LOGICĂ MATEMATICĂ și COMPUTAȚIONALĂ, SEMINAR DIN LOGICA PROPOZIȚIONALĂ CLASICĂ

MNEMONIC DIN CURS

Considerăm algebra Boole standard:

$$\mathcal{L}_2 = (L_2, \vee, \wedge, \bar{\ }, \leq, 0, 1).$$

$L_2 = \{0, 1\}$, cu $0 \neq 1$

<u>Def.:</u> <u>Interpretare</u> (<u>evaluare</u>, <u>semantică</u>): $h : V \to L_2$.

↳ "dă valori de adevăr variabilelor propoziționale."

<u>Prop.:</u> $\forall h : V \to L_2$, $\exists !\, \tilde{h} : E \to L_2$, $\tilde{h} \overset{not.}{=} h$, numită tot <u>interpretare</u> (<u>evaluare</u>, <u>semantică</u>)

a. î.:

$$\begin{tikzcd} V \arrow[r, hook] \arrow[dr, "h"'] & E \arrow[d, "\tilde{h}"] \\ & L_2 \end{tikzcd}$$

(a) $(\forall u \in V)(\tilde{h}(u) = h(u))$,

(i.e. $\tilde{h}|_V = h$).

(b) $(\forall \varphi \in E)(\tilde{h}(\neg\varphi) = \overline{\tilde{h}(\varphi)})$;

(c) $(\forall \varphi, \psi \in E)(\tilde{h}(\varphi \to \psi) = \tilde{h}(\varphi) \to \tilde{h}(\psi))$

Obs.: $\forall\, h: V \to L_2$, $\forall\, \varphi, \psi \in E$, au loc:

$$(d)\quad \tilde{h}(\varphi \vee \psi) = \tilde{h}(\varphi) \vee \tilde{h}(\psi),$$

$$(e)\quad \tilde{h}(\varphi \wedge \psi) = \tilde{h}(\varphi) \wedge \tilde{h}(\psi),$$

$$(f)\quad \tilde{h}(\varphi \leftrightarrow \psi) = \tilde{h}(\varphi) \leftrightarrow \tilde{h}(\psi).$$

Dem.:

$$(d)\quad \tilde{h}(\varphi \vee \psi) = \tilde{h}(\neg\varphi \to \psi) \overset{(c)}{=}$$

$$= \tilde{h}(\neg\varphi) \to \tilde{h}(\psi) \overset{(b)}{=}$$

$$= \overline{\tilde{h}(\varphi)} \to \tilde{h}(\psi) = \overline{\overline{\tilde{h}(\varphi)}} \vee \tilde{h}(\psi) =$$

$$= \tilde{h}(\varphi) \vee \tilde{h}(\psi).$$

$$(e)\quad \tilde{h}(\varphi \wedge \psi) = \tilde{h}(\neg(\varphi \to \neg\psi)) \overset{(b),(c)}{=}$$

$$= \overline{\tilde{h}(\varphi) \to \overline{\tilde{h}(\psi)}} = \overline{\overline{\tilde{h}(\varphi)} \vee \overline{\tilde{h}(\psi)}} =$$

$$= \overline{\overline{\tilde{h}(\varphi)}} \wedge \overline{\overline{\tilde{h}(\psi)}} = \tilde{h}(\varphi) \wedge \tilde{h}(\psi).$$

$$(f)\quad \tilde{h}(\varphi \leftrightarrow \psi) = \tilde{h}((\varphi \to \psi) \wedge (\psi \to \varphi))$$

$$\overset{(e)}{=} \tilde{h}(\varphi \to \psi) \wedge \tilde{h}(\psi \to \varphi) \overset{(c)}{=}$$

$$= (\tilde{h}(\varphi) \to \tilde{h}(\psi)) \wedge (\tilde{h}(\psi) \to \tilde{h}(\varphi)) =$$

$$= \tilde{h}(\varphi) \leftrightarrow \tilde{h}(\psi).$$

Def.: $h: V \to L_2$; $\varphi \in E$; $\Sigma \subseteq E$.

$$h \vDash \varphi \overset{def.}{\Longleftrightarrow} \tilde{h}(\varphi) = 1.$$

$$\vDash \varphi \overset{def.}{\Longleftrightarrow} (\forall\, f: V \to L_2)(f \vDash \varphi).$$

$h \vDash \Sigma \Leftrightarrow (\forall \gamma \in \Sigma)(h \vDash \gamma)$.

$\Sigma \vDash \varphi \Leftrightarrow (\forall f : V \to L_2)$
$(f \vDash \Sigma \Rightarrow f \vDash \varphi)$.

Obs.: $\varphi \in E$. Atunci: $\vDash \varphi \Leftrightarrow \emptyset \vDash \varphi$.

