Seminar 1 with solutions

Exercise 1. Determine which of the following claims are true for any two nonempty finite sets $A, B \subseteq \mathbb{N}$, and prove your claim.

(i) $A \cap B \subseteq A \cup B$;

(ii) $A \cap B \subsetneq A \cup B$;

(iii) $|A \cup B| = |A| + |B|$;

(iv) $|A \times B| = |A| \cdot |B|$;

(v) $|A^n| = |A|^n$, for any $n \geq 1$, where $A^n = A \underbrace{\times \ldots \times}_{n \text{ times}} A$.

Solution.

(i) The claim is true. Let $x \in A \cap B$. Then $x \in A$ **and** $x \in B$, therefore $x \in A$ **or** $x \in B$ which means that $x \in A \cup B$.

(ii) The claim is false. We provide a counterexample. Choose $A = B = \{1, 2, 3\}$. Then $A \cap B = A \cup B$, showing that the strict inclusion is in general false.

(iii) The claim is false. Take $A = \{1, 2\}$ and $B = \{2, 3\}$ as a counterexample.

(iv) The claim is true, and it follows from the definition of the cartesian product.

(v) The claim is true; we prove it by induction on n.
$\underline{n = 1}$: we obviously have that $\left|A^1\right| = |A|^1$.
$\underline{n \to n + 1}$: we assume the claim to be true for some n and we prove it for $n + 1$. We have

$$\left|A^{n+1}\right| = |A^n \times A| \overset{\text{by (iv)}}{=} |A^n| \cdot |A| \overset{\text{by the induction hyp.}}{=} |A|^n \cdot |A| = |A|^{n+1}.$$

Exercise 2. Is the proof given for the claim below correct? If not, why?

Claim. *For any set of n horses, where $n \geq 1$, all horses are of the same color.*

Proof. We argue by induction on n.
$\underline{n = 1}$: Obvious.
$\underline{n \to n+1}$:
Let $n \in \mathbb{N}$ and assume that in any set of n horses, all horses have the same color. Let H be a set of $n+1$ horses. We show that all horses in H have the same color. Pick some horse $h \in H$. Clearly, the set $H \setminus \{h\}$ has n elements, and thus, by the induction hypothesis, any two horses $H \setminus \{h\}$ have the same color. Pick some other $h' \neq h \in H$. By the same argument, all horses $H \setminus \{h'\}$ (a set which clearly contains h) have the same color. It follows that h has to have the same color as all the other horses in H, concluding the proof.

Solution. The error in the proof occurs in the inductive step $1 \to 2$ (note that this is different from the base case $n = 1$, which indeed holds trivially). At this step, the induction hypothesis reads as '*any horse has the same color as itself*'. If we consider some arbitrary set $H = \{h_1, h_2\}$ of two horses, the reasoning in the proof fails, because we cannot conclude from this that h_1 and h_2 have the same color.

Exercise 3. Give examples of relations R between elements of a set A such that:

(i) R is reflexive and symmetric but not transitive;

(ii) R is reflexive and transitive but not symmetric;

(iii) R is symmetric and transitive but not reflexive.

Solution.

(i) Take $A = \mathbb{R}$ and $xRy \Leftrightarrow |x - y| \leq 1$;

(ii) Take $A = \mathbb{N}$ and $xRy \Leftrightarrow x \leq y$;

(iii) Take $A = \mathbb{N}$ and $xRy \Leftrightarrow xy$ is odd.

Exercise 4. Prove or disprove the following claim: if G is a graph with n nodes, where $n \geq 2$, then it has at least two nodes that have the same degree.

Solution. Let G be a graph with n nodes for $n \geq 2$. It is clear that the degree of a node is less than or equal $n-1$, so the possible values for a degree are 0, 1, ..., $n-1$.
Claim: G cannot have both a node with degree 0 and a node with degree $n-1$.
Proof of claim: If G has a node with degree 0, that is, one that is not connected to any other node, than there cannot be a node of degree $n-1$, because that one would have to be connected to all other nodes.

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It follows that the degrees of the nodes in G are contained either in $\{0, 1, \ldots, n-2\}$ or in $\{1, 2, \ldots, n-1\}$, both sets of cardinality $n-1$. By the pigeonhole principle, at least two of the n nodes have have the same value.