Stationary Distributions: Example

Thursday, October 08, 2015

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Homework 2 due date extended to Monday, October 19 at 2 PM.

Next class is on Tuesday, October 13.

How should we choose epochs for the inspection protocol problem?

- One natural choice would be to choose each epoch to correspond to each manufactured product coming off the assembly line.
  - But it turns out to that one can formulate a somewhat simpler Markov chain model by choosing epochs to correspond to each inspection. Would need to include more states in the state space if we choose epochs corresponding to each product rather than each inspection, because then we would have to also encode memory of number of skipped products since last inspection.

If we choose each epoch to be an inspection, then what should the state space be?

- It suffices to choose the state of the system to simply be whether the system is in every-product-sampling mode or sparse-sampling mode, and when it is in every-sampling mode, how many good products has it seen up to and including the current inspection.
  - We can parameterize the every-sampling regime by an integer $\{0,1,2,\ldots,M-1\}$ corresponding to how many good products has the inspector seen since the last defect or the beginning of the run.
    - Assume the inspector is initialized in every-sampling mode.
  - We can represent the state where the inspector is in sparse-sampling mode as state M.

So we define the Markov chain state variable as:

$X_n =$ # good products seen in a row since the last defect or beginning (whichever is more recent) as of the end of the nth inspection, but cap it at M.

Let's write down the probability transition matrix:

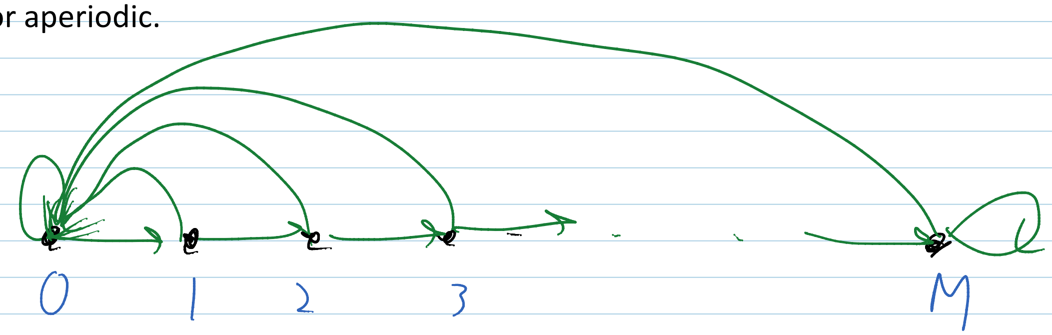
$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ M-1 \\ M \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & \cdots & M-1 & M \end{array} \\ \begin{pmatrix} p & 1-p & 0 & \cdots & 0 & 0 \\ p & 0 & 1-p & \cdots & & \\ p & 0 & 0 & 1-p & & \\ \vdots & & & & & \\ p & 0 & 0 & 0 \cdots & 0 & 1-p \\ p & 0 & 0 & 0 \cdots & 0 & 1-p \end{pmatrix} \end{array}$$

Could also define a stochastic update rule

$$X_{n+1} = \min\left[(X_n + Z_n) Z_n, M\right]$$

$$Z_n = \begin{cases} 1 & \text{w/ prob } 1-p \\ 0 & \text{w/ prob } p \end{cases}$$

Let's now think about solving for the stationary distribution in order to describe long-run properties of the Markov chain. But let's check first what the structure of the Markov chain is, i.e., is it irreducible and/or aperiodic.



It's irreducible. From any state i, I can successively increment from 1 to go to M, then I can go to 0, and then successively increment from 0 to go to any other state j.

It's aperiodic because period is a class property and so it must be the same for any state in an irreducible Markov chain, and state 0 can go to itself in 1 step so clearly 0, and therefore the whole MC, has period 1.

An irreducible, aperiodic FSDT MC has a unique stationary distribution, and the LLN for FSDT MC applies.

