ME591 - Assignment 6

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20196493

November 7, 2019

Code

The code used for completing this assignment is available on https://github.com/hermanjakobsen/Random-Data/tree/master/HW6

Problem 1a)

The Nyquist cut-off frequency is given by

$$f_c = \frac{f_s}{2} = \frac{2048}{2} = 1024 \text{ Hz} \tag{1}$$

The number of spectral lines over the frequency range 0 to the Nyquist cut-off frequency is then given by

$$n = \frac{f_c - 0}{df} = \frac{f_c}{1/T} = f_c = \underline{\underline{1024}} \tag{2}$$

1b)

The band-pass of the filter is

$$B = f_2 - f_1 = 756 - 500 = 256 \text{ Hz} \tag{3}$$

The center frequency of the band-pass is given as

$$f_0 = f_1 + \frac{B}{2} = 500 + \frac{256}{2} = 628 \text{ Hz} \tag{4}$$

Then, the modulating frequency is

$$f_m = f_0 - \frac{B}{2} = 628 - \frac{256}{2} = \underline{\underline{500 \text{ Hz}}} \tag{5}$$

which is equal to the left cut-off frequency f_1 of the band-pass filter.

1c)

The digital low pass filter should function as an interpolater. The original cut-off frequency f_c should be used for the filter. Further, using an FIR filter reduces the computational work, due to the output only being dependent of the input. In that way, only desired outputs (outputs that needs interpolation) need to be computed, compared to an IIR filter which is recursive and requires that all outputs must be calculated.

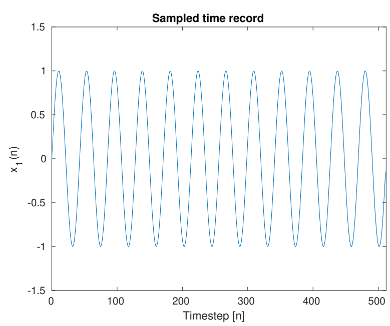
1d)

From the lecture slides, the decimation factor is given as

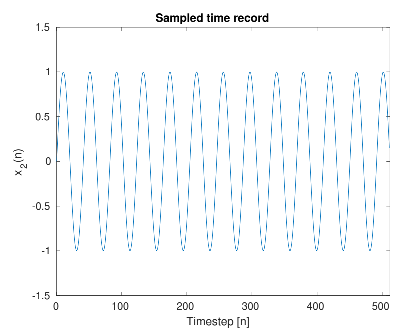
$$d = \frac{N/2}{k_2 - k_1} = \frac{N/2}{B} = \frac{2048/2}{256} = \underline{\underline{4}} \tag{6}$$

Problem 2a)

The sampled time records $x_1(n)$ and $x_2(n)$ are shown in fig. 1a and fig. 1b, respectively.



(a) $x_1(n)$



(b) $x_2(n)$

Figure 1: Sampled time records

2b)

To plot the autospectral density estimates, the MATLAB function *periodogram()* was used. The autospectral estimates, $\hat{G}_{x_1x_1}(f)$ and $\hat{G}_{x_2x_2}(f)$, are shown in fig. 2a and fig. 2b, respectively.

2c)

The finite Fourier transform of a signal can be viewed as the Fourier transform of an unlimited time history record multiplied by a rectangular time window. The spectral window of such a rectangle window consists of a larger main lobe and side lobes, where the side lobes allow power leakage at frequencies far away from the main lobe. However, this leakage problem will not occur if the data is periodic and the time record length is an exact number of periods. This is because the spectral window of the rectangle function is zero at frequencies $f = k/T_p$, where $k = 1, 2, 3, ...$ and T_p is the period of the data.

In this case the period of the data is 1 second. Hence for $x_1(n)$, the time record length is an exact number of periods, which is 12. This means that no power leakage will occur, something that can be verified by looking at fig. 2a. All the significant power is gathered as a spike at $f = 12$ Hz, while some small power leakage occurs at other frequencies due to computational errors and such.

