Nume și prenume: Avian Silviu-Gabriel
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Examen
Probabilități și statistică

1. $X \sim \begin{pmatrix} -5 & 1 \\ 0.24 & 0.76 \end{pmatrix}$, $Y \sim \begin{pmatrix} 0 & 6 \\ p_1 & p_2 \end{pmatrix}$, $p_1, p_2 \in (0,1)$

a) $\mathbb{P}(X=-5, Y=6) = 0.12$

$\mathbb{E}[X \mid Y=6] = -2$

X\Y	0	6	Σ
-5	0.12	0.12	0.24
1	$p_1-0.12$	$p_2-0.12$	0.76
Σ	p_1	p_2	

$$X \mid Y=6 \sim \begin{pmatrix} -5 & 1 \\ \mathbb{P}(X=-5 \mid Y=6) & \mathbb{P}(X=1 \mid Y=6) \end{pmatrix} \sim \begin{pmatrix} -5 & 1 \\ \frac{0.12}{p_2} & \frac{p_2-0.12}{p_2} \end{pmatrix}$$

$$\cancel{\mathbb{P}(X=-5 \mid Y=6)} = \frac{\mathbb{P}(Y=6 \mid X=-5)\cdot \mathbb{P}(X=-5)}{\mathbb{P}(Y=6)} = \quad 0.12$$

$$-5 \cdot \frac{0.12}{p_2} + 1 \cdot \frac{p_2-0.12}{p_2} = -2 \Leftrightarrow$$

$$\frac{-0.6+p_2-0.12}{p_2} = -2 \Leftrightarrow \frac{p_2-0.72}{p_2} = -2 \Leftrightarrow p_2 - 0.72 = -2p_2$$

$$\Leftrightarrow 3p_2 = 0.72 \Leftrightarrow p_2 = 0.24$$

$$p_1 = 1 - p_2 \Rightarrow p_1 = 1 - 0.24 = 0.76$$

Deci $p_1 = 0.76$ și $p_2 = 0.24$.

b) $X \sim \begin{pmatrix} -5 & 1 \\ 0.24 & 0.76 \end{pmatrix}$, $Y \sim \begin{pmatrix} 0 & 6 \\ 0.76 & 0.24 \end{pmatrix}$

$X+Y \in \{-5, 1, 7\}$

$\mathbb{P}(X+Y=-5) = \mathbb{P}(\{X=-5\} \cap \{Y=0\}) = \mathbb{P}(X=-5) \cdot \mathbb{P}(Y=0)$
$= 0.24 \cdot 0.76 = 0.1824$

$\mathbb{P}(X+Y=1) = \mathbb{P}((\{X=1\} \cap \{Y=0\}) \cup (\{X=-5\} \cap \{Y=6\})) =$
$= \mathbb{P}(X=1) \cdot \mathbb{P}(Y=0) + \mathbb{P}(X=-5) \cdot \mathbb{P}(Y=6)$
$= 0.76 \cdot 0.76 + 0.24 \cdot 0.24 = 0.5776 + 0.0576 = 0.6352$

$\mathbb{P}(X+Y=7) = \mathbb{P}(\{X=1\} \cap \{Y=6\}) = \mathbb{P}(X=1) \cdot \mathbb{P}(Y=6) =$
$= 0.76 \cdot 0.24 = 0.1824$

$X+Y \sim \begin{pmatrix} -5 & 1 & 7 \\ 0.1824 & 0.6352 & 0.1824 \end{pmatrix}$

$X-Y \in \{-5, -11, 1\}$

$\mathbb{P}(X-Y=-11) = \mathbb{P}(\{X=-5\} \cap \{Y=6\}) = \mathbb{P}(X=-5) \cdot \mathbb{P}(Y=6)$
$= 0.24 \cdot 0.24 = 0.0576$

$\mathbb{P}(X-Y=-5) = \mathbb{P}((\{X=-5\} \cap \{Y=0\}) \cup (\{X=1\} \cap \{Y=6\}))$
$= \mathbb{P}(X=-5) \cdot \mathbb{P}(Y=0) + \mathbb{P}(X=1) \cdot \mathbb{P}(Y=6)$
$= 0.24 \cdot 0.76 + 0.76 \cdot 0.24 = 0.1824 + 0.1824$
$= 0.3648$

