# Adaptive $hp$-FEM for Multiphysics Problems

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## Multiphysics Problems

Multiphysics problems belong to the most complicated problems that can be described by partial differential equations (PDEs). Typically they are nonlinear, time-dependent, and various physical fields interact with each other on vastly different spatial and/or temporal scales. Automatic adaptivity (mesh refinement in problematic regions) is absolutely necessary to capture important small-scale phenomena with sufficient accuracy. Yet, most engineering models are solved without automatic adaptivity and sufficient accuracy control today. In part, this is due to the fact that analytical error estimates are available for *simple* problems solved by *simple* methods only. In particular, error estimates working for a sufficiently wide scale of multiphysics problems and numerical methods are missing. In addition, many important models are being developed continuously. We believe that for many important problems, analytical error estimates will never exist. Therefore, we develop *computational error estimates* which are not sensitive to the problem type – they work equally well for single PDEs and PDE systems, linear and nonlinear problems, and stationary as well as time-dependent ones.

## Computational Techniques

Our computational methodology is based on extremely efficient (exponentially convergent) higher-order finite element methods ($hp$-FEM) and it relies on the following four basic techniques that we have devised during the last several years:

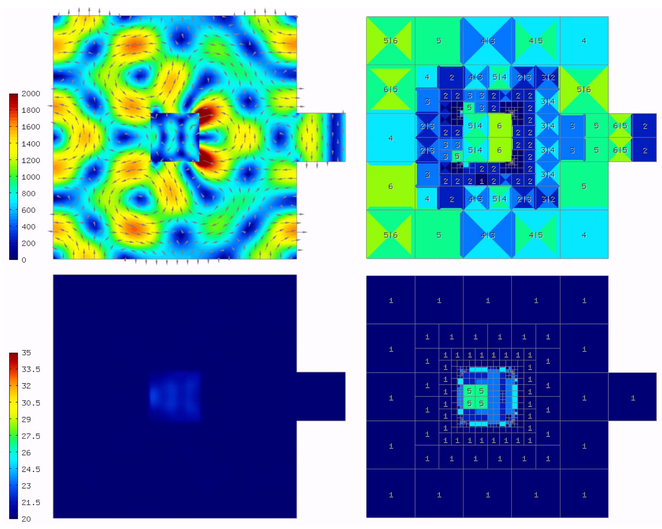
- Approximation with *arbitrary-level hanging nodes* that eliminates forced mesh refinements and allows to change scales of resolution very quickly.
- *Multimesh $hp$-FEM*. This technology allows us to approximate various components and physical fields in multiphysics problems on individual meshes equipped with mutually independent adaptivity mechanisms. The effect on the performance of the numerical method is tremendous.
- PDE-independent adaptivity algorithm based on *approximation pairs*. This idea is taken from embedded higher-order methods for ODEs: During the adaptivity proces, we maintain a pair of approximations with different orders of accuracy. The adaptivity is guided solely by discrepancies between global and local orthogonal projections.
- Space-time adaptivity based on the Rothe's method. The Rothe's method (less known sister of the Method of Lines) approximates a time-dependent PDE problem with one or more time-independent ones. We solve these time-independent problems using our universal spatial adaptivity algorithms. For every PDE, the adaptivity starts from a common coarse mesh and ends up with a locally refined mesh (different in every time step). Thus an effect of *simultaneous mesh refinement and coarsening* takes place.

## HERMES

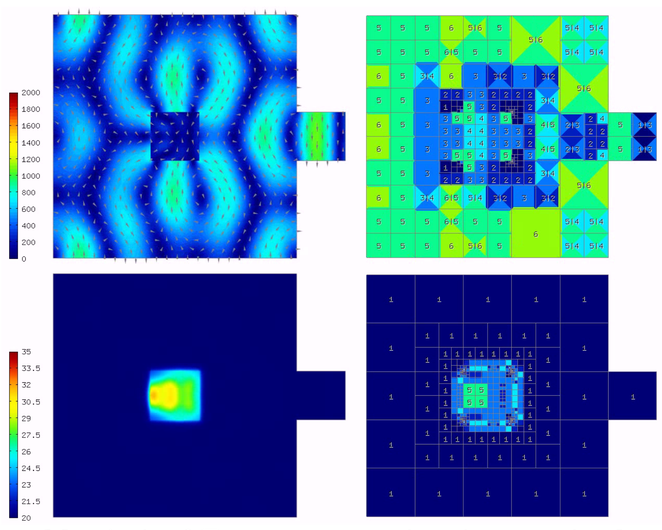
HERMES (Hierarchic Modular $hp$-FEM System) is a modular C++ library for finite element discretization for a wide range of engineering problems starting from simple linear PDEs to complex nonlinear time-dependent multiphysics coupled problems. The software is developed by the $hp$-FEM group at the University of Texas at El Paso (http://hpfem.math.utep.edu). In HERMES, all physical fields and solution components can be approximated on individual meshes equipped with mutually independent automatic adaptivity mechanisms. So far, one can use higher-order continuous elements (for $H^1$ problems) and higher-order edge elements (for $H$(curl) problems involving electric fields). We plan to implement higher-order $H$(div)-conforming elements soon. Various element types can be combined in one computation. Three-dimensional version of HERMES is being developed as well. The project is supported by the Grant Agency of the Czech Republic, Grant Agency of the Academy of Science of the Czech Republic, U.S. Department of Defense, and the NSF.