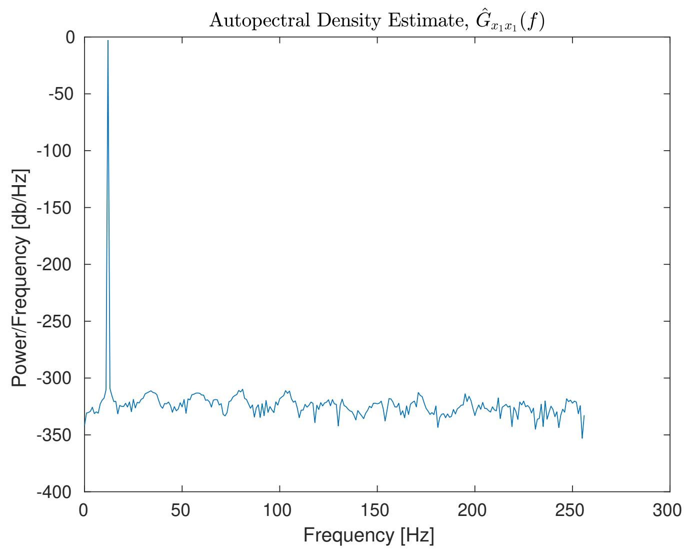
For $x_2(n)$, however, the time record is not an exact number of periods. Looking at fig. 2b, it can be seen that frequencies around $f = 12.5$ Hz has significant power contributions. This is due to power leakage. Ideally, all power should be contained at $f = 12.5$ Hz.

2d)

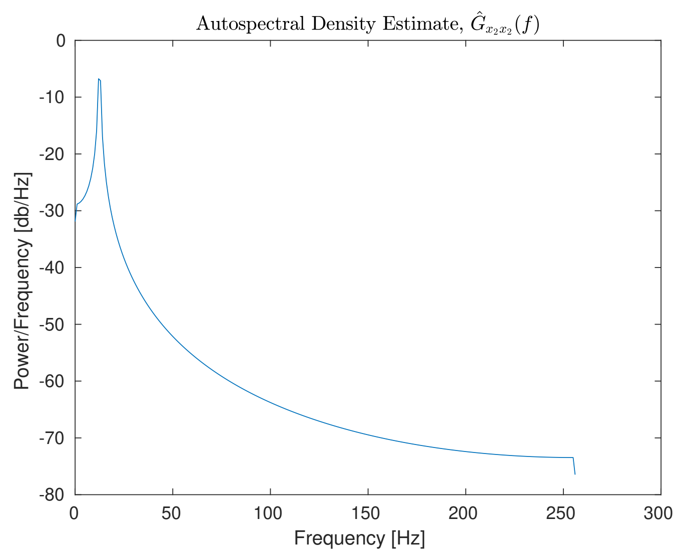
The autospectral density estimates, after applying the Hanning windows, are plotted in fig. 3a and fig. 3b. The Hanning windows were created using the *hann()* function in MATLAB.

The Hanning window function has its first zero crossing at $f = 2/T_p$. For $x_1(n)$, this means that the windowing function will contribute to power leakage at $f = 1/T_p$. This can be verified by looking at fig. 3a. The spike at $f = 12$ Hz has been widened out and the power contribution from the adjacent frequencies has been significantly increased. This is an unwanted effect. It is therefore important to evaluate the periodic data properly before applying a window function.

However, for $x_2(n)$ the autospectral density estimate is improved. After applying the Hanning window, the peak at $f = 12.5$ Hz has become steeper as well as a reduction in power from adjacent frequencies. For $x_2(n)$, the application of the Hanning window gave a better result.

