ME591 - Assignment 7

Herman Kolstad Jakobsen
20196493

November 22, 2019

Code

The code used for completing this assignment is available at
https://github.com/hermanjakobsen/Random-Data/tree/master/HW7

Problem 1a)

The instantaneous estimation of $\gamma^2_{xy}(f)$ is plotted in fig. 1. The instantaneous estimation of $\gamma^2_{xy}(f)$ is unity all over the frequency range due to poor estimation of the spectral density functions needed to calculate $\gamma^2_{xy}(f)$. The three spectral densities $S_{xx}(f)$, $S_{yy}(f)$ and $S_{xy}(f)$ are random variables. Hence, to get a more reliable estimation, each has to be averaged for multiple windows.

An analogy will be to look at the experiment to tell if a coin is fair in a toss. One single flip of the coin does not give enough information. The toss has to be repeated until you get a sufficient probability estimation of the coin's values. The same applies to the estimation of $\gamma^2_{xy}(f)$. A single measurement says nothing. The measurement needs to be repeated in order to get an estimation of the random variables used to calculate $\gamma^2_{xy}(f)$.

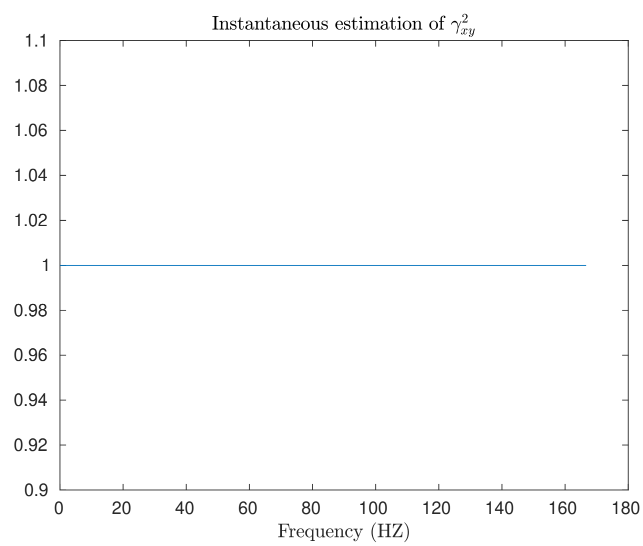


Figure 1: Instantaneous estimation of $\gamma^2_{xy}(f)$

1b)

The ensemble averaged estimation $\gamma^2_{xy}(f)$ was obtained using the MATLAB function *mscohere* with default parameters. The result is plotted in fig. 2. The sudden drops in $\gamma_{xy}^2(f)$ for frequencies lower than 100 Hz is the result of unlinearities in the system relating $x(t)$ and $y(t)$. The drops are clearly shown at approximately 5, 10, 55 and 65 Hz. These unlinearities might occur due to the resonance frequencies of the system. The reason for the estimation not being unity for the other frequencies under 100 Hz might be due to computation errors, and the fact that the coherence function is an estimate.

From 100 Hz and upwards, the coherence estimate has a general drop in its values. At higher frequencies, it is fair to assume that the rotor will impose some kind of low-pass filtering characteristics which will cause the autospectrum to decay sharply. However, the noise floor for data acquisition and recording equipment will generally not decay with increasing frequencies. It is therefore probable that the diminishing coherence at the higher frequencies is a result of extraneous measurement noise.

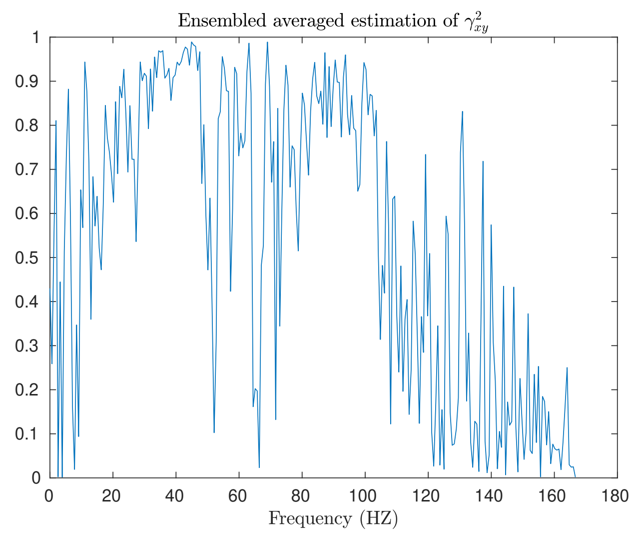
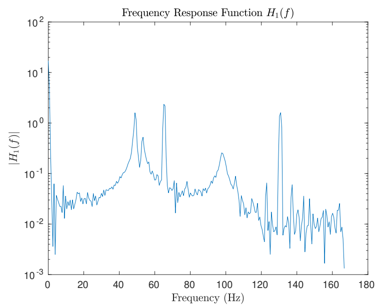


