Renewal Processes

Wednesday, December 16, 2015
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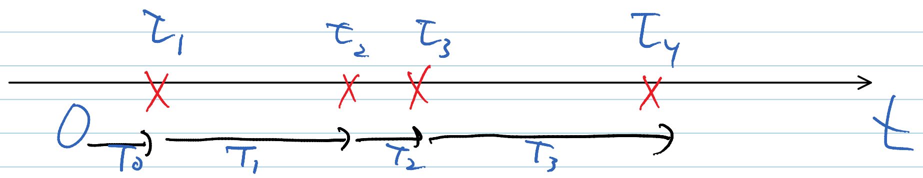
Reading:

- Karlin and Taylor Ch. 5
- Resnick Ch. 3

A renewal process is a generalization of the Poisson point process.



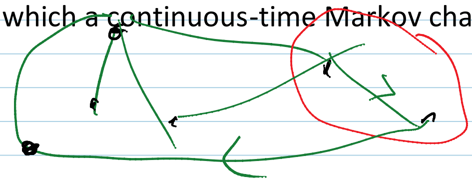
The Poisson point process is completely memoryless, and so the probability distribution for the time between incidents is exponentially distributed with some specified mean.

A renewal process is memoryless in a more restricted sense: the length of the time intervals between successive incidents are independent, i.e., $\{T_j\}_{j=1}^{\infty}$ but need not be exponential.

- The counting process associated with a renewal process (i.e., $N(t)$ is the number of incidents that happen up to time t) need not be a continuous-time Markov chain, as it was for the Poisson point process. That is, we do not lose memory between incidents, only at the moments when incidents happen.
- Normally we also take the inter-incident times $\{T_j\}_{j=1}^{\infty}$ to be iid.
- delayed renewal process is a variation where the time to first incident T_1 may have a different probability distribution than the other T_j.

Applications of renewal processes:

- equipment replacement with nonconstant hazard rate $= \frac{p_T(t)}{1-F_T(t)}$ = rate of failure at time t given survival up to time t
- the time between spikes (action potentials) sent by a neuron
- broadly define renewal processes as the times at which a continuous-time Markov chain visits some subset of states.
- a spatial version of renewal process for car following



Just as renewal processes generalize Poisson point process to require the memoryless property to hold only when an incident happens, one can generalize continuous-time Markov chains so they lose memory (other than current state) only when a transition happens. This relaxes the requirement that the time spent in a state is exponentially distributed, and can be a general probability distribution. This is called a semi-Markov process; not difficult to simulate but the analytical calculations are much more involved.

- queueing models with nonexponential service times
- continuous-time branching processes where the rate of branching depends on the age of an agent

Mathematical Specification of Renewal Processes

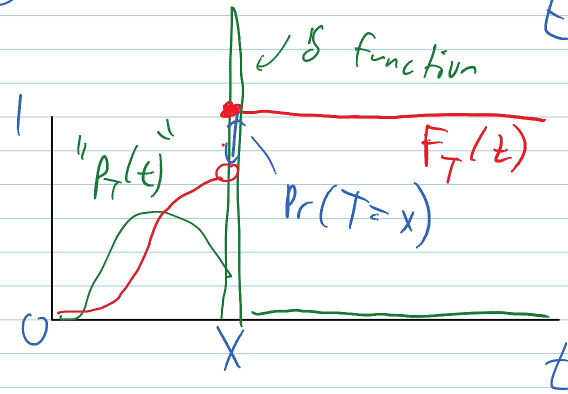
The key object in defining a renewal process is the CDF (cumulative distribution function) for the interincident time T:

$F_T(t) = \Pr(T \leq t)$

When the random variable T is continuous, then we can relate the CDF and PDF by:

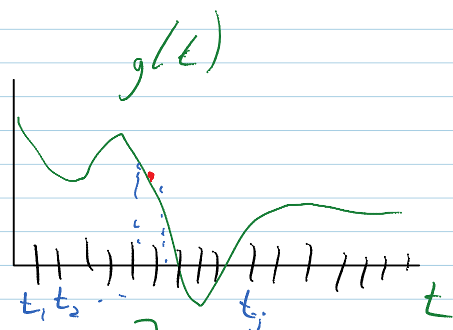
$p_T(t) = \frac{dF_T(t)}{dt}$ and $F_T(t) = \int_{-\infty}^{t} p_T(t')dt'$

But the CDF formulation is more general; in particular it allows for mixed or hybrid random variables that are both continuous and discrete.



