Logică Propoziţională Clasică – demonstraţii algebrice şi semantice

SEMINAR DE LOGICĂ MATEMATICĂ ŞI COMPUTAŢIONALĂ

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Exerciții: Să se dem. algebric că:

$(\forall \varphi, \psi, \chi \in E)$ (au loc:

(i) $\vdash (\varphi \wedge \psi) \rightarrow \psi$

(ii) $\vdash \varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$

(iii) $\vdash (\varphi \rightarrow (\psi \rightarrow \chi)) \leftrightarrow (\psi \rightarrow (\varphi \rightarrow \chi))$

REZOLVARE:

(i) și (ii) sunt de fapt necesare pt. construirea algebrei Lindenbaum–Tarski, pe baza căreia se fac demonstrațiile algebrice.

$(\forall \tau \in E)$ (not. cu $\hat{\tau}$ clasa lui τ în algebra Lindenbaum–Tarski $E/\sim$).

(i) $\alpha := (\varphi \wedge \psi) \rightarrow \psi \in E$.

$\vdash \alpha \overset{\text{lemă}}{\Leftrightarrow} \hat{\alpha} = 1$.

$\hat{\alpha} = (\hat{\varphi} \wedge \hat{\psi}) \rightarrow \hat{\psi} = 1$, pt. că $\hat{\varphi} \wedge \hat{\psi} \leq \hat{\psi}$ în algebra Boole $E/\sim$.

$\Rightarrow \vdash \alpha$.

(ii) $\beta := \varphi \to (\psi \to (\varphi \wedge \psi)) \in E$.

Not.: $x := \hat{\varphi}$, $y := \hat{\psi}$.

$\Rightarrow \hat{\beta} = \hat{\varphi} \to (\hat{\psi} \to (\hat{\varphi} \wedge \hat{\psi})) =$

$= x \to (y \to (x \wedge y)) =$

$= \overline{x} \vee \overline{y} \vee (x \wedge y) =$

$= (\overline{x} \vee \overline{y} \vee x) \wedge (\overline{x} \vee \overline{y} \vee y) =$

$= (1 \vee \overline{y}) \wedge (\overline{x} \vee 1) = 1 \wedge 1 = 1 \Rightarrow$

$\xRightarrow{\text{(lemă)}} \vdash \beta$.

(iii) $\gamma := (\varphi \to (\psi \to \chi)) \leftrightarrow (\psi \to (\varphi \to \chi)) \in E$.

Not.: $x := \hat{\varphi}$, $y := \hat{\psi}$, $z := \hat{\chi}$.

$\hat{\gamma} = (\hat{\varphi} \to (\hat{\psi} \to \hat{\chi})) \leftrightarrow (\hat{\psi} \to (\hat{\varphi} \to \hat{\chi})) =$

$= (x \to (y \to z)) \leftrightarrow (y \to (x \to z)) =$

$= (\overline{x} \vee \overline{y} \vee z) \leftrightarrow (\overline{y} \vee \overline{x} \vee z) =$

$= (\overline{x} \vee \overline{y} \vee z) \leftrightarrow (\overline{x} \vee \overline{y} \vee z) = 1$,

pt. că e o echiv. bool. între 2 termeni egali. $\xRightarrow{\text{(lemă)}} \vdash \gamma$.

Exerc.: Să se dem. semantic urm. regulă de deducție: $(\forall \Gamma_1, \Gamma_2, \Gamma_3 \subseteq E)$ $(\forall \varphi, \psi, \chi \in E)$ (avem:

$$\frac{\Gamma_1 \vdash \varphi \vee \psi;\ \Gamma_2 \cup \{\varphi\} \vdash \chi;\ \Gamma_3 \cup \{\psi\} \vdash \chi}{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \vdash \chi}$$

REZOLVARE:

Pp. că: $\begin{cases} \Gamma_1 \vdash \varphi \vee \psi \overset{\text{TCT}}{\Longleftrightarrow} \Gamma_1 \vDash \varphi \vee \psi; (*) \\ \Gamma_2 \cup \{\varphi\} \vdash \chi \overset{\text{TCT}}{\Longleftrightarrow} \Gamma_2 \cup \{\varphi\} \vDash \chi; (**) \\ \Gamma_3 \cup \{\psi\} \vdash \chi \overset{\text{TCT}}{\Longleftrightarrow} \Gamma_3 \cup \{\psi\} \vDash \chi; (***) \end{cases}$

Fie $h: V \to \mathcal{L}_2$, a.î. $h \vDash \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$

$\Leftrightarrow \begin{cases} h \vDash \Gamma_1 \overset{(*)}{\Longrightarrow} \tilde{h}(\varphi \vee \psi) = 1 \Leftrightarrow \underbrace{\tilde{h}(\varphi)}_{\in \mathcal{L}_2} \vee \underbrace{\tilde{h}(\psi)}_{\in \mathcal{L}_2} = 1 \\ h \vDash \Gamma_2 \quad (2) \\ h \vDash \Gamma_3 \quad (3) \end{cases}$

$\Updownarrow$ $\left\{ \begin{array}{l} \tilde{h}(\varphi) = 1 \\ \text{sau} \\ \tilde{h}(\psi) = 1 \end{array} \right.$

Dc. $\tilde{h}(\varphi) = 1$, at. $\overset{(2)}{\Longrightarrow}$ $\Rightarrow h \vDash \Gamma_2 \cup \{\varphi\} \overset{(**)}{\Longrightarrow} \tilde{h}(\chi) = 1$

Dc. $\tilde{h}(\psi) = 1$, at. $\overset{(3)}{\Longrightarrow} h \vDash \Gamma_3 \cup \{\psi\} \Rightarrow$ $\overset{(***)}{\Longrightarrow} \tilde{h}(\chi) = 1$.

$\Rightarrow$ În fiecare caz, $\tilde{h}(\chi) = 1 \Rightarrow$

$\Rightarrow \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \vDash \chi \overset{\text{TCT}}{\Longleftrightarrow} \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \vdash \chi.$

Exerc. Fie $\alpha, \beta, \gamma \in E$. Să se dem. că, în logica prop. clasică:

$\{\alpha, \beta \to \gamma, \gamma \to \neg\alpha\} \vdash \neg\beta \wedge \neg\gamma$:

(a) algebric; (b) semantic.

REZOLVARE:

Fie $\Sigma = \{\alpha, \beta \to \gamma, \gamma \to \neg\alpha\}$.

(a) Avem, pentru orice $\varphi \in E$: $\Sigma \vdash \varphi \Leftrightarrow \varphi/_{\sim_\Sigma} = 1_\Sigma$ în algebra Boole $(E/_{\sim_\Sigma}, \vee_\Sigma, \wedge_\Sigma, \leq_\Sigma, \bar{\cdot}^\Sigma, 0_\Sigma, 1_\Sigma)$.

Notând cu $\to_\Sigma$ implicația booleană în această algebră Boole și folosind faptul că, pentru orice elemente x, y ale unei algebre Boole B, avem: $x \to y = 1 \Leftrightarrow x \leq y$, obținem:

$$\alpha,\ \beta\to\gamma,\ \gamma\to\neg\alpha\in\Sigma\Rightarrow$$

$$\Rightarrow\left\{\begin{array}{l}
\Sigma\vdash\alpha\Leftrightarrow\alpha/_{\sim_\Sigma}=1_\Sigma;\ (*)\\
\Sigma\vdash\beta\to\gamma\Leftrightarrow 1_\Sigma=(\beta\to\gamma)/_{\sim_\Sigma}=\\
=\beta/_{\sim_\Sigma}\to_\Sigma\gamma/_{\sim_\Sigma}\Leftrightarrow\\
\Leftrightarrow\beta/_{\sim_\Sigma}\le_\Sigma\gamma/_{\sim_\Sigma};\ (**)\\
\Sigma\vdash\gamma\to\neg\alpha\Leftrightarrow 1_\Sigma=\\
=(\gamma\to\neg\alpha)/_{\sim_\Sigma}=\gamma/_{\sim_\Sigma}\to_\Sigma\\
\to_\Sigma\overline{(\alpha/_{\sim_\Sigma})}^{\Sigma}\Leftrightarrow\gamma/_{\sim_\Sigma}\le_\Sigma\\
\le_\Sigma\overline{(\alpha/_{\sim_\Sigma})}^{\Sigma}\Leftrightarrow\overline{\overline{(\alpha/_{\sim_\Sigma})}^{\Sigma}}^{\Sigma}\le\overline{(\gamma/_{\sim_\Sigma})}^{\Sigma}\Leftrightarrow\\
\Leftrightarrow\alpha/_{\sim_\Sigma}\le\overline{\gamma/_{\sim_\Sigma}}^{\Sigma}.\ (***)
\end{array}\right.$$

$$(*),\ (***)\Rightarrow 1_\Sigma=\overline{\gamma/_{\sim_\Sigma}}^{\Sigma}=$$

$$=(\neg\gamma)/_{\sim_\Sigma}.\ (1)$$

$$(**)\Leftrightarrow\overline{(\gamma/_{\sim_\Sigma})}^{\Sigma}\le_\Sigma\overline{(\beta/_{\sim_\Sigma})}^{\Sigma}.$$

