Optional Stopping Theorem; Stochastic epidemic Models

Monday, December 07, 2015
2:01 PM

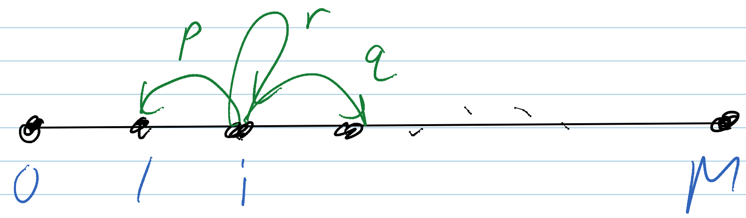
Homework 4 due Wednesday, December 16 at 5 PM.

If you are interested in an optional lecture next week (renewal processes, Markov Chain Monte Carlo, etc.) please fill out survey from course home page tomorrow or Wednesday (by 5 PM).

The OST (Optional Stopping/Sampling Theorem) looks like it just says you can't beat a fair game with a reasonable adaptive strategy, but it can also be exploited to streamline calculations when you can construct suitable martingales associated to your calculation.

1) Application of OST to discrete birth-death chains.

Let's consider birth-death chain on the state space $\{0,1,2,\ldots,M\}$ with constant (homogenous) probabilities for births and deaths ($p_i = p, q_i = q, r_i = r$):



We'll put absorbing boundary conditions at 0 and M.

We'll show how to use martingales and OST to obtain short calculations for the absorption probability and expected time until absorption. The whole game is to realize what kind of martingale will work. Here the martingales will be deus ex machina; we'll discuss later what to do if we aren't inspired to find the martingale.

a) Absorption probabilities:

$\{\mathcal{A}_n\}$

Claim: $Y_n = \left(\frac{p}{q}\right)^{X_n}$ is a martingale with respect to the filtration generated by the $\{X_n\}$, the birth-death chain.

In fact, Y_n is an example of a class of exponential martingales (or Wald's martingale) that related to taking the exponential of a suitable stochastic process (Karlin and Taylor Sec. 6.1)

Let's check martingale properties, then use it with the OST to calculate absorption probabilities.

Check martingale:

Y_n is measurable w.r.t. $\mathcal{A}_n$ clearly

$\mathbb{E}(Y_{n+1} | \mathcal{A}_n) = \mathbb{E}\left(\left(\frac{p}{q}\right)^{X_{n+1}} \Big| \mathcal{A}_n\right)$

$$\mathbb{E}(Y_{n+1} \mid \mathcal{A}_n) = \mathbb{E}\left(\left(\frac{p}{q}\right)^{X_{n+1}} \Big| \mathcal{A}_n\right)$$

$$X_{n+1} = \begin{cases} X_n + 1 & \text{w/ prob } q & Z_n = 1 \\ X_n & r & Z_n = 0 \\ X_n - 1 & p & Z_n = -1 \end{cases}$$

$$= X_n + Z_n$$

if $X_n \in \{1, 2, \ldots, M-1\}$

$X_{n+1} = X_n$ if $X_n \in \{0, M\}$

If $X_n \in \{0, M\}$

$$\mathbb{E}(Y_{n+1} \mid \mathcal{A}_n) = \mathbb{E}\left(\left(\frac{p}{q}\right)^{X_n} \Big| \mathcal{A}_n\right)$$

$$= \left(\frac{p}{q}\right)^{X_n} = Y_n$$

If $X_n \in \{1, 2, \ldots, M-1\}$

$$\mathbb{E}(Y_{n+1} \mid \mathcal{A}_n) = \mathbb{E}\left(q\left(\frac{p}{q}\right)^{X_n+1} + r\left(\frac{p}{q}\right)^{X_n} + p\left(\frac{p}{q}\right)^{X_n-1} \Big| \mathcal{A}_n\right)$$

Here the conditional average is averaging only over the randomness not contained in $\mathcal{A}_n$,

namely the noise (Z_n) at step n that determines where the Markov chain goes at step $n+1$. But X_n is measurable w.r.t. the σ —algebra being conditioned upon, so it behaves deterministically. In particular, there is nothing left to average:

$$= q\left(\frac{p}{q}\right)^{X_n+1} + r\left(\frac{p}{q}\right)^{X_n} + p\left(\frac{p}{q}\right)^{X_n-1}$$

$$= \left(\frac{p}{q}\right)^{X_n} \left(q \left(\frac{p}{q}\right) + r + p \left(\frac{p}{q}\right)^{-1} \right)$$

$$= \left(\frac{p}{q}\right)^{X_n} \left(\overbrace{p + r + q}^{1} \right)$$

$$\mathbb{E}(Y_{n+1} | \mathcal{U}_n) = Y_n$$

$\mathbb{E}|Y_n| < \infty$ follows easily from the fact that X_n is bounded.

Now we use the realization of this martingale together with the OST to calculate absorption probability.

Let's check the technical conditions for this martingale and the Markov time
$\tau = \min\{n > 0 : X_n \in \{0, M\}\}$:

1. $P(\tau < \infty) = 1$ because absorption from a finite communication class has probability one.
2. $\mathbb{E}|Y_\tau| < \infty$ because Y_n is a bounded martingale
3. $|\lim_{n\to\infty} \mathbb{E}(Y_n I(\tau > n))| \leq \lim_{n\to\infty} \mathbb{E}(R I(\tau > n))$ where R is a finite deterministic value, independent of n, such that $|Y_n| \leq R$ for all n (always exists for bounded martingale).

