Introduction to Stochastic Processes
MATH 6790–1 – Fall 2015
Homework 3

Due Monday, November 23 at 2:00 PM

This homework has 235 plus 25 points available but, as always, homeworks are graded out of 100 points. Full credit will generally be awarded for a solution only if it is both correctly and efficiently presented using the techniques covered in the lecture and readings, and if the reasoning is properly explained. If you used software or simulations in solving a problem, be sure to include your code, simulation results, and/or worksheets documenting your work. If you score more than 100 points, the extra points do count toward your homework total. Generally speaking, computations regarding statistical properties of Markov chains should be done using deterministic formulas from class and the texts (when appropriate) and not just through direct Monte Carlo simulation, unless otherwise instructed.

1 Application and Calculation Problems

1.1 Rat in a Markov Maze (75 points plus bonus)

Suppose you are conducting canonical behavioral experiments on rats in mazes. For example, you may be trying to see if the rat learns the maze after being put in it a few times. As a baseline against which to compare the experimental data, you wish to formulate a stochastic model for how the rat navigates the maze under the assumption the rat is completely dumb and unlearned. To test this out, you borrow Ratbert and start him in the lower right corner of the maze. Two features of the maze are a piece of cheese (left over from the colloquium refreshments) and some places where the rat receives a shock if it travels there (denoted by a large electrical bolt). Don't mind Sam in the upper left for the moment; he comes in later.

a. (**10 points plus bonus**) Formulate a Markov chain model for how Ratbert

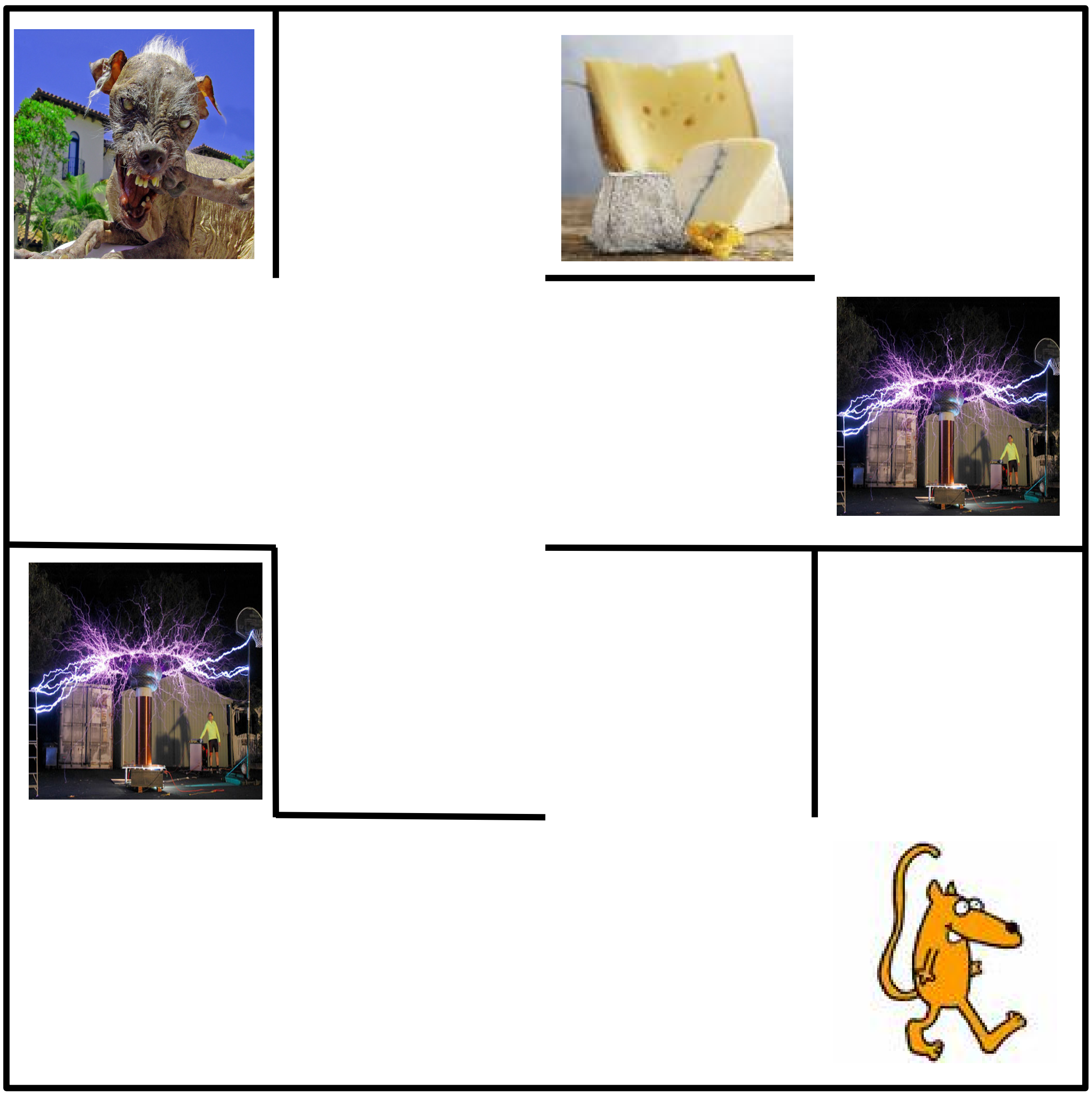

Figure 1. The maze defining Ratbert's world.

