

Homework 3 posted, due Monday, November 23.

Monte Carlo simulations of branching processes are relatively simple; look at the current state X_n ; take that many independent random samples from the random variable simulation for Y , add them up, get X_{n+1} .

But what about deterministic formulas for computing statistical properties of the branching process?

We gave the updating rule for the probability distribution (encoded as the probability generating function) of the state of the Markov chain, and that deterministic formula can be iterated forward from epoch $n = 0$.

Recall that moments of a random variable can be relatively easily obtained from the probability generating function. Let's see what this implies for the mean size of a branching process:

$$\begin{aligned}
 \mathbb{E} X_{n+1} &= \left. s \frac{d}{ds} P_{X_{n+1}}(s) \right|_{s=1} \\
 &= \left. s \frac{d}{ds} \left(P_{X_n} \left(P_Y(s) \right) \right) \right|_{s=1} \quad \text{chain rule} \\
 &= \left. s P'_Y(s) P'_{X_n} \left(P_Y(s) \right) \right|_{s=1} \\
 &= \underbrace{1 P'_Y(1)}_{\mathbb{E} Y} P'_{X_n} \left(\underbrace{P_Y(1)}_{=1} \right) \\
 &= \mathbb{E} Y \cdot \left(1 P'_{X_n}(1) \right)
 \end{aligned}$$

because $P_Y(1) = \sum_{j=0}^{\infty} p_j 1^j = \sum_{j=0}^{\infty} p_j = 1$
if prob dist not "defective"

$$\mathbb{E} X_{n+1} = \mathbb{E} Y \cdot \mathbb{E} X_n$$

by recursion:

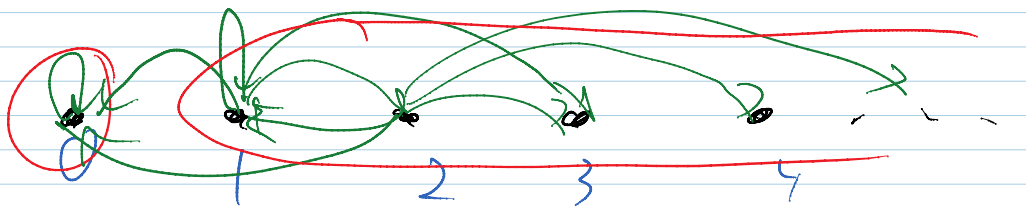
$$\mathbb{E} X_n = (\mathbb{E} Y)^n \mathbb{E} X_0$$

This is an intuitive relationship, but it's not trivial. There is a SDE analogue of this simple exponential growth model where one can get different results for the mean growth due to the fluctuations.

One can derive formulas for higher moments (variance, standard deviation, etc.) by similar means. A bit messier.

Long-Time Properties of Branching Processes

As usual we do a **topological analysis** of the Markov chain to begin to characterize its long-term properties.



Under the generic situation for the offspring distribution, namely:

- $p_0 > 0, p_j > 0$ for some $j \geq 2$

we have two communication classes: absorbing state at 0, transient class at $\{1, 2, \dots\}$

For the nongeneric cases:

- If $p_0 = 0, p_1 = 1, p_j = 0$ for $j > 1$: nothing happens, $X_n = X_0$.
- If $p_0 = 0, p_1 < 1$, then $X_n \rightarrow \infty$
- If $p_0 > 0, p_1 = 1 - p_0, p_j = 0$ for $j > 1$: then $X_n \downarrow 0$.

Now we'll just focus on the generic cases. Even though we know that all positive states are transient, there is still a question of the long-term property. Two options:

