

Welcome to CS166!

Why study data structures?

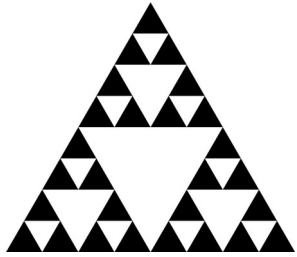
Why Study Data Structures?

- ***Expand your library of problem-solving tools.***
 - We'll cover a wide range of tools for a bunch of interesting problems. These come in handy, both IRL and in Theoryland.
- ***Learn new problem-solving techniques.***
 - We'll see some truly beautiful problem-solving strategies that work beyond just a single example.
- ***Challenge your intuition for the limits of efficiency.***
 - You'd be amazed how many times we'll take a problem you're sure you know how to solve and then see how to solve it faster.
- ***See the beauty of theoretical computer science.***
 - We'll cover some amazingly clever theoretical techniques in the course of this class. You'll love them.

Where is CS166 situated in
Stanford's CS sequence?

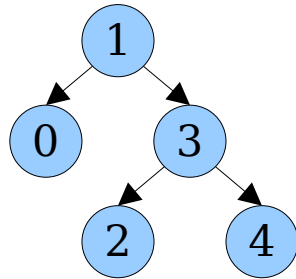
Our (Transitive) Prerequisites

CS106B / CS107



```
struct Node {  
    int value;  
    Node* left;  
    Node* right;  
};
```

make && gdb ./a.out



CS103

$$a_0 = 1 \quad a_{n+1} = 2a_n + n$$

Theorem: $a_n = 2^{n+1} - n - 1$.

Proof: By induction. As a base case, when $n = 0$, we have

$$2^{n+1} - n - 1 = 2^1 - 0 - 1 = 1 = a_0.$$

For the inductive step, assume that $a_k = 2^{k+1} - k - 1$. Then

$$\begin{aligned} a_{k+1} &= 2a_k + k \\ &= 2^{k+2} - 2k - 2 + k \\ &= 2^{(k+1)+1} - (k+1) - 1, \end{aligned}$$

as required. ■

CS109

$$\mathbb{E}\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n \mathbb{E}[X_i]$$

$$\Pr[X \geq c] \leq \frac{\mathbb{E}[X]}{c}$$

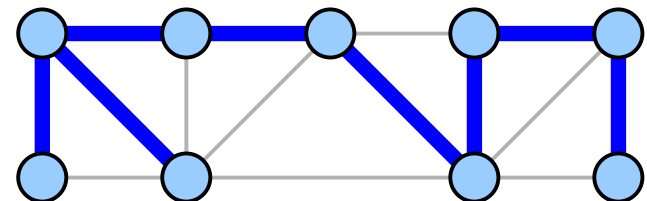
CS161

$$T(n) = aT(n/b) + O(n^d)$$

$$n^2 \log n^2 = O(n^3)$$

$$n^2 \log n^2 = \Omega(n^2)$$

$$n^2 \log n^2 = \Theta(n^2 \log n)$$



Who are we?

Course Staff

Keith Schwarz (htiek@cs.stanford.edu)

Amy Kanne
Bryan Cai

Sign up for EdStem!

Signup link in the “Assignment Policies” handout.

Zoom Etiquette

- Please feel free to ask us questions asynchronously in text chat! You can either post in all chat or just message me privately.
- If we pause and open for questions, feel free to unmute and ask questions! You can also raise your hand if you have something you'd like to say.
- As a courtesy, please mute your microphone if you aren't speaking.
- As a courtesy, please do not use the all-chat for side conversations; I monitor chat to see if there are questions and it can be a bit distracting.

The Course Website

<http://cs166.stanford.edu>

Course Requirements

- We plan on having four ***problem sets***.
 - Problem sets may be completed individually or in a pair.
 - They're a mix of written problems and C++ coding exercises.
 - You'll submit one copy of the problem set regardless of how many people worked on it.
 - Need to find a partner? Use EdStem, call into office hours, or send us an email.
- We plan of having five ***individual assessments***.
 - Similar to problem sets, except that they must be completed individually.
 - Course staff can answer clarifying questions, but otherwise it's up to you to work out how to solve them.
- We plan to have a final ***research project***.
 - We'll hammer out details in the next couple of weeks. Expect to work in a group, do a deep dive into a topic, and get lots of support from us.

Individual Assessment 0

- Individual Assessment 0 goes out today. It's due next Tuesday at 2:30PM Pacific time.
- This is mostly designed as a refresher of topics from the prerequisite courses CS103, CS107, CS109, and CS161.
- If you're mostly comfortable with these problems and are just "working through some rust," then you're probably in the right place!

Let's Get Started!

Range Minimum Queries

The RMQ Problem

- The *Range Minimum Query problem* (**RMQ** for short) is the following:

Given an array A and two indices $i \leq j$,
what is the smallest element out of
 $A[i], A[i + 1], \dots, A[j - 1], A[j]$?

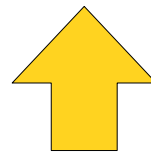
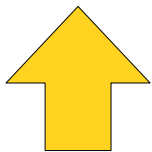
31	41	59	26	53	58	97	93
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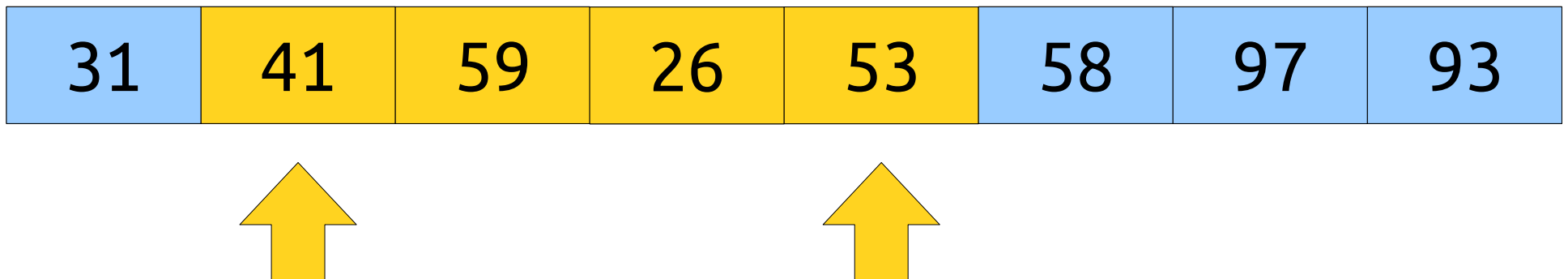
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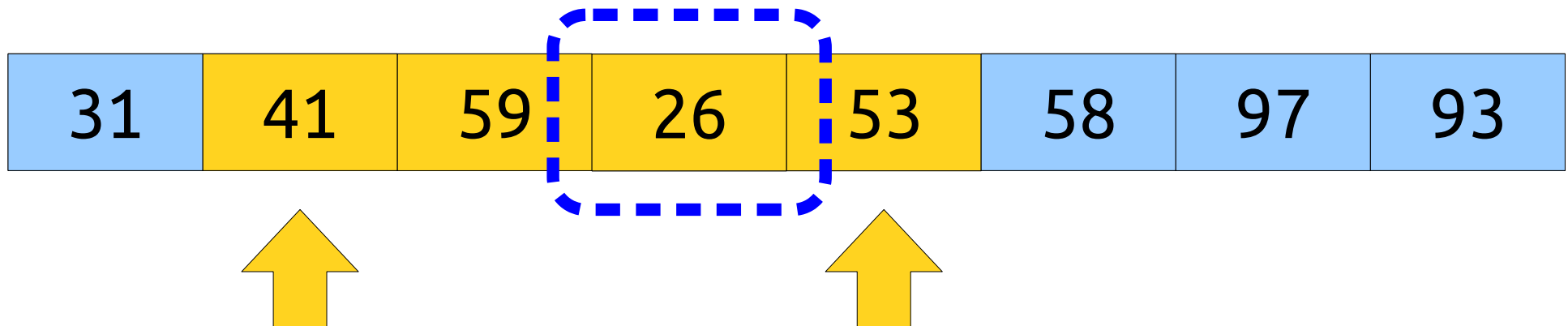
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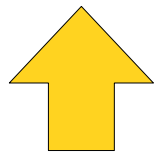
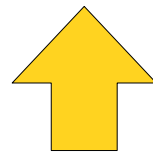


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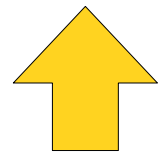
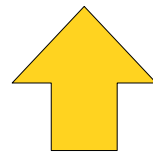


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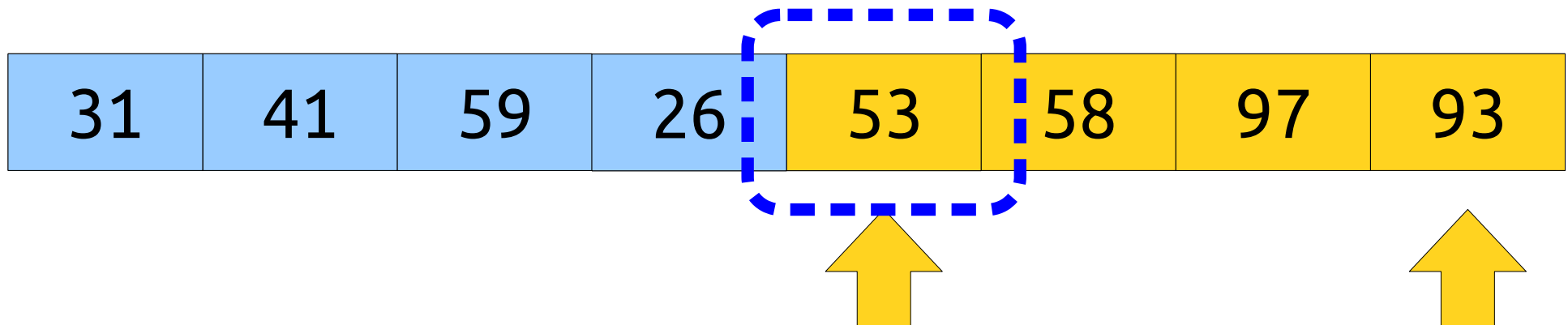
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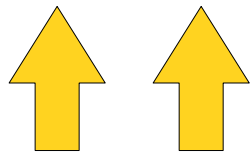


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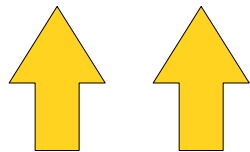


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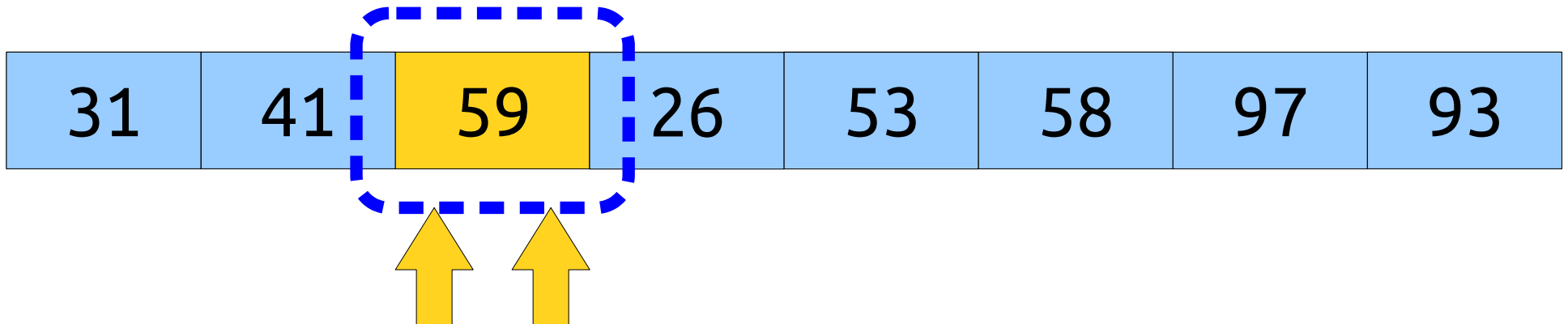
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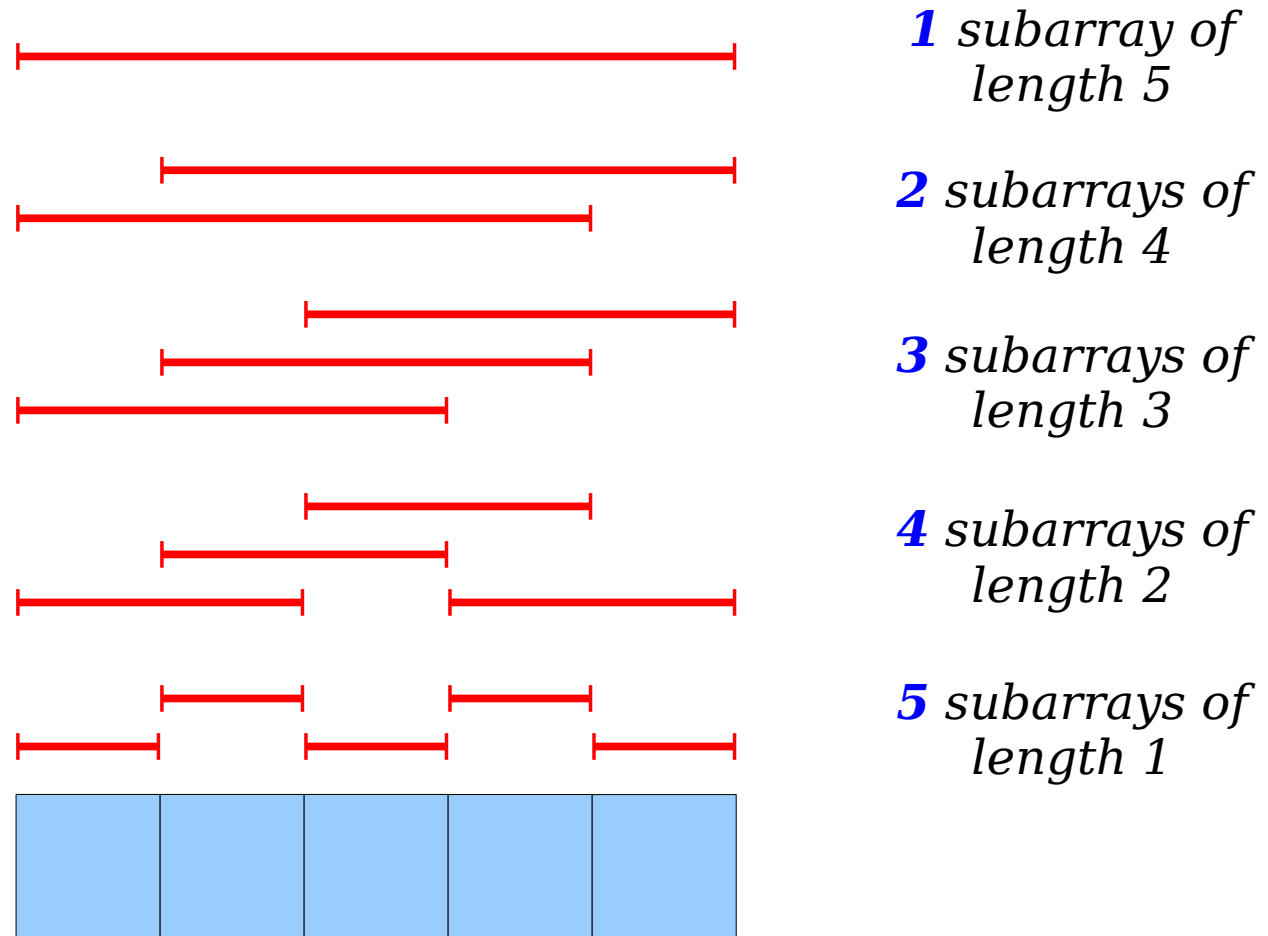
- Notation: We'll denote a range minimum query in array A between indices i and j as **RMQ_A(i, j)**.
- For simplicity, let's assume 0-indexing.

A Trivial Solution

- There's a simple $O(n)$ -time algorithm for evaluating $\text{RMQ}_A(i, j)$: just iterate across the elements between i and j , inclusive, and take the minimum!
- So... why is this problem at all algorithmically interesting?
- Suppose that the array A is fixed in advance and you're told that we're going to make multiple queries on it.
- Can we do better than the naïve algorithm?

An Observation

- In an array of length n , there are only $\Theta(n^2)$ distinct possible queries.
- Why?



A Different Approach

- There are only $\Theta(n^2)$ possible RMQs in an array of length n .
- If we precompute all of them, we can answer RMQ in time $O(1)$ per query.

