Media și momentele de ordin superior

<u>Def</u>: Fie $(\Omega, \mathcal{F}, \mathbb{P})$ un câmp de probabilitate și $X : \Omega \to \mathbb{R}$ v.a. discretă, definim media

$$\mathbb{E}[X] = \sum_x x\,\mathbb{P}(X = x) = \sum_x x f(x)$$

ori de câte ori $\sum |x| f(x) < \infty$. În cazul în care seria este ∞ atunci spunem că v.a. X <u>nu</u> are medie.

Exp: Aruncăm cu un zar

$X \in \{1, 2, 3, 4, 5, 6\}$

$\mathbb{P}(X = x) = 1/6$, deci $\mathbb{E}[X] = \sum_x x \mathbb{P}(X = x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \dots + 6 \cdot \frac{1}{6}$

$= 3,5$

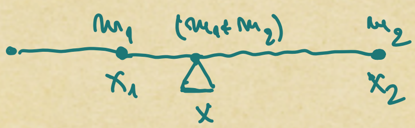
Exp: $X \sim \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$

$\mathbb{E}[X] = x_1 \cdot p_1 + x_2 p_2 + \dots + x_n \cdot p_n$

$X \sim \begin{pmatrix} -1 & 0 & 1 \\ 1/2 & 1/4 & 1/4 \end{pmatrix}$

$\Rightarrow \mathbb{E}[X] = -\frac{1}{2} + \frac{1}{4} = -\frac{1}{4}$

<u>Interpretare fizică</u>



$x M = x_1 m_1 + x_2 m_2$

$\bar{x} = x_1 \cdot \frac{m_1}{M} + x_2 \cdot \frac{m_2}{M}$

Proprietăți: 1) Dacă X este constantă $\Rightarrow$ media este acea constantă ($X = c \Rightarrow \mathbb{E}[X] = c$)

2) Dacă $X \geq 0$ atunci și $\mathbb{E}[X] \geq 0$ (pozitivitate)

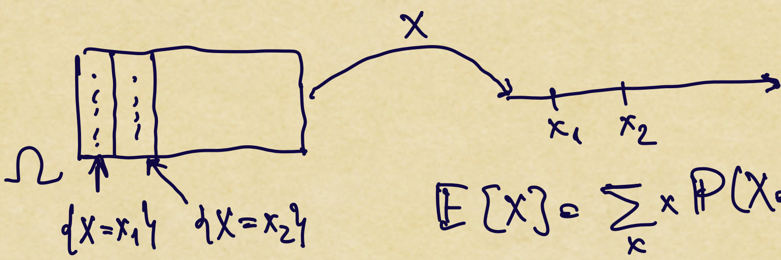
3) Dacă $X \geq Y$ atunci $\mathbb{E}[X] \geq \mathbb{E}[Y]$ (monotonie)

$(X(\omega) \geq Y(\omega)\ (\forall)\, \omega \in \Omega)$

4) Dacă X și Y v.a. discrete și $a, b \in \mathbb{R}$ atunci

$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\,\mathbb{E}[Y]$ (liniaritate)

Dem: $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$



$\{X = x\} = X^{-1}(x)$

$\mathbb{E}[X] = \sum_x x\,\mathbb{P}(X = x)$

$\{X = x\} = \{\omega \in \Omega \mid X(\omega) = x\}$

$\Rightarrow \mathbb{P}(X = x) = \sum_\omega \mathbb{P}(\{\omega\})$

Exp: Aruncăm cu o monedă de 10 ori

$X = \{$ nr de H din cele 10 aruncări $\}$

$\Omega = \{H, T\}^{10} \quad X \sim B(10, p)$

$$\mathbb{P}(X=k) = \binom{10}{k} p^k (1-p)^{10-k}$$

$\{X = k\} = \{(\omega_1, \dots, \omega_{10}) \mid \omega_i \in \{H, T\}$ și exact k sunt $H\}$

Deci, $$\mathbb{E}[X+Y] = \sum_{\omega} (X+Y)(\omega)\, \mathbb{P}(\{\omega\}) = \sum_{\omega} (X(\omega) + Y(\omega))\, \mathbb{P}(\{\omega\})$$
$$= \sum_{\omega} X(\omega)\, \mathbb{P}(\{\omega\}) + \sum_{\omega} Y(\omega)\, \mathbb{P}(\{\omega\})$$

5) (legătura dintre medie și prob.)