$\mathbb{P}(X-Y=1) = \mathbb{P}(\{X=1\} \cap \{Y=0\}) = \mathbb{P}(X=1) \cdot \mathbb{P}(Y=0)$
$= 0.76 \cdot 0.76 = 0.5776$

$$X-Y \sim \begin{pmatrix} -11 & -5 & 1 \\ 0.0576 & 0.3648 & 0.5776 \end{pmatrix}$$

$$2X^2 \sim \begin{pmatrix} 2 & 50 \\ 0.76 & 0.24 \end{pmatrix}, \quad 2Y^2 \sim \begin{pmatrix} 0 & 72 \\ 0.76 & 0.24 \end{pmatrix}$$

$$2X^2+2Y^2 \in \{2, 50, 74, 122\}$$

$$\mathbb{P}(2X^2+2Y^2=2) = \mathbb{P}(\{2X^2=2\} \cap \{2Y^2=0\}) = \mathbb{P}(2X^2=2)\cdot\mathbb{P}(2Y^2=0)$$

$$= 0.76 \cdot 0.76 = 0.5776$$

$$\mathbb{P}(2X^2+2Y^2=50) = \mathbb{P}(\{2X^2=50\} \cap \{2Y^2=0\}) = \mathbb{P}(2X^2=50)\cdot\mathbb{P}(2Y^2=0)$$

$$= 0.24 \cdot 0.76 = 0.1824$$

$$\mathbb{P}(2X^2+2Y^2=74) = \mathbb{P}(\{2X^2=2\} \cap \{2Y^2=72\}) = \mathbb{P}(2X^2=2)\cdot\mathbb{P}(2Y^2=72)$$

$$= 0.76 \cdot 0.24 = 0.1824$$

$$\mathbb{P}(2X^2+2Y^2=122) = \mathbb{P}(\{2X^2=50\} \cap \{2Y^2=72\}) = \mathbb{P}(2X^2=50)\cdot \mathbb{P}(2Y^2=72) =$$

$$= 0.24 \cdot 0.24 = 0.0576$$

$$2X^2+2Y^2 \sim \begin{pmatrix} 2 & 50 & 74 & 122 \\ 0.5776 & 0.1824 & 0.1824 & 0.0576 \end{pmatrix}$$

$$\mathbb{E}[X] = -5\cdot 0.24 + 1\cdot 0.76 = -1.2+0.76 = -0.44$$

$$\mathbb{E}[Y] = 0\cdot 0.76 + 6\cdot 0.24 = 1.44$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$$

$$X^2 \sim \begin{pmatrix} 1 & 25 \\ 0.76 & 0.24 \end{pmatrix}$$

$$\mathbb{E}[X^2] = 1\cdot 0.76 + 25\cdot 0.24 = 6.76$$

$$\mathrm{Var}(X) = 6.76 - (-0.44)^2 = 6.76 - 0.1936 = 6.5664$$

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2$$

$$Y^2 \sim \begin{pmatrix} 0 & 36 \\ 0.76 & 0.24 \end{pmatrix}$$

$$\mathbb{E}[Y^2] = 0 \cdot 0.76 + 36 \cdot 0.24 = 8.64$$

$$\mathrm{Var}(Y) = 8.64 - 1.44^2 = 8.64 - 2.0736 = 6.5664$$

$$\mathrm{Var}(2X - 2Y + 8) = \mathrm{Var}(2X - 2Y) = \mathrm{Var}(2X) + \mathrm{Var}(-2Y) =$$

$$= 2^2 \cdot \mathrm{Var}(X) + (-2)^2 \cdot \mathrm{Var}(Y) = 4 \cdot 6.5664 + 4 \cdot 6.5664 =$$

$$= 8 \cdot 6.5664 = 52.5312$$

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)} \cdot \sqrt{\mathrm{Var}(Y)}}$$

$$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

$$XY \sim \begin{pmatrix} -30 & 0 & 6 \\ 0.0576 & 0.76 & 0.1824 \end{pmatrix}$$

$$\mathbb{E}[XY] = -30 \cdot 0.0576 + 0 \cdot 0.76 + 6 \cdot 0.1824$$

$$= -1.728 + 1.0944 = -0.6336$$

$$\mathrm{Cov}(X,Y) = -0.6336 - (-0.44) \cdot 1.44$$

$$= -0.6336 + 0.6336 = 0$$

$$\mathrm{Cov}(X,Y) = 0 \Rightarrow \rho(X,Y) = 0$$

3.

