

Problem 1.1

- 1 repair person

- $(1 + R + 1)$ total cameras to be repaired. $= (R + 2)$ items.

└─ my camera

└─ R cameras in queue.

└─ the camera in which Aimee is working currently

* Note that, we can count this towards a full camera repair since the repair times are exponentially distributed and therefore Memoryless.

- τ = avg. repair time on one camera

- Repair time \sim exponential $(1/\tau)$

(a) I could solve part (a) using two different approaches to get the same answer.

Approach 1:

Let $X :=$ R.V. denoting the time it takes to repair $(R+2)$ items.

$T :=$ R.V. denoting the time after which you wish to return to check if your camera is repaired.

We know that, when each repair time $\sim \exp(1/\tau)$

then $X =$ sum of $(R+2)$ camera repair times

$\sim \text{erlang}(R+2, 1/\tau)$ ✓

We want the expression:

$$\underbrace{P(X \leq T)}_{\text{prob that you return and find your camera repaired.}} \geq p \quad (\text{where } p > 0.5).$$

prob that you return and find your camera repaired.

PTO.

$$P(X \leq T) = 1 - \sum_{n=0}^{R+1} \frac{1}{n!} e^{-T/\tau} (T/\tau)^n; \text{ for } X \sim \text{erlang}(R+2, 1/\tau)$$

Now let $p = 0.8$, then we need to solve for.

$$\sum_{n=0}^{R+1} \frac{1}{n!} e^{-T/\tau} (T/\tau)^n \leq 0.2 \quad \checkmark$$

I used matlab to solve this equation for the value of T/τ . The plot showing (T/τ) versus R is attached.

I observe that as R increases, (T/τ) increases.
 $\Rightarrow T$ increases as $\tau = \text{a constant}$.

better since finite sum

add to 1 because poisson pmf normalized

Approach 2:

Let the no. of camera repaired at time 0 be 0.

$N(t)$ = no. of camera repaired at time t .

T = the time after which you plan to return.

Then we are interested in the event,

$$P(N(T) - N(0) \geq R+2) \geq 0.8$$

With repair times $\sim \text{exp}(1/\tau)$, we have $N \sim \text{poisson}(t/\tau)$.

$$\therefore P(N(T) - N(0) \geq R+2) = \sum_{j=R+2}^{\infty} \frac{e^{-T/\tau} (T/\tau)^j}{j!} \geq 0.8$$

I used Matlab to solve this equation too and found the same graph as approach 1.

Conclusion: I should return after T time to check

if my camera is repaired if I want that I will get it repaired with probability 0.8. (Once we know τ , T is known from the graph attached).

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b) Suppose $\tau = 1$ hour. Let $p = 3/4$. Find t (in hours) for k ranging from 0 to 5. Summarized below:

k	0	1	2	3	4	5
t	2.69	3.92	5.109	6.27	7.42	8.55

(Calculated using Wolfram Alpha. See attachment.)

Calculating the optimal time if only 1 camera was being Fixed total gives $t \approx 1.38$, and doubling this gives ≈ 2.76 which is actually greater than our time for $k=0$. So just multiplying the optimal time ^{for 1} by $k+2$ overestimates the waiting time. We can also see that each additional camera adds about 1.2 hrs to the wait time, but this number decreases as k increases (≈ 1.13 for $k=5$)

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c) Using the case where $k=0$. What is the probability your camera is done in 90 minutes? In this case $P_{TR}(t) = \frac{1}{\tau} e^{-t/\tau} = e^{-t} \Rightarrow P(TR < 1.5) = \int_0^{1.5} e^{-t} dt = -e^{-t} \Big|_0^{1.5} = 1 - 1.5e^{-1.5} - e^{-1.5} \approx 0.4421$ (Not very good odds)

What is the probability she hasn't started on your camera? That is the probability the first camera took longer than 90 minutes. $P_{TR}(t) = \frac{1}{\tau} e^{-t/\tau} = e^{-t}$

$P(TR > 1.5) = \int_{1.5}^{\infty} e^{-t} dt = -e^{-t} \Big|_{1.5}^{\infty} = -(0 - e^{-1.5}) = e^{-1.5} \approx 0.2231$ (Significant chance)

According to the optimal strategy, you would wait ≈ 2.7 hours for a 75% chance. To be even more certain, 90% corresponds to ≈ 3.88 hr and 99% gives about 6.6 hrs.

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Following your friend's advice, there is a significant chance of arriving between 1 and 3 hours too early!

d) The following is a table of values for t_r and $E(t_r)$

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
t_r	3.92	5.109	6.27	7.42	8.55	9.68	10.802	11.91	13.01	14.11	15.21	16.31	17.41	18.51	19.61	20.71	21.81	22.91	24.01	25.11
$E(t_r)$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$t_r/E(t_r)$	1.3	1.27	1.24	1.22	1.21	1.2	1.19	1.18	1.17	1.16	1.15	1.14	1.13	1.12	1.11	1.1	1.09	1.08	1.07	1.06

For $p = 0.75$, we see the ratio of $t_r/E(t_r)$ approaches 1, so the optimal time approaches the expected time. I also attached Mathematica code to evaluate this ratio for larger k and plot it. Convergence to 1 is more apparent.

1. b)

Input interpretation:

$$0.75 = 1 - e^{-t} \sum_{j=1}^{0+2} t^{0+2-j} \times \frac{1}{(0+2-j)!}$$

$n!$ is the factorial function

Result:

$$0.75 = 1 - e^{-t} (t + 1)$$

Real solutions:

$$t \approx -0.898172$$

$$t \approx 2.69263$$

Input interpretation:

$$0.75 = 1 - e^{-t} \sum_{j=1}^{1+2} t^{1+2-j} \times \frac{1}{(1+2-j)!}$$

$n!$ is the factorial function

Result:

$$0.75 = 1 - e^{-t} \left(\frac{t^2}{2} + t + 1 \right)$$

Solution:

$$t \approx 3.9204$$

Input interpretation:

$$0.75 = 1 - e^{-t} \sum_{j=1}^{2+2} t^{2+2-j} \times \frac{1}{(2+2-j)!}$$

$n!$ is the factorial function

Result:

$$0.75 = 1 - \frac{1}{6} e^{-t} (t^3 + 3t^2 + 6t + 6)$$

Solutions:

$$t \approx -1.51173$$

$$t \approx 5.10943$$

Input interpretation:

$$0.75 = 1 - e^{-t} \sum_{j=1}^{3+2} t^{3+2-j} \times \frac{1}{(3+2-j)!}$$

$n!$ is the factorial function

Result:

$$0.75 = 1 - e^{-t} \left(\frac{t^4}{24} + \frac{t^3}{6} + \frac{t^2}{2} + t + 1 \right)$$

Numerical solution:

More digits

$$t \approx 6.27443069844469...$$

Input interpretation:

$$0.75 = 1 - e^{-t} \sum_{j=1}^{4+2} t^{4+2-j} \times \frac{1}{(4+2-j)!}$$

$n!$ is the factorial function

Result:

$$0.75 = 1 - e^{-t} \left(\frac{t^5}{120} + \frac{t^4}{24} + \frac{t^3}{6} + \frac{t^2}{2} + t + 1 \right)$$

Numerical solutions:

More digits

$$t \approx -2.10257918875657...$$

$$t \approx 7.42270183552009...$$

Input interpretation:

$$0.75 = 1 - e^{-t} \sum_{j=1}^{5+2} t^{5+2-j} \times \frac{1}{(5+2-j)!}$$

$n!$ is the factorial function

Result:

$$0.75 = 1 - \frac{1}{720} e^{-t} (t^6 + 6t^5 + 30t^4 + 120t^3 + 360t^2 + 720t + 720)$$

Numerical solution:

More digits

$$t \approx 8.55846679800003...$$

$f = \{\};$

$x = \{\};$

$\text{Do}[\text{sol} = \text{Solve}[1 - \text{Exp}[-t] * \text{Sum}[t^{(k+2-j)} * 1 / (k+2-j)!, \{j, k+2\}] == 0.75 \&\& t > 0, t];$

$\text{AppendTo}[x, k];$

$\text{AppendTo}[f, \text{sol}[[\text{All}, 1, 2]] / (k+2)];$
 $\{k, 0, 120\}]$

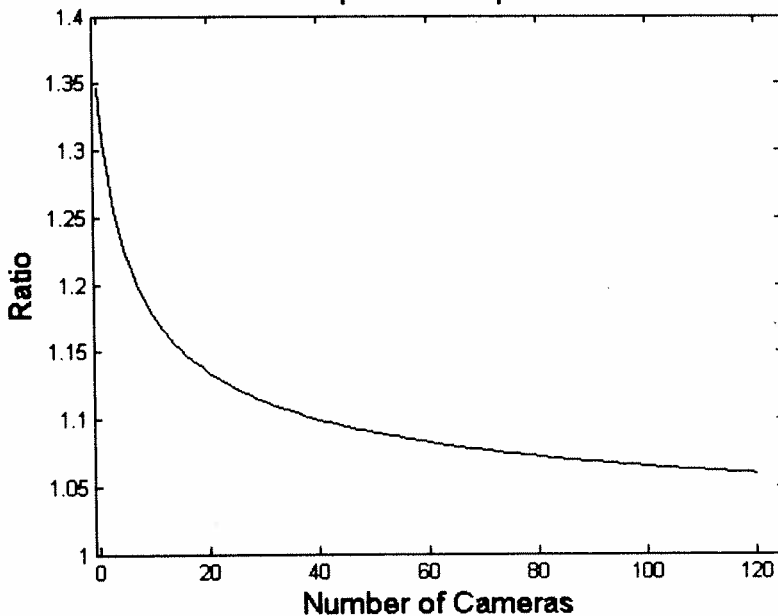
Mathematica

code to solve

transcendental for t

($k = 0$ to 120)

Ratio of Optimal to Expected Time



$$\rho = 3/4$$

Ratio of $\frac{t_R}{E(t_R)}$

(d) From numerical experiment the ratio is very close to 1.

(4)

$$\text{if } p < \frac{1}{2} \quad \frac{t^*}{\tau k} \sim 1 \quad \text{but} \quad \frac{t^*}{\tau k} < 1$$

$$\text{if } p > \frac{1}{2} \quad \frac{t^*}{\tau k} > 1 \quad \text{but close to 1}$$

$$\text{cdf of Gamma}(x; k, \tau) = 1 - \sum_{n=0}^{k-1} \frac{1}{n!} e^{-\frac{x}{\tau}} \left(\frac{x}{\tau}\right)^n \quad (\text{Erlang dist})$$

Also note that the Sum $\bar{T} = \bar{T}_0 + \dots + \bar{T}_k$ is the sum of iid expon distribution with mean 1 and Var 1 (if we let $\tau = 1$).

$$\Rightarrow \text{central limit} \Rightarrow Z = \frac{\bar{T} - k}{\sqrt{k}} \sim N(0, 1)$$

$$\Rightarrow \mathbb{P}\left\{Z \geq \frac{2}{3}\right\} = \mathbb{P}\left\{\Phi \geq \frac{2}{3}\right\}$$

This is achieved at roughly $\underline{Z} = 0.43 \Leftrightarrow \bar{T} = 0.43\sqrt{k} + 1$
 \downarrow this is irrelevant for $k \rightarrow \infty$.

Therefore the $t^* \sim k + x^* \sqrt{k}$.

where x^* is the point where $\Phi(x^*) = p$, where

Φ is the cdf of $N(0, 1)$.

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$$\Rightarrow x^* > 0 \text{ for } p > \frac{1}{2}, \quad x^* < 0 \text{ for } p < \frac{1}{2}$$

but the asymptotic behavior of t^* is not affected $t \sim k$

This also explains why if $p < \frac{1}{2}$ $\frac{t^*}{k} < 1$ in numerical simulations.
if $p > \frac{1}{2}$ $\frac{t^*}{k} > 1$

Problem 1.3

Answer (a): (based on scenario two)

Let's consider a town with a population of 'N' people on any given day. Today, the number of infected people happens to be ' X_n '. Any person in this town can interact with another person with a probability 'r' which is independent of their past interactions. This is scenario two as described in the homework assignment. For any interaction between a healthy person and an infected person, there is a probability 'p' that the healthy person gets infected. A person having the disease already can become healthy again with a probability 'q'. Now, we have to make some assumptions and clarify our model before we continue.

A 'day' in this case is our epoch and we assume all things that happen during a day happen simultaneously.

1. All interactions happen independently. If 'a' interacts with 'b' with a probability 'r', so does 'b' interact with 'c' with a probability 'r' and so on. The fact that one interaction happened does not affect the chances of any other interaction happening.
2. There can be only one interaction between two unique individuals. For instance, 'a' and 'b' would interact only once in a day. This is coherent with the consideration of any day being an epoch.
3. A healthy person may interact with any number of infected persons and each such interaction will carry a probability 'p' of infection. If any of these interactions successfully results in the infection, the status of this person will change at the end of the day. This means he will not be an infected person until the next day. For the 'rest of the day', this person will be counted as healthy.
4. If an infected person is cured with a probability 'q', this cure will take place at the end of the day. So if a person starts the day as infected, he will remain infected until the next day or the next epoch. Both 3 and 4 imply that the status of a person will change only in the next epoch. Note that as a consequence, it is not possible for a person to get infected and get cured in the same day or vice versa.

Now, my approach towards this problem would be to define the Markov Chain as follows.

$$X_n \in [0, N]; Z_n \in [1, N]$$
$$X_{n+1} = X_n + Z_n$$

$X_n \rightarrow$ Number of infected people at start of day n

$Z_n \rightarrow$ Number of new infections (I_n) – Number of cures (C_n) on day n

We can find the probability transition matrix without much hassle for this scenario and so that will be our approach. Let's start with finding out the probability that a given person 'a', who starts off healthy on day 'n' becomes infected on day 'n+1'.