$\Rightarrow 1_\Sigma = \overline{\beta/_{\sim_\Sigma}}^{\Sigma} = (\neg\beta)/_{\sim_\Sigma}$ (2)

(1), (2) $\Rightarrow (\neg\beta \wedge \neg\gamma)/_{\sim_\Sigma} =$
$= (\neg\beta)/_{\sim_\Sigma} \wedge_\Sigma (\neg\gamma)/_{\sim_\Sigma} = 1_\Sigma \wedge 1_\Sigma =$
$= 1_\Sigma \Rightarrow \Sigma \vdash \neg\beta \wedge \neg\gamma.$

! ATENȚIE: Nu aveam neapărat $E/_{\sim_\Sigma} = \{0_\Sigma, 1_\Sigma\}$ și nici măcar $0_\Sigma \neq 1_\Sigma$, așadar nu am fi putut efectua un raționament prin reducere la absurd.

(b) $\Sigma \vdash \neg\beta \wedge \neg\gamma \overset{(TCT)}{\Longleftrightarrow}$
$\overset{(TCT)}{\Longleftrightarrow} \Sigma \vDash \neg\beta \wedge \neg\gamma \overset{(def.)}{\Longleftrightarrow}$
$\overset{(def.)}{\Longleftrightarrow} (\forall\ h : V \to \mathcal{L}_2)$
$(h \vDash \Sigma \Rightarrow h \vDash \neg\beta \wedge \neg\gamma).$

Fie $h : V \to \mathcal{L}_2$ a.î.

$h \vDash \Sigma \overset{\text{def}}{\Leftrightarrow} (\forall\, \tau \in \Sigma)(h \vDash \tau) \Leftrightarrow$

$\overset{\text{def}}{\Leftrightarrow} (\forall\, \tau \in \Sigma)(\tilde{h}(\tau) = 1) \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \tilde{h}(\alpha) = 1 \quad (\bullet) \\ 1 = \tilde{h}(\beta \to \gamma) = \tilde{h}(\beta) \to \tilde{h}(\gamma) \Rightarrow \tilde{h}(\beta) \leq \tilde{h}(\gamma) \quad (\bullet\bullet) \\ 1 = \tilde{h}(\gamma \to \neg\alpha) = \tilde{h}(\gamma) \to \overline{\tilde{h}(\alpha)} \end{cases}$$

$\overset{(\bullet)}{=} \tilde{h}(\gamma) \to \overline{1} = \tilde{h}(\gamma) \to 0 \Rightarrow$

$\Rightarrow \tilde{h}(\gamma) \leq 0 \Leftrightarrow \tilde{h}(\gamma) = 0 \overset{(\bullet\bullet)}{\Rightarrow}$

$\Rightarrow \tilde{h}(\beta) \leq 0 \Leftrightarrow \tilde{h}(\beta) = 0.$

$\Rightarrow \tilde{h}(\neg\beta \wedge \neg\gamma) = \overline{\tilde{h}(\beta)} \wedge \overline{\tilde{h}(\gamma)} =$

$= \overline{0} \wedge \overline{0} = 1 \wedge 1 = 1 \Rightarrow h \vDash \neg\beta \wedge \neg\gamma.$

$\overset{\text{def}}{\Rightarrow} \Sigma \vDash \neg\beta \wedge \neg\gamma \overset{\text{TCT}}{\Leftrightarrow} \Sigma \vdash \neg\beta \wedge \neg\gamma.$

Întrucât $\tilde{h} : E \to \mathcal{L}_2 = \{0, 1\}$ cu $0 \neq 1$, aici puteam proceda şi prin reducere la absurd:

Pp. prin absurd că $\tilde{h}(\neg\beta \wedge \neg\gamma) \neq 1 \Leftrightarrow \tilde{h}(\neg\beta \wedge \neg\gamma) = 0 \Leftrightarrow$

$\Leftrightarrow \overline{\tilde{h}(\beta)} \wedge \overline{\tilde{h}(\gamma)} = 0 \Leftrightarrow$

$\Leftrightarrow \overline{\tilde{h}(\beta)} = 0 = \overline{1}$ sau $\overline{\tilde{h}(\gamma)} = 0 = \overline{1} \Leftrightarrow$

$\Leftrightarrow \tilde{h}(\beta) = 1$ sau $\tilde{h}(\gamma) = 1 \Leftrightarrow$

$\Leftrightarrow \tilde{h}(\gamma) = 1.$

întrucât, cf. (••), $\tilde{h}(\beta) = 1 \Rightarrow$

$\Rightarrow \tilde{h}(\gamma) = 1.$

Dar $\tilde{h}(\gamma) \leq \overline{\tilde{h}(\alpha)}$, pentru că $\tilde{h}(\gamma) \rightarrow \overline{\tilde{h}(\alpha)} = 1.$

$\Rightarrow 1 \leq \overline{\tilde{h}(\alpha)} \Leftrightarrow \overline{\tilde{h}(\alpha)} = 1 \Leftrightarrow$

$\Leftrightarrow \tilde{h}(\alpha) = \overline{1} = 0 \neq 1$ (în L_2). Abs cu (•).

$\Rightarrow \tilde{h}(\beta \wedge \neg\gamma) = 1 \overset{(def.)}{\Rightarrow} \Sigma \models \neg\beta \wedge \neg\gamma \Rightarrow$

$\overset{(TCT)}{\Leftrightarrow} \Sigma \vdash \neg\beta \wedge \neg\gamma.$

Exerc.: Fie $\alpha, \varphi, \psi, \chi \in E$, a.î. $\alpha = (\varphi \rightarrow \neg(\neg\psi \rightarrow \chi)) \leftrightarrow ((\psi \rightarrow \neg\varphi) \wedge (\chi \rightarrow \neg\varphi))$. Demonstrați că:

$\vdash \alpha$: (a) algebric, (b) semantic.

REZOLVARE:

(a) Fie $x := \hat{\varphi}$, $y := \hat{\psi}$ și $z := \hat{\chi}$ în $E/_{\sim}$.

$$\Rightarrow \hat{\alpha} = \left(\hat{\varphi} \to \overline{(\hat{\psi} \to \hat{\chi})}\right) \leftrightarrow$$
$$\leftrightarrow \left((\hat{\psi} \to \overline{\hat{\varphi}}) \wedge (\hat{\chi} \to \overline{\hat{\varphi}})\right) =$$
$$= \left(x \to \overline{(y \to z)}\right) \leftrightarrow \left((y \to \bar{x}) \wedge (z \to \bar{x})\right) =$$
$$= \left(x \to \overline{(\bar{y} \vee z)}\right) \leftrightarrow \left((\bar{y} \vee \bar{x}) \wedge (\bar{z} \vee \bar{x})\right) =$$
$$= \left(\bar{x} \vee \overline{(\bar{y} \vee z)}\right) \leftrightarrow \left((\bar{y} \wedge \bar{z}) \vee \bar{x}\right) \underset{\text{(de Morgan)}}{=}$$
$$= \left(\bar{x} \vee (\bar{y} \wedge \bar{z})\right) \leftrightarrow \left(\bar{x} \vee (\bar{y} \wedge \bar{z})\right) = 1,$$

fiind o echivalență booleană între doi termeni egali.

Deci $\hat{\alpha} = 1 \Leftrightarrow \vdash \alpha$.

(b) Demonstrăm că $\models \alpha$.

