Problem 1

The MATLAB code for this problem will be attached at the very end of this pdf.

1a)

A plot of the magnitude of the discrete Fourier transform of x_n is shown fig. 1. The real- and imaginary part of the discrete Fourier transform is shown in fig. 2 and fig. 3 respectively.

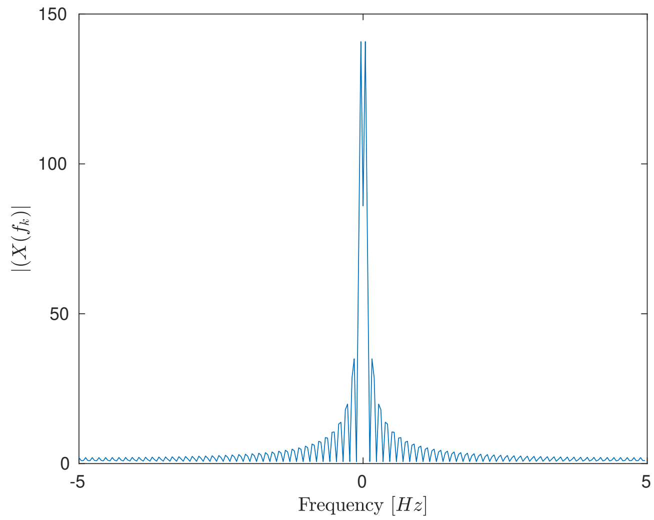


Figure 1: Magnitude of the discrete Fourier transform of x_n

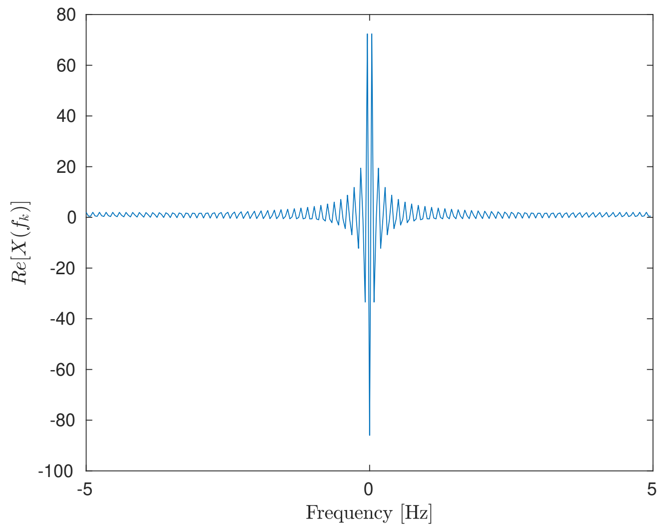


Figure 2: The real component of the discrete Fourier transform of x_n

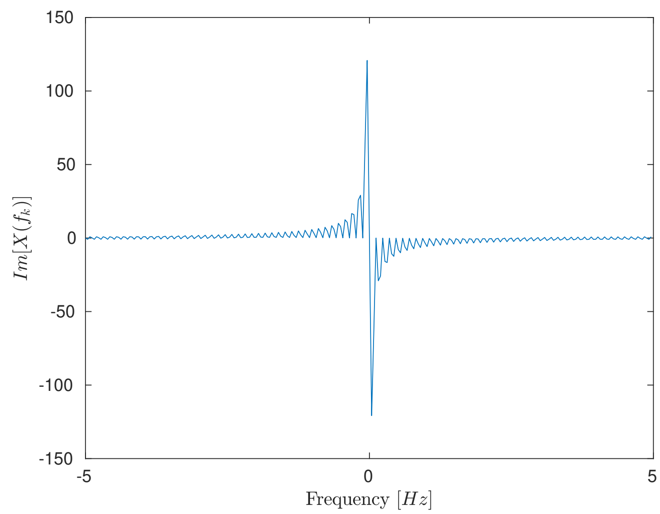


Figure 3: The imaginary component of the discrete Fourier transform of x_n

1b)

Both the real- and imaginary part of the discrete Fourier transform of x_n is plotted in fig. 4. The real part is symmetric around $f_{k=0}$, while the imaginary part is both symmetric around $f_{k=0}$ and flipped about the x-axis. This means that $X_{-k} = X_k^*$.

