac{|V_6|}{128}$$

(Given = $|V_6| \geq |V|/7$)

$|S| = X$,

$$\boxed{E[|S|] = \frac{|V|}{7} \times \frac{1}{128}}$$

We need to show that a RV has a value closer to its expectatn

L use chernoff

conditions of chernoff:

- independence of RV.

→ We apply Chernoff on X.

So, are X_r independent of each other.

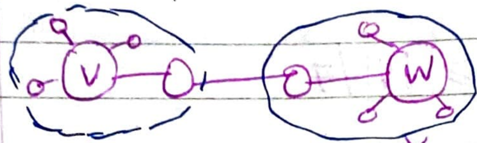
No. Because whether a vertice is in S or not dependens on whether its neighbour is in S or not.

So if a X_r ~~takes 1~~ is in S, some X_r are bound to ~~take value 0.~~ not be in S. So, not idenpendent.

Date
Page

we cannot apply Chernoff the way we understand / learnt it.

ANALYSIS: Consider,
(not experiment)

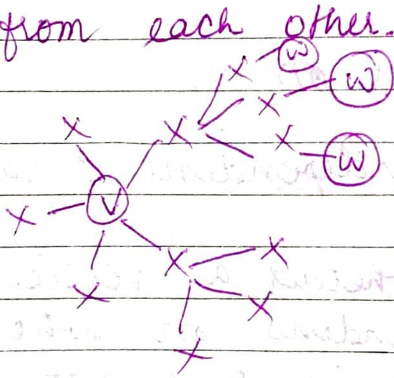


$X_v \leftarrow$ Independent $\rightarrow X_w$

because v & w have influence only on their neighbours

So, we consider only vertices of degree at most 6 and are at least a distance of 3 apart from each other.

→ How do we count the number of vertices that every two vertices are at distance 3 from each other.



So, we are throwing away at most 6 neighb, and for each of them we throw away 6 neighb.

Date
Page

So, $|V'|$ = vertices of degree at most 6 which have ~~every~~ 2 vertices have 3 distance apart neigh.

$$|V'| = \frac{|V_6|}{36}$$

So, we are able to find a subset $|V'|$ of vertices of degree at most 6 s.t. the actions of these vertices in the random experiments we do, are independent of each other.

Redefine RV $X_v = \begin{cases} 1 & \text{if } v \in S \quad \forall v \in V' \\ 0 & \text{o/w} \end{cases}$

are independent, bernoulli

$$X = \sum X_v$$

$$E[X] \geq \frac{|V'|}{7 \times 128} \geq \frac{|V|}{36 \times 7 \times 128}$$

Now, use chernoff to show that

$$Pr(X \leq E[X]/2)$$

is at most $\exp\left(-\frac{EX}{12}\right)$ which is polynomially small.

$\Rightarrow$ event $\left\{X \geq \frac{E[X]}{2}\right\}$ happens $\bar{c}$ high prob.

and $c = \frac{1}{72000}$ (after calculation)

Lecture 21 : 6th April

APPROXIMATION ALGOS

P = problems that can be solved in time that is polynomial in size of input

NP = time polynomial in size of input, using non-determinism

→ $P \subseteq NP$

but opposite not true

→ Evidence to $P \neq NP$ is coming from the existence of NP-complete problems.

NP-Complete Problems are

1. NP
2. every other problem in NP should be reducible in polynomial time to this problem.