Seminar 3 - 20.10.2021

Ex. 1: Să se studieze injectivitatea, surjectivitatea și bijectivitatea funcției $f:\mathbb{R}\to\mathbb{R}$,

$$f(x)=\begin{cases} x^2+m, & x\le 0 \quad f_1\\ mx, & 0<x<1 \quad f_2\\ m^2-x, & x\ge 1 \quad f_3\end{cases}$$

în funcție de parametrul real m.

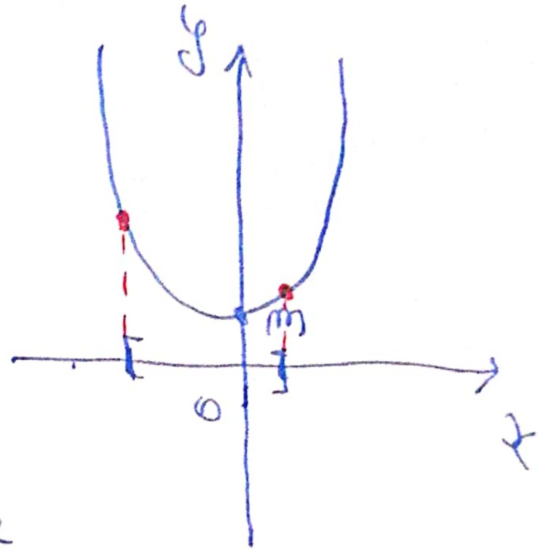
Rez:

$f_1:(-\infty,0]\to\mathbb{R},\ f_1(x)=x^2+m$

$$V\left(-\frac{b}{2a},-\frac{\Delta}{4a}\right)=V(0,m)$$

$\operatorname{Im} f_1=[m,\infty)$

f_1 injectivă



$f_2:(0,1)\to\mathbb{R},\ f_2(x)=mx$

$$\operatorname{Im} f_2=\begin{cases}(0,m), & m>0\\ (m,0), & m<0\\ 0, & m=0.\end{cases}\ \Bigg\}\ f_2 \text{ injectivă}$$

$f_3:[1,\infty)\to\mathbb{R},\ f_3(x)=m^2-x$

$\operatorname{Im} f_3=(-\infty, m^2-1]$

Avem 3 cazuri:

I. $m>0$

II. $m=0$

f nu este inj

f nu este bij

? surj

III. $m<0$

Cazul I : $m>0$

$Im f_1 = [m, \infty)$ $\qquad$ f_1, f_2, f_3 sunt injective.

$Im f_2 = (0, m)$

$Im f_3 = (-\infty, m^2-1]$

Studiem injectivitatea lui f :

! • f_1, f_2, f_3 inj

• $Im f_i \cap Im f_j = \Phi$, $\forall i,j \in \{1,2,3\}$.

$$f \text{ inj} \Leftrightarrow m^2-1 \leq 0 \Leftrightarrow \left.\begin{matrix} m \in [-1,1] \\ \underline{\underline{m>0}} \end{matrix}\right\} \Rightarrow m \in (0,1].$$

Studiem surjectivitatea lui f :

$Im f_1 \cup Im f_2 \cup Im f_3 = \mathbb{R}$

$[m, \infty) \cup (0, m) \cup (-\infty, m^2-1] = \mathbb{R}$

$(0, \infty) \cup (-\infty, m^2-1] = \mathbb{R}$.

$$\left.\begin{matrix} m^2-1 \geq 0 \\ m>0 \end{matrix}\right\} \Rightarrow m \geq 1.$$

În cazul $m>0$:

• f inj $\Leftrightarrow m \in (0,1]$

• f surj $\Leftrightarrow m \geq 1$.

• f bij $\Leftrightarrow m = 1$.

Ex. 2 : Calculați $\varphi(n)$ cu P.I.E., $\varphi : \mathbb{N} \to \mathbb{N}$ indicatorul lui Euler, $\varphi(n) = |\{a \in \mathbb{N} \mid (a,n) = 1,\ a \le n\}|$.