moves around the maze, stating clearly what you are assuming. Your model can be fairly simple (for normal credit), but you earn bonus points by studying more elaborate models.

b. (**10 points**) What is the average amount of time before Ratbert reaches the cheese?

c. (**5 points**) What is the expected number of shocks Ratbert receives before he reaches the cheese?

d. (**10 points**) What is the probability that Ratbert finds the cheese before getting shocked?

e. (**15 points**) Suppose that after Ratbert finds and eats the cheese, you release another rat, Sam, into the upper left hand corner of the maze to give Ratbert a fun surprise. Well, Sam's actually not a rat – he's the winner of the 2006 World's Ugliest Dog contest (search on Google Images for the full effect). He's also blind. Formulate a Markov chain model (it can be fairly simple), stating your assumptions, and compute how long you expect to have to wait until Sam finds Ratbert (or vice versa). For full credit, both Sam and Ratbert should be mobile in your model.

f. (**10 points**) Sam's tired now so you put him in his cage, and start over again with another experiment for Ratbert, who's as fresh and eager and clueless as ever. Suppose the cheese is replaced with a fresh piece (at the same location) every time Ratbert moves away from the location denoted by the cheese. Ratbert, who starts out hungry, will be satisfied and go to sleep after he has eaten C pieces of cheese. Derive a formula for the expected amount of time it takes for Ratbert to find rest, assuming he never learns anything about its maze world.

g. (**15 points**) Suppose you intervene in the experiment by having your graduate student, Catbert, watch Ratbert until he gets shocked for the first time. Catbert then disables the shock generator at the location where Ratbert was first shocked (while leaving the other shock location active). Calculate the average number of shocks before Ratbert enjoys his first piece of cheese.

1.2 I Was a Snowball in Hell (75 points)

Suppose you are organizing an event, and planning carefully, you send out a single broadcast email to a group of m people to advertise and prepare for the event. Each recipient will generate a random number of email responses, which we will model as independent and identically distributed with probability mass function $p^I = \{p^I_j\}_{j=0}^{\infty}$. Note that a recipient can generate more than one email response, i.e., if they have follow-up questions, change their mind, or say they have to get back to you about something and then later send further information. Each email you receive with regard to the event requires you to send a random number of new outgoing emails (possibly to different people), which we will model as independent and identically distributed with probability mass function $p^O = \{p_j^O\}_{j=0}^{\infty}$. Again, you might have to send multiple emails to deal with a single incoming email, i.e., to make arrangements with staff members, seek legal assistance, summon the police, etc. Each email you send out will in turn generate an independent and identically distributed number of responses with probability mass function p^I. Suppose the email exchanges continue in the same way, with the number of emails generated by each outgoing or incoming email being independent of each other. Pick reasonable probability mass functions p^I and p^O and compute:

a. (**10 points**) the mean and variance of the number of outgoing emails you must send out in the nth round of the exchange.

b. (**10 points**) the probability that the emailing regarding the event stops at the nth exchange.

c. (**10 points**) the probability that you are forever dealing with email regarding this event. (Email could still be generated after the event, from feedback and criticisms, and reimbursements, etc.)

Now suppose that with each round of your outgoing emails (not with *each* outgoing email, but with the batch of your responses to their responses to your original email, and with the batch of your responses to their responses to your responses to their responses to your original email, etc.), you must also add a random number of new outgoing emails (independently of how many incoming emails you received) to provide updates, and that this number of additional outgoing emails is governed by a probability mass function $p^N = \{p_j^N\}_{j=0}^{\infty}$.

d. (**15 points**) How would your answers to the previous questions be affected by this change?

e. (**5 points**) Suppose the probability mass functions p^I, p^O, and p^N are such that a stationary distribution exists for the number of emails sent out in each exchange. Obtain a relationship between the probability generating function $\mathcal{P}_\pi(s)$ for the stationary distribution and the probability generating functions associated with p^I, p^O, and p^N.

f. (**10 points**) Formally solve this equation for $\mathcal{P}_\pi(s)$ in terms of repeated compositions of the other probability generating functions.

g. (**15 points**) Provide some conditions under which the expression obtained in the previous part is a valid generating function of a well-defined probability distribution. You may need to apply the Continuity Theorem, which states that if $\mathcal{P}_n(s)$ are generating functions and $\mathcal{P}(s) = \lim_{n\to\infty} \mathcal{P}_n(s)$ in a pointwise sense, then $\mathcal{P}(s)$ is itself a generating function of a well-defined probability distribution provided $\mathcal{P}(1) = \lim_{s\uparrow 1} \mathcal{P}(s) = 1$. (See Resnick Section 1.5 for a proof). To get full credit for this part, you must make your arguments rigorous.