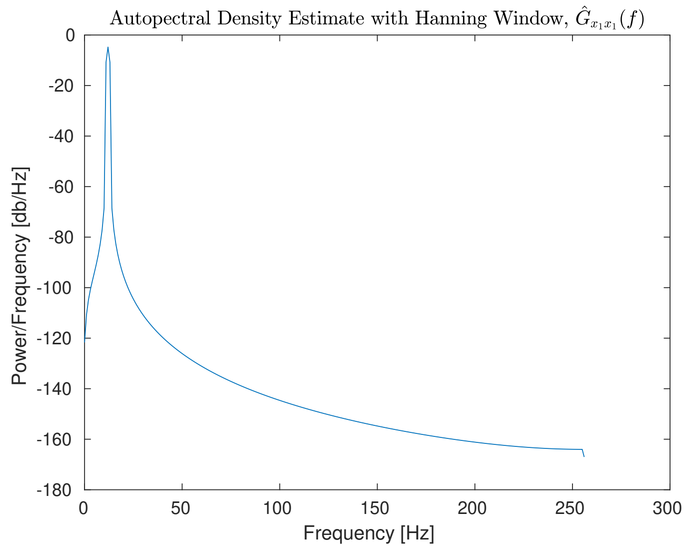


(a) $\hat{G}_{x_1 x_1}(f)$

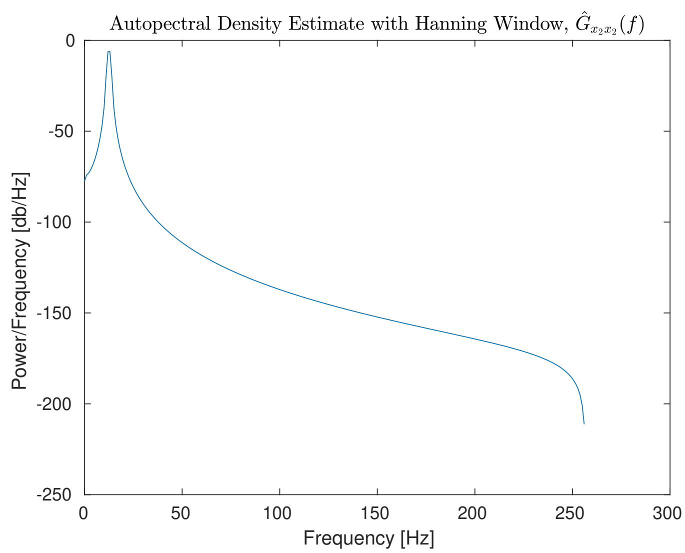


(b) $\hat{G}_{x_2 x_2}(f)$

Figure 2: Autospectral density estimates



(a) $\hat{G}_{x_1x_1}(f)$



(b) $\hat{G}_{x_2x_2}(f)$

Figure 3: Autospectral density estimates with applied Hanning windows

Problem 3a)

We are given

$$y(t) = sin(7\pi t) = sin(2\pi f_0 t) \quad , f_0 = \frac{7}{2} \text{ Hz} \tag{7}$$

The true spectrum of $y(t)$ is obtained using the ideal Fourier transform. So

$$\begin{aligned} Y(f) &= \int_{-\infty}^{\infty} y(t) e^{-j2\pi ft} dt \\ &= \int_{-\infty}^{\infty} sin(2\pi f_0 t) e^{-j2\pi ft} dt \\ &= \frac{1}{2j} \int_{-\infty}^{\infty} (e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}) e^{-j2\pi ft} dt \\ &= \frac{1}{2j} \int_{-\infty}^{\infty} \{e^{-j2\pi(f-f_0)t} - e^{-j2\pi(f+f_0)t}\} dt \\ &= \frac{1}{2j} \{\delta(f - f_0) - \delta(f + f_0)\} \end{aligned} \tag{8}$$

Hence, the true spectrum consists of two delta functions appearing at $f_0 = \pm 7/2$ Hz. It is physically impossible to plot the dirac-delta functions, so a solution is to plot the magnitude of the complex Fourier coefficients. The complex Fourier coefficients are plotted in fig. 4. It would also be possible to plot the magnitude of the Fourier transform, using the following relation

$$\frac{|X(f_k, T)|}{T} = |A_k| \tag{9}$$

with

$$T = \frac{1}{f_0} = \frac{2}{7} \tag{10}$$

3b)

The discrete finite Fourier Transform, $|Y(f_k, T|$, of $y(t)$ is plotted in fig. 5.

3c)

The discrete finite Fourier Transform, $|X(f_k, T|$, of the scaled Hanning windowed signal $x(t)$ is shown in fig. 6.

