Office hours:

- Mondays 5-6 PM (undergraduate class preference)
- Tuesdays 5-6 PM (this class preference)
- Thursdays 5-6 PM (free-for-all)

Reading:

- Karlin and Taylor Secs. 1.1-1.3

Homework 1 posted soon, due Friday, October 1 at 5 PM.

- www.rpi.edu/~kramep/Stoch/stoch2015.html

No class on Monday; next class is on Thursday, September 10.

Optional reading for applied probability:

- Bertsekas & Tsitsiklis, *Introduction to Probability*
- Ross, *A First Course in Probability*

Optional reading for rigorous, measure-theoretic probability:

- Kloeden and Platen, Secs. 1.1-1.3
- Shreve, *Stochastic Calculus for Finance Volume II*, Chapters 1-2
- Sinai, *Lectures on Probability Theory*
- Billiingsley, *Probability and Measure*

Probability theory was only properly mathematicized in the 1930s by Kolmogorov using measure theory

- separate mathematics from philosophy/intuition (modeling)
- measure theory is a subject in real analysis which is concerned with mathematical structures that map sets into numbers

The upshot of the measure-theoretic viewpoint is that a mathematically well-defined probability model consists of three ingredients:

1. Sample space Ω
    - note that this is not the same as the state space of a random variable

    This is an abstract space that encodes all possible outcomes of the components of the system of which we are uncertain, and which are relevant to the questions of interest. It's the "universe of possibilities."

    For example, suppose we want to consider the number of migrants entering the

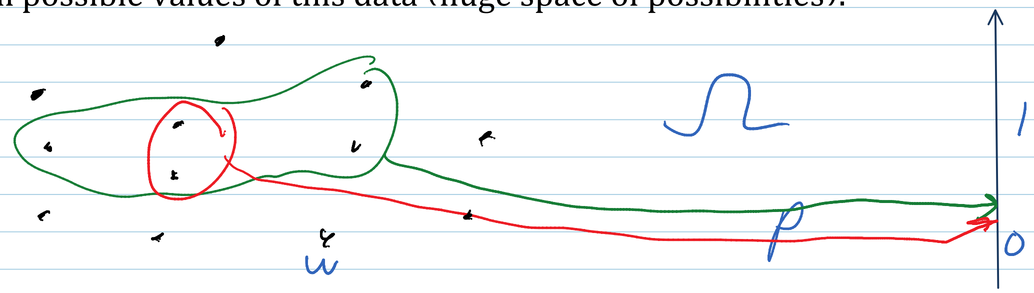
Budapest train station over the next month.

corrected after lecture

Sample space might look something like this: Every point $\omega \in \Omega$ (called an elementary outcome) has the following information:

- number of migrants leaving Syria on day n, n goes from 1 to 30
- for each migrant, their intended destination
- for each migrant, whether they successfully pass a milestone on their journey to Europe
- weather in Balkans for each day, 1..30

Ω is the space of all possible values of this data (huge space of possibilities).



2. σ −algebra $\mathcal{F}$ of "measurable sets" or "events"

For the case of discrete sample spaces (which will apply for most of the class), we can just take $\mathcal{F}$ to be the set of all subsets 2^{Ω}.

In more complex settings (particularly with continuous spaces), one has to pay more attention to this object for two reasons:

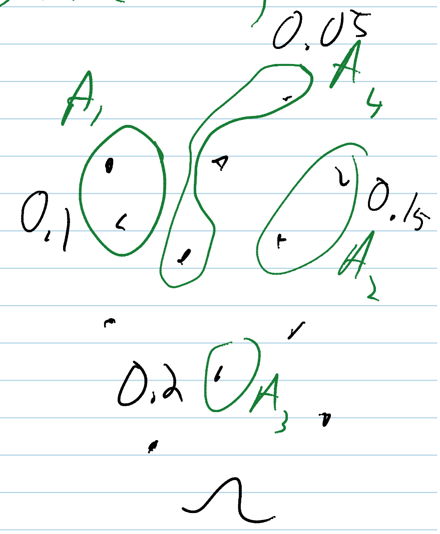
- technical mathematics; for continuous sample spaces, it is not possible to "measure" every possible subset in a satisfying way.
- also useful for encoding information flow, particularly in finance

A σ −algebra is a mathematical structure that has the property that it is closed under the operations of finite unions, countable unions, finite intersections, countable intersections, and complements.

