# ME453 - Homework 5

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ME453 Assignment 5 Herman K. Jakobsen

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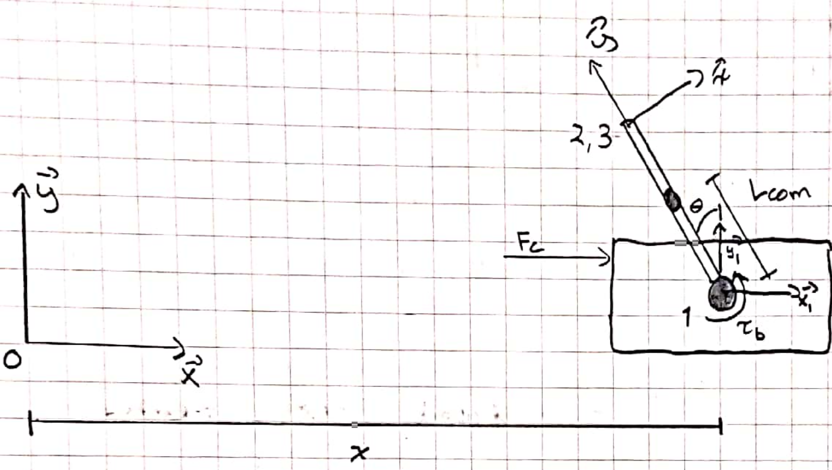
## Problem 1




We are given the boundary conditions

$$x_0 = 0 \,,\quad \dot{x}_0,\ \ddot{x}_0,\ \theta_0,\ \dot{\theta}_0,\ \ddot{\theta}_0$$

Will further assume that the end-effector is not affected by any external forces

$$f_3^3 = 0 \,,\quad \mu_3^3 = 0$$

Our general coordinates are defined as

$$q = \begin{bmatrix} x \\ \theta \end{bmatrix}$$

The first joint is a prismatic joint, while the second joint is revolute. Hence

$$\omega_0^0 = 0 \qquad \omega_1^1 = 0 \qquad \omega_2^2 = \dot{\theta} k$$

and

$$\dot{\omega}_0^0 = 0 \qquad \dot{\omega}_1^1 = 0 \qquad \dot{\omega}_2^2 = \ddot{\theta} k$$

where it has been defined that

$$\dot{\theta}_0 = \ddot{\theta}_0 = 0$$

since the base frame is not rotating.

The rotation matrices between the different frames are given as

$$R_1^0 = I, \quad R_2^1 = R_{z,\theta}, \quad R_3^2 = I$$

The vectors between the joints are

$$r_{1,c_1}^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad r_{0,1}^1 = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}, \quad r_{0,c_1}^1 = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}$$

$$r_{2,c_2}^2 = \begin{bmatrix} 0 \\ -L_{com} \\ 0 \end{bmatrix}, \quad r_{1,2}^2 = \begin{bmatrix} 0 \\ 2L_{com} \\ 0 \end{bmatrix}, \quad r_{1,c_2}^2 = \begin{bmatrix} 0 \\ L_{com} \\ 0 \end{bmatrix}$$

In order to account for the gravitational term, define

$$\ddot{p}_0^0 = \begin{bmatrix} \ddot{x}_0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}, \quad \ddot{x}_0 = 0 \text{ since the base frame is not moving}$$

Due to the first joint being prismatic we get

$$\ddot{p}_1^1 = R_0^1 \ddot{p}_0^0 + R_0^1 \begin{bmatrix} \ddot{x} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \ddot{x} \\ g \\ 0 \end{bmatrix}$$

and

$$\ddot{p}_{c_1}^1 = \ddot{p}_1^1 = \begin{bmatrix} \ddot{x} \\ g \\ 0 \end{bmatrix}$$

