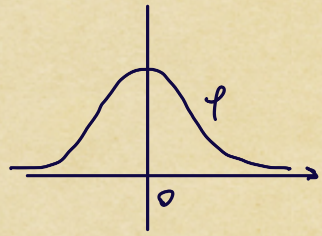
Repartiția normală

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}$$

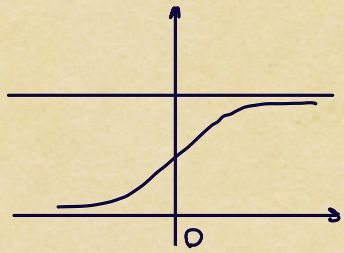


φ densitate

1) $\varphi(x) \geq 0$

2) $\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$

Sim față de $\varphi(x) = \varphi(-x)$ | $\Phi(x) = \int_{-\infty}^{x} \varphi(t)\,dt$



$X \sim N(0,1)$ normală standard $\quad \left(-e^{-\frac{x^2}{2}}\right)'$

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} x\,\varphi(x)\,dx = \int_{-\infty}^{\infty} \underbrace{\frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}_{\text{impară}}\,dx = 0$$

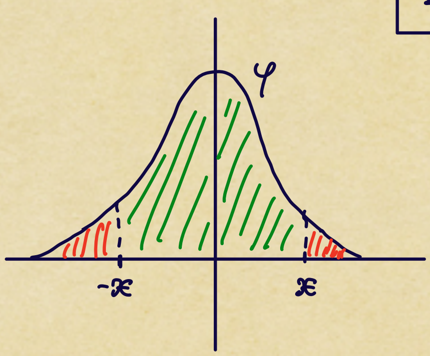
$$Var(X) = \mathbb{E}[X^2] - \underbrace{\mathbb{E}[X]}_{=0}^2$$

$$\mathbb{E}[X^2] = \int x^2 \varphi(x)\,dx = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \left(-e^{-\frac{x^2}{2}}\right)' dx =$$

$$= \underbrace{-\frac{x e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\Bigg|_{-\infty}^{+\infty}}_{=0} + \int_{-\infty}^{+\infty} -\frac{x e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\,dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} -e^{-\frac{x^2}{2}}\,dx = 1 \Rightarrow Var(X) = 1$$

Obs: Dacă $X \sim N(0,1)$ atunci $\mathbb{E}[X] = 0$ și $Var(X) = 1$

$$\boxed{\Phi(x) = 1 - \Phi(-x)}$$



$$\Phi(-x) = \int_{-\infty}^{x} \varphi(t)\,dt$$

sch. var: $u = -t$

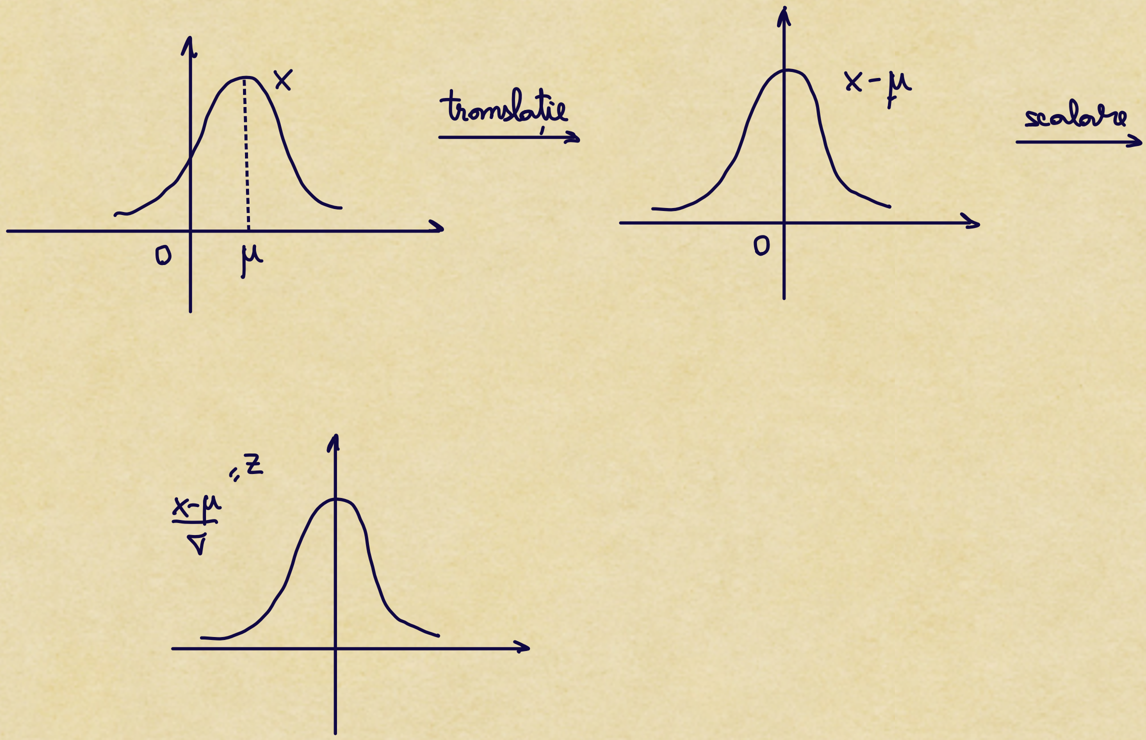
$$= \int_{+\infty}^{x} \varphi(-u)(-du) = \int_{x}^{+\infty} \varphi(-u)\,du = \int_{x}^{+\infty} \varphi(u)\,du =$$