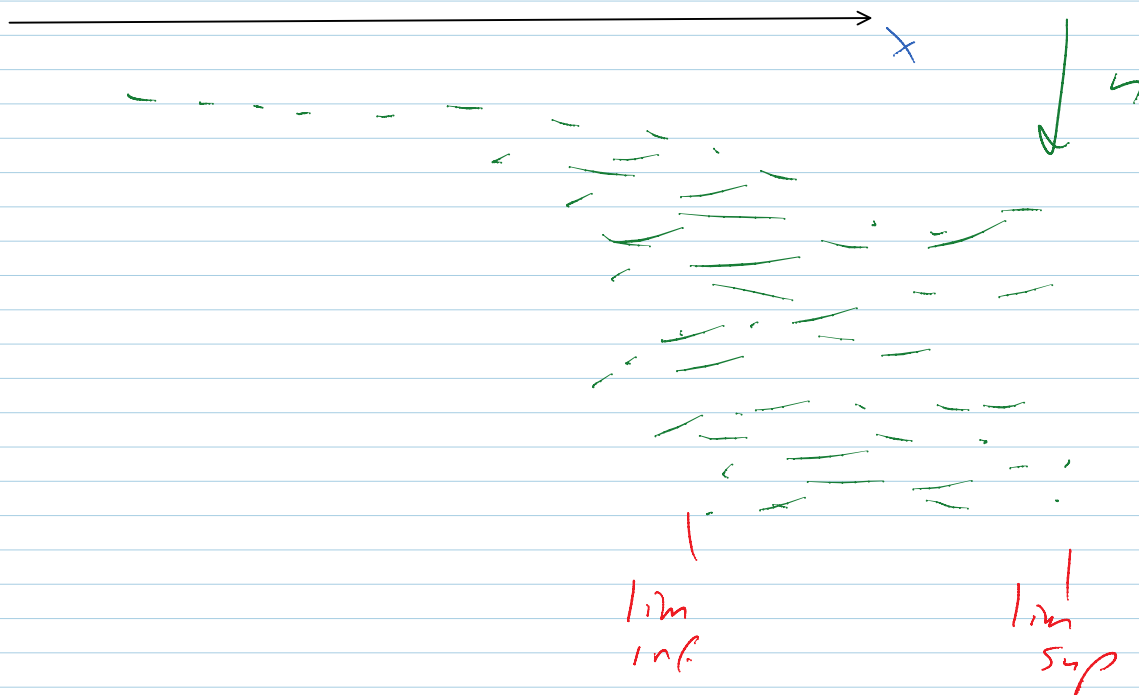
- $\lim_{n \rightarrow \infty} X_n = 0$ (extinction) or $\lim_{n \rightarrow \infty} X_n = \infty$ (explosive growth).
- To exclude other possibilities, note that transience is incompatible with

$$0 < \liminf_{n \rightarrow \infty} X_n < \infty \text{ having probability } > 0.$$

For, $X = \lim_{n \rightarrow \infty} X_n$

would be visited infinitely often, which has probability 0 for a transient state.

So then the main question is, for a given branching process, **what is the probability of extinction vs. the probability of explosive growth?**

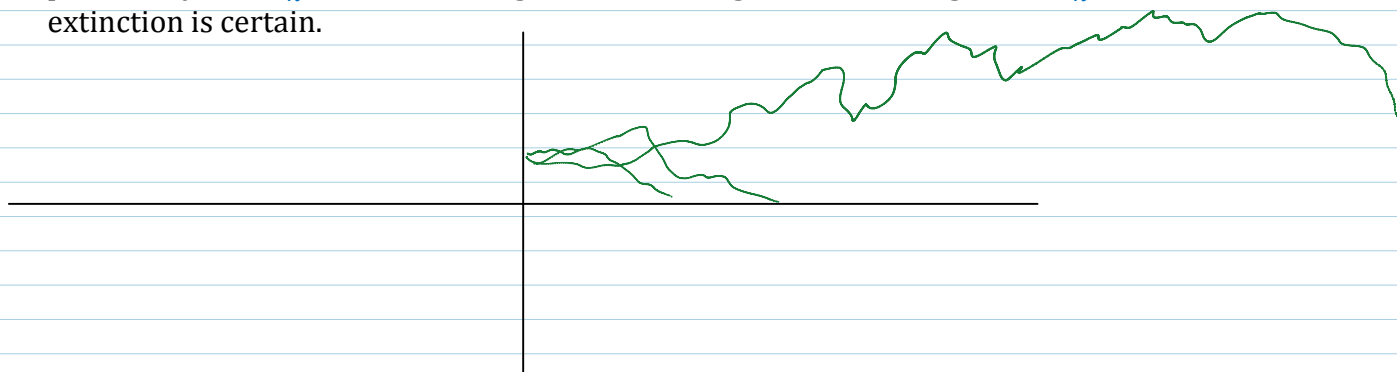


To this end, what does the formula for the mean tell us?

$$\mathbb{E}X_n = \mathbb{E}X_0 (\mathbb{E}Z)^n$$

We have three clear cases, depending on:

