ME591 - Homework 3

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Problem 1

Note: The MATLAB code for this problem can be found in the *.zip* folder.

1a)

The histograms and estimated probability distributions for the three cases with $m = 1$, $m = 5$ and $m = 15$ are shown in fig. 1, fig. 2 and fig. 3, respectively.

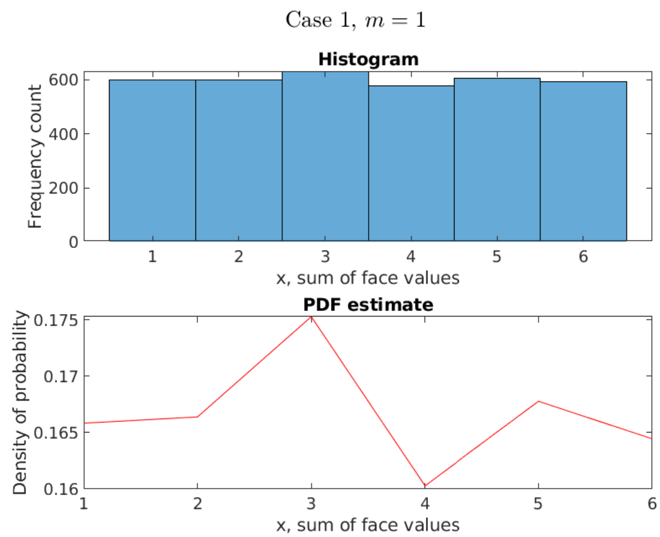

Figure 1: Histogram and estimated probability distribution for $m = 1$

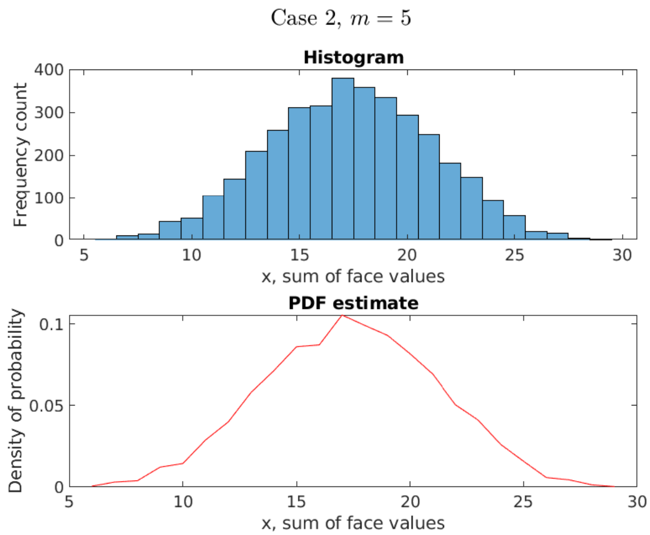

Figure 2: Histogram and estimated probability distribution for $m = 5$

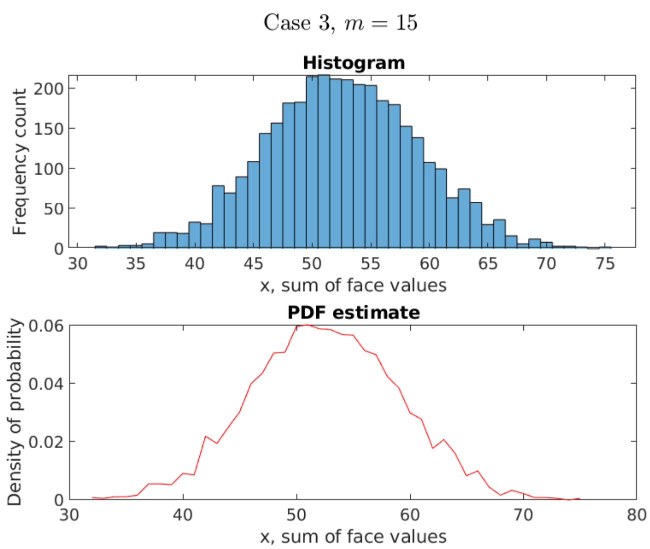

Figure 3: Histogram and estimated probability distribution for $m = 15$

1b)

The estimated errors for $m = 1$ and $m = 15$ are shown in fig. 4 and fig. 5, respectively.

