Curs 13

$f_{X,Y}(x,y)$ - densitatea comună

$$f_X(x) = \int f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int f_{X,Y}(x,y)\,dx$$

$$f_{X|A}(x) = \frac{f_X(x)}{\mathbb{P}(X\in A)} \quad \mathbb{P}(X\in B|A) = \int_B f_{X|A}(x)\,dx$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Formula prob. totale:

$$f_X(x) = \sum_{i=1}^{n} f_{X|A_i}(x)\,\mathbb{P}(A_i)$$

a) Y este o v.a. discretă $\{y_1, \dots, y_n\}$

X v.a. cont f_X

$$f_X(x) = \sum_{i=1}^{n} f_{X|Y}(x|y_i)\,\mathbb{P}(Y=y_i)$$

b) Y o v.a. cont f_Y

X v.a. cont f_X

$$f_X(x) = \int f_{X|Y}(x|y)\,f_Y(y)\,dy$$

Independența ~~v.a.~~ v.a.

$X \perp Y$

$$\mathbb{P}(X\in A, Y\in B) = \mathbb{P}(X\in A)\,\mathbb{P}(X\in B), \quad \forall A,B\subseteq \mathbb{R}$$

$A=(-\infty, x]$, $B(-\infty, y] \to \mathbb{P}(X\le x, Y\le y)=\mathbb{P}(X\le x)\mathbb{P}(Y\le y)$, $\forall x,y$

$$\int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,dv\,du=\int_{-\infty}^{x} f_X(u)\,du\int_{-\infty}^{y} f_Y(v)\,dv \quad \Big/ \text{ derivăm după } y \text{ și } x \text{ sau } x \text{ și } y$$

$$\frac{\partial^2}{\partial x\partial y}\int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,du\,du=\frac{\partial}{\partial x}\int_{-\infty}^{x} f_X(u)\,du\ \frac{\partial}{\partial y}\int_{-\infty}^{y} f_Y(v)\,dv$$

$$\frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\cdots\right)$$

$$\boxed{f_{X,Y}(x,y)=f_X(x)\,f_Y(y)}$$

(P) ~~Dacă~~ Fie X și Y sunt v.a. cu densitățile f_X și respectiv f_Y. Atunci $X \perp\!\!\!\perp Y \Leftrightarrow$

$\Leftrightarrow f_{X,Y}(x,y)=f_X(x)f_Y(y)$

(P) Fie X și Y două v.a. și g și h două fct.

Dacă $f_{X,Y}(x,y)=g(x)h(y)$, $\forall x,y$ atunci $X \perp\!\!\!\perp Y$

(P) Dacă X și Y sunt 2 v.a. independente atunci

$$E[g(X)h(Y)]=E[g(X)]E[h(Y)]$$

Obs: Dacă $g(x)$ și $h(y)=y$ atunci

$$E[XY]=E[X]E[Y]$$

$$Var(X+Y)=Var(X)+Var(Y)$$

Formula lui Bayes

X, Y două v.a. cont

$$\xrightarrow[f_X]{X} \boxed{\text{Zgomot}} \xrightarrow[f_{Y|X}(y|x)]{Y} \boxed{\text{inferență}} \xrightarrow{f_{X|Y}(x|y)}$$

semnal pe care nu îl observăm

semnal pe care îl observăm

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)\, f_Y(y)$$
$$= f_{Y|X}(y|x)\, f_X(x)$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{Y|X}(y|x)\, f_X(x)}{f_Y(y)} = \frac{f_{Y|X}(y|x)\, f_X(x)}{\int f_{Y|X}(y|x')\, f_X(x')\, dx'}$$

Caz hibrid

$$\xrightarrow[B]{\{0,1\}} \boxed{\text{Zgomot}} \xrightarrow{Y} \boxed{\text{inferență}} \xrightarrow{P(A|Y=y)}$$