16	18	33	98
0	1	2	3

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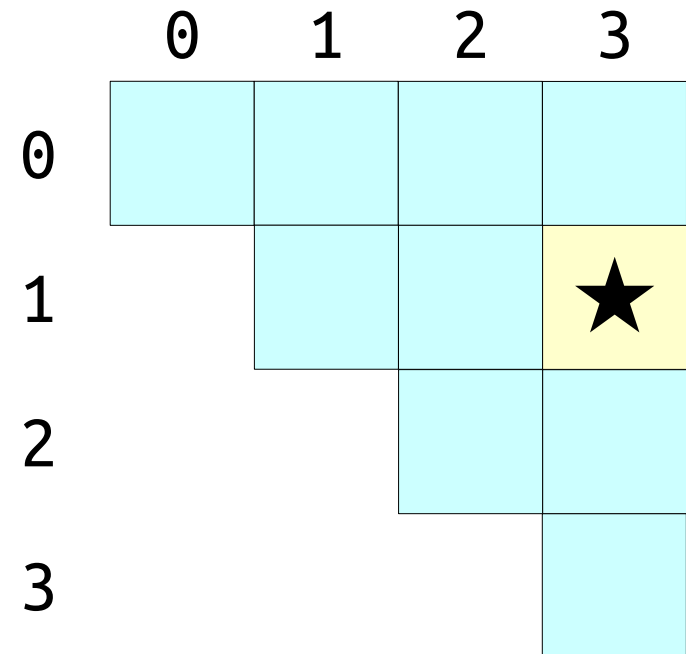
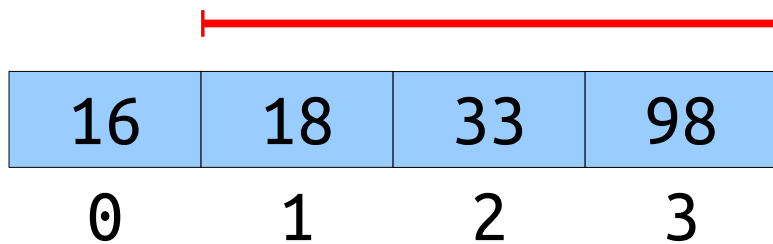
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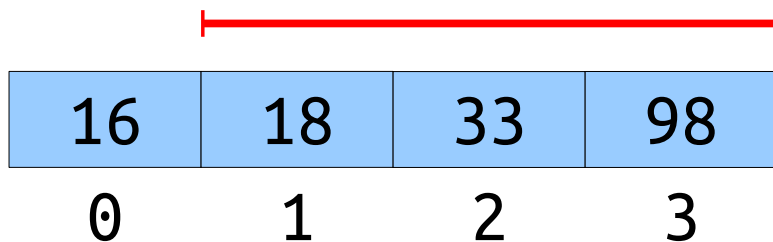
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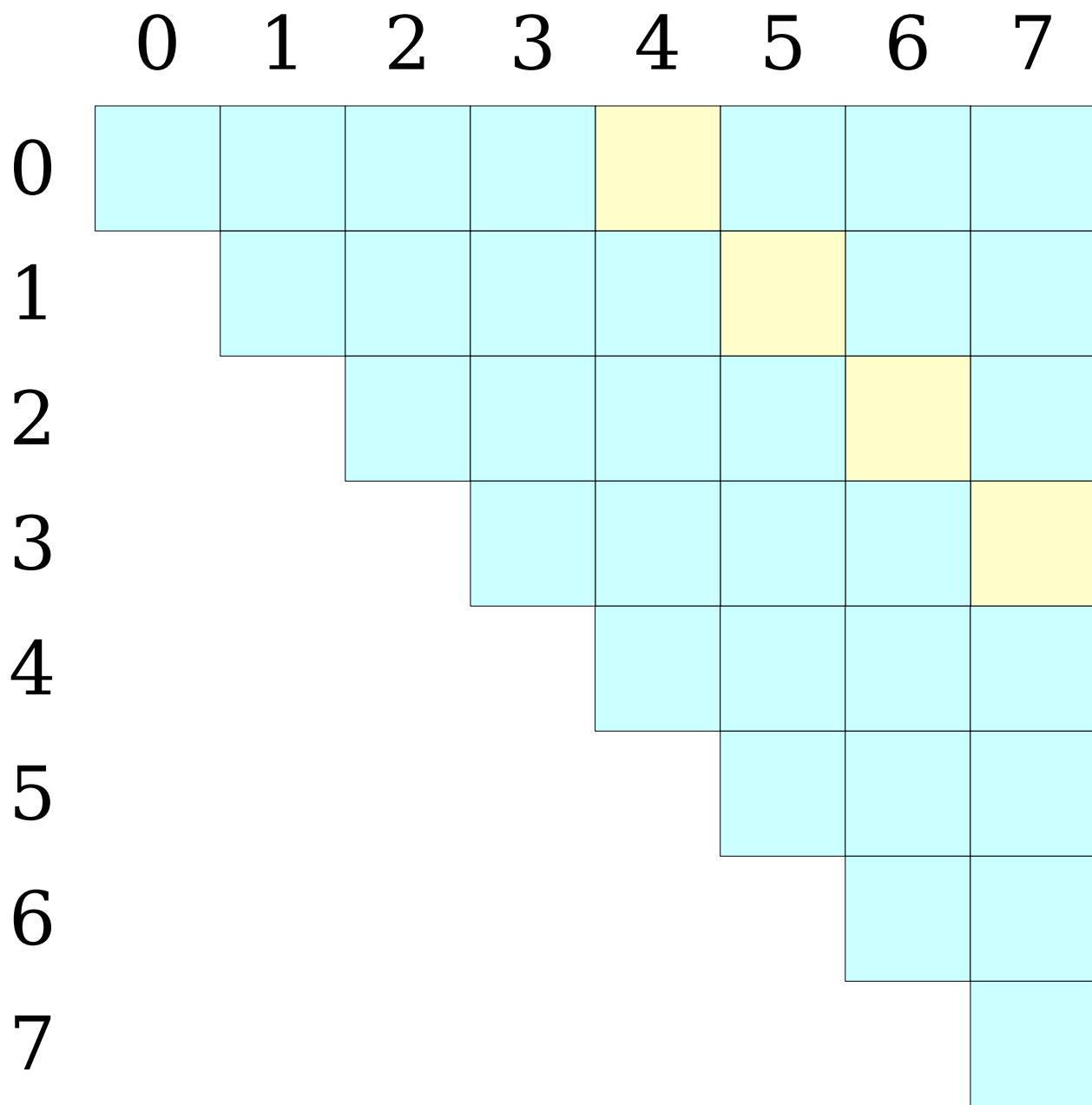
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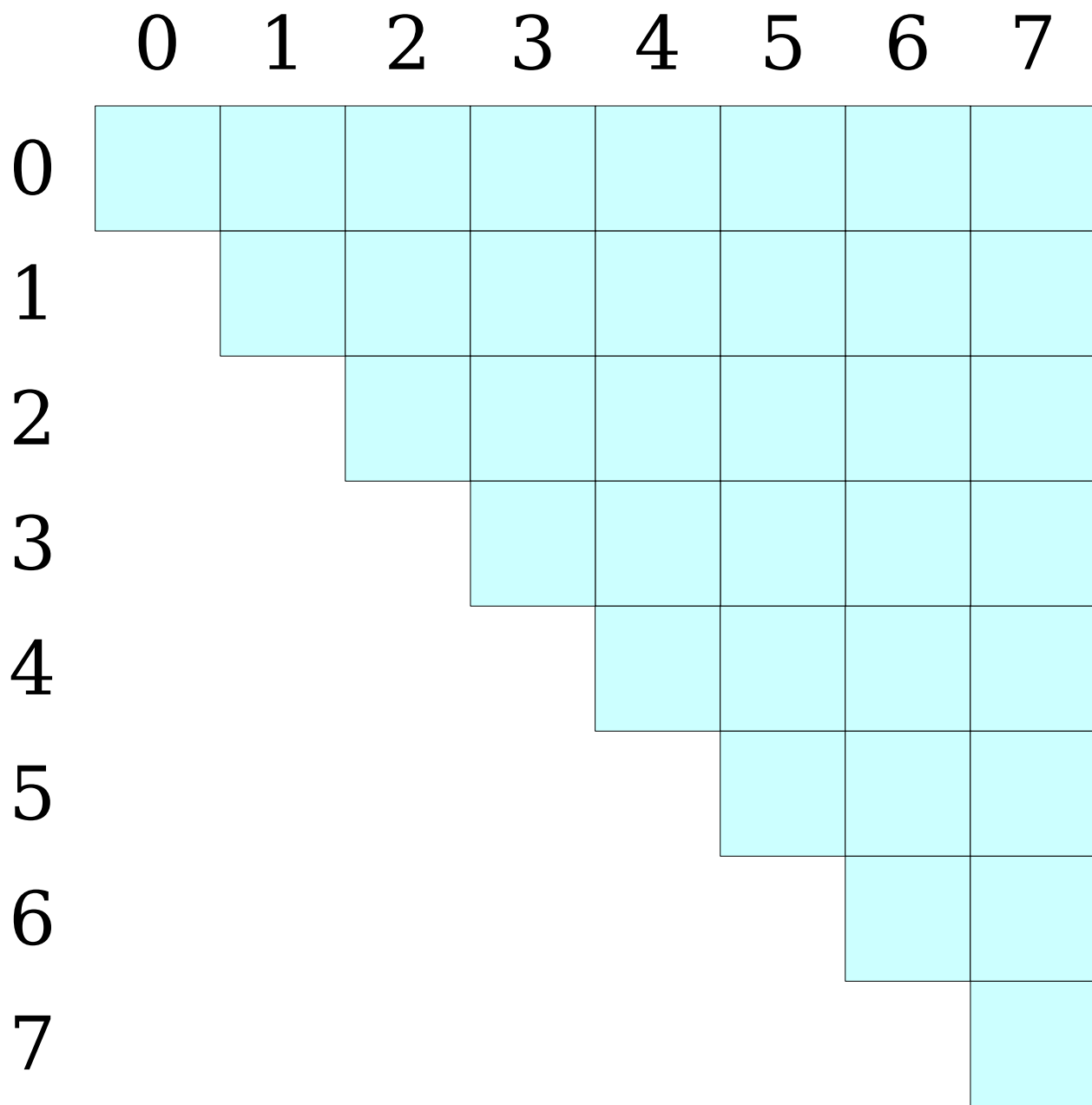
Building the Table

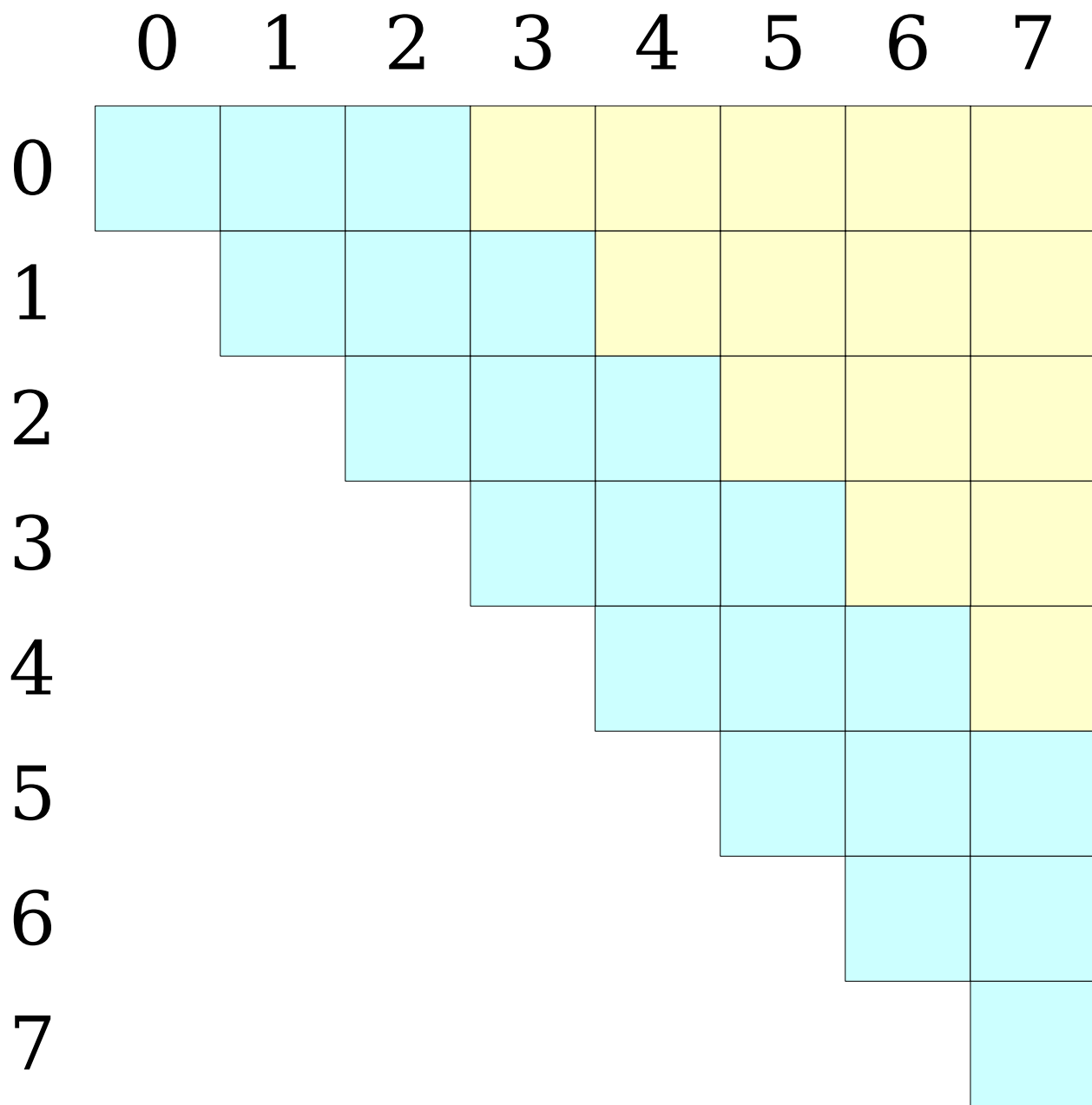
- One simple approach: for each entry in the table, iterate over the range in question and find the minimum value.
- How efficient is this?
 - Number of entries: $\Theta(n^2)$.
 - Time to evaluate each entry: $O(n)$.
 - Time required: $O(n^3)$.
- The runtime is $O(n^3)$ using this approach. Is it also $\Theta(n^3)$?

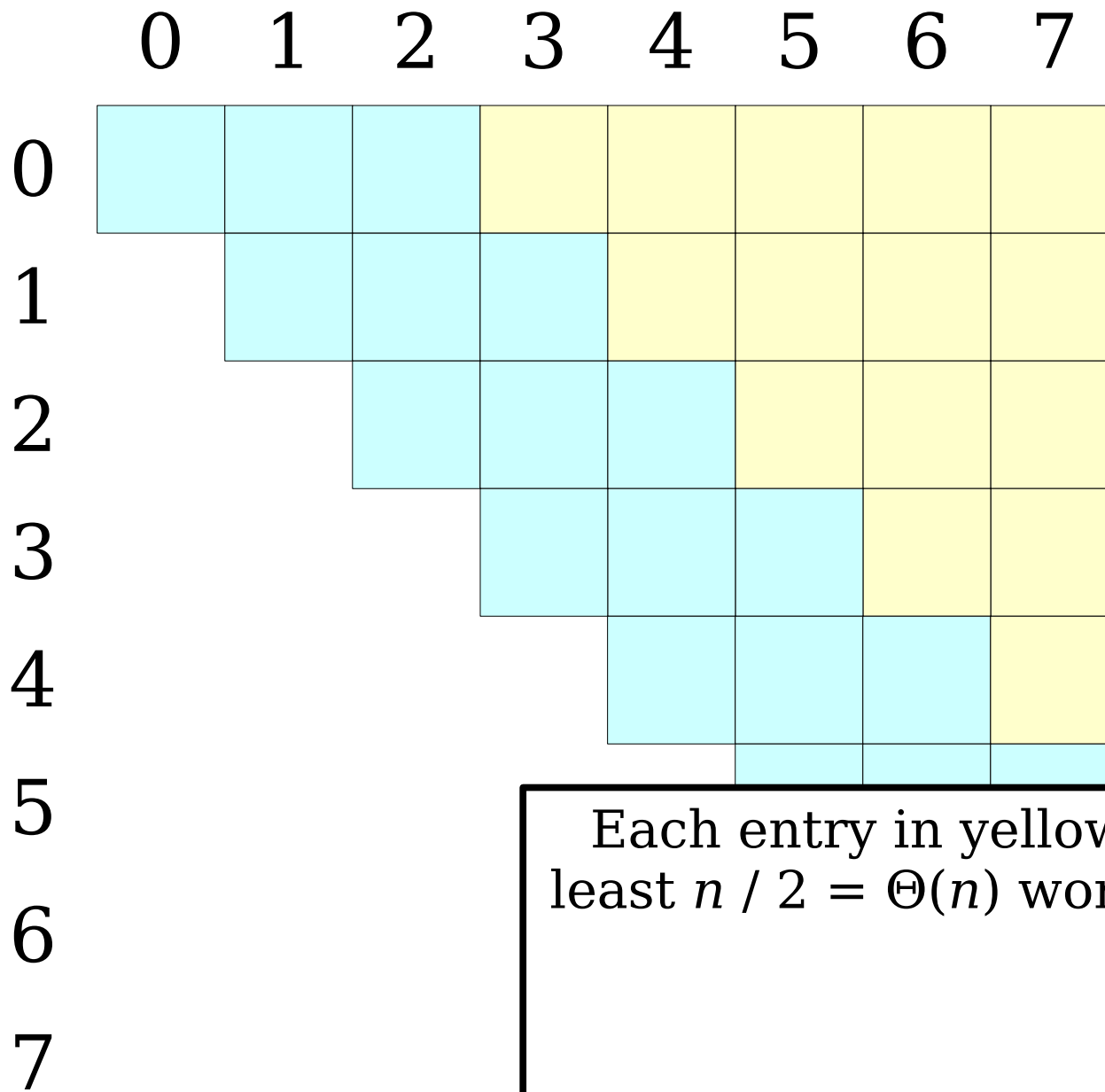
A diagram illustrating a staircase pattern of light blue squares. The squares are arranged in a grid that is 8 rows high and 8 columns wide. The first row has 8 squares, the second row has 7 squares, and so on, down to the eighth row which has 1 square. The squares are numbered 0 through 7 on the left side, corresponding to the rows.



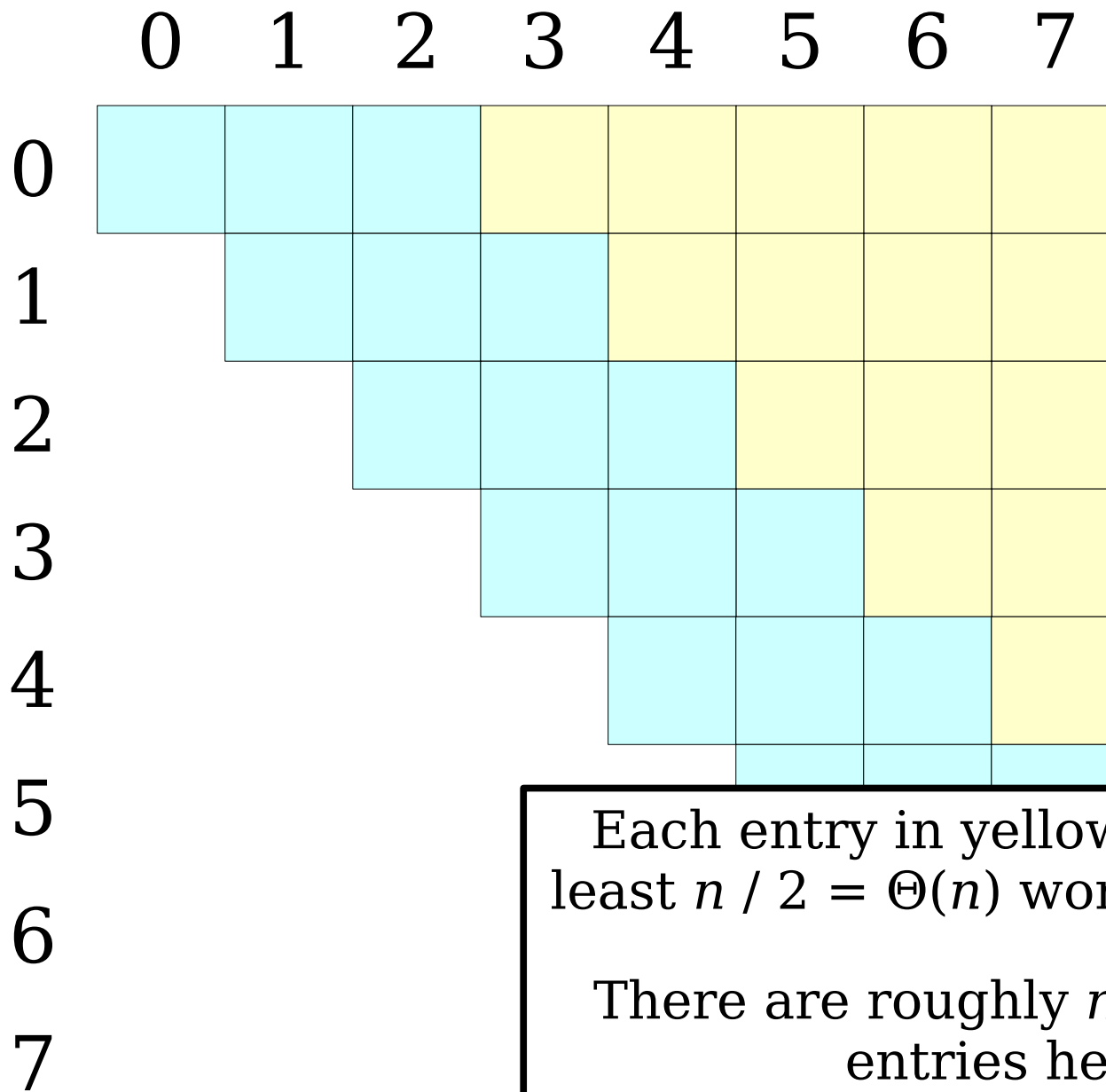
A 10x10 grid with a staircase pattern of cyan squares and three yellow squares. The cyan squares are located at (row, column) coordinates: (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), (1,10), (2,2), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), (2,9), (2,10), (3,3), (3,4), (3,5), (3,6), (3,7), (3,8), (3,9), (3,10), (4,4), (4,5), (4,6), (4,7), (4,8), (4,9), (4,10), (5,5), (5,6), (5,7), (5,8), (5,9), (5,10), (6,6), (6,7), (6,8), (6,9), (6,10), (7,7), (7,8), (7,9), (7,10), (8,8), (8,9), (8,10), (9,9), (9,10). The yellow squares are located at (1,6), (2,8), and (3,10).





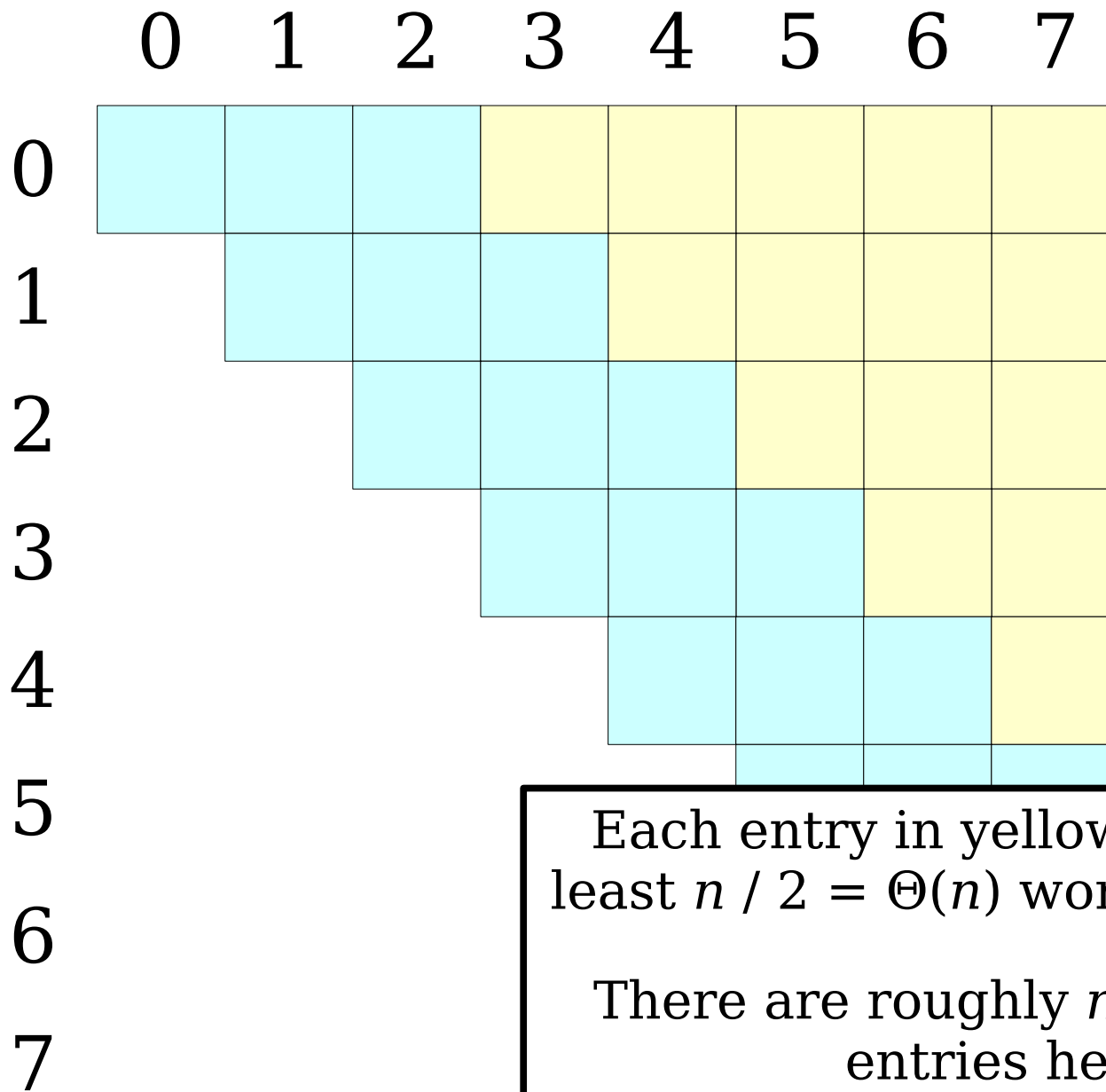


Each entry in yellow requires at least $n / 2 = \Theta(n)$ work to evaluate.