Fie $h : V \to \mathcal{L}_2$, arbitrară, fixată.

$$\tilde{h}(\alpha) = \left(\tilde{h}(\varphi) \to \overline{(\tilde{h}(\psi) \to \tilde{h}(\chi))}\right) \leftrightarrow$$
$$\leftrightarrow \left((\tilde{h}(\psi) \to \overline{\tilde{h}(\varphi)}) \wedge (\tilde{h}(\chi) \to \overline{\tilde{h}(\varphi)})\right).$$

De aici putem continua cu un

calcul boolean, ca la punctul (a), sau putem proceda astfel: notăm

$\beta := \varphi \to \neg(\neg\psi \to \chi) \in E$ şi

$\gamma := (\psi \to \neg\varphi) \wedge (\chi \to \neg\varphi) \in E$.

$$\tilde{h}(\alpha) = \tilde{h}(\beta) \leftrightarrow \tilde{h}(\gamma),$$

aşadar: $\tilde{h}(\alpha) = 1 \Leftrightarrow \tilde{h}(\beta) = \tilde{h}(\gamma)$.

Caz 1: $\tilde{h}(\beta) = 0 \Leftrightarrow$

$\Leftrightarrow \tilde{h}(\varphi) \to \overline{(\overline{\tilde{h}(\psi)} \to \tilde{h}(\chi))} = 0 \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \tilde{h}(\varphi) = 1 \\ \text{şi} \\ \overline{\overline{\tilde{h}(\psi)} \to \tilde{h}(\chi)} = 0 \end{cases} \Leftrightarrow$$

$\Leftrightarrow \overline{\tilde{h}(\psi)} \to \tilde{h}(\chi) = 1 \Leftrightarrow$

$$\Leftrightarrow \begin{cases} \overline{\tilde{h}(\psi)} = 0 \Leftrightarrow \tilde{h}(\psi) = 1, \\ \text{sau} \\ \tilde{h}(\chi) = 1. \end{cases}$$

$$\text{Aşadar: } \tilde{h}(\beta) = 0 \Leftrightarrow \begin{cases} \tilde{h}(\varphi) = \tilde{h}(\psi) = 1 \\ \text{sau} \\ \tilde{h}(\varphi) = \tilde{h}(\chi) = 1 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \tilde{h}(\psi)=1 \text{ și } \overline{\tilde{h}(\varphi)}=0 \\ \text{sau} \\ \tilde{h}(\chi)=1 \text{ și } \overline{\tilde{h}(\varphi)}=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \tilde{h}(\psi)\to\overline{\tilde{h}(\varphi)}=0 \\ \text{sau} \\ \tilde{h}(\chi)\to\overline{\tilde{h}(\varphi)}=0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \left(\tilde{h}(\psi)\to\overline{\tilde{h}(\varphi)}\right)\wedge\left(\tilde{h}(\chi)\to\overline{\tilde{h}(\varphi)}\right)=$$

$$=0. \Leftrightarrow \tilde{h}(\gamma)=0.$$

<u>Caz 2:</u> $\tilde{h}(\beta)=1. \overset{\text{caz 1}}{\Longleftrightarrow} \tilde{h}(\gamma)=1.$

$$\Rightarrow \tilde{h}(\beta)=\tilde{h}(\gamma) \Leftrightarrow \tilde{h}(\alpha)=1.$$

$$\Rightarrow \vDash \alpha. \overset{(TC)}{\Longleftrightarrow} \vdash \alpha.$$

ALTĂ METODĂ LA (b):

Amintesc că, în orice algebră Boole B, pt. orice $a\in B$, avem:

$a = a, \Leftrightarrow a \leftrightarrow a = 1;$ (1)

$0 \leq a, \Leftrightarrow 0 \to a = 1;$ (*)

$a \leq 1, \Leftrightarrow a \to 1 = 1;$ (**)

$a \to 0 = \bar{a} \vee 0 = \bar{a};$ (***)

$1 \to a = \bar{1} \vee a = 0 \vee a = a.$ (****)

Fie $h: V \to L_2 = \{0, 1\}$ arbitrară fixată.

$\tilde{h}(\alpha) = (\tilde{h}(\varphi) \to (\tilde{h}(\psi) \to \tilde{h}(\chi))) \leftrightarrow$

$\leftrightarrow ((\tilde{h}(\psi) \to \overline{\tilde{h}(\varphi)}) \wedge (\tilde{h}(\chi) \to \overline{\tilde{h}(\varphi)})),$

Caz 1: Dacă $\tilde{h}(\varphi) = 0, \Rightarrow \overline{\tilde{h}(\varphi)} = \bar{0} =$

$= 1 \Rightarrow \tilde{h}(\alpha) = (0 \to (\tilde{h}(\psi) \to \tilde{h}(\chi))) \leftrightarrow$

$\leftrightarrow ((\tilde{h}(\psi) \to 1) \wedge (\tilde{h}(\chi) \to 1)) \overset{(*),(**)}{=}$

$= 1 \leftrightarrow (1 \wedge 1) = 1 \leftrightarrow 1 \overset{(1)}{=} 1.$

Caz 2: Dacă $\tilde{h}(\varphi) = 1, \Rightarrow \overline{\tilde{h}(\varphi)} = \bar{1} =$

$= 0 \Rightarrow \tilde{h}(\alpha) = (1 \to (\tilde{h}(\psi) \to \tilde{h}(\chi))) \leftrightarrow$

$\leftrightarrow ((\tilde{h}(\psi) \to 0) \wedge (\tilde{h}(\chi) \to 0)) \overset{(****),(***)}{=}$

$= \overline{(\tilde{h}(\psi) \to \tilde{h}(\chi))} \leftrightarrow (\overline{\tilde{h}(\psi)} \wedge \overline{\tilde{h}(\chi)}) =$

$= \overline{\overline{\tilde{h}(\psi)} \vee \tilde{h}(\chi)} = \overline{\overline{\tilde{h}(\psi)}} \vee \overline{\tilde{h}(\chi)} =$

$= \tilde{h}(\psi) \wedge \overline{\tilde{h}(\chi)}$

$= (\tilde{h}(\psi) \wedge \overline{\tilde{h}(\chi)}) \leftrightarrow (\tilde{h}(\psi) \wedge \overline{\tilde{h}(\chi)}) \overset{(1)}{=} 1.$

$\Rightarrow (\forall h: V \to L_2)(h \vDash \alpha) \Leftrightarrow$

$\overset{(def)}{\Leftrightarrow} \vDash \alpha \overset{(TC)}{\Leftrightarrow} \vdash \alpha.$

Exerc.: Să se dem. semantic următoarea regulă de deducție: pt. orice $\Gamma \subseteq E$ și orice $\varphi, \psi, \chi \in E$, $\dfrac{\Gamma \cup \{\varphi\} \vdash \chi, \ \Gamma \cup \{\psi\} \vdash \chi}{\Gamma \cup \{\varphi \vee \psi\} \vdash \chi}$.

REZOLVARE:

Fie $\Gamma \subseteq E$ și $\varphi, \psi, \chi \in E$, a.î. $\begin{cases} \Gamma \cup \{\varphi\} \vdash \chi \\ \text{și} \\ \Gamma \cup \{\psi\} \vdash \chi. \end{cases} \overset{(TCT)}{\Longleftrightarrow} \begin{cases} \Gamma \cup \{\varphi\} \vDash \chi \ (1) \\ \text{și} \\ \Gamma \cup \{\psi\} \vDash \chi \ (2). \end{cases}$

$\Gamma \cup \{\varphi \vee \psi\} \vDash \chi$. — ??

Fie $h : V \to L_2$, a.î. $h \vDash \Gamma \cup \{\varphi \vee \psi\}. \Leftrightarrow \begin{cases} h \vDash \Gamma \ (*) \\ \text{și} \\ \tilde{h}(\varphi \vee \psi) = 1 \end{cases} \Leftrightarrow$

$\Leftrightarrow \tilde{h}(\varphi) \vee \tilde{h}(\psi) = 1.$

$\tilde{h}(\varphi), \tilde{h}(\psi) \in L_2 = \{0, 1\}.$ $\Rightarrow$

$\Rightarrow \tilde{h}(\varphi) = 1$ sau $\tilde{h}(\psi) = 1.$

Caz 1: dacă $\tilde{h}(\varphi) = 1, \overset{(*)}{\Rightarrow}$

$\overset{(*)}{\Rightarrow} h \vDash \Gamma \cup \{\varphi\} \overset{(1)}{\Rightarrow} \tilde{h}(\chi) = 1.$

Caz 2: dacă $\tilde{h}(\psi) = 1, \overset{(*)}{\Rightarrow}$

$\overset{(*)}{\Rightarrow} h \vDash \Gamma \cup \{\psi\} \overset{(2)}{\Rightarrow} \tilde{h}(\chi) = 1.$

