ME453 - Homework 4

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Problem 7

The inertia tensor is generally given by

$$\mathcal{I} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

The mass distribution of the body is symmetric with respect to the body attached frame. Hence, the cross products of inertia are zero. That is

$$\mathcal{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

The principal moments of inertia are calculated by first finding the inertia of the whole "cube", and then subtracting the inertia of the empty spaces next to the middle pole. Due to symmetry, the two empty spaces will contribute with equal inertia.

For the principal moment of inertia about the x-axis, we have

I_{xx} = Inertia whole cube − 2 · Inertia empty space

$$= \int_{-c-\frac{b}{2}}^{c+\frac{b}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (y^2+z^2)\rho \, dx\,dy\,dz \; - 2 \cdot \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{b}{2}}^{-\frac{c}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (y^2+z^2)\rho \, dx\,dy\,dz$$

Using MATLAB to compute the integrals yields

$$\underline{\underline{I_{xx} = \rho \, \frac{abc}{4} \left(3b^2 + 4bc + 3c^2\right)}}$$

About the y-axis, we have

$$I_{yy} = \int_{-c-\frac{b}{2}}^{c+\frac{b}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2+z^2)\rho \, dx\,dy\,dz \; - 2 \cdot \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{b}{2}}^{-\frac{c}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2+z^2)\rho \, dx\,dy\,dz$$

$$\underline{\underline{I_{yy} = \rho \, \frac{abc}{12} \left(3a^2 + 7b^2 + 12bc + 8c^2\right)}}$$

About the z-axis, we have

$$I_{zz} = \int_{-c-\frac{b}{2}}^{c+\frac{b}{2}} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2+y^2)\rho \, dx\,dy\,dz \; - 2 \cdot \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{b}{2}}^{-\frac{c}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2+y^2)\rho \, dx\,dy\,dz$$

$$\underline{\underline{I_{zz} = \rho \, \frac{abc}{12} \left(3a^2 + 2b^2 + c^2\right)}}$$

Problem 2a)

The choice of C is, for instance, not unique when the matrix contains elements that have terms containing multiplications of derivatives of q_i. These elements can be placed several places in C, depending on which derivative of q_i that is extracted.

An example with $q \in \mathbb{R}^2$ is as follows:

$$C(q,\dot{q})\dot{q} = \begin{bmatrix} m_1 q_1 \dot{q}_2 & m_2 q_2 \dot{q}_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} m_1 q_1 \dot{q}_2 \dot{q}_1 + m_2 q_2 \dot{q}_1 \dot{q}_2 \\ 1 \end{bmatrix}}_{A}$$

The matrix A can also be obtained by defining

$$C(q,\dot{q})\dot{q} = \begin{bmatrix} m_2 q_2 \dot{q}_2 & m_1 q_1 \dot{q}_1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} m_2 q_2 \dot{q}_2 \dot{q}_1 + m_1 q_1 \dot{q}_1 \dot{q}_2 \\ 1 \end{bmatrix} = A$$

For this example, there exists two C matrices that yield the same result. Hence, the choice of the matrix C is not unique.

This phenomenon can be extended into matrices of higher dimension.

For instance, for $q \in \mathbb{R}^3$ we have

$$C\dot{q} = \begin{bmatrix} m_1 \dot{q}_3 & 0 & 0 \\ 0 & m_2 \dot{q}_1 & 0 \\ 0 & 0 & m_3 \dot{q}_2 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} m_1 \dot{q}_3 \dot{q}_1 \\ m_2 \dot{q}_1 \dot{q}_2 \\ m_3 \dot{q}_2 \dot{q}_3 \end{bmatrix}$$

By instead defining

$$C = \begin{bmatrix} 0 & 0 & m_1 \dot{q}_1 \\ m_2 \dot{q}_2 & 0 & 0 \\ 0 & m_3 \dot{q}_3 & 0 \end{bmatrix}$$

we would have achieved the same result. If the elements had terms with multiplications of $\dot{q}_1$, $\dot{q}_2$ and $\dot{q}_3$, we would have even more choices of C that would give the same result.

