Consultație PS

6.02.2023

Ex 1 $X \sim \begin{pmatrix} -1 & 0 & 1 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$

Rep. $5X-2$, X^3, $X+X^2$, media și varianța

$\mathbb{P}(X < 1/8 \mid X \geq -1/8) = ?$

Sol: $Y = g(X)$

$g_1(x) = 5x-2$ $\quad Y_1 = 5X-2 \in \{-7, -2, 3\}$

$$5X-2 \sim \begin{pmatrix} -7 & -2 & 3 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$$

$$E[5X-2] = 5\,\mathbb{E}[X] - 2 = 5\cdot(-1\times 0.3 + 0\times 0.2 + 1\times 0.5) - 2$$

$$= 5\times 0.2 - 2$$

$$Var(5X-2) = 25\,Var(X) = 25\left(\mathbb{E}[X^2] - \mathbb{E}[X]^2\right)$$

0.2^2

$\mathbb{E}[X^2] = 0.8$ $\qquad X^2 \sim \begin{pmatrix} 0 & 1 \\ 0.2 & 0.8 \end{pmatrix}$

sau $\mathbb{E}[h(X)] = \sum_x h(x)\,\mathbb{P}(X=x)$

$$= (-1)^2 \times 0.3 + 0^2 \times 0.2 + 1^2 \times 0.5 = 0.8$$

$$X^3 \sim \begin{pmatrix} -1 & 0 & 1 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$$ este rep. la fel ca și X

$\mathbb{E}[X^3] = \mathbb{E}[X]$ și $Var(X^3) = Var(X)$

$$X^2 \sim \begin{pmatrix} 0 & 1 \\ 0.2 & 0.8 \end{pmatrix} \qquad X \sim \begin{pmatrix} -1 & 0 & 1 \\ 0.3 & 0.2 & 0.5 \end{pmatrix}$$

$X + X^2 \in \{0, 0, 2\}$

$$\underline{X + X^2} \sim \begin{pmatrix} 0 & 2 \\ 0.5 & 0.5 \end{pmatrix}$$

$$\begin{cases} X = -1 \\ (-1) + (-1)^2 = 0 \\ X = 0 \\ 0 + 0^2 = 0 \\ X = 1 \\ 1 + 1^2 = 2 \end{cases}$$

$\mathbb{P}(X + X^2 = 2) = \mathbb{P}(X = 1) = 0.5$

$\mathbb{E}[X + X^2] = 0 \times 0.5 + 2 \times 0.5 = 1$

altfel $\mathbb{E}[X + X^2] = \mathbb{E}[X] + \mathbb{E}[X^2] = 0.2 + 0.8 = 1$

sau

$$\mathbb{E}[\underbrace{X + X^2}_{h(x)}] = \underbrace{(-1 + (-1)^2)}_{h(-1)} \times 0.3 + \overbrace{(0 + 0^2)}^{h(0)} \times 0.2 + \underbrace{(1 + 1^2)}_{h(1)} \times 0.5 = 1$$

$$Var(\underbrace{X + X^2}_{Y}) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 \qquad Y \sim \begin{pmatrix} 0 & 2 \\ 0.5 & 0.5 \end{pmatrix}$$

$\mathbb{E}[Y] = 1$

$\mathbb{E}[Y^2] = 0^2 \times 0.5 + 2^2 \times 0.5 = 4 \times 0.5 = 2 \Rightarrow Var(Y) = 1$

$$\mathbb{P}(\underbrace{X < 1/8}_{A} \mid \underbrace{X \geq -1/8}_{B}) = \mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

$$= \frac{\mathbb{P}(X < 1/8 \text{ și } X \geq -1/8)}{\mathbb{P}(X \geq -1/8)}$$