$\Rightarrow \Gamma \cup \{\varphi \vee \psi\} \vDash \chi. \overset{(TCT)}{\Longleftrightarrow}$

$\overset{(TCT)}{\Longleftrightarrow} \Gamma \cup \{\varphi \vee \psi\} \vdash \chi.$

Exercițiu: Să se demonstreze semantic următoarea regulă de deducție: pentru orice $\Sigma_1, \Sigma_2, \Sigma_3 \in \mathcal{P}(E)$ și orice $\varphi, \psi, \chi \in E$:

$$\frac{\Sigma_1 \cup \{\varphi\} \vdash \psi,\ \Sigma_2 \cup \{\psi \wedge \chi\} \vdash \varphi,\ \Sigma_3 \cup \{\psi\} \vdash \chi}{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vdash \varphi \leftrightarrow (\psi \wedge \chi)}.$$

REZOLVARE: Fie $\Sigma_1, \Sigma_2, \Sigma_3 \in \mathcal{P}(E)$ și $\varphi, \psi, \chi \in E$, a.î. au loc deducțiile sintactice:

$$\begin{cases} \Sigma_1 \cup \{\varphi\} \vdash \psi \overset{(TCT)}{\Leftrightarrow} \Sigma_1 \cup \{\varphi\} \vDash \psi; \ (1) \\ \Sigma_2 \cup \{\psi \wedge \chi\} \vdash \varphi \overset{(TCT)}{\Leftrightarrow} \Sigma_2 \cup \{\psi \wedge \chi\} \vDash \varphi; (2) \\ \Sigma_3 \cup \{\psi\} \vdash \chi \overset{(TCT)}{\Leftrightarrow} \Sigma_3 \cup \{\psi\} \vDash \chi. \ (3) \end{cases}$$

Fie $h: V \to L_2$ a.î. $h \vDash \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \Leftrightarrow h \vDash \Sigma_1, h \vDash \Sigma_2, h \vDash \Sigma_3$.

Caz 1: Dacă $\tilde{h}(\varphi) = 1, \xRightarrow{(h \vDash \Sigma_1)}$
$\Rightarrow h \vDash \Sigma_1 \cup \{\varphi\} \xRightarrow{(1)} \tilde{h}(\psi) = 1 \xRightarrow{(h \vDash \Sigma_3)}$
$\Rightarrow h \vDash \Sigma_3 \cup \{\psi\} \xRightarrow{(3)} \tilde{h}(\chi) = 1. \Rightarrow$
$\Rightarrow \tilde{h}(\varphi \leftrightarrow (\psi \wedge \chi)) = \tilde{h}(\varphi) \leftrightarrow (\tilde{h}(\psi) \wedge \tilde{h}(\chi)) =$
$= 1 \leftrightarrow (1 \wedge 1) = 1 \leftrightarrow 1 = 1.$

Caz 2: Dacă $\tilde{h}(\varphi) = 0$:
Presupunem prin absurd că $\tilde{h}(\psi \wedge \chi) = 1 \xRightarrow{(h \vDash \Sigma_2)}$
$h \vDash \Sigma_2 \cup \{\psi \wedge \chi\} \xRightarrow{(2)} \tilde{h}(\varphi) = 1$ ↯ $\Rightarrow$
$\Rightarrow \tilde{h}(\psi \wedge \chi) = 0 \Rightarrow \tilde{h}(\varphi \leftrightarrow (\psi \wedge \chi)) =$
$= \tilde{h}(\varphi) \leftrightarrow \tilde{h}(\psi \wedge \chi) = 0 \leftrightarrow 0 = 1.$
$\Rightarrow \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vDash \varphi \leftrightarrow (\psi \wedge \chi) \overset{(TCT)}{\Leftrightarrow} \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vdash \varphi \leftrightarrow (\psi \wedge \chi).$

Exerc. (tip examen (PT. 1 PUNCT) (2020–2021)), din logica propozitionala clasica:
Fie $\alpha \in T$ și $\beta, \gamma \in E$.
Dem. că mulțimea $\{\alpha \to \beta, (\beta \vee \gamma) \to \neg\alpha\}$ e inconsistentă.

RESOLVARE:

Pp. prin absurd că această mulțime e consistentă. $\overset{(prop.)}{\Leftrightarrow}$ Această mulțime e satisfiabilă. $\overset{(def.)}{\Leftrightarrow}$

$\overset{(def.)}{\Leftrightarrow} (\exists\, h : V \to \mathcal{L}_2)\left(h \vDash \{\alpha \to \beta, (\beta \vee \gamma) \to \neg\alpha\}\right)$
$\Leftrightarrow$

$h \vDash \alpha \to \beta$ și $h \vDash (\beta \vee \gamma) \to \neg\alpha$

$$\overset{(def.)}{\Leftrightarrow} \begin{cases} 1 = \tilde{h}(\alpha \to \beta) = \tilde{h}(\alpha) \to \tilde{h}(\beta) \\ \text{și} \\ 1 = \tilde{h}((\beta \vee \gamma) \to \neg\alpha) = \\ \quad = (\tilde{h}(\beta) \vee \tilde{h}(\gamma)) \to \overline{\tilde{h}(\alpha)} \end{cases} \Leftrightarrow$$

(proprietate a artea ... algebrelor Boole) $\Leftrightarrow$

$$\begin{cases} \tilde{h}(\alpha) \le \tilde{h}(\beta) \quad (1) \\ \text{și} \\ \tilde{h}(\beta) \vee \tilde{h}(\gamma) \le \overline{\tilde{h}(\alpha)} \quad (2) \end{cases}$$

Conform ipotezei, $\vdash \alpha, \overset{(TC)}{\Rightarrow} \vDash \alpha.$

$\Rightarrow \tilde{h}(\alpha) = 1 \overset{(1)}{\Longrightarrow} \tilde{h}(\beta) = 1$

$\Downarrow(2)$

$\tilde{h}(\beta) \vee \tilde{h}(\gamma) \leq \overline{1} = 0 \Leftrightarrow$

$\Leftrightarrow \tilde{h}(\beta) \vee \tilde{h}(\gamma) = 0.$

$\Rightarrow 0 = 1 \vee \tilde{h}(\gamma) = 1$ în $\mathcal{L}_2$. $\Rightarrow$

$\Rightarrow$ Mulțimea $\{\alpha \to \beta, (\beta \vee \gamma) \to \neg\alpha\}$ e inconsistentă.

Exerc. (tip examen (PT. 1 PUNCT) 2020–2021), din logica propozițională clasică)

Dem. că, pt. orice $\Sigma, \Delta \in \mathcal{P}(E)$ și orice $\alpha, \beta, \gamma \in E$, are loc următoarea regulă de deducție în logica propozițională clasică:

$$\frac{\Sigma \vdash (\alpha \vee \beta) \to \gamma, \quad \Delta \vdash \neg\alpha \to \neg\gamma}{\Sigma \cup \Delta \vdash \alpha \leftrightarrow \gamma}.$$

REZOLVARE:

Fie $\Sigma, \Delta \in \mathcal{P}(E)$ și

$\alpha, \beta, \gamma \in E$ astfel încât:

$$\begin{cases} \Sigma \vdash (\alpha \vee \beta) \rightarrow \gamma \\ \text{și} \\ \Delta \vdash \neg\alpha \rightarrow \neg\gamma \end{cases} \overset{(TCT)}{\Longleftrightarrow} \begin{cases} \Sigma \vDash (\alpha \vee \beta) \rightarrow \gamma \quad (I) \\ \text{și} \\ \Delta \vDash \neg\alpha \rightarrow \neg\gamma. \quad (II) \end{cases}$$

Fie $h : V \rightarrow \mathcal{L}_2$ a.î.

$$h \vDash \Sigma \cup \Delta. \overset{(def.)}{\Longleftrightarrow} \begin{cases} h \vDash \Sigma \quad (*) \\ \text{și} \\ h \vDash \Delta. \quad (**) \end{cases}$$

$(I), (*) \overset{(def.)}{\Longrightarrow} h \vDash (\alpha \vee \beta) \rightarrow \gamma \overset{(def.)}{\Longleftrightarrow}$
$\overset{(def.)}{\Longleftrightarrow} 1 = \tilde{h}((\alpha \vee \beta) \rightarrow \gamma) =$
$= (\tilde{h}(\alpha) \vee \tilde{h}(\beta)) \rightarrow \tilde{h}(\gamma) \overset{\text{(propr. aritm. alg. Boole)}}{\Longleftrightarrow}$
$\Leftrightarrow \tilde{h}(\alpha) \vee \tilde{h}(\beta) \leq \tilde{h}(\gamma). \quad (1)$