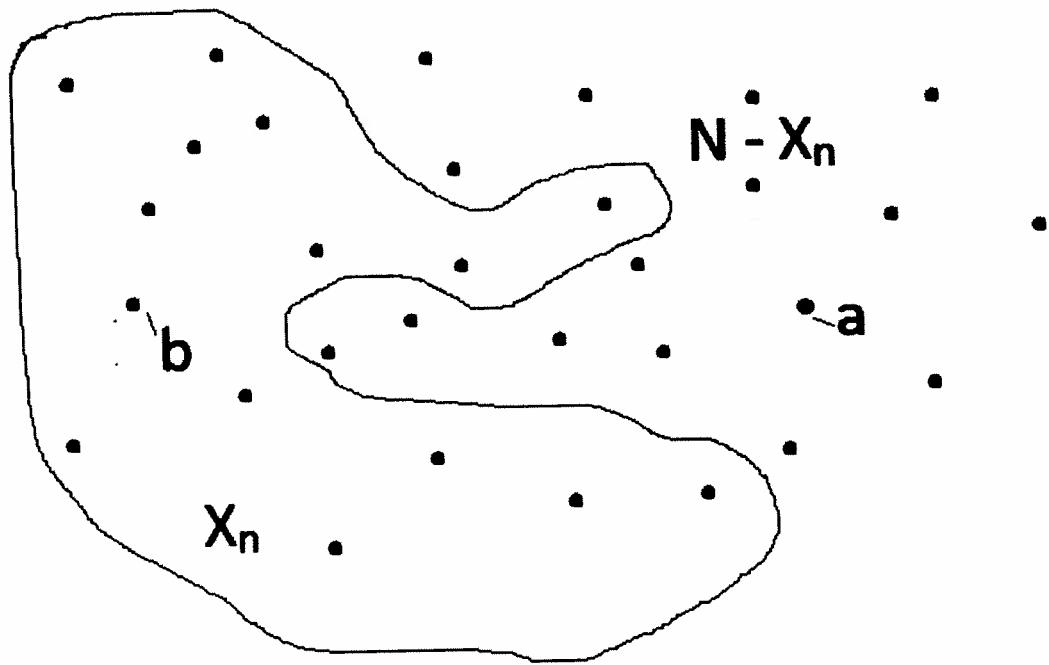


Figure 1: A 'town' of Population 'N' and number of infected people 'X_n' on a given day.

So 'a' can possibly interact with any, some or all of the Infected people numbered X_n. For any such interaction, he will probably get infected (with 'p' probability). He needs at least one such unfortunate interaction to happen. The number of interactions we are looking at are X_n. I have formulated the probability by enumerating all possible scenarios, which include from at least one to at most X_n interactions happen in a day and within each of those scenarios, at least one and at most all interactions are (unfortunately) successful in causing an infection. However, I find it easier to present the calculation of this probability by knowing the probability that 'a' does not get infected throughout the day and saying:

$$\Pr(a \text{ is infected at end of day } n) = 1 - \Pr(a \text{ is not infected at end of day } n)$$

So, the probability that 'a' is not infected is calculated by saying that 'a' either interacts with no one and obviously doesn't get infected, or he interacts with one infected person and doesn't get infected, or interacts with two, and doesn't get affected in both cases, ... and so on until the case when he interacts with all X_n infected persons and doesn't get infected in all those cases. Essentially, since each interaction is independent and has its own probability 'r', it's basically a sum of binomial distribution probabilities with the extra condition that in each successful interaction, no infection occurs with probability (1-p).

$$\begin{aligned} \Pr(a \text{ is not infected at end of day } n) &= (1-r)^{X_n} + \binom{X_n}{1} r(1-r)^{X_n-1}(1-p) + \binom{X_n}{2} r^2(1-r)^{X_n-2}(1-p)^2 \\ &+ \dots + \binom{X_n}{X_n} r^{X_n}(1-r)^0(1-p)^{X_n} \\ &= \sum_{i=0}^{X_n} \binom{X_n}{i} r^i(1-r)^{(X_n-i)}(1-p)^i \end{aligned} \quad (1)$$

So,

$$Pr(a \text{ is infected at end of day } n) = 1 - \sum_{i=0}^{X_n} \binom{X_n}{i} r^i (1-r)^{(X_n-i)} (1-p)^i \quad (2)$$

Now, if we are interested in the probability that 'm' people get infected in that same day, we can say that we first select m people from $N - X_n$ and then say all of them get infected, raising the above probability to a power of m. This event could happen for all possible selections of m people from $N - X_n$ healthy people. So we have the following:

$$\begin{aligned} Pr(I_n = m | X_n) \\ = \binom{N - X_n}{m} \left[\sum_{i=0}^{X_n} \binom{X_n}{i} r^i (1-r)^{(X_n-i)} (1-p)^i \right]^{N-X_n-m} \left[1 - \sum_{i=0}^{X_n} \binom{X_n}{i} r^i (1-r)^{(X_n-i)} (1-p)^i \right]^m \\ 0 \leq m \leq N - X_n \end{aligned}$$

That's basically the probability that we have m infections at the end of day n given that there are X_n infected people on that day. Now if we want to know that probability that there are Z_n more / less infected people at the end of day n, we also need to account for any infected people getting cured at the end of day n. The probability for that is a simple binomial distribution where we need say k successful events (in this case, k people getting cured) occur from a sample of X_n infected people or possibilities.

$$Pr(C_n = k | X_n) = \binom{X_n}{k} q^k (1-q)^{X_n-k} \quad 0 \leq k \leq X_n$$

In order to derive the probability transition matrix, we need to find the following probability:

$$P_{ij} = Pr(X_{n+1} = X_n + Z_n | X_n)$$

So, if we have m infections and k cures at the end of the day, we have $Z_n = m - k$. Let's lose k and use Z_n in the probability for cures and consequently know that if Z_n infected people are to be added / subtracted from the nth day number of X_n , we need all possible combinations of infections are cures such that $Z_n = m - k$.

This means,

$$Pr(C_n = m - Z_n | X_n) = \binom{X_n}{m - Z_n} q^{m-Z_n} (1-q)^{X_n-m+Z_n} \quad Z_n \leq m \leq X_n + Z_n$$

And so finally we have,

$$Pr(X_{n+1} = X_n + Z_n | X_n) = \sum_{m=\max(0, Z_n)}^{\min(N-X_n, X_n+Z_n)} Pr(I_n = m | X_n) * Pr(C_n = m - Z_n | X_n) \quad (3)$$

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This is our probability transition matrix for a population of N people. I have plugged the above equations into a MATLAB script (CalcPrZn.m) which is attached with the homework. This calculates each element of this matrix given the previous number of infected people X_n and the expected change in the number for the next day Z_n . This answers part (a).