CDFs are universal descriptors of arbitrary random variables, no matter how nasty, whereas PMFs only work for discrete random variables, and PDFs only work for continuous random variables. Note that one can in practice work with hybrid/mixture random variables using "generalized PDFs" where the jumps in the CDF (discrete components) are represented in the PDF by Dirac delta functions.

CDFs are also complete descriptors of a random variable.



How do we compute expectations of a random variable in terms of its CDF?

Stieltjes integral:

$$\mathbb{E}g(T) = \int_0^\infty g(t)dF_T(t) \approx \sum_j g(t_j^*)\left[F(t_{j+1}) - F(t_j)\right]$$

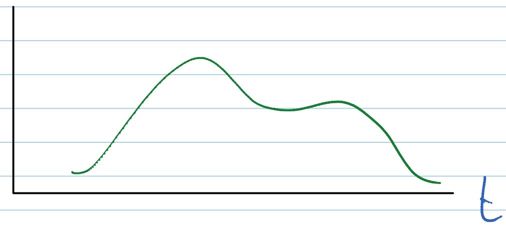
For continuous rvs:

$$dF_T(t) = p_T(t)\,dt$$

In particular, assuming as we are that $T \geq 0$, one can write the mean of a random variable in terms of the CDF without a Stieltjes integral:

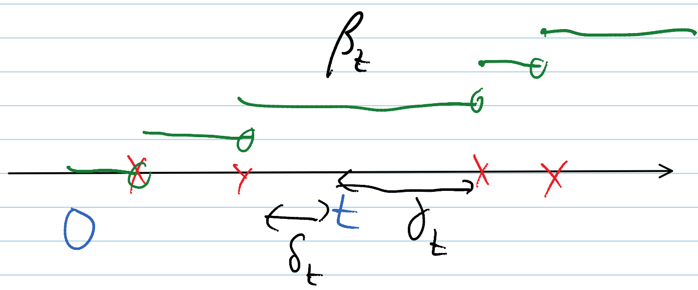
$\mathbb{E}\,T = \int_0^\infty (1 - F_T(t))\,dt$

This follows from integration by parts.



Some key quantities associated to renewal processes

- Counting process
  - $N(t) = \sum_{j=1}^{\infty} I_{\{t \geq \tau_j\}}$
- Current life: δ_t
- Residual life: γ_t
- Total life: β_t



For the special case of a Poisson point process, one can explicitly compute the probability distribution for all these quantities. We already saw that the counting process has a Poisson distribution with mean rt where $r = 1/\langle T\rangle$ is the rate of the Poisson point process.

Also, by the memoryless property, one can show that γ_t is exponentially distributed with mean $\mu = \langle T\rangle$.

Also, because the Markov property works backward in time as well as forward, so the current life δ_t has a exponential distribution which is cut off at the value t corresponding to no incident having occurred over $[0, t]$. (That is, δ_t is a hybrid random variable which has a probability $e^{-\frac{t}{\mu}}$ to take the value t.) But if we take t large enough, this cut off plays a small role, and $\lim_{t\to\infty}\langle\delta_t\rangle = \mu$.

Consequently, we also have that $\lim_{t\to\infty}\langle\beta_t\rangle = \lim_{t\to\infty}\langle\delta_t\rangle + \langle\gamma_t\rangle = \mu + \mu = 2\mu$.

This gives rise to the "Poisson paradox."

- length-biased sampling is the resolution

Now we turn to how more general renewal processes can be analyzed.

- simulation is straightforward if one knows how to simulate T

The basic strategy is the following three steps:

1. Apply a renewal argument to obtain a (recursive) integral equation for the statistical quantity of interest.
2. Arrange this integral equation so that the renewal theorem can be applied to it.
3. Apply the renewal theorem and obtain the resulting behavior at large times.

Alternatively, in some simple situations, you can solve the integral equation using Laplace transforms to get finite-time behavior but this is not typical.

