Câmp de probabilitate. Operații cu evenimente
Formule de calcul

exp. aleator $\rightarrow (\Omega, \mathcal{F})$

Ω: sp. stărilor

mulțimea ev. elementare

$\mathcal{F} \subseteq \mathcal{P}(\Omega)$ mulțimea ev. posibile

$$\begin{cases} a)\ \Omega \in \mathcal{F} \\ b)\ A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F} \\ c)\ A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F} \\ c')\ (A_n)_n \subset \mathcal{F} \Rightarrow \bigcup_n A_n \in \mathcal{F} \end{cases}$$

a), b), c) $\Rightarrow$ algebră

a), b) + c') $\Rightarrow$ σ-algebră (sigma)

$$\mathbb{P}: \mathcal{F} \rightarrow [0,1]$$
$$A \rightarrow p$$

Pp. că avem un experiment aleator și un eveniment A de interes. Repetăm experimentul (în condiții similare) de un nr. mare de ori N.

Notăm $N(A)$ - nr de realizări ale lui A.

$\frac{N(A)}{N}$ - frecv. relativă de realizare a lui A

$N(A) \in \{0, \dots N\}$

$\frac{N(A)}{N} \in [0,1]$

$\mathbb{P}(A) \approx \lim_{N \to \infty} \frac{N(A)}{N}$

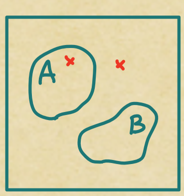
$\mathbb{P}(A) \in [0,1]$

Dacă $A = \Omega \Rightarrow N(A) = N \Rightarrow \frac{N(\Omega)}{N} = 1 \Rightarrow \mathbb{P}(\Omega) = 1$

$P(A) \in [0,1]$

$\mathbb{P}(\Omega) = 1$

Pp $A, B \in \mathcal{F}$, $A \cap B = \phi$



Ω

$A, B \in \mathcal{F}$, $A \cap B = \phi$

$A \cup B \in \mathcal{F}$

$N(A \cup B) = \underbrace{N(A) + N(B)}_{\text{pt că } A, B \text{ disjuncte}}$ $\Big/ : N$

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ (finit aditivitate)

Def: O funcție $\mathbb{P}: \mathcal{F} \to [0,1)$ care verifică

a) $\mathbb{P}(\Omega)=1$

b) $(\forall)\,(A_n)_n \subseteq \mathcal{F}$ disjuncte două câte două (σ-aditivitate)

$$\mathbb{P}\Big(\bigcup_n A_n\Big) = \sum_{i=1}^{n} \mathbb{P}(A_n)$$

s.n. măsură de probabilitate pe $(\Omega, \mathcal{F})$ (pe scurt, probabilitate)

Experiment aleator $\to$ $(\Omega, \mathcal{F}, \mathbb{P})$ câmp de probabilitate

Exp: a) Aruncatul cu banul

$\Omega = \{H, T\}$

$\mathcal{F} = \mathcal{P}(\Omega) = \{\emptyset, \{H\}, \{T\}, \Omega\}$

$\mathbb{P}: \mathcal{F} \to [0,1)$

$\mathbb{P}(\Omega) = 1$, $\mathbb{P}(\emptyset) = 0$

$\mathbb{P}(\{H\}) = p \in [0,1) \Rightarrow \mathbb{P}(\{T\}) = 1-p$

pt o monedă echilibrată: $p = 1/2$

b) Aruncatul cu zarul

$\Omega = \{1,2,3,4,5,6\}$

$\mathcal{F} = \mathcal{P}(\Omega)$ $\quad 2^6$ elemente

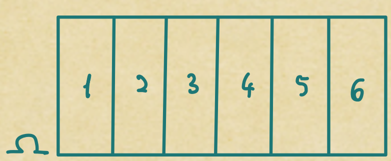
$\{0,1\}^\Omega = \{f: \Omega \to \{0,1\}\}$

$\hookrightarrow A^B = \{f: B \to A\}$

$\mathbb{P}: \mathcal{F} \to [0,1)$

$\mathbb{P}(\Omega) = 1$, $\mathbb{P}(\emptyset) = 0$

$\mathbb{P}(\{i\}) = p_i \in [0,1)$, $i \in \{1, \dots, 6\}$



$\Omega = \{1\} \cup \{2\} \cup \dots \cup \{6\}$

Prop: a) $P(\phi) = 0$

$\Omega \cup \phi = \Omega \Rightarrow P(\Omega \cup \phi) = P(\Omega) = 1$

$\Omega \cap \phi = \phi \qquad \underbrace{\mathbb{P}(\Omega)}_{1} + \mathbb{P}(\phi) = 1 \Rightarrow \mathbb{P}(\phi) = 0$

(aproape bine)