Note: I did not know what error values and plots to expect. Hence, there is a good chance that the plots are incorrect. The MATLAB code generating these plots can be found in the *.zip* folder.

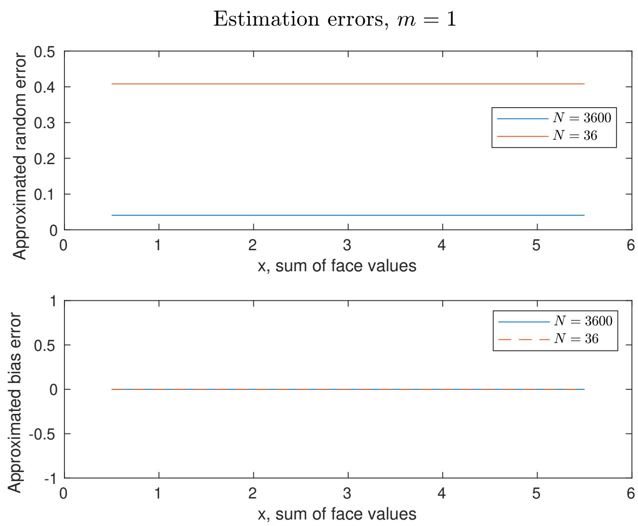

Figure 4: Estimation errors for $m = 1$

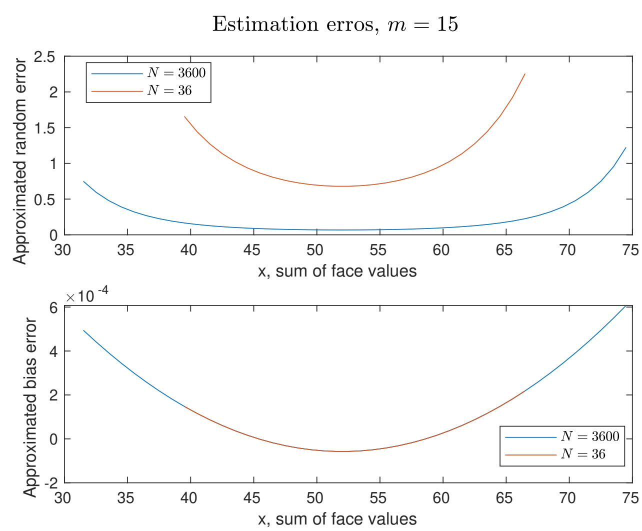

Figure 5: Estimation errors for $m = 15$

1c)

Looking at fig. 1, fig. 2 and fig. 3, the Central Limit Theorem is clearly shown. For both cases where $m = 5$ and $m = 15$, the estimated probability distribution tends towards a normal distribution. However, as the number of class intervals increases in fig. 3, the estimated probability density gets more rough. The probability density estimate in fig. 2 is smoother in comparison.

The histogram in fig. 1 shows almost an ideal uniform distribution, but small deviations are shown for especially $x = 3$ and $x = 4$. This is in compliance with the estimated errors in fig. 4. The bias error is zero, which means no error is expected. However, some random error leads to a deviation between the estimated and the ideal probability distribution. From fig. 4, it is also shown that a higher number of data samples results in a smaller random error.

As seen in fig. 5, a higher number of class intervals results in a higher random error. Further, the random error has its highest values at the end-values of x. Since the approximated bias error is independent of the number of samples, it is equal for both cases with $N = 36$ and $N = 3600$. Similar to the case with $m = 1$, the random error decreases as the number of data samples increases.

According to the slides given in lecture the bias error for $m = 15$ should be smaller than the bias error for $m = 1$, since we have more class intervals for $m = 15$. The reason for this not being true can be the fact that the probability distribution for $m = 1$ is uniform. The phenomenon of more class intervals resulting in smaller bias error could maybe be seen by comparing the errors of $m = 5$ and $m = 15$?

