Finite Horizon Calculations

Thursday, September 24, 2015

2:01 PM

Homework 1 due Friday, October 2 at 5 PM.

- no office hours on Tuesday, September 29 or Thursday, October 1

Continuing with the queueing model from last time. One can formulate alternative FSDT MC models for the single server queueing problem by choosing the epochs in a different way. Namely, we can define epochs in terms of the occurrence of certain incidents rather than necessarily as a fixed time step. So can formulate queueing models, for example, with epochs defined as the moment of the nth departure (completed service) or nth arrival of demand.

We'll show how to formulate the model for the case when the nth epoch corresponds to the nth completed service, but the idea can be similarly applied to the other choice mentioned.

X_n = length of the queue after the nth service has been completed

The key statistical parameters we need to formulate the MC for this choice of epoch is the probability distribution $\{p_j\}_{j=0}^{\infty}$ for the number of demands that arrive during a service period (based on histogram of historical data perhaps).

Then the probability transition matrix for the FSDT MC based on service completions: State space $S = \{0,1,2,\dots,M-1\}$ because can't have M items in the queue right after service completion which depleted queue by 1.

$$1 - \sum_{j=0}^{M-2} p_j$$

$$P = \begin{array}{c|c} & \begin{array}{cccccc} 0 & 1 & 2 & \cdots & M-2 & M-1 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ \\ M-1 \end{array} & \begin{pmatrix} p_0 & p_1 & p_2 & \cdots & p_{M-2} & \sum_{j=M-1}^{\infty} p_j \\ p_0 & p_1 & p_2 & \cdots & p_{M-2} & \sum_{j=M-1}^{\infty} p_j \\ 0 & p_0 & p_1 & \cdots & p_{M-3} & \sum_{j=M-2} p_j \\ & 0 & p_0 & p_1 & & \\ & & 0 & p_0 & p_1 \ddots & \\ & 0 & & 0 & \ddots \; p_0 & 1-p_0 \end{pmatrix} \end{array}$$

Stochastic update rule?

$$X_{n+1} = \min\left((X_n - 1)_+ + Z_n,\; M-1\right)$$

Z_n iid [illegible] dist $\{p_j\}^{\infty}$

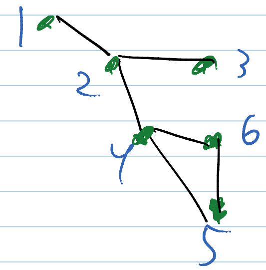
Z_n iid, w/ prob dist $\{p_j\}_{j=0}^{\infty}$

$\parallel$ # arrivals during service period

Another example:

Random Walk on a Graph (Lawler Ch. 1)

At each epoch, an agent at a node can stay at same node or move to a neighboring node.



$$P = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & p_{11} & p_{12} & 0 & 0 & 0 & 0 \\ 2 & p_{21} & p_{22} & p_{23} & p_{24} & 0 & 0 \\ 3 & & & & & & \\ 4 & & & & & & \\ 5 & & & & & & \\ 6 & & & & & & \end{array}$$

Random walks on graphs can be models for:

- actual physical motion between certain regions by people and/or animals
- electronic excitation of atoms
- configurational changes in biomolecules (Christof Schuette, Berlin)
- PageRank type algorithms to characterize relative importance of nodes in an internet or other kinds of network
    - David F. Gleich (2015), "PageRank Beyond the Web," *SIAM Review*, **57** (3), 321-364.

State space = $\{1,2,\ldots,M\}$ is the set of nodes, however labeled.

Typically one will specify the model in terms of a probability transition matrix P where P_{ij} is the probability that an agent at node i will move to node j in the next epoch.

- $P_{ii} > 0$ allowed, unless one defines the epochs in terms of the moment at which the state changes (this gives rise to a so-called embedded Markov chain)
- $P_{ij} \neq P_{ji}$ allowed
- $\sum_{j=1}^{M} P_{ij} = 1$

Absent some sort of regular structure of the graph and the transition probabilities, the stochastic update rule would be awkward.