Fie $A \in \mathcal{F}$ eveniment

$$\mathbb{1}_A = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases} \qquad \mathbb{P}(\mathbb{1}_A = 1) = \mathbb{P}(A)$$

$$\mathbb{1}_A \sim \begin{pmatrix} 0 & 1 \\ 1-\mathbb{P}(A) & \mathbb{P}(A) \end{pmatrix} \qquad \mathbb{E}(\mathbb{1}_A) = \mathbb{P}(A)$$

6) Fie X v.a discretă și $g: \mathbb{R} \to \mathbb{R}$ și $Y = g(X)$. Atunci:

$$\mathbb{E}[g(X)] = \sum_{x} g(x)\, \mathbb{P}(X=x)$$

$$\mathbb{E}[Y] = \sum_{y} y\, \mathbb{P}(Y=y)$$

$$\begin{tikzcd} \Omega \arrow[r, "X"] \arrow[rr, bend right, "g \circ X"'] & \mathbb{R} \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

$$\mathbb{P}(Y=y) = \mathbb{P}(g(X)=y) = \mathbb{P}(X \in g^{-1}(y))$$

$$g^{-1}(y) = \{x \mid g(x) = y\} \qquad = \sum_{x \in g^{-1}(y)} \mathbb{P}(X=x)$$

$$\mathbb{E}[g(X)] = \mathbb{E}[Y] = \sum_{y} y \sum_{x \in g^{-1}(y)} \mathbb{P}(X=x) = \sum_{x} g(x)\, \mathbb{P}(X=x)$$

Exp: $X \sim \begin{pmatrix} -2 & -1 & 1 & 3 \\ 1/4 & 1/8 & 1/2 & 1/8 \end{pmatrix} \quad Y = X^2$ (Pentru examen!)

Met 1: $Y \in \{1, 4, 9\}$

$$\mathbb{P}(Y=1) = \mathbb{P}(X=-1) + \mathbb{P}(X=1) = 1/8 + 1/2 = 5/8$$

$$Y = X^2 \sim \begin{pmatrix} 1 & 4 & 9 \\ \frac{5}{8} & \frac{2}{8} & \frac{1}{8} \end{pmatrix}$$

$$\mathbb{E}[X^2] = \frac{5}{8} + \frac{8}{8} + \frac{9}{8} = \frac{22}{8}$$

Met 2: $g(x) = x^2$

$$\mathbb{E}[X^2] = (-2)^2 \cdot \frac{1}{4} + (-1)^2 \frac{1}{8} + 1^2 \cdot \frac{1}{2} + 3^2 \cdot \frac{1}{8} = \frac{22}{8}$$

7) Fie X și Y 2 v.a. independente. Atunci

$$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$$

Dacă g, h - funcții atunci $g(X)$ și $h(Y)$ sunt independente
at.

$$\mathbb{E}[g(X) \cdot h(Y)] = \mathbb{E}[g(X)] \cdot \mathbb{E}[h(Y)]$$

$Y \sim \begin{pmatrix} 0 & 2 \\ 1/2 & 1/2 \end{pmatrix}$ $X \perp\!\!\!\perp Y$

$$\mathbb{E}[X^7 Y^9] = \mathbb{E}[X^7] \cdot \mathbb{E}[Y^9]$$

Obs! În general, $\mathbb{E}[X \cdot Y] \neq \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Def: Fie X o v.a. discretă, numim moment de ordin $k \geq 1$, ca fiind $\mathbb{E}[X^k]$.

Se numește moment de ordin k centrat în $a \in \mathbb{R}$, $\mathbb{E}[(X-a)^k]$ și moment central de ordin k, $\mathbb{E}[(X-\mathbb{E}[X])^k]$

Def: Varianța sau dispersia v.a. X este momentul centrat de ordin 2 și se notează cu:

$$Var(X) = \mathbb{E}[(X-\mathbb{E}[X])^2]$$

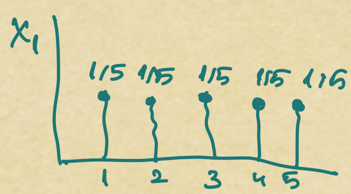
Obs!: Arată gradul de împrăștiere a obs. față de medie.

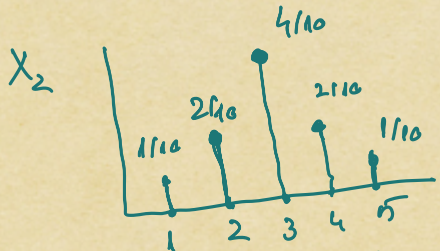
Exp: $X_1 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1/5 & 1/5 & 1/5 & 1/5 & 1/5 \end{pmatrix}$ $X_3 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix}$

$X_2 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1/10 & 2/10 & 4/10 & 2/10 & 1/10 \end{pmatrix}$ $X_4 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$