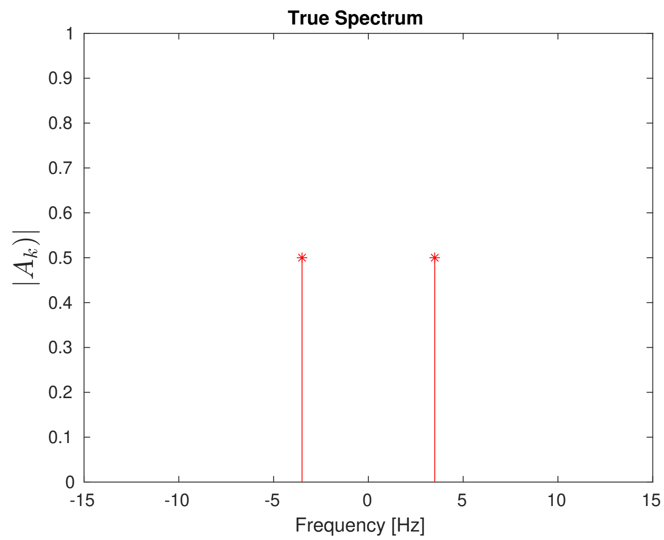


Figure 4: Fourier coefficients of $y(t)$

3d)

Comparing fig. 5 and fig. 6, it can be concluded that the Hanning window reduces the spectral leakage. Especially, the power leakage at $f = 0$ is greatly reduced in fig. 6. Ideally, the two plots should consist of only two spikes at $f = \pm 7/2$, and fig. 6 resembles this the most. In conclusion, the Hanning window improves the discrete finite Fourier transform. The Hanning window has positive effect due to the time record not being an exact number of periods of the data.

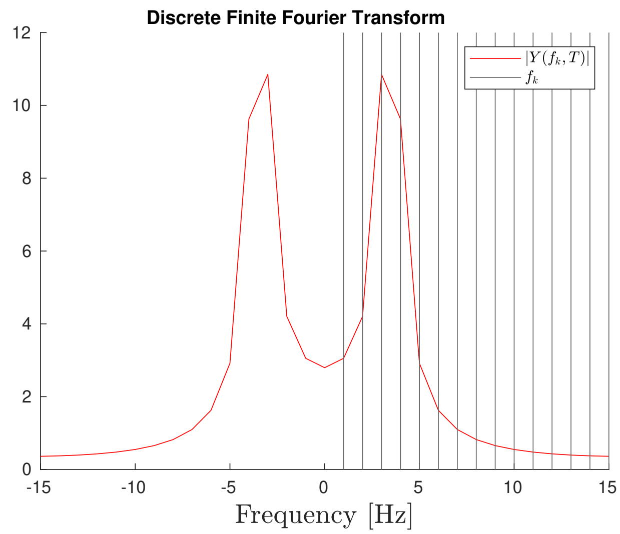


Figure 5: Discrete finite Fourier transform of $y(t)$

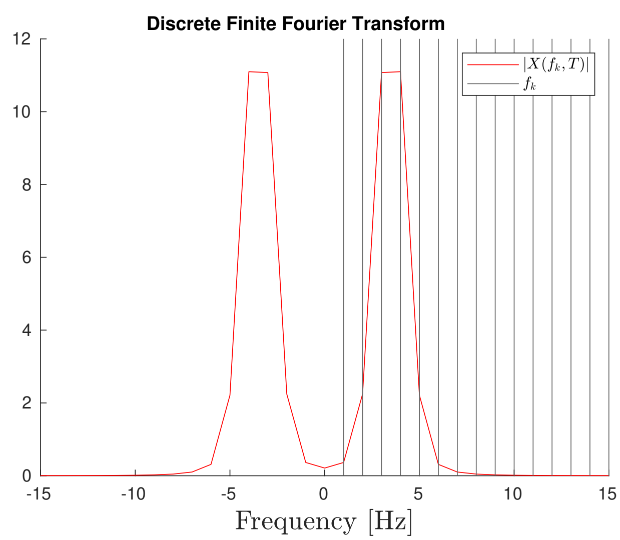


Figure 6: Discrete finite Fourier transform of scaled Hanning windowed signal $x(t)$

Problem 4a)

The power spectrum using the rectangular window is plotted in fig. 7.