2 Numerical Computations

2.1 Branching Out in Two Ways (60 plus 25 bonus points)

One way to extend the branching process model is to label the agents by types. For simplicity, consider a model with just two types of agents, for example "normals" and "mutants," or two particle types produced in a fission reaction, or virus strains replicating themselves and mutating into each other. For generality, we refer to agents of Type A and Type B. As in the standard Galton-Watson branching process, we assume that at each epoch, each agent generates a random number of offspring independently of all other agents and all other epochs. Moreover, at each epoch, each agent of Type A generates a random number $Y^{(AA)}$ of offspring of Type A and $Y^{(AB)}$ offspring of Type B, with joint probability distribution

$$p^{(A)}(y^{(AA)}, y^{(AB)}) = P(Y^{(AA)} = y^{(AA)}, Y^{(AB)} = y^{(AB)}).$$

Similarly, at each epoch, each agent of Type B generates a random number $Y^{(BA)}$ of offspring of Type A and $Y^{(BB)}$ offspring of Type B with joint probability distribution

$$p^{(B)}(y^{(BA)}, y^{(BB)}) = P(Y^{(BA)} = y^{(BA)}, Y^{(BB)} = y^{(BB)}).$$

Note that the number of each type of offspring produced by a given branching event need not be independent.

a. (**15 points**) Develop a computer program to conduct Monte Carlo simulations of this branching process with two types of agents. For full credit, your program should be general, in the sense that it can simulate any branching process specified by user-defined joint probability distributions $p^{(A)}$ and $p^{(B)}$ for the number of offspring of each type arising from each of the two types of parents. You may assume a finite maximum number of offspring of each type per individual is prescribed by the model.

b. (**10 points**) Use your program to simulate several qualitatively different scenarios (whole population goes extinct, one agent type takes over, both types coexist and flourish, etc.). Can you draw any inferences about how the mean number of offspring of each type affects which scenario will be observed?

c. (**10 bonus points**) Design a branching process model with two types corresponding to a realistic application and simulate what happens for various choices of the underlying parameters. Draw conclusions about the implications of your model as appropriate.

d. (**15 points**) Develop deterministic formulas for the probabilities that the descendants produced by a single agent of a given type will be extinct after n

epochs, and write a computer program to compute these probabilities for the examples you simulated in Part b. Compare your theoretical extinction probabilities with results of Monte Carlo simulations from Part b.

e. (**20 points**) Develop equations for the probabilities that the descendants produced by a single agent of a given type will eventually become extinct, and compute these probabilities for the examples you simulated in Part b. The equations you obtain will, like the case for a single agent, not necessarily have unique solutions. For full credit, explain and prove with rigor which solution is the appropriate one when multiple solutions are possible.

f. (**15 bonus points**) Derive criteria involving $\mu^{(ij)} = \langle Y^{(ij)} \rangle$ (with $i, j \in \{A, B\}$) which guarantee that any initial finite population of agents (possibly mixed between the two types) will eventually go extinct. You earn more bonus points the more precisely general your criteria are.

3 Theoretical Problems

3.1 Another Criterion for Transience (25 points)

Let P be the transition probability matrix for a Markov chain on a state space S. We say a real-valued function f defined on S is subharmonic at state $i \in S$ with respect to P if

$$\sum_{j \in S} P_{ij} f(j) \geq f(i).$$

This corresponds to the notion of subharmonicity in elliptic partial differential equation when the Markov chain is a random walk on a fine grid.

Fix a state $k \in S$.

a. (**5 points**) Let $\mathcal{A}$ be the set of all functions f with the properties:

- $f(k) = 0$,
- $0 \leq f(j) \leq 1$ for all $j \in S$,
- f is subharmonic at all $j \neq k$ with respect to P. Let

$$f^*(j) = \sup_{f \in \mathcal{A}} f(j).$$

Show that $f^* \in \mathcal{A}$.

b. (**10 points**) Show that for all $i \neq k$,

$$\sum_{j \in S} P_{ij} f^*(j) = f^*(i),$$

so that f^* is "harmonic" with respect to P.

c. (**10 points**) Use the above to conclude that if we are given an irreducible Markov chain with transition probability matrix P, and we can construct a function f such that:

- $f(k) = 0$ for some state $k \in S$,
- $0 \leq f(j) \leq 1$ for all $j \in S$,
- $f(j) > 0$ for some $j \in S$,
- f is subharmonic at all $j \neq k$ with respect to P

then the Markov chain must be transient.