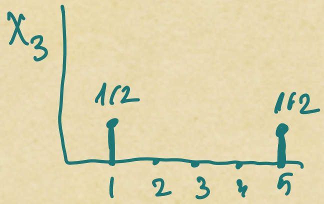
$$\mathbb{E}[X_1] = \mathbb{E}[X_2] = \mathbb{E}[X_3] = \mathbb{E}[X_4] = 3$$

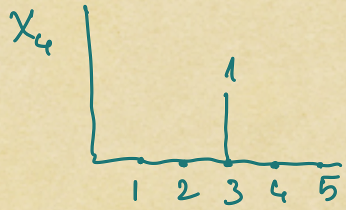
$$Var(X_1) = \mathbb{E}[(X_1-3)^2] = \frac{(1-3)^2}{5} + \frac{(2-3)^2}{5} + \dots + \frac{(5-3)^2}{5} = \frac{10}{5} = 2$$





$$Var(X_2) = \mathbb{E}[(X_2-3)^2] = \frac{(1-3)^2}{10} + \frac{(2-3)^2 \cdot 2}{10} + \cancel{\frac{(3-3)^2 \cdot 4}{10}} + \frac{(4-3)^2 \cdot 2}{10} + \frac{(5-3)^2}{10} = \frac{12}{10}$$





$Var(X_4) = 0.$

$$Var(X_3) = \frac{(1-3)^2}{2} + \frac{(5-3)^2}{2} = \frac{8}{2} = 4$$

Propietăți ale varianței: 1) Dacă $X = c$ (cst.) $\Rightarrow Var(X) = 0.$

2) $Var(X) \geq 0$

(translație) 3) Dacă X v.a. și $a \in \mathbb{R}$ atunci $Var(a+X) = Var(X)$

(scalare) 4) Dacă X v.a. și $b \in \mathbb{R}^*$ atunci $Var(b \cdot X) = b^2 \cdot Var(X)$

$$Var(bX) = \mathbb{E}[(bX - \mathbb{E}[bX])^2]$$
$$= \mathbb{E}[b^2(X - \mathbb{E}[X])^2]$$

5) $Var(a+bX) = b^2 \cdot Var(X)$; $(\forall)\ a, b \in \mathbb{R}$

6) $Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$

$$\mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2 - 2X\mathbb{E}[X] + \mathbb{E}[X]^2]$$
$$= \mathbb{E}[X^2] - 2\,\mathbb{E}[X\mathbb{E}[X]] + \mathbb{E}[X]^2$$
$$= \mathbb{E}[X^2] - 2\mathbb{E}[X]^2 + \mathbb{E}[X]^2$$

7) X și Y v.a. independente

$$Var(X+Y) = Var(X) + Var(Y)$$

<u>Def</u>: Fie X, Y - v.a. se numește covarianța lui X și Y:

$$Cov(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$

(dacă este 0, at. X, Y sunt necorelate)

În general, $Var(X+Y) = Var(X) + Var(Y) - 2\,Cov(X,Y)$

<u>Def</u>: Se numește abatere standard:

$$SD(X) = \sqrt{Var(X)}$$

$\sigma^2 \underset{(not.)}{=}$ Varianță

$\sigma \underset{(not.)}{=}$ Abatere standard

<u>Exemple de calcul al mediei și varianței</u>

(1) $X \sim B(p) \iff X \sim \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$

$\mathbb{E}[X] = p$

$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = p - p^2 = p(1-p)$

(2) $X \sim B(n,p)$ $\quad \mathbb{P}(X=k) = \binom{n}{k} p^k \cdot (1-p)^{n-k}$

$$\mathbb{E}[X] = \sum_{k=0}^{n} k \cdot \mathbb{P}(X=k) = \sum_{k=0}^{n} k \binom{n}{k} p^k \cdot (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \cancel{k} \cdot \frac{n!}{\underset{(k-1)!}{\cancel{k!}}\,(n-k)!} p^k \cdot (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} p^k \cdot (1-p)^{n-k}$$

$$= p \cdot n \cdot \underbrace{\boxed{\sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}}}_{= 1}$$

$$\sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}$$

$$\sum_{l=0}^{n} \binom{n-1}{l} p^{l} (1-p)^{n-1-l} = (p+1-p)^{n-1}$$

$\Rightarrow \mathbb{E}[X] = n \cdot p$

$$X = \underbrace{X_1 + \ldots + X_n}$$

$$\mathbb{E}[X] = \mathbb{E}[\downarrow] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n]$$

$$Var(X) = Var(X_1 + \ldots + X_n)$$

$$= \sum_{i=1}^{n} Var(X_i) = np(1-p)$$

(3) Hipergeometrică $HG(n, N, M)$

$$\mathbb{P}(X=k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$$

X_j - la extragerea j avem bilă neagră $X_j = 1$

bilă albă $X_j = 0$

$X = X_1 + \ldots + X_n$ $\qquad \mathbb{P}(X_j = 1) = \frac{M}{N}$

Extrageri fără întoarcere

$$\mathbb{E}[X] = \mathbb{E}[X_1] + \ldots + \mathbb{E}[X_n] = n \cdot \frac{M}{N}$$

