ME591 - Assignment 9

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Problem 7a)

Given a zero mean Gaussian input, so

$$p(v) = \frac{p(x)}{|2x|} = \frac{p(x=\sqrt{|v|})}{2\sqrt{|v|}} = \frac{1}{2\sigma_x\sqrt{2\pi|v|}} e^{-\frac{|v|}{2\sigma_x^2}}$$

Hence, the output has also a zero mean Gaussian probability distribution. Then

$$R_W(0) = \sigma_v^2 + \mu_v^2 = \sigma_v^2 = E[(v-\mu_v)^2] = E[v^2]$$

Further

$$E[v^2] = E[x|x|\cdot x|x|] = E[x^4]$$

Define

$$x_1, x_2, x_3, x_4 = x$$

Using the hint for the fourth-order moment of the zero mean value Gaussian inputs we obtain

$$\begin{aligned}
E[x^4] &= E[x_1 x_2 x_3 x_4] \\
&= E[x_1x_2]E[x_3x_4] + E[x_1x_3]E[x_2x_4] + E[x_1x_4] + E[x_2x_3] \\
&= 3E[x^2]E[x^2] \\
&= 3E[(x-\mu_x)^2]E[(x-\mu_x)^2] \\
&= 3\sigma_x^2\sigma_x^2 \\
&= 3\sigma_x^4 \quad \blacksquare
\end{aligned}$$

7b)

The error is given as

$$e'(t) = v - ax$$

Then

$$E\left[e'(t)^2\right] = E\left[(v-ax)^2\right]$$
$$= E\left[v^2 - 2axv + a^2x^2\right]$$
$$= E[v^2] - 2a\,E[xv] + a^2 E[x^2]$$
$$= E[x^4] - 2a\,E[xv] + a^2 E[x^2] \quad , \; E[x^2] = \sigma_x^2$$
$$= 3\sigma_x^4 - 2a\,E[xv] + a^2\sigma_x^2$$

where

$$E[xv] = E\left[x^2\,|x|\right] = 2E\left[x^3\right]_{x>0}$$

Using the second hint, we obtain

$$2E[x^3] = 2\int_0^\infty x^3 \cdot \frac{1}{\sigma_x\sqrt{2\pi}}\, e^{-\frac{x^2}{2\sigma_x^2}}\, dx$$

$$= \frac{1}{\sigma_x\sqrt{2\pi}}\, 2\int_0^\infty x^3 e^{-\frac{1}{2\sigma_x^2}x^2}\, dx$$

$$= \frac{1}{\sigma_x\sqrt{2\pi}} \cdot \frac{\Gamma(1)}{\left(\frac{1}{2\sigma_x^2}\right)^2}$$

$$= \frac{1}{\sigma_x\sqrt{2\pi}} \cdot 4\sigma_x^4$$

$$= 2\sigma_x^3\sqrt{\frac{2}{\pi}}$$

So

$$E\left[e'(t)^2\right] = 3\sigma_x^4 - 2a \cdot 2\sigma_x^3\sqrt{\frac{2}{\pi}} + a^2\sigma_x^2$$
$$= 3\sigma_x^4 - 4a\sigma_x^3\sqrt{\frac{2}{\pi}} + a^2\sigma_x^2$$
$$:= Q$$

Minimization of the mean square error yields

$$\frac{dQ}{da} = -4\sigma_x^3\sqrt{\frac{2}{\pi}} + 2a\sigma_x^2 = 0$$

$$\Rightarrow \quad a = 2\sigma_x\sqrt{\frac{2}{\pi}}$$

Ultimately

$$v(t) = x(t)\,|x(t)| \approx 2\sigma_x\sqrt{\frac{2}{\pi}}\,x$$

1c)