5 telefoane
2 defecte

$X \backslash Y$	0	1	2	3	Σ
1	0	1/10	1/10	2/10	4/10
2	0	1/10	2/10	0	3/10
3	1/10	2/10	0	0	3/10
Σ	1/10	4/10	3/10	2/10	

$\mathbb{P}(X=1, Y=1) = \mathbb{P}(X=1) \cdot \mathbb{P}(Y=1) = \frac{2}{5} \cdot \frac{1}{4} = 1/10$

$\mathbb{P}(X=1, Y=2) = \mathbb{P}(X=1) \cdot \mathbb{P}(Y=2) = \frac{2}{5} \cdot \frac{\cancel{3}}{4} \cdot \frac{1}{\cancel{3}} = 1/10$

$\mathbb{P}(X=1, Y=3) = \mathbb{P}(X=1) \cdot \mathbb{P}(Y=3) = \frac{2}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} \cdot \frac{2}{2} = 2/10$

(indiferent dacă penultimul telefon este defect sau nu, vom ști care din ultimele 2 telefoane rămase este cel defect)

$\mathbb{P}(X=2, Y=1) = \mathbb{P}(X=2) \cdot \mathbb{P}(Y=1) = \cancel{\frac{2}{5}} \cdot \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{\cancel{3}} = 1/10$

$\mathbb{P}(X=2, Y=2) = \mathbb{P}(X=2) \cdot \mathbb{P}(Y=2) = \frac{\cancel{3}}{5} \cdot \frac{2}{4} \cdot \frac{2}{\cancel{3}} \cdot \frac{2}{2} = 2/10$

(același caz ca la $X = 1, Y = 3$)

$\mathbb{P}(X=3, Y=0) = \mathbb{P}(X=3) \cdot \mathbb{P}(Y=0) = \frac{\cancel{3}}{5} \cdot \frac{\cancel{3}2}{4} \cdot \frac{\cancel{2}1}{\cancel{3}} = 1/10$

(am găsit 3 telefoane non-defecte, știu că ultimele 2 sunt defecte, din ipoteză)

$\mathbb{P}(X=3, Y=1) = \mathbb{P}(X=3) \cdot \mathbb{P}(Y=1) = \frac{\cancel{3}}{5} \cdot \frac{2}{4} \cdot \frac{2}{\cancel{3}} \cdot \frac{2}{2} = 2/10$

(același caz ca la $X=1, Y=3$; când al 3-lea telefon este defect)

$p_X(x) \sim \begin{pmatrix} 1 & 2 & 3 \\ 4/10 & 3/10 & 3/10 \end{pmatrix}$; $p_Y(y) \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1/10 & 4/10 & 3/10 & 2/10 \end{pmatrix}$

b) $\mathbb{E}[X] = \cancel{\ldots}\ 1\cdot 4/10 + 2\cdot 3/10 + 3\cdot 3/10 = \frac{4+6+9}{10} = 19/10$

$\mathbb{E}[Y] = 0\cdot 1/10 + 1\cdot 4/10 + 2\cdot 3/10 + 3\cdot 2/10 = \frac{4+6+6}{10} = 16/10$

$X^2 \sim \begin{pmatrix} 1 & 4 & 9 \\ 4/10 & 3/10 & 3/10 \end{pmatrix} \quad Y^2 \sim \begin{pmatrix} 0 & 1 & 4 & 9 \\ 1/10 & 4/10 & 3/10 & 2/10 \end{pmatrix}$

$\mathbb{E}[X^2] = 1\cdot 4/10 + 4\cdot 3/10 + 9\cdot 3/10 = \frac{4+12+27}{10} = 43/10$

$\mathbb{E}[Y^2] = 0\cdot 1/10 + 1\cdot 4/10 + 4\cdot 3/10 + 9\cdot 2/10 = \frac{4+12+18}{10} = 34/10$