$X \in \{-1, 0, 1\}$

$\{X < 1/8 \text{ și } X \geq -1/8\} = \{X = 0\}$

$\{X \geq -1/8\} = \{X = 0\} \cup \{X = 1\}$

$$\mathbb{P}(X < 1/8 \mid X \geq -1/8) = \frac{\mathbb{P}(X = 0)}{\mathbb{P}(X = 0) + \mathbb{P}(X = 1)} = \frac{0.2}{0.2 + 0.5} = \frac{2}{7}$$

$$X \sim \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_m \\ p_1 & p_2 & p_3 & & p_m \end{pmatrix}$$

$$Y \sim \begin{pmatrix} y_1 & y_2 & \dots & y_n \\ q_1 & q_2 & \dots & q_n \end{pmatrix}$$

$$X + Y \in \{x_i + y_j, \ i \in \{1, \dots, m\}, \ j \in \{1, \dots, n\}\}$$

$$X \sim \begin{pmatrix} -1 & 0 & 1 \end{pmatrix} \quad Y \sim \begin{pmatrix} -2 & 0 & 2 \end{pmatrix}$$

$X + Y \in \{-3, -1, 1, -2, 0, 2, -1, 1, 3\}$

$X + Y \in \{-3, -2, -1, 0, 1, 2, 3\}$

$$\mathbb{P}(X+Y=-1) = \mathbb{P}(X=-1, Y=0) + \mathbb{P}(X=1, Y=-2)$$

$$\mathbb{P}(X+Y=a) = \sum_{x+y=a} \mathbb{P}(X=x, Y=y)$$

X\Y	-2	0	2
-1			
0			
1			

$X \sim \begin{pmatrix} -1 & 0 & 1 \\ \_ & \_ & \_ \end{pmatrix}$

$Y \sim \begin{pmatrix} -2 & 0 & 2 \\ \_ & \_ & \_ \end{pmatrix}$

$X \perp\!\!\!\perp Y$

Ex 2 (X,Y)

X\Y	2	4	6
0	0.1	0.2	0.1
1	0.1	0.1	0.1
2	0.1	0.1	0
3	0.05	0	0.05

Tabelul de valori reprez. introduce o rep comună?

a) Rep. marginale pt X și Y, $\mathbb{E}[X]$, $Var(X)$, $\mathbb{E}[Y]$, $Var(Y)$

b) Rep cond. a lui Y la $X=1$
X la $Y=4$

c) Calculați media cond. $\underline{\mathbb{E}[Y|X]}$ și $Var(Y|X)$

d) Calculați coef. de corelație

e) Verificați dacă are loc egalitatea:

$$Var(Y) = \mathbb{E}[Var(Y|X)] + Var(\mathbb{E}[Y|X])$$

Sol:

X \ Y	2	4	6	Σ
0	0.1	0.2	0.1	0.4
1	0.1	0.1	0.1	0.3
2	0.1	0.1	0	0.2
3	0.05	0	0.05	0.1
Σ	0.35	0.4	0.25	

$f(x,y) = \mathbb{P}(X=x, Y=y)$

Trebuie să verificăm i) $f(x,y) \geq 0$, $\forall x,y$

ii) $\sum_{x,y} f(x,y) = 1$

$1+4a = 1 \Rightarrow a = 0$

a) Rep. marginală a lui X (Σ linii)

$$X \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0.4 & 0.3 & 0.2 & 0.1 \end{pmatrix}$$

Rep. marginală a lui Y (Σ coloane)

$$Y \sim \begin{pmatrix} 2 & 4 & 6 \\ 0.35 & 0.4 & 0.25 \end{pmatrix}$$

Coef. de corelație:

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}}$$

$$Cov(X,Y) = \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])]$$
$$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

$$\mathbb{E}[XY] = \sum_{x,y} xy\, \mathbb{P}(X=x, Y=y)$$

$$\mathbb{E}[h(X,Y)] = \sum_{x,y} h(x,y)\, \mathbb{P}(X=x, Y=y)$$

$$= 0\times2\times0.1 + 0\times4\times0.2 + 0\times6\times0.1 +$$
$$1\times2\times0.1 + 1\times4\times0.1 + 1\times6\times0.1 +$$
$$2\times2\times0.1 + 2\times4\times0.1 + 2\times6\times0 +$$
$$3\times2\times0.05 + 3\times4\times0 + 3\times6\times0.05 = \ldots$$

b) Rep. cond. a lui Y la $\underline{X=1}$

$X \backslash Y$	2	4	6	
—	—	—		
1	$0.1/0.3$	$0.1/0.3$	$0.1/0.3$	0.3

$$Y \mid X=1 \sim \begin{pmatrix} 2 & 4 & 6 \\ \frac{0.1}{0.3} & \frac{0.1}{0.3} & \frac{0.1}{0.3} \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$$

$$Y \mid X=3 \sim \begin{pmatrix} 2 & 4 & 6 \\ \frac{0.05}{0.1} & 0 & \frac{0.05}{0.1} \end{pmatrix} = \begin{pmatrix} 2 & 4 & 6 \\ \frac{1}{2} & 0 & 1/2 \end{pmatrix}$$

$X \mid Y=4$

X \ Y	2	4	6
0		0.2/0.4	
1		0.1/0.4	
2		0.1/0.4	
3		0/0.4	
		0.4	

$$X \mid Y=4 \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{0.2}{0.4} & \frac{0.1}{0.4} & \frac{0.1}{0.4} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 \end{pmatrix}$$

c) $\mathbb{E}[Y \mid X]$ este o var. aleatoare care ia val. $\mathbb{E}[Y \mid X=x]$

$$\mathbb{E}[Y \mid X=0] = 2\,\mathbb{P}(Y=2 \mid X=0) + 4\,\mathbb{P}(Y=4 \mid X=0) + 6\,\mathbb{P}(Y=6 \mid X=0) = \underline{4}$$

$$\mathbb{E}[Y \mid X=1] = 2 \times 1/3 + 4 \times 1/3 + 6 \times 1/3 = \underline{4}$$

$$\mathbb{E}[Y \mid \underline{X=2}] = \underline{3} \qquad \mathbb{E}(Y \mid X=3) = \underline{4}$$

$$\underline{\mathbb{E}[Y|X]} \sim \begin{pmatrix} 3 & 4 \\ \mathbb{P}(X=2) & \mathbb{P}(X\neq 2) \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0.2 & 0.8 \end{pmatrix}$$

$Var(Y|X) = ?$ este tot o v.a care ia valorile $Var(Y|X=x)$

$$Var(Y|X=0) = \mathbb{E}[Y^2|X=0] - \underbrace{\mathbb{E}[Y|X=0]}^2$$

$$= \left(2^2 \times \mathbb{P}(Y=2|X=0) + 4^2 \times \mathbb{P}(Y=4|X=0) + 6^2 \times \mathbb{P}(Y=6|X=0)\right) - \mathbb{E}[Y|X=0]^2$$

$$= 2^2 \times \frac{0.1}{0.4} + 4^2 \times \frac{0.2}{0.4} + 6^2 \times \frac{0.1}{0.4} - 16$$

$$= 1 + 8 + 9 - 16 = \underline{2}$$

$Var(Y|X=1) = \underline{2.66}$

$Var(Y|X=2) = \underline{1}$

$Var(Y|X=3) = \underline{4}$

$$\| \begin{pmatrix} 1 & 2 & 2.66 & 4 \\ 0.2 & 0.4 & 0.3 & 0.1 \end{pmatrix}$$

$$\underline{Var(Y|X)} \sim \begin{pmatrix} 1 & 2 & 2.66 & 4 \\ \mathbb{P}(X=2) & \mathbb{P}(X=0) & \mathbb{P}(X=1) & \mathbb{P}(X=3) \end{pmatrix}$$

$$\underline{Var(Y)} = \mathbb{E}[\underline{Var(Y|X)}] + Var(\underline{\mathbb{E}[Y|X]})$$

Ex 3

$$f(x) = \begin{cases} 1/x, & x \in [1-c, 1+c] \\ 0, & \text{altfel} \end{cases}$$

$c \in (0,1)$

a) Det. const c a.î. f să fie densitate

b) $X \sim f$, $\mathbb{E}[X]$ și $Var(X)$

c) Calc. f. de rep. și să scrieți un cod R care să trateze f. de rep.

