Countable state birth-death chain

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Where does the equation $x = Qx$ come from in the decisive test for transience?

Lemma (which is of some interest in its own right):
Pick a reference state $i_* \in S$. Then define

$\beta_j = P(T_{i_*}(1) = \infty | X_0 = j)$. Form the column vector β with entries $\{\beta_j\}_{j \neq i_*}$.

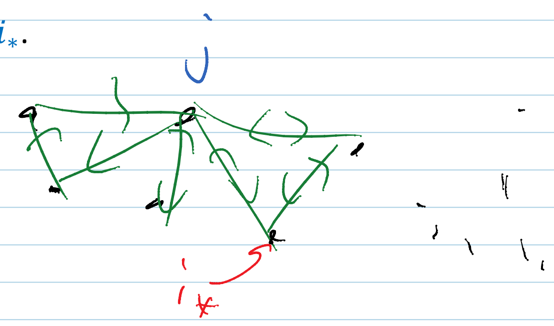
Form the matrix Q by deleting row and column corresponding to the reference state i_* from the probability transition matrix P.

Then β is the maximal solution to the equation $Qx = x$ with the property that $0 \leq x_j \leq 1$ for $j \in S \setminus \{i_*\}$.

- Maximal solution means that any other solution y to the equation with the stated properties must also satisfy $y_j \leq \beta_j$ for all $j \in S \setminus \{i_*\}$.

Note that β_j is the probability that, starting from

state j, the MC never reaches the reference state i_*.



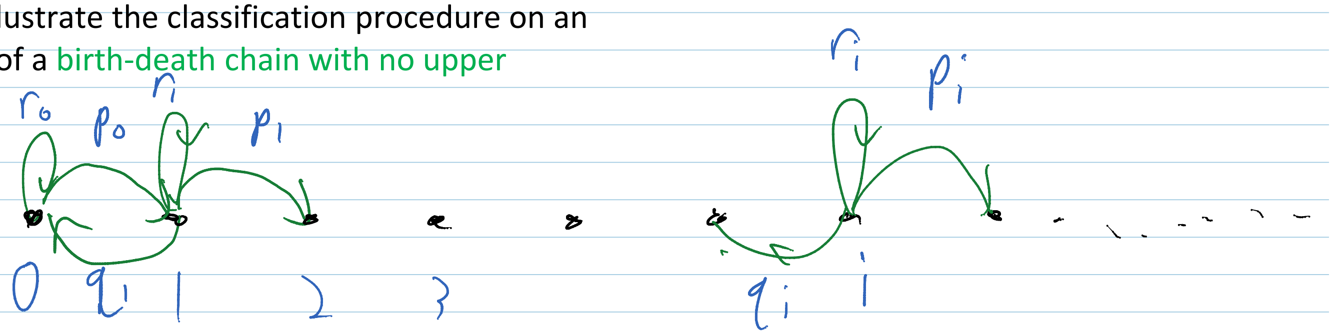
Once this lemma is established, we note that

recurrence is equivalent to $\beta = 0$.

Sketch of the proof of the lemma:

Define a vector $U = 1 - \beta$; its entries U_j correspond to the probability that the Markov chain starting from state j will at some point hit the reference state i_*. One does the same trick as we did for finite state Markov chains; modify the Markov chain to make i_* absorbing, and then set up a calculation to be absorbed in state i_* which is essentially the same kind of absorption probability calculation as we did before. The main difference is that here we have only one absorbing state; this would be trivial for finite state case. Unwinding the resulting formula and expressing it in terms of β gives $Q\beta = \beta$, Showing that β is the maximal solution of the equation $Qx = x$ is just an analysis trick based on a comparison argument.

We will illustrate the classification procedure on an example of a birth-death chain with no upper bound.



Same definitions as for finite-state birth-death chain, but state space is now $S = \{0,1,2,\dots\} = \mathbb{Z}_{\geq 0}$

Let's take the irreducible case, where $p_i > 0, q_i > 0$ for $i \geq 1, p_0 > 0$. It's irreducible because one can proceed between any two states by moving one step at a time toward the other state.

Let's classify this countable-state birth death chain by following the classification procedure. We have an irreducible infinite state MC, so no freebies.