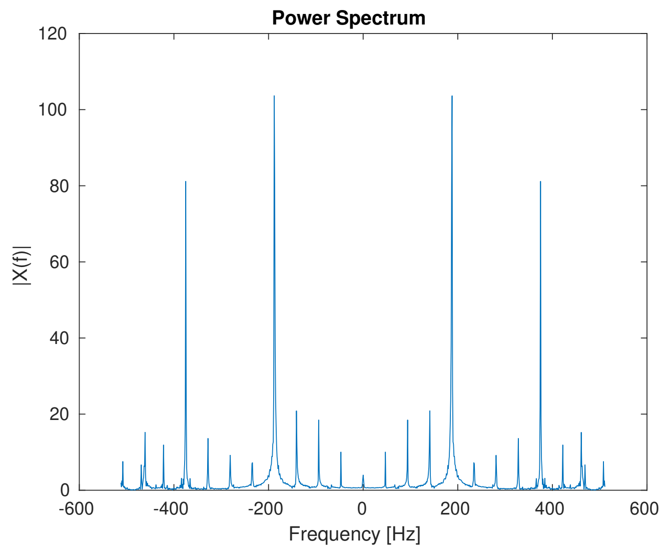


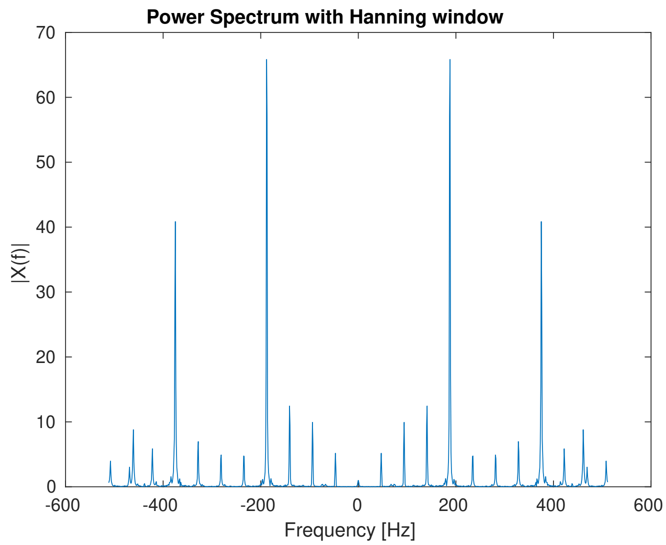
Figure 7: Power spectrum using rectangular window

4b)

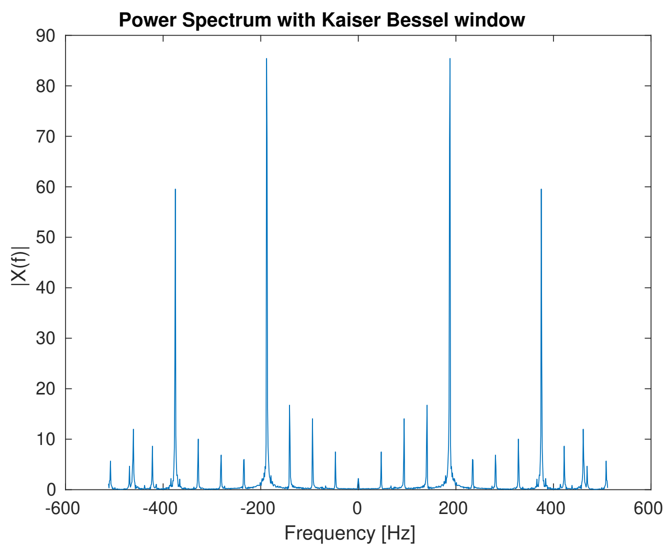
The power spectrum using the Hanning window is shown in fig. 8a, while the power spectrum using the Kaiser Bessel window is shown in fig. 8b. The Kaiser Bessel window was chosen with $\beta = 2.5$.

4c)

Looking at fig. 7 small lobes around the spikes can be observed. The lobes are not desired, and is a result of power leakage. By inspecting fig. 8a and fig. 8b, it can be seen that the lobes are reduced. However, as an unwanted side-effect the magnitude of the spikes has been reduced. Applying a window function then introduces a choice between keeping the magnitude of the power spectrum and reducing the power leakage - reducing the power leakage will reduce the magnitude of the power spectrum. The Hanning window suppresses the lobes the most, but also reduces the magnitude the most. Choosing a larger β for the Kaiser Bessel window will give a similar response.



(a) Power spectrum with applied Hanning window



(b) Power spectrum with applied Kaiser Bessel window

Figure 8: Power spectrum with applied window functions