THEOREM *Induction* $\triangleq$ $Inv \land Next \Rightarrow Inv'$

⟨1⟩1. SUFFICES ASSUME: 1. *Inv*
2. *Next*
PROVE: Inv'

PROOF: Obvious.

⟨1⟩2. CASE: $\land\ x > y$
$\land\ x' = x - y$
$\land\ y' = y$

⟨2⟩1. $TypeOK'$

PROOF: ⟨1⟩1.1 and the definitions of *Inv* and *TypeOK* imply that x and y are in $Nat \setminus \{0\}$. By case assumption ⟨1⟩2, this implies that x' and y' are in $Nat \setminus \{0\}$, proving ⟨2⟩1.

⟨2⟩2. $GCDInv'$

⟨3⟩1. $GCD(y', x') = GCD(y, x)$

PROOF: ⟨1⟩1.1 and the definitions of *Inv* and *TypeOK* imply that x and y are in $Nat \setminus \{0\}$, so ⟨3⟩1 follows from case assumption ⟨1⟩2 and GCD3 (substituting y for m and x for n).

⟨3⟩2. $GCD(x', y') = GCD(x, y)$

PROOF: ⟨1⟩1.1, the definitions of *Inv* and *TypeOK*, and ⟨2⟩1 imply that x, y, x', and y' are in $Nat \setminus \{0\}$, so ⟨3⟩2 follows from ⟨3⟩1 and GCD2.

⟨3⟩3. Q.E.D.

PROOF: ⟨3⟩2, ⟨1⟩1.1, and the definitions of *Inv* and *GCDInv* imply ⟨2⟩2.

⟨2⟩3. Q.E.D.

PROOF: By ⟨2⟩1, ⟨2⟩2, and definition of *Inv*.

⟨1⟩3. CASE: $\land\ y > x$
$\land\ y' = y - x$
$\land\ x' = x$

⟨2⟩1. $TypeOK'$

PROOF: ⟨1⟩1.1 and the definitions of *Inv* and *TypeOK* imply that x and y are in $Nat \setminus \{0\}$. By case assumption ⟨1⟩3, this implies that x' and y' are in $Nat \setminus \{0\}$, proving ⟨2⟩1.

⟨2⟩2. $GCDInv'$

⟨3⟩1. $GCD(x', y') = GCD(x, y)$

PROOF: ⟨1⟩1.1 and the definitions of *Inv* and *TypeOK* imply that x and y are in $Nat \setminus \{0\}$, so ⟨3⟩1 follows from case assumption ⟨1⟩3 and GCD3 (substituting x for m and y for n).

⟨3⟩2. Q.E.D.

PROOF: $\langle 3\rangle 1$, $\langle 1\rangle 1.1$, and the definitions of *Inv* and *GCDInv* imply $\langle 2\rangle 2$.

$\langle 2\rangle 3$. Q.E.D.

PROOF: By $\langle 2\rangle 1$, $\langle 2\rangle 2$, and definition of *Inv*.

$\langle 1\rangle 4$. Q.E.D.

PROOF: By $\langle 1\rangle 1.2$ and the definition of *Next*, the cases $\langle 1\rangle 2$ and $\langle 1\rangle 3$ are exhaustive.

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