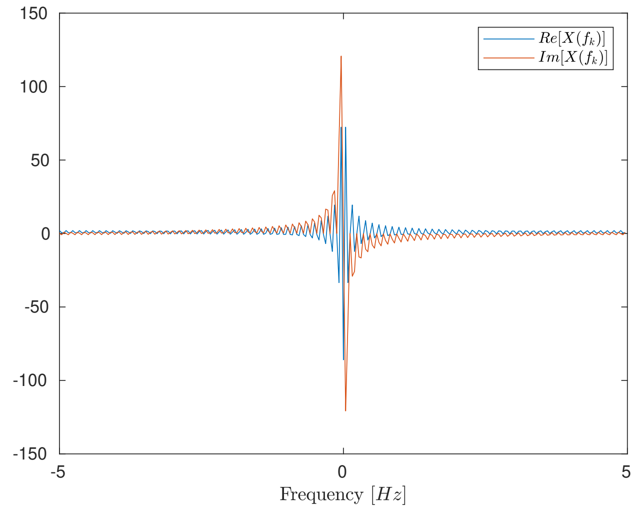


Figure 4: Discrete Fourier transform of x_n

1c)

Unfortunately, I did not manage to do this task correctly. My thought was that sampling up to N would result in a discrete Fourier transform showing $X(f_k)$ for $k = -\frac{N}{2}$ to $k = \frac{N}{2}$. Then, in order to obtain $X(f_s)$, $s = \frac{N}{2} + 1$ to N, the number of samples must be increased to $2N$. To my understanding, the discrete Fourier transform would result in $X(f_k)$ for $k = -N$ to $k = N$ However, this was not the case, and the desired plot never not obtained. The result is shown in fig. 5.

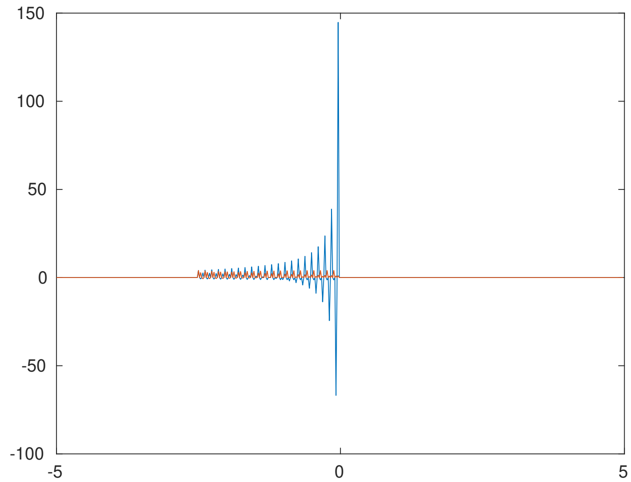


Figure 5: Failed attempt to prove $X_{N-k} = X_{-k}$, $k = 0$ to $\frac{N}{2}$

1d)

By only keeping the smallest frequencies, the inverse discrete Fourier transform resulted in the curve $\hat{x}_n$ shown fig. 6. However, the discrete Fourier transform did not result in a completely real-valued sequence. The reason remains unclear. The chosen frequencies for the inverse discrete Fourier transfrom are shown in fig. 7. Note that the x-axis in this plot is meaningless.