2b)

Want to show that

$$\dot{D} - 2C = N$$

where N is skew-symmetric, meaning that

$$N + N^T = 0 \iff -n_{ij} = n_{ji}$$

First off, the time derivative of an element in the "mass matrix" is

$$\frac{d}{dt}(d_{ij}) \overset{\substack{\text{chain}\\ \text{rule}}}{=} \sum_{k=1}^{n} \frac{\partial d_{ij}}{\partial q_k}\frac{dq_k}{dt} = \sum_{k=1}^{n} \frac{\partial d_{ij}}{\partial q_k}\dot{q}_k$$

Then

$$\begin{aligned}
n_{ij} &= \dot{d}_{ij} - 2c_{ij} \\
&= \sum_{k=1}^{n} \frac{\partial d_{ij}}{\partial q_k}\dot{q}_k - 2\left\{\sum_{k=1}^{n} c_{ijk}\,\dot{q}_k\right\} \\
&= \sum_{k=1}^{n} \left\{\frac{\partial d_{ij}}{\partial q_k} - 2c_{ijk}\right\}\dot{q}_k \\
&= \sum_{k=1}^{n} \left\{\cancel{\frac{\partial d_{ij}}{\partial q_k}} - \left(\cancel{\frac{\partial d_{ij}}{\partial q_k}} + \frac{\partial d_{ik}}{\partial q_j} - \frac{\partial d_{jk}}{\partial q_i}\right)\right\}\dot{q}_k \\
&= \sum_{k=1}^{n} \left\{\frac{\partial d_{ik}}{\partial q_j} - \frac{\partial d_{jk}}{\partial q_i}\right\}\dot{q}_k
\end{aligned}$$

Ultimately

$$n_{ji} = \sum_{k=1}^{n} \left\{\frac{\partial d_{jk}}{\partial q_i} - \frac{\partial d_{ik}}{\partial q_j}\right\}\dot{q}_k = -\sum_{k=1}^{n} \left\{\frac{\partial d_{ik}}{\partial q_j} - \frac{\partial d_{jk}}{\partial q_i}\right\}\dot{q}_k = -n_{ij}$$

$\blacksquare$

2c)

The kinetic energy of a manipulator is given by

$$K = \frac{1}{2}\dot{q}^T D(q)\dot{q}$$

Its time derivate is

$$\frac{dK}{dt} = \frac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \frac{1}{2}\ddot{q}^T D(q)\dot{q} + \frac{1}{2}\dot{q}^T D(q)\ddot{q}$$

$$= \frac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \dot{q}^T D(q)\ddot{q}$$

The principle of conservation of energy yields

$$\frac{dK}{dt} = \dot{q}^T(\tau - G(q))$$

Hence

$$\frac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \dot{q}^T D(q)\ddot{q} = \dot{q}^T(\tau - G(q))$$

From the equations of motion we have the following for any C

$$\tau = D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q)$$

So

$$\frac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} + \cancel{\dot{q}^T D(q)\ddot{q}} = \dot{q}^T\left(\cancel{D(q)\ddot{q}} + C(q,\dot{q})\dot{q} - \cancel{G(q)} + \cancel{G(q)}\right)$$

$$\frac{1}{2}\dot{q}^T \dot{D}(q)\dot{q} = \dot{q}^T C(q,\dot{q})\dot{q}$$

Ultimately

$$\dot{q}^T \dot{D}(q)\dot{q} - 2\dot{q}^T C(q,\dot{q})\dot{q} = 0$$

$$\dot{q}^T\left(\dot{D}(q) - 2C(q,\dot{q})\right)\dot{q} = 0$$

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Problem 3a)

Define x and θ as generalized coordinates.

Will then find the energy of the system. The total energy will consist of kinetic and potential energy for both the cart and bar.

Denote h as the height from the center of mass to the ground.