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There are roughly $n^2 / 8 = \Theta(n^2)$ entries here.



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Total work required: $\Theta(n^3)$

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- **Claim:** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

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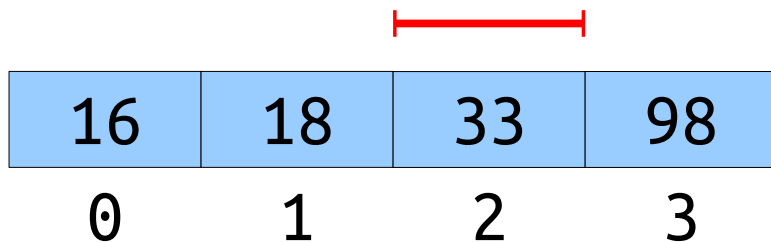


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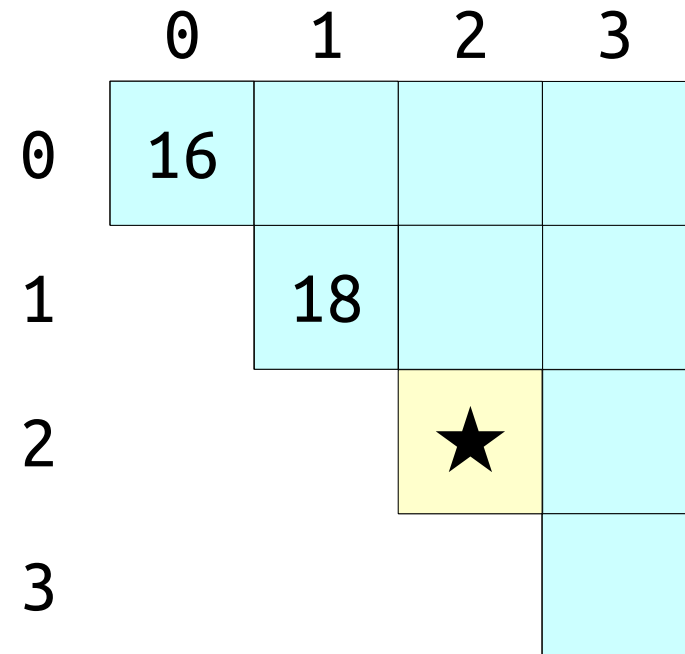
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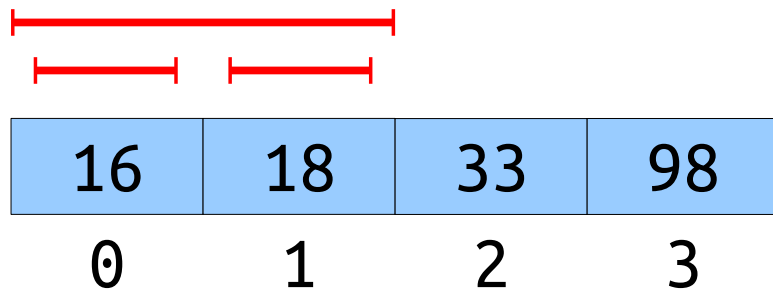


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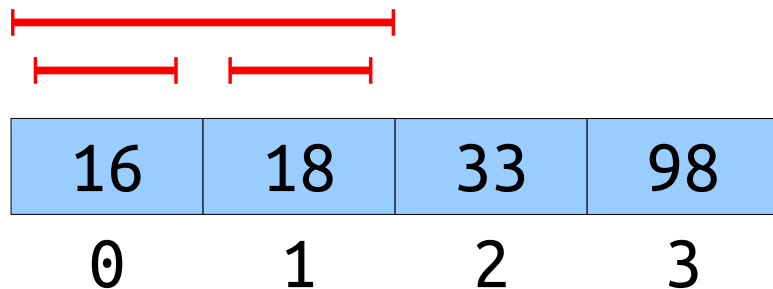
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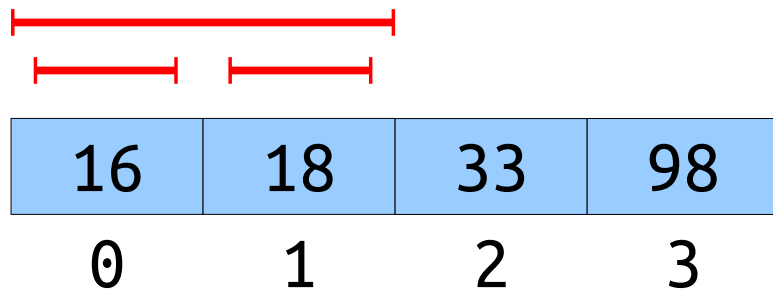
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A diagram above the table shows a red horizontal line with vertical end caps spanning from index 0 to index 3. Below this line, there are two shorter red horizontal lines with vertical end caps, one spanning from index 0 to index 1, and another spanning from index 1 to index 3. This illustrates the decomposition of the subarray [0, 3] into [0, 1] and [1, 3].

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0	16	16		
1		18	★	
2			33	
3				98

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- **Claim:** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.




A diagram above the array shows a red horizontal line spanning from index 0 to 3. Below this line, two shorter red horizontal lines are shown, one from index 0 to 1 and another from index 1 to 3, illustrating the decomposition of the subarray [0, 3] into [0, 1] and [1, 3].

16	18	33	98
0	1	2	3

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16	18	33	98
0	1	2	3

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A diagram showing a horizontal array of four blue boxes containing the values 16, 18, 33, and 98. Above the boxes, red horizontal lines with vertical end-caps indicate subarray boundaries. A long red line spans from the start of the first box to the end of the third box. Below it, two shorter red lines span from the start of the second box to the end of the second box, and from the start of the third box to the end of the third box.

16	18	33	98
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A diagram showing a horizontal array of four blue boxes containing the values 16, 18, 33, and 98. Above the boxes, red horizontal lines with vertical end-caps indicate subarray decomposition. A long red line spans from the start of the 16 box to the end of the 98 box. Below it, two shorter red lines span from the start of the 18 box to the end of the 33 box, and from the start of the 33 box to the end of the 98 box.

16	18	33	98
0	1	2	3

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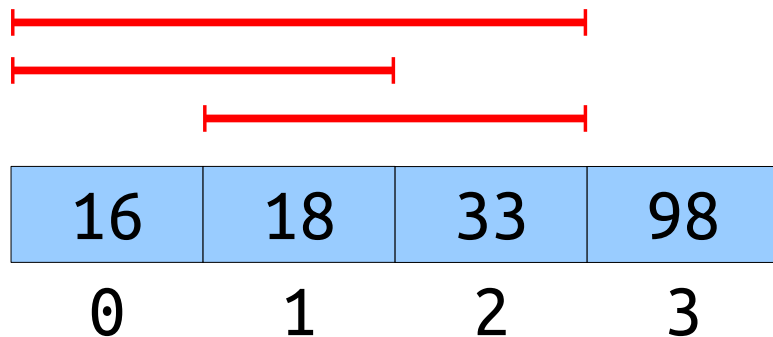


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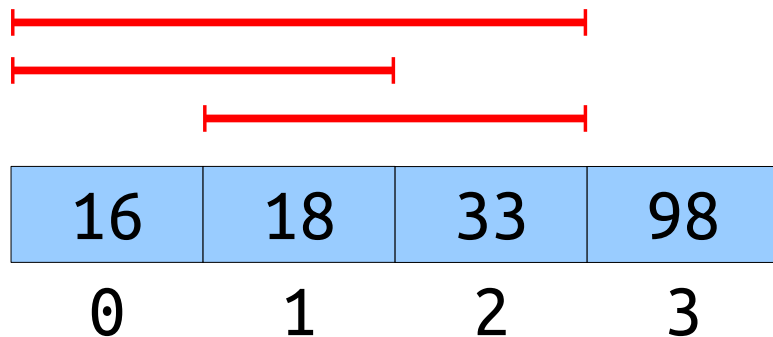
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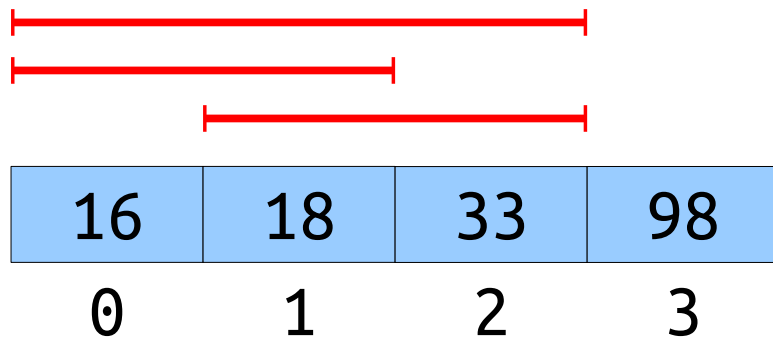
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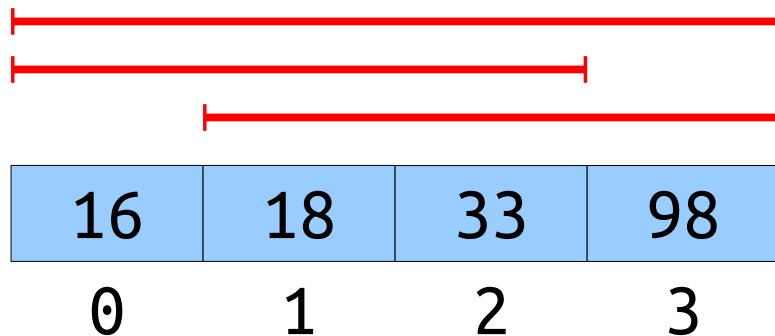


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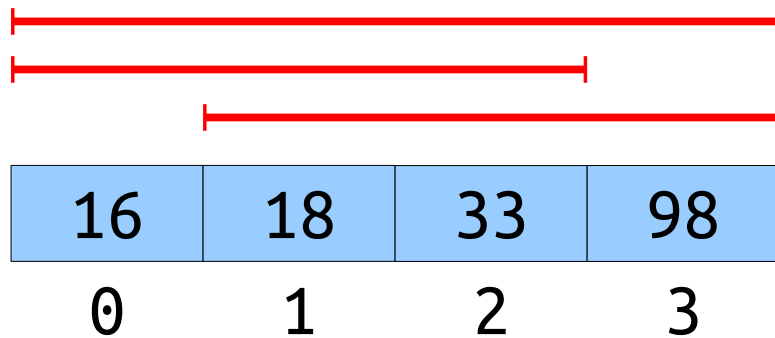
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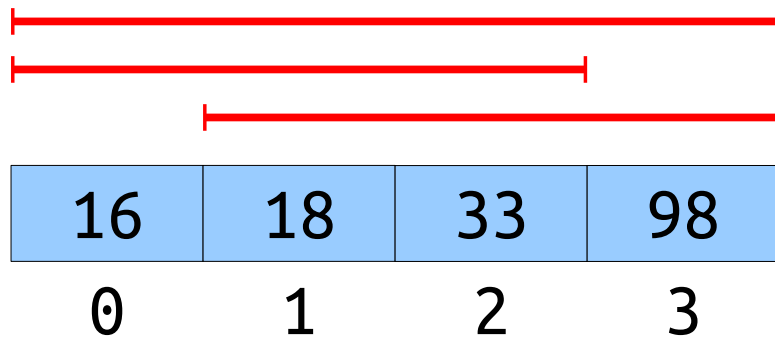
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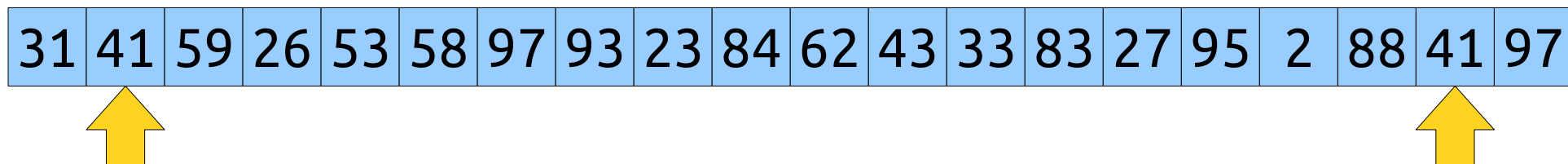
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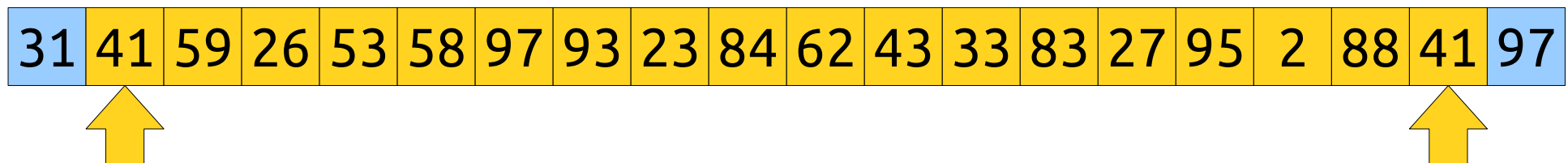
Some Notation

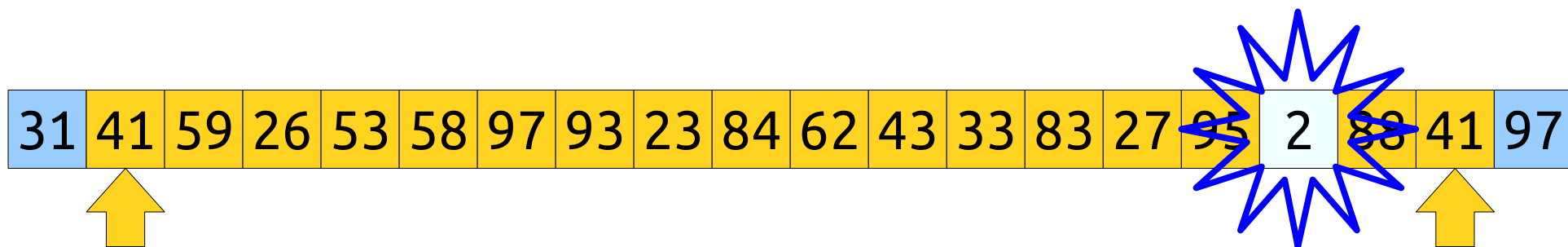
- We'll say that an RMQ data structure has time complexity $\langle p(n), q(n) \rangle$ if
 - preprocessing takes time at most $p(n)$ and
 - queries take time at most $q(n)$.
- We now have two RMQ data structures:
 - $\langle O(1), O(n) \rangle$ with no preprocessing.
 - $\langle O(n^2), O(1) \rangle$ with full preprocessing.
- These are two extremes on a curve of tradeoffs: no preprocessing versus full preprocessing.
- **Question:** *Is there a “golden mean” between these extremes?*

Another Approach: ***Block Decomposition***

31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97
----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	---	----	----	----







31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97
----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	---	----	----	----

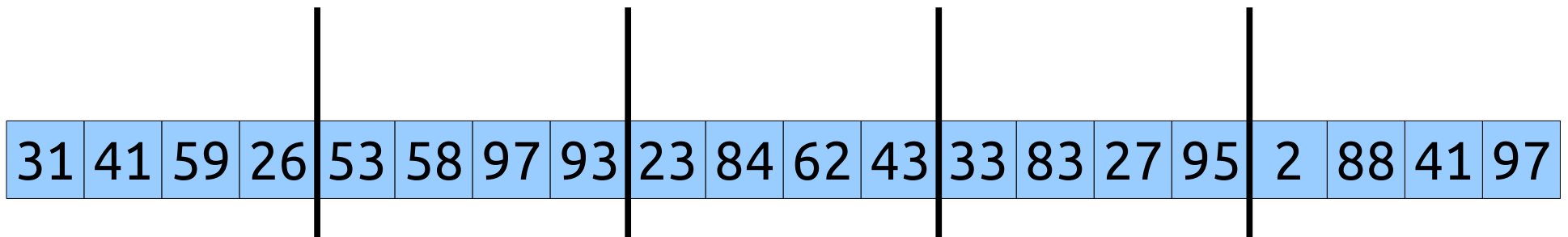
A Block-Based Approach

- Split the input into $O(n / b)$ blocks of some “block size” b .

31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97
----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	---	----	----	----

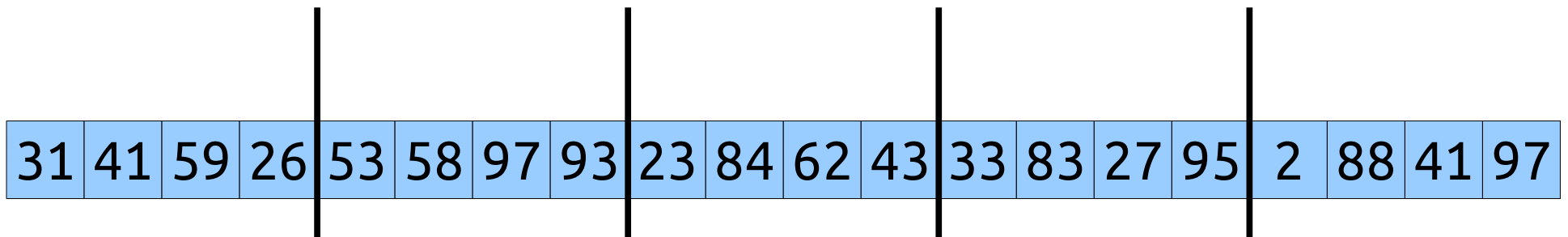
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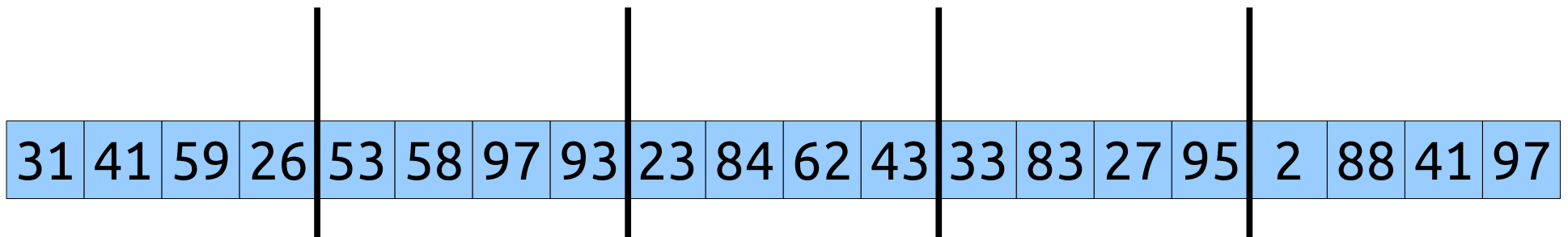
A Block-Based Approach

- Split the input into $O(n / b)$ blocks of some “block size” b .
 - Here, $b = 4$.



A Block-Based Approach

- Split the input into $O(n / b)$ blocks of some “block size” b .
 - Here, $b = 4$.
- Compute the minimum value in each block.



