ME591 - Assignment 8

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ME 591 Assignment 8 Herman K. Jakobsen

Problem 1)

From the figure, and the fact that output noise is correlated with the input signal, we have

$$G_{xy} = G_{xv} + G_{xn}$$

Then

$$\gamma_{xy}^2 = \frac{|G_{xy}|^2}{G_{xx} G_{yy}} = \frac{|G_{xv} + G_{xn}|^2}{G_{xx}(G_{vv} + G_{nn})} \geq \frac{|G_{xv}|^2}{G_{xx}(G_{vv} + G_{nn})}.$$

Since the system of interest is perfectly linear, we have

$$\gamma_{xv}^2 = \frac{|G_{xv}|^2}{G_{xx} G_{vv}} = 1$$

Hence

$$\frac{|G_{xv}|^2}{G_{xx}(G_{vv} + G_{nn})} = \frac{\frac{|G_{xv}|^2}{G_{xx} G_{vv}}}{1 + \frac{G_{nn}}{G_{vv}}} = \frac{1}{1 + \frac{G_{nn}}{G_{vv}}}$$

Then

$$\gamma_{xy}^2 \geq \frac{1}{1 + \frac{G_{nn}}{G_{vv}}}$$

$$\frac{\gamma_{xy}^2}{G_{vv}} \geq \frac{1}{G_{vv} + G_{nn}}$$

$$\gamma_{xy}^2 (G_{vv} + G_{nn}) \geq G_{vv}$$

$$\gamma_{xy}^2 G_{yy} \geq G_{yy} - G_{nn}$$

$$G_{nn}(f) \geq (1 - \gamma_{xy}^2) G_{yy} \quad \blacksquare$$

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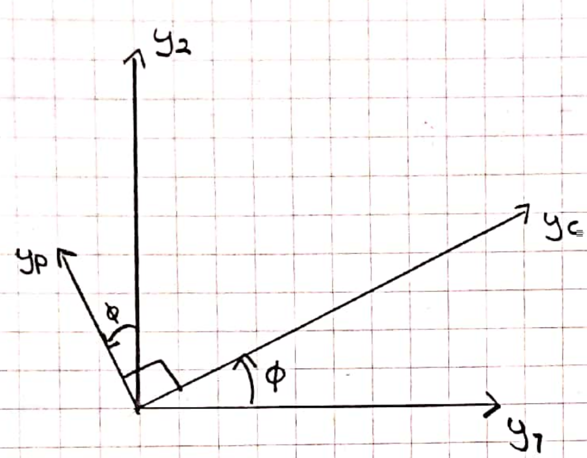
Problem 2a)

The measurements can be illustrated as follows



The following geometric relations can then be derived

$$y_c = y_1 \cos\phi + y_2 \sin\phi$$

$$y_p = -y_1 \sin\phi + y_2 \cos\phi$$

Can then use these relations to express the in-line spectrum $G_{cc}(f)$ and the perpendicular spectrum $G_{pp}(f)$.

$$\begin{aligned} G_{cc}(f) &= 2\,\mathcal{F}\{R_{cc}(\tau)\} \\ &= 2\,\mathcal{F}\{R_{11}(\tau)\cos^2\phi + R_{22}(\tau)\sin^2\phi + (R_{12}(\tau) + R_{21}(\tau))\sin\phi\cos\phi\} \\ &= G_{11}(f)\cos^2\phi + G_{22}(f)\sin^2\phi + (G_{12}(f) + G_{21}(f))\sin\phi\cos\phi \\ &= G_{11}(f)\cos^2\phi + G_{22}(f)\sin^2\phi + 2\,\mathrm{Re}[G_{12}(f)]\sin\phi\cos\phi \quad , G_{12} = G_{21}^* \end{aligned}$$

$$\begin{aligned} G_{pp}(f) &= 2\,\mathcal{F}\{R_{pp}(\tau)\} \\ &= 2\,\mathcal{F}\{R_{11}(\tau)\sin^2\phi + R_{22}(\tau)\cos^2\phi - (R_{12}(\tau) + R_{21}(\tau))\sin\phi\cos\phi\} \\ &= G_{11}(f)\sin^2\phi + G_{22}(f)\cos^2\phi - 2\,\mathrm{Re}[G_{12}(f)]\sin\phi\cos\phi \end{aligned}$$