Final example:

Inventory Model (Karlin and Taylor Sec. 2.3)

Imagine that a business with unpredictable demand is able to store a maximum inventory of M items in stock. Each business day, there is an unpredictable demand with given probability distribution $p_j = P(D = j)$. The restocking policy of the business is, at the end of every business day, check the stock and if there is fewer than s items remaining, then order enough new items to fill up the available inventory. Assume that this replacement order is fulfilled at 2 AM the next business day.

What will be the dynamics of the available stock over time?

Let's take each business day as an epoch.

X_n will be defined as the number of items in stock at the end of the nth business day.
(beginning of business day is also OK choice, but slightly different model)
State space $S = \{0, \dots, M\}$

Probability transition matrix possible but seems easier to just formulate stochastic update rule:

$$X_{n+1} = \begin{cases} (X_n - Z_n)_+ & \text{if } X_n \geq s \\ (M - Z_n)_+ & \text{if } X_n < s \end{cases}$$

where Z_n is the demand that occurs on day $n+1$,
with probability ditsribution $\{p_j\}_{j=0}^{\infty}$.

Finite Horizon Statistics for FDST Markov Chains

Now that we've defined Markov chain models, we want to compute properties of their behavior. We'll develop some deterministic approaches to complement direct stochastic (Monte Carlo) simulation.

Our goal is to answer questions like, in the context of the inventory problem:

- Over the next two weeks, how much unfulfilled demand will there be?
- What is the long term average of my stock?
- Average shipping cost (the fraction of days that I need to reorder).

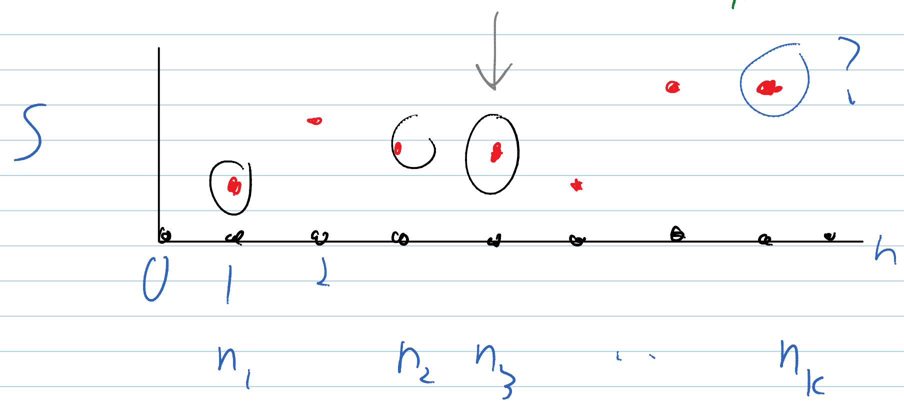
We'll begin by addressing these kinds of questions over a finite time horizon, i..e, a finite number of epochs into the future.

To develop the basic formulas for doing these finite-horizon calculations, we start with the following formulation of the Markov property:

$$P(X_{n_k} = j_k \mid X_{n_1} = j_1, X_{n_2} = j_2, \ldots, X_{n_{k-1}} = j_{k-1})$$

$$= P(X_{n_k} = j_k \mid X_{n_{k-1}} = j_{k-1})$$

for $n_1 < n_2 < \ldots < n_k$



This form of the Markov property can be derived by conditional probability calculations from the original definition of the Markov property.

Now that means if we want to calculate any finite dimensional distribution of the Markov chain:

$$P(\underbrace{X_{n_1} = j_1, X_{n_2} = j_2, \ldots,}_{A} \underbrace{X_{n_k} = j_k}_{B})$$

def'n cond prob

$$= P(\underbrace{X_{n_k} = j_k}_{B} \mid \underbrace{X_{n_1} = j_1, X_{n_2} = j_2, \ldots, X_{n_{k-1}} = j_{k-1}}_{A})$$

$$\times P(\underbrace{X_{n_1} = j_1, X_{n_2} = j_2, \ldots, X_{n_{k-1}} = j_{k-1}}_{A})$$

