Martingales

Thursday, December 03, 2015

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References: Karlin and Taylor Ch. 6

Lawler Sec. 5.1-5.3

Homework 4 due date extended to Wednesday, December 16 at 5 PM.

We say that a random variable Z is measurable with respect to a $\sigma-algebra$ $\mathcal{G}$ if for any reasonable (Borel) set B,

the event $\{Z \in B\} \in \mathcal{G}$.

In the context of filtrations, a random variable Z being measurable w.r.t $\mathcal{A}_n$

means in practice that Z is completely determined by the information available at epoch n (as encoded by the filtration).

In particular, when we said that we want the filtration $\{\mathcal{A}_n\}_{n=0}^{\infty}$

to be such that the martingale Y_n is adapted to it, the key property is that Y_n is measurable w.r.t. $\mathcal{A}_n$ for all n.

Key formulas involving this concept:

1) If Z is measurable w.r.t. the σ $-$algebra $\mathcal{G}$

then: $$\mathbb{E}(f(Z)|\mathcal{G}) = f(Z)$$ for any deterministic f.

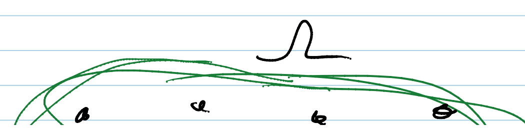
1) More generally, if Z is measurable w.r.t. the σ $-$algebra $\mathcal{G}$

then for any deterministic f,g and any other random variable Y:

$$\mathbb{E}(f(Z)g(Y)|\mathcal{G}) = f(Z)\,\mathbb{E}(g(Y)|\mathcal{G})$$

The intuition behind these formulas is that if Z is measurable w.r.t. a σ-algebra, then conditioning on that σ $-$algebra makes Z behave deterministically.

$$\mathbb{E}(f(Z)|\mathcal{G})$$

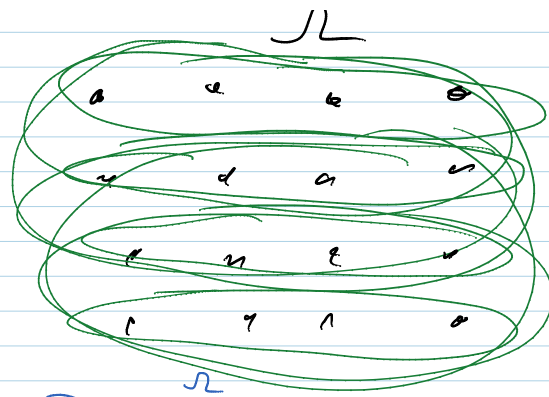


that σ −algebra makes Z behave deterministically.

$$\mathbb{E}(f(Z)|\mathcal{I})$$

this would average over columns and give a result that depends on row, which is random under the full probability model.

Ω



$\mathcal{F} = 2^{\Omega}$

Elementary examples of martingales (Karlin and Taylor Sec. 6.7)

1) If we are given a sequence of independent (but not necessarily identically distributed) rvs $\{X_n\}_{n=1}^{\infty}$ with $\mathbb{E}|X_n| < \infty$, then we can construct the following martingale: $Y_n = \sum_{n'=1}^{n}(X_{n'} - \mathbb{E}X_{n'})$ with respect to the filtration generated by the $\{X_n\}_{n=1}^{\infty}$.

   The filtration generated by a stochastic process is the smallest filtration such that the stochastic process is measurable with respect to it. In other words, the filtration contains precisely the information content generated by the stochastic process.

   Proof: The adapted part is true by construction (Y_n only refers to $\{X_{n'}\}_{n' \leq n}$.)

$$\mathbb{E}(Y_{n+1} | \mathcal{A}_n)$$

filtration generated by $\{X_n\}$

$$= \mathbb{E}\left(\sum_{n'=1}^{n+1} (X_{n'} - \mathbb{E}X_{n'}) \Big| \mathcal{A}_n\right)$$

$$= \mathbb{E}\left(\sum_{n'=1}^{n} (X_{n'} - \mathbb{E}X_{n'}) + (X_{n+1} - \mathbb{E}X_{n+1}) \Big| \mathcal{A}_n\right)$$

$$= \underbrace{\sum_{n'=1}^{n} (X_{n'} - \mathbb{E}X_{n'})}_{\text{measurable w.r.t. } \mathcal{A}_n} + \mathbb{E}(X_{n+1} - (\mathbb{E}X_{n+1}) | \mathcal{A}_n)$$

$$= Y \quad + \mathbb{E}(X \quad (\mathbb{E}X \quad))$$

$$= Y_n + \mathbb{E}(X_{n+1} - \mathbb{E}X_{n+1})$$

because X_{n+1} independent of $\mathcal{U}_n$.

