TRILINOS Tutorial

Heat equation:

$$\rho c_p \frac{\partial T}{\partial t} - \nabla \cdot (k\nabla T) - \dot{Q} = 0, \qquad (x,y) \in \Omega$$

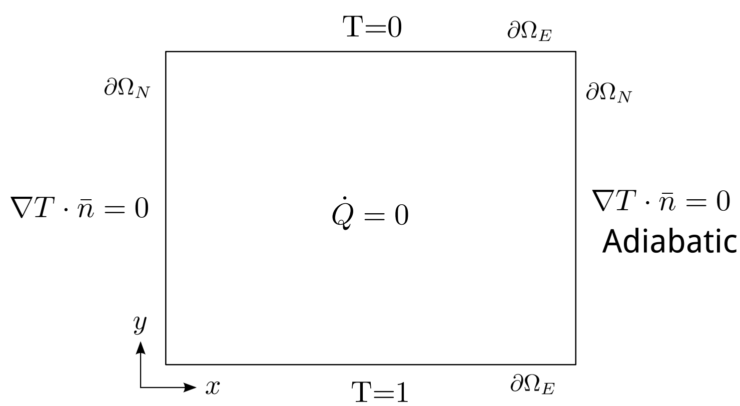
Essential boundary conditions

$$T(x,y) = \alpha(x,y) \qquad (x,y) \in \partial\Omega_E$$

Natural boundary conditions

$$k(x,y)\frac{\partial T}{\partial \bar{n}} = k\nabla T \cdot \bar{n} = \beta(x,y) \qquad (x,y) \in \partial\Omega_N$$

Simpler problem for the sake of illustration:
steady state head conduction in a rectangle, no sources or sinks



$$\nabla \cdot (k\nabla T) = 0$$

$$T(x,0) = 1, T(x,1) = 0$$

Galerkin: Multiply by test function v and integrate over Ω.

$$\iint_\Omega [\nabla \cdot (k\nabla T)]v \; dxdy = 0$$

Product rule for ∇:

$$\nabla \cdot (vk\nabla T) = (\nabla v) \cdot (k\nabla T) + v\nabla \cdot (k\nabla T)$$

$$v\nabla \cdot (k\nabla T) = \nabla \cdot (vk\nabla T) - (\nabla v) \cdot (k\nabla T)$$

$$\iint_\Omega [\nabla \cdot (k\nabla T)]v \; dxdy = \iint_\Omega \nabla \cdot (vk\nabla T) \; dxdy - \iint_\Omega (\nabla v) \cdot (k\nabla T) dxdy$$

Divergence theorem:

$$\iint_{\Omega} \nabla \cdot a \; dxdy = \int_{\partial\Omega} a \cdot \bar{n} dS$$

So

$$\iint_{\Omega} \nabla \cdot (vk\nabla T) \; dxdy = \int_{\partial\Omega} vk\nabla T \cdot \bar{n} \; dS$$

We now have

$$\iint_{\Omega} \nabla v \cdot k\nabla T \; dxdy - \int_{\partial\Omega} vk\nabla T \cdot \bar{n} \; dS = 0$$

$$\nabla T \cdot \bar{n} = 0 \text{ on } \partial\Omega_N$$

Finite Element Problem

$$\text{Let } T(x,y) = u(x,y) \approx U(x,y) = \sum_{j=1}^{N} c_j \phi_j(x,y)$$

$$v(x,y) \approx V(x,y) = \sum_{j=1}^{N} d_j \phi_j(x,y)$$

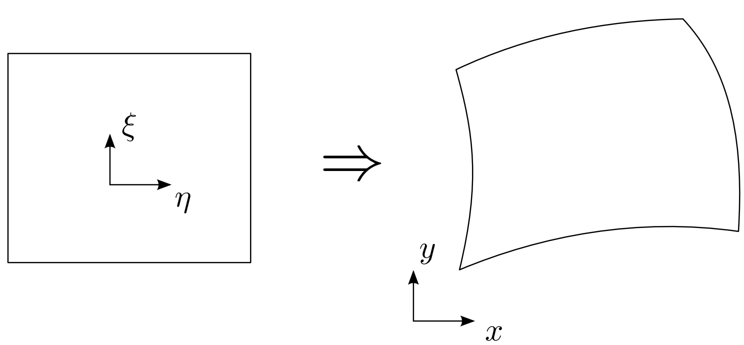
where ϕ_j are FE basis functions.

$$\iint \nabla v \cdot k\nabla T \; dxdy = \sum_{E} \iint_{\Omega_e} \nabla V \cdot k\nabla U \; dxdy = 0$$

$$\nabla V \cdot \nabla U = v_x u_x \cdot v_y u_y$$

$$\iint_{\Omega_e} [k(V_x U_x + V_y U_y)] \; dxdy$$

Canonical Element Transformation



$$\text{Let } U_0 = U(x(\xi,\eta), y(\xi,\eta))$$

$$V_0 = V(x(\xi,\eta), y(\xi,\eta))$$

$$J = \begin{bmatrix} x_\xi & x_\eta \\ y_\xi & y_\eta \end{bmatrix} \qquad det J = x_\xi y_\eta - x_\eta y_\xi$$

$$V_x = \frac{\partial V_0}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial V_0}{\partial \eta}\frac{\partial \eta}{\partial x} = V_{0_\eta}\eta_x + V_{0_\eta}\eta_x$$

$$\iint_{\Omega_0} [k(V_{0_\xi}\xi_x + V_{0_\eta}\eta_x)(U_{0_\xi}\xi_x + U_{0_\eta}\eta_x) + k(V_{0_\xi}\xi_y + V_{0_\eta}\eta_y)(U_{0_\xi}\xi_y + U_{0_\eta}\eta_y)] \det J_e \; d\xi d\eta$$

$$\iint_{\Omega_0} [k(\xi_x^2 + \xi_y^2)V_{0_\xi}U_{0_\xi} + k(\xi_x\eta_x + \xi_y\eta_y)(V_{0_\xi}U_{0_\eta} + V_{0_\eta}U_{0_\xi}) + k(\eta_x^2 + \eta_y^2)V_{0_\eta}U_{0_\eta}] \det J_e d\xi d\eta$$

$$\text{Let } U_0(\xi,\eta) = c_e^T N(\xi,\eta) = N^T(\xi,\eta)c_e$$
$$V_0(\xi,\eta) = d_e^T N(\xi,\eta) = N^T(\xi,\eta)d_e$$

$$\begin{aligned} = \iint_{\Omega_0} [\quad & k \quad (\xi_x^2 + \xi_y^2) d_e^T N_\xi N_\xi^T c_e + \\ & k \quad (\xi_x\eta_x + \xi_y\eta_y)[d_e^T N_\xi N_\eta^T c_e + d_e^T N_\eta N_\xi^T c_e] + \\ & k \quad (\eta_x^2 + \eta_y^2) d_e^T N_\eta N_\eta^T c_e] \det J_e \; d\xi d\eta = d_e^T K_e c_e \end{aligned}$$

$$K_e = \iint_{\Omega_0} [k(\xi_x^2 + \xi_y^2)N_\xi N_\xi^T + k(\xi_x\eta_x + \xi_y\eta_y)[N_\xi N_\eta^T + N_\eta N_\xi^T] + k(\eta_x^2 + \eta_y^2)N_\eta N_\eta^T] \det J_e \; d\xi d\eta$$

$$\sum_E d_e^T K_e c_e = 0$$
$$KC = 0$$