$$= \int_{x}^{+\infty} \varphi(u)\,du = 1 - \int_{\infty}^{x} \varphi(u)\,du$$

Def: Spunem că v.a $X \sim N(\mu, \sigma^2)$ dacă admite densitatea de repartiție

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

Ⓟ Dacă $X \sim N(\mu, \sigma^2)$ atunci $\exists\, Z \sim N(0,1)$ a.î. $X = \mu + \sigma Z$



Repartiție normală

$$X \sim N(\underbrace{\mu}_{\text{media}}, \underbrace{\sigma^2}_{\text{varianța}}) \Rightarrow \mathbb{E}[X] = E[\mu + \sigma Z] \cdot \mu + \sigma\, \mathbb{E}[Z] = \mu$$

$Z \sim N(2,1)$

$$Var(X) = Var(\mu + \sigma Z) = \sigma^2 Var(Z) = \sigma^2$$

$$F(x) = \mathbb{P}(X \le x) = \mathbb{P}(\mu + \sigma Z \le x)$$
$$= \mathbb{P}\left(Z \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$
$$f(x) = \frac{d}{dx} F(x) = \frac{d}{dx} \Phi\left(\frac{x-\mu}{\sigma}\right) = \varphi\left(\frac{x-\mu}{\sigma}\right) \frac{1}{\sigma}$$

$$(f \circ g)' = f'(g) \cdot g'$$

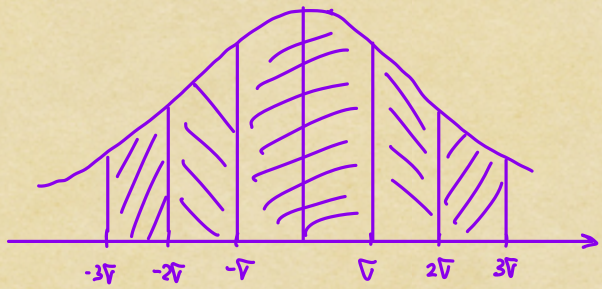
Ⓟ (Prop 68-95-99,7%)