## Microwave Heating

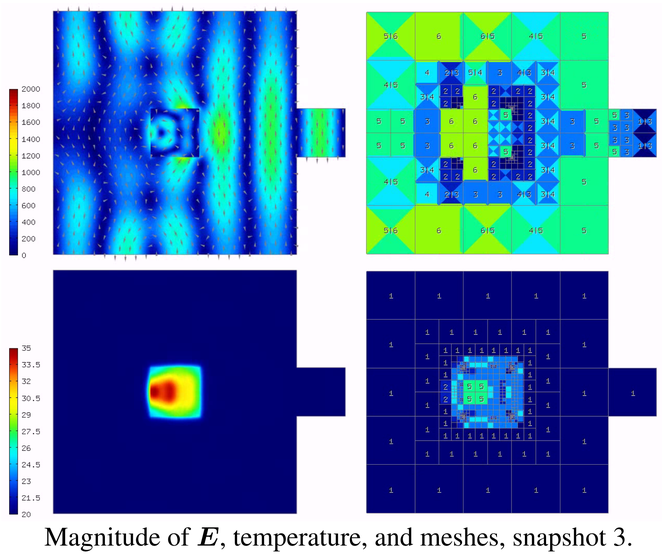
Electromagnetic waves are generated using a sinusoidal current in a small cavity (on the right). They enter the large cavity containing an object with temperature-dependent electric permittivity and thermal conductivity. The electric field deposits energy into the object. Its temperature rises, and the changing material properties influence the electric field.



Magnitude of $\boldsymbol{E}$, temperature, and meshes, snapshot 1.



Magnitude of $\boldsymbol{E}$, temperature, and meshes, snapshot 2.



Magnitude of $\boldsymbol{E}$, temperature, and meshes, snapshot 3.

## Flame Propagation

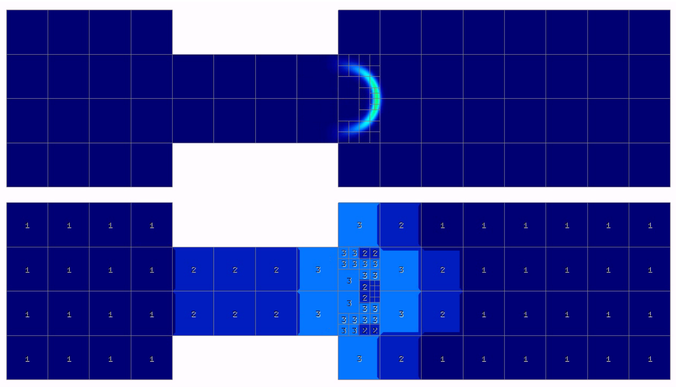
This flame propagation problem is described via a pair of mutually coupled nonlinear parabolic equations

$$\frac{\partial \theta}{\partial t} - \Delta\theta = \omega(\theta, Y),$$
$$\frac{\partial Y}{\partial t} - \frac{1}{\mathrm{Le}}\Delta Y = -\omega(\theta, Y).$$

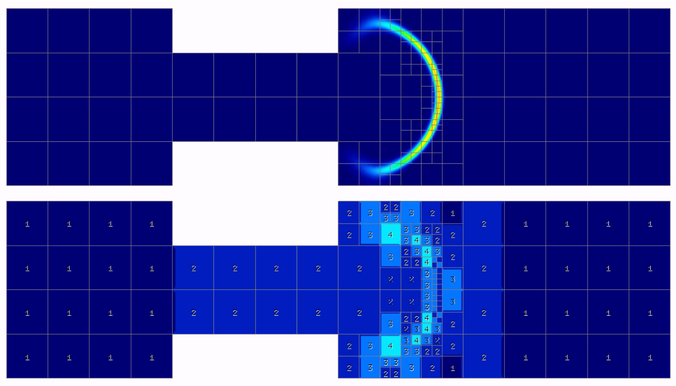
Here, $\theta \in [0,1]$ is the dimensionless temperature and $Y \in [0,1]$ the concentration. The reaction rate (flame intensity) $\omega$ is defined by the Arrhenius law

$$\omega(\theta, Y) = \frac{\beta^2}{2\mathrm{Le}} Y e^{\frac{\beta(\theta-1)}{1+\alpha(\theta-1)}}.$$