Then, the potential energies are

$$V_c = m_c g h_c = m_c g \cdot 0 = 0$$

$$V_b = m_b g h_b = m_b g \cdot L_{com} \cos\theta$$

The cart has no rotational energy, so its kinetic energy is only translational energy. Hence

$$K_c = \frac{1}{2} m_c v_c^T v_c$$

where

$$p_c = \begin{bmatrix} x \\ 0 \end{bmatrix} \quad \text{and} \quad v_c = \dot{p}_c = \begin{bmatrix} \dot{x} \\ 0 \end{bmatrix}$$

so

$$K_c = \frac{1}{2} m_c \dot{x}^2$$

The position and translational velocity for the bar is

$$p_b = \begin{bmatrix} x - L_{com} \sin\theta \\ L_{com} \cos\theta \end{bmatrix}$$

$$v_b = \dot{p}_b = \begin{bmatrix} \dot{x} - \dot{\theta} L_{com} \cos\theta \\ -\dot{\theta} L_{com} \sin\theta \end{bmatrix}$$

The angular velocity of the bar is simply

$$\omega_b = \begin{bmatrix} 0 \\ 0 \\ \dot\theta \end{bmatrix}$$

The kinetic energy of the bar is then

$$K_b = \frac{1}{2} m_b V_b^T V_b + \frac{1}{2} \omega_b^T I_b \omega_b$$

$$= \frac{1}{2} m_b \left[(\dot x - \dot\theta L_{com} \cos\theta)^2 + (\dot\theta L_{com} \sin\theta)^2 \right] + \frac{1}{2} \begin{bmatrix} 0 & 0 & \dot\theta \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_b \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot\theta \end{bmatrix}$$

$$= \frac{1}{2} m_b \left(\dot x^2 - 2\dot x \dot\theta L_{com} \cos\theta + \dot\theta^2 L_{com}^2 \cos^2\theta + \dot\theta^2 L_{com}^2 \sin^2\theta \right) + \frac{1}{2} \dot\theta^2 I_b$$

$$= \frac{1}{2} m_b \left(\dot x^2 - 2\dot x \dot\theta L_{com} \cos\theta + \dot\theta^2 L_{com}^2 \right) + \frac{1}{2} I_b \dot\theta^2.$$

The total energy of the system is then

$$V = V_c + V_b = m_b g L_{com} \cos\theta$$

$$K = K_c + K_b = \frac{1}{2} m_c \dot x^2 + \frac{1}{2} m_b \left(\dot x^2 - 2\dot x \dot\theta L_{com} \cos\theta + \dot\theta^2 L_{com}^2 \right) + \frac{1}{2} I_b \dot\theta^2$$

Now, the Lagrangian is defined as

$$L = K - V = \frac{1}{2} m_c \dot x^2 + \frac{1}{2} m_b \left(\dot x^2 - 2\dot x \dot\theta L_{com} \cos\theta + \dot\theta^2 L_{com}^2 \right) + \frac{1}{2} I_b \dot\theta^2 - m_b g L_{com} \cos\theta$$

Since F_c imposes a change in x and τ_b imposes a change in θ, these forces can be regarded as actuator forces.

The equations of motion are then given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i} \right) - \frac{\partial L}{\partial q_i} = \tau_i$$

So

$$\frac{\partial L}{\partial \dot{x}} = m_c \dot{x} + \frac{1}{2} m_b \left(2\dot{x} - 2\dot{\theta} L_{com} \cos\theta \right)$$

$$= (m_c + m_b)\dot{x} - m_b L_{com} \cos\theta \, \dot{\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (m_c + m_b)\ddot{x} - \left(m_b L_{com} \cos\theta \, \ddot{\theta} - m_b L_{com} \sin\theta \, \dot{\theta}^2 \right)$$

$$= (m_c + m_b)\ddot{x} - m_b L_{com} \cos\theta \, \ddot{\theta} + m_b L_{com} \sin\theta \, \dot{\theta}^2$$