Answer (b)

In order to find the expected number of infected people on a given day n , we need to find the probability distribution on that day. For instance, our probability distribution would look something like the below expression. Know that $\overline{\phi^n}$ is a column vector of row length $N+1$ and gives the probability that on day n there are either zero infected people OR 1 infected person - and so on to - OR N infected people.

$$\phi_i^n = \Pr(X_n = i - 1)$$

The probability transition matrix we derived above can be expressed in the following form. This is a matrix of size

$$P_{ij} = \Pr(X_{n+1} = j - 1 | X_n = i - 1)$$

If we knew the probability distribution vector $\overline{\phi^n}$ for day n , then we can find the vector for day $n+1$ by the matrix multiplication shown below. The superscript T indicates transpose.

$$(\overline{\phi^{n+1}})^T = (\overline{\phi^n})^T \times \underline{P} \quad (4)$$

This can be explained by knowing that,

$$\Pr(A) = \Pr(B) \times \Pr(A|B) + \Pr(B^c) \times \Pr(A|B^c)$$

Any given row of the resulting vector on the left hand side looks as below. This makes sense.

$$\phi_j^{n+1} = \Pr(X_{n+1} = j - 1) = \sum_{i=0}^N \Pr(X_n = i - 1) \times \Pr(X_{n+1} = j - 1 | X_n = i - 1)$$

Now, once we know the probability distribution vector, which is basically the probability that a given value of the variable X_{n+1} is obtained, we can find the average value of this variable on the $n+1^{th}$ day by saying the following¹.

$$(X_n)_{exp} = \sum_{i=1}^N (i - 1) \times \phi_i^n \quad (5)$$

In order to get the probability distribution as a function of time or number of days elapsed, we start with a known distribution $\overline{\phi^1}$. Then we march in time as follows. Once the probability distribution is known, we use equation (5) to get the expected number of infected people.

$$(\overline{\phi^n})^T = (\overline{\phi^1})^T \times (\underline{P})^{n-1} \quad (6)$$

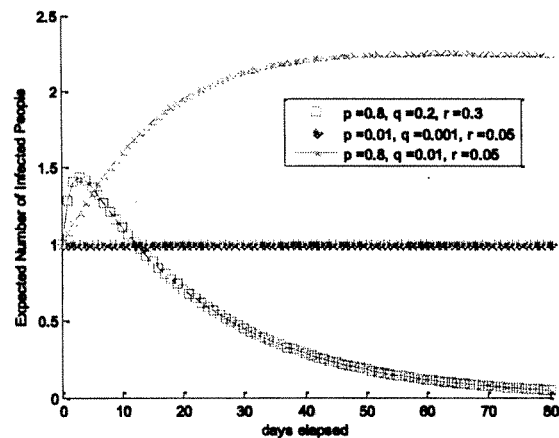
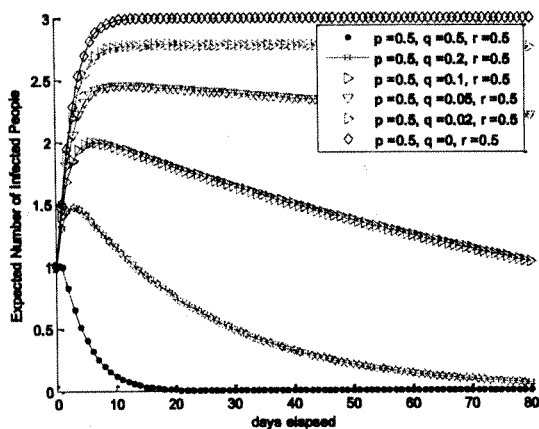
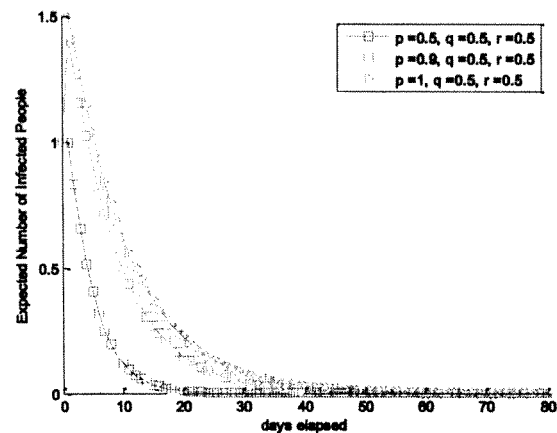
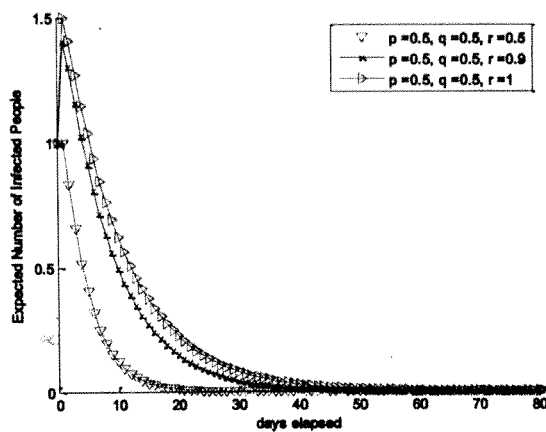
¹ Abhishek Choudhary explained me that I could find the expected value in this manner given the discrete probability distribution.

All these calculation steps were incorporated in a MATLAB script called *EvaluateDisease.m* that called the subroutines *CalcPrZn.m*, *PrInfection.m* and *PrCure.m*.

Answer (f) Part 1

This is f) is Monte Carlo

Shown below are some results. I was interested in knowing the conditions under which an epidemic outbreak would occur in the population. This implies, I would like the expected number of infected people to plateau out at N (or 3 in this case) after some days. Obviously, the probability of infection given an interaction, p would have a major role. Further, I would be interested in knowing the impact of the severity of the disease. For instance, if it was a case of the flu, people would get cured fairly quick, so I could choose a higher value of q . What if the community did not interact as often? What is the impact of r ? Shown below are some plots for different values of r , p and q .



(Please make a note that for the above results, I marched through time initializing my day from 'zero' and not 1 in the above results.)

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CalcPrZn.m

```
function PrZn = CalcPrZn(N,Xn,r,p,q,Zn)
```

```
PrZn = 0;
```

```
%disp(strcat('max',num2str(max(0,Zn))));
```

```
%disp(strcat('min',num2str(min(N-Xn,Xn+Zn))));
```

```
for m = max(0,Zn):min(N-Xn,Xn+Zn)
```

```
    PrZn = PrZn + PrInfection(N, Xn, r, p, m)*PrCure(Xn,q,m-Zn);
```

```
    %disp(PrZn);
```

```
    %disp(m);
```

```
end
```

EvaluateDisease.m

```

clear ExpInf Pij Phi TransPhi i j N;
N = 10;
%r = 0.5;
%p = 0.5;
%q = 0.5;
InitialPhi = [0 1 0 0 0 0 0 0 0 0];
% How many steps to take?
steps = 80;
days = zeros(steps+1,1);
days(:,1) = 0:1:steps;

ExpInf = zeros(steps+1,1);
Pij = zeros(N+1,N+1);

for i = 1:N+1
    for j = 1:N+1
        Pij(i,j) = CalcPrZn(N,i-1,r,p,q,j-1);
    end
end

for i = 1:steps+1
    if i <= 1
        Phi(:,i) = InitialPhi;
    else
        TransPhi = transpose(Phi(:,i-1))*Pij;
        Phi(:,i) = transpose(TransPhi);
    end

    for j = 1:N+1
        ExpInf(i,1) = ExpInf(i,1) + Phi(j,i)*(j-1);
    end
end

%{
clf;
figure(1);
for j = 1:N+1
    hold on;
    plot(days,Phi(j,:),char(allLines(j)));
    legendPhi(j) = cellstr(strcat('P(Infected = ',num2str(j-1),')'));
}

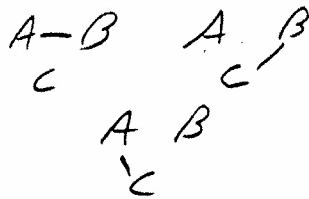
```

EvaluateDisease.m

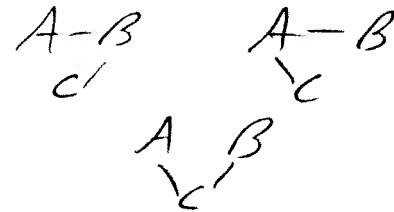
```
    xlabel('days');  
    ylabel('Probability');  
end  
  
legend(legendPh1);  
  
%}
```

Let's make a few notes about the matrix that we constructed (although we won't go through the derivation of every line here because it would take too long to type...). It suffices to show our approach for the first submatrix (which we did) and explain that the other submatrices follow a similar approach. As mentioned before, the probabilities in this model depend on the number of interacting pairs in that group. So, when we see values of $\frac{1}{3}$ pop up, it is simply because there are three possibilities for the single and double interacting pairs. The way we thought about it was that there are three people A, B, C , which gives us three possibilities for one interacting pair and three possibilities for two interacting pairs. Thus, there is a $\frac{1}{3}$ chance that each possibility below is what happens.