Given that

$$w' = ax$$

Then

$$R_{w'w'}(0) = E[w'^2] = E[(ax)^2] = E[a^2x^2] = a^2E[x^2]$$

$$= a^2\sigma_x^2$$

$$= 4\sigma_x^2 \cdot \frac{2}{\pi}\sigma_x^2$$

$$= \frac{8}{\pi}\sigma_x^4 \approx 2{,}54\,\sigma_x^4$$

The ratio between the approximated value and the true value is

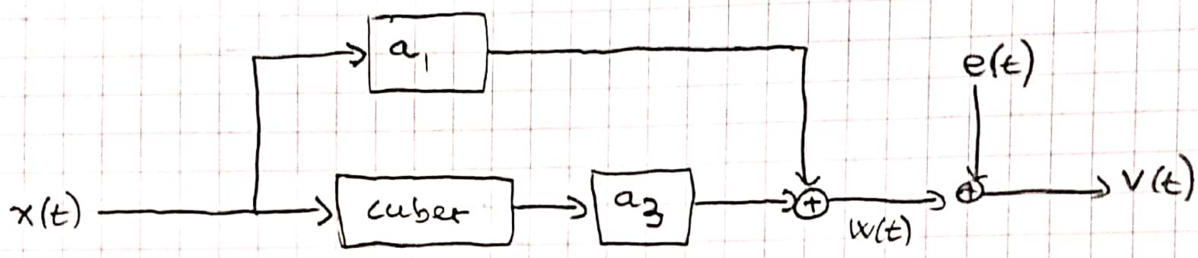
$$\frac{8/\pi}{3} \approx 0{,}85$$

The approximation differs by about 15% from the correct result. The first-order approximation can therefore be regarded as a decent approximation.

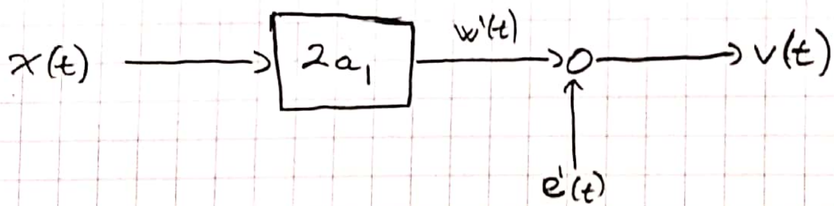
Increasing the order of the approximation will give a drastical improvement in the accuracy.

1d)

The third-order polynomial approximation can be illustrated as



The first-order polynomial approximation can be illustrated as



The first-order approximation reflects the linear part of the nonlinear output. Hence,

$$\gamma_{xw'} = 1 - \gamma_{xw}$$

?

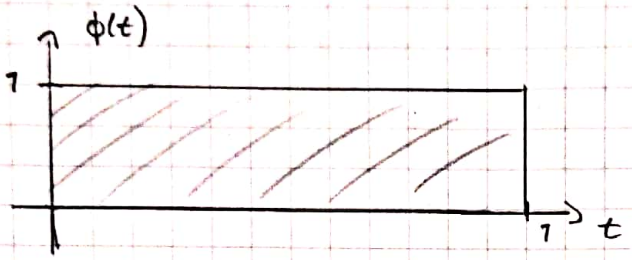
7e)

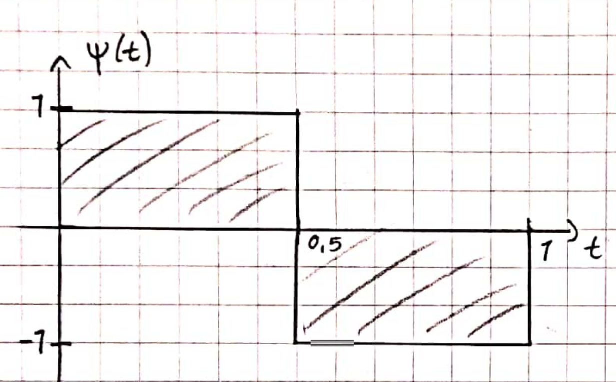
- One disadvantage is clearly the loss of accuracy when using the first-order approximation. The third-order approximation is significantly more accurate.

+ One advantage is linearity. Using a first-order approximation imposes a linear relation between input and output, a trait that is always desirable.