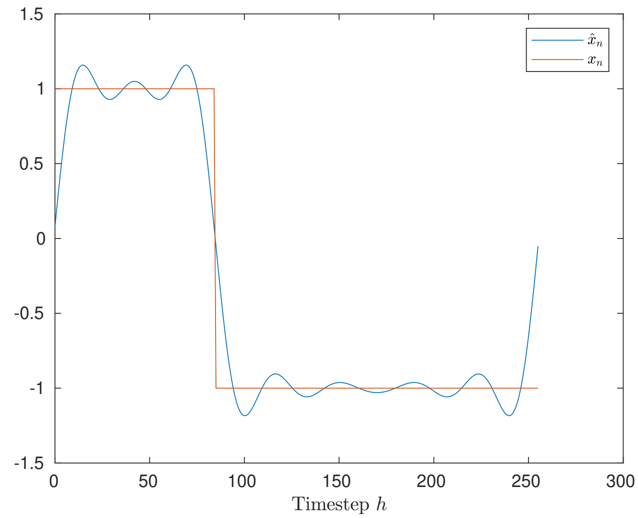


Figure 6: Inverse discrete Fourier transform of selected frequencies $\hat{x}_n$ and the sampled sequence of data x_n.

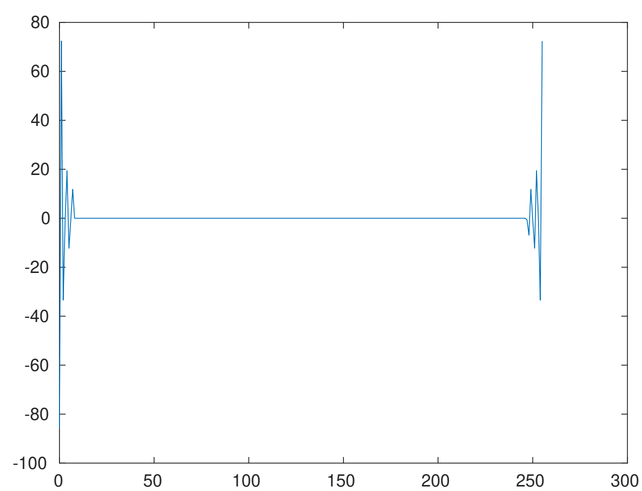


Figure 7: Chosen frequencies for the inverse discrete Fourier transform.

1e)

The reason why x_n and $\hat{x}_n$ are different is the removal of frequencies. The curve $\hat{x}_n$ is lacking its high-frequency components, and the effect is shown in fig. 6 by the curve failing to keep a constant value. Adding and subtracting different high-frequency components to $\hat{x}_n$ would enable the possibility of the curve holding a constant value. The end-points where the $\hat{x}_n$ goes from -1 to 1 is also affected by the removal of high-frequency components. Ultimately, the complex Fourier series expansion identity does not hold for $\hat{x}_n$.

2a)

From the two plots, we see that we can get either 1 or -1 as a function value. Resembles a coin-toss.

Therefore

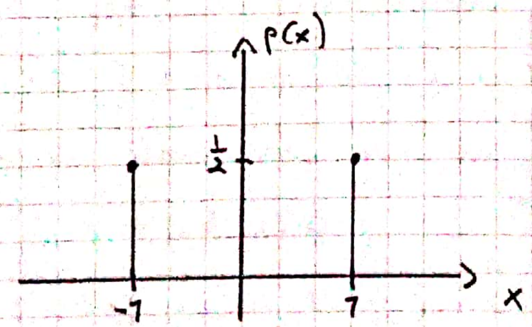
$$p(x=1) = \frac{4}{8} = \frac{1}{2}$$

and

$$p(x=-1) = \frac{4}{8} = \frac{1}{2}$$

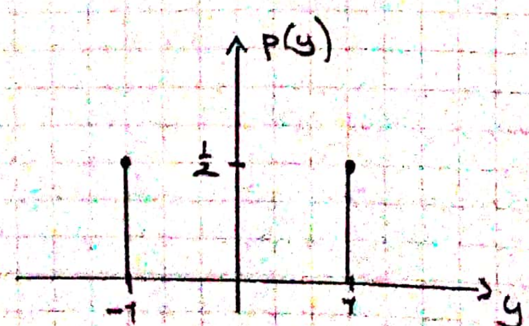
This can be summarized as

$$p(x) = \left\{\begin{matrix} \frac{1}{2}, & x=-1 \\ \frac{1}{2}, & x=1 \end{matrix}\right\} = \underline{\underline{\frac{1}{2}}}$$