Now we solve for the stationary distribution:

$$\vec{\pi}^T P = \vec{\pi}^T$$

$$\vec{\pi} = \begin{pmatrix} \pi_0 \\ \pi_1 \\ \vdots \\ \pi_M \end{pmatrix}$$

$$\left(\pi_0 \quad \pi_1 \quad \pi_2 \quad \cdots \quad \pi_M\right) \left(p \quad 1-p, \right.$$

$$(\pi_0 \ \pi_1 \ \pi_2, \ldots, \pi_M) \begin{pmatrix} p & 1-p & & & \\ p & & 1-p & & \\ p & & & 1-p & \\ \vdots & & & & \\ \vdots & & & & 1-p \\ p & & & & 1-p \end{pmatrix}$$

$$= (\pi_0, \pi_1, \pi_2, \ldots, \pi_M)$$

This gives us a system of M+1 equations

$j=0$: $\quad p\left(\sum_{i=0}^{M} \pi_i\right) = \pi_0$

$1 \le j \le M-1$: $\quad (1-p)\pi_{j-1} = \pi_j$

$j=M$: $\quad (1-p)\pi_{M-1} + (1-p)\pi_M = \pi_M$

Note that we will eventually want $\sum_{i=0}^{M} \pi_i = 1$.
Let's apply this to the $j = 0$ equation:

$$\pi_0 = p$$

$$\pi_j = p(1-p)^j \quad \text{for } 1 \le j \le M-1$$

by recursion

$$(1-p)\pi_{M-1} = p\pi_M$$

$$\pi_M = \frac{1-p}{p}\pi_{M-1}$$

$$\pi_M = (1-p)^M$$

$$\vec{\pi} = \Big(\ p, \ \Big)$$

$$\vec{\pi} = \begin{pmatrix} p \\ p(1-p) \\ p(1-p)^2 \\ \vdots \\ \vdots \\ p(1-p)^{M-1} \\ (1-p)^M \end{pmatrix}$$

One can check this is properly normalized $\sum_{i=0}^{M} \pi_i = 1$. With other calculation approaches, one might have a free constant in the solution which is then chosen so this normalization holds.

Could we have calculated this using the stochastic update rule instead?
$X_{n+1} = f(X_n, Z_n)$

The equation $\pi^T = \pi^T P$ translates to:

$P(X_{n+1} = j) = P(X_n = j) = \pi_j$

$P(X_{n+1} = j) = \sum_{i \in S} P(X_n = i) P(X_{n+1} = j | X_n = i)$ by law of total probaiblity
Plugging in:

$$\pi_j = \sum_{i \in S} \pi_i P(f(i, Z_n) = j | X_n = i)$$

and by independence of X_n, Z_n we have:

$$\pi_j = \sum_{i \in S} \pi_i P(f(i, Z_n) = j)$$

Now we will address the key questions about cost and quality of the inspection protocol.

Let's first address the issue of cost:

$$C = \frac{\#products\ inspected}{\#products\ assembled} = \underset{\substack{\uparrow \\ \text{long run}}}{\lim_{n \to \infty}} \frac{\#products\ inspected\ before\ n+1st\ inspection}{\#products\ assembled\ before\ n+1st\ inspection}$$

$$= \lim_{n \to \infty} \frac{n}{\sum_{n'=1}^{n} f(X_{n'})}$$

where $f(x) = \begin{cases} 1 & \text{if } x \in \{0, 1, 2, \ldots, M-1\} \end{cases}$

where $f(x) = \begin{cases} 1 & \text{if } x \in \{0,1,2,\ldots,M-1\} \\ r & \text{if } x = M \end{cases}$

The $f(X_n')$ is the number of products which emerge between just before inspection n' and just before inspection $n'+1$.

$$C = \lim_{n\to\infty} \frac{1}{\frac{1}{n}\sum_{n'=1}^{n} f(X_{n'})} = \frac{1}{\lim_{n\to\infty}\frac{1}{n}\sum_{n'=1}^{n} f(X_{n'})} = \frac{1}{\sum_{j=0}^{M} \pi_j f(j)}$$

where π is the stationary distribution, by LLN for FSDT MC.