Let's first try Track 2a, the decisive test for positive recurrence. Begin by looking for a stationary distribution ν.

$$\vec{\nu}^T P = \vec{\nu}^T$$

$$\nu_j \geq 0$$

$$P = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & r_0 & p_0 & 0 & 0 & 0 \\ 1 & q_1 & r_1 & p_1 & 0 & 0 \\ 2 & 0 & q_2 & r_2 & p_2 & \\ 3 & 0 & 0 & q_3 & r_3 & p_3 \\ 4 & & & & & \end{array}$$

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$$P_{ij} = \begin{cases} p_i & \text{if } j = i+1 \\ q_i & \text{if } j = i-1 \\ r_i & \text{if } j = i \\ 0 & \text{else} \end{cases}$$

$$\sum_{k \in S} \pi_k P_{kj} = \pi_j$$

$$\pi_0 r_0 + \pi_1 q_1 = \pi_0$$

$$\pi_0 p_0 + \pi_1 r_1 + \pi_2 q_2 = \pi_1$$

$$\vdots$$

$$\pi_{i-1} p_{i-1} + \pi_i r_i + \pi_{i+1} q_{i+1} = \pi_i \quad \text{for } i \geq 1$$

$$\vdots$$

Could solve by recursion, but there's also an easier way...look for detailed balance solution. (Not guaranteed to work, but if it does, it gives correct answer.)

$$\pi_i P_{ij} = \pi_j P_{ji} \quad \text{for all } i, j \in S$$

By the way, detailed balance often works for one-dimensional calculations. Let's just see how the mathematics works out:

Detailed balance translates to the following:

$$\pi_i p_i = \pi_{i+1} q_{i+1} \quad \text{for } i = 0, 1, 2, \ldots$$

Solving this gives:

$$\pi_{i+1} = \frac{p_i}{q_{i+1}} \pi_i$$

q_{i+1}

So by induction:

$$\nu_i = \nu_0 \left(\prod_{k=0}^{i-1} \frac{p_k}{q_{k+1}} \right)$$

This self-consistently satisfies all detailed-balance relations, so we can take it as the invariant measure, which as usual, has an undetermined constant $\nu_0 > 0$.

Can this invariant measure be normalized? If so, we have a stationary distribution and the MC is positive recurrent. If not, then the MC is not positive recurrent.

So the irreducible birth-death chain is positive recurrent if and only if $\sum_{i=1}^{\infty}(\prod_{k=0}^{i-1} \frac{p_k}{q_{k+1}}) < \infty$

If the sum is infinite, we can't tell whether the MC is null recurrent or transient, so we proceed to apply the decisive test for transience.

Pick a reference state, convenient to take $i_* = 0$,
but any choice is fine.

$$Q = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & \cdots \\ \hline 1 & r_1 & p_1 & 0 & 0 & \cdots \\ 2 & q_2 & r_2 & p_2 & 0 & \cdots \\ 3 & 0 & q_3 & r_3 & p_3 & 0 \cdots \\ 4 & & & \ddots & \ddots & \ddots \\ \vdots & & & & & \end{array}$$

$$\text{or} \quad Q_{ij} = \begin{cases} p_i & \text{if } j = i+1 \\ q_i & \text{if } j = i-1 \\ r_i & \text{if } j = i \\ 0 & \text{else} \end{cases}$$

$$\text{for } i, j \geq 1$$

Look for solutions to the equation $Qx = x$

$$\sum_{k=1}^{\infty} Q_{ik} x_k = x_i \qquad \text{for } i = 1, 2, \ldots$$

$$r_1 x_1 + p_1 x_2 = x_1$$

$$q_2 x_1 + r_2 x_2 + p_2 x_3 = x_2$$

$$\vdots$$

$$q_i x_{i-1} + r_i x_i + p_i x_{i+1} = x_i \qquad \text{for } i \geq 2$$

These equations can be solved by recursion or by solving in terms of difference variables $v_i = x_{i+1} - x_i$ as we did for the absorption probability calculation for finite-state birth-death chain. Applying these ideas, we get in the same way:

$$x_j = x_1 \left(\sum_{k=1}^{j} \gamma_k \right)$$

$$\text{where } \gamma_k = \prod_{m=1}^{k} \frac{q_m}{p_m} = \frac{p_0}{r_{k-1} p_{k-1}}$$

Transience is equivalent to the existence of a nonnegative, nonzero, bounded solution which is equivalent to: $\sum_{k=1}^{\infty} \gamma_k < \infty$.

So now putting both tests together, we have the following classification of countable-state birth-death chains:

Define $\nu_j = \prod_{i=0}^{j-1} \frac{p_i}{q_{i+1}}$

Then:

- If $\sum_{j=1}^{\infty} \nu_j < \infty$, then positive recurrent
- If $\sum_{j=1}^{\infty} (p_j \nu_j)^{-1} < \infty$, then transient
- Otherwise, null recurrent

Just plug in the transition probabilities for the MC under consideration, so for example for random walks with $p_j = p, q_j = q, r_j = r$, we would find: $\nu_j = \rho^j$ where $\rho = \frac{p}{q}$.

So if $\rho < 1$, positive recurrent.
If $\rho > 1$, transient
If $\rho = 1$, null recurrent.