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Problem 2a)

Given that

$$y(t) = x(t-\tau_0)$$

The autocorrelation is then

$$R_{yy}(\tau) = E\left[y(t)\,y(t+\tau)\right]$$
$$= E\left[x(t-\tau_0)\;x(t-\tau_0+\tau)\right]$$

Defining

$$u = t-\tau_0$$

yields

$$R_{yy}(\tau) = E\left[x(u)\;x(u+\tau)\right]$$
$$= R_{xx}(\tau)$$

where $x(t)$ is defined as white noise. The autocorrelation function for $x(t)$ is

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(f)\, e^{j2\pi f\tau}\, df = \frac{G}{2}\int_{-\infty}^{\infty} e^{j2\pi f\tau}\, df = \frac{G}{2}\,\delta(\tau)$$

Hence

$$\underline{R_{yy}(\tau) = \frac{G}{2}\,\delta(\tau)}$$

The cross correlation of input and output is

$$R_{xy}(\tau) = E[x(t)\, y(t+\tau)]$$

$$= E[x(t)\, x(t-\tau_0+\tau)]$$

Defining

$$v = \tau - \tau_0$$

So

$$R_{xy}(v+\tau_0) = E[x(t)\; x(t+v)]$$

$$= R_{xx}(v)$$

Substituting back the variable yields

$$\underline{\underline{R_{xy}(\tau) = R_{xx}(\tau-\tau_0) = \frac{G}{2}\,\delta(\tau-\tau_0)}}$$

The cross correlation of output and input is

$$R_{yx}(\tau) = E[y(t)\; x(t+\tau)]$$

$$= E[x(t-\tau_0)\; x(t+\tau)]$$

Defining

$$u = t - \tau_0$$

Substitution gives

$$R_{yx}(\tau) = E[x(u)\; x(u+\tau+\tau_0)]$$

$$= R_{xx}(\tau+\tau_0)$$

Hence

$$\underline{\underline{R_{yx}(\tau) = \frac{G}{2}\,\delta(\tau+\tau_0)}}$$

2b)

The cross spectral density function of input and output is

$$S_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau) e^{-j2\pi f\tau} d\tau$$

$$= \frac{G}{2} \int_{-\infty}^{\infty} \delta(\tau-\tau_0) e^{-j2\pi f\tau} d\tau$$

$$= \frac{G}{2} \int_{-\infty}^{\infty} e^{j2\pi f(\tau-\tau_0)} e^{-j2\pi f\tau} d\tau$$

$$= \frac{G}{2} \int_{-\infty}^{\infty} e^{-j2\pi f\tau_0} d\tau$$

$$\underline{\underline{S_{xy}(f) = \frac{G}{2} e^{-j2\pi f\tau_0}}}$$

The auto spectral density function is

$$S_{yy}(f) = \int_{-\infty}^{\infty} R_{yy}(\tau) e^{-j2\pi f\tau} d\tau$$

$$= \frac{G}{2} \int_{-\infty}^{\infty} \delta(\tau) e^{-j2\pi f\tau} d\tau$$

$$\underline{\underline{S_{yy}(f) = S_{xx}(f) = \frac{G}{2}}}$$

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The cross spectral density function of output and input is

$$S_{yx}(f) = \int_{-\infty}^{\infty} R_{yx}(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$= \frac{G}{2} \int_{-\infty}^{\infty} \delta(\tau + \tau_0)\, e^{-j2\pi f\tau}\, d\tau$$

$$S_{yx}(f) = \frac{G}{2}\, e^{j2\pi f\tau_0}$$

2c)

Given the following condition

$$|S_{xy}(f)| = |S_{yx}(f)|$$

Inserting the spectral density functions yields

$$\left|\frac{G}{2} e^{-j2\pi f \tau_0}\right| = \left|\frac{G}{2} e^{j2\pi f \tau_0}\right|$$

$$\Rightarrow \quad \left|\frac{G}{2}\right| \left|e^{-j2\pi f \tau_0}\right| = \left|\frac{G}{2}\right| \left|e^{j2\pi f \tau_0}\right|$$

$$\frac{G}{2} = \frac{G}{2}$$

The condition is satisfied independent of the frequency.