Further forward recursion yields

$$\ddot{p}_2^2 = R_1^2 \ddot{p}_1^1 + \dot{\omega}_2^2 \times r_{1,2}^2 + \omega_2^2 \times (\omega_2^2 \times r_{1,2}^2)$$

$$= R_{z,-\theta} \ddot{p}_1^1 + \dot{\omega}_2^2 \times r_{1,2}^2 + \omega_2^2 \times (\omega_2^2 \times r_{1,2}^2)$$

$$= \begin{bmatrix} \ddot{x}\cos\theta + g\sin\theta \\ -\ddot{x}\sin\theta + g\cos\theta \\ 0 \end{bmatrix} + \begin{bmatrix} -2\ddot{\theta}L_{com} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\dot{\theta}^2 L_{com} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \ddot{x}\cos\theta + g\sin\theta - 2\ddot{\theta}L_{com} \\ -\ddot{x}\sin\theta + g\cos\theta + 2\dot{\theta}^2 L_{com} \\ 0 \end{bmatrix}$$

$$\ddot{p}^2_{c2} = \ddot{p}^2_2 + \dot{\omega}^2_2 \times r^2_{2,c_2} + \omega^2_2 \times (\omega^2_2 \times r^2_{2,c_2})$$

$$= \begin{bmatrix} \ddot{x}\cos\theta + g\sin\theta - 2\ddot{\theta}L_{com} \\ -\ddot{x}\sin\theta + g\cos\theta + 2\dot{\theta}^2 L_{com} \\ 0 \end{bmatrix} + \begin{bmatrix} \ddot{\theta}L_{com} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\dot{\theta}^2 L_{com} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \ddot{x}\cos\theta + g\sin\theta - \ddot{\theta}L_{com} \\ -\ddot{x}\sin\theta + g\cos\theta + \dot{\theta}^2 L_{com} \\ 0 \end{bmatrix}$$

Backwards recursion then yields

$$f^2_2 = R^2_3 f^3_3 + m_2 \ddot{p}^2_{c_2}$$

$$= m_b \ddot{p}^2_{c2}$$

$$= \begin{bmatrix} \ddot{x} m_b\cos\theta + m_b g\sin\theta - m_b\ddot{\theta}L_{com} \\ -\ddot{x} m_b \sin\theta + m_b g\cos\theta + m_b\dot{\theta}^2 L_{com} \\ 0 \end{bmatrix}$$

$$\mu^2_2 = R^2_3 \mu^3_3 - f^2_2 \times r^2_{1,c_2} + (R^2_3 f^3_3) \times r^2_{2,c_2} + \hat{\omega}^2_2 I_b \omega^2_2 + I_b \dot{\omega}^2_2$$

$$= -f^2_2 \times r^2_{1,c_2} + \hat{\omega}^2_2 I_b \omega^2_2 + I_b \dot{\omega}^2_2$$

$$= -\begin{bmatrix} 0 \\ 0 \\ \ddot{x} m_b L_{com}\cos\theta + m_b g L_{com}\sin\theta - m_b\ddot{\theta}L^2_{com} \end{bmatrix} + \begin{bmatrix} 0 & -\dot{\theta} & 0 \\ \dot{\theta} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_b \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_b \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ \ddot{\theta} \end{bmatrix}$$

$$= -\begin{bmatrix} 0 \\ 0 \\ \ddot{x} m_b L_{com} \cos\theta + m_b g L_{com} \sin\theta - m_b \ddot{\theta} L_{com}^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ I_b \ddot{\theta} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ -\ddot{x} m_b L_{com} \cos\theta - m_b g L_{com} \sin\theta + (m_b L_{com}^2 + I_b)\ddot{\theta} \end{bmatrix}$$

The first equation of motion is then

$$\tau_b = \mu_2^{2^T} z_1 \qquad , z_1^T = [0 \quad 0 \quad 1]$$

$$\underline{\underline{(m_b L_{com}^2 + I_b)\ddot{\theta} - m_b L_{com} \cos\theta \, \ddot{x} - m_b g L_{com} \sin\theta = \tau_b}}$$

