Well-Founded Relations

An operator $\succ$ is called a *partial order* on a set $\mathcal{N}$ iff it satisfies the following two conditions:

Irreflexivity $\forall\, n \in \mathcal{N} \,:\, \neg\,(n \succ n)$

Transitivity $\forall\, m, n, p \in \mathcal{N} \,:\, (m \succ n) \land (n \succ p) \Rightarrow (m \succ p)$

A partial order $\succ$ on $\mathcal{N}$ is called a *total order* iff it also satisfies the condition:

Completeness $\forall\, m, n \in \mathcal{N} \,:\, (m \succ n) \lor (n \succ m) \lor (m = n)$

A partial order $\succ$ on a set $\mathcal{N}$ is said to be *well-founded* iff there is no infinite descending chain of the form:

$$n_1 \succ n_2 \succ n_3 \succ \ldots$$

with all the n_i in $\mathcal{N}$. This condition can be expressed formally in terms of functions as

$$\neg\,\exists\, f \in [\mathit{Nat} \rightarrow \mathcal{N}] \,:\, \forall\, i \in \mathit{Nat} \,:\, f[i] \succ f[i+1]$$

Any partial order on a finite set is obviously well-founded. The relation $>$ is a well-founded total order on the set *Nat* of natural numbers. A well-founded partial (or total) order $\succ$ on a set $\mathcal{N}$ is also a well-founded partial (or total) order on any subset of $\mathcal{N}$.

A useful well-founded total order is the relation $\succ_k$ on k-tuples of natural numbers, defined by letting

$$\langle\, a_1, \ldots, a_k \,\rangle \succ_k \langle\, b_1, \ldots, b_k \,\rangle$$

iff there exists i in $1\,..\,k$ such that $a_i > b_i$ and $a_j = b_j$ for all j in $1\,..\,(i-1)$. Since a k-tuple of natural numbers is a function from $1\,..\,k$ to *Nat*, this definition can be written formally as

$$\begin{array}{lcl} a \succ_k b & \triangleq & \land\; a \in [1\,..\,k \rightarrow \mathit{Nat}] \\ & & \land\; b \in [1\,..\,k \rightarrow \mathit{Nat}] \\ & & \land\; \exists\, i \in 1\,..\,k \,:\, \begin{array}[t]{l} \land\; a[i] > b[i] \\ \land\; \forall j \in 1\,..\,(i-1) \,:\, a[j] > b[j] \end{array} \end{array}$$

(This isn't a TLA$^+$ definition because we can't write $\succ_k$ in TLA$^+$; we would have to define the operator for a particular value of n.)

We can generalize these relations $\succ_k$ to the well-founded total order $\succ$ on the set of all finite sequences of natural numbers by defining $m \succ n$ to be true iff either (i) sequence m is longer than sequence n or (ii) they both have length k and $m \succ_k n$. The TLA$^+$ definition of $\succ$ is easily written using the operators *Seq* and *Len* defined in the standard *Sequences* module.

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