Dem.:

$\emptyset \vDash \varphi \overset{(def)}{\Leftrightarrow} (\forall f : V \to L_2)(f \vDash \emptyset \Rightarrow$
$\Rightarrow f \vDash \varphi)$. (I)

$(\forall f : V \to L_2)(f \vDash \emptyset)$.

Fie $h : V \to L_2$.

$h \vDash \emptyset \overset{(def)}{\Leftrightarrow} (\forall \gamma \in \emptyset)(h \vDash \gamma) \Leftrightarrow$

$\Leftrightarrow \underbrace{(\forall \gamma)(\underbrace{\gamma \in \emptyset}_{\text{fals}, \forall \gamma} \Rightarrow h \vDash \gamma)}_{\text{adevărat}, \forall \gamma} \to$ adevărat (*)

Cf. (I), avem (scriem echivalent „$\Rightarrow$"):

$\emptyset \vDash \varphi \Leftrightarrow (\forall f : V \to L_2)$
$(\underbrace{f \nvDash \emptyset}_{\text{fals}, \forall f : V \to L_2} \text{ sau } f \vDash \varphi) \Leftrightarrow$

$\Leftrightarrow (\forall f : V \to L_2)(f \models \varphi) \overset{(def.)}{\Leftrightarrow} \models \varphi$.

Teor.: $\varphi \in E$, $\Sigma \subseteq E$. Atunci:

(TC) $\vdash \varphi \Leftrightarrow \models \varphi$.

(TCT) $\Sigma \vdash \varphi \Leftrightarrow \Sigma \models \varphi$.

Obs.: Știm că, pt. orice $\varphi \in E$, $\vdash \varphi \Leftrightarrow \emptyset \vdash \varphi$. Din acest fapt și obs. anterioară, se observă că (TC) este caz particular al (TCT) (cazul $\Sigma = \emptyset$).

Corolar al (TC): Noncontradicția logicii propoziționale clasice:

$(\nexists \varphi \in E)(\vdash \varphi$ și $\vdash \neg\varphi)$.

Def.: $\Sigma \subseteq E$. ex.: $\{\varphi, \neg\varphi\}$, $\varphi \in E$; ex.: E

Σ → inconsistentă $\overset{def}{\Leftrightarrow} (\forall \varphi \in E)(\Sigma \vdash \varphi)$;

Σ → consistentă $\overset{def}{\Leftrightarrow} (\exists \varphi \in E)(\Sigma \nvdash \varphi)$.

ex.: $\emptyset$, mulț. axiomelor, T. (v. corolarul anterior) → (și obs.: Σ → consist. $\Leftrightarrow (\nexists \varphi \in E)(\Sigma \vdash \varphi$ și $\Sigma \vdash \neg\varphi)$).

Prop.*: $\Sigma \subseteq E$, $\varphi \in E$. Atunci:

(1) $\Sigma \cup \{\varphi\} \longrightarrow$ inconsistentă $\Leftrightarrow$
$\Leftrightarrow \Sigma \vdash \neg\varphi$;

(2) $\Sigma \cup \{\neg\varphi\} \longrightarrow$ inconsistentă $\Leftrightarrow$
$\Leftrightarrow \Sigma \vdash \varphi$.

Prop.:** $\Sigma \subseteq E$. Atunci: $\Sigma \longrightarrow$ consistentă $\Leftrightarrow$
$\Leftrightarrow (\exists h : V \longrightarrow L_2)(h \vDash \Sigma)$.

DEMONSTRAŢII SEMANTICE

Exerc.: Considerăm următorul text:

(a) dacă nu plouă atunci, în cazul când ies la plimbare, nu trec pe la cafenea.

(b) dacă nu plouă, atunci ies la plimbare.

(c) Trec pe la cafenea.

(d) Plouă.

Să se dem. că din (a), (b), (c) se deduce (d).

REZOLVARE: (V. ALTE DOUĂ METODE LA SF. ACESTUI §!)

Considerăm trei variabile propoziţionale $\underbrace{p, q, r}_{\text{2 câte 2 distincte}} \in V$, cărora le dăm următoarele valori:

p: "plouă"; q: "ies la plimbare";
r: "trec pe la cafenea".

Atunci frazelor de la punctele (a), (b), (c), (d) le corespund urm. enunțuri:

(a): $\varphi := \neg p \to (q \to \neg r) \in E$

(b): $\psi := \neg p \to q \in E$

(c): $r \in E$

(d): $p \in E$.

Avem de dem. că: $\{\varphi, \psi, r\} \models p$. Not. $\Sigma := \{\varphi, \psi, r\} \subset E$.

$\Sigma \models p$.