Sol: a) f este densitate de rep.

i) $f(x) \geq 0$, $\forall x$

ii) $\int f(x)\,dx = 1$

$f(x) \geq 0$ este adevărată pt că $c \in (0,1)$ și $\pm c \geq 0$

și cum $x \geq 1 - c \geq 0 \Rightarrow \frac{1}{x} > 0$

$$\int f(x)\,dx = 1 \Leftrightarrow \int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{+\infty} \frac{1}{x} \mathbb{1}_{[1-c,1+c)}(x)\,dx$$

$$= \int_{1-c}^{1+c} \frac{1}{x}\, dx = \ln\left(\frac{1+c}{1-c}\right) = 1$$

$$\Leftrightarrow \frac{1+c}{1-c} = e \Leftrightarrow \boxed{c = \frac{e-1}{e+1}}$$

b) $\mathbb{E}[X]$, $Var(X)$

$$\mathbb{E}[X] = \int x f(x)\, dx = \int_{1-c}^{1+c} \cancel{x} \cdot \frac{1}{\cancel{x}}\, dx = 2c = 2\frac{e-1}{e+1}$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\mathbb{E}[X^2] = \int x^2 f(x)\, dx = \int_{1-c}^{1+c} x^{\cancel{2}} \frac{1}{\cancel{x}}\, dx = \left. x^2/2 \right|_{1-c}^{1+c}$$

$$= \frac{(1+c)^2 - (1-c)^2}{2} = 2c = 2\frac{e-1}{e+1}$$

$$Var(X) = 2c - (2c)^2$$

c) Fct de rep $F(x)$

$$F(x) = \int_{-\infty}^{x} f(t)\, dt = \int_{-\infty}^{x} \frac{1}{t} \mathbb{1}_{[1-c, 1+c]}(t)\, dt$$

Dacă $x < 1-c$ atunci $F(x) = 0$

$x > 1+c$ atunci $F(x) = 1$

Dacă $x \in [1-c, 1+c]$

$$F(x) = \int_{1-c}^{x} \frac{1}{t}\, dt = \ln|t| \Big|_{1-c}^{x} = \ln\left(\frac{x}{1-c}\right)$$


```
cdf_F <- function(x, c = 0.5){
    ifelse(x < 1-c, 0,
        ifelse(x > 1-c, 1, log(x/(1-c))))
}
t <- seq(-1, 2, length.out = 200)
plot(t, cdf_F(t), type = "l")
```

Ex 4 (X, Y) admite densitatea

$$f(x,y) = \begin{cases} k(x+y+1), & x \in [0,1],\ y \in [0,2] \\ 0, & \text{altfel} \end{cases}$$

a) Să se det k

b) Să se det densitățile marginale

c) Să se verifice dacă X și Y sunt indep

d) Să se det. densitățile cond. $f_{X|Y}$ și $f_{Y|X}$

Sol: a) $f(x,y)$ densitate de prob. $\begin{cases} f(x,y) \geq 0 \\ \iint f(x,y)\,dx\,dy = 1 \end{cases}$

$f \geq 0 \Rightarrow k \geq 0$

$$\iint f(x,y)\,dx\,dy = 1 \iff \int_0^1 \int_0^2 k(x+y+1)\,dy\,dx = 1$$

$$\iff \int_0^1 k\left(xy + \frac{y^2}{2} + y\right)\Big|_0^2 dx = 1$$

$$\iff \int_0^1 k\left(2x + \frac{4}{2} + 2\right) dx = 1$$

$$\iff k\left(2x^2/2 + 4x\right)\Big|_0^1 = 1 \iff (1+4)k = 1 \iff \boxed{k = 1/5}$$