   $|\lim_{n\to\infty} \mathbb{E}(Y_n I(\tau > n))| \leq \lim_{n\to\infty} \mathbb{E}(R I(\tau > n)) \leq R \lim_{n\to\infty} \mathbb{E}(I(\tau > n))$
   $= R \lim_{n\to\infty} P(\tau > n) = R \times 0 = 0$ by the first property and continuity of probability.
   More generally, the third property will follow from the first property for any bounded martingale. Lawler Sec. 5.4 also shows that the third property will more generally hold for "uniformly integrable" martingales, which need not be bounded, but have strong control on the probability to be large.

So then the OST implies: $\mathbb{E}Y_\tau = \mathbb{E}Y_0$.

What does this give us? If we start at $X_0 = i$, then
$\mathbb{E}Y_0 = Y_0 = \left(\frac{p}{q}\right)^i$.

Meanwhile:
$\mathbb{E}Y_\tau = P(X_\tau = 0)\left(\frac{p}{q}\right)^0 + P(X_\tau = M)\left(\frac{p}{q}\right)^M$

But we also know that $P(X_\tau = 0) + P(X_\tau = M) = 1$.

Solve this system of two equations for 2 unknowns,
by, i.e., substitution:

$$P(X_\tau=0)\cdot 1 + P(X_\tau=M)\left(\frac{p}{q}\right)^M = \left(\frac{p}{q}\right)^i$$

$$P(X_\tau=0) + (1-P(X_\tau=0))\left(\frac{p}{q}\right)^M = \left(\frac{p}{q}\right)^i$$

Solve:

$$P(X_\tau=0\,|\,X_0=i) = \frac{\left(\frac{p}{q}\right)^i - \left(\frac{p}{q}\right)^M}{1-\left(\frac{p}{q}\right)^M}$$

$$P(X_\tau=M\,|\,X_0=i) = \frac{1-\left(\frac{p}{q}\right)^i}{1-\left(\frac{p}{q}\right)^M}$$

Those agree with our previous formulas from solving the system of linear equations; we had a simpler problem to solve by using the martingale and OST.

b) Now let's repeat the process for expected time until absorption:

$Y_n = X_n + (p-q)n$ will be the useful martingale here. This is just a special case of the martingale constructed in the previous class as the sum of independent random variables (namely the Z_n describing the changes between epochs).

Let's check the conditions for the OST; here the martingale isn't bounded (due to dependence on n). Nonetheless, we can show using elementary techniques for FSDTMC that $P(\tau > n)$ decays geometrically (variation of the infinite monkeys argument). Together with the bound $|Y_n| \leq M + (p-q)n$, one can verify all conditions.

Now taking this as shown, we apply the OST:

$\mathbb{E}Y_\tau = \mathbb{E}Y_0$

Unwind:

RHS: $\mathbb{E}Y_0 = Y_0 = i$ if $X_0 = i$.

LHS: $\mathbb{E}Y_\tau = \mathbb{E}(X_\tau + (p-q)\tau) = \mathbb{E}X_\tau + (p-q)\mathbb{E}\tau$

$$\mathbb{E}X_\tau = 0\,P(X_\tau = 0) + M\,P(X_\tau = M) = M\,P(X_\tau = M)$$

Equating these expressions give:

$$M\,P(X_\tau = M \mid X_0 = i) + (p-q)\,\mathbb{E}\tau = i$$

$$\mathbb{E}\tau = \frac{i - M\,P(X_\tau = M \mid X_0 = i)}{p-q}$$

$$\mathbb{E}(\tau \mid X_0 = i) = \frac{i - M\left(\frac{1-\left(\frac{p}{q}\right)^i}{1-\left(\frac{p}{q}\right)^M}\right)}{p-q}$$

When $p = q$, this construction won't work, but we can use L'Hopital rule on this expression to get the correct answer: $\mathbb{E}(\tau \mid X_0 = i) = \frac{i(M-i)}{2}$.

Key question: How do we construct the right martingale to make the OST magic work?

1. For the absorption probability calculation, this does not involve time elapsed until absorption so we want a martingale that doesn't involve n. Just want Y_n to be a suitable function of X_n.
2. For the expected time until absorption, we can think ahead to what the OST would give and realize that we should have a martingale that grows linearly with n, i.e., $Y_n = cn + g(X_n)$ where c is some nonzero constant.

But how do we figure out how to build a martingale with these desired properties? No free lunch. Either:

- experiment with broad classes of martingales, or play around until something works
- if you try to derive the martingale systematically by writing out the martingale condition, together with the desired dependence on n, you get a problem that's as hard to solve as the direct calculation.

In particular, if we try to generalize the martingale approach to birth-death chains with variable birth-death rates (p_i, q_i, r_i), playing around with generalizing the above martingales didn't work too well for me.

More sophisticated example for applying martingales to compute answers to Markov chain questions:

- Karlin and Taylor Sec. 6.4: pricing American options
- Karlin and Taylor Sec. 6.6c: probability distribution for the time between branching events in a continuous-time version of a branching process