$$\frac{\partial L}{\partial x} = 0$$

and

$$\frac{\partial L}{\partial \dot{\theta}} = \frac{1}{2} m_b \left(-2\dot{x} L_{com} \cos\theta + 2\dot{\theta} L_{com}^2 \right) + I_b \dot{\theta}$$

$$= m_b L_{com}^2 \dot{\theta} + I_b \dot{\theta} - m_b L_{com} \cos\theta \, \dot{x}$$

$$= (m_b L_{com}^2 + I_b)\dot{\theta} - m_b L_{com} \cos\theta \, \dot{x}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = (m_b L_{com}^2 + I_b)\ddot{\theta} - \left(m_b L_{com} \cos\theta \, \ddot{x} - m_b L_{com} \sin\theta \, \dot{\theta}\dot{x} \right)$$

$$= (m_b L_{com}^2 + I_b)\ddot{\theta} - m_b L_{com} \cos\theta \, \ddot{x} + m_b L_{com} \sin\theta \, \dot{\theta}\dot{x}$$

$$\frac{\partial L}{\partial \theta} = m_b L_{com} \sin\theta \, \dot{\theta}\dot{x} + m_b g L_{com} \sin\theta$$

The equations of motion are then

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = F_c$$

$$\Rightarrow \quad \underline{(m_c + m_b)\ddot{x} - m_b L_{com} \cos\theta \, \ddot{\theta} + m_b L_{com} \sin\theta \, \dot{\theta}^2 = F_c}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = \tau_b$$

$$\Rightarrow \quad \underline{(m_b L_{com}^2 + I_b)\ddot{\theta} - m_b L_{com} \cos\theta \, \ddot{x} - m_b g L_{com} \sin\theta = \tau_c}$$

The "mass matrix" $D(q)$ and the gravitational vector $G(q)$ can be obtained by looking at the EOMs. Thus

$$D(q) = \begin{bmatrix} m_c + m_b & -m_b L_{com} \cos\theta \\ -m_b L_{com} \cos\theta & m_b L_{com}^2 + I_b \end{bmatrix}$$

$$G(q) = \begin{bmatrix} 0 \\ -m_b g L_{com} \sin\theta \end{bmatrix}$$

The coriolis/centripental matrix $C(q, \dot{q})$ can in this case also be obtained by looking at the EOMs. However, the Christoffel symbols

$$c_{ijk} = \frac{1}{2}\left(\frac{\partial d_{kj}}{\partial q_i} + \frac{\partial d_{ki}}{\partial q_j} - \frac{\partial d_{ij}}{\partial q_k} \right)$$

will be used to obtain the matrix. So

$$c_{111} = \frac{1}{2}\left(\frac{\partial d_{11}}{\partial x} + \frac{\partial d_{11}}{\partial x} - \frac{\partial d_{11}}{\partial x} \right) = 0$$

$$c_{112} = \frac{1}{2}\left(\frac{\partial d_{21}}{\partial x} + \frac{\partial d_{21}}{\partial x} - \frac{\partial d_{11}}{\partial \theta} \right) = 0$$

$$c_{121} = \frac{1}{2}\left(\frac{\partial d_{12}}{\partial x} + \frac{\partial d_{11}}{\partial \theta} - \frac{\partial d_{12}}{\partial x} \right) = 0$$

$$c_{122} = \frac{1}{2}\left(\frac{\partial d_{22}}{\partial x} + \frac{\partial d_{21}}{\partial \theta} - \frac{\partial d_{12}}{\partial \theta} \right) = \frac{1}{2}\left(m_b L_{com} \sin\theta - m_b L_{con} \sin\theta \right) = 0$$

$$c_{211} = \frac{1}{2}\left(\frac{\partial d_{11}}{\partial \theta} + \frac{\partial d_{12}}{\partial x} - \frac{\partial d_{21}}{\partial x} \right) = 0$$