$$\int_0^2 \int_0^1 k(x+y+1)\,dx\,dy = 1$$

b) Densitatile marginale

$$f_X(x) = \int f(x,y)\,dy$$

$$f_Y(y) = \int f(x,y)\,dx$$

$$\underline{f_X(x)} = \int f(x,y)\,dy = \int \frac{1}{5}(x+y+1)\,\mathbb{1}_{[0,1]}(x)\,\mathbb{1}_{[0,2]}(y)\,dy$$

$$= \mathbb{1}_{[0,1]}(x) \int_0^2 \frac{1}{5}(x+y+1)\,dy$$

$$= \mathbb{1}_{[0,1]}(x)\,\frac{1}{5}\left(xy + y^2/2 + y\right)\Big|_0^2$$

$$= \frac{2x+4}{5}\,\mathbb{1}_{[0,1]}(x)$$

$$\underline{f_Y(y)} = \int f(x,y)\,dx = \int_0^1 \frac{x+y+1}{5}\,dx\,\mathbb{1}_{[0,2]}(y)$$

$$= \frac{2y+3}{10}\,\mathbb{1}_{[0,2]}(y)$$

c) Observăm că

$$f(x,y) \neq f_X(x)\cdot f_Y(y) \Rightarrow X \text{ și } Y \text{ nu sunt indep.}$$

d) Densitățile condiționate

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{\frac{x+y+1}{5}\,\mathbb{1}_{[0,1]}(x)\,\mathbb{1}_{[0,2]}(y)}{\frac{2y+3}{10}\,\mathbb{1}_{[0,2]}(y)}$$

$$= \begin{cases} \dfrac{2(x+y+1)}{2y+3}, & x\in[0,1],\ y\in[0,2] \\[2ex] 0, & \text{altfel} \end{cases}$$

În mod similar se determină

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

Dacă vrem $E[X|Y=y] = \int x\, f_{X|Y}(x|y)\,dx$

$$= \int_0^1 x\,\frac{2(x+y+1)}{2y+3}\,dx \quad \text{pt } y\in[0,2]$$

o fct de Y

Determinați $\mathbb{E}[X|Y]$ ← v.a

$$\mathbb{E}[X|Y] = g(Y) \quad \text{unde } g(y) = \underline{\mathbb{E}[X|Y=y]}$$

$$\mathbb{P}(X \in A, Y \in B) = \mathbb{P}((X,Y) \in A \times B)$$

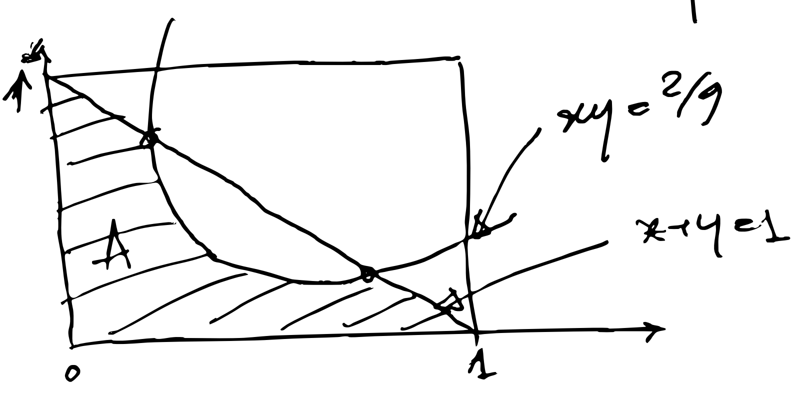
$$= \iint_{A \times B} f(x,y)\, dx\, dy$$

Ex5 Să se determine prob. ca suma a 2 nr. alese la întâmplare în [0,1] să nu depășească valoarea 1 iar produsul lor să nu depășească 2/9.