$$= Y_n + 0$$

$$\mathbb{E}(Y_{n+1} | \mathcal{U}_n) = Y_n$$

Technical property:

$$\mathbb{E}|Y_n| = \mathbb{E}\left(\left|\sum_{n'=1}^{n} (X_{n'} - \mathbb{E}X_{n'})\right|\right)$$

triangle inequality

$$\leq \mathbb{E}\left(\sum_{n'=1}^{n} |X_{n'} - \mathbb{E}X_{n'}|\right)$$

triangle ineq

$$\leq \mathbb{E}\left(\sum_{n'=1}^{n} |X_{n'}| + |\mathbb{E}X_{n'}|\right)$$

$|\mathbb{E}Y| \leq \mathbb{E}|Y|$ Hölder or sth simple

$$\leq \sum_{n'=1}^{n} \left(\mathbb{E}|X_{n'}| + \mathbb{E}|X_{n'}|\right)$$

$$\mathbb{E}|Y_n| < \infty$$

Therefore Y_n is a martingale w.r.t. the indicated filtration.

2) Suppose we are given a collection of independent (not necessarily identically distributed) rvs $\{X_n\}_{n=1}^{\infty}$ such that $\mathbb{E}X_n = 0, \sigma_n^2 = \mathbb{E}X_n^2 < \infty$

Then: $Y_n \equiv (\sum_{n'=1}^{n} X_{n'})^2 - \sum_{n'=1}^{n} \sigma_{n'}^2$ will be a martingale w.r.t. the filtration generated by the $\{X_n\}_{n=1}^{\infty}$

We see here a typical theme which is that martingales looks like "corrections" to some operation on a stochastic process.

Again, Y_n is clearly adapted to the filtration $\mathcal{A}_n$ generated by the $\{X_n\}$ for the same reason as before.

Main property:

$$\mathbb{E}(Y_{n+1} \mid \mathcal{A}_n) = \mathbb{E}\Big(\Big(\underbrace{\sum_{n'=1}^{n+1} X_{n'}}_{\left(\sum_{n'=1}^{n} X_{n'} + X_{n+1}\right)}\Big)^2 - \sum_{n'=1}^{n+1} \sigma_{n'}^2 \,\Big|\, \mathcal{A}_n \Big)$$

$$= \mathbb{E}\Big(\underbrace{\Big(\sum_{n'=1}^{n} X_{n'}\Big)^2}_{\text{measurable wrt } \mathcal{A}_n} + 2 X_{n+1} \underbrace{\sum_{n'=1}^{n} X_{n'}}_{\text{measurable wrt } \mathcal{A}_n} + X_{n+1}^2 \,\Big|\, \mathcal{A}_n \Big) - \sum_{n'=1}^{n+1} \sigma_{n'}^2$$

$$= \Big(\sum_{n'=1}^{n} X_{n'}\Big)^2 + 2\Big(\sum_{n'=1}^{n} X_{n'}\Big)\mathbb{E}(X_{n+1} \mid \mathcal{A}_n) + \mathbb{E}(X_{n+1}^2 \mid \mathcal{A}_n) - \sum_{n'=1}^{n+1} \sigma_{n'}^2$$

Use independence of $X_{n+1}, \mathcal{A}_n$.

$$= \Big(\sum_{n'=1}^{n} X_{n'}\Big)^2 + 2\Big(\sum_{n'=1}^{n} X_{n'}\Big)\mathbb{E}X_{n+1} + \mathbb{E}X_{n+1}^2 - \sum_{n'=1}^{n+1} \sigma_{n'}^2$$

$$= \Big(\sum_{n'=1}^{n} X_{n'}\Big)^2 + 2\Big(\sum_{n'=1}^{n} X_{n'}\Big) 0 + \sigma_{n+1}^2$$

$$= \left(\sum_{n'=1} X_{n'}\right) + 2\left(\sum_{n'=1} X_{n'}\right) 0 + \sigma_{n+1}^2$$

$$- \sum_{n'=1}^{n+1} \sigma_{n'}^2$$

$$= \left(\sum_{n'=1}^{n} X_{n'}\right)^2 - \sum_{n'=1}^{n} \sigma_{n'}^2$$

$$\mathbb{E}(Y_{n+1} | \mathcal{U}_n) = Y_n$$

One can check $\mathbb{E}|Y_n| < \infty$ by triangle inequality and/or Schwarz inequality arguments as in the previous example.

3) Let's construct a martingale associated to a branching process.

$X_{n+1} = \sum_{k=1}^{X_n} Z_{n,k}$ where the $\{Z_{n,k}\}$ are iid random variables with mean $m = \mathbb{E}Z_{n,k}$.