(4) Poisson $X \sim Poi(\lambda)$ $\quad \mathbb{P}(X=k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!}$

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k\,\mathbb{P}(X=k) = \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} e^{-\lambda} = \lambda$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\mathbb{E}[X^2] = \sum_{k=1}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} (k+1) \frac{\lambda^k}{k!} =$$

$$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} + \lambda e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}} = \lambda^2 + \lambda$$

$$\Rightarrow \mathrm{Var}(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

(5) $X \sim \mathrm{Geom}(p)$

$$\mathbb{P}(X=k) = (1-p)^{k-1} \cdot p$$

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k \cdot p(1-p)^{k-1} = \sum_{k=1}^{\infty} k \cdot p q^{k-1} = p \cdot \sum_{k=1}^{\infty} k \cdot q^{k-1}$$

$$= p \cdot \sum_{k=1}^{\infty} (q^k)'$$

$$= p \cdot \left(\sum_{k=1}^{\infty} q^k\right)'$$

$$= p \cdot \left(\frac{q}{1-q}\right)' = \frac{p}{(1-q)^2}$$

$$= \frac{1}{p}$$

$$\mathrm{Var}(X) = \frac{1-p}{p^2}$$

<u>Variabile aleatoare continue</u>

<u>Def</u>: $(\Omega, \mathcal{F}, \mathbb{P})$ un c.p. și $X: \Omega \to \mathbb{R}$ v.a. V.A. X este continuă (absolut continuă) dacă există $f: \mathbb{R} \to \mathbb{R}_+$ cu prop.

$$\mathbb{P}(X \in A) = \int_A f(x)\,dx, \ (\forall) A \subseteq \mathbb{R}$$

<u>Obs</u>: Dacă $A = (a, b)$

$$\mathbb{P}(a < X < b) = \int_a^b f(x)\,dx$$

$\mathbb{P}(X \in (a,b))$

f - s.n. densitate de repartiție

(P): Dacă f e densitate de repartiție

$$1)\ f \geq 0$$

$$2)\ \int_{\mathbb{R}} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$$

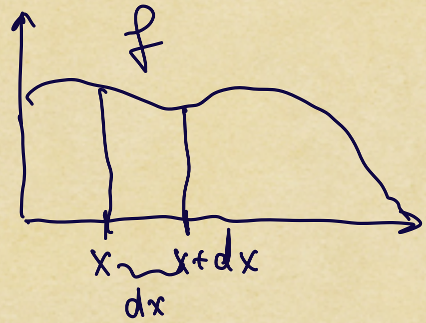
$$\mathbb{P}(X \in \mathbb{R}) = \mathbb{P}(\Omega) = 1$$

Obs! $\mathbb{P}(X = a) = \int_a^a f(x)\,dx = 0.$

$A = \{a\}$

$$\mathbb{P}(a < X < b) = \mathbb{P}(a \leq X < b)$$
$$= \mathbb{P}(a \leq X \leq b)$$
$$= \mathbb{P}(a < X \leq b)$$

Interpretare:



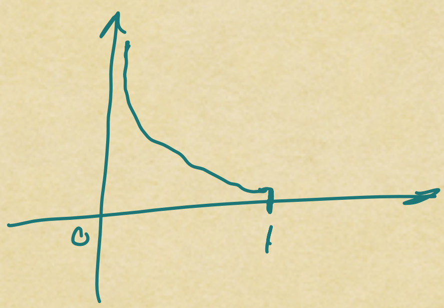
$$\mathbb{P}(X \in (x, x+dx)) = \int_x^{x+dx} f(t)\,dt$$

Dacă dx mic $f(x)(x + dx - x) = dx \cdot f(x)$

$$\Rightarrow f(x) = \frac{\mathbb{P}(X \in (x, x+dx))}{dx}$$ (Probabilitatea pe unitatea de lungime)

Exp: $f(x) = \begin{cases} \frac{1}{2\sqrt{x}}, & 0 < x < 1 \\ 0, & \text{altfel} \end{cases}$

$f(x) \geq 0$



$$\int_{\mathbb{R}} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^1 \frac{1}{2\sqrt{x}}\,dx = 1$$

$\sum$ (discret) $\longrightarrow \int$ (continuu)

$f(x) = \mathbb{P}(X = x) \longrightarrow 0$

$$\mathbb{P}(X \in A) = \sum_{x \in A} \mathbb{P}(X = x) \longrightarrow \int_A f(x)\,dx$$