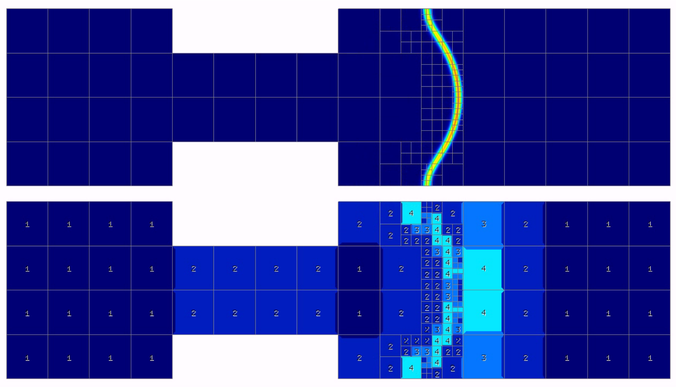
We use the Lewis number $\mathrm{Le} = 1$, gas expansion coefficient $\alpha = 0.8$, and the nondimensional activation energy $\beta = 10$. The following figures show snapshots of the flame intensity $\omega$ and the corresponding meshes.



Reaction rate and mesh, snapshot 1.



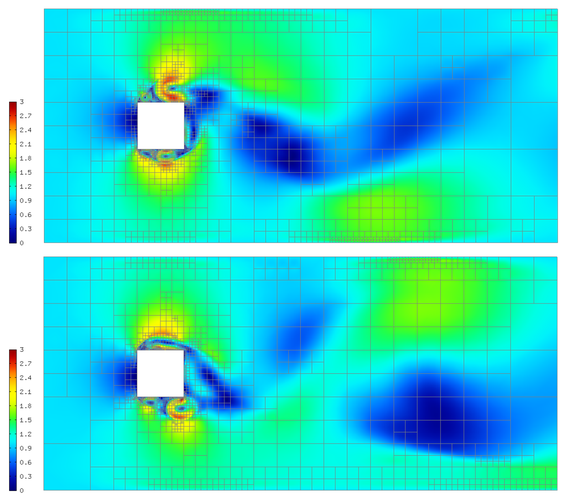
Reaction rate and mesh, snapshot 2.



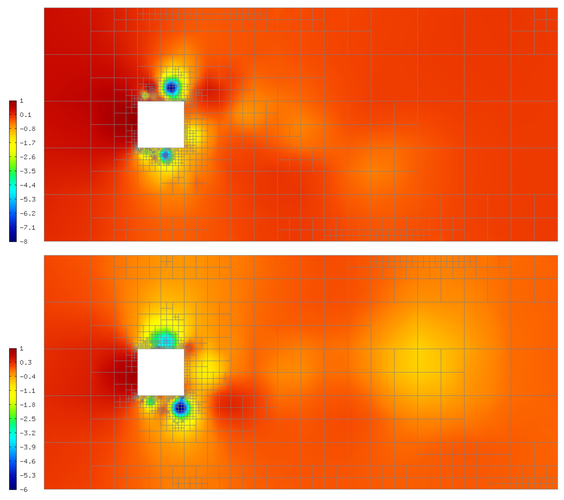
Reaction rate and mesh, snapshot 3.

## Thermally-Condurtive Flow

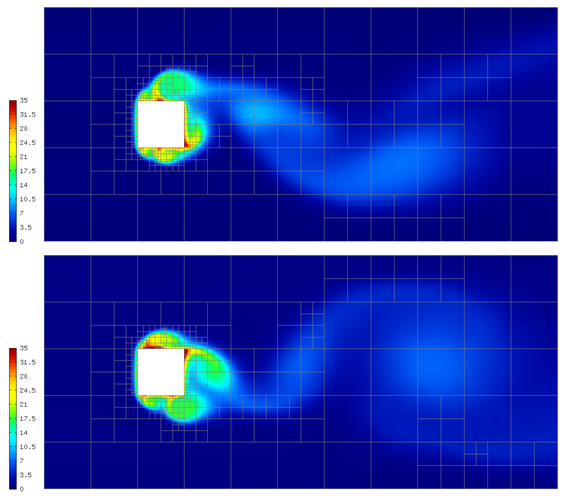
We use unsteady incompressible Navier-Stokes equations to model the flow of viscous flow past a heated obstacle. This flow does not have a steady solution and develops the so-called *Von Karman vortices*. The Reynolds number is Re = $10^5$. The velocity, pressure, and temperature are approximated on individual meshes that evolve in time independently of each other.



Flow velocity.



Flow pressure.



Flow temperature.