For $y(t)$ we have the same situation

$$p(y) = \left\{\begin{matrix} \frac{1}{2}, & y=-1 \\ \frac{1}{2}, & y=1 \end{matrix}\right\} = \underline{\underline{\frac{1}{2}}}$$

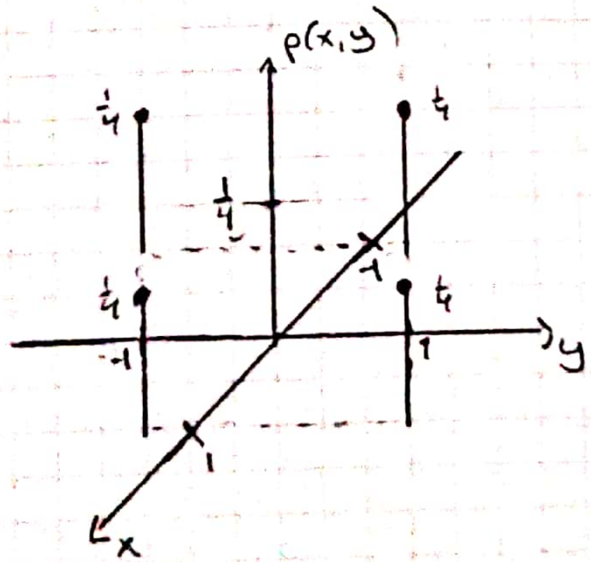


For the joint probability, we have the following scenarios

y \ x	-1	1
-1	$p(x=-1, y=-1)$	$p(x=1, y=-1)$
1	$p(x=-1, y=1)$	$p(x=1, y=1)$

By counting the data points, we get

y \ x	-1	1
-1	$\frac{2}{8} = \frac{1}{4}$	$\frac{2}{8} = \frac{1}{4}$
1	$\frac{2}{8} = \frac{1}{4}$	$\frac{2}{8} = \frac{1}{4}$



- Sorry for my poor 3D-drawing ! :)

2b)

Two processes are independent if

$$p(\tilde{x},\tilde{y}) = p(\tilde{x})\,p(\tilde{y})$$

For our processes, we have

$$p(x,y) = \frac{1}{4}$$

and

$$p(x)\,p(y) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$$

Can see that

$$p(x,y) = p(x)\,p(y) = \frac{1}{4}$$

Hence, the two processes are statistical independent.

2c)

Two processes are uncorrelated if

$$E[\tilde{x}\tilde{y}] = E[\tilde{x}]\,E[\tilde{y}]$$

For the given processes, we have

$$E[x] = -1\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = 0$$

$$E[y] = -1\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = 0$$

$$E[xy] = \sum_x \sum_y x\,y\,p(x,y)$$

$$= (-1)\cdot(-1)\cdot\frac{1}{4} + 1\cdot(-1)\cdot\frac{1}{4} + (-1)\cdot 1\cdot\frac{1}{4} + 1\cdot 1\cdot\frac{1}{4}$$

$$= \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4} = 0$$

So

$$E[xy] = E[x]\,E[y] = 0$$

Hence, the two processes are uncorrelated

Independent processes $\Rightarrow$ uncorrelated, so this checks out!

2c)

Two processes are orthogonal if

$$E[(\tilde{x}+\tilde{y})^2] = E[\tilde{x}^2] + E[\tilde{y}^2]$$

For our two processes, we have

$$E[x^2] = (-1)^2 \cdot \tfrac{1}{2} + 1^2 \cdot \tfrac{1}{2} = 1$$

$$E[y^2] = (-1)^2 \cdot \tfrac{1}{2} + 1^2 \cdot \tfrac{1}{2} = 1$$

$$E[(x+y)^2] = (-1-1)^2 \cdot \tfrac{1}{4} + (-1+1)^2 \cdot \tfrac{1}{4} + (1-1)^2 \cdot \tfrac{1}{4} + (1+1)^2 \cdot \tfrac{1}{4}$$

$$= 4 \cdot \tfrac{1}{4} + 4 \cdot \tfrac{1}{4}$$

$$= 2$$

So

$$E[(x+y)^2] = E[x^2] + E[y^2] = 2$$

and the processes are therefore orthogonal.