1. $\mathbb{E}Y < 1$ (subcritical) implies that $\lim_{n \rightarrow \infty} \mathbb{E}X_n = 0$. This precludes there being a positive probability that $X_n \rightarrow \infty$ because that would give infinite weight to $\mathbb{E}X_n$ at large n . So extinction is forced in this case.
2. $\mathbb{E}Y > 1$ (supercritical) implies that $\lim_{n \rightarrow \infty} \mathbb{E}X_n = \infty$. This is consistent with positive probability of extinction and/or explosive growth.
3. $\mathbb{E}Y = 1$ (critical) implies that $\mathbb{E}X_n = \mathbb{E}X_0$. This is also incompatible with a positive probability that $X_n \rightarrow \infty$ because again that would give infinite weight to $\mathbb{E}X_n$. So extinction is certain.



What's left to do is characterize the probability of extinction in the "supercritical" case $\mathbb{E}Y > 1$.

Rather than use the absorption probability formulas for general Markov chains, which involve probability transition matrices, we will derive a special formula for branching processes using probability generating functions (which exploit the special structure of branching processes).

Define: $a(k) = P(X_n = 0 \text{ for some } n > 0 | X_0 = k)$

$$\begin{aligned}
 a(k) &= P\left(\bigcap_{i=1}^k \left\{ \begin{array}{l} \text{descendants of initial agent \# } i \\ \text{go extinct} \end{array} \right\}\right) \\
 &= \prod_{i=1}^k P\left(\left\{ \begin{array}{l} \text{descendants of initial agent \# } i \\ \text{go extinct} \end{array} \right\}\right) \quad \text{independent} \\
 &= \prod_{i=1}^k a(1) \\
 a(k) &= (a(1))^k
 \end{aligned}$$

We will now define $a = a(1)$, and we can compute the probability to go extinct starting from k agents simply from a using the above formula ($a(k) = a^k$). Now need to just solve for a .

We will set up an equation for a based on first-step analysis, similar to what we did for absorption in general Markov chains, but now adapted to branching processes.

$$\begin{aligned}
 a &= P\left(\bigcup_{n=1}^{\infty} \{X_n = 0\} \mid X_0 = 1\right) \\
 &\quad \text{Law of Total probability, partition on } X_1 \\
 a &= \sum_{j=0}^{\infty} P\left(\bigcup_{n=1}^{\infty} \{X_n = 0\} \mid X_1 = j, X_0 = 1\right) \\
 &\quad \times P(X_1 = j \mid X_0 = 1) \\
 &= P\left(\bigcup_{n=1}^{\infty} \{X_n = 0\} \mid X_1 = 0, X_0 = 1\right)
 \end{aligned}$$

$$= P\left(\bigcup_{n=1}^{\infty} \{X_n = 0\} \mid X_1 = 0, X_0 = 1\right) \\ P(X_1 = 0 \mid X_0 = 1)$$

$$+ \sum_{j=1}^{\infty} P\left(\bigcup_{n=1}^{\infty} \{X_n = 0\} \mid X_1 = j, X_0 = 1\right) \\ P(X_1 = j \mid X_0 = 1)$$

$$= 1 \cdot p_0 + \sum_{j=1}^{\infty} P\left(\bigcup_{n=2}^{\infty} \{X_n = 0\} \mid X_1 = j, X_0 = 1\right) \times p_j$$

\uparrow
 since $X_1 = 0$
 impossible
 given $X_1 = j > 0$

$$= 1 \cdot p_0 + \sum_{j=1}^{\infty} P\left(\bigcup_{n=2}^{\infty} \{X_n = 0\} \mid X_1 = j\right) p_j$$