$+ N(0,1)$

$P(Y=y)=0$

$$P(A|Y=y) = \lim_{dy\to 0} P(A|Y\in(y, y+dy))$$

$$\left(y-\frac{dy}{2}, y+\frac{dy}{2}\right)$$

$$= \lim_{dy\to 0} \frac{P(A\cap Y\in(y, y+dy))}{P(Y\in(y, y+dy))} =$$

$$= \lim_{dy\to 0} \frac{P(A)P(Y\in(y, y+dy)|A)}{P(Y\in(y, y+dy))} =$$

$$= \lim_{dy\to 0} \frac{P(A)\cdot \int_y^{y+dy} f_{Y|A}(u)\,du}{\int_y^{+dy} f_Y(u)\,du} = \lim_{dy\to 0} \frac{P(A) f_{Y|A}(y)\,dy}{f_Y(y)\,dy}$$

$$\boxed{\mathbb{P}(A \mid Y=y) = \frac{\mathbb{P}(A)\, f_{Y|A}(y)}{f_Y(y)}}$$

$$f_Y(y) = f_{Y|A}(y)\,\mathbb{P}(A) + f_{Y|A^c}(y)\,\mathbb{P}(A^c)$$

~~Formula lui Bayes~~

Formula prob totale

X\Y	discret	cont
discret	$\mathbb{P}(X=x) = \sum_y \mathbb{P}(X=x \mid Y=y)\,\mathbb{P}(Y=y)$	$\mathbb{P}(X=x) = \int \mathbb{P}(X=x \mid Y=y)\, f_Y(y)\,dy$
cont	$f_X(x) = \sum_y f_{X|Y}(x \mid y)\,\mathbb{P}(Y=y)$	$f_X(x) = \int f_{X|Y}(x \mid y)\, f_Y(y)\,dy$

Formula lui Bayes

X\Y	discret	cont
discret	$\mathbb{P}(Y=y \mid X=x) = \frac{\mathbb{P}(X=x \mid Y=y)\,\mathbb{P}(Y=y)}{\mathbb{P}(X=x)}$	$f_{Y|X}(y \mid x) = \frac{\mathbb{P}(X=x \mid Y=y)\, f_Y(y)}{\mathbb{P}(X=x)}$
cont.	$\mathbb{P}(Y=y \mid X=x) = \frac{f_{X|Y}(x \mid y)\,\mathbb{P}(X=y)}{f_X(x)}$	$f_{Y|X}(y \mid x) = \frac{f_{X|Y}(x \mid y)\, f_Y(y)}{f_X(x)}$

Ex A și B, durata de viață A $Exp(\lambda_0)$

— " — B $Exp(\lambda_1)$

$\lambda_0 < \lambda_1$

P. să primim un telefon de la A cu prob p_0 și de la B cu $p_1 = 1 - p_0$

Fie T durata de viață a telefonului primit.

a) Fct de rep. și densitatea lui T

b) Vrem să găsim prob. ca tel. să fie primit de la B știind că T=t.

T v.a. cont.

Fie I v.a.

$$\begin{cases} 0, & \text{dacă tel A} \\ 1, & \text{dacă tel B} \end{cases}$$

$P(I=0)=p_0$

$P(I=1)=p_1=1-p_0$

$T|I=0 \sim Exp(\lambda_0)$

$T|I=1 \sim Exp(\lambda_1)$

$$P(T\le t)=\underbrace{P(T\le t|I=0)}_{1-e^{-\lambda_0 t}}\underbrace{P(I=0)}_{p_0}+\underbrace{P(T\le t|I=1)}_{1-e^{-\lambda_1 t}}\underbrace{P(I=1)}_{1-p_0}=$$

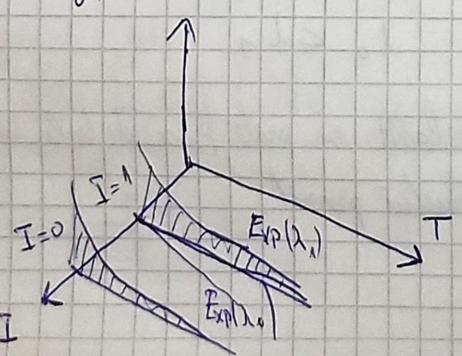
$Exp(\lambda)$ $\quad f(x)=\lambda e^{-\lambda x}$

$F(x)=1-e^{-\lambda x}$

$$=\left(1-e^{-\lambda_0 t}\right)p_0+\left(1-e^{-\lambda_1 t}\right)(1-p_0)$$

$$f_T(t)=\frac{d}{dt}F_T(t)=\lambda_0 e^{-\lambda_0 t}p_0+\lambda_1 e^{-\lambda_1 t}(1-p_0),\ t>0$$

(T, I)



b) $P(I=1 \mid T=t) = \dfrac{f_{T|I}(t|1)\, P(I=1)}{f_T(t)} =$

$$= \frac{\lambda_1 e^{-\lambda_1 t}(1-p_0)}{\lambda_0 e^{-\lambda_0 t} p_0 + \lambda_1 e^{-\lambda_1 t}(1-p_0)}$$