Fie $h : V \to \mathcal{L}_2$, a.î. $h \models \Sigma \Leftrightarrow$

$$\Leftrightarrow \begin{cases} 1 = \tilde{h}(\varphi) = \overline{\tilde{h}(p)} \to \left(\tilde{h}(q) \to \overline{\tilde{h}(r)}\right) \\ 1 = \tilde{h}(\psi) = \overline{\tilde{h}(p)} \to \tilde{h}(q) \\ 1 = \tilde{h}(r) \end{cases}$$

$$\Rightarrow 1 = \overline{\tilde{h}(p)} \to \left(\tilde{h}(q) \to \overline{1}\right) = \overline{\tilde{h}(p)} \to \left(\tilde{h}(q) \to 0\right) = \overline{\overline{\tilde{h}(p)}} \vee \overline{\tilde{h}(q)} \vee 0 = \tilde{h}(p) \vee \overline{\tilde{h}(q)}. \quad (*)$$

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Dar avem și:

$$1 = \overline{\tilde{h}(p)} \to \tilde{h}(q) = \overline{\overline{\tilde{h}(p)}} \vee \tilde{h}(q) = \tilde{h}(p) \vee \tilde{h}(q) \quad (**)$$

$$(*), (**) \Rightarrow 1 = 1 \wedge 1 = \left(\tilde{h}(p) \vee \overline{\tilde{h}(q)}\right) \wedge \left(\tilde{h}(p) \vee \tilde{h}(q)\right) = \tilde{h}(p) \vee \left(\overline{\tilde{h}(q)} \wedge \tilde{h}(q)\right) = \tilde{h}(p) \vee 0 = \tilde{h}(p).$$

Deci $\tilde{h}(p) = 1$, prin urmare $\Sigma \vDash p$.

METODA II DE REZOLVARE PENTRU Exerc. / pg. 6:

Folosind notațiile p, q, r, φ, ψ din prima metodă:
Fie $h : V \to L_2$, arbitrară. După cum arată tabelul semantic alăturat, dacă $h \vDash \{\varphi, \psi, r\}$, adică $\tilde{h}(\varphi) = \tilde{h}(\psi) = \tilde{h}(r) = 1$, atunci $\tilde{h}(p) = 1$.

Așadar:
$\{\varphi, \psi, r\} \vDash p$

(Ne interesează doar valorile lui h în $p, q, r \in V$;
$\exists$ o infinitate de interpretări h cu $\tilde{h}(\varphi) = \tilde{h}(\psi) \overset{(h : V \to L_2)}{=} \tilde{h}(r) = 1$, toate acestea cu $\tilde{h}(p) = 1$).

$\tilde{h}(p)$	1	1	1	1	0	0	0	0
$\tilde{h}(q)$	1	1	0	0	1	1	0	0
$h(r) = \tilde{h}(r)$	1	0	1	0	1	0	1	0
$\tilde{h}(\neg p)$	0	0	0	0	1	1	1	1
$\tilde{h}(\neg r)$	0	1	0	1	0	1	0	1
$\tilde{h}(q \to \neg r)$	0	1	1	1	0	1	1	1
$\tilde{h}(\psi)$	1	1	1	1	1	1	0	0
$\tilde{h}(\varphi)$	1	1	1	1	0	1	1	1

METODA III DE REZOLVARE PENTRU EXERCIȚIUL PG. 6: Folosind notațiile p, q, r, φ, ψ din prima metodă, să demonstrăm sintactic că $\{\varphi, \psi, r\} \vdash p$.

$\{\varphi, \psi, r\} \vdash \varphi = \neg p \rightarrow (q \rightarrow \neg r)$

$\{\varphi, \psi, r\} \vdash \psi = \neg p \rightarrow q$

$\{\varphi, \psi, r\} \vdash (\neg p \rightarrow (q \rightarrow \neg r)) \rightarrow$
$\rightarrow ((\neg p \rightarrow q) \rightarrow (\neg p \rightarrow \neg r))$ (A_2)

$\{\varphi, \psi, r\} \vdash (\neg p \rightarrow q) \rightarrow (\neg p \rightarrow \neg r)$ (MP)

$\{\varphi, \psi, r\} \vdash \neg p \rightarrow \neg r$ (MP)

$\{\varphi, \psi, r\} \vdash (\neg p \rightarrow \neg r) \rightarrow (r \rightarrow p)$ (A_3)

$\{\varphi, \psi, r\} \vdash r \rightarrow p$ (MP)

$\{\varphi, \psi, r\} \vdash r$.

$\{\varphi, \psi, r\} \vdash p$ (MP).

Așadar, într-adevăr, avem:

$\{\varphi, \psi, r\} \vdash p \overset{(TCT)}{\Longleftrightarrow} \{\varphi, \psi, r\} \vDash p$.