Further backwards recursion yields

$$f_1^1 = R_2^1 f_2^2 + m_c \ddot{p}_{c_1}^1$$

$$= \begin{bmatrix} \ddot{x} m_b \cos^2\theta + m_b g \cancel{\cos\theta \sin\theta} - m_b \ddot{\theta} \cos\theta L_{com} + \ddot{x} m_b \sin^2\theta - m_b g \cancel{\cos\theta \sin\theta} + m_b \dot{\theta}^2 L_{com} \sin\theta \\ \ddot{x} m_b \cancel{\cos\theta \sin\theta} + m_b g \sin^2\theta - m_b \ddot{\theta} L_{com} \sin\theta - \ddot{x} m_b \cancel{\sin\theta \cos\theta} + m_b g \cos^2\theta + m_b \dot{\theta}^2 L_{com} \cos\theta \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} m_c \ddot{x} \\ m_c g \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \ddot{x} m_b - m_b \ddot{\theta} L_{com} \cos\theta + m_b \dot{\theta}^2 L_{com} \sin\theta + m_c \ddot{x} \\ m_b g - m_b \ddot{\theta} L_{com} \sin\theta + m_b \dot{\theta}^2 L_{com} \cos\theta + m_c g \\ 0 \end{bmatrix}$$

Since joint 7 is prismatic, with a force acting along the x-axis, we obtain

$$F_c = f_1^{1\,T} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

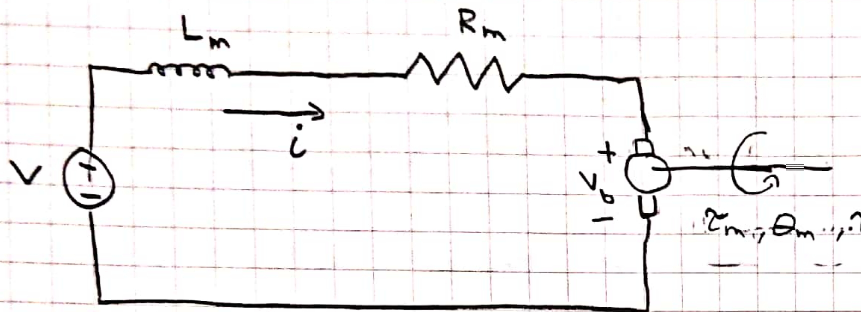
Hence, the second equation of motion is

$$\underline{\underline{(m_b + m_c)\ddot{x} - m_b L_{com} \cos\theta \, \ddot{\theta} + m_b L_{com} \sin\theta \, \dot{\theta}^2 = F_c}}$$

It can be seen that the two EOMs are equal to the ones found using Lagrange in Homework 4.

Problem 2a)

The electrical side of the motor can be illustrated as




Kirchoffs law yields

$$-V + L_m\frac{di}{dt} + R_m i + V_b = 0$$

$$\Rightarrow \quad L_m \frac{di}{dt} + R_m i = V - V_b$$

The variable $V_b$ symbolizes back-EMF and is given by

$$V_b = K_E \dot{\theta}_m$$

Hence

$$\boxed{L_m \frac{di}{dt} + R_m i = V - K_E \frac{d\theta_m}{dt}}$$

Will further assume that rotational speed is given in $\left[\frac{rad}{s}\right]$. Thus

$$K_e = \frac{1}{346}\left[\frac{V}{rpm}\right] = \frac{1}{346} \cdot \frac{1}{2\pi/60}\left[\frac{V}{rad/s}\right]$$