$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 43/10 - (19/10)^2 = 4.3 - 3.61 = 0.69$

$\mathrm{Var}(Y) = \mathbb{E}[Y^2] - (\mathbb{E}[Y])^2 = 34/10 - (16/10)^2 = 3.4 - 2.56 = 0.84$

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\cdot\sqrt{\mathrm{Var}(Y)}}$$

$\mathrm{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\cdot\mathbb{E}[Y]$

$XY \sim \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 6 & 9 \\ 0.1 & 0.16 & 0.24 & 0.2 & 0.09 & 0.15 & 0.06 \end{pmatrix}$

$\mathbb{E}[XY] = 0\cdot 0.1 + 1\cdot 0.16 + 2\cdot 0.24 + 3\cdot 0.2 + 4\cdot 0.09 + 6\cdot 0.15 + 9\cdot 0.06$

$= 0.16 + 0.48 + 0.6 + 0.36 + 0.9 + 0.54$

$= 3.04$

$\mathrm{Cov}(X,Y) = 3.04 - \cancel{19} \ 1.9\cdot 1.6 = 0$

$\rho(X,Y) = 0$

c) $X|Y=2 \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{1/10}{3/10} & \frac{2/10}{3/10} & \frac{0}{3/10} \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$

$$\mathbb{E}[X|Y=2] = 1\cdot 1/3 + 2\cdot 2/3 + 3\cdot 0 = 5/3$$

$$Var(X|Y=2) = \mathbb{E}[(X|Y=2)^2] - (\mathbb{E}[X|Y=2])^2$$

$$(X|Y=2)^2 \sim \begin{pmatrix} 1 & 4 & 9 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

$$\mathbb{E}[(X|Y=2)^2] = 1\cdot 1/3 + 4\cdot 2/3 + 9\cdot 0 = 9/3 = 3$$

$$Var(X|Y=2) = 3 - (5/3)^2 = \frac{27}{9} - \frac{25}{9} = \frac{2}{9}$$

4. $$f(x) = \begin{cases} \frac{x}{100} e^{-\frac{x^2}{200}}, & x \geq 0 \\ 0, & \text{altfel} \end{cases}$$

$$F(x) = \int_0^x \frac{t}{100} e^{-\frac{t^2}{200}} dt = -e^{-\frac{t^2}{200}} \Big|_0^x = 1 - e^{-\frac{x^2}{200}}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x^2}{200}}, & x \geq 0 \end{cases}$$

$$F^{-1}(x) = y \quad | \cdot F$$

$$x = F(y)$$

$$x = 1 - e^{-\frac{y^2}{200}}$$

$$\cancel{e^{-\frac{x^2}{200}}} \; e^{-\frac{y^2}{200}} = 1 - x$$

$$-\frac{y^2}{200} = \ln(1-x)$$

$$\frac{y^2}{200} = \ln\left(\frac{1}{1-x}\right)$$

$$y^2 = 200 \ln\left(\frac{1}{1-x}\right)$$

$$y = 10\sqrt{2}\sqrt{\ln\left(\frac{1}{1-x}\right)}$$

$$\Rightarrow F^{-1}(x) = 10\sqrt{2}\sqrt{\ln\left(\frac{1}{1-x}\right)}$$

$$F^{-1}(0.75) = 10\sqrt{2}\sqrt{\ln\frac{1}{0.25}} = 10\sqrt{2}\sqrt{\ln 4}$$

$$F^{-1}(0.25) = 10\sqrt{2}\sqrt{\ln\frac{1}{0.75}} = 10\sqrt{2}\sqrt{\ln\frac{100}{75}}$$

$$F^{-1}(0.75) - F^{-1}(0.25) = 10\sqrt{2}\left(\sqrt{\ln 4} - \sqrt{\ln\frac{20}{15}}\right)$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\mathbb{E}[X] = \int_0^\infty x f(x)\,dx = \int_0^\infty \frac{x^2}{100} e^{-\frac{x^2}{200}}\,dx$$

$$\cancel{= -\frac{x}{2} e^{-\frac{x^2}{200}}\Big|_0^\infty + \frac{1}{2}\int_0^\infty e^{-\frac{x^2}{200}}\,dx}$$