Dacă $X \sim N(\mu, \sigma^2)$ atunci

$\mathbb{P}(|X-\mu| \leq \sigma) \simeq 68\%$

$\mathbb{P}(|X-\mu| \leq 2\sigma) \simeq 95\%$

$\mathbb{P}(|X-\mu| \leq 3\sigma) \simeq 99{,}7\%$



<u>Ex:</u> $X \sim N(-1, 4)$

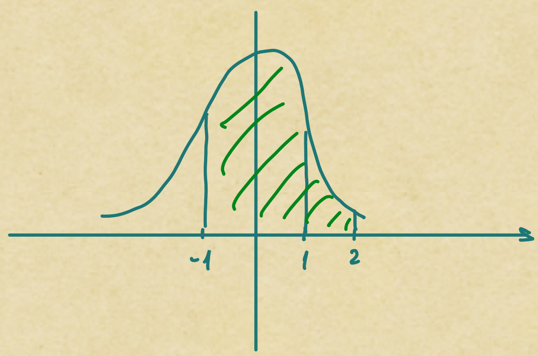
$\mathbb{P}(|X| < 3)$

<u>Pas 1</u> Standardizare

$\mathbb{P}(-3 < X < 3) = \mathbb{P}(-3-(-1) < X-(-1) < 3-(-1)) =$

$= \mathbb{P}(-2 < X+1 < 4)$ $\quad (/\sigma)$

$= \mathbb{P}\left(-\frac{2}{2} < \frac{X+1}{2} < \frac{4}{2}\right) = \mathbb{P}\left(-1 < \frac{X+1}{2} < 2\right) \quad \sim N(0,1)$



$\mathbb{P}(-1 \leq Z \leq 1) \simeq 0{,}68$

$\mathbb{P}(-2 \leq Z \leq 2) \simeq 0{,}95$

$= \underbrace{\mathbb{P}(-1 \leq Z \leq 1)}_{\simeq 0{,}68} + \underbrace{\mathbb{P}(1 \leq Z \leq 2)}_{\simeq \frac{0{,}95-0{,}68}{2}}$

Ex2 $Y \sim N(0,1)$, $X = |Y|$

$\mathbb{E}[X]$, $Var[X]$, $f(x)$

$$\mathbb{E}[|Y|] = \int_{-\infty}^{+\infty} |x| \varphi(x)\, dx = 2 \int_0^{\infty} \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = \sqrt{\frac{2}{\pi}} \left(-e^{-\frac{x^2}{2}}\right)\Big|_0^{\infty} = \sqrt{\frac{2}{\pi}}$$

$$Var(|Y|) = \mathbb{E}[|Y|^2] - \mathbb{E}[|Y|]^2$$

$$= \mathbb{E}[Y^2] - \frac{2}{x} = 1 - \frac{2}{\pi}$$

$$F_X(x) = \mathbb{P}(X \le x) = \mathbb{P}(|Y| \le x)$$

Dacă $x < 0 \Rightarrow F_X(x) = 0$

$\Phi(-x) = 1 - \Phi(x)$

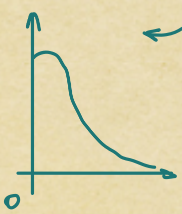
Dacă $x > 0 \Rightarrow F_X(x) = \mathbb{P}(-x \le Y \le x)$

$$= \Phi(x) - \Phi(-x)$$

$$= 2\Phi(x)$$

se dă la examen

$$F_X(x) = \begin{cases} 0 & , x < 0 \\ 2\Phi(x) - 1 & , x > 0 \end{cases}$$



$$f_X(x) = \begin{cases} 0 & , x < 0 \\ 2\varphi(x), & x > 0 \end{cases}$$

Repartiții comune, marginale și condiționate

X, Y doux v.a. $(\Omega, \mathcal{F}, \mathbb{P})$

$\mathbb{P}((X,Y) \in A \times B)$

$\mathbb{P}(X \in A)$ sau $\mathbb{P}(Y \in B)$

$\mathbb{P}(X \in A \mid Y \in B)$

1) Cazul discret

Fie $(\Omega, \mathcal{F}, \mathbb{P})$ un c.p. și $X: \Omega \to \mathbb{R}$, $Y: \Omega \to \mathbb{R}$

$$X(\Omega) = \{x_1, x_2, \dots, x_m\}$$

$$Y(\Omega) = \{y_1, y_2, \dots, y_n\}$$

Perechea $(X, Y): \Omega \to \mathbb{R}^2$

$$\omega \mapsto (X(\omega), Y(\omega))$$

$$(X,Y)(\Omega) = \{(x_i, y_j) \mid i = \overline{1,m},\ j = \overline{1,n}\} \to m \times n \text{ valori}$$