1 interacting pair



2 interacting pairs



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We also see some strange looking terms like $(1 - (1 - p)^2)$. Literally this means that a person doesn't not get infected by two other people. What this means practically is that a susceptible individual gets infected from one of the two infectious individuals. That concludes our Markov chain model for this scenario using a probability transition matrix.

d) (5 points)

So, what makes this particular Markov chain model so awkward? For Case (2), the Markov chain model works wonderfully because of the nature of the interactions between all of the people. However, in Case (1) people have a probability r of being an interacting pair which means they either interact every day or they don't. So, what does this mean for our model? As was mentioned before, the state space for our model was a set of vectors. We needed to use vectors because we had to store information about the number of interacting pairs in addition to the number of infectious individuals. This modification to the state space makes our model rather awkward.

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Another thing to observe is that this Markov chain is reducible. By this, it is meant that all of the states do not communicate with each other. If we were dealing with an irreducible Markov chain (like in Case (2)) all states communicate with each other and the Markov chain model itself is much less awkward.

e) (20 points)

Next we want to determine how many people in our population are infectious, on average, on the n^{th} day. We will choose the same parameters that we chose for part b), namely $p = 0.7$, $q = 0.3$, and $r = 0.8$, in addition to 1 person being initially infected. The important thing to note here is that we have four different subgroups in our state space (each corresponding to a different number of interacting pairs). We know that these different subgroups cannot communicate with one another (as

d.

There are two reasons why the first scenario is more awkward. First, it needs to discuss by different situations. Second, which is more important, as mentioned above, is that for one of those situations, the simple state space does not carry enough information needed to build a Markov chain, and thus we need to take a more complicated state space.

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e.

A MATLAB code is attached for this problem (see Appendix).

For doing this question, I calculate the average number of people infected by the following formula:

$$E(N) = \sum_{i=1}^4 E(N_i)p(i)$$

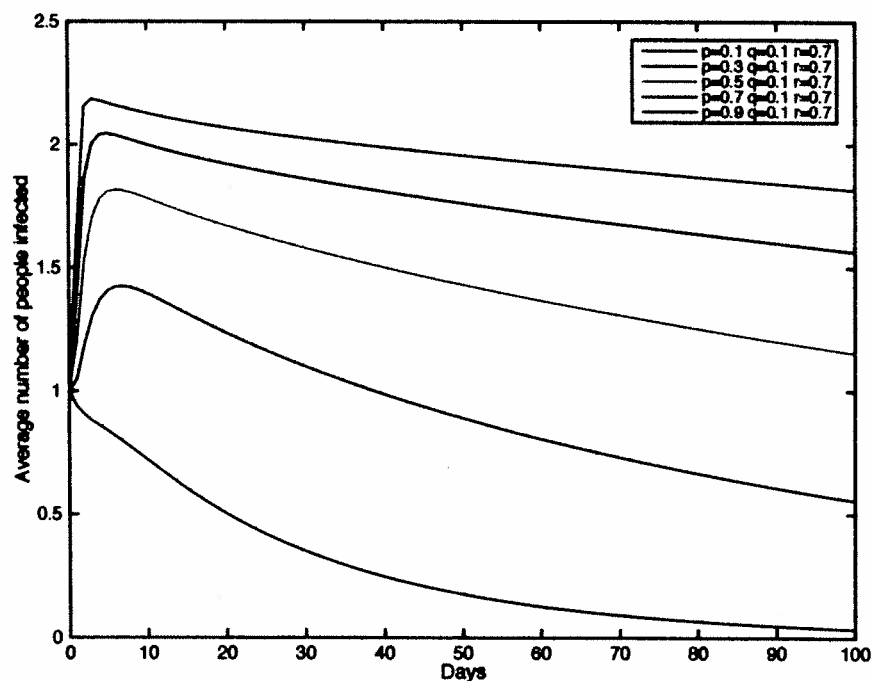
In the formula, random variables N denotes the number of people being infected, and random variable N_i denotes that the number of people being infected in the i th model above, $p(i)$ denotes the probability that i th model is happen. For model 1,2 and 4, we are doing almost the same things as those in question b. For model 3, the meaning of the state vector has been changed. Hence the average number of people infected is going to be $k_i \times (0,1,1,2,2,3)$, and the start state vector k_0 is going to be:

$$k_0 = (0 \quad 1/3 \quad 2/3 \quad 0 \quad 0 \quad 0)$$

Same as part b, k_n could be either updated from k_0 all the way to it, or use de formula $k_n = k_0 \times P^n$