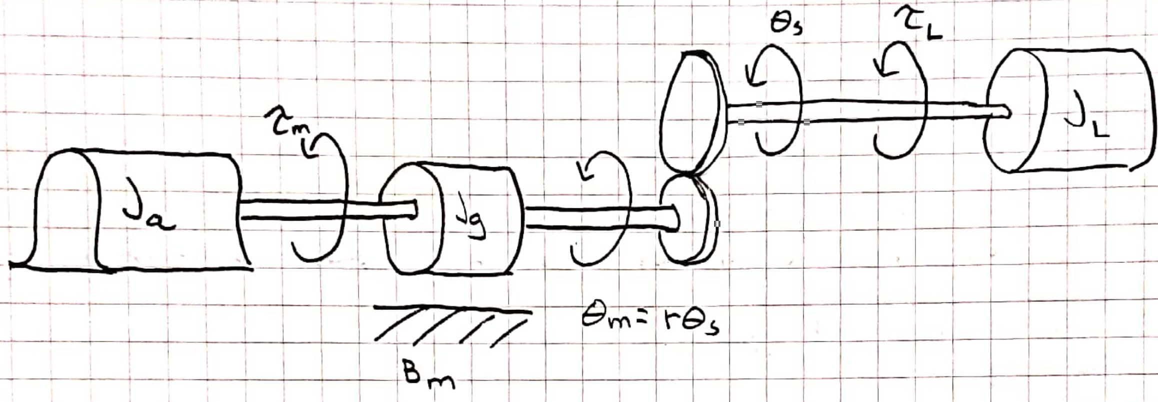
$$K_e = 0{,}028$$

Inserting numeric values yields

$$10^{-3} \cdot 0{,}065 \frac{di}{dt} + 0{,}386\, i = 48 - 0{,}028 \frac{d\theta_m}{dt}$$

$$6{,}5 \cdot 10^{-5} \frac{di}{dt} + 0{,}386\, i = 48 - 0{,}028 \frac{d\theta_m}{dt}$$

The mechanical side of the motor can be illustrated as




Torque balance yields

$$(J_a + J_g)\ddot{\theta}_m + B_m \dot{\theta}_m = \tau_m - \frac{\tau_L}{r}$$

where the generated torque is

$$\tau_m = K_T i$$

Hence

$$(J_a + J_g)\ddot{\theta}_m + B_m \dot{\theta}_m = K_T i - \frac{\tau_L}{r}$$

The sum of the gear and actuator inertia can be defined as

$$J_m = J_a + J_g$$

Hence

$$\boxed{J_m \ddot{\theta}_m + B_m \dot{\theta}_m = K_T i - \frac{\tau_L}{r}}$$

Assuming that the rotor speed is given in $\left[\frac{rad}{s}\right]$ the unit of $B_m$ must be

$$[Nm\, s] = \left[\frac{kg\, m^2}{s}\right]$$

The no load speed given in $\left[\frac{rad}{s}\right]$ is

$$\text{No load speed} = 16500\ [rpm] = 16500 \cdot \frac{2\pi}{60} \left[\frac{rad}{s}\right]$$

Further, we have

$$[V] = \left[\frac{kg\ m^2}{s^3\ A}\right]$$

Therefore, the unit of the viscous friction constant can be obtained by

$$[V] \cdot [A] \cdot [s^2] = \left[\frac{kg\ m^2}{s}\right]$$

which translates to

$$B_m = \frac{\text{Nominal voltage} \cdot \text{No load current}}{(\text{No load speed})^2}$$

$$= \frac{48 \cdot 10^{-3} \cdot 356}{\left(16500 \cdot \frac{2\pi}{60}\right)^2} = \underline{5{,}72 \cdot 10^{-6}}$$

Inserting numeric values into the mechanical differential equation yields

$$\underline{\underline{10^{-7} \cdot 33{,}3\ \ddot{\theta}_m + 5{,}72 \cdot 10^{-6}\ \dot{\theta}_m + 10^{-3} \cdot 27{,}6\ i - \frac{\tau_L}{r}}}$$

2b)

The assumptions can be summarized as

$$J_g = 0$$

$$\dot{i} = 0$$

$$\ddot{\theta}_m = 0$$

Applying these assumptions yields the following model

$$R_m i = V - K_E \dot{\theta}_m$$

$$B_m \dot{\theta}_m = K_T i - \frac{\tau_L}{r}$$

The relationship between the motor speed and joint speed is defined as

$$\theta_m = r \theta_s$$

Applying this gives

$$R_m i = V - K_E r \dot{\theta}_s$$

$$B_m r \dot{\theta}_s = K_T i - \frac{\tau_L}{r}$$

The motor is restricted by its maximum voltage and current output. Will therefore separate the variables in two equations

$$i = \frac{1}{K_T} \left( B_m r \dot{\theta}_s + \frac{\tau_L}{r} \right)$$

Inserting this into the first equation yields

$$\frac{R_m}{K_T} \left( B_m r \dot{\theta}_s + \frac{\tau_L}{r} \right) = V - K_E \dot{\theta}_r$$

$$V = \frac{R_m B_m}{K_T} r \dot{\theta}_s + \frac{R_m \tau_L}{K_T r} + K_E r \dot{\theta}_s$$

