Homework 10

CAAM 378
Rice University

Due Friday, November 9 at 5pm

For pledged problems you may not consult any resources other than the instructor, your notes, the text, the TA and the graders. For unpledged problems you may consult these resources plus students currently enrolled in CAAM 378.

1. (Unpledged) Consider a cutting stock problem with a raw of length 80 inches and the finals and demands described in this table:

Final Length	Demand
52 inches	6
29 inches	8
27 inches	10
21 inches	12

   Start with the patterns that only cut a single final from each raw. Use the column generation approach to satisfy the demand with the fewest raws. Solve the restricted master problem and knapsacks using matlab.

2. (Pledged) Consider a very long roll of wallpaper that repeats its pattern every yard. Four sheets of wallpaper must be cut from the roll. With reference to the beginning of the wallpaper (point 0), the beginning and end of each sheet are located as shown in the following table:

Sheet	Beginning (Yards)	End (Yards)
1	0.3	0.7
2	0.4	0.8
3	0.2	0.5
4	0.7	0.9

   Formulate a model to determine the order in which the sheets should be cut to minimize the total amount of wasted paper. Assume that a final cut is made to bring the roll back to the beginning of the pattern.

3. (Unpledged) Let $G = (V, E)$, (where $|V| = n$) be a complete graph on which you wish to solve the TSP. Consider the following heuristic for the TSP:

$$\begin{aligned} \min \quad & \sum_{(i,j)\in E} c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j=1}^{n} x_{ij} = 1 \text{ for all } i \in \{1, \dots, n\} \\ & \sum_{i=1}^{n} x_{ij} = 1 \text{ for all } j \in \{1, \dots, n\} \\ & x_{ij} \in \{0, 1\} \text{ for all } (i, j) \in \{1, \dots, n\}^2. \end{aligned}$$

Assume $n = 6$. Provide an example of a graph with edge costs c such that the above model has a unique optimal solution that contains exactly two subtours of length 3. Then, provide another example such that the above model has a unique optimal solution that is a Hamiltonian cycle–that is, a cycle that visits every node exactly once. The following link has more information about Hamiltonian cycles:

`http://mathworld.wolfram.com/HamiltonianCycle.html`

4. (Unpledged) Recall the following traveling salesman problem (TSP) subtour elimination constraints:

$$\sum_{i,j\in S} x_{ij} \le |S| - 1 \text{ for all } S \subset N.$$

Calculate precisely how many such constraints there are for a 5-city TSP and a 12-city TSP. Assume a complete graph. You may use a computer for these calculations.

5. (Unpledged, bonus question)

   (a) Let $a \in \mathbb{R}^n$ have non-negative entries, and let $b \in \mathbb{R}$ be strictly positive. Let $\mathcal{C} = \{C_1, \dots, C_L\}$ denote the set of all minimal covers of the knapsack inequality $a^{\mathrm{T}} x \le b$. Prove that

$$\left\{x \in \{0,1\}^n \mid a^{\mathrm{T}} x \le b\right\} = \left\{x \in \{0,1\}^n \;\middle|\; \sum_{i\in C_\ell} x_i \le |C_\ell| - 1 \text{ for all } \ell = 1, \dots, L\right\}.$$

   That is, prove that a binary vector satisfies the knapsack inequality if and only if it satisfies all the minimal cover inequalities for that knapsack inequality.

   (b) Let $a_1, a_2 \in \mathbb{R}^n$ have non-negative entries, and let $b_1, b_2 \in \mathbb{R}$ be strictly positive. For $i = 1, 2$, let

$$S_i = \left\{x \in \{0,1\}^n \mid a_i^{\mathrm{T}} x \le b_i\right\}$$

(note that S_i contains only binary vectors). Prove that $S_1 = S_2$ if and only if the inequalities $a_1^{\mathrm{T}} x \leq b_1$ and $a_2^{\mathrm{T}} x \leq b_2$ have the same minimal cover inequalities.

These results provide a rigorous, general proof for the first part of question 1 on the previous homework.