This checks out since $E[xy] = E[x]E[y] = 0$.

3)

Given that the random variables are uncorrelated

$$E[xy] = E[x]\,E[y] \quad , \quad \rho_{xy} = 0$$

Want to show that the variables are statistically independent

$$p(x,y) = p(x)\,p(y)$$

where we know that

$p(x,y)$ is a 2-dimensional normal distribution.

So

$$p(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho_{xy}}}\, e^{\frac{-1}{2(1-\rho_{xy}^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho_{xy}\cdot\frac{x-\mu_x}{\sigma_x}\cdot\frac{y-\mu_y}{\sigma_y} + \left(\frac{y-\mu_y}{\sigma_2}\right)^2\right]}$$

Inserting $\rho_{xy} = 0$ yields

$$p(x,y) = \frac{1}{2\pi\sigma_x\sigma_y}\, e^{-\frac{1}{2}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]}$$

$$= \frac{1}{2\pi\sigma_x\sigma_y}\, e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \cdot e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_x}\, e^{-\frac{1}{2}\left(\frac{x-\mu_x}{\sigma_x}\right)^2} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_y}\, e^{-\frac{1}{2}\left(\frac{y-\mu_y}{\sigma_y}\right)^2}$$

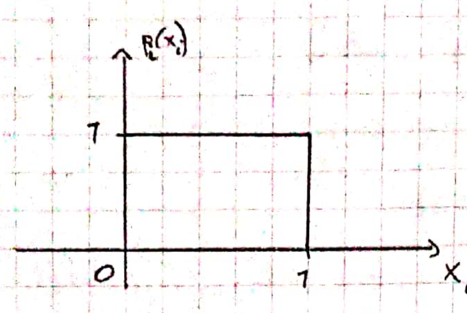
$$= p(x)\cdot p(y)$$

where $p(x)$ and $p(y)$ are also normally distributed.

Hence, the random variables are independent.

4a)

Given that $x_i(t)$ are uniformly distributed over $0 \le x_i \le 1$



$$p_i(x_i) = \frac{1}{b-a}, \quad \begin{matrix} b=1 \\ a=0 \end{matrix}$$

So

$$p(y) = \int_{-\infty}^{\infty} p_1(\xi)\, p_2(y-\xi)\, d\xi$$

Convolution integral is generally given by

$$p(y) = 0, \quad y \le y_{10} + y_{20}$$

$$p(y) = \int_{\max(y_{10},\, y-y_{11})}^{\min(y_{21},\, y-y_{10})} p_1(\xi)\, p_2(y-\xi)\, d\xi, \quad y_{10}+y_{20} \le y \le y_{11}+y_{21}$$

$$p(y) = 0, \quad y \ge y_{11}+y_{21}$$

where

$$y_{10} = y_{20} = 0$$

$$y_{11} = y_{21} = 1$$

This gives two cases where the integrand is non-zero.
For $0 \le y \le 1$:

$$p(y) = \int_0^y 1 \cdot d\xi = y$$

For $1 \le y \le 2$:

$$p(y) = \int_{y-1}^{1} 1 \cdot d\xi = 1 - (y-1) = 2 - y$$

Ultimately

$$p(y) = \begin{cases} y, & 0 \le y \le 1 \\ 2-y, & 1 \le y \le 2 \end{cases}$$

4b)

Given the value

$$z(t) = x_1(t) + x_2(t) + x_3(t)$$

This can be rewritten as

$$z(t) = y(t) + x_3(t)$$

Using the formula for $p(y)$ yields

$$p(z) = \int_{-\infty}^{\infty} p_3(\xi)\, p_y(z-\xi)\, d\xi \quad , \quad \begin{cases} 0 \le z-\xi \le 1 \\ \Downarrow \\ z-1 \le \xi \le z \end{cases}$$

The functions will convolute in three different cases,

For $0 \le z \le 1$:

$$p(z) = \int_{\max(0,\, z-1)}^{\min(1,\, z)} \xi \, d\xi = \int_0^z \xi \, d\xi = \underline{\tfrac{1}{2} z^2}$$

For $1 \le z \le 2$:

The functions will convolute on two intervals $(0,1)$ and $(1,2)$.