<u>Media unei fct. de v.a.</u>

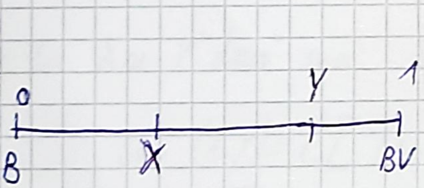
X, Y două v.a. $f_{X,Y}(x,y)$ și $g: \mathbb{R}^2 \to \mathbb{R}$

$$\boxed{E[g(X,Y)] = \iint g(x,y)\, f_{X,Y}(x,y)\, dx\, dy}$$

În particular,

$$E[XY] = \iint xy\, f_{X,Y}(x,y)\, dx\, dy$$

<u>Ex</u>



$X, Y \sim U[0,1]$ indep.

$E\big[|X-Y|\big] =$

$= \iint |x-y|\, f_{X,Y}(x,y)\, dx\, dy$

$E[|X-Y|] = \iint |x-y|\, \mathbb{1}_{[0,1]}(x)\, \mathbb{1}_{[0,1]}(y)\, dx\, dy$

$= \int_0^1 \int_y^1 (x-y)\, dx\, dy + \int_0^1 \int_0^y (y-x)\, dx\, dy = \cancel{\frac{x^2}{2} \int_0^1 \left(\frac{y^2}{2} - x\right)\Big|_0^y dx} =$

$= \int_0^1 \left(yx - \frac{x^2}{2}\right)\Big|_0^y dy = \int_0^1 y^2 - \frac{y^2}{2}\, dy = \int_0^1 \frac{y^2}{2}\, dy = \frac{y^3}{3}\cdot\frac{1}{2}\Big|_0^1 = \frac{1}{3}$

<u>Medie condiționată</u>

X v.a. cont și A ~~...~~ ev. $\mathbb{P}(A)>0$

$$E[X|A] = \int x\, f_{X|A}(x)\,dx$$

Dacă $A = \{Y=y\}$

$$E[X|Y=y] = \int x\, f_{X|Y}(x|y)\,dx$$

Formula prob. totale

$$f_X(x) = \sum_{i=1}^{n} f_{X|A_i}(x)\,P(A_i) \quad \Big| \cdot x, \text{ apoi integrez.}$$

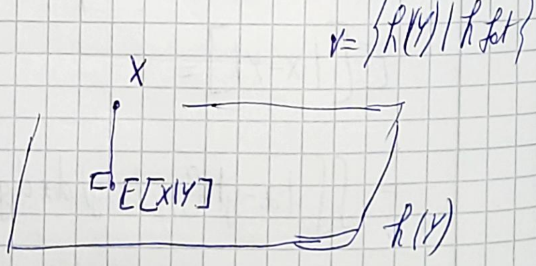
$$\boxed{E[X] = \sum_{i=1}^{n} E[X|A_i]\,P(A_i)}$$