$$= \int_0^\infty \frac{x}{2}\left(-e^{-\frac{x^2}{200}}\right)'\,dx$$

$$= -\frac{x}{2} e^{-\frac{x^2}{200}}\Big|_0^\infty + \frac{1}{2}\int_0^\infty e^{-\frac{x^2}{200}}\,dx$$

$$\lim_{x\to\infty} \frac{x}{2} e^{-\frac{x^2}{200}} = \lim_{x\to\infty} \frac{x}{2e^{\frac{x^2}{200}}} \overset{\frac{\infty}{\infty}}{\underset{L'H}{=}} \lim_{x\to\infty} \frac{1}{2e^{\frac{x^2}{200}}\cdot\frac{x}{100}}$$

$$\mathbb{E}[X] = \frac{1}{2}\int_0^\infty e^{-\frac{x^2}{200}}\,dx \overset{y=\frac{x}{10}}{=} 5\int_0^\infty e^{-\frac{y^2}{2}}\,dy = 5\,\frac{\sqrt{2\pi}}{2}$$

$$\mathbb{E}[X^2] = \int_0^\infty x^2 f(x) = \int_0^\infty \frac{x^3}{100} e^{-\frac{x^2}{200}}\,dx \overset{y=x^2}{=} \int_0^\infty \frac{1}{200} u e^{-\frac{u}{200}}\,du$$

$$dx = \frac{1}{2x}du$$

$$= \frac{1}{200}\cdot\int_0^\infty u e^{-\frac{u}{200}}\,du = 200$$

$$\mathrm{Var}(X) = 200 - \left(5\,\frac{\sqrt{2\pi}}{2}\right)^2 \approx 200 - 12.53 \approx 187.47$$

2. $\exists \sqrt{\mathbb{E}[X]} \Rightarrow \mathbb{E}[X] \geq 0$

Fie $y = \mathbb{E}[X] \Rightarrow \begin{cases} y \geq \sqrt{y}, & y \in \cancel{[0,1)}\,[1,+\infty) \\ y \leq \sqrt{y}, & y \in [0,1) \end{cases}$

Deci, $\mathbb{E}[X] \ ? \ \sqrt{\mathbb{E}[X]}$, deoarece depinde de $\mathbb{E}[X]$

$\mathbb{E}[\sin^2(X)] + \mathbb{E}[\cos^2(X)] = \mathbb{E}[\sin^2(X) + \cos^2(X)] = \mathbb{E}[1] = 1$

Deci, $\mathbb{E}[\sin^2(X)] + \mathbb{E}[\cos^2(X)] = 1$.

$\mathbb{E}[\log(X)] \geq \log(\mathbb{E}[X])$

$\mathbb{P}(X \leq Y) \ ? \ \mathbb{P}(Y \geq X)$, deoarece nu știm nimic de Y, raportat la X

$\mathbb{P}(X + Y > 10) \leq \mathbb{P}(X > 5 \text{ sau } Y > 5)$, deoarece include mai multe cazuri favorabile

$\mathbb{E}[X^2(X^2+1)] \ ? \ \mathbb{E}[X^2(Y^2+1)]$, deoarece nu știm nimic de Y în raport cu X.

$\mathbb{E}\left[\frac{1}{X}\right] \geq \frac{1}{\mathbb{E}[X]}$

$\mathbb{E}\left[\frac{X}{Y}\right] = \mathbb{E}\left[X \cdot \frac{1}{Y}\right] = \mathbb{E}[X] \cdot \mathbb{E}[Y^{-1}]$

$\mathbb{E}\left[\frac{X}{Y}\right] \geq \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}$

6. ~~$r \approx 0.48$~~

~~În Florescă~~

5. X - votează Cîțu

Y - votează Orban

$\mathbb{P}(X) = 0.01$

$\mathbb{P}(Y) = 0.99$

$X \sim \mathbb{P}(636)$

$\lambda = 636$

$$\mathbb{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!} = e^{-636} \cdot \frac{636^k}{k!}$$

$\mathbb{E}[X] = \cancel{60}\, 636 = \lambda$

$Var(X) = 636 = \lambda$