by Markov property

$$= p_0 + \sum_{j=1}^{\infty} a(j) p_j$$

\uparrow
 by time shift $n \rightarrow n+1$

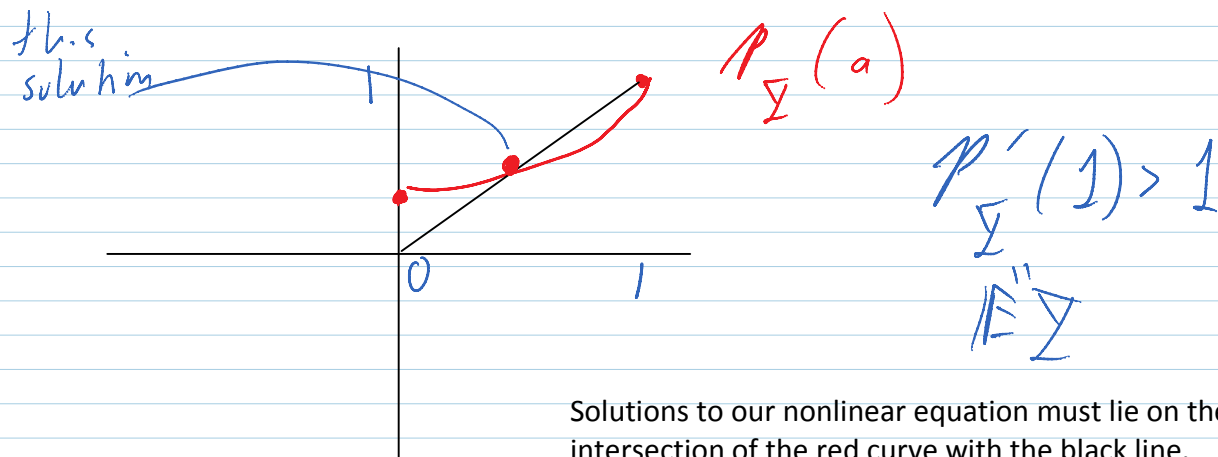
$$= p_0 + \sum_{j=1}^{\infty} p_j a^j$$

$$= \sum_{j=0}^{\infty} p_j a^j$$

$$q = P_{\mathbb{P}}(q)$$

We have a **nonlinear equation** for the absorption probability a in terms of the probability generating function for the offspring. But this equation could have many or no solutions, so how do we know how to solve?

Let's plot a graph, and recall we are in the generic case where $p_0 > 0, p_j > 0$ for some $j \geq 2$. Therefore, the pgf for Y must satisfy:



$$p_Y(0) = p_0 > 0$$

$$p_Y(1) = 1$$

$$p'_Y(s) = \sum_{j=1}^{\infty} j p_j s^{j-1} > 0 \quad \text{increasing}$$

$$p''_Y(s) = \sum_{j=2}^{\infty} j(j-1) p_j s^{j-2} > 0 \quad \text{convex "up"}$$

The properties imply that necessarily there will be two intersections, one at $a = 1$ and one satisfying $0 < a < 1$, provided that

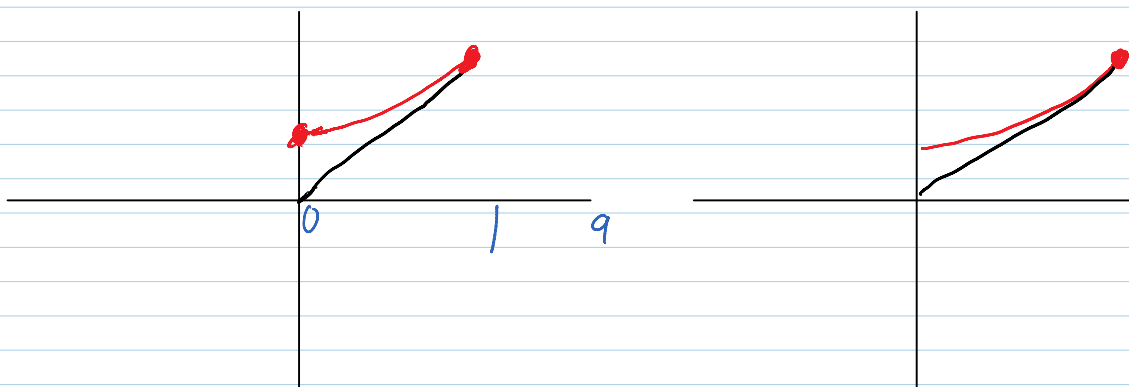
$$p'_Y(1) > 1.$$

But that's the case we're interested in, namely $EY > 1$.

Our arguments based on first-step analysis also work for the critical and subcritical cases, but we know the answer there has to be $a = 1$.

$$EY < 1$$

$$EY = 1$$



Need to give argument for why the solution for the absorption probability in supercritical case is the **nontrivial solution** $0 < a < 1$; next time.