$$\boxed{E[X] = \int E[X|Y=y]\, f_Y(y)\,dy}$$

<u>Def</u>: Fie $g(y) = E[X|Y=y]$. Atunci v.a. $E[X|Y] = g(Y)$

<u>Prop</u>: a) $E[E[X|Y]] = E[X]$

b) $Var(E[X|Y]) =$



$$E[X|Y] = \arg\min_{g} E[(X-g(Y))^2]$$
$$E[(X-g(Y))]$$

$$Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_m \end{pmatrix}$$

$$h = a_0 + a_1 Y_1 + \dots + a_m Y_m$$

$$E[(X-a)^2]$$
$$E[|X-a|]$$

$$Var(X) = Var(E[X|Y]) + E[Var(X|Y)]$$

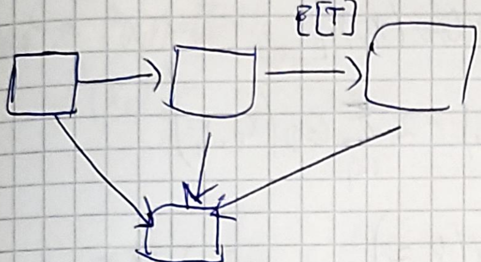
$N =$

$X_1, X_2, \dots$

$T = X_1 + X_2 + \dots + X_N$

$\omega_1 \quad N(\omega_1) = 10$

$T(\omega_1) = X_1^{(\omega_1)} + \dots + X_{10}^{(\omega_1)}$



Covarianța și corelație

Def: Fie X și Y două v.a. Se n. covarianța dintre X și Y

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$$

În particular, $X = Y \Rightarrow Cov(X, \cancel{Y}) = Var(X)$

P $Cov(X,Y) = E[XY] - E[X]E[Y]$

Def: Spunem că v.a. X și Y sunt necorelate dacă $Cov(X,Y) = 0$

Altfel spus, $E[XY] = E[X]E[Y]$

Obs: Dacă $X \perp Y \Rightarrow X$ și Y sunt necorelate

EXP: $X \sim N(0,1)$, $Y = X^2$ $\Rightarrow E[X]E[Y] = 0$, $E[XY] = E[X^3] = 0$ $\Rightarrow X$ și Y sunt necorelate

X și Y nu sunt indep!

$E[X^3]$: integrală din fct. impară $= 0$

Prop:

a) $Cov(X,X) = Var(X)$

b) $Cov(X, a) = 0$, a const.

c) $Cov(a+bX, Y) = b\,Cov(X,Y)$

d) $Cov(X,Y) = Cov(Y,X)$

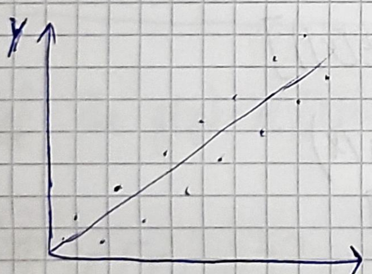
e) $Var(X+Y) = Var(X)+Var(Y)+2\,Cov(X,Y)$

$$Var(X_1+\dots+X_n) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i,X_j)$$

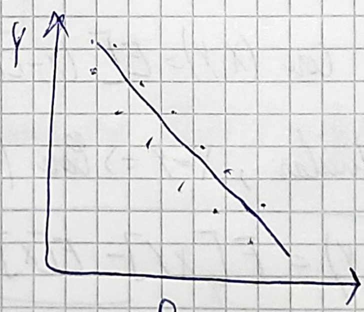
f) $Cov(X+Y,Z) = Cov(X,Z)+Cov(Y,Z)$

Def. Fie X și Y două v.a. și definim coeficientul de corelație dintre X și Y

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\cdot\sqrt{Var(Y)}}$$

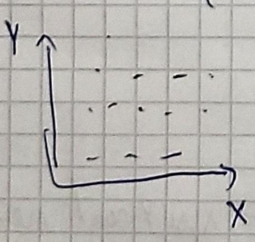


$\rho>0$



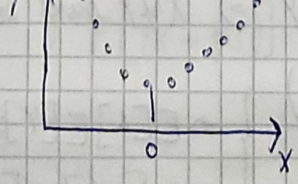
$\rho<0$

$\rho=0$ (indep.)



$-1<\rho<1$

$\rho=0$ (necorelate)



Prop: $\rho \in [-1, 1]$. Dacă $\rho = 1$ (sau -1) atunci $X = a + bY$ ($Y = a + bX$) aproape sigur

$$\mathbb{P}(X = a + bY) = 1 \quad \text{(a.s.)}$$

Dem: X, Y v.a. $\mathbb{E}[X] = \mu_{\cancel{x}}$, $Var(X) = \sigma^2_{\cancel{x}}$, $E[Y] = \mu_y$, $Var(Y) = \sigma_y^2$

μ_X $\quad$ σ_x^2

P. $\underline{\mu_x = \mu_y = 0}$ și $\sigma_x^2 = \sigma_y^2 = 1$

Dacă $\Rightarrow$ $\rho(X, Y) = \mathbb{E}[X, Y]$

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)\, Var(Y)}} = \frac{E[(X - \mu_x)(Y - \mu_y)]}{\sigma_x \, \sigma_y}$$

$$\mathbb{E}\left[\left(\frac{X - \mu_x}{\sigma_x}\right)\left(\frac{Y - \mu_y}{\sigma_y}\right)\right]$$

v.a. normalizate

Știm că $E[(X + \lambda Y)^2] \geq 0, \forall \lambda \in R$

$$\lambda^2 E[Y^2] + 2\lambda E[XY] + E[X^2] \geq 0 \quad \cancel{\forall \lambda \in R} \quad \forall \lambda$$

$$\cancel{\Delta = 4\lambda^2} \quad \Delta = 4\, E[XY]^2 - 4 E[X^2] E[Y^2] \leq 0$$

$$\mathbb{E}[XY]^2 \leq \underbrace{E[X^2]}_{=1} \underbrace{E[Y^2]}_{=1} \Rightarrow E[XY]^2 \leq 1 \Rightarrow \rho(X, Y)^2 \leq 1 \Rightarrow |\rho(X, Y)| \leq 1$$