Sol: $X, Y \sim \mathcal{U}([0,1])$ indep

$$\mathbb{P}(X+Y \le 1,\ XY \le 2/9) = ?$$

$$= \mathbb{P}((X,Y) \in A), \quad A = \{(x,y) \in [0,1]^2 \mid x+y \le 1,\ xy \le 2/9\}$$



$$\mathbb{P}(X+Y\le 1, XY\le 2/9) = \iint f(x,y)\,dx\,dy$$

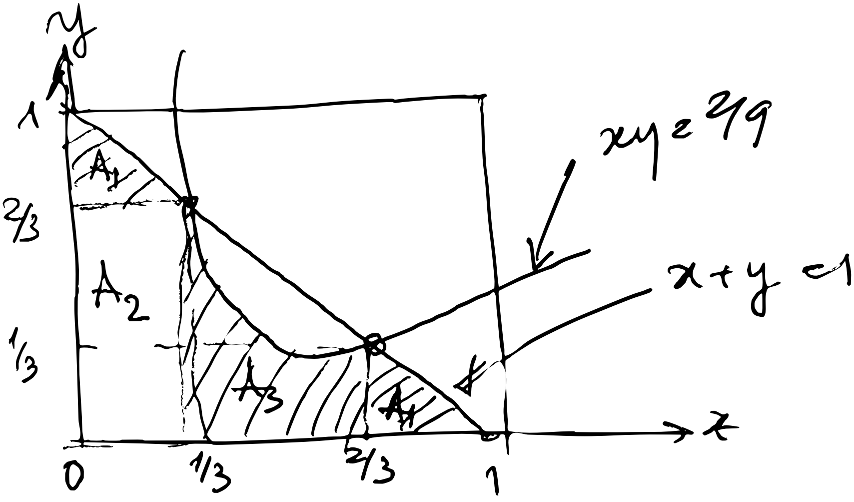
Densitatea comună $f(x,y)$ a A este

$$f(x,y) = f_X(x) f_Y(y) \quad \text{din indep}$$

$$= \mathbb{1}_{[0,1)}(x)\ \mathbb{1}_{[0,1)}(y) = \mathbb{1}_{[0,1)^2}(x,y)$$

$$\mathbb{P}(X+Y\le 1, XY\le 2/9) = \iint_A \mathbb{1}_{[0,1)^2}(x,y)\,dx\,dy$$

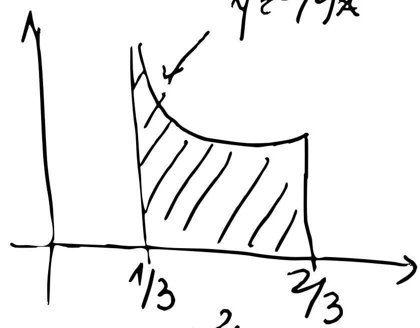
$$= \text{Aria}(A)$$



Punctele de intersecție:
$$\begin{cases} x+y=1 \\ xy = 2/9 \end{cases} \iff \begin{cases} x = 1-y \\ (1-y)y = 2/9 \end{cases}$$

$$y_{1,2} = \begin{cases} \frac{1}{3} \\ 2/3 \end{cases} \implies x_{1,2} = \begin{cases} 2/3 \\ 1/3 \end{cases}$$

$$\text{Aria}(A) = 2A_1 + A_2 + A_3$$



$$A_1 = \frac{1/3 \cdot 1/3}{2} = 1/18$$

$$A_2 = \frac{1}{3} \times \frac{2}{3} = 2/9$$

$$A_2 = \int_{1/3}^{2/3} \frac{2}{9x}\,dx = \frac{2}{9} \ln x \Big|_{1/3}^{2/3} = \frac{2}{9} \ln(2)$$

$$\underline{\text{Aria}(A) = 1/3 + \frac{2}{9} \ln 2}$$

Ex $X, Y \sim \mathcal{N}(0,1)$ indep

Calculați corelația dintre cea mai mică și cea mai mare valoare dintre ale 2 măsurători