Hence, the condition is fulfilled in the following frequency region

$$\underline{\underline{-\infty < f < \infty}}$$

2d)

Given the following condition

$$S_{xy}(f) = S_{yx}(f)$$

Inserting the spectral density functions yields

$$\frac{G}{2} e^{-j2\pi f \tau_0} = \frac{G}{2} e^{j2\pi f \tau_0}$$

$$-f = f$$

The condition is only satisfied when

$$\underline{\underline{f = 0}}$$

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Problem 3a)

From the figure, the autospectral density is defined as

$$S_{xx}(f) = \begin{cases} \frac{1}{B^2} f + \frac{1}{B}, & -B \le f \le 0 \\ -\frac{1}{B^2} f + \frac{1}{B}, & 0 \le f \le B \end{cases}$$

The mean square is then given by

$$R_{xx}(0) = \int_{-\infty}^{\infty} S_{xx}(f)\, e^{j2\pi f \cdot 0}\, df$$

$$= \int_{-\infty}^{\infty} S_{xx}(f)\, df$$

$$= \int_{-B}^{0} \frac{1}{B^2} f + \frac{1}{B}\, df + \int_{0}^{B} \left(-\frac{1}{B^2} f + \frac{1}{B}\right) df$$

$$= \frac{1}{2B^2}\left[f^2\right]_{-B}^{0} + \frac{1}{B}\left[f\right]_{-B}^{0} - \frac{1}{2B^2}\left[f^2\right]_{0}^{B} + \frac{1}{B}\left[f\right]_{0}^{B}$$

$$= \frac{1}{2B^2} \cdot (-B^2) + \frac{1}{B} \cdot B - \frac{1}{2B^2} \cdot B^2 + \frac{1}{B} \cdot B$$

$$= -\frac{1}{2} + 1 - \frac{1}{2} + 1$$

$$\underline{\underline{\Psi_x^2 = R_{xx}(0) = 1}}$$

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3b)

From the previous task, the auto-spectral density function is defined as

$$S_{xx}(f) = \begin{cases} \frac{f}{B^2} + \frac{1}{B} , & -B \leq f \leq 0 \\ -\frac{f}{B^2} + \frac{1}{B} , & 0 \leq f \leq B \end{cases}$$

The autocorrelation function is given by

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} S_{xx}(f)\, e^{j2\pi f \tau}\, df$$

$$= \int_{-B}^{0} \left(\frac{f}{B^2} + \frac{1}{B}\right) e^{j2\pi f\tau}\, df + \int_{0}^{B} \left(-\frac{f}{B^2} + \frac{1}{B}\right) e^{j2\pi f\tau}\, d\tau$$

$$= \frac{1}{B^2}\int_{-B}^{0} f e^{j2\pi f\tau}\, df + \frac{1}{B}\int_{-B}^{0} e^{j2\pi f\tau}\, df - \frac{1}{B^2}\int_{0}^{B} f e^{j2\pi f\tau}\, d\tau + \frac{1}{B}\int_{0}^{B} e^{j2\pi f\tau}\, d\tau$$

where

$$\int_{a}^{b} f e^{j2\pi f\tau}\, df = \left[f \cdot \frac{1}{j2\pi\tau}\, e^{j2\pi f\tau} \right]_{a}^{b} - \frac{1}{j2\pi\tau}\int_{a}^{b} e^{j2\pi f\tau}\, df$$

$$= \frac{1}{j2\pi\tau}\left[f\, e^{j2\pi f\tau}\right]_{a}^{b} + \frac{1}{(2\pi\tau)^2}\left[e^{j2\pi f\tau}\right]_{a}^{b}$$