Ultimately

$$V = \left( \frac{R_m B_m}{K_T} + K_E \right) r \dot{\theta}_s + \frac{R_m \tau_L}{K_T r}$$

Due to the restrictions we have

$$\frac{B_m}{K_T} r\dot{\theta}_s + \frac{\tau_L}{K_T r} \leq I_{max}$$

$$\left(\frac{R_m B_m}{K_T} + k_E\right) r\dot{\theta}_s + \frac{R_m \tau_L}{K_T r} \leq V_{max}$$

The maximum and minimum gear ratio will depend on the extreme operating points. Hence, only the first and last data point will be used in further analysis.

For the first data point we obtain

$$\left(\frac{0{,}386 \cdot 5{,}72 \cdot 10^{-6}}{10^{-3} \cdot 27{,}6} + 0{,}028\right) r \cdot 0{,}49 + \frac{0{,}386 \cdot 127{,}5}{10^{-3} \cdot 27{,}6 \cdot r} \leq 48$$

$$\frac{5{,}72 \cdot 10^{-6}}{10^{-3} \cdot 27{,}6} \cdot 0{,}49\, r + \frac{127{,}5}{10^{-3} \cdot 27{,}6 \cdot r} \leq 40$$

which reduces to

$$1{,}02 \cdot 10^{-4}\, r + 4{,}62 \cdot 10^{3} \cdot \frac{1}{r} \leq 40$$

$$0{,}014\, r + 1{,}78 \cdot 10^{3} \cdot \frac{1}{r} \leq 48$$

The expressions can be rewritten into

$$1{,}02 \cdot 10^{-4}\, r^2 - 40 r + 4619{,}57 \leq 0$$

$$0{,}014\, r^2 - 48 r + 1783{,}15 \leq 0$$

For the last data point we obtain

$$\frac{5{,}72\cdot 10^{-6}}{10^{-3}\cdot 27{,}6}\cdot 13{,}31\,r + \frac{28{,}8}{10^{-3}\cdot 27{,}6\,r} \leq 30$$

$$\left(\frac{0{,}386\cdot 5{,}72\cdot 10^{-6}}{10^{-3}\cdot 27{,}6} + 0{,}028\right)\cdot 13{,}31\,r + \frac{0{,}386\cdot 28{,}8}{10^{-3}\cdot 27{,}6\,r} \leq 48$$

Which reduces to

$$2{,}76\cdot 10^{-3}\,r^2 - 30r + 1043{,}48 \leq 0$$

$$0{,}374\,r^2 - 48r + 402{,}78 \leq 0$$

Hence, the maximum and minimum gear ratio are given by the following inequalities

$$1{,}02\cdot 10^{-4}\,r^2 - 30r + 4619{,}57 \leq 0 \qquad (1)$$

$$0{,}014\,r^2 - 48r + 1783{,}15 \leq 0 \qquad (2)$$

$$2{,}76\cdot 10^{-3}\,r^2 - 30r + 1043{,}48 \leq 0 \qquad (3)$$

$$0{,}374\,r^2 - 48r + 402{,}78 \leq 0 \qquad (4)$$

From plotting the inequalities, we can see that the minimum and maximum gear ratio is determined by the smallest solution of $(1)=0$ and the largest solution of $(4)=0$, respectively.

Computing the solutions using MATLAB yields

$$\underline{\underline{r_{min} = 115{,}23}} \quad \text{and} \quad \underline{\underline{r_{max} = 119{,}32}}$$

Note that the answer is not completely precise due to roundoff errors in the calculations.