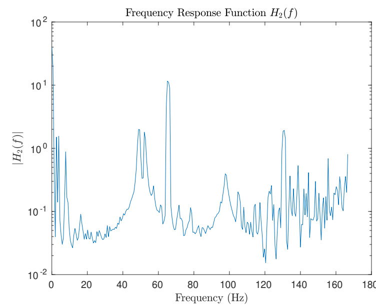
Figure 2: Ensemble averaged estimation of $\gamma_{xy}^2(f)$

Problem 2a)

The conventional Frequency Response Function (FRF) estimation $H_1(f)$ is shown in fig. 3a, while the inverse FRF estimation $H_2(f)$ is shown in fig. 3b.



(a) Conventional FRF estimation

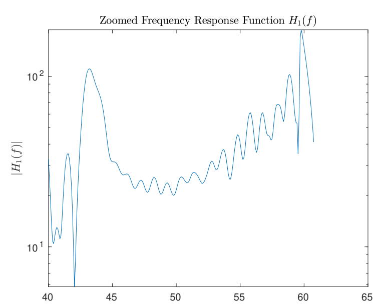


(b) Inverse FRF estimation

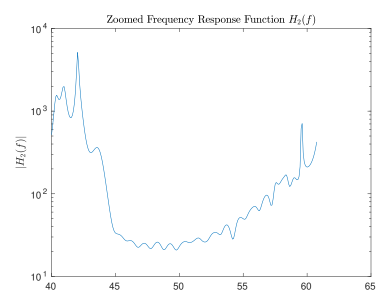
Figure 3: FRF estimation

2b)

The DSP System Toolbox from MATLAB was used to zoom the plots of the FRF estimations. The zoomed plots for $H_1(f)$ and $H_2(f)$ are showed in fig. 4a and fig. 4b, respectively. It was unsure what to expect from these plots, so there is a good possibility that they are wrong and that the DSP System Toolbox has been used incorrectly. The polar plots for $H_1(f)$ and $H_2(f)$ are showed in fig. 5a and fig. 5b, respectively.

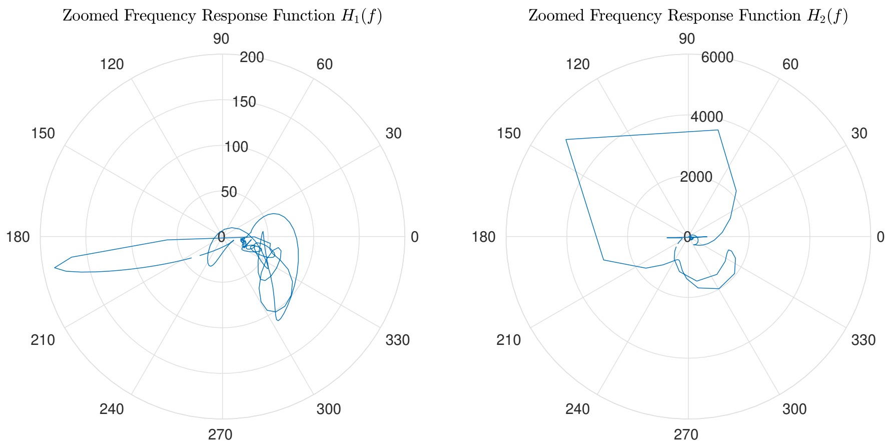


(a) Zoomed conventional FRF estimation



(b) Zoomed inverse FRF estimation

Figure 4: Zoomed FRF estimation



(a) Zoomed conventional FRF estimation (b) Zoomed inverse FRF estimation

Figure 5: Polar plot of zoomed FRF estimation

2c)

From fig. 3 it can be concluded that the system has several resonance frequencies. Since $H_2(f)$ gives a better estimate around resonance frequencies, it is beneficial to look at fig. 3b when deciding these frequencies. Hence, the system has resonance frequencies at approximately 5, 10, 45, 65, 100 and 130 Hz.