We then have

$$A(f) = \frac{G_{cc}(f)}{G_{pp}(f)}$$

$$= \frac{G_{11}\cos^2\phi + G_{22}\sin^2\phi + 2\,\mathrm{Re}[G_{12}]\sin\phi\cos\phi}{G_{11}\sin^2\phi + G_{22}\cos^2\phi - 2\,\mathrm{Re}[G_{12}]\sin\phi\cos\phi}$$

The dominant direction is obtained by maximizing $A(f)$. Hence, differentiation yields

$$\frac{\partial A}{\partial \phi} = \frac{[G_{11}\cdot(-\sin 2\phi) + G_{22}(\sin 2\phi) + 2\,\mathrm{Re}[G_{12}]\cos 2\phi]\cdot[G_{11}\sin^2\phi + G_{22}\cos^2\phi - 2\,\mathrm{Re}[G_{12}]\sin\phi\cos\phi]}{(\cdots)}$$

$$- \frac{[G_{11}(\sin 2\phi) + G_{22}(-\sin 2\phi) - 2\,\mathrm{Re}[G_{12}]\cos 2\phi]\cdot[G_{11}\cos^2\phi + G_{22}\sin^2\phi + 2\,\mathrm{Re}[G_{12}]\sin\phi\cos\phi]}{(\cdots)}$$

Since we want to maximize, we set

$$\frac{\partial A}{\partial \phi} = 0$$

Can then only look at the numerator, since the denominator will never make the expression equal to zero. So

$$[-G_{11}\sin 2\phi + G_{22}\sin 2\phi + 2\,\mathrm{Re}[G_{12}]\cos 2\phi]\cdot[G_{11}\sin^2\phi + G_{22}\cos^2\phi - 2\,\mathrm{Re}[G_{12}]\sin\phi\cos\phi]$$

$$- [G_{11}\sin 2\phi - G_{22}\sin 2\phi - 2\,\mathrm{Re}[G_{12}]\cos 2\phi]\cdot[G_{11}\cos^2\phi + G_{22}\sin^2\phi + 2\,\mathrm{Re}[G_{12}]\sin\phi\cos\phi]$$

$$= 0$$

$$
\begin{aligned}
&- G_{11}^2 \sin 2\phi \sin^2\phi - \cancel{G_{11} \sin 2\phi\, G_{22} \cos^2\phi} + \cancel{G_{11} \sin 2\phi\, 2\,\mathrm{Re}[G_{12}] \sin\phi \cos\phi} \\
&+ \cancel{G_{22} \sin 2\phi\, G_{11} \sin^2\phi} + G_{22}^2 \sin 2\phi \cos^2\phi - \cancel{G_{22} \sin 2\phi\, 2\,\mathrm{Re}[G_{12}] \sin\phi \cos\phi} \\
&+ 2\,\mathrm{Re}[G_{12}] \cos 2\phi\, G_{11} \sin^2\phi + 2\,\mathrm{Re}[G_{12}] \cos 2\phi\, G_{22} \cos^2\phi - \cancel{4\,\mathrm{Re}[G_{12}]^2 \cos 2\phi \cos\phi \sin\phi} \\
&- G_{11}^2 \sin 2\phi \cos^2\phi - \cancel{G_{11} \sin 2\phi\, G_{22} \sin^2\phi} - \cancel{G_{11} \sin 2\phi\, 2\,\mathrm{Re}[G_{12}] \sin\phi \cos\phi} \\
&+ \cancel{G_{22} \sin 2\phi\, G_{11} \cos^2\phi} + G_{22}^2 \sin 2\phi \sin^2\phi + \cancel{G_{22} \sin 2\phi\, 2\,\mathrm{Re}(G_{12}) \sin\phi \cos\phi} \\
&+ 2\,\mathrm{Re}[G_{12}] \cos 2\phi\, G_{11} \cos^2\phi + 2\,\mathrm{Re}[G_{12}] \cos 2\phi\, G_{22} \sin^2\phi + \cancel{4\,\mathrm{Re}[G_{12}]^2 \cos 2\phi \sin\phi \cos\phi} \\
&= 0
\end{aligned}
$$