3. Probability measure:

$$P: \mathcal{F} \to [0,1]$$

||
measurable
subsets
(can be 2^Ω for discrete Ω)



with the following axioms:

- $P(A) \in [0,1]$ for any $A \in \mathcal{F}$
- $P(\Omega) = 1$
- The probability measure is finitely and countably additive, meaning that given any collection $\{A_i\}_{i\in I} \subseteq \mathcal{F}$

  where I is a finite or countable index set which have the property of being disjoint or mutually exclusive

  $$A_i \cap A_{i'} = \emptyset \text{ for } i \neq i'$$

  then $P(\cup_{i\in I} A_i) = \sum_{i\in I} P(A_i)$

  ||
  Probability
  (A_1 or A_2 or A_3 ...)

$$P\left(\bigcup_{i=1}^{4} A_i\right) = 0.1 + 0.2 + 0.15 + 0.05 = 0.5$$

Any choice of these three objects consistent with the above constraints gives a probability space or probability model $(\Omega, \mathcal{F}, P)$

All the modeling goes into how these mathematical structures are chosen to represent a given system. Once this probability model is specified, then the calculations regarding the properties of the model can be put on mathematically rigorous and unequivocal foundations.

How does the modeling step (setting up the probability model) work in practice?

- sample space: define the relevant uncertain inputs and parameters that are relevant
- σ −algebra: almost always implicitly taken to be the richest possible
- probability measure: implicitly defined by giving a sufficiently detailed model to calculate any desired probability

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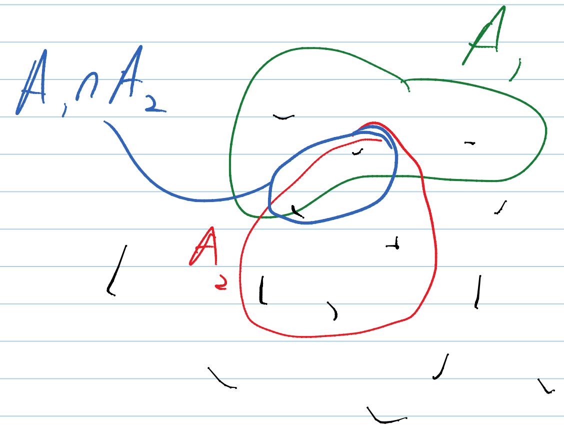
From the axioms of probability theory, one can deduce some fundamental relations:



$$P(\emptyset) = 0$$

$$P(A^c) = 1 - P(A)$$

A^c = not A

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$A_1 \cup A_2$: A_1 or A_2

$$P(A_1 \cap A_2) = \ldots ?$$

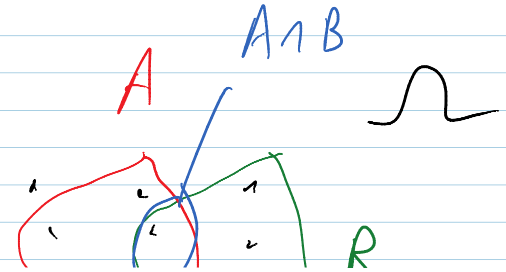
$A_1 \cap A_2$: A_1 and A_2

Besides the above relationship between ``and'' and ``or'' probabilities, there is no general way to compute the probability of the union or intersection of two events without further information about their relationship to each other. To make progress then, we need to develop a mechanism that describes the relationship between two events:

Conditional Probability

For any events $A, B \in \mathcal{F}$

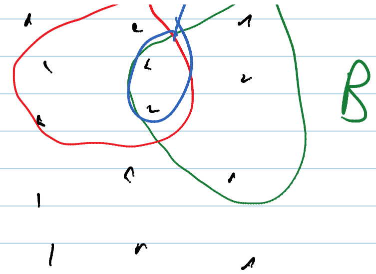
we define the conditional probability of event A given event B as follows:



$P(A|B) = \frac{P(A\cap B)}{P(B)}$ provided $P(B) > 0$.

$P(A|B) = \frac{\quad}{P(B)}$ provided $P(B) > 0$.

unknown known info



A very useful fundamental relationship between conditional probabilities is given by Bayes' rule:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$