Inegalitatea Cauchy-Schwartz

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\| \qquad \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \left(\sum a_i^2\right)\left(\sum b_i^2\right)$$

$$X \sim \begin{pmatrix} a_1 & \cdots & a_n \\ \frac{1}{n} & & \frac{1}{n} \end{pmatrix}$$

$$Y \sim \begin{pmatrix} b_1 & \cdots & b_n \\ \frac{1}{n} & \cdots & \frac{1}{n} \end{pmatrix}$$

$\lambda = -1$ $\rho = 1$

$E[(X-Y)^2]=0$

$X = Y$

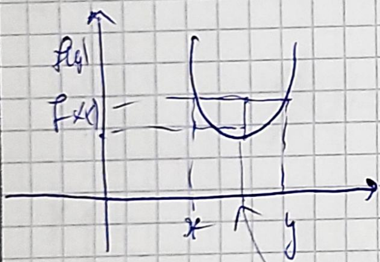
Inegalități și teoreme limită

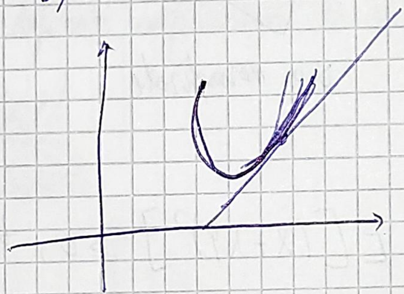
(T) (Inegalitatea Cauchy-Schwartz) Fie X și Y v.a. cu $Var(X) < \infty$, $Var(Y) < \infty$. Atunci

$$|E[XY]| \le \sqrt{E[X^2]E[Y^2]}$$

Inegalitatea lui Jensen

a) Fct. convexă

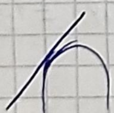




$tx+(1-t)y, \ t \in [0,1]$

$$f(tx+(1-t)y) \le t f(x) + (1-t) f(y)$$

b) Concavă



Fct concavă: $\forall x, y, \ \forall t \in [0,1]$

$$f(tx+(1-t)y) \ge t f(x) + (1-t) f(y)$$

(T) (Ineg Jensen) Fie X v.a. și g o fct convexă. Atunci:

$$E[g(X)] \ge g(E[X])$$

Dacă g este concavă:

$$E[g(X)] \le g(E[X])$$

Obs: $Var(X) \ge 0$

$$E[X^2] \ge E[X]^2$$

(T) (Ineg lui Markove)

Fie X v.a. pozitivă. Atunci

$$\mathbb{P}(X \ge a) \le \frac{E[X]}{a}$$

Dem

$$Y = \begin{cases} 0, & X < a \\ a, & X \ge a \end{cases} \quad \text{atunci}$$

$$E[Y] = a\mathbb{P}(X \ge a)$$

$$Y \le X \Rightarrow E[Y] \le E[X]$$

Ex

$$X \sim U([0,1])$$

$$\mathbb{P}(X > 2) \le \frac{1}{4}$$

$$\mathbb{P}(X \ge 1) \le \frac{1}{2}$$

$$\mathbb{P}\left(X \ge \frac{1}{2}\right) \le 1$$

(P) Ineg ~~Chebysheve~~ Chebyshev - Cebîșev)

Fie X v.a. $E[X] = \mu < \infty$, $Var(X) = \sigma^2 < \infty$

$$\mathbb{P}(|X - \mu| \ge a) \le \frac{Var(X)}{a^2}, \ \forall a > 0$$

$$Y = (X-\mu)^2$$

$$\mathbb{P}(Y \geq a^2) \leq \frac{E[Y]}{a^2} = \frac{Var(X)}{a^2}$$

<u>Aleg</u> $a = k\sigma$

$$\mathbb{P}(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$$

(Ineg Chernoff)

Fie X v.a., $a > 0$, $t > 0$

$$\mathbb{P}(X \geq a) \leq \frac{E[e^{tX}]}{e^{ta}} \quad \forall a, t$$