Funcţii Caracteristice

SEMINAR DE LOGICĂ MATEMATICĂ ŞI COMPUTAŢIONALĂ

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Exerc.: Să se dem. asociativitatea lui Δ, folosind funcții caracteristice.

REZOLVARE:

Fie A, B, C mulțimi.

$A \Delta (B \Delta C) \overset{?}{=} (A \Delta B) \Delta C$.

Fie $T = A \cup B \cup C \cup \{0\}$. $\Rightarrow T \neq \emptyset$ și $A \subseteq T$, $B \subseteq T$, $C \subseteq T$. $\Rightarrow A \Delta (B \Delta C) \subseteq T$ și $(A \Delta B) \Delta C \subseteq T$.

Pt. orice $X \subseteq T$, fie χ_X funcția caracteristică a lui X raportat la T.

$$\chi_{A\Delta(B\Delta C)} = \chi_A + \chi_{B\Delta C} - 2\cdot\chi_A\cdot\chi_{B\Delta C} = \chi_A + \chi_B + \chi_C - 2\cdot\chi_B\cdot\chi_C - 2\cdot\chi_A\cdot(\chi_B + \chi_C - 2\chi_B\chi_C) = \chi_A + \chi_B + \chi_C - 2\chi_A\chi_B - 2\chi_A\chi_C - 2\chi_B\chi_C + 4\chi_A\chi_B\chi_C. \quad (*)$$

$$\chi_{(A\Delta B)\Delta C} = \chi_{A\Delta B} + \chi_C - 2\chi_{A\Delta B}\chi_C = \chi_A + \chi_B - 2\chi_A\chi_B + \chi_C - 2(\chi_A + \chi_B - 2\chi_A\chi_B)\cdot\chi_C = \chi_A + \chi_B + \chi_C - 2\chi_A\chi_B - 2\chi_A\chi_C - 2\chi_B\chi_C + 4\chi_A\chi_B\chi_C. \quad (**)$$

Altfel: știm că Δ e comutativă, așadar: $(A\Delta B)\Delta C = C\Delta(A\Delta B)$, deci:

$$\chi_{(A\Delta B)\Delta C} = \chi_{C\Delta(A\Delta B)} \overset{(*),\ \text{tripletul } (A,B,C) \to (C,A,B)}{=\!=} \chi_C + \chi_A + \chi_B - 2\chi_C\chi_A - 2\chi_C\chi_B - 2\chi_A\chi_B + 4\chi_C\chi_A\chi_B \overset{(*)}{=} \chi_{A\Delta(B\Delta C)}$$

$(*), (**) \Rightarrow \chi_{A\Delta(B\Delta C)} = \chi_{(A\Delta B)\Delta C} \Leftrightarrow A\Delta(B\Delta C) = (A\Delta B)\Delta C$.

Exerc. (legile de distrib. generalizate pt. $\cup$ şi $\cap$ de mulţimi):

$A \to$ mulţime,

$I, J \to$ mulţimi, $I \neq \emptyset$, $J \neq \emptyset$,

$(A_i)_{i \in I}$, $(B_j)_{j \in J} \to$ familii de mulţimi.

Dem. că:

(1) $A \cup \left(\bigcap_{j \in J} B_j\right) = \bigcap_{j \in J} (A \cup B_j)$;

(2) $A \cap \left(\bigcup_{j \in J} B_j\right) = \bigcup_{j \in J} (A \cap B_j)$;

(3) $\left(\bigcap_{i \in I} A_i\right) \cup \left(\bigcap_{j \in J} B_j\right) = \bigcap_{i \in I} \bigcap_{j \in J} (A_i \cup B_j)$

$\left(= \bigcap_{j \in J} \bigcap_{i \in I} (A_i \cup B_j)\right)$;

(4) $\left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{j \in J} B_j\right) = \bigcup_{i \in I} \bigcup_{j \in J} (A_i \cap B_j)$

$\left(= \bigcup_{j \in J} \bigcup_{i \in I} (A_i \cap B_j)\right)$.

REZ: Fie $T := A \cup \bigcup_{i \in I} A_i \cup \bigcup_{j \in J} B_j \cup \{0\} \neq \emptyset$, şi $(\forall M \subseteq T)(\chi_M :=$

:= fct. caract. a lui M raportat la T).

(1) Notăm: $f := \chi_{A\cup\left(\bigcap_{j\in J} B_j\right)} : T \to \{0,1\}$ și $g := \chi_{\bigcap_{j\in J}\left(A\cup B_j\right)} : T \to \{0,1\}$. ~~$f \underset{?}{=} g$.~~

Fie $x \in T$.

$f(x) = \max\{\chi_A(x), \min\{\chi_{B_j}(x) \mid j\in J\}\}$

$g(x) = \min\{\max\{\chi_A(x), \chi_{B_j}(x)\} \mid j \in J\}$.

<u>Caz 1:</u> $\min\{\chi_{B_j}(x) \mid j \in J\} = 0 \Leftrightarrow (\exists\ j_0 \in J)\ (\chi_{B_{j_0}}(x) = 0) \Rightarrow f(x) = \max\{\chi_A(x), 0\} = \chi_A(x)$.

$g(x) \underset{?}{=} \chi_A(x)$.

$$g(x) = \min\{\max\{\chi_A(x), \chi_{B_j}(x)\} \mid j \in J\} \overset{(j_0 \in J)}{\leq} \max\{\chi_A(x), \chi_{B_{j_0}}(x)\} =$$

$$= \max\{\chi_A(x), 0\} = \chi_A(x)$$

$$g(x) = \min\{\overbrace{\max\{\chi_A(x), \chi_{B_j}(x)\}}^{\geq \chi_A(x)} \mid j \in J\} \geq \min\{\chi_A(x) \mid j \in J\} = \chi_A(x)$$

$$\Rightarrow g(x) = \chi_A(x) = f(x).$$

<u>Case 2:</u> $\min\{\chi_{B_j}(x) \mid j \in J\} = 1$

$$\Leftrightarrow (\forall j \in J)(\chi_{B_j}(x) = 1) \Rightarrow f(x) =$$

$$= \max\{\chi_A(x), 1\} = 1.$$

$$g(x) = \min\{\max\{\chi_A(x), 1\} \mid j \in J\} = \min\{1 \mid j \in J\} = 1 = f(x).$$

$$\Rightarrow (\forall x \in T)(f(x) = g(x)) \Leftrightarrow f = g$$

$$\Leftrightarrow \chi_{A \cup \left(\bigcap\limits_{j \in J} B_j\right)} = \chi_{\bigcap\limits_{j \in J}(A \cup B_j)} \Leftrightarrow$$

$$\Leftrightarrow A \cup \left(\bigcap_{j \in J} B_j\right) = \bigcap_{j \in J}\left(A \cup B_j\right).$$

(2) Analog.

(3) $\Leftarrow$ (1), „astfel"

$$\left(\bigcap_{i \in I} A_i\right) \cup \left(\bigcap_{j \in J} B_j\right) \overset{(1)}{=} \bigcap_{j \in J}\left(\left(\bigcap_{i \in I} A_i\right) \cup B_j\right) \overset{(1)}{=} \bigcap_{j \in J}\bigcap_{i \in I}\left(A_i \cup B_j\right).$$

$$\Big\| \; (1)$$

$$\bigcap_{i \in I}\left(A_i \cup \left(\bigcap_{j \in J} B_j\right)\right) \overset{(1)}{=} \bigcap_{i \in I}\bigcap_{j \in J}\left(A_i \cup B_j\right).$$

(4) $\Leftarrow$ (2), analog.