$$- G_{11}^2 \sin 2\phi + G_{22}^2 \sin 2\phi + 2\,\mathrm{Re}[G_{12}] \cos 2\phi\, G_{11} + 2\,\mathrm{Re}[G_{12}] \cos 2\phi\, G_{22} = 0$$

$$- G_{11}^2 \tan 2\phi + G_{22}^2 \tan 2\phi + 2\,\mathrm{Re}[G_{12}]\, G_{11} + 2\,\mathrm{Re}[G_{12}]\, G_{22} = 0$$

$$\tan 2\phi \left(G_{22}^2 - G_{11}^2 \right) = -2\,\mathrm{Re}[G_{12}] \left(G_{11} + G_{22} \right)$$

Ultimately

$$\tan 2\phi_0 = - \frac{2\,\mathrm{Re}[G_{12}] \left(G_{11} + G_{22} \right)}{G_{22}^2 - G_{11}^2}$$

Or by isolating ϕ_0

$$\phi_0(f) = \frac{1}{2} \cdot \tan^{-1} \left[- \frac{2\,\mathrm{Re}[G_{12}(f)] \left(G_{11}(f) + G_{22}(f) \right)}{G_{22}^2(f) - G_{11}^2(f)} \right]$$

2b)

Define

$$z = y_1 + j y_2$$

The spectral density function is defined as

$$G_{ij}(f) = G_{x_i x_j}(f) = \frac{2}{T} E\left[X_i^*(f)\, X_j(f)\right]$$

So

$$G_{\bar{z}z}(f) = \frac{2}{T} E\left[\left(Z^*(f)\right)^* Z(f)\right]$$

$$= \frac{2}{T} E\left[Z(f)\, Z(f)\right]$$

Then

$$G_{\bar{z}z}(f) = 2\, \mathcal{F}\{R_{\bar{z}z}(\tau)\}$$

$$= 2\, \mathcal{F}\left\{ R_{11}^2 + j\left(R_{12} + R_{21}\right) - R_{22}^2 \right\}$$

$$= 2\,($$

?

Problem 3a)

From the diagram we can conclude

$$\gamma_{1y}^2 = A + C = 0{,}5$$

$$\gamma_{2y}^2 = B + C = 0{,}6$$

Hence, the maximum contributions of input x_1 and input x_2 is $0{,}5$ and $0{,}6$, respectively. Stated differently

$$x_{1,\,max.\,cont.} = 0{,}5$$

$$x_{2,\,max.\,cont.} = 0{,}6$$

Further, the minimum contributions given by the two inputs are

$$\gamma_{1y\cdot 2}^2 \quad \text{and} \quad \gamma_{2y\cdot 1}^2$$

respectively. We then have the following relation

$$G_{nn} = \left(1 - \gamma_{1y}^2 - \gamma_{2y\cdot 1}^2\right) G_{yy}$$

Thus

$$\gamma_{2y\cdot 1}^2 = 1 - \gamma_{1y}^2 - \frac{G_{nn}}{G_{yy}} = 1 - 0{,}5 - 0{,}1 = 0{,}4$$

For input x_1, we have

$$\gamma_{1y\cdot 2}^2 = 1 - \gamma_{2y}^2 - \frac{G_{nn}}{G_{yy}} = 1 - 0{,}6 - 0{,}1 = 0{,}3$$

Ultimately

contribution x_1 : $[0{,}3 \,,\, 0{,}5]$

contribution x_2 : $[0{,}4 \,,\, 0{,}6]$

3b)

Generally, we have

$$\gamma^2_{y:x} = \gamma^2_{1y} + \gamma^2_{2y\cdot 1} \quad \text{or} \quad \gamma^2_{y:x} = \gamma^2_{2y} + \gamma^2_{1y\cdot 2}$$

depending on which input that precedes the other.