31	41	59	26	53	58	97	93	23	84	62	43	33	83	27	95	2	88	41	97
----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	---	----	----	----

A Block-Based Approach

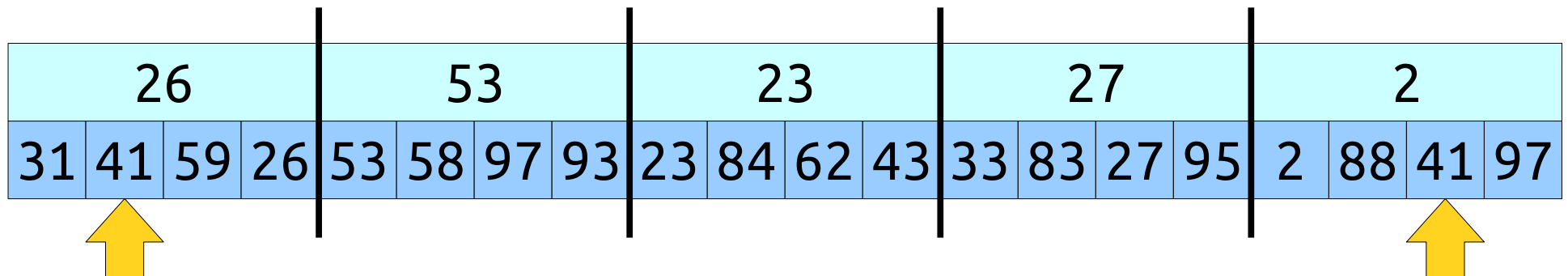
- Split the input into $O(n / b)$ blocks of some “block size” b .
 - Here, $b = 4$.
- Compute the minimum value in each block.

26				53				23				27				2			
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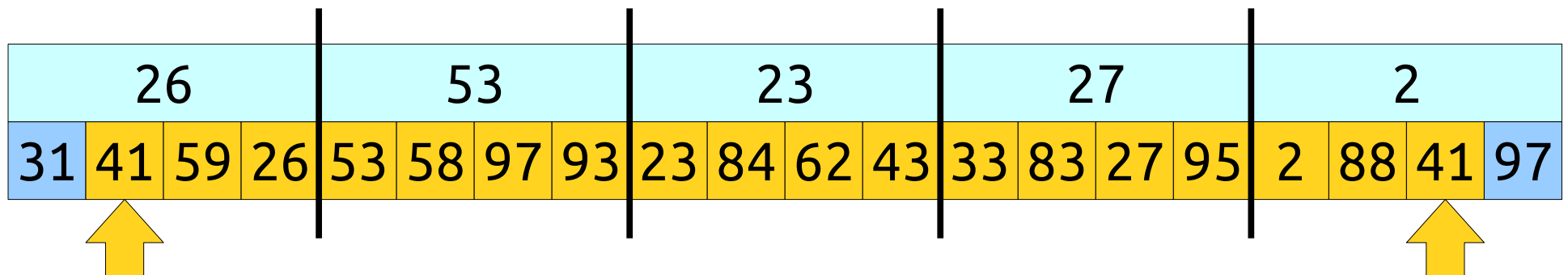


The diagram illustrates an array of 20 numbers divided into 5 blocks of size 4. The minimum value for each block is shown above it. Yellow arrows point to the first and last elements of the array.

Block	Elements	Minimum
1	31, 41, 59, 26	26
2	53, 58, 97, 93	53
3	23, 84, 62, 43	23
4	33, 83, 27, 95	27
5	2, 88, 41, 97	2

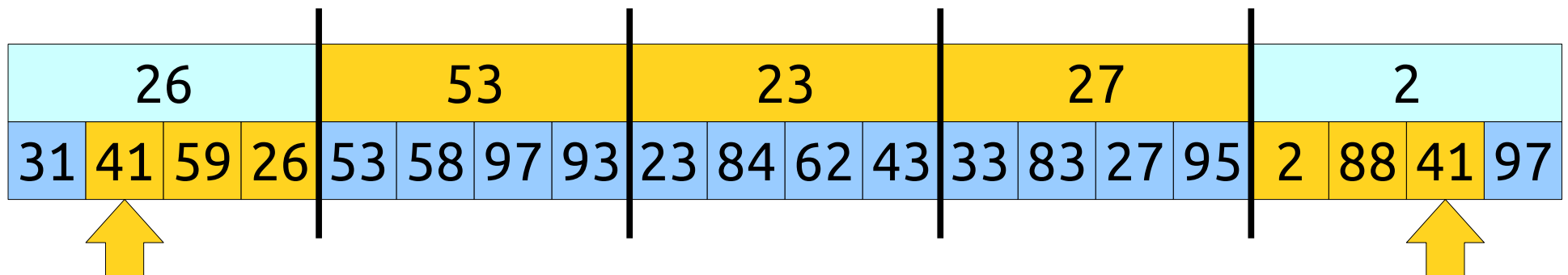
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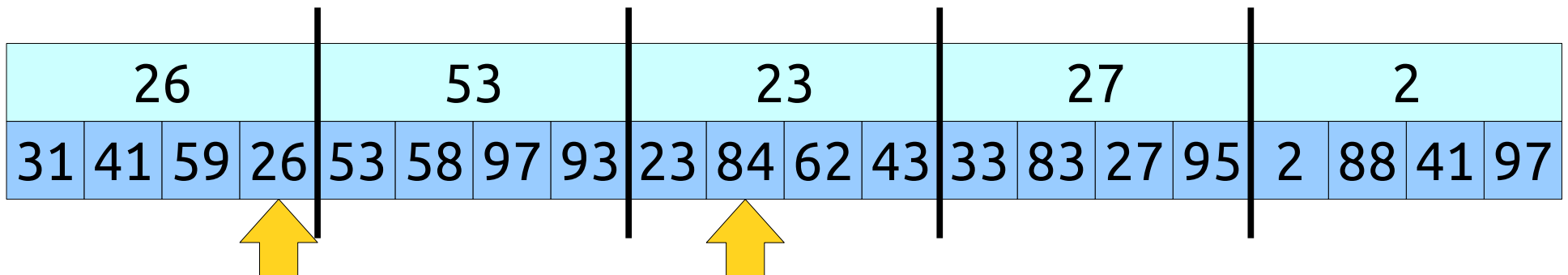
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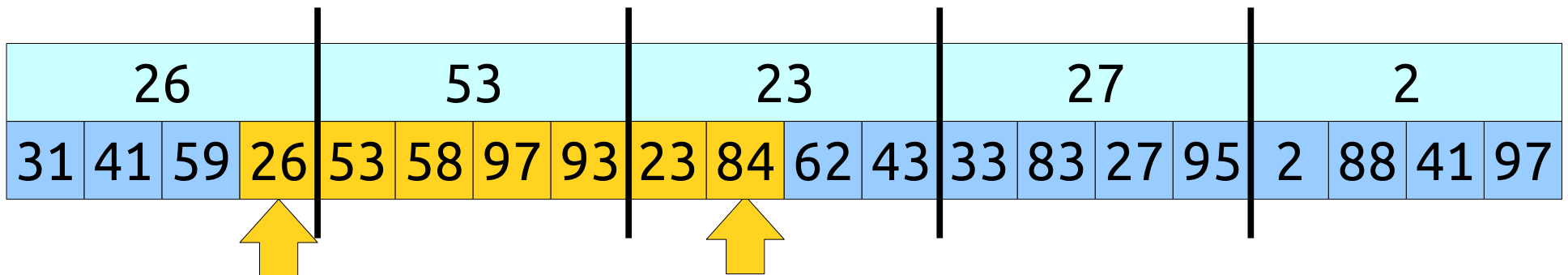
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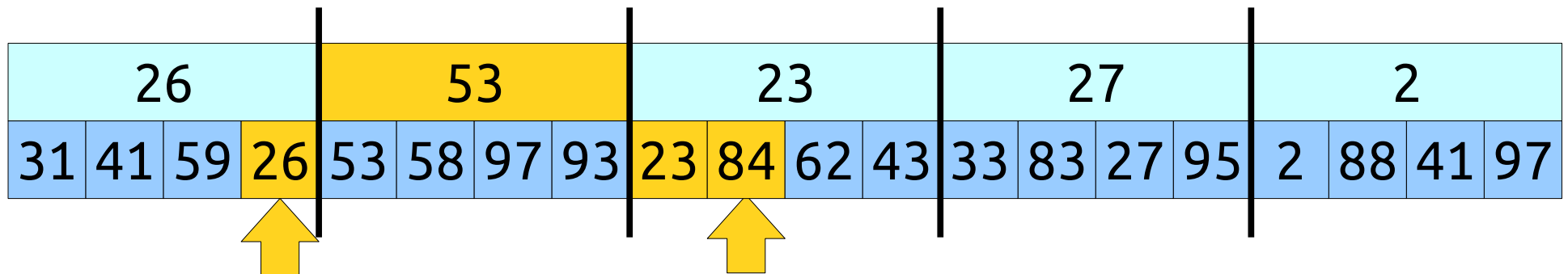
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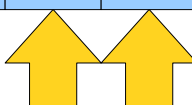
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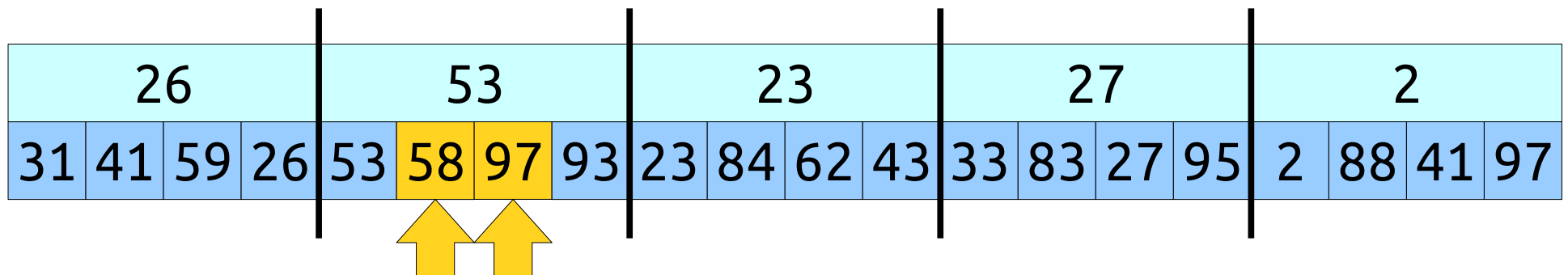
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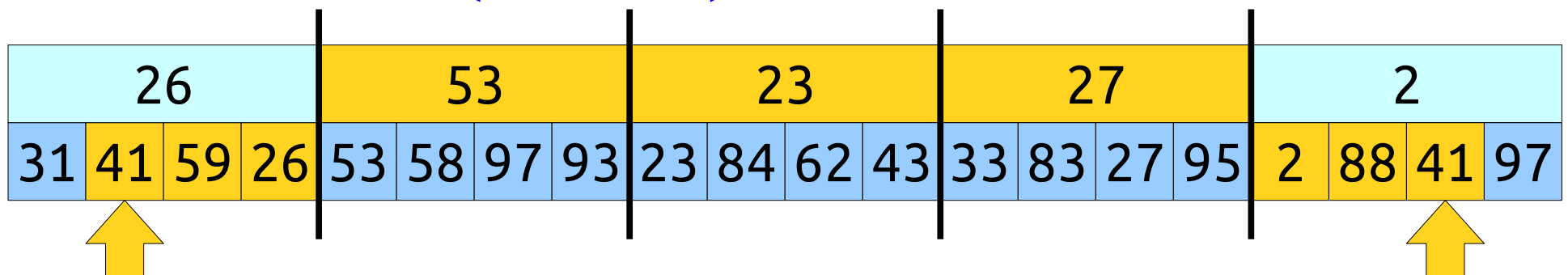
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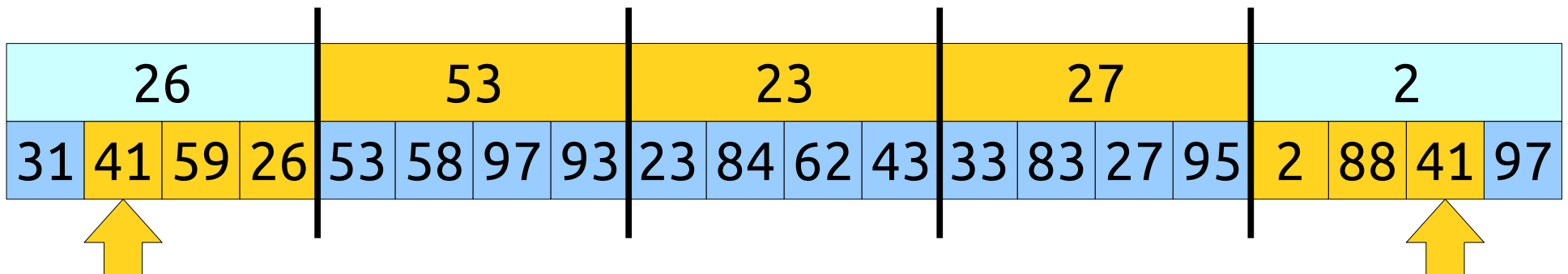
Analyzing the Approach

- Let's analyze this approach in terms of n and b .
- Preprocessing time:
 - $O(b)$ work on $O(n / b)$ blocks to find minima.
 - Total work: **$O(n)$** .
- Time to evaluate $\text{RMQ}_A(i, j)$:
 - $O(1)$ work to find block indices (divide by block size).
 - $O(b)$ work to scan inside i and j 's blocks.
 - $O(n / b)$ work looking at block minima between i and j .
 - Total work: **$O(b + n / b)$** .



Intuiting $O(b + n / b)$

- As b increases:
 - The b term rises (more elements to scan within each block).
 - The n / b term drops (fewer blocks to look at).
- As b decreases:
 - The b term drops (fewer elements to scan within a block).
 - The n / b term rises (more blocks to look at).
- Is there an optimal choice of b given these constraints?



Optimizing b

- What choice of b minimizes $b + n / b$?

Formulate a hypothesis, but
***don't post anything in chat
just yet.***

Optimizing b

- What choice of b minimizes $b + n / b$?

Now, ***private chat me your best guess.***

Not sure? Just answer “??”

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- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

Optimizing b

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$$\frac{d}{db}(b+n/b) = 1 - \frac{n}{b^2}$$

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- Setting the derivative to zero:

$$1 - n/b^2 = 0$$

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- Setting the derivative to zero:

$$\begin{aligned} 1 - n/b^2 &= 0 \\ 1 &= n/b^2 \end{aligned}$$

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$$1 = n/b^2$$

$$b^2 = n$$

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- In that case, the runtime is

$$O(b + n / b)$$

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$$O(b + n / b) = O(n^{1/2} + n / n^{1/2})$$

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$$O(b + n / b) = O(n^{1/2} + n / n^{1/2}) = O(n^{1/2} + n^{1/2})$$

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- Asymptotically optimal runtime is when $b = n^{1/2}$.
- In that case, the runtime is

$$O(b + n / b) = O(n^{1/2} + n / n^{1/2}) = O(n^{1/2} + n^{1/2}) = \mathbf{O(n^{1/2})}$$

Summary of Approaches

- Three solutions so far:
 - Full preprocessing: $\langle O(n^2), O(1) \rangle$.
 - Block partition: $\langle O(n), O(n^{1/2}) \rangle$.
 - No preprocessing: $\langle O(1), O(n) \rangle$.
- Modest preprocessing yields modest performance increases.
- **Question:** Can we do better?

A Second Approach: ***Sparse Tables***

An Intuition

- The $\langle O(n^2), O(1) \rangle$ solution gives fast queries because every range we might look up has already been precomputed.
- This solution is slow overall because we have to compute the minimum of every possible range.
- **Question:** Can we still get constant-time queries without preprocessing all possible ranges?

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26	26	26	26	26
1		41	41	26	26	26	26	26
2			59	26	26	26	26	26
3				26	26	26	26	26
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26	26	26	26	26
1		41	41	26	26	26	26	26
2			59	26	26	26	26	26
3				26	26	26	26	26
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7


	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93


An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
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5						58	58	58
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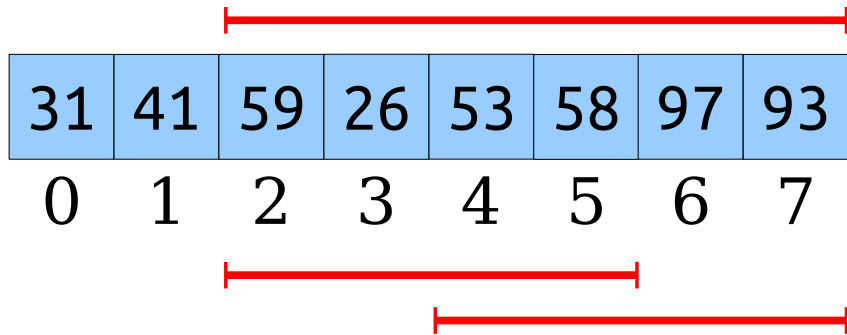
An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
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4					53	53	53	53
5						58	58	58
6							97	93
7								93

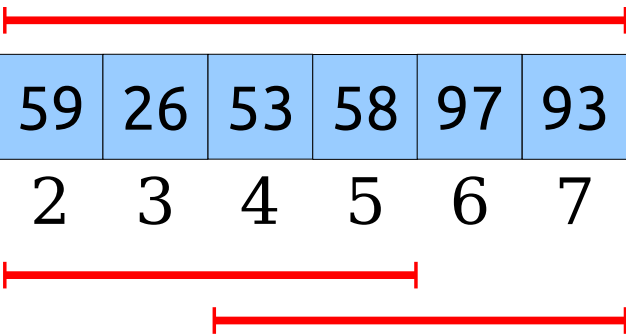
An Observation



	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7




	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

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31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93


An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

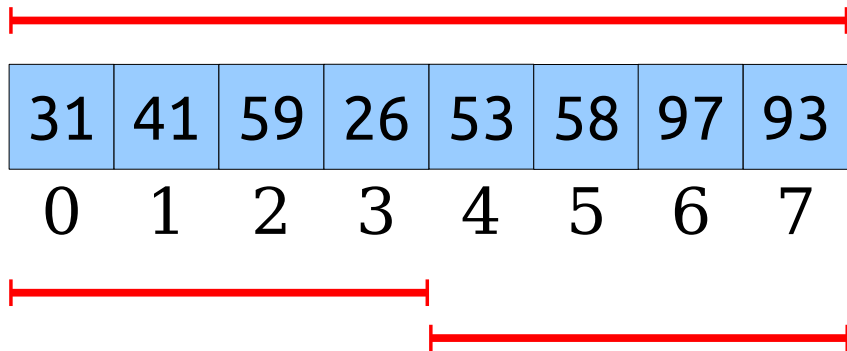
An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				★
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

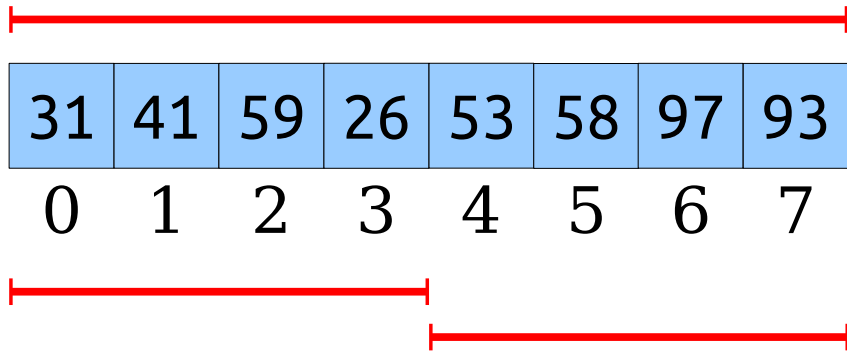
An Observation



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0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				★
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation



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	0	1	2	3	4	5	6	7
0	31	31	31	26				★
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2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

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	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
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4					53	53	53	53
5						58	58	58
6							97	93
7								93

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	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

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0	1	2	3	4	5	6	7

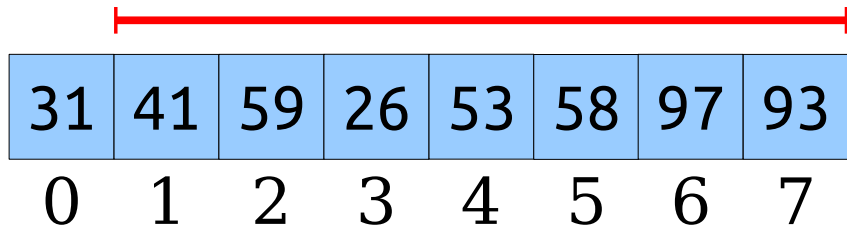
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2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation

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0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

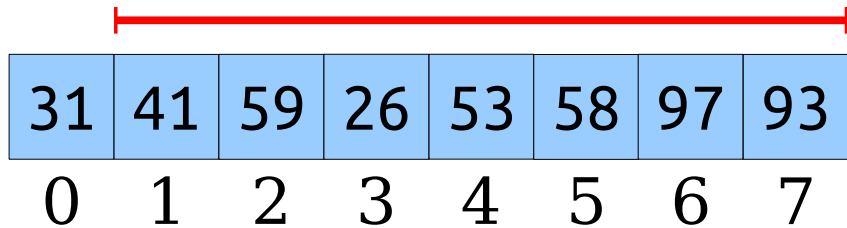
An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
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4					53	53	53	
5						58	58	58
6							97	93
7								93