Renewal Argument

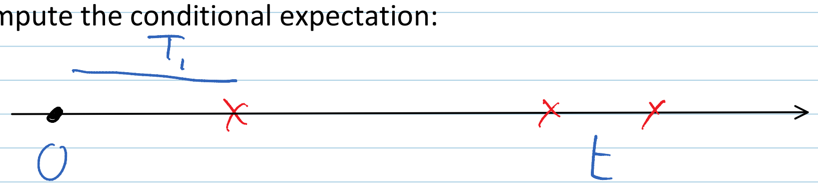
This is a generalization of the first-step analysis for Markov chains. We instead will condition on the time of the first incident, which is a time at which memory is lost.

Let's apply this first to the mean of the counting process: $M(t) = \mathbb{E}N(t)$

In general one applies the law of total expectation by partitioning on the time of the first incident:

$$M(t) = \mathbb{E}N(t) = \int_0^\infty \mathbb{E}[N(t) \mid T_1 = t'] \, dF_T(t')$$

Now compute the conditional expectation:



$$\mathbb{E}[N(t) \mid T_1 = t'] = \begin{cases} M(t-t') + 1 & \text{if } t' \leq t \\ 0 & \text{else} \end{cases}$$

Plug into the law of total expectation:

$$M(t) = \int_0^t [M(t-t') + 1] \, dF_T(t') + \int_t^\infty 0 \, dF_T(t')$$

So we have a recursive integral equation:

$$M(t) = \int_0^t \left(M(t-t') + 1 \right) dF_T(t')$$

Sometimes this can be solved analytically via Laplace transform, but can be hard to invert.

For the purpose of determining the statistical properties at long time, the renewal theorem gives a systematic way for doing this, but it requires that we put the recursive integral equation into a special form called the renewal equation:

$$M(t) = \int_0^t dF_T(t') + \int_0^t \left(M(t-t') + 1 \right) dF_T(t')$$

$$\underbrace{\int_0^t dF_T(t')}_{\displaystyle = F_T(t) - \cancel{F_T(0)}^{\to 0}}$$

$$M(t) = F_T(t) + \int_0^t \left(M(t-t') + 1 \right) dF_T(t')$$

We can now apply the "renewal theorem" to this integral equation, which we now formulate. One of the requirements below is that we have to exclude the special case when $F_T(t)$ is arithmetic, meaning that the possible values of T fall on some discrete lattice including the origin.

Renewal Theorem

Suppose that $F(t)$ is a nondecreasing function with $F(0) = 0$, F is not arithmetic, and $\lim_{t\to\infty} F(t) = 1$.

Then the solution $A(t)$ to the following integral equation (the renewal equation):

$$A(t) = a(t) + \int_0^t A(t-t')\, dF(t')$$

has the following properties:

1. If $a(t)$ is bounded, then the solution $A(t)$ is unique and bounded on finite time intervals.
2. The solution to the integral equation can be expressed as

$$A(t) = a(t) + \int_0^t a(t-t')\, dM(t')$$

where M(t) is the renewal function, the solution to:

$$M(t) = F(t) + \int_0^t M(t-t')dF(t')$$

Note this is exactly the integral equation we had for the mean of the counting process $\mathbb{E}N(t)$. This says that $M(t)$ acts as a sort of Green's function for the general renewal equation.

3. If a(t) is integrable ($\int_0^\infty |a(t')|dt' < \infty$)

then $\lim_{t\to\infty} A(t) = (\int_0^\infty a(t')dt')/\mu$

where $\mu = \int_0^\infty (1 - F(t'))dt'$

4. $\lim_{t\to\infty} M(t) - M(t-s) = \frac{s}{\mu}$

Applying the renewal theorem to the mean of the counting process, we can just use the 4th point to see that at long time $M(t) - M(t-s) = s/\mu$ where μ is just the mean time between incidents.

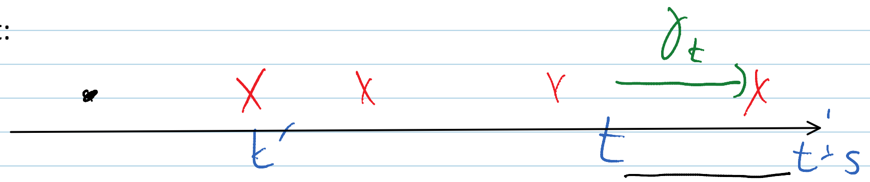
A slightly less simple example:

Let's derive a formula for the probability distribution of the residual life of a general renewal process, at long time:

$A_s(t) = P(\gamma_t > s)$

This could be done by direct conditional probability techniques, but we'll illustrate the 3-step strategy that we outlined above.