Further

$$\gamma^2_{2y\cdot 1} = \frac{|G_{2y\cdot 1}|^2}{G_{22\cdot 1}\, G_{yy}}$$

$$= \frac{\left| G_{2y} - \frac{G_{21}}{G_{11}} \cdot G_{1y} \right|^2}{(1-\gamma^2_{12})\, G_{22}\, G_{yy}}$$

$$= \frac{\left| G_{2y} - \frac{G_{21}}{G_{11}} \cdot G_{1y} \right|^2}{\left(1 - \frac{|G_{12}|^2}{G_{11} G_{22}}\right) G_{22}\, G_{yy}}$$

$$= \frac{\left| G_{2y} - \frac{G_{21}}{G_{11}} G_{1y} \right|^2}{\left(G_{22} - \frac{|G_{12}|^2}{G_{11}}\right) G_{yy}} \leq \frac{|G_{2y}|^2}{G_{22}\, G_{yy}} = \gamma^2_{2y}$$

Can therefore conclude that

$$\gamma^2_{y:x} = \gamma^2_{1y} + \gamma^2_{2y\cdot 1} \leq \gamma^2_{1y} + \gamma^2_{2y}$$

The same result holds by changing the preceding input

$$\gamma^2_{y:x} = \gamma^2_{2y} + \gamma^2_{1y\cdot 2} \leq \gamma^2_{2y} + \gamma^2_{1y}$$

Ultimately

$$\gamma^2_{y:x} \leq \gamma^2_{1y} + \gamma^2_{2y} \quad \blacksquare$$

3c)

Have the fraction

$$\frac{1-\gamma_{1y}^2}{1-\gamma_{2y}^2}$$

Both γ_{1y}^2 and γ_{2y}^2 can contain a contribution from the other input. This contribution will be equal for γ_{1y}^2 and γ_{2y}^2, and can be represented as c as in the problem description.

So the above fraction can be rewritten into

$$\frac{1-\gamma_{1y}^2}{1-\gamma_{2y}^2} = \frac{1-(c+\gamma_{1y\cdot 2}^2)}{1-(c+\gamma_{2y\cdot 1}^2)}$$

$$= \frac{(1-c)-\gamma_{1y\cdot 2}^2}{(1-c)-\gamma_{2y\cdot 1}^2}$$

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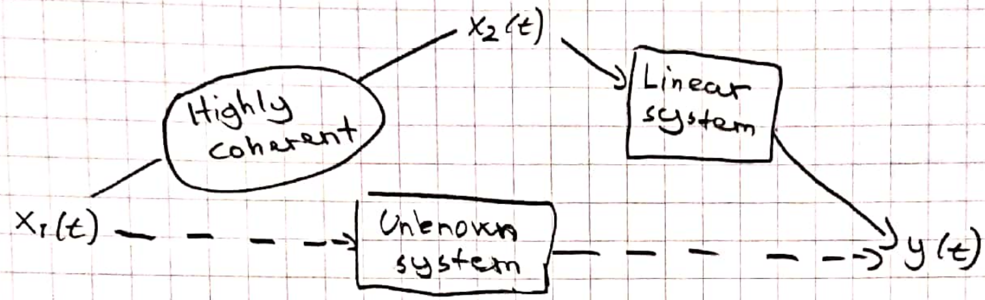
3d)



Assume that a coherence function value near unity is computed between x_1 and y. One could then assume that there is a physical linear system relating these two variables as input and output. Now, suppose there is a third variable x_2, which is highly coherent with x_1 and also passes through a linear system to make up y. Here, the high coherence computed between x_1 and y might be only a reflection of the fact that $x_2(t)$ is highly coherent with x_1, and x_2 is related via a linear system to y. There might be no direct physical system between x_1 and y at all.