$$c_{212} = \frac{1}{2}\left(\frac{\partial d_{21}}{\partial \theta} + \frac{\partial d_{22}}{\partial x} - \frac{\partial d_{21}}{\partial \theta} \right) = \frac{1}{2}\left(m_b L_{com} \sin\theta - m_b L_{com} \sin\theta \right) = 0$$

$$c_{221} = \frac{1}{2}\left(\frac{\partial d_{12}}{\partial \theta} + \frac{\partial d_{12}}{\partial \theta} - \frac{\partial_{22}}{\partial x} \right) = \frac{1}{2}\left(m_b L_{com} \sin\theta + m_b L_{com} \sin\theta \right) = m_b L_{com} \sin\theta$$

$$c_{222} = 0$$

The elements of the matrix is then given by

$$c_{kj} = \sum_{i=1}^{n=2} c_{ijk} \dot{q}_i$$

Hence

$$c_{11} = 0$$

$$c_{12} = m_b L_{com} \sin\theta \dot{\theta}$$

$$c_{21} = 0$$

$$c_{22} = 0$$

which gives the following matrix

$$C(q, \dot{q}) = \begin{bmatrix} 0 & m_b L_{com} \sin\theta \dot{\theta} \\ 0 & 0 \end{bmatrix}$$

The equations of motion can then be stated as

$$D(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau$$

$$\underbrace{\begin{bmatrix} m_c + m_b & -m_b L_{com} \cos\theta \\ -m_b L_{com} \cos\theta & m_b L_{com}^2 + I_b \end{bmatrix}}_{D(q)} \begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & m_b L_{com} \sin\theta \dot{\theta} \\ 0 & 0 \end{bmatrix}}_{C(q,\dot{q})} \begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -m_b g L_{com} \sin\theta \end{bmatrix}}_{G(q)} = \begin{bmatrix} F_c \\ \tau_c \end{bmatrix}$$

3b)

Want to find the regressor matrix $Y(q,\dot{q},\ddot{q})$ such that

$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = Y(q,\dot{q},\ddot{q})\,\phi$$

Must first define the elements in ϕ. Listing the elements in the EOM matrices yields

$$d_{11} = \underbrace{m_c + m_b}_{\phi_1}$$

$$d_{12} = d_{21} = -\underbrace{m_b L_{com}}_{\phi_2} \cos\theta$$

$$d_{22} = \underbrace{m_b L_{com}^2 + I_b}_{\phi_3}$$

$$d_{11} = \phi_1 \qquad d_{12} = d_{21} = -\phi_2 \cos\theta \qquad d_{22} = \phi_3$$

and

$$g_1 = 0$$

$$g_2 = -\phi_2\, g \sin\theta$$

$$c_{12} = \phi_2 \sin\theta\, \dot{\theta}$$

The EOMs can then be expressed as

$$\begin{bmatrix} \phi_1 & -\phi_2\cos\theta \\ -\phi_2\cos\theta & \phi_3 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} 0 & \phi_2 \sin\theta\,\dot{\theta} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ -\phi_2 g \sin\theta \end{bmatrix}$$

$$= \begin{bmatrix} \phi_1 \ddot{x} - \phi_2 \cos\theta\, \ddot{\theta} + \phi_2 \sin\theta\, \dot{\theta}^2 \\ -\phi_2 \cos\theta\, \ddot{x} + \phi_3 \ddot{\theta} - \phi_2 g \sin\theta \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \ddot{x} & \sin\theta\,\dot{\theta}^2 - \cos\theta\,\ddot{\theta} & 0 \\ 0 & -\cos\theta\,\ddot{x} - g\sin\theta & \ddot{\theta} \end{bmatrix}}_{Y(q,\dot{q},\ddot{q})} \underbrace{\begin{bmatrix} \phi_1 = m_c + m_b \\ \phi_2 = m_b L_{com} \\ \phi_3 = m_b L_{com}^2 + I_b \end{bmatrix}}_{\phi}$$