The polar plots of the zoomed FRF in fig. 5 are quite different from each other. I do not know what to expect from these types of plots, so it may be a good chance that they are not correct. However, fig. 5b has a much higher magnitude compared to fig. 5a. Since $H_2(f)$ is a better estimation around resonance frequencies, and gives such a high magnitude, it can be concluded that the system has a resonance frequency in the frequency range $40 \sim 60$ Hz. The resonance peak also imposes a significant phase shift in $H_2(f)$. The conventional estimation $H_1(f)$ has a generally low phase shift and magnitude, except one spike whcich goes to approximately 180°. This could be a sign of a resonance frequency.

The higher frequency region is an anti-resonance region and $H_1(f)$ will therefore give a better estimation. Looking at fig. 3a the magnitude is generally low for this region, disregarding the one peak. Hence, it can be said that the system works as a low-pass filter for higher frequencies. Moreover, both the estimations in fig. 3 are noisy for the this region, which can be a sign of extraneous measurement noise.

Problem 3a)

By definition we have

$$H_1(f) = \frac{S_{xy}(f)}{S_{xx}(f)} \quad \text{and} \quad H_2(f) = \frac{S_{yy}(f)}{S_{yx}(f)} \tag{1}$$

So

$$\frac{H_1}{H_2}(f) = \frac{S_{xy}/S_{xx}}{S_{yy}/S_{yx}}(f) = \frac{S_{xy}S_{yx}}{S_{xx}S_{yy}}(f) \tag{2}$$

where it is known that

$$S_{yx}(f) = S^*_{xy}(f) \tag{3}$$

Hence

$$\frac{H_1}{H_2}(f) = \frac{S_{xy}S^*_{xy}}{S_{xx}S_{yy}}(f) = \frac{|S_{xy}|^2}{S_{xx}S_{yy}} = \gamma^2_{xy}(f) \tag{4}$$

3b)

Given the system in fig. 6, we can define $m(t)$ as extraneous input measurement noise, $n(t)$ as extraneous output measurement noise and $u(t)$ and $v(t)$ as true input and output signals respectively. The conventional FRF estimation $H_1(f)$

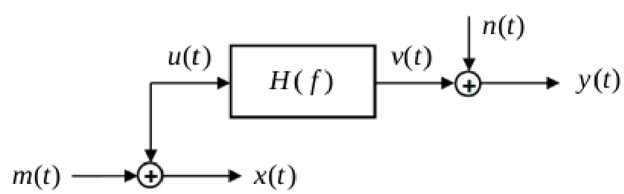


Figure 6: Single-input/single-output system with extraneous noise

assumes no input measurement noise and can then be expressed as

$$H_1(f) = \frac{S_{xy}(f)}{S_{xx}(f)} = \frac{S_{uv}}{S_{uu}\left(1 + \frac{S_{mm}(f)}{S_{uu}(f)}\right)} = H(f)\left(\frac{1}{1+\alpha(f)}\right) \tag{5}$$

where

$$\alpha(f) = \frac{S_{mm}(f)}{S_{uu}(f)} \tag{6}$$

First off, the conventional FRF estimation gives unbiased phase estimates. The conventional estimation, however, will always underestimate the true FRF magnitude which can be seen from eq. (5). This is especially the case when the noise to signal ratio $\alpha(f)$ is large. Therefore, the estimation $H_1(f)$ gives a good estimation of the FRF near anti-resonance regions, where $\alpha(f)$ is small. The estimation works good when only the output is affected by noise.