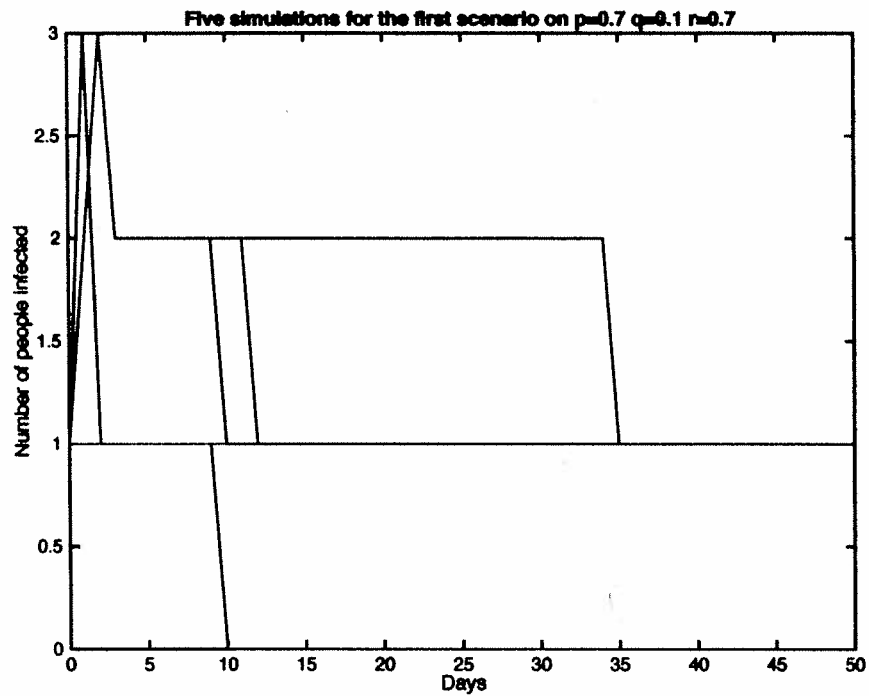
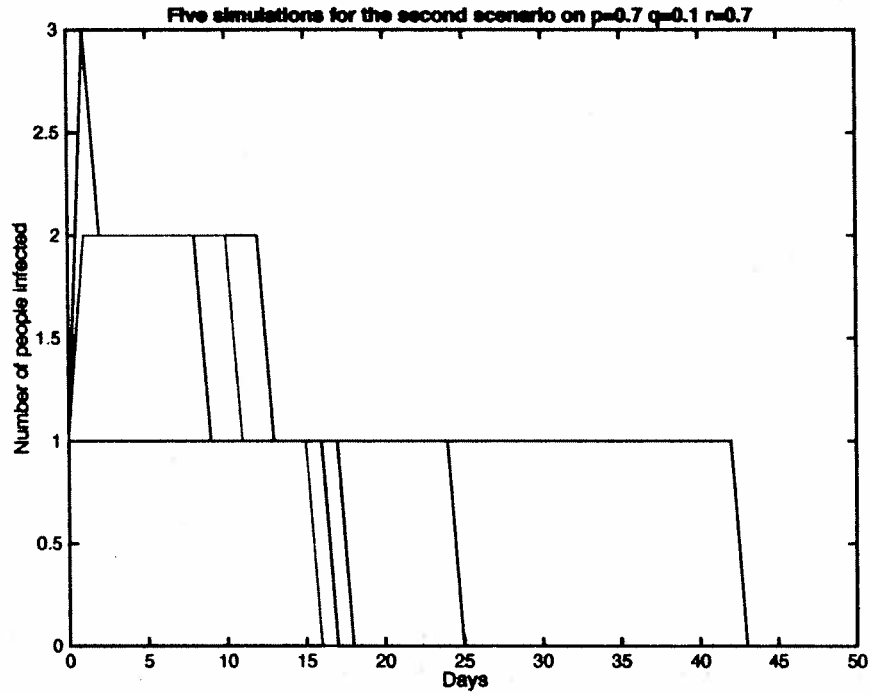
The result of the update code is shown in the following figure.

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f.

Four MATLAB codes are attached for this problem (see Appendix).



For the second scenario, the simulation is simple. Let vector k_i denotes the state vector in i th day. As mentioned, $k_0 = [0,1,0,0]$. Based on it, we can calculate k_1 .

```

p43=3*q*(1-q)^2;
p44=(1-q)^3;
P=[p11,p12,p13,p14;p21,p22,p23,p24;p31,p32,p33,p34;p41,p42,p43,p44];
for i=1:1:4
    for j=1:1:4
        P(i,j)=simplify(P(i,j));
    end;
end;

% Draw picture for certain choices
for i=1:1:5
    P0=double(subs(P,{p,q,r},{0.2*i-0.1,0.1,0.7}));
    K=[0,1,0,0];
    kte(1)=1;
    for j=2:1:101
        K=K*P0;
        kte(j)=K*[0,1,2,3]';
    end;
    plot(d,kte);
    hold on;
end;
xlabel('Days');
ylabel('Average number of people infected');
legend('p=0.1 q=0.1 r=0.7','p=0.3 q=0.1 r=0.7','p=0.5 q=0.1 r=0.7','p=0.7 q=0.1 r=0.7','p=0.9 q=0.1 r=0.7');

```

4. MATLAB code for Question 1.3.e

% Model for the 1st scenario

```
clear;
```

```
clc;
```

% Initialize

```
d=0:1:100;
```

```
kte=zeros(1,101);
```

```
syms p q r;
```

% Three-point and point-line with point infected situation

```
r1=(1-r)^3+r*(1-r)^2;
```

```
p111=1;
```

```
p112=0;
```

```
p121=q;
```

```
p122=1-q;
```

```
P1=[p111,p112;p121,p122];
```

% Point-line with one of the line infected situation

```
r2=2*r*(1-r)^2;  
p211=1;  
p212=0;  
p213=0;  
p221=q*(1-p);  
p222=(1-q)*(1-p)+q*p;  
p223=(1-q)*p;  
p231=q^2;  
p232=2*q*(1-q);  
p233=(1-q)^2;  
P2=[p211,p212,p213;p221,p222,p223;p231,p232,p233];
```

% Two-line situation

```
r3=3*r^2*(1-r);  
% States are different  
% 1 - no one is infected  
% 2 - connection is infected  
% 3 - one of the terminals is infected  
% 4 - connection and one of the terminals are infected  
% 5 - both of the terminals are infected  
% 6 - all of the three are infected  
p311=1;  
p312=0;  
p313=0;  
p314=0;  
p315=0;  
p316=0;  
p321=q*(1-p)^2;  
p322=(1-q)*(1-p)^2;  
p323=2*q*p*(1-p);  
p324=2*(1-q)*p*(1-p);  
p325=q*p^2;  
p326=(1-q)*p^2;  
p331=q*(1-p);  
p332=q*p;  
p333=(1-q)*(1-p);  
p334=(1-q)*p;  
p335=0;  
p336=0;  
p341=q^2*(1-p);  
p342=q*(1-q)*(1-p);  
p343=q^2*p+q*(1-q)*(1-p);  
p344=q*(1-q)*p+(1-q)^2*(1-p);
```

```

p345=q*(1-q)*p;
p346=(1-q)^2*p;
p351=q^2*(1-p)^2;
p352=q^2*(1-(1-p)^2);
p353=2*q*(1-q)*(1-p)^2;
p354=2*q*(1-q)*(1-(1-p)^2);
p355=(1-q)^2*(1-p)^2;
p356=(1-q)^2*(1-(1-p)^2);
p361=q^3;
p362=q^2*(1-q);
p363=2*q^2*(1-q);
p364=2*q*(1-q)^2;
p365=q*(1-q)^2;
p366=(1-q)^3;
P3=[p311,p312,p313,p314,p315,p316;p321,p322,p323,p324,p325,p326;p331,p332,p
333,p334,p335,p336;p341,p342,p343,p344,p345,p346;p351,p352,p353,p354,p355,p
356;p361,p362,p363,p364,p365,p366];

```

% Triangle situation

```

r4=r^3;
p411=1;
p412=0;
p413=0;
p414=0;
p421=q*(1-p)^2;
p422=q^2*p*(1-p)+(1-q)*(1-p)^2;
p423=q*p^2+(1-q)^2*p*(1-p);
p424=(1-q)*p^2;
p431=q^2*(1-p)^2;
p432=q^2*(1-(1-p)^2)+2*q*(1-q)*(1-p)^2;
p433=2*q*(1-q)*(1-(1-p)^2)+(1-q)^2*(1-p)^2;
p434=(1-q)^2*(1-(1-p)^2);
p441=q^3;
p442=3*q^2*(1-q);
p443=3*q*(1-q)^2;
p444=(1-q)^3;
P4=[p411,p412,p413,p414;p421,p422,p423,p424;p431,p432,p433,p434;p441,p442,p
443,p444];