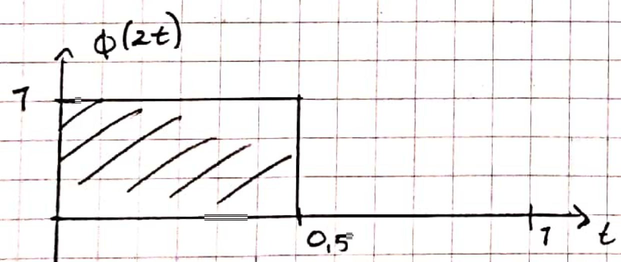
Problem 2a)

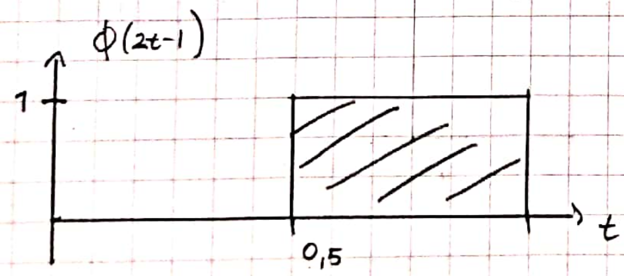
From slide 11 Lecture 23, we have the following for the Haar wavelets





From the two plots, we can derive the following





The last two plots are in compliance with the definition of the scaling function and Haar wavelet.

2b)

The definition of orthogonal processes is

$$E[xy] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, p(x,y)\, dx\, dy = 0$$

Hence

$$\int_{-\infty}^{\infty} \psi(t)\,\phi(t)\, dt = \int_{-\infty}^{\infty} \Big(\phi(2t) - \phi(2t-1)\Big)\Big(\phi(2t) + \phi(2t-1)\Big)\, dt$$

$$= \int_{-\infty}^{\infty} \phi^2(2t) - \phi^2(2t-1)\, dt$$

$$= \int_{0}^{1} \phi^2(2t)\, dt - \int_{0}^{1} \phi^2(2t-1)\, dt$$

In the second term the scaling function is moved with a time offset. However, the integral of the two terms will be equal on the given time intervals. Hence,

$$\int_{0}^{1} \phi^2(2t)\, dt - \int_{0}^{1} \phi^2(2t-1)\, dt = 0$$

which proves that $\psi(t)$ is orthogonal to $\phi(t)$. ∎

2c)

Have that

$$\int_0^1 \psi(2t-1)\, dt$$

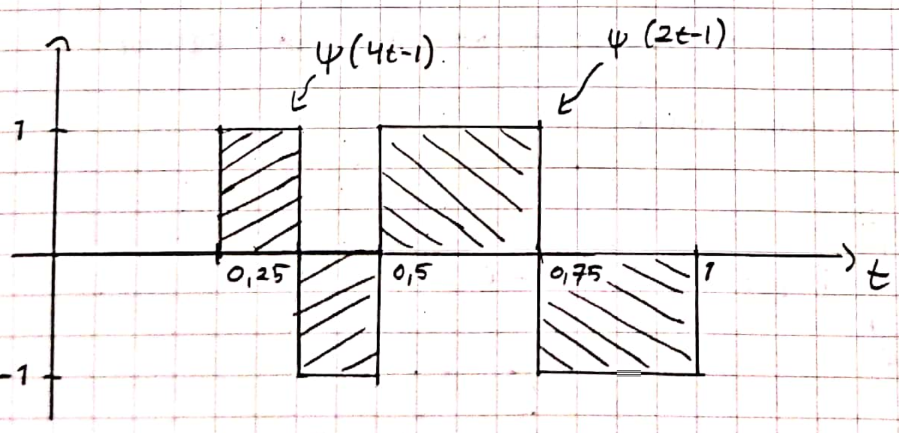
The Haar wavelet $\psi(t)$ is defined as an even function on the interval $[0,1]$. Hence, it is trivial to conclude that

$$\int_0^1 \psi(2t-1)\, dt = 0 \quad \blacksquare$$

Further

$$\int_0^1 \psi(2t-1)\, \psi(4t-1)\, dt$$

A geometric interpretation yields



It is trivial to see that the $\psi(2t-1)=0$ wherever $\psi(4t-1)\neq 0$ and vice versa. Hence