1. Renewal argument:



$$A_s(t) = P(\gamma_t > s) = \int_0^\infty P(\gamma_t > s \mid T_1 = t')\, dF_T(t')$$

$= A_s(t-t')$

$$P(\gamma_t > s \mid T_1 = t') = \begin{cases} \underbrace{P(\gamma_{t-t'} > s)}_{= A_s(t-t')} & \text{if } 0 < t' < t \\ 0 & \text{if } t < t' < t+s \\ 1 & \text{if } t' > t+s \end{cases}$$

$$A_s(t) = \int_0^t A_s(t-t')\,dF_T(t') + \int_t^{t+s} 0\,dF_T(t') + \int_{t+s}^{\infty} 1\,dF_T(t')$$

$$= \int_0^t A_s(t-t')\,dF_T(t') + 0 + F_T(\infty) - F_T(t+s)$$

$$A_s(t) = \int_0^t A_s(t-t')\,dF_T(t') + 1 - F_T(t+s)$$

2. Write in the form of a renewal equation to prepare for applying the renewal theorem.

   We see that this can be done with $F(t) = F_T(t)$, $a(t) = 1 - F_T(t+s)$.

3. Check the technical conditions of the renewal theorem, particularly the third point regarding the long-time behavior:

$$\int_0^\infty |a(t')|\,dt' = \int_0^\infty |1 - F_T(t'+s)|\,dt'$$

$$= \int_0^\infty (1 - F_T(t'+s))\,dt'$$

$$u = t'+s$$

$$= \int^\infty (1 - F(u))\,du$$

$$= \int_s^\infty (1 - F_T(u))\,du$$

$$< \int_0^\infty (1 - F_T(u))\,du = \mathbb{E}T$$

which is finite whenever $\mathbb{E}T = \mu < \infty$

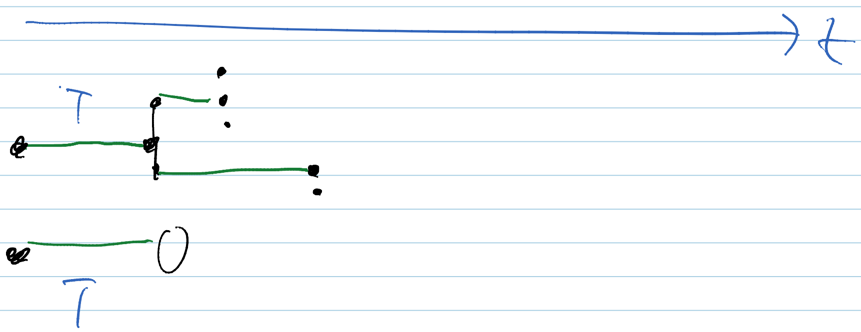
So then the renewal theorem tells us:

$$\lim_{t \to \infty} A_t(s) = \frac{\int_0^\infty a(t')\,dt'}{\mu}$$

$$\lim_{t \to \infty} P(\gamma_t > s) = \frac{\int_0^\infty (1 - F_T(t'+s))\,dt'}{\mu}$$

$$= \frac{\int_s^\infty (1 - F_T(t'))\,dt'}{\mu}$$

Karlin and Taylor Sec. 5.8B does a very interesting application of this renewal argument/renewal theorem approach to compute the mean population size of a continuous time branching process for which the time between birth and branching of an agent is given by an arbitrary probability distribution, i.e., CDF $F_T(t)$.



The integral equation you get from the renewal argument doesn't quite satisfy the conditions of the renewal theorem, so you have to twist the integral equation a little bit by introducing exponential factors e^{rt} with unknown growth rate r into the integral equation. There is a unique growth rate r which will cause the renewal theorem to work, and it satisfies a sort of eigenvalue problem for the integral equation. And that growth rate r is then shown to describe the mean growth rate of the population.