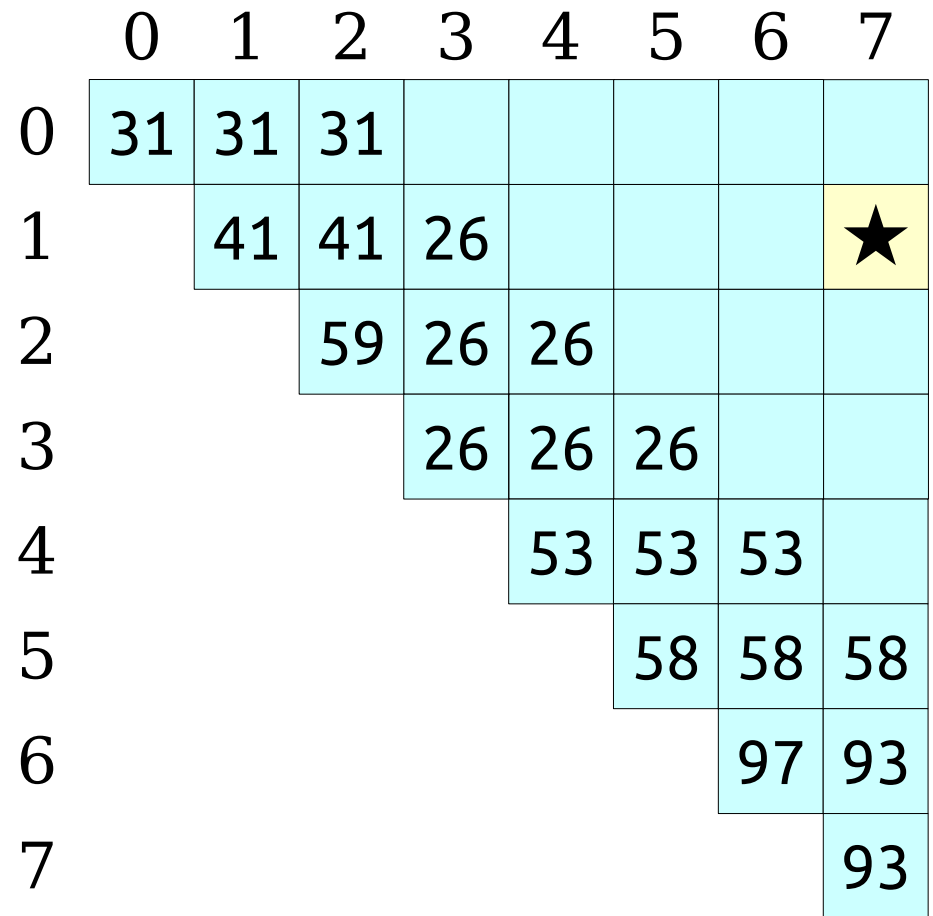
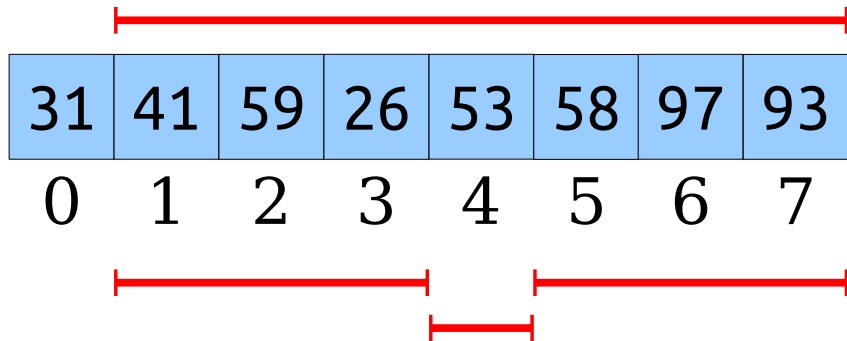
An Observation



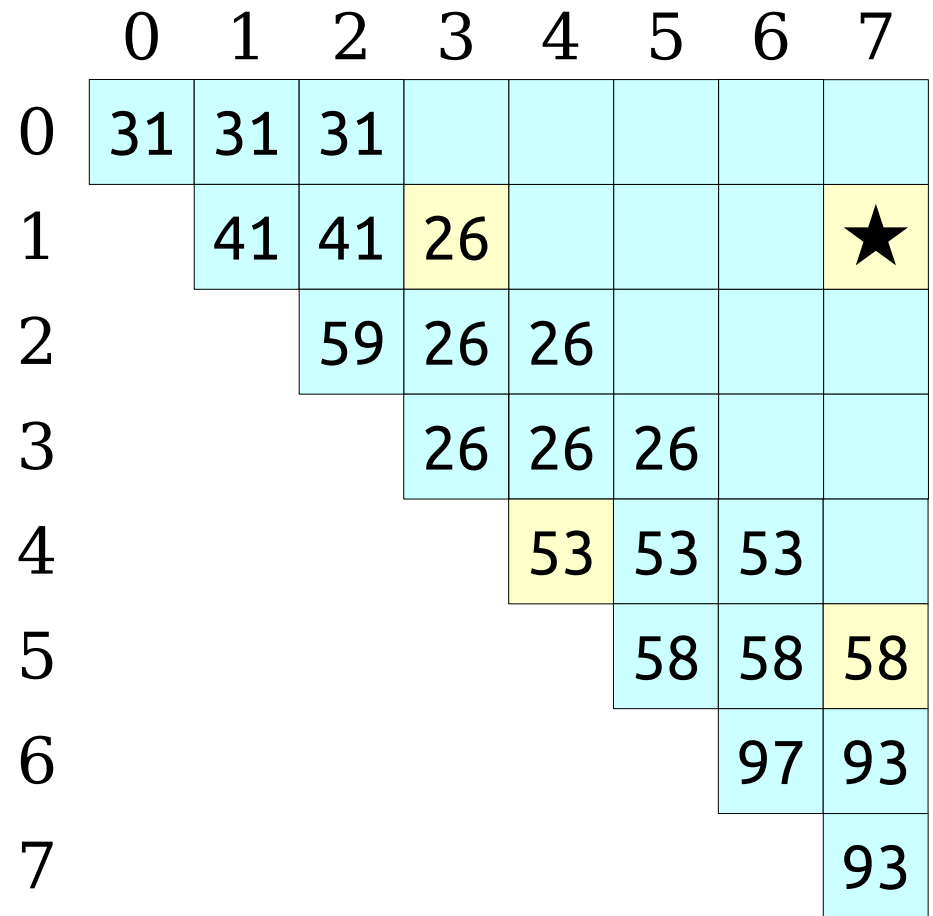
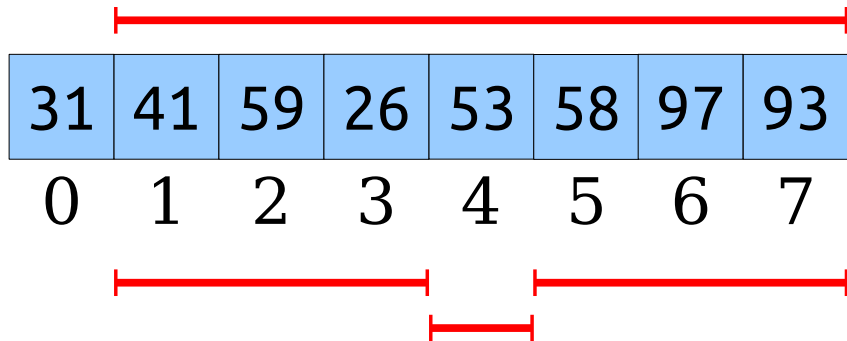
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0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				★
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation



An Observation



An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

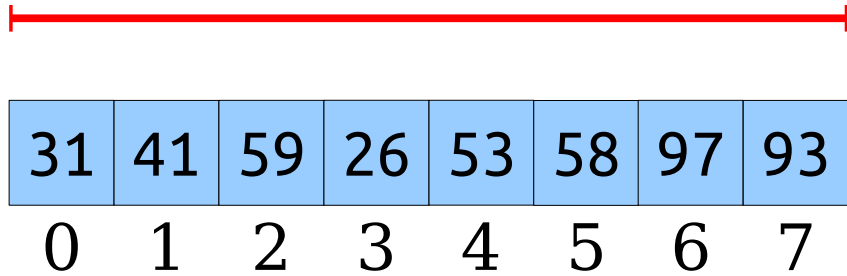
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0	31							
1		41						
2			59					
3				26				
4					53			
5						58		
6							97	
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

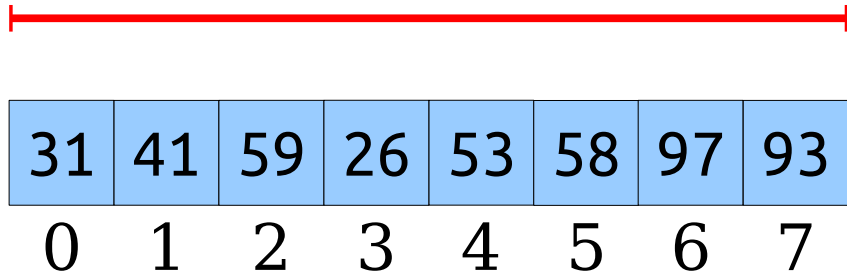
	0	1	2	3	4	5	6	7
0	31							
1		41						
2			59					
3				26				
4					53			
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6							97	
7								93

An Observation



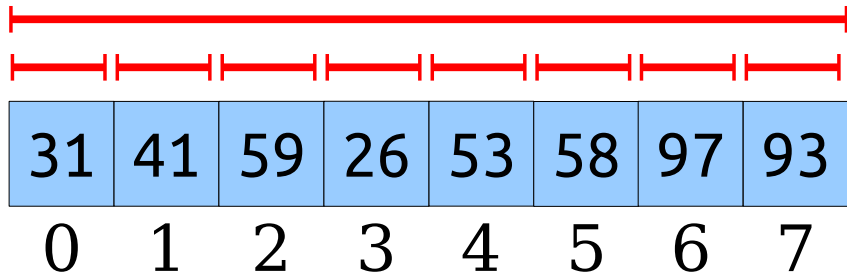
	0	1	2	3	4	5	6	7
0	31							
1		41						
2			59					
3				26				
4					53			
5						58		
6							97	
7								93

An Observation



	0	1	2	3	4	5	6	7
0	31							★
1		41						
2			59					
3				26				
4					53			
5						58		
6							97	
7								93

An Observation

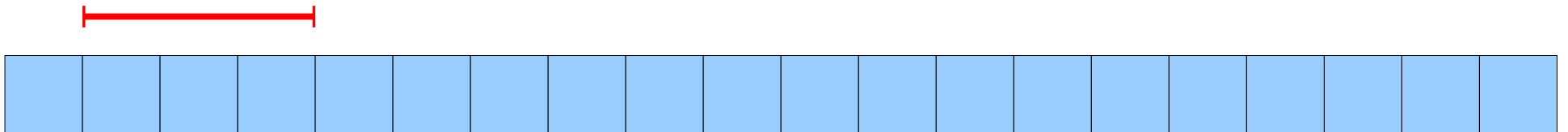
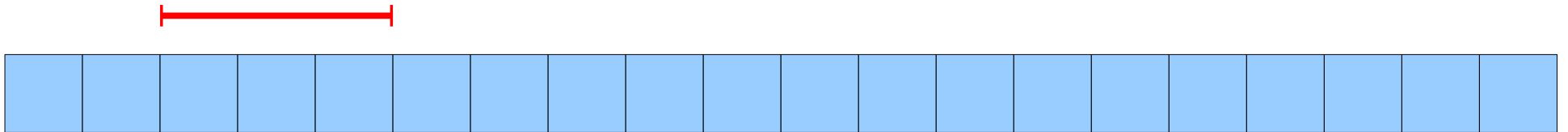
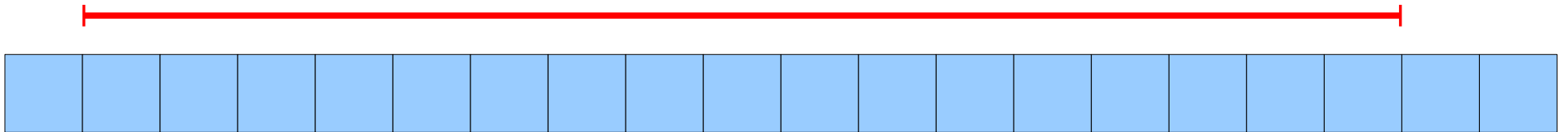
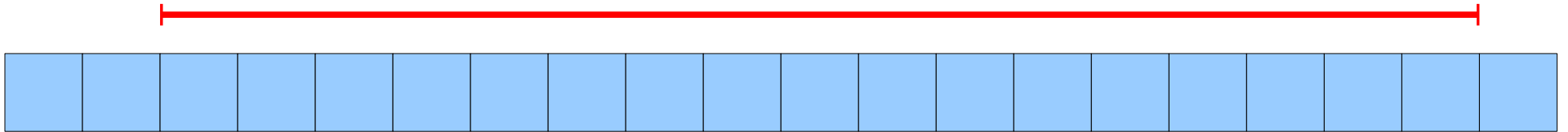


	0	1	2	3	4	5	6	7
0	31							★
1		41						
2			59					
3				26				
4					53			
5						58		
6							97	
7								93

The Intuition

- It's still possible to answer any query in time $O(1)$ without precomputing RMQ over all ranges.
- If we precompute the answers over too many ranges, the preprocessing time will be too large.
- If we precompute the answers over too few ranges, the query time won't be $O(1)$.
- **Goal:** Precompute RMQ over a set of ranges such that
 - There are $o(n^2)$ total ranges, but
 - there are enough ranges to support $O(1)$ query times.

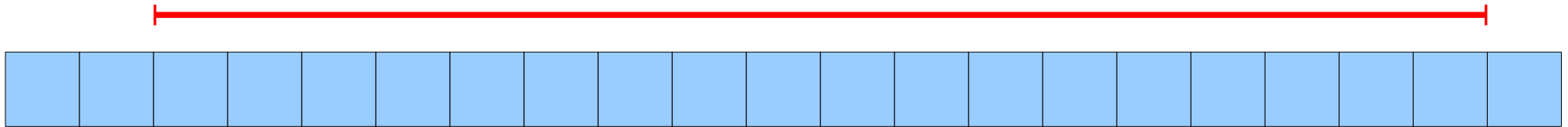
Some Observations



The Approach

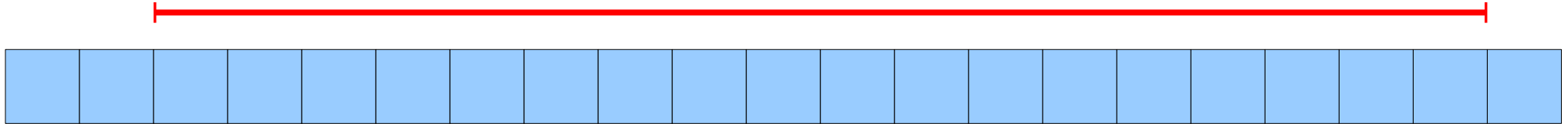
- For each index i , compute RMQ for ranges starting at i of size 1, 2, 4, 8, 16, ..., 2^k as long as they fit in the array.
 - Gives both large and small ranges starting at any point in the array.
 - Only $O(\log n)$ ranges computed for each array element.
 - Total number of ranges: $O(n \log n)$.
- **Claim:** Any range in the array can be formed as the union of two of these ranges.

Creating Ranges

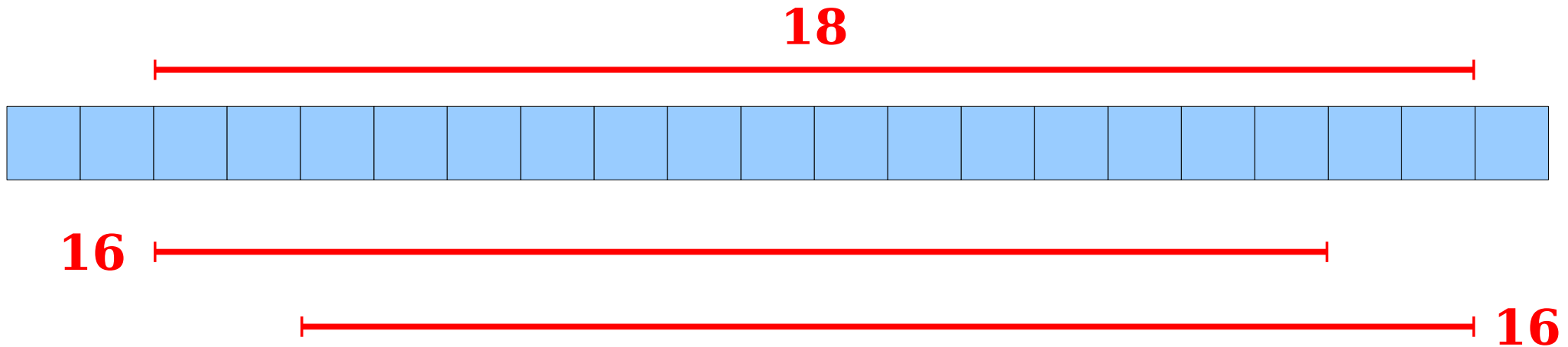


Creating Ranges

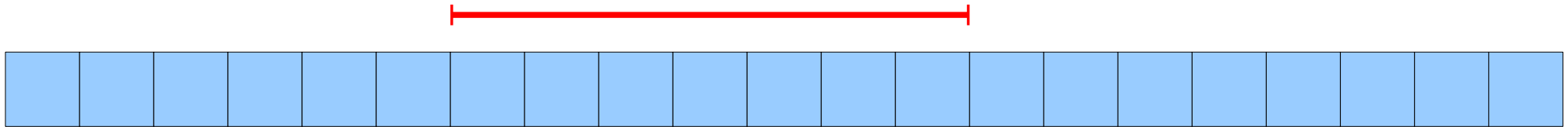
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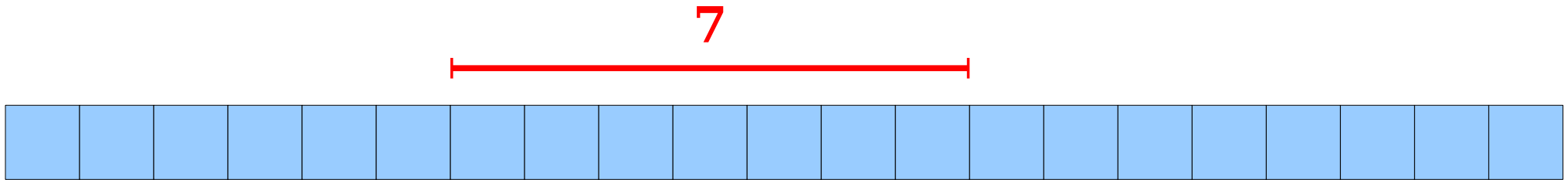
Creating Ranges



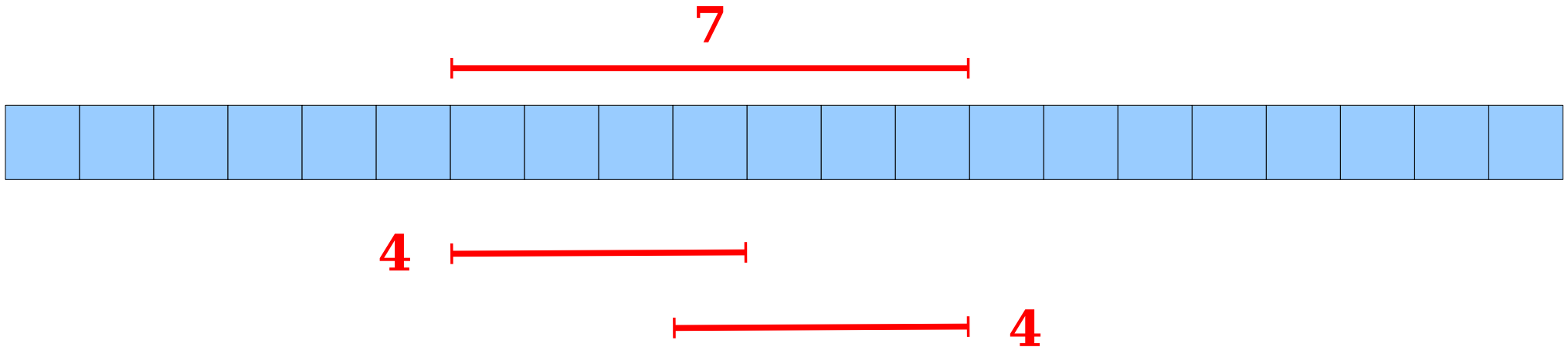
Creating Ranges



Creating Ranges



Creating Ranges



Doing a Query

- To answer $\text{RMQ}_A(i, j)$:
 - Find the largest k such that $2^k \leq j - i + 1$.
 - With the right preprocessing, this can be done in time $O(1)$; you'll figure out how in an upcoming assignment.
 - The range $[i, j]$ can be formed as the overlap of the ranges $[i, i + 2^k - 1]$ and $[j - 2^k + 1, j]$.
 - Each range can be looked up in time $O(1)$.
 - Total time: **$O(1)$** .