Rez : $n = p_1^{a_1} \dots p_k^{a_k}$, p_i prime distincte, $a_i \ge 1$, $a_i \in \mathbb{N}$.

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right).$$

Calculăm $|\{a \in \mathbb{N} \mid (a,n) \ne 1,\ a \le n\}| = n - \varphi(n)$.

$A_i = \{a \in \mathbb{N} \mid (a,n) \vdots p_i,\ a \le n\}$, $i = \overline{1,k}$

$$|A_i| = \left[\frac{n}{p_i}\right] \underset{p_i \mid n}{=} \frac{n}{p_i}$$

$$\{a \in \mathbb{N} \mid (a,n) \ne 1,\ a \le n\} = A_1 \cup A_2 \cup \dots \cup A_k$$

$$|A_1 \cup A_2 \cup \dots \cup A_k| \overset{P.I.E.}{=} \sum_{i=1}^{k} |A_i| - \sum_{1 \le i < j \le k} |A_i \cap A_j| + \sum_{i<j<l} |A_i \cap A_j \cap A_l| + \dots + (-1)^{k-1} \left|\bigcap_{i=1}^{k} A_i\right|$$

$$|A_i \cap A_j| = \frac{n}{p_i p_j}$$

$$|A_i \cap A_j \cap A_l| = \frac{n}{p_i p_j p_l}$$

$\vdots$

$$\left|\bigcap_{i=1}^{k} A_i\right| = \frac{n}{\prod_{i=1}^{k} p_i}$$

$$|A_1 \cup \dots \cup A_k| = \sum_{i=1}^{k} \frac{n}{p_i} - \sum_{i<j} \frac{n}{p_i p_j} + \sum_{i<j<l} \frac{n}{p_i p_j p_l} - \dots + (-1)^{k-1} \frac{n}{\prod_{i=1}^{k} p_i}$$

$$\varphi(n) = n - \left|\bigcup_{i=1}^{k} A_i\right| = n - \sum_i \frac{n}{p_i} + \sum_{i<j} \frac{n}{p_i p_j} - \dots + (-1)^k \frac{n}{p_1 \cdots p_k}$$

$$= n\underbrace{\left(1 - \sum_i \frac{1}{p_i} + \sum \frac{1}{p_i p_j} - \dots + (-1)^k \frac{1}{p_1 \dots p_k}\right)}$$

$$\stackrel{?}{=} n \underbrace{\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)}.$$

Egalitatea se dem. prin inducție după k.

Ex. 3 : Fie $f: A \to B$ o funcție. Arătați că :

a. f surj $\Leftrightarrow \exists\, g: B \to A$ a.î. $f \circ g = 1_B$

$1_B : B \to B$, $1_B(b) = b$, $\forall\, b \in B$

b. f inj $\Leftrightarrow \exists\, h: B \to A$ a.î. $h \circ f = 1_A$

Rez :

a. „$\Leftarrow$" Pp. că $\exists\, g: B \to A$ a.î. $f \circ g = 1_B$.

Vrem f surj. ($\mathrm{Im} f = B$)

$$\mathrm{Im}(f \circ g) \stackrel{?}{=} \mathrm{Im}(1_B) = B$$

$$\mathrm{Im}(f\circ g) = (f \circ g)(B) = f(g(B)) = f(\mathrm{Im}\, g) \subseteq f(A) = \mathrm{Im} f \subseteq B$$

$$\mathrm{Im}\, g \subseteq A$$

$$B = f(\mathrm{Im}\, g) \subseteq f(A) = \mathrm{Im} f \subseteq B$$

$\Rightarrow \mathrm{Im} f = B$.

SAU : f surj cu def. ($\forall\, b \in B\ \exists\, a \in A$ a.î. $f(a) = b$).