Erroneous low coherence function would occur if x_2 was lowly coherent with y, but highly coherent with x_1, while x_1 is highly coherent with y (if this is even possible). Then, x_1 would impose an erroneous low coherence on x_2.

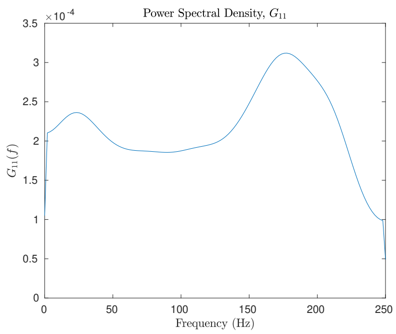
The partial coherence function gives the percentage of the spectrum of the total output record due to the conditioned input record.

Problem 4

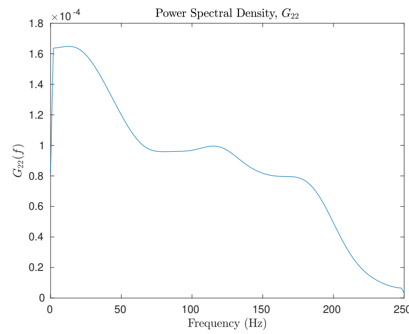
Code used to solve this problem is available at
https://github.com/hermanjakobsen/Random-Data/tree/master/HW8

4a)

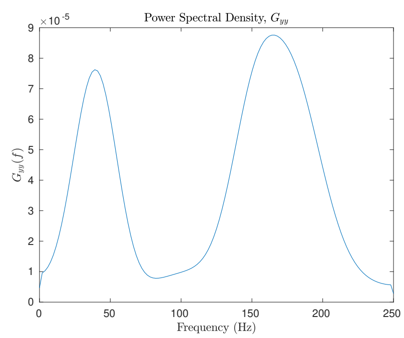
The power spectral densitiy functions $G_{11}(f)$, $G_{22}(f)$ and $G_{yy}(f)$ are shown in fig. 1a, fig. 1b and fig. 1c, respectively.