So

$$p(z) = \int_{\max(0,z-1)}^{\min(1,z)} \xi \, d\xi + \int_{\max(1,\, z-1)}^{\min(2,\, z)} 2-\xi \, d\xi$$

$$= \int_{z-1}^{1} \xi \, d\xi + \int_{1}^{z} 2-\xi \, d\xi$$

$$= \left[\tfrac{1}{2}\xi^2\right]_{z-1}^{1} + \left[2\xi - \tfrac{1}{2}\xi^2\right]_{1}^{z}$$

$$= \tfrac{1}{2} - \tfrac{1}{2}(z-1)^2 + 2z - \tfrac{1}{2}z^2 - 2 + \tfrac{1}{2}$$

$$= \tfrac{1}{2} - \tfrac{1}{2}\left(z^2 - 2z + 1\right) + 2z - \tfrac{1}{2}z^2 - 2 + \tfrac{1}{2}$$

$$= -\tfrac{1}{2}\left(z^2 - 2z + 1\right) + 2z - \tfrac{1}{2}z^2 - 1$$

$$\underline{p(z) = -\tfrac{1}{2}\left(2z^2 - 6z + 3\right)}$$

For $2 \leq z \leq 3$

The functions convolve on the interval $(1,2)$, so

$$p(z) = \int_{\max(1,z-1)}^{\min(2,z)} 2 - \xi \, d\xi$$

$$= \int_{z-1}^{2} 2 - \xi \, d\xi = \left[2\xi - \frac{1}{2}\xi^2 \right]_{z-1}^{2}$$

$$= 2 \cdot 2 - \frac{4}{2} - 2 \cdot (z-1) + \frac{1}{2} \cdot (z-1)^2$$

$$= 4 - \cancel{2} - 2z + \cancel{2} + \frac{1}{2}(z^2 - 2z + 1)$$

$$= \frac{1}{2}(z^2 - 6z + 9)$$

$$\underline{p(z) = \frac{1}{2}(z-3)^2}$$

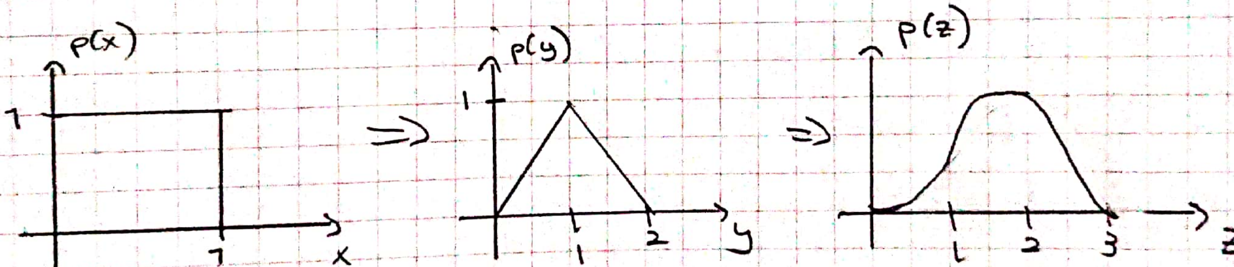
Summarized

$$p(z) = \begin{cases} \frac{1}{2} z^2 , & 0 \leq z \leq 1 \\ -\frac{1}{2}(2z^2 - 6z + 3) , & 1 \leq z \leq 2 \\ \frac{1}{2}(z-3)^2 , & 2 \leq z \leq 3 \end{cases}$$

4c)

The central limit theorem states that when independent random variables are added, their sum will tend toward a normal distribution, even though the original variables themselves are not normally distributed.