„$\Rightarrow$" f surj. Vrem să găsim o funcție $g: B \to A$ a.î. $f \circ g = 1_B$.

f surj $\Rightarrow \forall b \in B \ \exists a \in A$ a.î. $f(a) = b$

$\forall b \in B$ considerăm $f^{-1}(\{b\}) \neq \emptyset$.

Pt. fiecare b alegem „preimagine" $a_b \in f^{-1}(\{b\})$ fixat.

Definim $g: B \to A$, $g(b) = a_b$, $\forall b \in B$.

$(f \circ g)(b) = f(a_b) = b$, $\forall b$.

Exemplu: $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, $f(a,b) = a$.

f surj.

$$f^{-1}(\{a\}) = \{(x,y) \in \mathbb{N} \times \mathbb{N} \mid f(x,y) = a\}$$
$$= \{(x,y) \in \mathbb{N} \times \mathbb{N} \mid x = a\}$$
$$= \{(a,y) \mid y \in \mathbb{N}\} = \{a\} \times \mathbb{N}.$$

$g: \mathbb{N} \to \mathbb{N} \times \mathbb{N}$

$g_1(n) = (n, n)$

$g_2(n) = (n, 1)$

$g_3(n) = (n, n + 243)$

b. f inj $\Leftrightarrow \exists h: B \to A$ a.î. $h \circ f = 1_A$.

„$\Leftarrow$" $h: B \to A$, $h \circ f = 1_A$. Vrem f inj.

Fie $a_1, a_2 \in A$ a.î. $f(a_1) = f(a_2) = b$

$$\Rightarrow \underset{h(b)}{\underset{\parallel}{h(f(a_1))}} = \underset{h(b)}{\underset{\parallel}{h(f(a_2))}} \Rightarrow \underset{a_1}{\underset{\parallel}{(h \circ f)(a_1)}} = \underset{a_2}{\underset{\parallel}{(h \circ f)(a_2)}}$$

$\Rightarrow f$ inj.

"$\Rightarrow$" f inj. Vrem să construim $h: B \to A$ a.î. $h \circ f = 1_A$.

f inj $\Rightarrow (\forall a_1, a_2 \in A,\ a_1 \neq a_2 \Rightarrow f(a_1) \neq f(a_2))$

$\forall b \in \operatorname{Im} f\ \exists!\ a_b \in A$ a.î. $f(a_b) = b$.

$$h: B \to A,\quad h(b) = \begin{cases} a_b, & b \in \operatorname{Im} f \\ a_0, & b \notin \operatorname{Im} f \end{cases}$$

unde $a_0 \in A$ arbitrar

Obs: $\underset{|\operatorname{Im} f|}{\underset{\parallel}{|A|}} \leq |B|$.

$h \circ f = 1_A \quad (h \circ f)(a_b) = h(f(a_b)) = h(b) = a_b$ ok.

$\operatorname{Im}(h \circ f) = h(\operatorname{Im} f)$

Obs: $f: A \to B$ inj $\Rightarrow f_1: A \to \operatorname{Im} f$ bij.

f_1 este inversabilă $\Rightarrow \exists!\ f_1^{-1}: \operatorname{Im} f \to A$.

$h: B \to A$, $\underset{\text{restricția lui } h \text{ la } \operatorname{Im} f}{\underset{\parallel}{h|_{\operatorname{Im} f}}} : \operatorname{Im} f \to A = f_1^{-1}$

Obs pt. a: $f|_{\operatorname{Im} g}: \operatorname{Im} g \to B$ bij.

T: Arătați că mulțimile $(0,1)$, (c,d), $c,d \in \mathbb{R}$, $c<d$, $\mathbb{R}$ și $\mathbb{R}_+^*$ sunt echipotente.

Găsiți bij. între
- $(0,1)$ și (c,d)
- (c,d) și $\mathbb{R}$
- $\mathbb{R}$ și $\mathbb{R}_+^*$

Hint: Folosiți funcții elementare (cls. a^x - a).