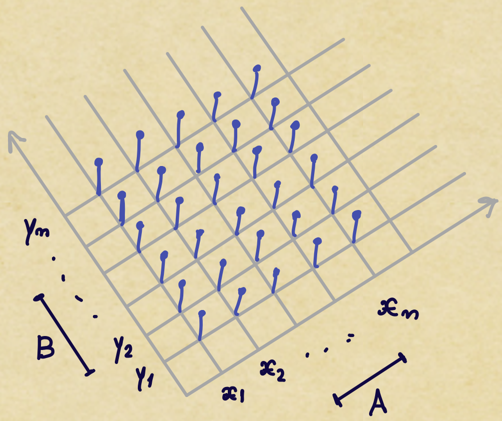
Funcția de masă a (X, Y)

$$f_{X,Y}(x, y) = \mathbb{P}(X = x, Y = y), \ \forall x \in \{x_1, \dots, x_m\},\ y \in \{y_1, \dots, y_n\}$$

$(p_{X,Y}(x,y))$

Prop: a) $f_{X,Y}(x,y) \geq 0$, $\forall x, y$

b) $\sum_{x \in X(\Omega)} \sum_{y \in Y(\Omega)} f_{X,Y}(x,y) = 1$



$A \subseteq \mathbb{R}$
$B \subseteq \mathbb{R}$

$$\mathbb{P}((X,Y) \in A \times B) =$$

$$= \sum_{x \in X(\Omega) \cap A} \sum_{y \in Y(\Omega) \cap B}$$

$$\mathbb{P}((X,Y) \in C) = \sum_{\substack{(x,y) \in X(\Omega) \times Y(\Omega) \\ (x,y) \in C}} f_{X,Y}(x,y)$$

Reamintim

X discretă, $f_X(x) = \mathbb{P}(X = x)$

$$\mathbb{P}(X \in A) = \sum_{x \in X(\Omega) \cap A} f_X(x)$$

$$\parallel$$

$$(\mathbb{P} \circ X^{-1})(A)$$

$$\mathbb{P}(X \in A) = \mathbb{P}(\{X \in A\} \cap \Omega)$$

$$= \mathbb{P}(X \in A, Y \in \mathbb{R})$$

$$= \mathbb{P}(X \in A \bigcup_y \{Y = y\}) = \mathbb{P}(\bigcup_y \{X \in A, Y = y\}) = \sum_y \mathbb{P}(X \in A, Y = y)$$

$$\mathbb{P}(X=x)=\sum_{y}(X=x, Y=y)$$

$$\boxed{f_X(x)=\sum_{y} f_{X,Y}(x,y)} \to \text{fct de masă a lui } X \quad \text{rep marginală a lui } X$$

$$\boxed{f_Y(y)=\sum_{x} f_{X,Y}(x,y)} \to \text{repr. marginală pt } Y$$

Fie X o v.a. adevărată și $A \in \mathcal{F}$ $\quad \mathbb{P}(A)>0$

$$\underbrace{\mathbb{P}(X=x \mid A)}_{f_{X|A}(x)}=\frac{\mathbb{P}(\{X=x\}\cap A)}{\mathbb{P}(A)}$$

Dacă $A=\{Y=y\}$ atunci $\underbrace{\mathbb{P}(X=x \mid Y=y)}_{f_{X|Y}(x|y)}=\dfrac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)}=\dfrac{f_{X,Y}(x,y)}{f_Y(y)}$

$$\boxed{f_{X|Y}(x|y)=\frac{f_{X,Y}(x,y)}{f_Y(y)}} \qquad \boxed{f_{Y|X}(y|x)=\frac{f_{X,Y}(x,y)}{f_X(x)}} \to \text{fct de masă condiționată a lui } Y \text{ la } X$$

<u>Ex</u>

X \ Y	y_1	y_2	...	...	y_j	...	...	y_n	Σ
x_1									
x_2									
$\vdots$									
$\vdots$					$f_{X,Y}(x,y)$				
x_i									$f_X(x_i)$
$\vdots$									
$\vdots$									
x_m									
Σ					$f_Y(y)$				