$$C = \frac{1}{\sum_{j=0}^{M-1} p(1-p)^j \cdot 1 + \underbrace{(1-p)^M r}_{j=M \text{ term}}}$$

$$= \frac{1}{\frac{\cancel{p}\left(1-(1-p)^M\right)}{1-\cancel{(1-}p)} + (1-p)^M r}$$

$$C = \frac{1}{1+(r-1)(1-p)^M}$$

Now let's compute long run quality, or lack thereof:

$$B = \frac{\#\, defective\ products\ shipped}{\#products\ shipped}$$

$$= \lim_{n\to\infty} \frac{\#\, defective\ products\ shipped\ up\ to\ inspection\ n+1}{\#products\ shipped\ up\ to\ inspection\ n+1}$$

("up to" does not include)

$$= \lim_{n\to\infty} \frac{\sum_{n'=1}^{n} D_{n'}}{\sum_{n'=1}^{n} S_{n'}}$$

where $S_{n'}$ is the number of products shipped from the nth inspection up to but not including the $(n+1)$st inspection. $D_{n'}$ is the number of defective shipped products shipped from the nth inspection up to but not including the $(n+1)$st inspection.

$S_{n'} = g(X_{n'})$

where $g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & 1 \le x \le M-1 \\ r & x \ge M \end{cases}$

$D_{n'} = h(X_{n'})$: $D_{n'}$ *is not* strictly determined by X_n alone; it also involves an additional random variable/source of uncertainty. But this is easy to handle because the additional random variables can be generated independnetly of the Markov chain.

So if we define $\{\xi_n\}_{n=0}^{\infty}$ to be iid random variables with a binomial distribution based on $r-1$ trials with success probability p:

$$P(\xi_n = j) = \binom{r-1}{j} p^j (1-p)^{r-1-j}$$

Then we can write:

$D_{n'} = h(X_{n'}, \xi_{n'})$

where $h(x,z) = \begin{cases} 0 & x = 0 \\ 0 & 1 \le x \le M-1 \\ z & x \ge M \end{cases}$

So now we can write the quality badness measure:

$$B = \lim_{n\to\infty} \frac{\sum_{n'=1}^{n} h(X_{n'}, \xi_{n'})}{\sum_{n'=1}^{n} g(X_{n'})} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right) \sum_{n'=1}^{n} h(X_{n'}, \xi_{n'})}{\left(\frac{1}{n}\right) \sum_{n'=1}^{n} g(X_{n'})} =$$

$$\frac{\lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{n'=1}^{n} h(X_{n'},\xi_{n'})}{\lim_{n\to\infty}\left(\frac{1}{n}\right)\sum_{n'=1}^{n} g(X_{n'})}$$

For the numerator, think of $Y_n = (X_n, \xi_n)$ as an augmented Markov chain, and its stationary distribution is just the product of π for X and the binomial distribution for ξ. Then we can apply LLN for MC to both numerator and denominator.

$$B = \frac{\sum_{x=0}^{M}\sum_{z=0}^{r-1} h(x,z)\pi_x \binom{r-1}{z} p^z(1-p)^{r-1-z}}{\sum_{x=0}^{M} g(x)\pi_x}$$

$$= \frac{\sum_{x=0}^{M-1}\sum_{z=0}^{r-1} 0\times\pi_x \binom{r-1}{z} p^z(1-p)^{r-1-z} + \sum_{z=0}^{r-1} z\,\pi_M \binom{r-1}{z} p^z(1-p)^{r-1-z}}{\sum_{x=1}^{M-1}\pi_x + r\pi_M}$$

$= \frac{\pi_M(r-1)p}{(1-\pi_0-\pi_M)+r\pi_M}$ using the formula for the mean of a binomial distribution in numerator, and normalization of π in denominator.

$$= \frac{(1-p)^M(r-1)p}{1-p+(r-1)(1-p)^M}$$

$$B = \frac{(1-p)^{M-1}(r-1)p}{1+(r-1)(1-p)^{M-1}}$$