$(II), (**) \overset{(def.)}{\Longrightarrow} h \vDash \neg\alpha \rightarrow \neg\gamma \overset{(def.)}{\Longleftrightarrow} 1 =$
$= \tilde{h}(\neg\alpha \rightarrow \neg\gamma) = \overline{\tilde{h}(\alpha)} \rightarrow \overline{\tilde{h}(\gamma)} \overset{\text{(propr. aritm. alg. Boole)}}{\Longleftrightarrow}$
$\overline{\tilde{h}(\alpha)} \leq \overline{\tilde{h}(\gamma)} \Leftrightarrow \tilde{h}(\gamma) \leq \tilde{h}(\alpha) \quad (\odot)$

$(1) \Longrightarrow \tilde{h}(\alpha) \leq \tilde{h}(\alpha) \vee \tilde{h}(\beta) \leq \tilde{h}(\gamma) \Rightarrow$
$\Longrightarrow \tilde{h}(\alpha) \leq \tilde{h}(\gamma). \quad (\odot\odot)$

$(\odot), (\odot\odot) \Longrightarrow \tilde{h}(\alpha) = \tilde{h}(\gamma) \overset{\text{(propr. aritm. alg. Boole)}}{\Longleftrightarrow}$
$\Leftrightarrow 1 = \tilde{h}(\alpha) \leftrightarrow \tilde{h}(\gamma) = \tilde{h}(\alpha \leftrightarrow \gamma)$
$\overset{(def.)}{\Longleftrightarrow} \Sigma \cup \Delta \vDash \alpha \leftrightarrow \gamma \overset{(TCT)}{\Longleftrightarrow}$
$\overset{(TCT)}{\Longleftrightarrow} \Sigma \cup \Delta \vdash \alpha \leftrightarrow \gamma.$

Exerc.: Dem. că, pentru orice $\alpha, \beta, \gamma \in E$, are loc, în logica prop. clasică:

$$\vdash (\alpha \to \beta) \to (\alpha \to (\gamma \to \beta)).$$

REZOLVARE:

METODA I: ALGEBRIC: În algebra Lindenbaum–Tarski $E/_\sim$, are loc:

$$((\alpha \to \beta) \to (\alpha \to (\gamma \to \beta)))/_\sim =$$

$$= (\alpha/_\sim \to \beta/_\sim) \to (\alpha/_\sim \to (\gamma/_\sim \to \beta/_\sim))$$

$$= (\overline{\alpha/_\sim} \vee \beta/_\sim) \to (\overline{\alpha/_\sim} \vee \overline{\gamma/_\sim} \vee \beta/_\sim) =$$

$= 1$, întrucât $\overline{\alpha/_\sim} \vee \beta/_\sim \leq$
$\leq \overline{\alpha/_\sim} \vee \overline{\gamma/_\sim} \vee \beta/_\sim$, așadar, conform unei leme din curs,

$$\vdash (\alpha \to \beta) \to (\alpha \to (\gamma \to \beta)).$$

METODA A II-A: SEMANTIC:

Fie $h: V \to \mathcal{L}_2$, arbitrară.

Putem efectua un calcul boolean ca mai sus sau putem proceda prin reducere la absurd, folosind faptul că aici valorile booleene sunt în $\mathcal{L}_2$: pp. prin absurd că

$$\tilde{h}((\alpha\to\beta)\to(\alpha\to(\gamma\to\beta)))=0 \Leftrightarrow$$

$$\Leftrightarrow \tilde{h}(\alpha\to\beta)\to\tilde{h}(\alpha\to(\gamma\to\beta))=0$$

$$\Leftrightarrow \tilde{h}(\alpha\to\beta)=1 \text{ și } \tilde{h}(\alpha\to(\gamma\to\beta))=0 \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \tilde{h}(\alpha)\to\tilde{h}(\beta)=1 \quad (*) \\ \text{și} \\ \tilde{h}(\alpha)\to(\tilde{h}(\gamma)\to\tilde{h}(\beta))=0 \Leftrightarrow \end{cases}$$

$$\Leftrightarrow \begin{cases} \tilde{h}(\alpha)=1 \\ \text{și} \\ \tilde{h}(\gamma)\to\tilde{h}(\beta)=0 \Leftrightarrow \begin{cases} \tilde{h}(\gamma)=1 \text{ și} \\ \tilde{h}(\beta)=0. \end{cases} \end{cases} \Rightarrow$$

$$\Rightarrow \tilde{h}(\alpha)\to\tilde{h}(\beta)=1\to 0=0, \stackrel{(*)}{\Rightarrow} 0=1$$

în $\mathcal{L}_2$, ↯ $\Rightarrow$

$$\Rightarrow \tilde{h}((\alpha\to\beta)\to(\alpha\to(\gamma\to\beta)))=1 \xrightarrow{(h\to \text{arb.})}$$

$$\Rightarrow \models (\alpha\to\beta)\to(\alpha\to(\gamma\to\beta)) \stackrel{(TC)}{\Leftrightarrow}$$

$\overset{TD}{\Leftrightarrow} \vdash (\alpha \to \beta) \to (\alpha \to (\gamma \to \beta))$.

Exerc.: Fie $\alpha, \beta, \gamma \in E$ şi $\Sigma = \{\alpha \to (\beta \to \gamma), \beta \wedge \neg\gamma\}$. Să se dem. că, în logica prop. clasică:

(1) dacă $\alpha, \beta, \gamma \in V$ şi $\alpha \neq \beta \neq \gamma$, atunci Σ e consistentă;

(2) dacă $(\exists \varphi \in E)(\alpha = \varphi \vee \neg\varphi)$ atunci Σ e inconsistentă.

REZOLVARE:

Fie $h: V \to \mathcal{L}_2$. Avem:

$$h \models \Sigma \Leftrightarrow \begin{cases} 1 = \tilde{h}(\alpha \to (\beta \to \gamma)) = \\ = \tilde{h}(\alpha) \to (\tilde{h}(\beta) \to \tilde{h}(\gamma)) \\ \text{şi} \\ 1 = \tilde{h}(\beta \wedge \neg\gamma) = \tilde{h}(\beta) \wedge \overline{\tilde{h}(\gamma)} \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \overline{\tilde{h}(\alpha)} \vee \overline{\tilde{h}(\beta)} \vee \tilde{h}(\gamma) = 1 \\ \text{şi} \\ \tilde{h}(\beta) = 1 \text{ şi } \tilde{h}(\gamma) = \overline{\overline{\tilde{h}(\gamma)}} = \overline{1} = 0 \end{cases} \Leftrightarrow$$

$$\Leftrightarrow \begin{cases} \tilde{h}(\beta) = 1, \tilde{h}(\gamma) = 0 \text{ şi} \\ \overline{\tilde{h}(\alpha)} \vee \overline{1} \vee 0 = 1 \Leftrightarrow \tilde{h}(\alpha) = \overline{\overline{\tilde{h}(\alpha)}} = \overline{1} = 0 \end{cases}$$

$\Leftrightarrow \tilde{h}(\alpha) = \tilde{h}(\gamma) = 0$ şi $\tilde{h}(\beta) = 1$. (*)

(1) Dacă $\alpha, \beta, \gamma \in V$ şi $\alpha \neq \beta \neq \gamma$, atunci $\exists$ (chiar o infinitate de interpretări) $h : V \to \mathcal{L}_2$ cu $\begin{cases} h(\alpha) = h(\gamma) = 0 \\ \text{şi} \\ h(\beta) = 1 \end{cases} \Leftrightarrow$

$\Leftrightarrow \begin{cases} \tilde{h}(\alpha) = \tilde{h}(\gamma) = 0 \\ \text{şi} \\ \tilde{h}(\beta) = 1 \end{cases} \overset{(*)}{\Longleftrightarrow} h \models \Sigma \overset{(def.)}{\Longrightarrow}$

$\Rightarrow \Sigma \to$ satisfiabilă $\overset{(prop)}{\Longleftrightarrow}$

$\Leftrightarrow \Sigma \to$ consistentă.