$$\int_0^1 \psi(2t-1)\, \psi(4t-1)\, dt = 0 \quad \blacksquare$$

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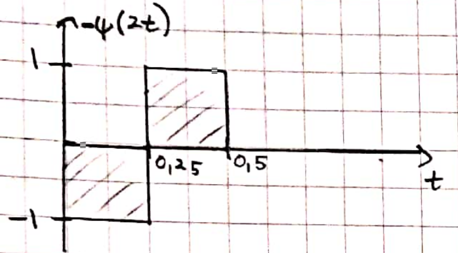
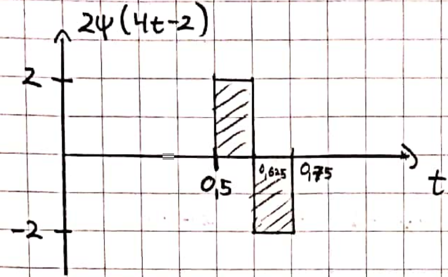
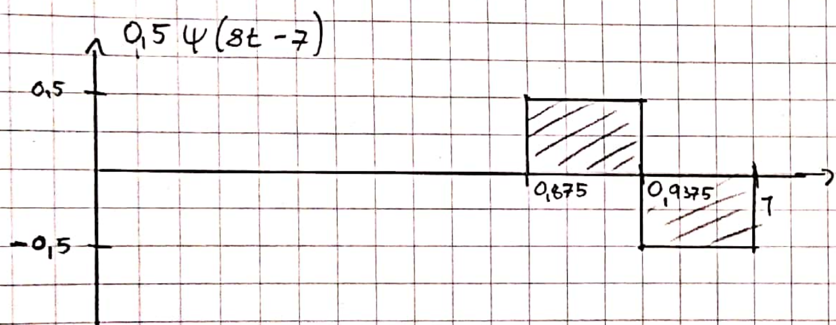
2d)

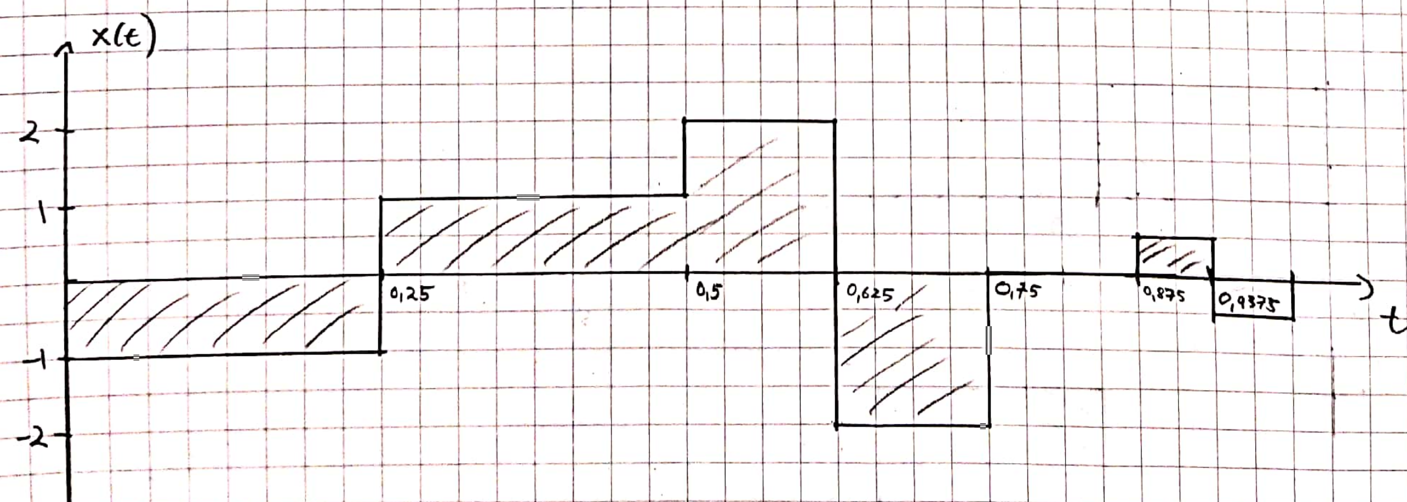
Plotting the different terms independently yields







