Introduction to Stochastic Processes
MATH 6660–1 – Fall 2015
Homework 2

Due Mpnday, October 19 at 2:00 PM

This homework has 170 points available but, as always, homeworks are graded out of 100 points. Full credit will generally be awarded for a solution only if it is both correctly and efficiently presented using the techniques covered in the lecture and readings, and if the reasoning is properly explained. If you used software or simulations in solving a problem, be sure to include your code, simulation results, and/or worksheets documenting your work. Note though that if analytical methods are available to answer a problem, you will not receive much credit if you only provide Monte Carlo simulations of the solution. If you score more than 100 points, the extra points do count toward your homework total.

1 Application and Calculation Problems

1.1 Machine Management (40 points)

Suppose a certain industrial operation requires the use of m machines of a certain type. Suppose each machine has some probability of breaking during a given week, independently of how old the machine is. However, the chance of breaking increases with workload, and the week's workload must be distributed over whatever machines are operational (not broken). Each Monday, an order is sent in to replace each currently broken machine with a new machine, and this new machine will arrive the following Monday.

a. (**10 points**) Create a finite-state discrete-time Markov chain model for the number of machines in operation each Monday. Explain your assumptions and reasoning clearly.

b. (**10 points**) Suppose the company incurs a cost c_b per week for each of the m machine which does not remain in operation throughout the week (due to needing to run overtime on other machines, etc.). Compute the average cost per week arising from broken machines in the long run. If you cannot find an analytic expression, compute numerical answers for a few interesting choices of parameters. (Suppose the probability of a machine breaking during a week is not very small.)

c. (**10 points**) The management now considers keeping r machines in reserve to replace machines that break during a week (so that the plant does not need to wait until Monday for new machines to arrive). That means that every Monday, the company orders enough machines to bring their total number of operational machines up to $m + r$, though only m are in use at any given time. Modify your finite-state, discrete-time Markov chain model to incorporate this reserve policy, explaining your reasoning and assumptions carefully.

d. (**10 points**) Suppose as above that the company incurs a cost c_b per week for each machine below m in operation. Suppose the cost of replacing a broken machine with one in reserve is c_r, and each machine held in reserve costs c_s per week in storage and maintenance costs. Pick some reasonable choices for the parameters in your model, and study how the total average cost per week for managing the broken machines (in the long run) depends on the number r kept in reserve. Can you recommend an optimal value of r from your model?

1.2 Lost Sales (50 points)

Consider the inventory model discussed in class, where a maximum of M products could be stored. On day n, a random number D_n of product is demanded, and these orders are fulfilled immediately if enough product is available. If the product is unavailable, the sale is lost; the customer goes elsewhere and does not return. Reordering decisions are made at the end of the day as follows: If the amount of inventory is below some number s, then the store manager orders enough new product to fill up the storage, so that M products are available the next morning (overnight delivery!). We set up a Markov chain model for the store's inventory in class.

a. (**15 points**) Describe how you would compute the long-run average number amount of lost sales per day (number of customers who try to buy the product but request it when it is not available). Be as precise as you can, allowing $\{D_n\}_{n=0}^{\infty}$ to be independent, identically distributed random variables with arbitrary probability distribution on the nonnegative integers.

b. (**20 points**) Implement your approach from part a via a deterministic computer code which allows the user to input the inventory management parameters s

and M, as well as an arbitrary probability distribution over an arbitrary finite state space of nonnegative integers for the demand variables $\{D_n\}_{n=0}^{\infty}$.

c. (**15 points**) Choose at least 3 different pairs of parameters (s, M) (making sure to vary each parameter at least once) for a fixed demand model, and compare your deterministic calculation for the long-run average of lost sales per day with the results of Monte Carlo simulations.

   Note that the proper way to estimate a statistic $\theta = \langle f(Y) \rangle$, where Y is any random variable, by Monte Carlo simulation, is to define the estimator:

$$\hat{\theta} \equiv \frac{1}{N} \sum_{j=1}^{N} f(Y_j)$$

   where $\{Y_j\}_{j=1}^{N}$ is an ensemble of iid samples (realizations) of the random variable Y. In Monte Carlo estimation, one should always also estimate the standard error:

$$\mathrm{SE}_{\theta} \equiv \sqrt{\frac{1}{N(N-1)} \sum_{j=1}^{N} (f(Y_j) - \hat{\theta})^2},$$

   which is an estimate for the sampling error of the estimator $\hat{\theta}$. Note that the statistic of interest θ is deterministic, but the estimator $\hat{\theta}$ is random, since it depends on which N random samples were taken. SE_{θ} is an estimate of the fluctuations in the estimator $\hat{\theta}$ when different ensembles of N random samples are taken. The result of a Monte Carlo estimator for a statistic θ should be represented with some indication of its uncertainty, and one way to do this is to report $\hat{\theta} \pm \mathrm{SE}_{\theta}$.

2 Theoretical Problems

2.1 Backward Markov Chains (60 points)

Develop the idea that Markov chains can be run backwards in time as well as forward, so long as time is run away from any imposed condition (like an initial condition).

a. (**10 points**) First show that for any Markov chain $\{X_n\}_{n=0}^{\infty}$ and any $m > 0$, $n > 1$:

$$P(X_{n-1} = j | X_n = i_n, X_{n+1} = i_{n+1}, \ldots, X_{n+m} = i_{n+m}) = P(X_{n-1} = j | X_n = i_n).$$

   whenever the left hand side is well-defined.

b. (**10 points**) Now choose a particular discrete-time Markov chain with at least 5 states, and simulate a statistically stationary process governed by this Markov chain over the epochs $n = 0$ through $n = 100$ by initializing it at $n = 0$ with a stationary distribution $\boldsymbol{\pi}$ of the Markov chain:

$$P(X_0 = j) = \pi_j.$$

c. (**15 points**) Next, show how one can equivalently simulate the statistically stationary Markov chain over these epochs by starting at $n = 100$ and simulating the Markov chain backward. For full credit you must provide an explanation for the algorithm which can work for an arbitrary finite state discrete-time Markov chain initialized at $n = 0$ with a stationary distribution. Visually compare your backward and forward simulations; do they look like they are producing equivalent trajectories?

d. (**10 points**) Provide some quantitative computational tests to demonstrate that the forwards and backwards algorithms really are producing equivalent trajectories.

e. (**5 points**) Extend your Markov chain algorithm so that it runs from epoch $n = 100$ backward to $n = -100$ and explain what happens as the epochs switch from positive to negative numbers.

f. (**10 points**) How would the above change if the Markov chain were not initialized with a stationary distribution?

2.2 First Return Time in Stationary Distribution (20 points)

Let $\{X_n\}_{n=0}^{\infty}$ be a finite state, discrete time, homogenous Markov chain, and assume it is initialized with a stationary distribution $\boldsymbol{\pi}$: $\mathrm{Prob}\,\{X_0 = j\} = \pi_j$ for $j \in S$. Let

$$\tau = \min\{n : X_n = X_0 \text{ and } n \geq 1\}$$

be the first time at which the Markov chain revisits its initial state. Calculate how its average value $\langle \tau \rangle$ depends on the structure of the Markov chain, its transition probability matrix, and the stationary distribution $\boldsymbol{\pi}$ (for the cases where it is not uniquely determined).