(2) Dacă $(\exists \varphi \in E)(\alpha = \varphi \vee \neg\varphi)$, $\overset{\text{(PTE, apoi TC, sau direct)}}{\Longrightarrow} \forall h : V \to \mathcal{L}_2$, avem $\tilde{h}(\alpha) = \tilde{h}(\varphi \vee \neg\varphi) = \tilde{h}(\varphi) \vee \overline{\tilde{h}(\varphi)} =$

$= 1 \overset{(*)}{\Longrightarrow} (\forall h : V \to \mathcal{L}_2)(h \not\models \Sigma) \Leftrightarrow$

$\overset{(def)}{\Longleftrightarrow} \Sigma \to$ nesatisfiabilă $\overset{(prop)}{\Longleftrightarrow} \Sigma \to$ inconsistentă.

<u>Exerc.:</u> Să se dem. că, pt. orice $\alpha, \beta, \gamma \in E$ şi orice $\Sigma, \Delta \in \mathcal{P}(E)$, are loc, în

logica propozițională clasică:

$$\frac{\Sigma \vdash \alpha \to (\beta \vee \gamma),\ \Delta \vdash \beta \to \gamma}{\Sigma \cup \Delta \vdash \alpha \to \gamma}.$$

REZOLVARE:

Fie $\alpha, \beta, \gamma \in E$ și $\Sigma, \Delta \in \mathcal{P}(E)$, astfel încât:

$$\begin{cases} \Sigma \vdash \alpha \to (\beta \vee \gamma) \\ \text{și} \\ \Delta \vdash \beta \to \gamma \end{cases} \overset{(TCT)}{\Longleftrightarrow} \begin{cases} \Sigma \vDash \alpha \to (\beta \vee \gamma) & (*) \\ \text{și} \\ \Delta \vDash \beta \to \gamma. & (**) \end{cases}$$

Fie $h: V \to \mathcal{L}_2$ a. î. $h \vDash \Sigma \cup \Delta$. $\overset{(def)}{\Longleftrightarrow} \begin{cases} h \vDash \Sigma & (\bullet) \\ \text{și} \\ h \vDash \Delta. & (\bullet\bullet) \end{cases}$

$(*), (\bullet) \overset{(def)}{\Longrightarrow} 1 = \tilde{h}(\alpha \to (\beta \vee \gamma)) = \tilde{h}(\alpha) \to (\tilde{h}(\beta) \vee \tilde{h}(\gamma)). \Leftrightarrow$

$\Leftrightarrow \tilde{h}(\alpha) \leq \tilde{h}(\beta) \vee \tilde{h}(\gamma).$ (1)

$(**), (\bullet\bullet) \overset{(def)}{\Longrightarrow} 1 = \tilde{h}(\beta \to \gamma) = \tilde{h}(\beta) \to \tilde{h}(\gamma) \Leftrightarrow \tilde{h}(\beta) \leq \tilde{h}(\gamma) \Leftrightarrow$

$\Leftrightarrow \tilde{h}(\beta) \vee \tilde{h}(\gamma) = \tilde{h}(\gamma).$ (2)

$(1), (2) \Rightarrow \tilde{h}(\alpha) \leq \tilde{h}(\gamma) \Leftrightarrow$

$\Leftrightarrow \tilde{h}(\alpha) \to \tilde{h}(\gamma) = 1 \Leftrightarrow$

$\Leftrightarrow \tilde{h}(\alpha \to \gamma) = 1 \xrightarrow[\text{cu } h \vDash \Sigma \cup \Delta)]{(h: V \to \mathcal{L}_2 \text{ arbitrară}}$

$\Rightarrow \Sigma \cup \Delta \vDash \alpha \to \gamma \overset{(TCT)}{\Longleftrightarrow}$

$\overset{(TCT)}{\Longleftrightarrow} \Sigma \cup \Delta \vdash \alpha \to \gamma.$

Prop.*: $\Sigma \subseteq E$, $\varphi \in E$. Atunci:

(1) $\Sigma \cup \{\varphi\} \rightarrow$ inconsistentă $\Leftrightarrow$
$\Leftrightarrow \Sigma \vdash \neg\varphi$,

(2) $\Sigma \cup \{\neg\varphi\} \rightarrow$ inconsistentă $\Leftrightarrow$
$\Leftrightarrow \Sigma \vdash \varphi$,

Prop.**: $\Sigma \subseteq E$. Atunci: $\Sigma \rightarrow$ consistentă $\Leftrightarrow (\exists h : V \rightarrow L_2)(h \vDash \Sigma)$.

Exerc.: Considerăm următorul text:

(a) Dacă nu am chef și îmi displace materia predată, atunci nu mă duc la curs.

(b) Nu îmi displace materia predată, pentru că am chef.

(c) Mă duc la curs, dacă ni se dau subiectele de examen.

(d) Dacă nu mă duc la curs, atunci ni se dau subiectele de examen.

(e) Îmi displace materia predată.

Să se arate că acest text este inconsistent (sinonim în limbaj natural: $\rightarrow$ „contradictoriu").

REZOLVARE:

Fie $p, q, r, s \in V$, 2×2 distincte, cărora le dăm următoarele valori:

p: "îmi displace materia predată"

q: "am chef"

r: "mă duc la curs"

s: "ni se dau subiectele de examen"

Atunci enunțurile care corespund frazelor din textul dat sunt:

(a) $\varphi = (\neg q \wedge p) \rightarrow \neg r \in E$

(b) $\psi = q \rightarrow \neg p \in E$

(c) $\chi = s \rightarrow r \in E$

(d) $\gamma = \neg r \rightarrow s \in E$

(e) $p \in V \subseteq E$.

Not. $\Sigma = \{\varphi, \psi, \chi, \gamma, p\} \subset E$.

Avem de dem. că Σ este inconsistentă.

Metoda I: Cf. Prop. *, e suficient să arătăm că: $\Sigma \setminus \{p\} \vdash \{\neg p\}$.

Nu vom aplica, aici, această metodă.

Metoda II: Aplicăm Prop. **:

Pp. prin absurd că Σ este consistentă. $\overset{\text{Prop.**}}{\Longleftrightarrow} (\exists\, h: V \rightarrow L_2)\ (h \vDash \Sigma)$.

Fie $h: V \rightarrow L_2$, a.î. $h \vDash \Sigma \Leftrightarrow$

$$\Leftrightarrow \tilde{h}(\varphi) = \tilde{h}(\psi) = \tilde{h}(\chi) = \tilde{h}(\gamma) =$$

$$= \tilde{h}(p) = 1. \Leftrightarrow$$ Au loc:

$$1 = \tilde{h}(\varphi) = \tilde{h}((\neg q \wedge p) \to \neg r) =$$

$$= \left(\overline{\tilde{h}(q)} \wedge \tilde{h}(p)\right) \to \overline{\tilde{h}(r)} =$$

$$= \overline{\left(\overline{\tilde{h}(q)} \wedge \tilde{h}(p)\right)} \vee \overline{\tilde{h}(r)} =$$

$$= \overline{\overline{\tilde{h}(q)}} \vee \overline{\tilde{h}(p)} \vee \overline{\tilde{h}(r)} =$$

$$= \tilde{h}(q) \vee \overline{\tilde{h}(p)} \vee \overline{\tilde{h}(r)} \quad (1)$$

$$1 = \tilde{h}(\psi) = \tilde{h}(q \to \neg p) =$$

$$= \tilde{h}(q) \to \overline{\tilde{h}(p)} =$$

$$= \overline{\tilde{h}(q)} \vee \overline{\tilde{h}(p)} \quad (2)$$

$$1 = \tilde{h}(\chi) = \tilde{h}(s \to r) =$$

$$= \tilde{h}(s) \to \tilde{h}(r) = \overline{\tilde{h}(s)} \vee \tilde{h}(r) \quad (3)$$

$$1 = \tilde{h}(\gamma) = \tilde{h}(\neg r \to s) =$$

$$= \overline{\tilde{h}(r)} \to \tilde{h}(s) = \quad (4)$$

$$= \overline{\overline{\tilde{h}(r)}} \vee \tilde{h}(s) = \tilde{h}(r) \vee \tilde{h}(s)$$

$$1 = \tilde{h}(p) \quad (5)$$

$(3),(4) \Rightarrow 1 = 1 \wedge 1 = (\tilde{h}(r) \vee$
$\vee \overline{\tilde{h}(s)}) \wedge (\tilde{h}(r) \vee \tilde{h}(s)) =$
$= \tilde{h}(r) \vee \underbrace{(\overline{\tilde{h}(s)} \wedge \tilde{h}(s))}_{=0} = \tilde{h}(r)$

$\xRightarrow{(1)} 1 = \tilde{h}(q) \vee \overline{\tilde{h}(p)} \vee \underbrace{\overline{1}}_{=0} =$
$= \tilde{h}(q) \vee \overline{\tilde{h}(p)} \quad (6).$

$(2),(6) \Rightarrow 1 = 1 \wedge 1 = (\overline{\tilde{h}(q)} \vee$
$\vee \overline{\tilde{h}(p)}) \wedge (\tilde{h}(q) \vee \overline{\tilde{h}(p)}) =$
$= \underbrace{(\overline{\tilde{h}(q)} \wedge \tilde{h}(q))}_{=0} \vee \overline{\tilde{h}(p)} = \overline{\tilde{h}(p)}$

$\Rightarrow \tilde{h}(p) = \overline{\overline{\tilde{h}(p)}} = \overline{1} = 0 \xRightarrow{(5)}$
$\Rightarrow 0 = 1$ ✗, pentru că $0 \neq 1$
în $\mathcal{L}_2 \Rightarrow \Sigma$ este inconsistentă.