```

% Draw picture for certain choices

for i=1:5

```

P10=double(subs(P1,{p,q},{0.2*i-0.1,0.1}));
P20=double(subs(P2,{p,q},{0.2*i-0.1,0.1}));
P30=double(subs(P3,{p,q},{0.2*i-0.1,0.1}));

```

```

P40=double(subs(P4,{p,q},{0.2*I-0.1,0.1}));
r10=double(subs(r1,{r},{0.7}));
r20=double(subs(r2,{r},{0.7}));
r30=double(subs(r3,{r},{0.7}));
r40=double(subs(r4,{r},{0.7}));
K1=[0,1];
K2=[0,1,0];
K3=[0,1/3,2/3,0,0,0];
K4=[0,1,0,0];
kte(1)=1;
for j=2:1:101
    K1=K1*P10;
    K2=K2*P20;
    K3=K3*P30;
    K4=K4*P40;

kte(j)=r10*K1(2)+r20*(K2(2)+K2(3)*2)+r30*((K3(2)+K3(3))+(K3(4)*K3(5))*2+K3(6)*3)
+r40*(K4(2)+K4(3)*2+K4(4)*3);
end;
plot(d,kte);
hold on;
end;
xlabel('Days');
ylabel('Average number of people infected');
legend('p=0.1 q=0.1 r=0.7','p=0.3 q=0.1 r=0.7','p=0.5 q=0.1 r=0.7','p=0.7 q=0.1
r=0.7','p=0.9 q=0.1 r=0.7');

```

5. MATLAB code for Question 1.3.f for second scenario

% Simulation for the 2nd scenario

clear;

cic;

% Initialize

d=0:1:50;

kte=zeros(1,51);

syms p q r;

% Form the transition matrix

p11=1;

p12=0;

p13=0;

p14=0;

p21=q*((1-r)^2+2*r*(1-r)*(1-p)+r^2*(1-p)^2);

p22=q*(2*r*(1-r)*p+2*r^2*p*(1-p))+(1-q)*((1-r)^2+2*r*(1-r)*(1-p)+r^2*(1-p)^2);

f) Scenario: Everyday each pair of three (n) people has a probability of interacting that day, independently of how they have interacted in the past.

(1) Let our n people to be A_1, \dots, A_n at time point t. $X_t^1, X_t^2, \dots, X_t^n$ describes their state

$X_t^k = 1 \Rightarrow$ At time t, A_k is ill
 $X_t^k = 0 \Rightarrow$ At time t, A_k is healthy.
 $N_t = \sum_{k=1}^n X_t^k$ is the number of people infected on time t.
 (2) Initialize:

$$X_0^1 = 1, X_0^2 = X_0^3 = \dots = X_0^n = 0$$

(3) Updating Rule:

where we know the value of $X_t^k, k=1, \dots, n$, then we use following method to get X_{t+1}^k :

$$(1) X_{t+1}^k = X_t^k, k=1, \dots, n$$

we scan from $k=1$ to $k=n$
 (ii) If $X_t^k = 0$, $X_{t+1}^k = k+1$

2° If $X_t^k = 1$,

we scan from $k'=1$ to $k'=n$
 If $X_{t+1}^{k'} = 1$, nothing happens;
 Else If $X_{t+1}^{k'} = 0$, $X_{t+1}^k = 0$,
 generating $w_1 \sim U(0,1)$, if $w_1 < q$, $X_{t+1}^k = 1$
 Else $X_{t+1}^k = 0$, nothing happens.

(iii) generating $w_2 \sim U(0,1)$, if $w_2 < q$, $X_{t+1}^k = 0$, else nothing changes
 If $X_{t+1}^k = k+1$, Else we return to (ii)

$$(4) \quad X_k^{m+1} = X_k \quad k=1, \dots, m$$

$m := M$, if $m > M$, we quit
else come back to (3)

② Scenario: For each pair of the ~~(3)~~ 3 (n) people, that pair has a probability to be an interacting pair.

(1) Let our n people to be A_1, \dots, A_n , X_1^t, \dots, X_n^t describes their state at time t .

$X_k^t = 1 \Leftrightarrow$ At time t , A_k is ill.

$X_k^t = 0 \Leftrightarrow$ At time t , A_k is healthy.

$N_t = \sum_{k=1}^n X_k^t$ is the number of people infected on n -th day.

(2) Initialize: $X_1^0 = 1, X_2^0 = X_3^0 = \dots = X_n^0 = 0$

~~(3) Updating rule:~~

~~we know the value of $X_k^m, k=1, 2, \dots, n$~~

~~(i) $X_k = r_k = X_k^m, k=1, \dots, n$~~

~~for~~

Determine which pairs are interacting:

For $i=1:n$

For $j=i+1:n$

Generate $R \sim U(0,1)$

if $R < r_{ij}$

Else, $r_{ij} = 0$

End

End

Let $r_{ij} = r_{ji}$ if $j < i$

$r_{ii} = 0$

(In fact r_{ii} has no influence)

(4) Updating: From X_k^m , ~~nothing~~ get X_k^{m+1}

(i) $X_k = \text{old} = X_k^m, k=1, \dots, n$

(ii) ~~start from k=1~~ For $k=1:n$

~~(iii)~~ If $y_k = 0$, nothing happens

Else if $y_k = 1$:

Do For $k'=1:n$

If $r_{k'k} = 0$, nothing happens.

Else if $X_{k'} = 1$, nothing happens

Else, Generate $W_1 \sim U(0,1)$,

If $W_1 < p$, $X_{k'} = 1$

Else, nothing happens

~~End~~ End

End

(iii) Generate $W_2 \sim U(0,1)$

If $W_2 < q$, $X_k^{k'} = 0$

Else nothing happens.

~~(4)~~ $X_k^{m+1} = X_k, m := m+1$

(5) If $m > M$ (M is our ending time)
quit;

Else do (4) again.