$f_X(x_i) \to \sum_{y} f_{X,Y}(x_i,y)$

$\to \mathbb{P}(X=x_i, Y=y_j)$

$f_Y(y) \to \sum_{x} f_{X,Y}(x,y_j)$

$$X \sim \begin{pmatrix} x_1 & x_2 & \dots x_i \dots & x_m \\ & & f_X(x_i) & \end{pmatrix}$$

$$f_X(x_i)=\sum_{j=1}^{n} f_{X,Y}(x_i,y_j)$$

$$X \mid Y=y_j \sim \begin{pmatrix} x_1 & x_2 & \dots & x_m \\ & f_{X,Y}(x_i,y_j) & & \end{pmatrix}$$

$$\swarrow$$

$$\frac{f_{X,Y}(x_i,y_j)}{f_Y(y_j)}$$

Ex: Prof răspunde greșit 1/4 din cazuri indep de întrebare

0, 1 sau 2 întreb cu 1/3

X - nr de întrebări $\in \{0,1,2\}$

Y - nr de răspunsuri greșite $\in \{0,1,2\}$

$\Rightarrow (X,Y) : \Omega \to \mathbb{R}^2$

$(x,y) \in \{0,1,2\}^2$

X \ Y	0	1	2	
0	1/3	0	0	1/3
1	1/4	1/12	0	1/3
2	3/16	2/16	$\frac{1}{3}\frac{1}{16}$	1/3

$$\mathbb{P}(X=0, Y=0) = 1/3 = \underbrace{\mathbb{P}(X=0)}_{=1/3} \cdot \underbrace{\mathbb{P}(Y=0 \mid X=0)}_{=1} = 1/3$$

$$\mathbb{P}(X=1, Y=1) = \mathbb{P}(X=1)\,\mathbb{P}(Y=1 \mid X=1) = 1/3 \times 1/4 = 1/12$$

$$\mathbb{P}(X=1, Y=0) = \mathbb{P}(X=1)\,\mathbb{P}(Y=0 \mid X=1) = 1/3 \times 3/4 = 1/4$$

$$\mathbb{P}(X=2, Y=2) = \mathbb{P}(X=2)\mathbb{P}(Y=0 \mid X=2) =$$

$$= 1/3 \times 3/4 \times 3/4 = 9/48 = 3/16$$

$$\mathbb{P}(X=2, Y=1) = \mathbb{P}(X=2)\,\mathbb{P}(Y=1 \mid X=2) \to$$ Care din ele e corectă? $\binom{2}{1}$

$$= 1/3 \times \binom{2}{1} 1/4 * 3/4 =$$

Formula probabilității totale

$B, A_1, A_2, \dots, A_n \in \mathcal{F}$, $A_1, A_2, \dots, A_n$ formam o part pe Ω

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \mid A_i)\,\mathbb{P}(A_i)$$

Dacă $B = \{X = x\} \Rightarrow \mathbb{P}(X=x) = \sum_{i=1}^{n} \underbrace{\mathbb{P}(X=x \mid A_i)}_{f_{X|A}(x)} \mathbb{P}(A_i)$

$$A = \{Y = y_i\} \Rightarrow \mathbb{P}(X=x) = \sum_{i=1}^{n} \mathbb{P}(X=x \mid Y=y_i)\,\mathbb{P}(Y=y_i)$$

$$f_X(x) = \sum_{i=1}^{n} f_{X|Y}(x \mid y_i)\, f_Y(y_i)$$

$$\mathbb{P}(A\cap B) = \mathbb{P}(A)\mathbb{P}(B|A)$$
$$= \mathbb{P}(B)\mathbb{P}(A|B)$$
$$A = \{X=x\},\ B=\{Y=y\}$$

$$\mathbb{P}(X=x,\ Y=y) = \mathbb{P}(X=x)\mathbb{P}(Y=y|X=x)$$
$$= \mathbb{P}(Y=y)\mathbb{P}(X=x|Y=y)$$

$$\boxed{\begin{aligned} f_{X,Y}(x,y) &= f_X(x) f_{Y|X}(y|x) \\ &= f_Y(y) f_{X|Y}(x|y)\end{aligned}}$$

<u>Formula lui Bayes</u>

$$\mathbb{P}(X=x|Y=y) = $$
$$= \frac{\mathbb{P}(X=x)\mathbb{P}(Y=y,X=x)}{\mathbb{P}(Y=y)} =$$
$$= \frac{\mathbb{P}(X=x)\mathbb{P}(Y=y|X=x)}{\sum_{x'}\mathbb{P}(X=x')\mathbb{P}(Y=y|X=x')}$$

$$\boxed{f_{X|Y}(x|y) = \frac{f_X(x) f_{Y|X}(y|x)}{\sum_{x'} f_X(x') f_{Y|X}(y|x')}}$$

Ex: O găină depune N ouă, $N \sim Pois(\lambda)$. Pp că fiecare eclozează cu prob $p\in(0,1)$ indep de celelalte.

