Introduction to Stochastic Processes
MATH 6660–1 – Fall 2015
Homework 1

Due Friday, October 2 at 2:00 PM

This homework has 140 points plus bonus points available but, as always, homeworks are graded out of 100 points. Full credit will generally be awarded for a solution only if it is both correctly and efficiently presented using the techniques covered in the lecture and readings, and if the reasoning is properly explained. If you used software or simulations in solving a problem, be sure to include your code, simulation results, and/or worksheets documenting your work. Note though that if analytical methods are available to answer a problem, you will not receive much credit if you only provide Monte Carlo simulations of the solution. If you score more than 100 points, the extra points do count toward your homework total.

1 Practice Problems

These problems will not be graded for credit, but they would be worthwhile to work out for extra practice.

Lawler Exercises 1.1, 1.2

Karlin & Taylor Elementary Problems 1.2, 1.8, 1.9, 1.10, 1.11, 1.13, Problem 1.12, 1.18

2 Application and Calculation Problems

2.1 The Dreams of the Nineties are Alive. . . (40 points)

Suppose that you are on vacation in the Pacific Northwest, and you stop in the town of Portlandia, Oregon to bring in a camera for repair at Fred and Carrie's Sustainable Camera Ecoshack, which refuses to work with respect to a schedule because they view planning and such procedures as "so corporate." Instead, they simply have their one repair person (who happens to be Aimee Mann) work on one camera at a time, in the order in which they are brought in. Fred and Carrie tell you that Aimee is currently working on another person's broken camera, that after that there are k cameras that will be worked on before yours, and that the average time to repair a camera is τ. You guess from looking at the cameras and watching Aimee's work attitude that the time to complete each repair job could be reasonably modeled as a collection of independent, identically distributed, exponentially distributed random variables. Since you are on vacation, you don't want to just sit and wait in the shop. On the other hand, you don't want to spend longer than you need to because you're hoping to move on to Seattle sooner rather than later. So you decide to go hang out at the Blunderbuss Music and Film Festival and return later to pick up the camera.

a. (**10 points**) You'd like to plan the *least* amount of time to spend in town so that when you return to the camera repair shop, there is at least a probability p (a value greater than $1/2$ that you choose according to your risk tolerance level) that the repair on your camera is complete. After how long should you return to check on your camera?

b. (**5 points**) Suppose the average camera repair time is one hour, choose a reasonable value for p, and evaluate your optimal waiting time for $k = 0, \ldots, 5$. Comment on your answers.

c. (**10 points**) Good day: Suppose when you arrive at the shop with your broken camera, Aimee is in the middle of working on a camera but there are no other cameras waiting so your camera will be repaired as soon as Aimee is finished working on the current camera. You start thinking about the optimal time to return based on your previous calculations, but your non-technical companion smacks you and says, obviously if the average camera repair time is one hour, and Aimee is in the middle of repair on a camera, you should simply return in 90 minutes. If you follow this strategy, what is the probability that the repair on your camera will be done when you return? What is the probability that Aimee hasn't even started working on your camera when you return? How long would you have waited before returning according to your optimal strategy worked out above? How much better is your strategy than your companion's simply reasoned strategy?

d. (**15 points**) Bad day: Suppose when you arrive at the shop with your broken camera, not only is Aimee working on a camera but a large number k of cameras are also waiting to be repaired. For your particular p value, describe the behavior of the *ratio* of your planned time to return to the expected (average) time for completion of repair on your camera as the number k of cameras waiting to be repaired becomes large. Describe more generally how this ratio behaves for a general fixed value of p as k becomes large. Interpret your findings, connecting them to any mathematical principles of which you may be aware.

2.2 Leaving the Line (15 points)

In the standard queueing model discussed in class (and in Karlin & Taylor, *A First Course in Stochastic Processes*, Section 2.2.C), service requests arrived independently of the length of the queue.

a. (**15 points**) Suppose that the incoming service requests may decide not to wait in the queue (and either look for a different server or just wait for later), and are more likely to do so the longer the queue is when it arrives. (We assume here the service requestors can see or are informed about the length of the queue.) Construct a finite-state discrete-time Markov chain model for such a queue, explaining your assumptions and reasoning. Formulate a model based on epochs which are regularly spaced small intervals of time as well as epochs defined as the moments when a service request is completed.

2.3 What Disease is in Your Network? (85 points plus bonus points)

This question has more interest when posed in the context of a large social network, but as an exercise consider a sample of three people in a community who may or may not interact with each other on a given day. On the first day, exactly one of them is infected with a communicable disease while the others are healthy. Every day that an infected person interacts with a healthy person, the healthy person has a probability p of becoming infected. Also, every day an infected person has a probability q of recovering and becoming healthy again. Assume that this disease confers no immunity, so an infected person who recovers can become infected again.

For the following questions, consider each of these two different scenarios:

- For each pair of the three people in question, that pair has a probability r of being an interacting pair, meaning they interact with each other every day.

- Every day, each pair of the three people has a probability r of interacting that day, independently of how they have interacted in the past.

a. (**10 points**) Formulate a Markov chain model, either via a stochastic update rule or probability transition matrix, for one of these scenarios. If you find that one scenario is easier to model as a Markov chain than the other; you are encouraged to develop the model for the easier scenario.

b. (**20 points plus 5 bonus points**) For the scenario you modeled, derive a formula or write a computer code to answer the question: What is the average number of the three people under consideration who are infected on the nth day? Illustrate your formula or code with some calculations for certain choices of parameters in the model. As always unless otherwise noted, a Monte Carlo simulation is not an acceptable method for answering this question. The bonus points are for efficiency in your calculation.

c. (**10 points**) Now formulate a Markov chain model, either via a stochastic update rule or probability transition matrix, for the scenario you have not considered so far.

d. (**5 points**) Explain, using mathematical concepts from the class when relevant, what is awkward about this Markov chain model.

e. (**20 points**) Develop an *efficient* means of answering part b for this second scenario. For full credit, relate your method of calculation to mathematical concepts from the class whenever possible.

f. (**20 points plus bonus**) Provide some direct stochastic simulations of your model, i.e., day-to-day histories of how the disease progresses among the three people under consideration. Be sure to include some nontrivial examples (i.e., not just the original sick person becoming healthy with nothing else happening). Consider both scenarios for how people interact. For bonus points simulate this disease model also on a more interesting (larger) network.

g. (**10 bonus points**) How would your results and conclusions for part b change if an individual became immune to the disease when they recover from an infection?