(a) $G_{11}(f)$



(b) $G_{22}(f)$

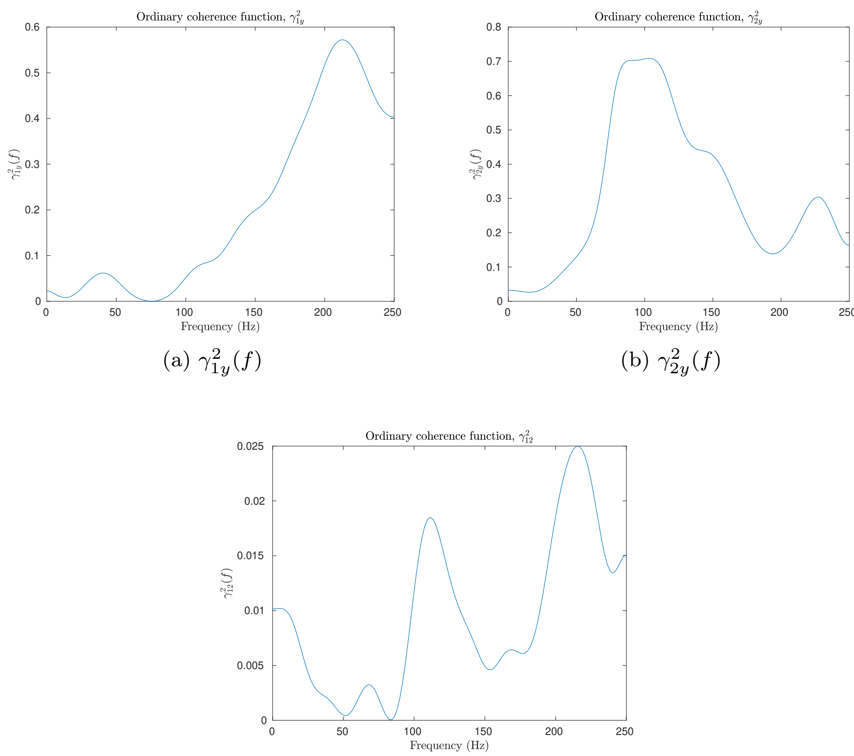


(c) $G_{yy}(f)$

Figure 1: Power spectral density functions

4b)

The ordinary coherence functions $\gamma^2_{1y}(f)$, $\gamma^2_{2y}(f)$ and $\gamma^2_{12}(f)$ are shown in fig. 2a, fig. 2b and fig. 2c, respectively.



(a) $\gamma^2_{1y}(f)$

(b) $\gamma^2_{2y}(f)$

(c) $\gamma^2_{12}(f)$

Figure 2: Ordinary coherence functions

4c)

The multiple coherence function $\gamma^2_{y:x}(f)$ is shown in fig. 3.

4d)

The frequency response functions $H_{1y}(f)$, $H_{2y}(f)$, $H_{1z}(f)$ and $H_{2z}(f)$ are shown in fig. 4a, fig. 4b, fig. 4c and fig. 4d, respectively.

4e)

The constant-parameter linear system functions $L_{1y}(f)$ and $L_{12}(f)$ are plotted in fig. 5a and fig. 5b, respectively.

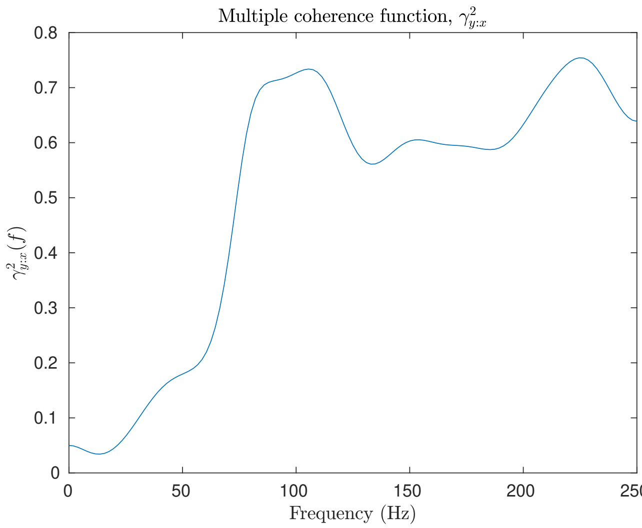


Figure 3: Multiple coherence function

4f)

The conditioned spectral density function $G22 \cdot 1(f)$, $G2y \cdot 1(f)$ and $Gyy \cdot 1(f)$ are plotted in fig. 6a, fig. 6b and fig. 6c, respectively.