Observe the following simple sketches of the probability functions in the previous tasks.



As we can see, by adding more and more uniform distributed variables, the distribution will tend toward a normal distribution. This is in compliance with the central limit theorem.

Listing 1: Code for Problem 1

```
clc, clear, close all;
%% Data preparation
N = 256; % Samples
h = 0.1; % Sampling interval
T = N*h; % Period

xn = zeros(1,N); % Data sequence
arrayIndex = 1;

% Create/gather datapoints for xn
for t=0:h:T-h
    if (t < 1/3*T-h)
        xn(arrayIndex) = 1;
    else
        xn(arrayIndex) = -1;
    end
    arrayIndex = arrayIndex + 1;
end


%% Task 1a)
Xpure = fft(xn, N); % Compute DFT of x
% Rearrange output from fft
X = zeros(1,length(xn));
X(1:128) = Xpure(129:256);
X(129:256) = Xpure(1:128);

mag = abs(X); % Magnitude

fs = 1/h; % Samples per second
binVals = [-N/2 : N/2-1];
faxHz = binVals*fs/N; % Frequency x-axis

figure(1);
plot(faxHz, mag);
hold on;
xlabel('Frequency [$Hz$]', 'interpreter', 'latex');
ylabel('$|(X(f_k)|$', 'interpreter', 'latex');
hold off;

figure(2);
plot(faxHz, real(X));
hold on;
xlabel('Frequency [Hz]', 'interpreter', 'latex');
```

```
ylabel('$Re[X(f_k)]$', 'interpreter', 'latex');
hold off;

figure(3);
plot(faxHz, imag(X));
hold on;
xlabel('Frequency [$Hz$]', 'interpreter', 'latex');
ylabel('$Im[X(f_k)]$', 'interpreter', 'latex');
hold off;


%% Task 1b)
figure(4);
plot(faxHz, real(X));
hold on;
plot(faxHz, imag(X));
legend('$Re[X(f_k)]$', '$Im[X(f_k)]$', 'interpreter', '
    latex');
xlabel('Frequency [$Hz$]', 'interpreter', 'latex');
hold off;


%% Task 1c ???)
Nc = 256 + N; % Samples

xnc = zeros(1,Nc); % Data sequence
arrayIndex = 1;

% Create/gather datapoints for xnc
for period=1:2
    for t=0:h:T-h
        if (t < 1/3*T-h)
            xnc(arrayIndex) = 1;
        else
            xnc(arrayIndex) = -1;
        end
        arrayIndex = arrayIndex + 1;

    end
end

figure(5);
XcPure = fft(xnc);
% Rearrange output from fft
Xc = zeros(1, length(xnc));
Xc(1:256) = XcPure(257:512);
```

```
Xc(257:512) = XcPure(1:256);

binVals = [-length(Xc)/2 : length(Xc)/2-1];
faxHz = binVals*fs/length(Xc);

Xmink = zeros(1, length(Xc));
Xmink(128:256) = Xc(128:256);
Xnmink = zeros(1, length(Xc));
Xnmink(128:256) = Xc(384:512);

binVals = [-length(Xc)/2 : length(Xc)/2-1];
faxHz = binVals*fs/length(Xc);

plot(faxHz, Xmink);
hold on;
plot(faxHz, Xnmink);
hold off;


%% Task 1d)
figure(6);
Xhat = zeros(1,256);
Xhat(1:8) = Xpure(1:8);
Xhat(248:256) = Xpure(248:256);
binVals = [0 : length(Xhat)-1];
xhat = ifft(Xhat);
plot(binVals, xhat);
hold on;
plot(binVals, xn);
xlabel('Timestep $h$', 'interpreter', 'latex');
legend('$\hat{x}_n$', '$x_n$', 'interpreter', 'latex');
```