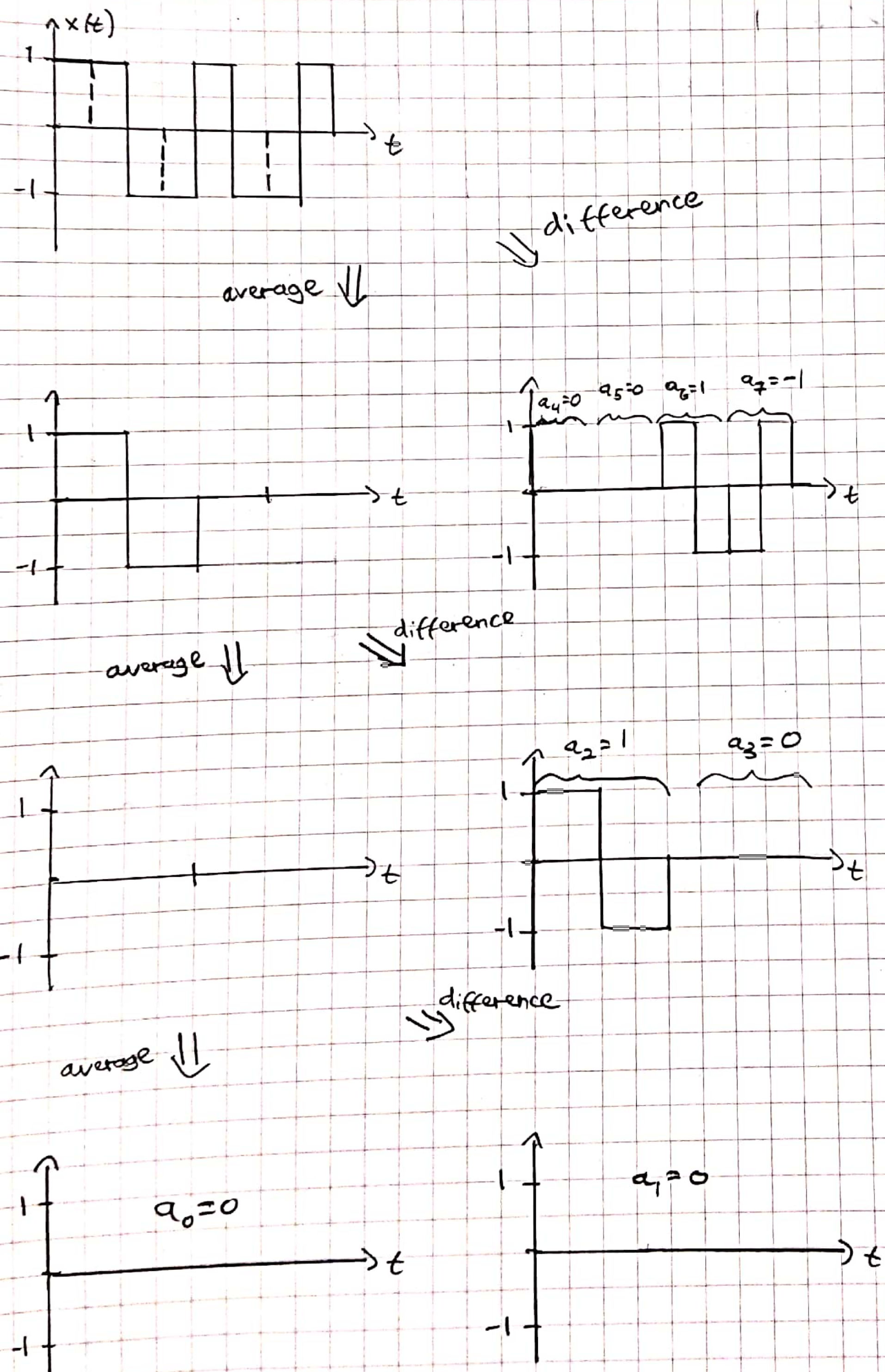
Adding these plots together gives



Problem 3)

The Haar wavelet coefficients can be obtained from the pulse function by dividing the data into average and difference.