So

$$R_{xx}(\tau) = \frac{1}{B^2}\left(\frac{1}{j2\pi\tau}\left[f e^{j2\pi f\tau}\right]_{-B}^{0} + \frac{1}{(2\pi\tau)^2}\left[e^{j2\pi f\tau}\right]_{-B}^{0}\right)$$

$$+ \frac{1}{B}\cdot\frac{1}{j2\pi\tau}\left[e^{j2\pi f\tau}\right]_{-B}^{0}$$

$$- \frac{1}{B^2}\left(\frac{1}{j2\pi\tau}\left[f e^{j2\pi f\tau}\right]_{0}^{B} + \frac{1}{(2\pi\tau)^2}\left[e^{j2\pi f\tau}\right]_{0}^{B}\right)$$

$$+ \frac{1}{B}\cdot\frac{1}{j2\pi\tau}\left[e^{j2\pi f\tau}\right]_{0}^{B}$$

$$= \frac{1}{B^2}\left[\frac{1}{j2\pi\tau}\left(B e^{-j2\pi B\tau}\right) + \frac{1}{(2\pi\tau)^2}\left(1 - e^{-j2\pi B\tau}\right)\right]$$

$$+ \frac{1}{j2\pi B\tau}\left(1 - e^{-j2\pi B\tau}\right)$$

$$- \frac{1}{B^2}\left[\frac{1}{j2\pi\tau}\left(B e^{j2\pi B\tau}\right) + \frac{1}{(2\pi\tau)^2}\left(e^{j2\pi B\tau} - 1\right)\right]$$

$$+ \frac{1}{j2\pi B\tau}\left(e^{j2\pi B\tau} - 1\right)$$

$$= \cancel{\frac{1}{j2\pi B\tau} e^{-j2\pi B\tau}} + \frac{1}{(2\pi B\tau)^2} - \frac{1}{(2\pi B\tau)^2} e^{-j2\pi B\tau}$$

$$+ \cancel{\frac{1}{j2\pi B\tau}} - \cancel{\frac{1}{j2\pi B\tau} e^{-j2\pi B\tau}}$$

$$- \frac{1}{j2\pi B\tau} e^{j2\pi B\tau} - \frac{1}{(2\pi B\tau)^2} e^{j2\pi B\tau} + \frac{1}{(2\pi B\tau)^2}$$

$$+ \frac{1}{j2\pi B\tau} e^{j2\pi B\tau} - \frac{1}{j2\pi B\tau}$$

$$= \frac{2}{(2\pi B\tau)^2} - \frac{1}{(2\pi B\tau)^2}\left(e^{j2\pi B\tau} + e^{-j2\pi B\tau}\right)$$

Ultimately

$$R_{xx}(\tau) = \frac{1}{2(\pi B\tau)^2} - \frac{1}{2(\pi B\tau)^2}\cos(2\pi B\tau)$$

$$R_{xx}(\tau) = \frac{1}{2(\pi B\tau)^2}\left(1 - \cos(2\pi B\tau)\right)$$

The rate of change is given by the derivative. Hence

$$\frac{d}{d\tau}R_{xx}(\tau) = \frac{1}{2(\pi B)^2}\frac{d}{dt}\left[\tau^{-2}\left(1-\cos(2\pi B\tau)\right)\right]$$

$$= \frac{1}{2(\pi B)^2}\left[-2\tau^{-3}\left(1-\cos(2\pi B\tau)\right) + \tau^{-2}\,2\pi B\sin(2\pi B\tau)\right]$$

$$= \frac{1}{\pi B\tau^2}\sin(2\pi B\tau) - \frac{1}{(\pi B)^2\tau^3}\left(1-\cos(2\pi B\tau)\right)$$

For $B=C$:

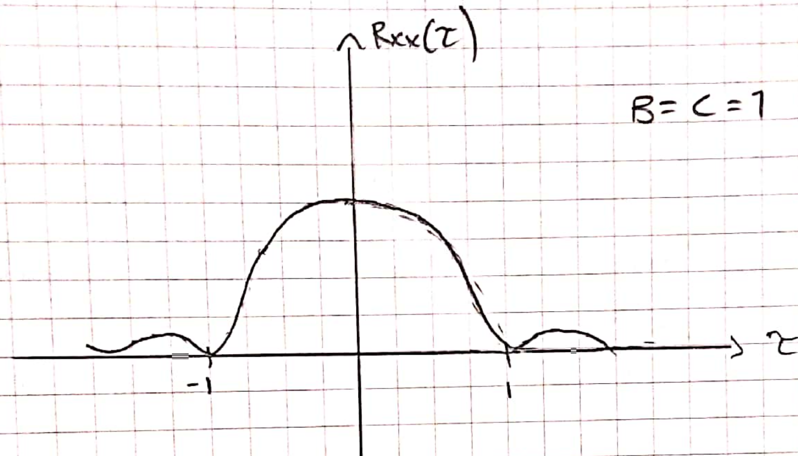
$$\frac{d}{d\tau}R_{xx}(\tau; B=C) = \frac{1}{\pi C\tau^2}\sin(2\pi C\tau) - \frac{1}{(\pi C)^2\tau^3}\left(1-\cos(2\pi C\tau)\right)$$

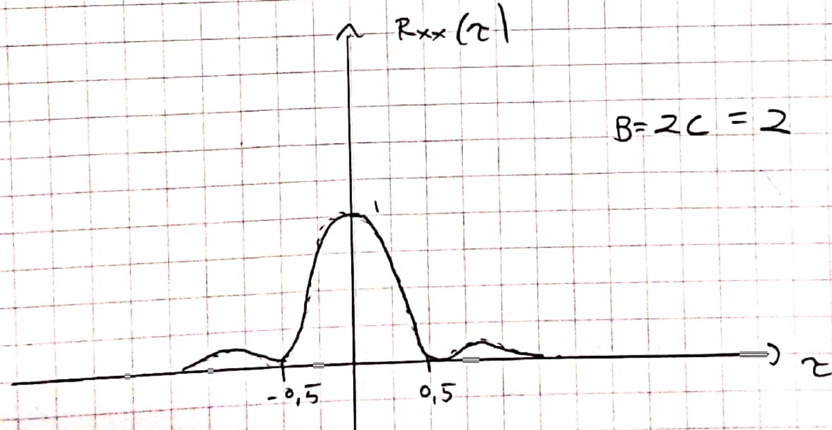
For $B=2C$:

$$\frac{d}{d\tau}R_{xx}(\tau; B=2C) = \frac{1}{2\pi C\tau^2}\sin(4\pi C\tau) - \frac{1}{(2\pi C)^2\tau^3}\left(1-\cos(4\pi C\tau)\right)$$

The derivatives give no obvious concluding answer regarding the decay.

However, setting $B = 1$, and sketching the graphs yield





It can be seen that the autocorrelation decays faster with $B = 2C$!

Other values for B were also tested out, and the same observations were made.

3c)

The autocorrelation function is

$$R_{xx}(\tau) = \frac{1}{2(\pi B \tau)^2}\left(1 - \cos(2\pi B \tau)\right)$$

So

$$\lim_{B \to 0} R_{xx}(\tau) = \lim_{B \to 0} \frac{1}{2(\pi B \tau)^2}\left(1 - \cos(2\pi B \tau)\right)$$

$$\overset{\text{L'Hopital}}{=} \lim_{B \to 0} \frac{1}{4(\pi \tau)^2 B} \cdot 2\pi\tau \sin(2\pi B \tau) = \lim_{B \to 0} \frac{\sin(2\pi B \tau)}{2\pi \tau B} = \lim_{u \to 0} \frac{\sin(u)}{u}$$

This limit is a known result, which can be obtained by using the squeeze theorem. Hence

$$\lim_{B \to 0} R_{xx}(\tau) = 1$$

Further

$$\lim_{B \to \infty} R_{xx}(\tau) = \lim_{B \to \infty} \frac{1}{2(\pi B \tau)^2}\left(1 - \cos(2\pi B \tau)\right)$$

$$\overset{\text{L'Hopital}}{=} \lim_{B \to \infty} \frac{\sin(2\pi B \tau)}{2\pi \tau B}$$

Using the squeeze theorem, we obtain

$$\lim_{B \to \infty} R_{xx}(\tau) = 0$$