Then $Y_n = \frac{X_n}{m^n}$ will be a martingale w.r.t. the filtration generated by the $\{X_n\}_{n=1}^{\infty}$ (or also w.r.t. the filtration generated by the $\{Z_{n-1,k}\}_{n=1}^{\infty}$)

Check: Y_n is again clearly adapted to these filtrations because it is determined by the sequence of random variables generating the filtration.

$$\mathbb{E}(Y_{n+1} | \mathcal{U}_n)$$ either filtration

$$= \mathbb{E}\left(\frac{1}{m^{n+1}} X_{n+1} \middle| \mathcal{U}_n\right)$$

$$= \mathbb{E}\left(\frac{1}{m^{n+1}} \sum_{k=1}^{X_n} Z_{nk} \middle| \mathcal{U}_n\right)$$ X_n measurable

$$\mathbb{E}\Big(\frac{1}{m^{n+1}}\sum_{k=1}^{X_n} Z_{n,k}\,\Big|\,\mathcal{A}_n\Big)$$

X_n measurable w.r.t. $\mathcal{A}_n$

$$= \frac{1}{m^{n+1}} \sum_{k=1}^{X_n} \mathbb{E}(Z_{n,k} \mid \mathcal{A}_n)$$

$$= \frac{1}{m^{n+1}} \sum_{k=1}^{X_n} \mathbb{E} Z_{n,k}$$

because $Z_{n,k}$ independent of $\mathcal{A}_n$

$$= \frac{1}{m^{n+1}} \sum_{k=1}^{X_n} m$$

$$= \frac{1}{m^{n+1}}\, m X_n = \frac{X_n}{m^n}$$

$$\mathbb{E}(Y_{n+1} \mid \mathcal{A}_n) = Y_n \quad \checkmark$$

$$\mathbb{E}|Y_n| = \mathbb{E}\left|\frac{X_n}{m^n}\right| = \frac{1}{m^n}\mathbb{E}|X_n|$$

$$= \frac{1}{m^n}\mathbb{E}X_n = \frac{m^n}{m^n} = 1 < \infty$$

4) $$Y_n = X_n - \mathbb{E}(X_n \mid \mathcal{A}_{n-1})$$

can be shown to be a martingale (Doob's martingale process) for any stochastic process X_n with $\mathbb{E}|X_n| < \infty$. (Karlin and Taylor Sec. 6.1k)

But this is mostly useless in practice.

What's the point in constructing martingales?

- building models
- shortcuts to problem solving
- making stochastic analysis either

Model building: Martingales are supposed to represent the dynamics of a fair game. Financial models for complete markets (i.e. fair market conditions) often refer to martingales. Actually in finance, one usually builds models based on discounted prices as martingales (because there is a background mean growth/interest rate).

Simplifying calculations: There are two key results that allow us to sometimes do difficult calculations relatively simply when we can construct an appropriate martingale:

- Martingale convergence theorem: Under weak technical conditions, martingales will converge to some random variable at large epochs. (see Karlin and Taylor Sec. 6.5).
- Optional Stopping/Sampling Theorem (OST) (Lawler Sec. 5.3, Karlin and Taylor Sec. 6.3-6.4)

Given a martingale $\{Y_n\}_{n=1}^{\infty}$ w.r.t to a filtration $\mathcal{A}_n$
and a Markov time τ w.r.t. the same filtration, suppose the following 3 conditions are satisifed:

1. $P(\tau < \infty) = 1$
2. $\mathbb{E}|Y_\tau| < \infty$
3. $\lim_{n\to\infty} \mathbb{E}(Y_n I_{\{\tau > n\}}) = 0$

Then $\mathbb{E}Y_\tau = \mathbb{E}Y_0$.

First, how do we understand the result? Notice that by iteration of conditional expectations, one can show that $\mathbb{E}Y_n = \mathbb{E}Y_0$ for any martingale $\{Y_n\}_{n=1}^{\infty}$. The OST extends this property to times τ that are random but Markov (causal).

A Markov time τ w.r.t. a filtration $\mathcal{A}_n$
is a random variable with the property that:

$$\{\tau \le n\} \in \mathcal{A}_n \qquad \text{for all } n = 1,2,\ldots.$$

Intuitively, a Markov time denotes the time of an incident whose "specialness" is known at the time it happens.

- for example, the first time to reach a state, or the first time to accumulate some value, … are Markov times
- the third time to reach some state is also Markov time, or the first time to reach state i after reaching state j, etc.
- the last time to reach some state is not a Markov time

For practical purposes, Markov times often refer to the time at which one can execute some optional power. So what the OST says is that in a fair game, no executable strategy will give you an expected return better than normal.

The conditions for the OST, while seeming technical, are important to rule out pathological cases such as: Let τ be the first time that $Y_\tau = 1$ for a standard random walk starting from $Y_0 = 0$. This example satisfies the first 2 conditions but not the third:

I_A is an indicator random variable which is: $I_A(\omega) = 1$ if $\omega \in A$

I_A is an indicator random variable which is:

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{else} \end{cases}$$