Riguros:

$$\left.\begin{array}{ll} A_n = \phi & \text{Pp } \mathbb{P}(\phi) > 0 \\ \bigcup_n A_n = \phi & \text{b) } \underbrace{\mathbb{P}(\phi)} = \underbrace{\sum_n \mathbb{P}(\phi)}_{\infty} \end{array}\right\} \text{contradicție}$$

b) $\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i)$, $A_1, \ldots, A_n$ disjuncte 2 câte 2

c) $A \in \mathcal{F} \Rightarrow \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$

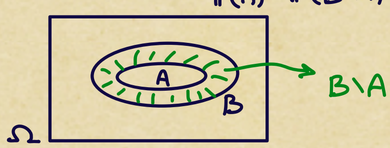
$A \cap A^c = \phi$

$A \cup A^c = \Omega \Rightarrow \mathbb{P}(A \cup A^c) = \mathbb{P}(\Omega) = 1$

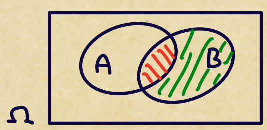
$\mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c)$

d) $A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

$\mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A)$



e) $A, B \in \mathcal{F}$, $\mathbb{P}(A \cup B) = ?$



$A \cup B = A \cup (B \setminus A)$

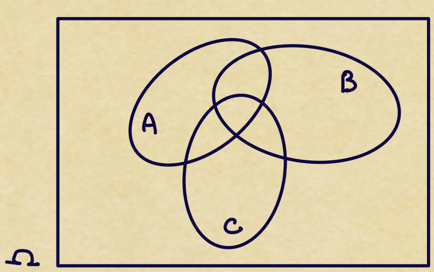
$A \cap (B \setminus A) = \phi$

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A)$

$= \mathbb{P}(A) + \underbrace{\mathbb{P}(B \setminus (A \cap B))}_{\mathbb{P}(B) - \mathbb{P}(A \cap B)}$

e') $A, B, C \qquad \mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap C) - \mathbb{P}(B \cap C) - \mathbb{P}(A \cap B) +$

$+ \mathbb{P}(A \cap B \cap C)$



f) formula lui Poincaré

$$\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) + \ldots +$$
$$+ (-1)^{n+1} \mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n)$$

$$\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$$
$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$$

$\mathbb{P}(\{H\}) = p \in (0,1)$

$A = \{$ va pica H mai devreme sau mai târziu $\}$

$\mathbb{P}(A) = 1$

$A_n = \{$ vom obține H în n aruncări $\}$

$A = \bigcup_n A_n$

$\bigcup_n A_n = \lim_n A_n$

$\mathbb{P}(\lim_n A_n) = \lim_n \underbrace{\mathbb{P}(A_n)}_{1-(1-p)^n}$

Modelul clasic de probabilitate (Modelul lui Laplace)

Fie $N \geq 1$, $N \in \mathbb{N}$ și considerăm un experiment aleator cu N rezultate posibile

$\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$

$\mathcal{F} = \mathcal{P}(\Omega)$ (2^N elemente)

$\mathbb{P}: \mathcal{F} \to [0,1]$ $\quad \mathbb{P}(\{\omega_i\}) = \frac{1}{N}$, $i \in \{1, \ldots, N\}$ echirepartiție

Fie $A \in \mathcal{F}$

reuniune finită

$$\mathbb{P}(A) = \mathbb{P}\Big(\bigcup_{\omega_i \in A} \{\omega_i\}\Big) = \sum_{\omega_i \in A} \mathbb{P}(\{\omega_i\}) = \frac{1}{N} \sum_{\omega_i \in A} 1 = \frac{|A|}{N} = \frac{|A|}{|\Omega|}$$

$A = \{\omega_1, \omega_7, \omega_9\}$

$$= \frac{\text{nr cazuri favorabile}}{\text{nr cazuri posibile}}$$

a) Formula sumei

A, B finite disjuncte $\Rightarrow |A \cup B| = |A| + |B|$

oarecare $\Rightarrow |A \cup B| = |A| + |B| - |A \cap B|$

Principiul includerii-excluderii

$A_1, A_2, \ldots, A_n$ finite

$$|A_1 \cup A_2 \cup \ldots \cup A_n| = \sum_{i=1}^{n} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{n+1} |A_1 \cap A_2 \ldots \cap A_n|$$

Apl: $\varphi(n)$ - nr de nr prime cu $n \leq n$

fct. Euler

$$f(n) : n \prod_{p<n} \left(1 - \frac{1}{p}\right)$$

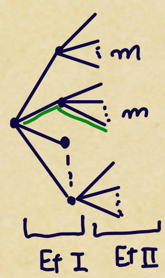
↑

Aplicație pt laborator

b) Formula produs

A, B finite $\qquad A \times B = \{(a,b) \mid a \in A, b \in B\}$

$$|A \times B| = \underset{(a,b)}{|A| \cdot |B|}$$

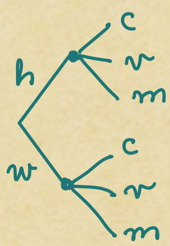


$n \times m$

Ex înghețată

arome: c, v, m - 3

cornet: h, w - 2



$6 = 2 \times 3$