Precomputing the Ranges

- There are $O(n \log n)$ ranges to precompute.
- Using dynamic programming, we can compute all of them in time $O(n \log n)$.

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	2^0	2^1	2^2	2^3
0				
1				
2				
3				
4				
5				
6				
7				

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0	1	2	3	4	5	6	7

	2^0	2^1	2^2	2^3
0				
1				
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	2^0	2^1	2^2	2^3
0				
1				
2				
3		★		
4				
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6				
7				

Precomputing the Ranges

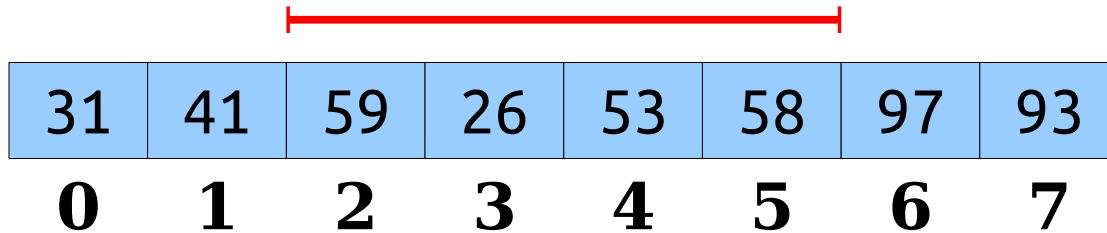
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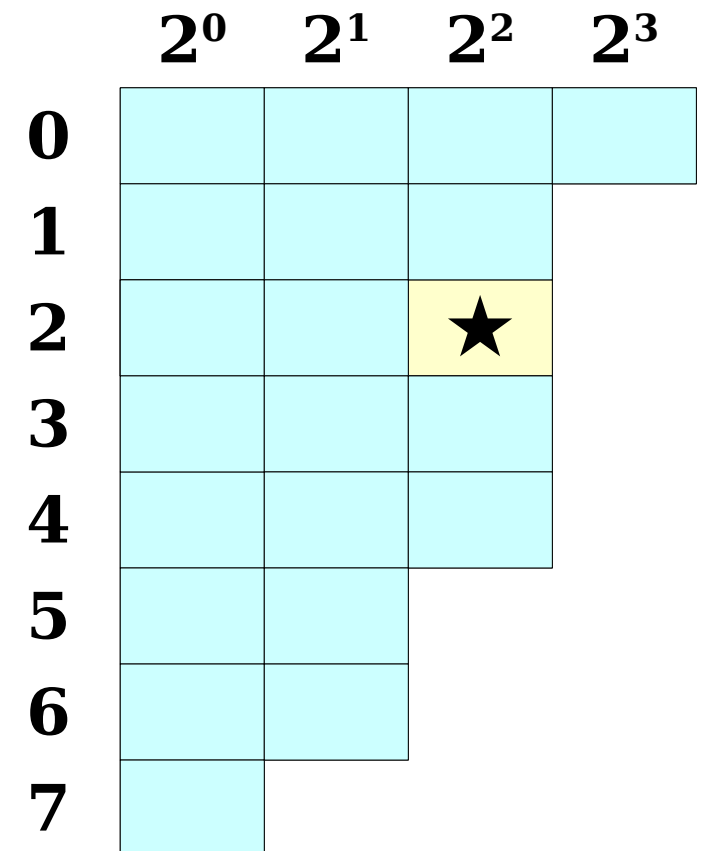
	2^0	2^1	2^2	2^3
0				
1				
2			★	
3				
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	2^0	2^1	2^2	2^3
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Precomputing the Ranges

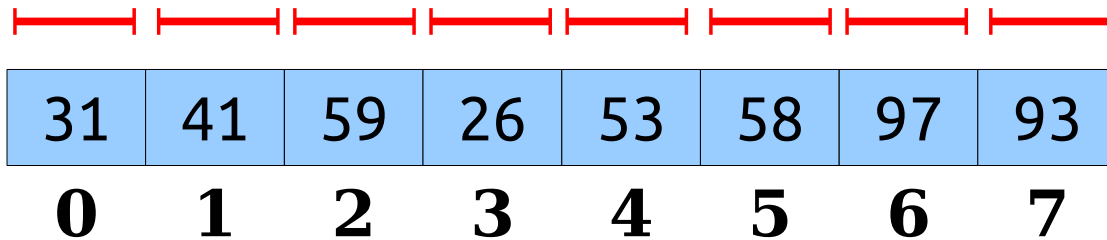
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	2^0	2^1	2^2	2^3
0				
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7				

Precomputing the Ranges

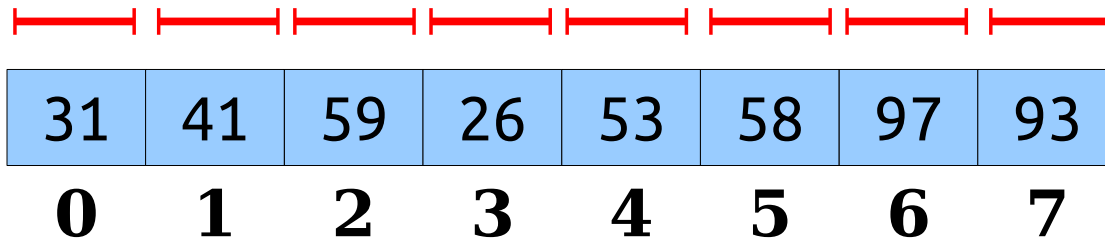
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	2^0	2^1	2^2	2^3
0				
1				
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	2^0	2^1	2^2	2^3
0	31			
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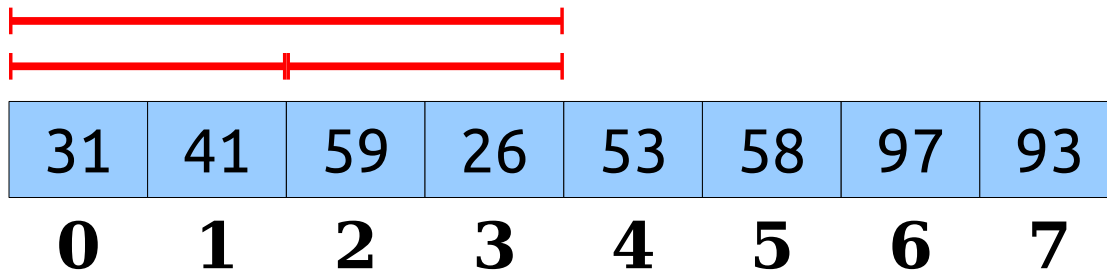


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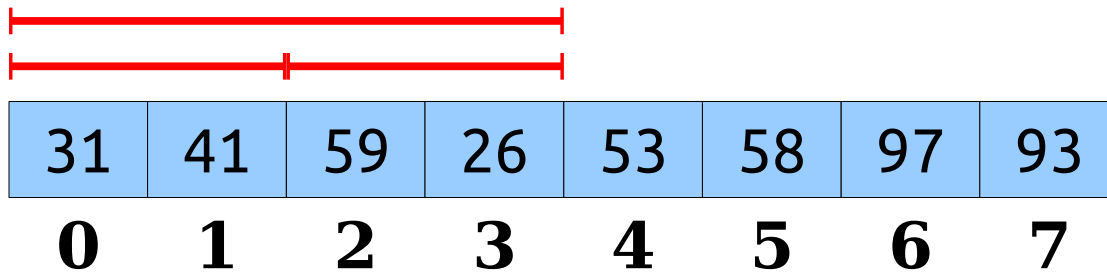
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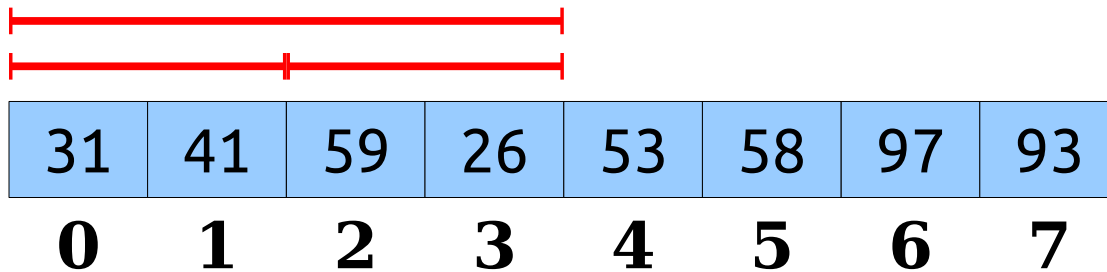
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2	59	26	26	
3	26	26	26	
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Sparse Tables

- This data structure is called a ***sparse table***.
- It gives an $\langle \mathbf{O(n \log n)}, \mathbf{O(1)} \rangle$ solution to RMQ.
- This is asymptotically better than precomputing all possible ranges!

The Story So Far

- We now have the following solutions for RMQ:
 - Precompute all: $\langle O(n^2), O(1) \rangle$.
 - Sparse table: $\langle O(n \log n), O(1) \rangle$.
 - Blocking: $\langle O(n), O(n^{1/2}) \rangle$.
 - Precompute none: $\langle O(1), O(n) \rangle$.
- ***Can we do better?***

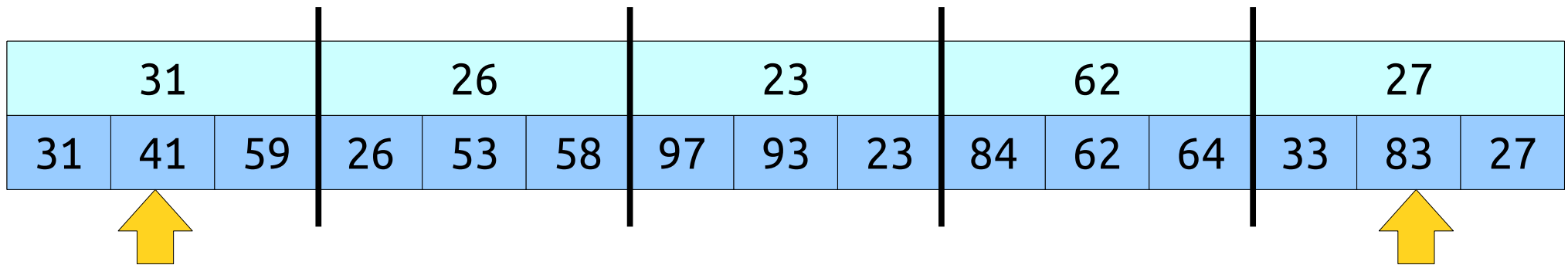
A Third Approach: ***Hybrid Strategies***

Blocking Revisited

31			26			23			62			27		
31	41	59	26	53	58	97	93	23	84	62	64	33	83	27

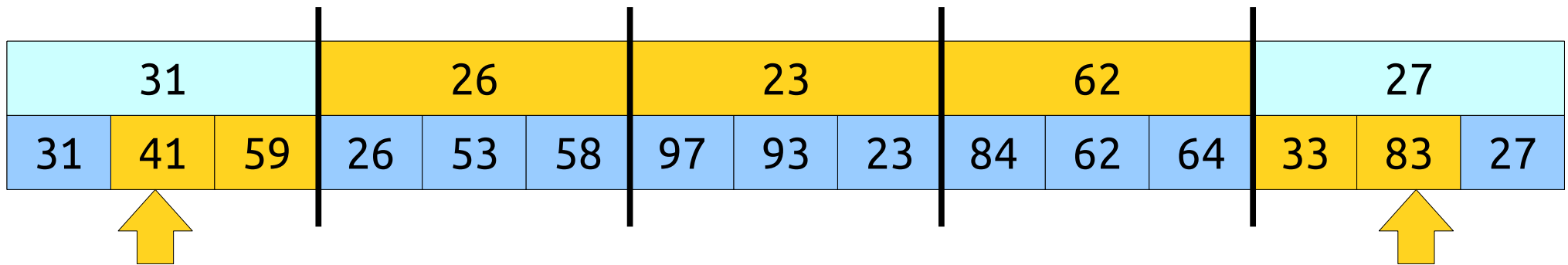
Blocking Revisited

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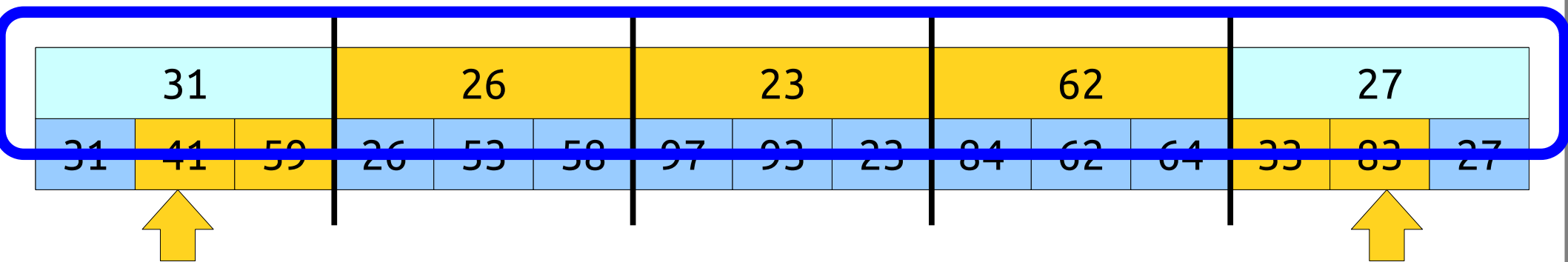


The diagram illustrates a blocking merge sort process. It shows a 15-element array divided into five segments by vertical lines. The top row of the table shows the segment keys: 31, 26, 23, 62, and 27. The bottom row shows the actual elements: 31, 41, 59, 26, 53, 58, 97, 93, 23, 84, 62, 64, 33, 83, and 27. Yellow arrows point to the elements 41 and 83, which are the first elements of the second and fourth segments respectively.

Blocking Revisited

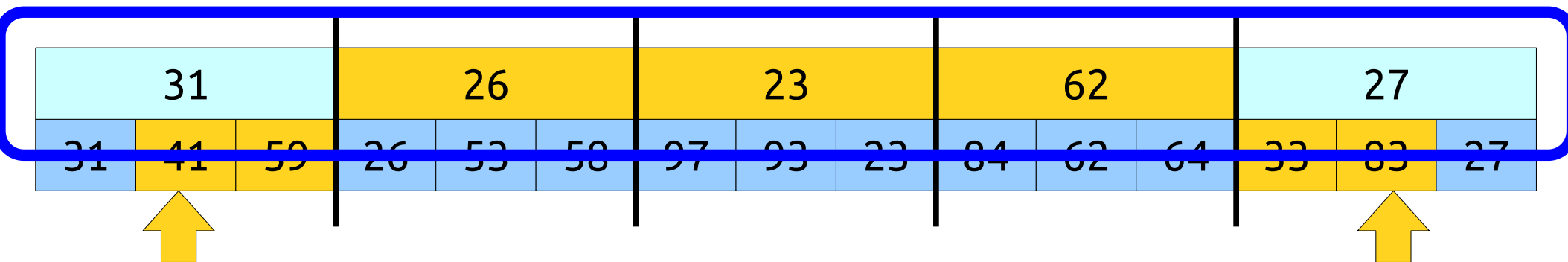


Blocking Revisited

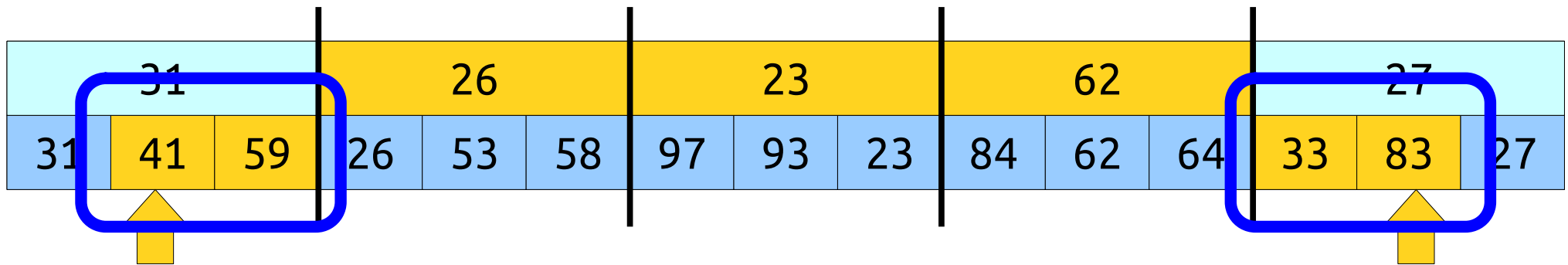


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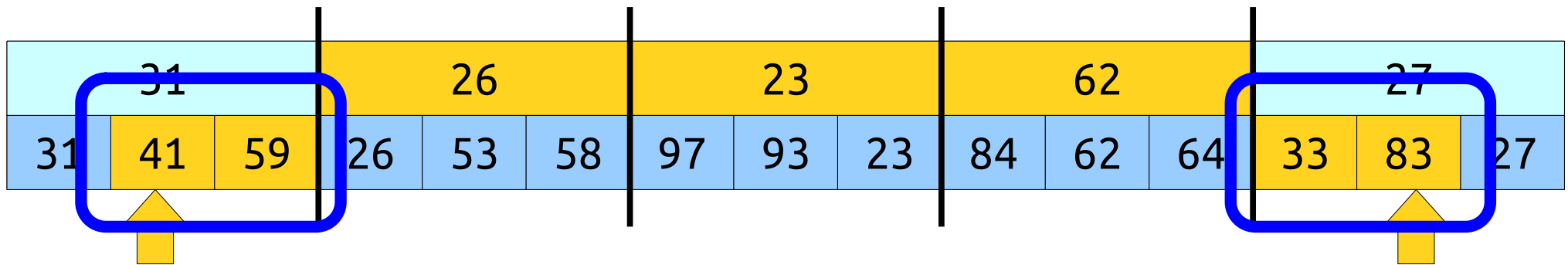
This is just RMQ on the block minima!



Blocking Revisited



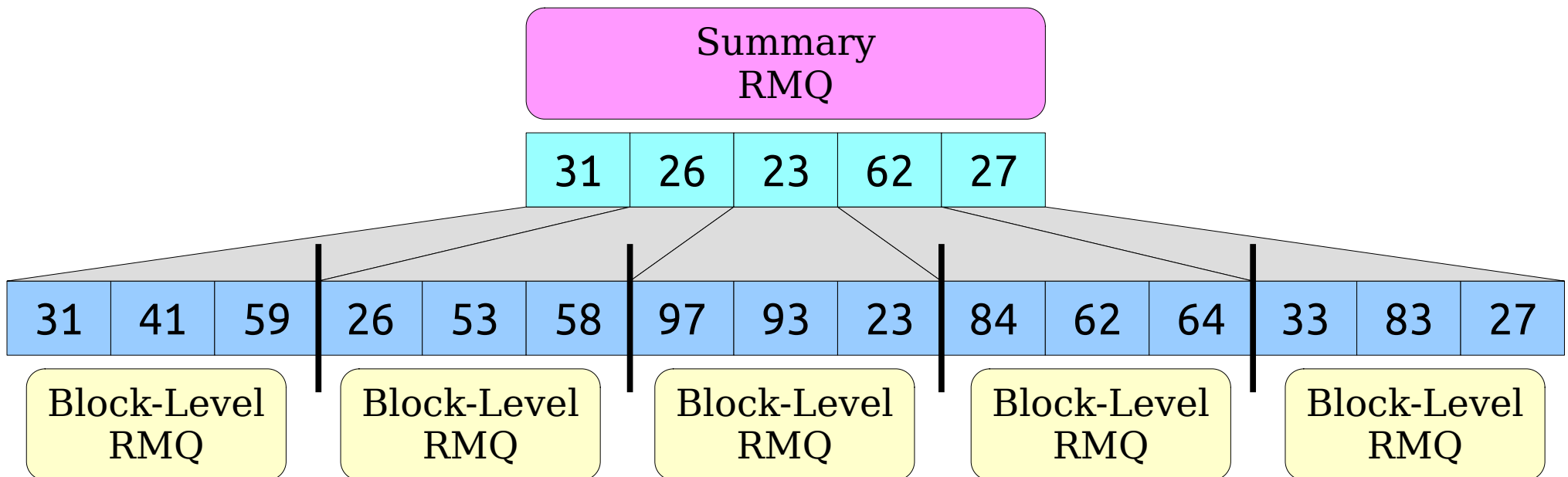
Blocking Revisited



*This is just RMQ
inside the blocks!*

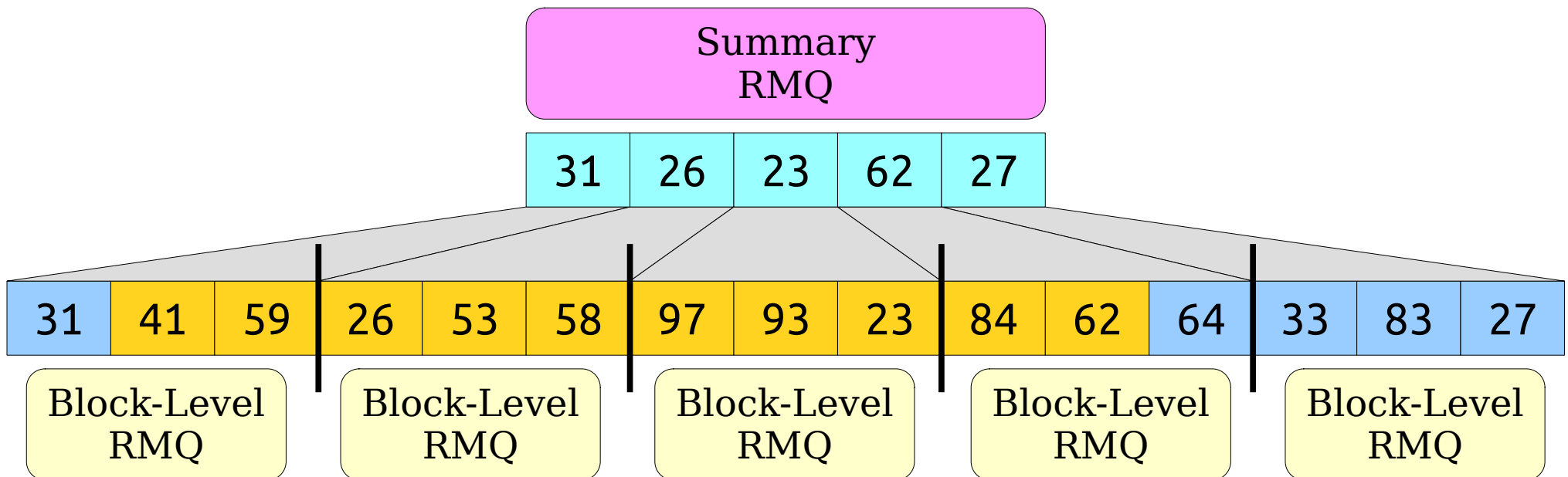
The Framework

- Split the input into blocks of size b .
- Form an array of the block minima.
- Construct a “summary” RMQ structure over the block minima.
- Construct “block” RMQ structures for each block.
- Aggregate the results together.