Sol: $\rho(\underbrace{\min(X,Y), \max(X,Y)}_{\text{nu sunt independente}}) = ?$

$M = \max(X,Y)$

$L = \min(X,Y)$

$$\rho(M,L) = ? \quad \frac{Cov(M,L)}{\sqrt{Var(M)}\sqrt{Var(L)}}$$

Obs: $x, y \in \mathbb{R}$

$$\max(x,y) + \min(x,y) = x + y$$

$\max(x,y) - \min(x,y) = |x-y|$

$\mathbb{E}[M+L] = \mathbb{E}[M] + \mathbb{E}[L]$

$\parallel$

$\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y] = 0$

$\mathbb{E}[M-L] = \mathbb{E}[M] - \mathbb{E}[L]$

$\parallel$

$\mathbb{E}[|X-Y|]$

$X, Y \sim N(0,1)$ indep

$X-Y \sim N(0,2)$

$\parallel$

$Z\sqrt{2}$, $Z \sim N(0,1)$

$$\mathbb{E}[|X-Y|] = \sqrt{2}\, \mathbb{E}[|Z|] =$$

$$= \sqrt{2} \int |z| \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$$

$$= 2\sqrt{2} \int_0^\infty \frac{z}{\sqrt{2\pi}} e^{-z^2/2} dz = \frac{2\sqrt{2}}{\sqrt{2\pi}}$$

$$= \frac{2}{\sqrt{\pi}}$$

$$\begin{cases} \mathbb{E}[M] + \mathbb{E}[L] = 0 \\ \mathbb{E}[M] - \mathbb{E}[L] = 2/\sqrt{\pi} \end{cases} \Rightarrow \begin{array}{l} \mathbb{E}[M] = \frac{1}{\sqrt{\pi}} \\ \mathbb{E}[L] = -\frac{1}{\sqrt{\pi}} \end{array}$$

$$\mathrm{Cov}(M,L) = \mathbb{E}[ML] - \mathbb{E}[M]\,\mathbb{E}[L]$$

$$= \mathbb{E}[XY] - \frac{1}{\sqrt{\pi}}\left(-\frac{1}{\sqrt{\pi}}\right)$$

$$= \mathbb{E}[XY] + \frac{1}{\pi}$$

$$\overset{\text{indep}}{=} \underbrace{\mathbb{E}[X]}_{0}\,\mathbb{E}[Y] + \frac{1}{\pi} = \frac{1}{\pi}$$

$$\mathrm{Var}(\underbrace{M+L}_{X+Y}) = \mathrm{Var}(M) + \mathrm{Var}(L) + 2\,\mathrm{Cov}(M,L)$$

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) \quad \text{din indep}$$

$$= 2$$

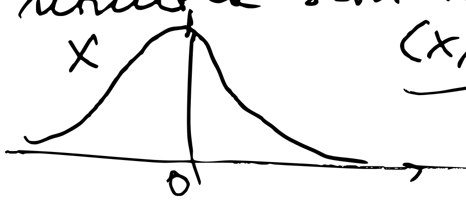
Atunci $\mathrm{Var}(M) + \mathrm{Var}(L) + \dfrac{2}{\pi} = 2$

$$\mathrm{Var}(M) + \mathrm{Var}(L) = 2 - 2/\pi$$

Legătura dintre max și min:

$$\max(x,y) = -\min(-x,-y)$$

Pentru normala standard, din simetrie $\underline{(X,Y)}$ este rep ca $\underline{(-X,-Y)}$



$$\mathrm{Var}(\max(X,Y)) = \mathrm{Var}(-\min(-X,-Y))$$

$$= \mathrm{Var}(\min(-X,-Y))$$

$$\overset{\text{simetrie}}{=} \mathrm{Var}(\min(X,Y)) \Rightarrow$$

$$\left.\begin{array}{l} \mathrm{Var}(M) = \mathrm{Var}(L) \\ \mathrm{Var}(M) + \mathrm{Var}(L) = 2 - 2/\pi \end{array}\right\} \Rightarrow$$

$$\mathrm{Var}(M) = \mathrm{Var}(L) = 1 - 1/\pi$$

$$\rho(M,L) = \frac{\frac{1}{\pi}}{1 - 1/\pi} = \frac{1}{\pi - 1}$$