Code

```
function res=Simulate_Senriol(M,n,p,q,r)
%senrio is whether people interact is independent of the past
%res(i) stands for the number of infected people i-th day
%M is the end time
%n is the number of people
%p,q,r is the same as our quesiton
res=zeros(M,1);
T=linspace(1,M,M);
% Initialize x
x(1)=1;
for i=2:n
    x(i)=0;
end

%update
for t=1:M
    record=x;
    x_record=x;
    for k=1:n
        if record(k)==1
            for k2=1:n
                if x_record(k2)==0
                    w=rand();
                    if w<p*r
                        x_record(k2)=1;
                    end
                end
            end
            w=rand();
            if w<q
                x_record(k)=0;
            end
        end
    end
    x=x_record;
    res(t)=0;
    for i=1:n
        res(t)=res(t)+x(i);
    end
end

plot(T,res);
xlabel('time');
ylabel('People infected');
title(sprintf('p=%e,q=%e,r=%e,n=%i,M=%i',p,q,r,n,M));
```

```
function difference=SS1_run
```

```
M=20;
n=15;
p=0.5;q=0.5;r=0.5;
figure(1)
```

```

Simulate_Senriol(M,n,p,q,r);

M=20;
n=15;
p=0.9;q=0.1;r=0.9;
figure(2)
Simulate_Senriol(M,n,p,q,r);

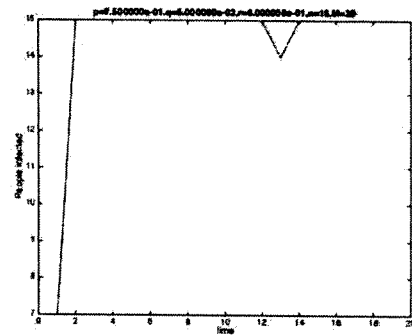
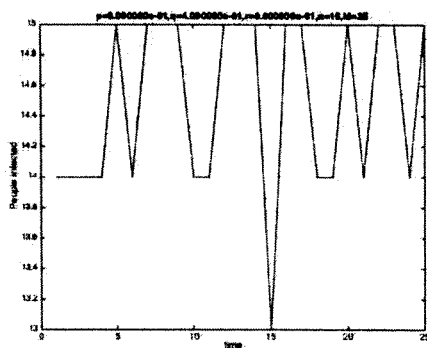
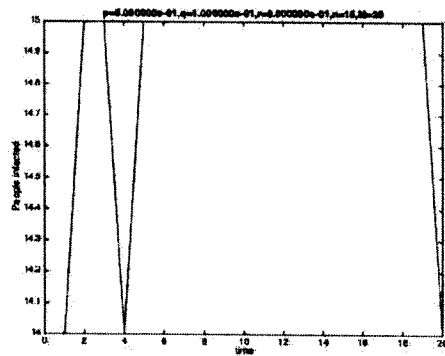
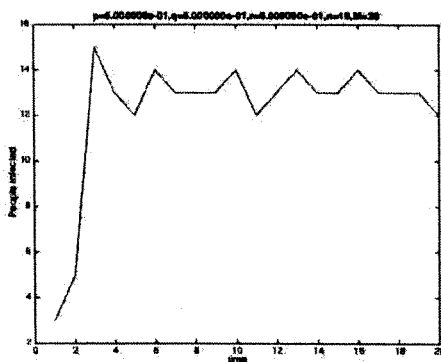
M=25;
n=15;
p=0.9;q=0.4;r=0.9;
figure(3)
Simulate_Senriol(M,n,p,q,r);

M=20;
n=15;
p=0.75;q=0.05;r=0.8;
figure(4)
Simulate_Senriol(M,n,p,q,r);

N=200;
record=0;
M=20;
n=3;
p=0.75;q=0.2;r=0.8;
for i=1:N
    figure(6);
    res=Simulate_Senriol(M,n,p,q,r);
    record=record+res(M);
end
difference=record/N-Average_b(M,p,q,r);

```

simulation for the first scenario



Let N_{20}^k be the result of the k -th ~~the~~ ^{infected} ~~average~~ number on the 20-th day for

$$\left| \frac{\sum_{k=1}^{200} N_{20}^k}{200} - N_{20} \right| = 4.567692675833790e-01$$

This is not very big implying our simulation to be reasonable.

```

function res=Simulate_Senrio2(M,n,p,q,r)
%senrio is whether people interact is independent of the past
%M is the end time, n is the number of people
%res(i) stands for the number of infected people i-th day

res=zeros(M,1);
T=linspace(1,M,M);
% Initialize x
x(1)=1;
for i=2:n
    x(i)=0;
end

meet=zeros(n,n);
% Decide which pairs are interactin pairs
for i=1:n
    for j=i+1:n
        w0=rand();
        if w0<r
            meet(i,j)=1;
        end
    end
end
for i=1:n
    for j=1:i
        meet(i,j)=meet(j,i);
    end
end

%update
for t=1:M
    record=x;
    x_record=x;
    for k=1:n
        if record(k)==1
            for k2=1:n
                if x_record(k2)==0
                    if meet(k,k2)==1
                        w=rand();
                        if w<p
                            x_record(k2)=1;
                        end
                    end
                end
            end
        end
        w=rand();
        if w<q
            x_record(k)=0;
        end
    end
    x=x_record;
    res(t)=0;
    for i=1:n
        res(t)=res(t)+x(i);
    end
end

```

end

```
plot(T,res);  
xlabel( 'n' );  
ylabel('People infected');  
title( sprintf('p=%e,q=%e,r=%e',p,q,r) );
```

function difference=SS2_run

```
M=20;  
n=15;  
p=0.5;q=0.5;r=0.5;  
figure(1)  
Simulate_Senrio2(M,n,p,q,r);
```

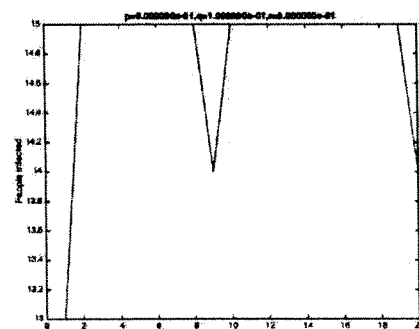
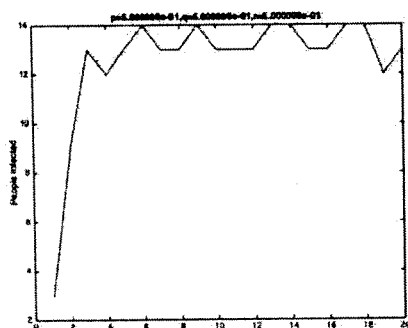
```
M=20;  
n=15;  
p=0.9;q=0.1;r=0.9;  
figure(2)  
Simulate_Senrio2(M,n,p,q,r);
```

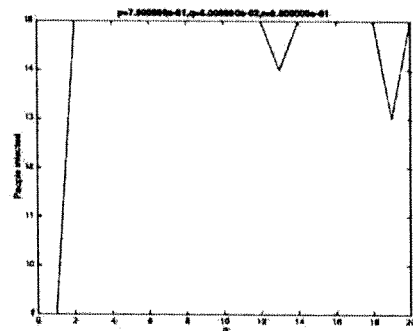
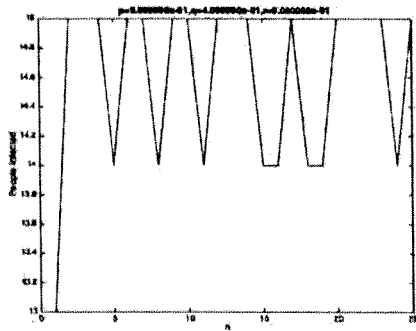
```
M=25;  
n=15;  
p=0.9;q=0.4;r=0.9;  
figure(3)  
Simulate_Senrio2(M,n,p,q,r);
```

```
M=20;  
n=15;  
p=0.75;q=0.05;r=0.8;  
figure(4)  
Simulate_Senrio2(M,n,p,q,r);
```

```
N=200;  
record=0;  
M=20;  
n=3;  
p=0.75;q=0.2;r=0.8;  
for i=1:N  
    figure(6);  
    res=Simulate_Senrio2(M,n,p,q,r);  
    record=record+res(M);  
end  
difference=record/N-Average_e(M,p,q,r);
```

Simulation for the second





$$3.104764245782425e-01 = \left| N_{20} - \frac{\sum_{k=1}^{200} N_{20}^k}{200} \right|$$

N_{20}^k is the number of people infected on the 20-th day for the k -th simulation.

This difference is not so large, so our simulation seems to be reasonable.

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