Am efectuat un calcul boolean puţin diferit faţă de cel din Seminarul IX – partea a II-a. Referitor la notaţii: puteam observa faptul că $\tilde{h}(v)=h(v)$ pentru fiecare $v\in\{p,q,r,s\}$.

Metoda I:

$\Sigma \setminus \{p\} \vDash \neg p$: Fie $h: V \to \mathcal{L}_2$, a.î. $h \vDash \Sigma \setminus \{p\}$. $\Leftrightarrow$

$$\Leftrightarrow \begin{cases} 1 = \tilde{h}(\varphi) = \left(\overline{\tilde{h}(q)} \wedge \tilde{h}(p)\right) \to \overline{\tilde{h}(r)} \\ 1 = \tilde{h}(\psi) = \tilde{h}(q) \to \overline{\tilde{h}(p)} \quad (*) \\ 1 = \tilde{h}(\chi) = \tilde{h}(s) \to \tilde{h}(r) = \overline{\tilde{h}(s)} \vee \tilde{h}(r) \\ 1 = \tilde{h}(\xi) = \overline{\tilde{h}(r)} \to \tilde{h}(s) = \end{cases}$$

$= \overline{\overline{\tilde{h}(r)}} \vee \tilde{h}(s) = \tilde{h}(r) \vee \tilde{h}(s) = \tilde{h}(s) \vee \tilde{h}(r)$

$\Rightarrow 1 = 1 \wedge 1 = \left(\overline{\tilde{h}(s)} \vee \tilde{h}(r)\right) \wedge \left(\tilde{h}(s) \vee \tilde{h}(r)\right) = \left(\overline{\tilde{h}(s)} \wedge \tilde{h}(s)\right) \vee \tilde{h}(r) = 0 \vee \tilde{h}(r) = \tilde{h}(r). \Rightarrow$

$\Rightarrow 1 = \left(\overline{\tilde{h}(q)} \wedge \tilde{h}(p)\right) \to \overline{1} = \left(\overline{\tilde{h}(q)} \wedge \tilde{h}(p)\right) \to 0. \underset{0 \to 0 = 1}{\overset{1 \to 0 = 0}{\Rightarrow}}$

$\Rightarrow 0 = \overline{\tilde{h}(q)} \wedge \tilde{h}(p). \quad (**)$

Pp. obs. că $\tilde{h}(p) = 1. \Rightarrow$

$$\Rightarrow \begin{cases} \text{din } (**) \Rightarrow \overline{\tilde{h}(q)} = 0 \Leftrightarrow \overline{\overline{\tilde{h}(q)}} = \overline{0} \Leftrightarrow \tilde{h}(q) = 1, \\ \text{din } (*) \Rightarrow 1 = \tilde{h}(q) \to \overline{1} = \tilde{h}(q) \to 0 \underset{0 \to 0 = 1}{\overset{1 \to 0 = 0}{\Rightarrow}} \tilde{h}(q) = 0. \end{cases} \Rightarrow$$

$\Rightarrow 1 = 0$ în $\mathcal{L}_2 \to$ contradicție. $\Rightarrow \tilde{h}(p) = 0. \Rightarrow$

$\Rightarrow \tilde{h}(\neg p) = \overline{\tilde{h}(p)} = \overline{0} = 1. \Rightarrow \Sigma \setminus \{p\} \vDash \neg p$

$\overset{(TC)}{\Leftrightarrow} \Sigma \setminus \{p\} \vdash \neg p. \overset{(prop^*)}{\Leftrightarrow} \Sigma \to$ inconsistentă.

Exerc: Să se demonstreze algebric că: $(\forall\, \varphi, \psi, \chi \in E)$

$$\vdash (\varphi \to (\psi \to \chi)) \leftrightarrow (\psi \to (\varphi \to \chi))$$

REZ: $(\forall\, \alpha \in E)(\vdash \alpha \overset{prop.}{\Longleftrightarrow} \hat{\alpha} = 1$ în $E/\sim$ = algebra Lindenbaum – Tarski a logicii propoziționale clasice),

Știm că $E/\sim$ este o algebră Boole.

Fie $\varphi, \psi, \chi \in E$.

Not: $x := \hat{\varphi}$, $y := \hat{\psi}$, $z := \hat{\chi}$ în $E/\sim$.

$$\left((\varphi \to (\psi \to \chi)) \leftrightarrow (\psi \to (\varphi \to \chi))\right)^{\wedge}$$

// not. alternativă

$$\widehat{(\varphi \to (\psi \to \chi)) \leftrightarrow (\psi \to (\varphi \to \chi))} =$$

$$= (\hat{\varphi} \to (\hat{\psi} \to \hat{\chi})) \leftrightarrow (\hat{\psi} \to (\hat{\varphi} \to \hat{\chi})) =$$

transformarea conectorilor logici în operații booleene ale lui $E/\sim$

$$= (x \to (y \to z)) \leftrightarrow (y \to (x \to z)) =$$

$$= (\bar{x} \vee \bar{y} \vee z) \leftrightarrow \underbrace{(\bar{y} \vee \bar{x} \vee z)}_{\substack{\parallel \\ \bar{x} \vee \bar{y} \vee z}} = 1.$$

în $E/\sim \overset{\text{(Lemă)}}{\Longrightarrow} \vdash (\varphi\to(\psi\to\chi))\leftrightarrow(\psi\to(\varphi\to\chi))$.

Exerc.: Să se demonstreze algebric și semantic că: $(\forall\, \varphi,\psi\in E)$

$\vdash (\neg\neg\varphi\wedge\psi)\to(\neg\psi\vee\varphi)$.

REZ.: Not.: $\underset{\substack{\cap\\ E}}{\alpha} := (\neg\neg\varphi\wedge\psi)\to(\neg\psi\vee\varphi)$.

Fie $\varphi,\psi,\chi\in E$.

Algebric: Not.: $x := \hat{\varphi}$, $y := \hat{\psi}$ în $E/\sim$

$\hat{\alpha} = (\overline{\overline{\hat{\varphi}}}\wedge\hat{\psi})\to(\overline{\hat{\psi}}\vee\hat{\varphi}) =$

$= (\overline{\overline{x}}\wedge y)\to(\overline{y}\vee x) =$

$= (x\wedge y)\to(\overline{y}\vee x) = 1$, pt. că:

$x\wedge y\le x\le \overline{y}\vee x$, prin urmare

$x\wedge y\le\overline{y}\vee x$, prin tranzitivitate.

Deci, în $E/\sim$, $\hat{\alpha} = 1 \overset{\text{(Lemă)}}{\Longleftrightarrow} \vdash \alpha$.

Semantic: Dem. că: $\vDash \alpha$.

Fie $h: V\to\mathcal{L}_2$ arbitrară, fixată.

$\tilde{h}: E\to\mathcal{L}_2$.

$\mathcal{L}_2 = \{0, 1\} =$ alg. Boole standard

(aici) falsul adevărul

alg. Boole a valorilor de adevăr.

→ dă valori de adevăr variabilelor propoziționale

→ calculează valorile de adevăr ale enunțurilor pe baza valorilor de adevăr date de h variabilelor propoziționale.