X nr de ouă care au eclozat $\qquad X+Y=N$

Y nr de ouă care nu au eclozat

Vrem să det rep (X,Y) și rep marginală și să verificăm $X \perp Y$

$$\mathbb{P}(X=i,\ Y=j) = \sum_{n=0}^{\infty}\mathbb{P}(X=i,\ Y=j|N=n)\mathbb{P}(N=n) =$$
$$= \mathbb{P}(X=i,\ Y=j|N=i+j)\mathbb{P}(N=i+j)$$

$$\mathbb{P}(N=n) = e^{-\lambda}\frac{\lambda^n}{n!}$$

$$X|N=n \sim B(n,p)$$
$$Y|N=n \sim B(n,1-p)$$

Dacă $i+j\neq n \Rightarrow \mathbb{P}(X=i,\ Y=j|N=n) = 0$

$$\mathbb{P}(X=i,\ Y=j|N=i+j) = \mathbb{P}(X=i|N=i+j) = \binom{i+j}{i}p^i(1-p)^j$$
$$= \mathbb{P}(Y=j|N=i+j)$$

$$\mathbb{P}(X=i,\ Y=j) = \binom{i+j}{i}p^i(1-p)^j e^{-\lambda}\frac{\lambda^{i+j}}{(i+j)!} = \frac{(i+j)!}{i!\,j!}p^i(1-p)^j e^{-\lambda(p+1-p)}\frac{\lambda^{i+j}}{(i-j)!} =$$

$$= e^{-\lambda p}\frac{p^i\lambda^i}{i!}e^{-\lambda(1-p)}$$

. . . .

$$\Rightarrow X\sim Pois(\lambda p)$$
$$Y\sim Pois(\lambda(1-p))$$

$$\mathbb{P}(X=i,\ Y=j) = \mathbb{P}(X=i)\mathbb{P}(Y=j),\ \forall i,j \qquad X\perp Y$$

Media unei fct de v.a

X v.a. $\Omega \to \mathbb{R}$ $\quad g:\mathbb{R}\to\mathbb{R}$

$$\mathbb{E}[g(X)] = \sum_{x} g(x)\,\mathbb{P}(X=x)$$

$X, Y: \Omega \to \mathbb{R}$, $g:\mathbb{R}^2\to\mathbb{R}$

$$\boxed{\mathbb{E}[g(X,Y)] = \sum_{x,y} g(x,y)\,\mathbb{P}(X=x, Y=y)}$$

X, Y

$$\mathbb{E}[XY] = \sum_{x,y} xy\;\mathbb{P}(X=x, Y=y)$$

Ex: Rep marginale X, Y, rep $X|Y=0$

$Y|X=1$, $\mathbb{E}[X,Y]$, $\mathbb{E}[2X+3Y]$

$X \backslash Y$	-1	0	2	Σ
1	1/18	3/18	2/18	1/3
2	2/18	0	3/18	1/3
3	0	4/18	3/18	1/3
	3/18	7/18	8/18	

$$X \sim \begin{pmatrix} 1 & 2 & 3 \\ 6/18 & 5/18 & 7/18 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & 2 \\ 3/18 & 7/18 & 8/18 \end{pmatrix}$$

$$X|Y=0 \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{3/18}{7/18} & 0 & \frac{4/18}{7/18} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \frac{3}{7} & 0 & \frac{4}{7} \end{pmatrix}$$

$$Y|X=1 \sim \begin{pmatrix} -1 & 0 & 2 \\ \frac{1/18}{6/18} & \frac{3/18}{6/18} & \frac{2/18}{6/18} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 \\ \frac{1}{6} & \frac{3}{6} & \frac{2}{6} \end{pmatrix}$$

$$\mathbb{E}[XY] = \frac{1}{18}(-1) + 2\cdot\frac{2}{18} + (-2)\cdot\frac{2}{18} + 4\cdot\frac{3}{18} + 6\cdot\frac{3}{18}$$

$$\mathbb{E}[2X+3Y] = 2\,\mathbb{E}[X] + 3\,\mathbb{E}[Y]$$

$$= \sum (2x+3y)\,\mathbb{P}(X=x, Y=y)$$