Markov property

$$= P(X_{n_k} = j_k \mid X_{n_{k-1}} = j_{k-1})$$

$$\times P(X_{n_1} = j_1, X_{n_2} = j_2, \ldots, X_{n_{k-1}} = j_{k-1})$$

Notice that this expresses the joint distribution of the state variable at k epochs $(n_1, n_2, \dots, n_k)$ in terms of the joint distribution of the state variable at $k-1$ epochs $(n_1, n_2, \dots, n_{k-1})$. So repeating this procedure, and invoking induction/recursion, we obtain:

$$P(X_{n_1} = j_1, X_{n_2} = j_2, \dots, X_{n_k} = j_k)$$
$$= P(X_{n_k} = j_k \mid X_{n_{k-1}} = j_{k-1})$$
$$\times P(X_{n_{k-1}} = j_{k-1} \mid X_{n_{k-2}} = j_{k-2})$$
$$\vdots$$
$$\times P(X_{n_2} = j_2 \mid X_{n_1} = j_1)$$
$$\times P(X_{n_1} = j_1)$$

Therefore, the joint distribution of the state variable of a Markov chain at any k epochs can be computed in terms of quantities that involve only 1 epoch or 2 epochs (at a time).

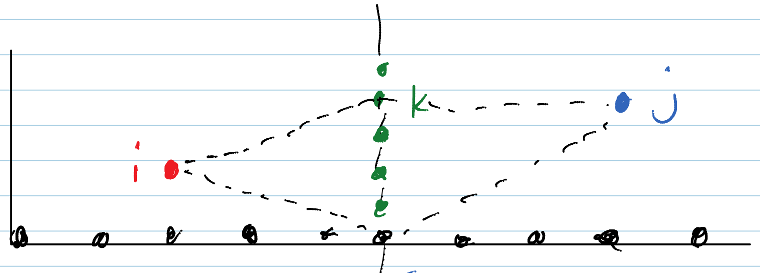
So how do we compute these 1-epoch and 2-epoch statistics? The only difficulty is that the epochs for the 2-epoch statistics need not be adjacent.

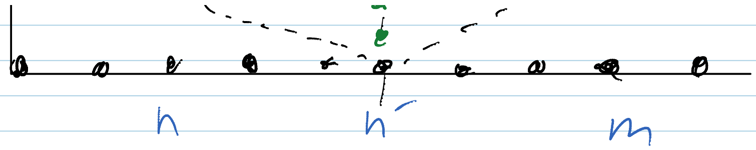
But we can readily overcome this by using the

Chapman-Kolmogorov equation

For any $n < n' < m$,

$$P(X_m = j \mid X_n = i) = \sum_{k \in S} P(X_m = j \mid X_{n'} = k) P(X_{n'} = k \mid X_n = i)$$





Proof is just combining the law of total probability (with a partition given by $\{X_{n'} = k\}_{k\in S}$) and the Markov property.

This allows us to compute the 2-epoch conditional probabilities in terms of the probability transition matrix by applying the Chapman-Kolmogorov equation recursively for either $n' = n+1$ or $n' = m-1$. The outcome of this recursion procedure is the following formula, in matrix form:

$$P(X_m = j \mid X_n = i) = \left(P^{(n)} P^{(n+1)} \cdots P^{(m-1)}\right)_{ij}$$

For time-homogenous MCs,

$$= \left(P^{m-n}\right)_{ij}$$

Also, by similar argument:

$$P(X_m = j) = \sum_{k\in S} P(X_0 = k)\, P(X_m = j \mid X_0 = k)$$

$$= \left(\vec{\phi}^{\,T} \cdot P^m\right)_j$$

$$\text{where} \quad \phi_k = P(X_0 = k)$$

Note the sums implied by the matrix multiplication are precisely the sums at intermediate epochs that arise from the Chapman-Kolmogorov equation (for the 2-epoch transition probabilities) or the law of conditional probability (for the 1-epoch probability distribution).