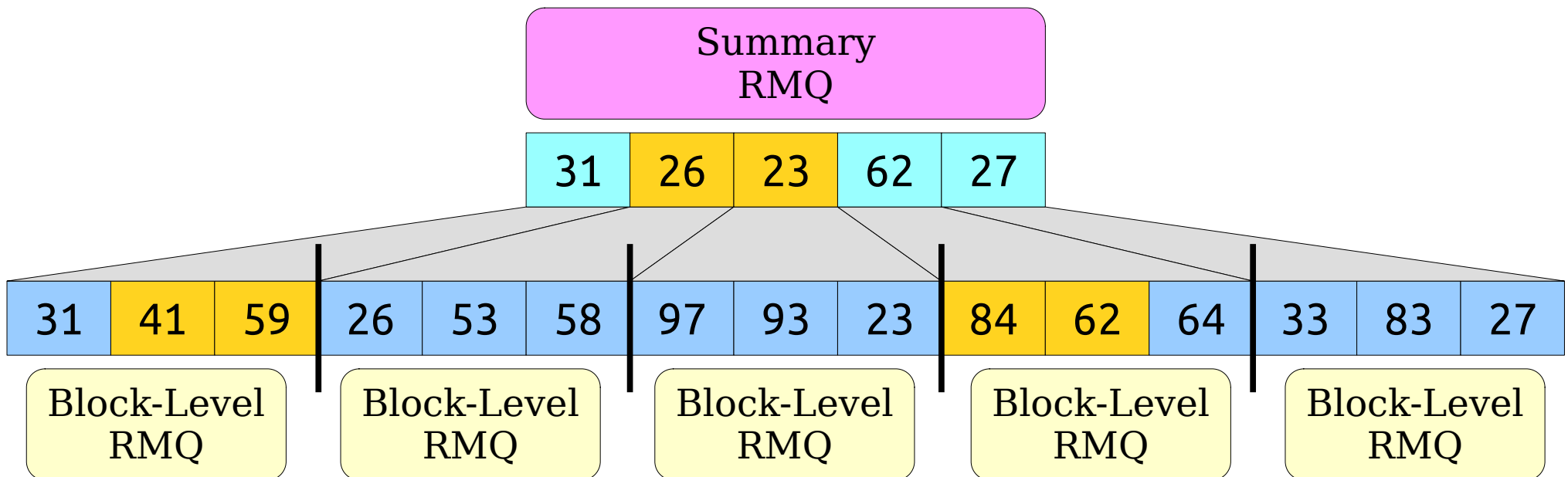
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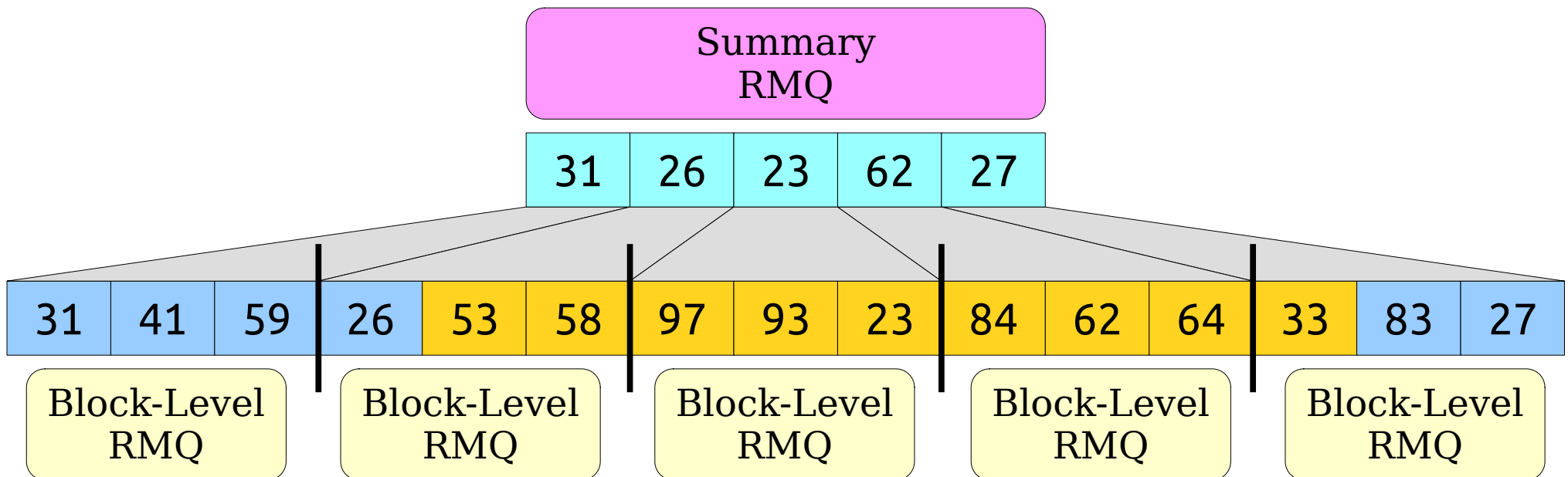
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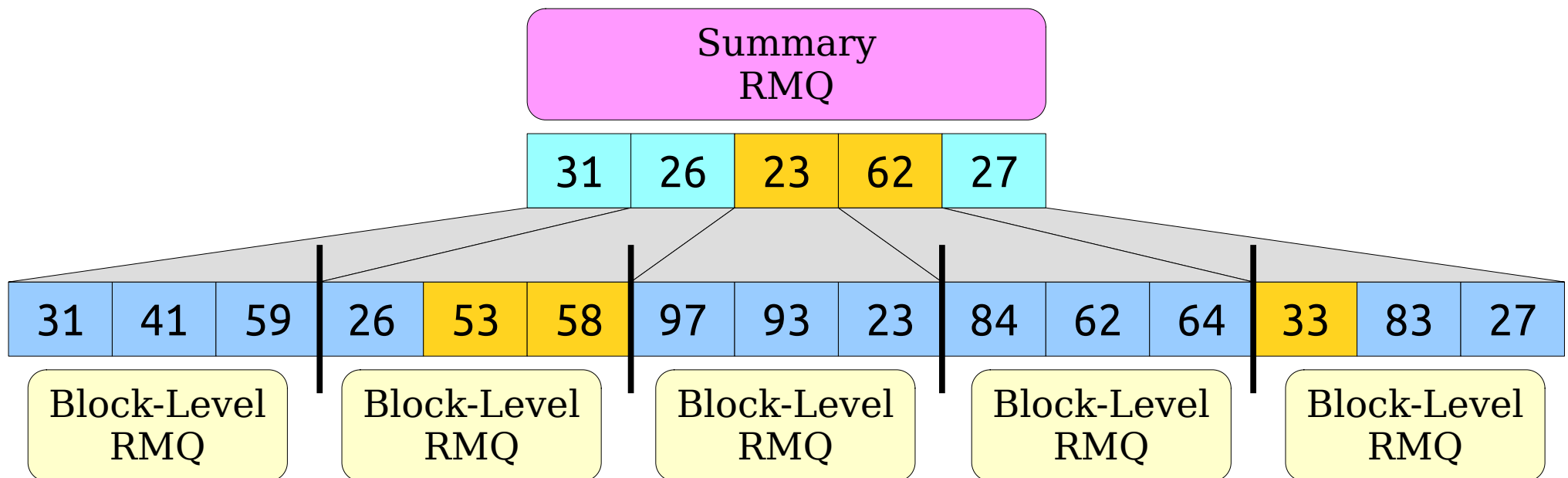
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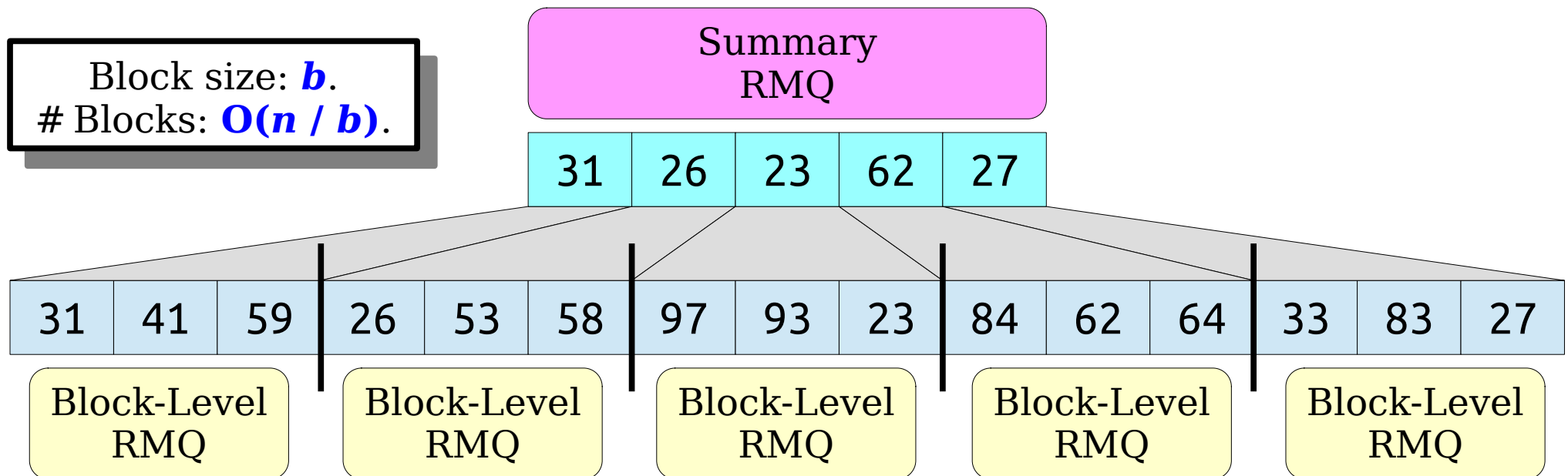
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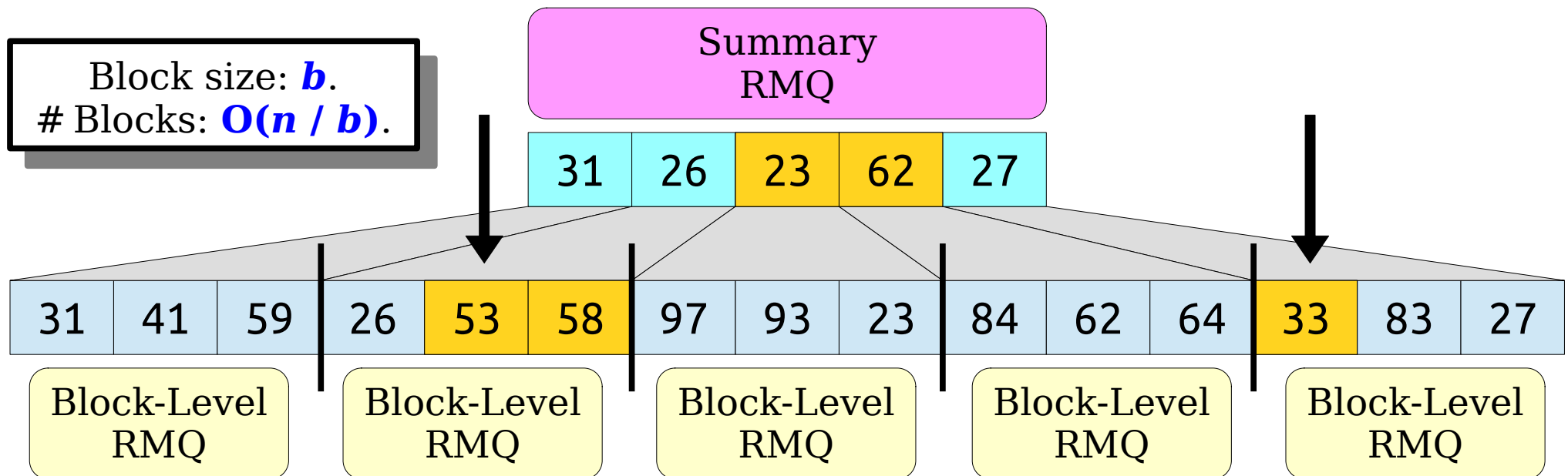
Analyzing Efficiency

- Suppose we use a $\langle p_1(n), q_1(n) \rangle$ -time RMQ for the summary RMQ and a $\langle p_2(n), q_2(n) \rangle$ -time RMQ for each block, with block size b .
- What is the preprocessing time for this hybrid structure?
 - $O(n)$ time to compute the minima of each block.
 - $O(p_1(n / b))$ time to construct RMQ on the minima.
 - $O((n / b) p_2(b))$ time to construct the block RMQs.
- Total construction time is $O(n + p_1(n / b) + (n / b) p_2(b))$.



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- What is the query time for this hybrid structure?
 - $O(q_1(n / b))$ time to query the summary RMQ.
 - $O(q_2(b))$ time to query the block RMQs.
- Total query time: $O(q_1(n / b) + q_2(b))$.



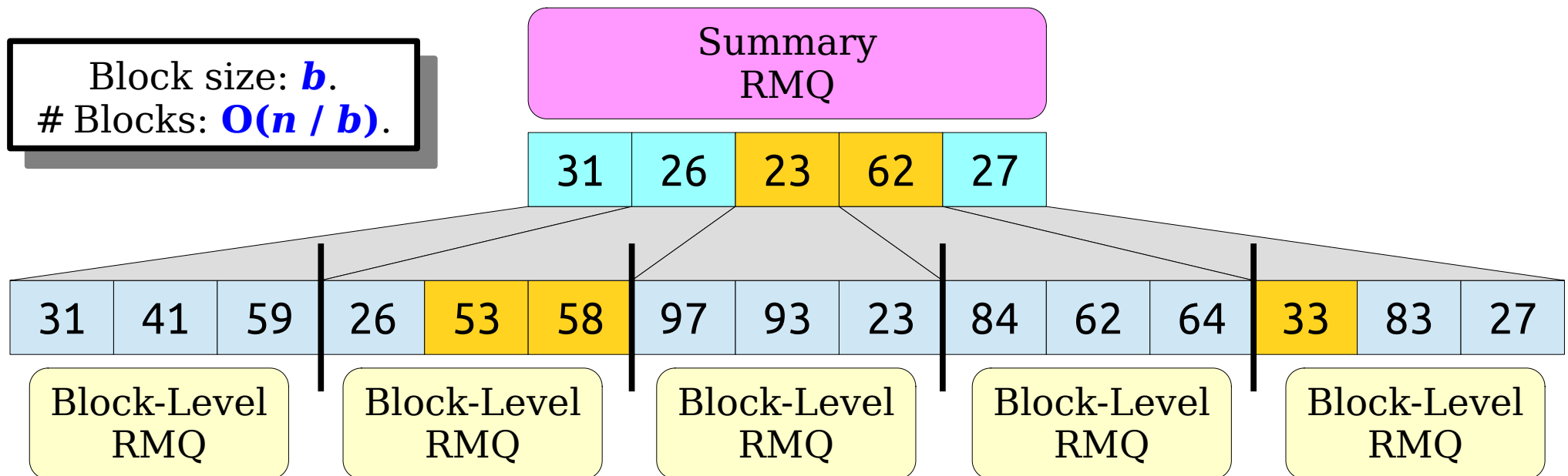
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- Hybrid preprocessing time:

$$O(n + p_1(n / b) + (n / b)p_2(b))$$

- Hybrid query time:

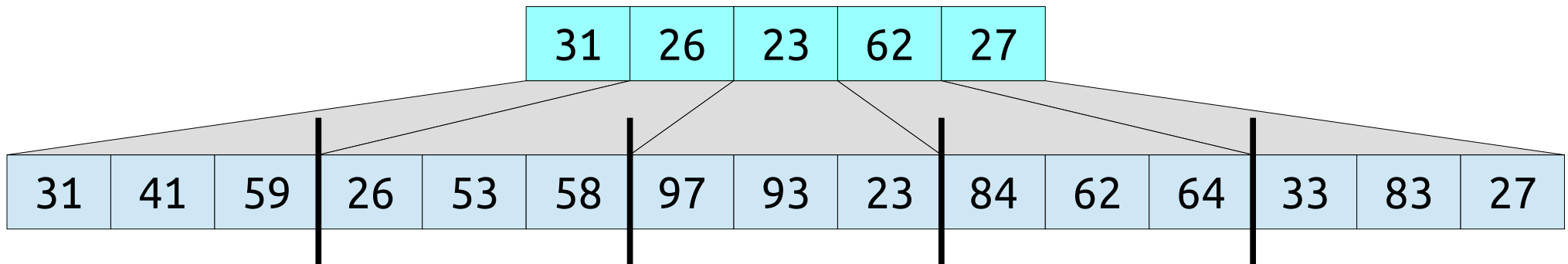
$$O(q_1(n / b) + q_2(b))$$



A Sanity Check

- The $\langle O(n), O(n^{1/2}) \rangle$ block-based structure from earlier uses this framework with the $\langle O(1), O(n) \rangle$ no-preprocessing RMQ structure and $b = n^{1/2}$.

Do no further preprocessing
than just computing the
block minima.



Don't do anything fancy per
block. Just do linear scans
over each of them.

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- Looks good so far!