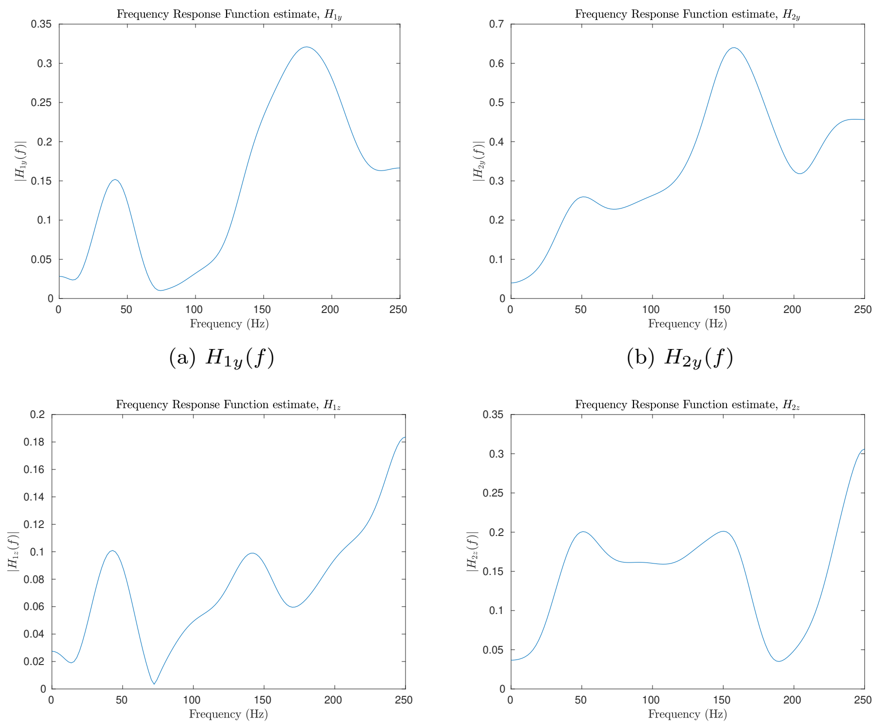
4g)

The partial coherence functions $\gamma^2_{2y\cdot 1}(f)$ and $\gamma^2_{1y\cdot 2}(f)$ are plotted in fig. 7a and fig. 7b, respectively.

4h)

The contributions of x_1 and x_2, as energy sources, to the output PSD $G_{yy}(f)$ can be found by comparing fig. 6c and fig. 1c. In fig. 6c, the plot can be interpreted as the output spectrum with y as output that is not affected by x_1. In other words, the noise output spectrum for a single-input/single-output model with x_1 as the input and y as the output. That being said, the main difference between the two plots are the peak that occurs at $100 \sim 200$ Hz. It is larger in magnitude in fig. 1c, which implies that x_1 contributes with significant energy in the mentioned frequency range.

In fig. 6b, the cross-spectral density between x_2 and y, when the contributions from x_1 are removed, is shown. The cross-spectral density is relatively high for frequencies up to 150 Hz and negative afterwards. It can therefore be



(a) $H_{1y}(f)$ (b) $H_{2y}(f)$

(c) $H_{1z}(f)$ (d) $H_{2z}(f)$

Figure 4: Frequency response functions

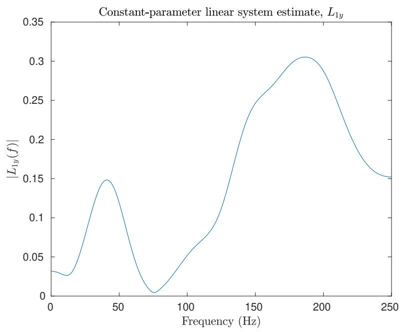
concluded that x_2 will contribute with energy for frequencies up to 150 Hz. For the higher frequencies, the input will drain energy from the output.

The remaining energy that is not caused by x_1 nor x_2 can be a consequence of noise.

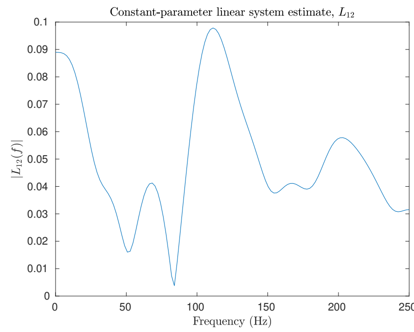
4i)

Comparing fig. 7a and fig. 7b, we can see that the spectrum of the total output record y is heavily influenced by x_2 for the middle frequencies and heavily influenced by x_1 for the higher frequencies. This is in compliance with the observations in the previous sub-task. The input x_2 stands for 70% of the output spectrum at the middle frequencies, while the input x_1 stands for 50% of the output spectrum at the higher frequencies.