Hence



The Haar wavelet coefficients are

$$a_2 = 1 \quad , \quad a_6 = 1 \quad , \quad a_7 = -1$$

while the rest are equal to zero. The pulse function can then be stated as

$$x(t) = \psi(2t) + \psi(4t-2) - \psi(4t-3)$$

which corresponds to the given plot.

Problem 4

Code used to solve this problem is available at
https://github.com/hermanjakobsen/Random-Data/tree/master/HW9

4a)

The data $x(i)$ is shown in fig. 1, while the spectrogram of the data is shown in fig. 2.

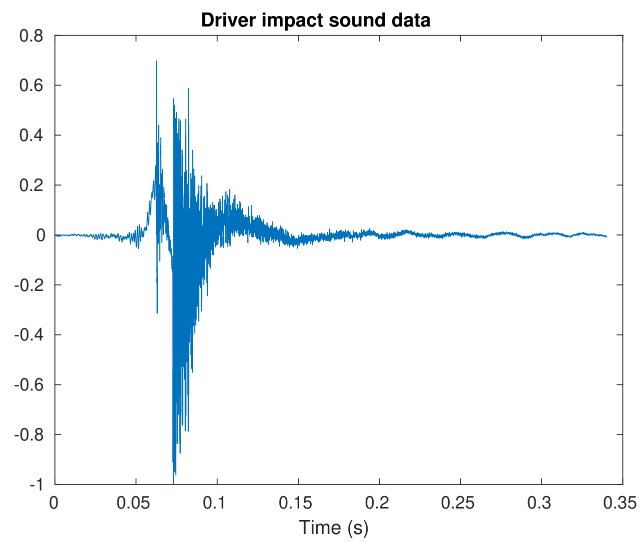


Figure 1: Driver impact sound data

4b)

A spectrogram is a visual representation of the spectrum of frequencies of a signal as it varies with time. As time passes, fig. 2 shows that the spectrum consists of gradually higher frequency components. This can be somewhat verified by looking at fig. 1, where the start of the data can be compared to the end of the data. The tail can be characterized as more noisy, which reflects higher frequency components.

Due to large amplitude variations in fig. 1, it can be concluded that the ball was hit at around 50 ms. At the same time instance in fig. 2, the power of the higher frequency components starts to increase. This is due to the sound of the hit consisting of higher frequency components. After some time, only the high frequency components prevails and characterizes the spectrum. Looking at fig. 1, the high frequency components can be because of noise, since the sound of the hit only lasts for about 100 ms.

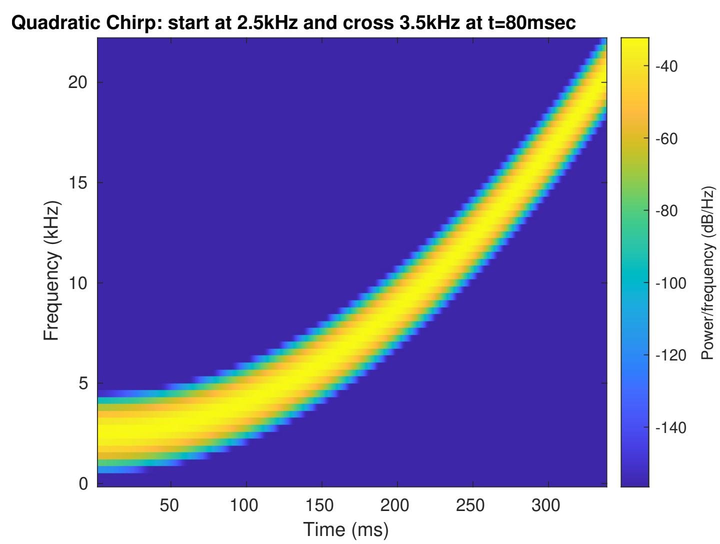


Figure 2: Spectrogram of driver impact sound data