

Answer

To prove $\vdash \Box(x = 1) \Rightarrow \Box(\Box(x = 1))$ we must prove

$$\llbracket \Box(x = 1) \Rightarrow \Box(\Box(x = 1)) \rrbracket(\sigma)$$

for an arbitrary behavior σ . Here is the proof.

1. SUFFICES: $\llbracket \Box(x = 1) \rrbracket(\sigma) \Rightarrow \llbracket \Box(\Box(x = 1)) \rrbracket(\sigma)$

PROOF: Because $\llbracket F \Rightarrow G \rrbracket(\sigma)$ is defined to equal $\llbracket F \rrbracket(\sigma) \Rightarrow \llbracket G \rrbracket(\sigma)$, for any F , G , and σ .

2. SUFFICES ASSUME: $\llbracket \Box(x = 1) \rrbracket(\sigma)$ and τ a suffix of σ

PROVE: $\llbracket \Box(x = 1) \rrbracket(\tau)$

PROOF: By step 1 and the definition of $\llbracket \Box F \rrbracket(\sigma)$, with $F \leftarrow \Box(x = 1)$.

3. SUFFICES ASSUME: ρ a suffix of τ

PROVE: $\llbracket x = 1 \rrbracket(\rho)$

PROOF: By step 2 and the definition of $\llbracket F \rrbracket(\sigma)$, with $F \leftarrow x = 1$.

4. Q.E.D.

PROOF: ρ is a suffix of σ by steps 2 and 3 (since a suffix of a suffix of σ is a suffix of σ). Hence $\llbracket x = 1 \rrbracket(\rho)$ is true by the assumption $\llbracket \Box(x = 1) \rrbracket(\sigma)$ of step 2 and the definition of \Box .

Observe that replacing $(x = 1)$ by F in the proof shows that $\Box F \Rightarrow \Box \Box F$ is a theorem, for any formula F .