Looking at fig. 1b and fig. 6a, it can be concluded that the spectrums are almost identical. Hence, the autospectrum of x_2 is barely influenced by x_1. This can be a sign that the two inputs are uncorrelated. The suspicion of uncorrelation between x_1 and x_2 is further strengthened by observing fig. 5b. The constant-parameter linear system represents how much of x_2 that is due to



(a) $L_{1y}(f)$

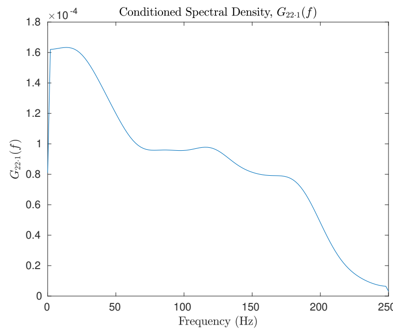


(b) $L_{12}(f)$

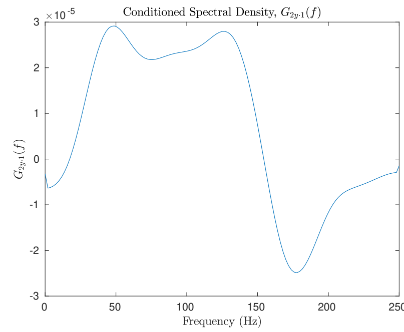
Figure 5: Constant-parameter linear system functions

x_1. Not much, in this case. Another evidence is to compare fig. 7a and fig. 7b with fig. 2a and fig. 2b. The coherence functions are almost identical to each other, which implies that the inputs are uncorrelated. Lastly, the coherence function in fig. 2c is small. Based on this, it can be concluded that the inputs are uncorrelated. Hence, $\gamma^2_{y:x}(f) = \gamma^2_{1y}(f) + \gamma^2_{2y}(f)$ which can be verified by looking at the plots.

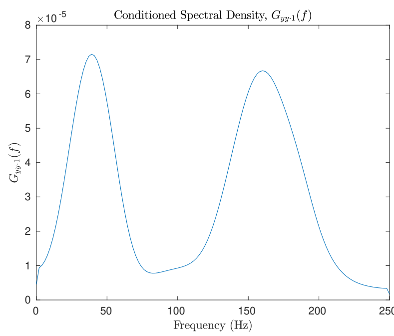
The frequency response functions in fig. 4 have a generally higher magnitude for output y compared to output z, so the inputs have generally a bigger impact for output y. Also, the frequency response functions regarding input x_2 are generally larger than the responses regarding x_1.



(a) $G_{22\cdot 1}(f)$

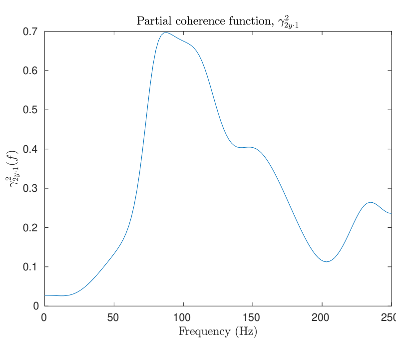


(b) $G_{2y\cdot 1}(f)$

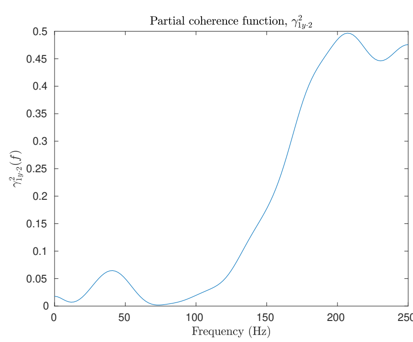


(c) $G_{yy\cdot 1}(f)$

Figure 6: Conditioned spectral density functions



(a) $\gamma^2_{2y\cdot 1}(f)$



(b) $\gamma^2_{1y\cdot 2}(f)$

Figure 7: Partial coherence functions