3c)

Given the definitions from fig. 6, the inverse FRF estimation $H_2(f)$ assumes no output measurement noise and can be expressed as

$$H_2(f) = \frac{S_{yy}(f)}{S_{yx}(f)} = \frac{S_{vv}(f)\left(1 + \frac{S_{nn}(f)}{s_{vv}(f)}\right)}{S_{vu}(f)} = H(f)(1 + \beta(f)) \tag{7}$$

where

$$\beta(f) = \frac{S_{nn}(f)}{S_{vv}(f)} \tag{8}$$

Similar to $H_1(f)$, the estimation $H_2(f)$ also gives unbiased phase estimates. From eq. (7) it can be seen that the inverse estimate always overestimate the true FRF magnitude, especially when the noise to signal ratio $\beta(f)$ is large. The estimate $H_2(f)$ will normally give a good estimation of the FRF near resonance regions. The estimation works good when only the input is affected by noise.

Comparing the two estimates $H_1(f)$ and $H_2(f)$, the conventional FRF estimation $H_1(f)$ is usually more used in practice. This is because the input signal is often controllable and the noise to signal ratio $\alpha(f)$ can therefore be reduced. The output noise, however, includes not only the measurement noise, but also the unknown system nonlinearity, ultimately making $\beta(f)$ uncontrollable.

3d)

The coherence function is normally less than 1 due to the following reasons:

- Extraneous noise is present in the measurements.
- The measurement $y(t)$ is an output due to an input $x(t)$ as well to other inputs.
- The system relating $x(t)$ and $y(t)$ is not linear.
- Bias and random errors are present in the spectral estimates.

3e)

If the system is linear with constant system parameters, the measurements are free of noise and the spectral estimations are free of computational erros, we have

$$\gamma_{xy}^2(f) = \frac{|S_{xy}|^2}{S_{xx}S_{yy}} = \frac{|H(f)|^2 S_{xx}^2(f)}{S_{xx}(f)|H(f)|^2 S_{xx}(f)} = 1 \tag{9}$$

If $x(t)$ and $y(t)$ are completely unrelated, the coherence function will be zero.

3f)

The estimations $H_1(f)$ and $H_2(f)$ are better than the estimation $\hat{H}(f)$ since $H_1(f)$ and $H_2(f)$ account for output noise and input noise respectively. The estimation $\hat{H}(f)$ is defined only as the ratio between the Fourier transform of the output divided by the Fourier transform of the input, which makes it prone to measurement noise. Measurement noise in the output, input or both input and output will result in a poor estimation by $\hat{H}(f)$.

Problem 4a)

Will assume that there is no measurement noise, since it was not stated in the task. Hence

$$H_1(f) = \frac{G_{xy}(f)}{G_{xx}(f)} = H(f)\left(\frac{1}{1+\alpha(f)}\right) = H(f), \quad \alpha(f) = 0 \tag{10}$$

which means that the conventional FRF estimation is equal to the true FRF. Inserting the spectral density functions yields

$$H(f) = \frac{4/f^3 - j\left(4/f^2\right)}{2f^2} = \frac{2}{f^5} - j\frac{2}{f^4} \tag{11}$$

The gain factor is given by

$$\begin{aligned}
|H(f)| &= \sqrt{\mathrm{Re}\{H(f)\}^2 + \mathrm{Im}\{H(f)\}^2} \\
&= \sqrt{{\frac{2}{f^5}}^{2} + {\frac{2}{f^4}}^{2}} \\
&= \sqrt{\frac{4}{f^{10}} + \frac{4}{f^8}} \\
&= \underline{\underline{2 \cdot \sqrt{\frac{1}{f^{10}} + \frac{1}{f^8}}}}
\end{aligned} \tag{12}$$

4b)

The phase factor is given by

$$\begin{aligned}
\theta &= \mathrm{atan2}(\mathrm{Im}\{H(f)\}, \mathrm{Re}\{H(f)\}) \\
&= \underline{\underline{\mathrm{atan2}\left(-\frac{4}{f^2}, \frac{4}{f^3}\right)}}
\end{aligned} \tag{13}$$

4c)

A time-delay is equivalent to a phase shift. Hence, the time-delay through the system is

$$T_d = \theta = \underline{\underline{\mathrm{atan2}\left(-\frac{4}{f^2}, \frac{4}{f^3}\right)}} \tag{14}$$