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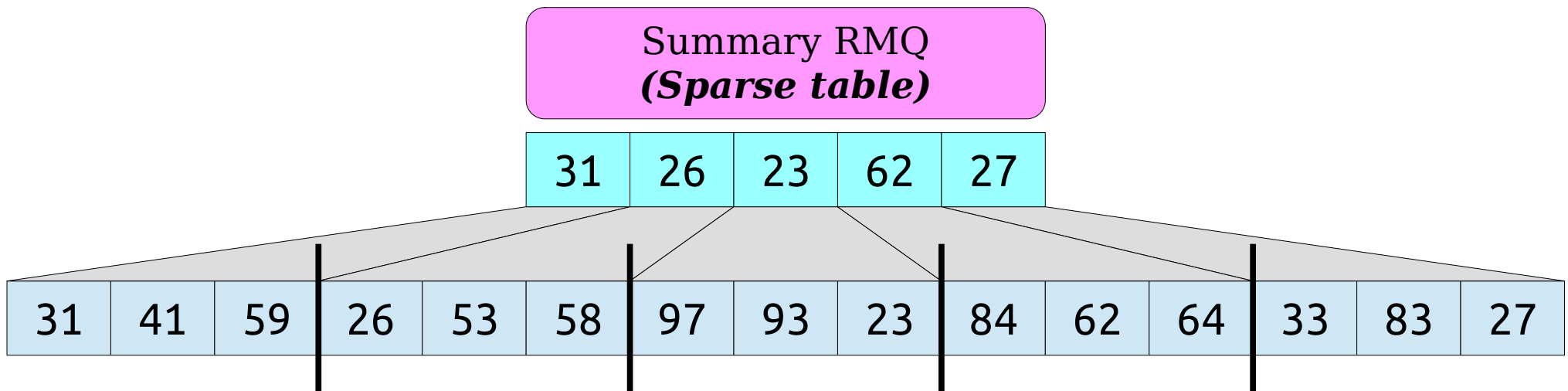
An Observation

- We can use any data structures we'd like for the summary and block RMQs.
- Suppose we use an $\langle O(n \log n), O(1) \rangle$ sparse table for the summary RMQ.
- If the block size is b , the time to construct a sparse table over the (n / b) blocks is **$O((n / b) \log (n / b))$** .
- **Cute trick:** If **$b = \Theta(\log n)$** , the time to construct a sparse table over the minima is

$$\begin{aligned} & O((n / \log n) \log(n / \log n)) \\ &= O((n / \log n) \log n) && \textit{(O is an upper bound)} \\ &= \mathbf{O(n)}. && \textit{(logs cancel out)} \end{aligned}$$

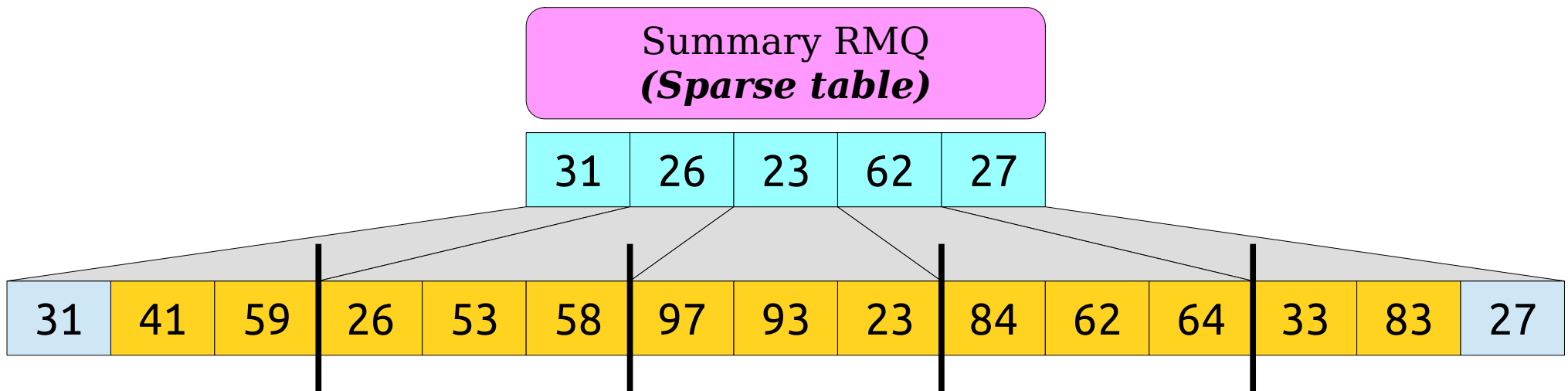
One Possible Hybrid

- Set the block size to $\log n$.
- Use a sparse table for the summary RMQ.
- Use the “no preprocessing” structure for each block.



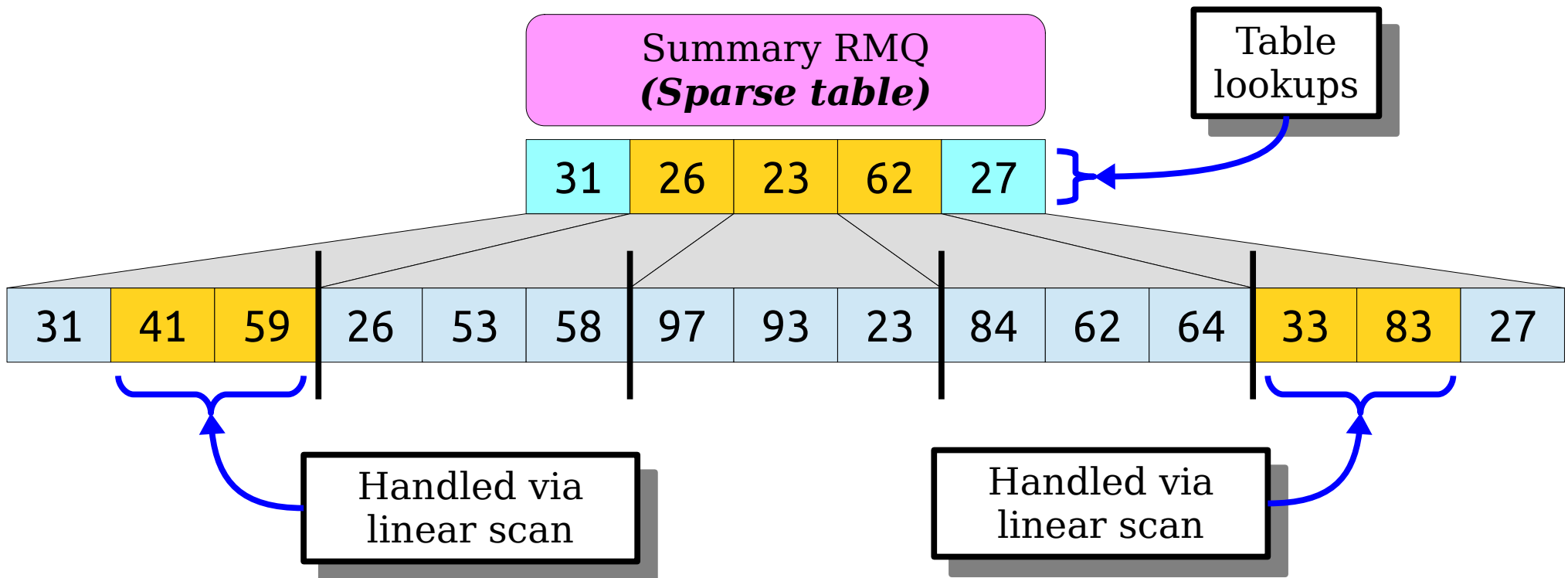
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- Preprocessing time:

$$\begin{aligned} & O(n + p_1(n / b) + (n / b) p_2(b)) \\ & = O(n + n + n / \log n) \end{aligned}$$

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$$\begin{aligned} & O(n + p_1(n / b) + (n / b) p_2(b)) \\ &= O(n + n + n / \log n) \\ &= \mathbf{O(n)} \end{aligned}$$

For Reference

$$p_1(n) = O(n \log n)$$

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One Possible Hybrid

- Set the block size to $\log n$.
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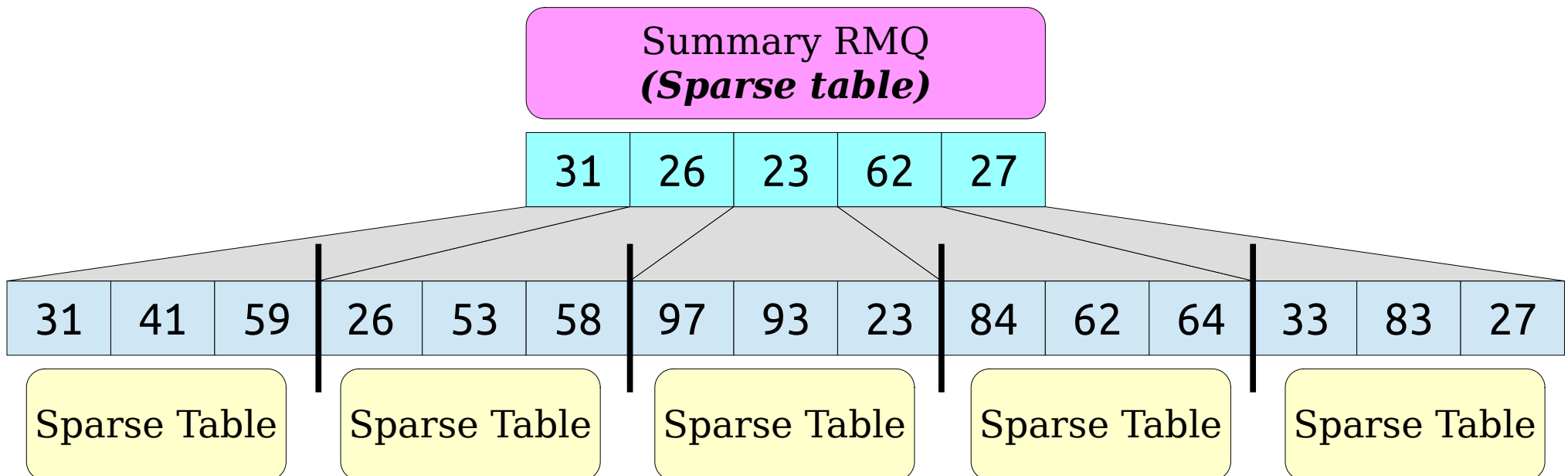
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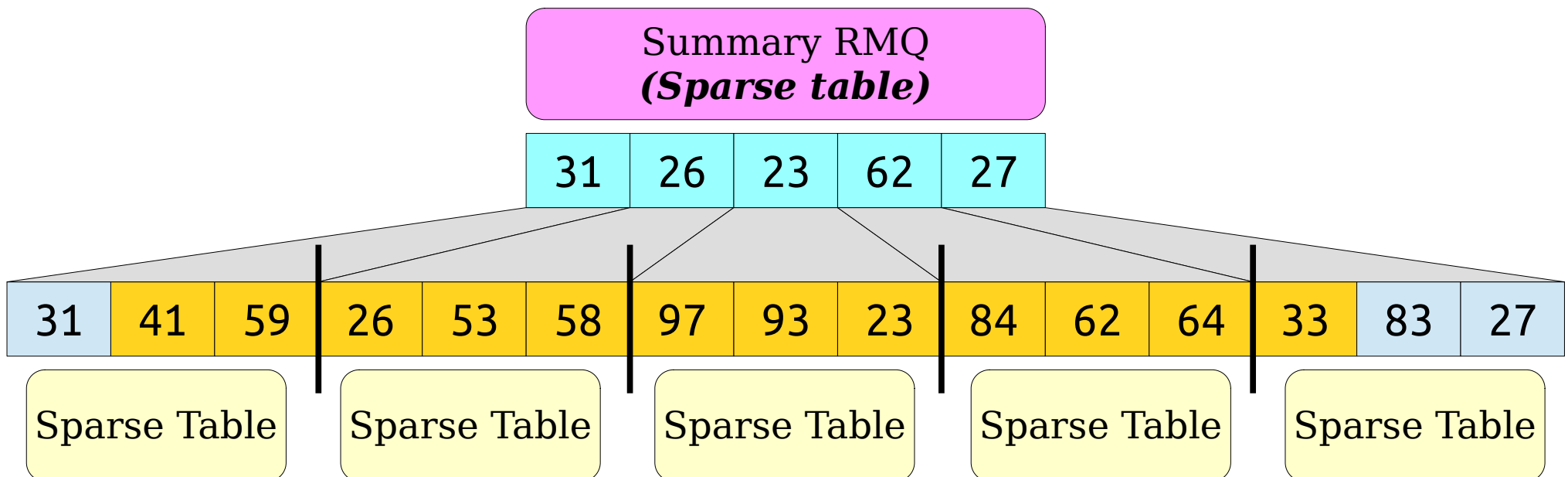
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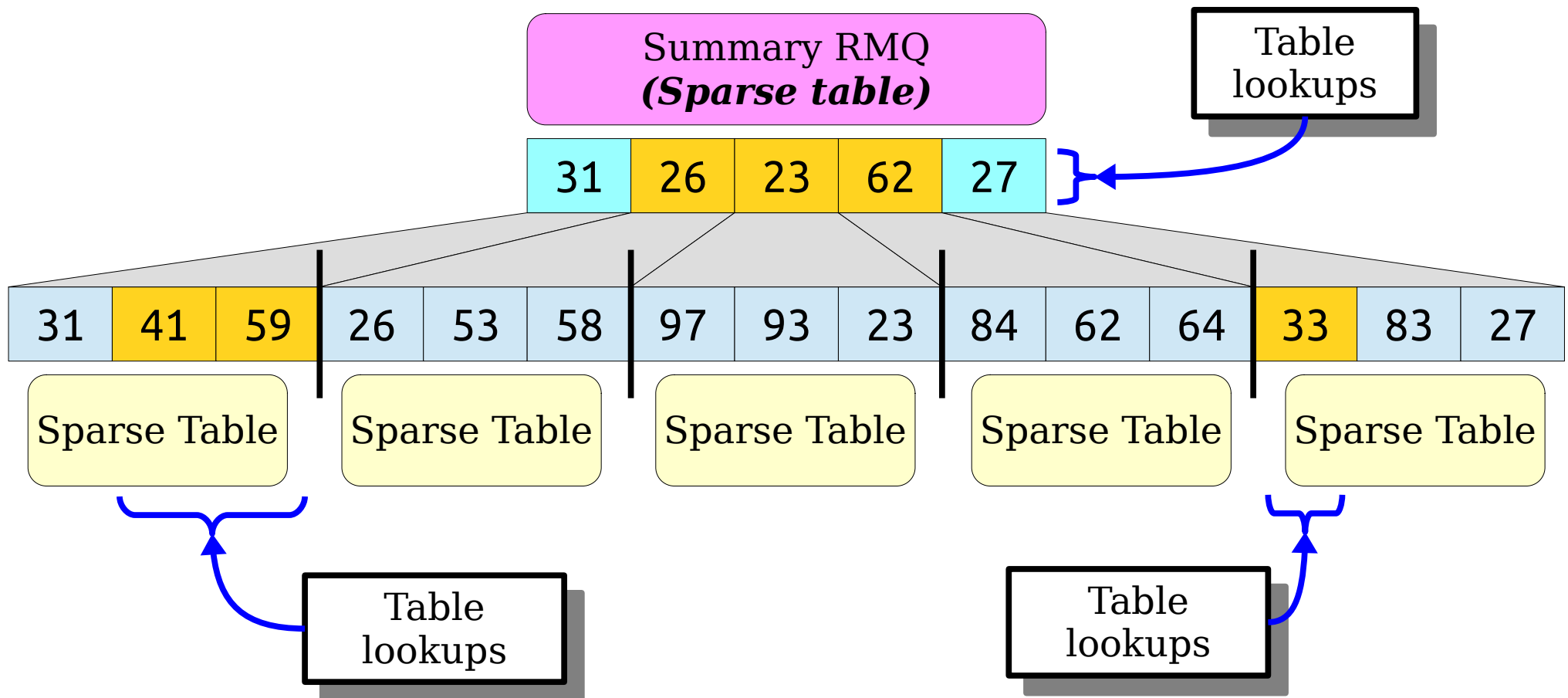
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- We have an $\langle \mathbf{O(n \log \log n)}, \mathbf{O(1)} \rangle$ solution to RMQ!

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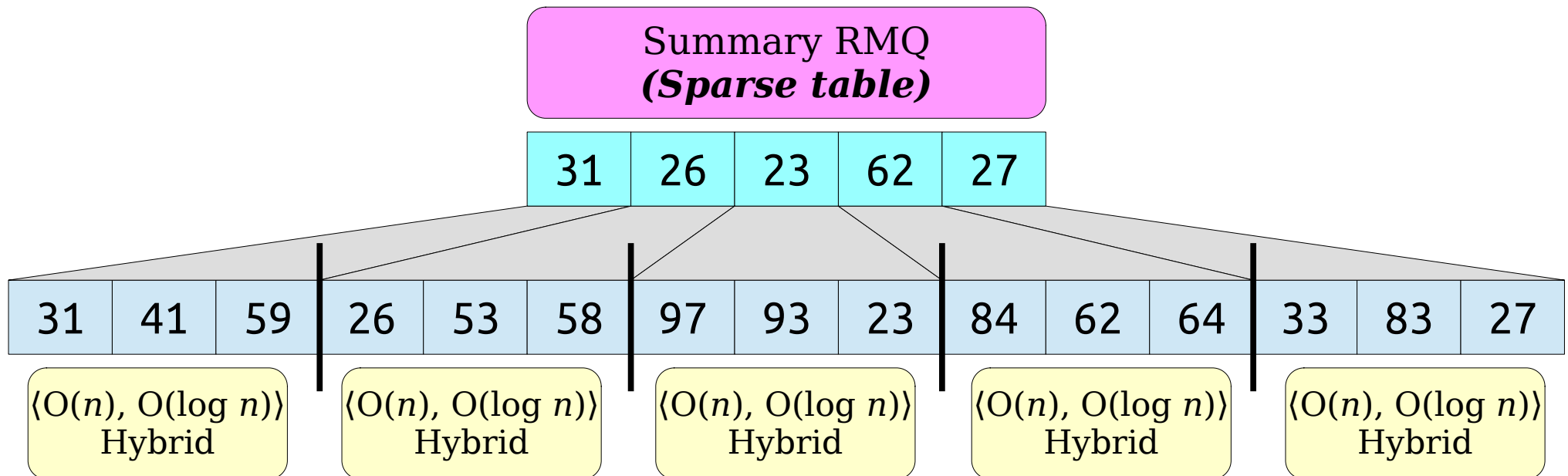
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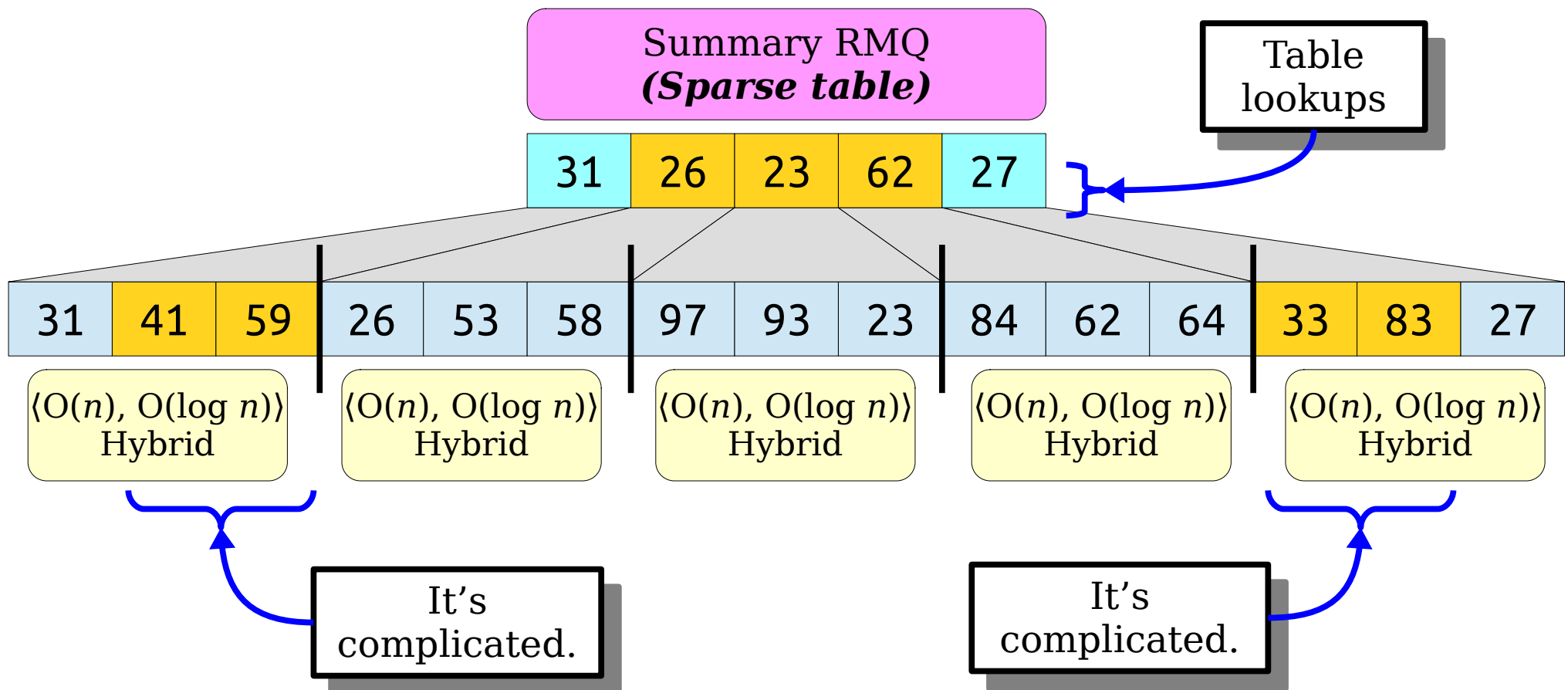
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Where We Stand

- We've seen a bunch of RMQ structures today:
 - No preprocessing: $\langle O(1), O(n) \rangle$
 - Full preprocessing: $\langle O(n^2), O(1) \rangle$
 - Block partition: $\langle O(n), O(n^{1/2}) \rangle$
 - Sparse table: $\langle O(n \log n), O(1) \rangle$
 - Hybrid 1: $\langle O(n), O(\log n) \rangle$
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 - Hybrid 3: $\langle O(n), O(\log \log n) \rangle$

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Is there an $\langle O(n), O(1) \rangle$ solution to RMQ?

Yes!

Next Time

- ***Cartesian Trees***
 - A data structure closely related to RMQ.
- ***The Method of Four Russians***
 - A technique for shaving off log factors.
- ***The Fischer-Heun Structure***
 - A clever, asymptotically optimal RMQ structure.