În algebra Boole standard, $\mathcal{L}_2$, avem:

a	b	$a \to b$
0	0	1
0	1	1
1	0	0
1	1	1

Folosind acest lucru, calculăm

$$\tilde{h}(\alpha) = \underbrace{(\tilde{h}(\varphi) \wedge \tilde{h}(\psi))}_{\overset{not.}{=} T_1(\varphi,\psi)} \to \underbrace{(\tilde{h}(\psi) \to \tilde{h}(\varphi))}_{\overset{not.}{=} T_2(\varphi,\psi)}$$

cu ajutorul acestui tabel semantic:

$\tilde{h}(\varphi)$	$\tilde{h}(\psi)$	$T_1(\varphi,\psi)$	$T_2(\varphi,\psi)$	$\tilde{h}(\alpha)$
0	0	0	1	1
0	1	0	0	1
1	0	0	1	1
1	1	1	1	1

$\Longrightarrow (\forall h : V \to L_2 = \{0,1\})(\tilde{h}(\alpha) = 1) \overset{(def)}{\Longleftrightarrow} \models \alpha \overset{(TC)}{\Longleftrightarrow} \vdash \alpha.$

Exerc.: Să se dem. semantic următoarea regulă de deducție:

$$(\forall \Gamma \subseteq E)(\forall \varphi, \psi, \gamma \in E)\ \frac{\Gamma \cup \{\varphi\} \vdash \gamma,\ \Gamma \cup \{\psi\} \vdash \gamma}{\Gamma \cup \{\varphi \vee \psi\} \vdash \gamma}.$$

REZ.: Fie $\Gamma \subseteq E$ și $\varphi, \psi, \gamma \in E$,

a.î.: $\begin{cases} \Gamma \cup \{\varphi\} \vdash \gamma \overset{(TC)}{\Longleftrightarrow} \Gamma \cup \{\varphi\} \models \gamma \quad (1) \\ \text{și} \\ \Gamma \cup \{\psi\} \vdash \gamma \overset{(TC)}{\Longleftrightarrow} \Gamma \cup \{\psi\} \models \gamma \quad (2). \end{cases}$

Dem. că: $\Gamma \cup \{\varphi \vee \psi\} \models \gamma$.

Fie $h : V \to \mathcal{L}_2$, a.î. $h \models \Gamma \cup \{\varphi \vee \psi\}$

$\Leftrightarrow \begin{cases} h \models \Gamma \quad (*) \\ \text{și} \\ \tilde{h}(\varphi \vee \psi) = 1 \Leftrightarrow \tilde{h}(\varphi) \vee \tilde{h}(\psi) = 1 \end{cases}$

în $\mathcal{L}_2 \Leftrightarrow \tilde{h}(\varphi)=1$ sau $\tilde{h}(\psi)=1$.

Caz 1: dacă $\tilde{h}(\varphi)=1, \overset{(*)}{\Rightarrow}$

$\overset{(*)}{\Rightarrow} h \vDash \Gamma \cup \{\varphi\} \overset{(1)}{\Rightarrow} \tilde{h}(\chi)=1$.

Caz 2: dacă $\tilde{h}(\psi)=1, \overset{(*)}{\Rightarrow}$

$\overset{(*)}{\Rightarrow} h \vDash \Gamma \cup \{\psi\} \overset{(2)}{\Rightarrow} \tilde{h}(\chi)=1$.

$\Rightarrow (\forall h : V \to \mathcal{L}_2)(h \vDash \Gamma \cup \{\varphi \vee \psi\} \Rightarrow$

$\Rightarrow \tilde{h}(\chi)=1) \overset{(def)}{\Leftrightarrow} \Gamma \cup \{\varphi \vee \psi\} \vDash \chi \overset{(TCT)}{\Leftrightarrow}$

$\overset{(TCT)}{\Leftrightarrow} \Gamma \cup \{\varphi \vee \psi\} \vdash \chi$.

Exerc.: Să se dem. semantic urm. regula de deducție: $(\forall \Sigma_1, \Sigma_2, \Sigma_3 \subseteq E)(\forall \varphi, \psi, \chi \in E)$

$$\left(\frac{\Sigma_1 \cup \{\varphi\} \vdash \psi,\ \Sigma_2 \cup \{\psi \wedge \chi\} \vdash \varphi,\ \Sigma_3 \cup \{\psi\} \vdash \chi}{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vdash \varphi \leftrightarrow (\psi \wedge \chi)}\right)$$

REZ.: Fie $\Sigma_1, \Sigma_2, \Sigma_3 \subseteq E$ și $\varphi, \psi, \chi \in E$, a.î. au loc deducțiile sintactice:

$$\begin{cases} \Sigma_1 \cup \{\varphi\} \vdash \psi \overset{(TCT)}{\iff} \Sigma_1 \cup \{\varphi\} \vDash \psi. \quad (1) \\ \Sigma_2 \cup \{\psi \wedge \chi\} \vdash \varphi \overset{(TCT)}{\iff} \Sigma_2 \cup \{\psi \wedge \chi\} \vDash \varphi. \quad (2) \\ \Sigma_3 \cup \{\psi\} \vdash \chi \overset{(TCT)}{\iff} \Sigma_3 \cup \{\psi\} \vDash \chi. \quad (3) \end{cases}$$

Dem. că: $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vDash \varphi \leftrightarrow (\psi \wedge \chi)$.

Fie $h : V \to \mathcal{L}_2$, a.î. $h \vDash \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \iff \begin{cases} h \vDash \Sigma_1 & (*) \\ h \vDash \Sigma_2 & (**) \\ h \vDash \Sigma_3 & (***) \end{cases}$

Caz 1: Dacă $\tilde{h}(\varphi) = 1 \overset{(*)}{\Rightarrow} h \vDash \Sigma_1 \cup \{\varphi\} \overset{(1)}{\Rightarrow} \tilde{h}(\psi) = 1 \overset{(***)}{\Rightarrow} h \vDash \Sigma_3 \cup \{\psi\} \overset{(3)}{\Rightarrow} \tilde{h}(\chi) = 1. \Rightarrow \tilde{h}(\varphi \leftrightarrow (\psi \wedge \chi)) = \tilde{h}(\varphi) \leftrightarrow (\tilde{h}(\psi) \wedge \tilde{h}(\chi)) = 1 \leftrightarrow (1 \wedge 1) = 1 \leftrightarrow 1 = 1.$

Caz 2: Dacă $\tilde{h}(\varphi) = 0$:

Pp. prin absurd că $\tilde{h}(\psi \wedge \chi) = 1 \overset{(**)}{\Rightarrow} h \vDash \Sigma_2 \cup \{\psi \wedge \chi\} \overset{(2)}{\Rightarrow} \tilde{h}(\varphi) = 1.$ $\xRightarrow{(\tilde{h}(\varphi)=0)} 0 = 1$ în $\mathcal{L}_2$ ↯. $\Rightarrow$

$\Rightarrow \tilde{h}(\psi \wedge \chi) = 0. \Rightarrow \tilde{h}(\varphi \leftrightarrow (\psi \wedge \chi)) = \tilde{h}(\varphi) \leftrightarrow \tilde{h}(\psi \wedge \chi) = 0 \leftrightarrow 0 = 1.$

$\Rightarrow (\forall h : V \to \mathcal{L}_2)(h \models \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$
$\Rightarrow \tilde{h}(\varphi \leftrightarrow (\psi \wedge \chi)) = 1) \overset{\text{def.}}{\Leftrightarrow}$
$\overset{\text{def.}}{\Leftrightarrow} \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \models \varphi \leftrightarrow (\psi \wedge \chi)$
$\overset{\text{TCT}}{\Leftrightarrow} \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vdash \varphi \leftrightarrow (\psi \wedge \chi).$

<u>Exerc.:</u> Să se dem. semantic urm. reg. de ded.: $(\forall \Sigma_1, \Sigma_2, \Sigma_3 \subseteq E)$
$(\forall \varphi, \psi, \chi \in E)$

$$\frac{\Sigma_1 \cup \Sigma_2 \vdash (\varphi \vee \psi) \leftrightarrow (\chi \to \neg\psi), \quad \Sigma_1 \cup \Sigma_3 \cup \{\varphi \leftrightarrow \chi\} \vdash \psi}{\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vdash \chi \to \psi}$$

REZ.: Fie $\Sigma_1, \Sigma_2, \Sigma_3 \subseteq E$ și $\varphi, \psi, \chi \in E$, a.î.:

$$\begin{cases} \Sigma_1 \cup \Sigma_2 \vdash (\varphi \vee \psi) \leftrightarrow (\chi \to \neg\psi) \\ \text{și} \\ \Sigma_1 \cup \Sigma_3 \cup \{\varphi \leftrightarrow \chi\} \vdash \psi \end{cases}$$

$$\overset{\text{TCT}}{\Leftrightarrow} \begin{cases} \Sigma_1 \cup \Sigma_2 \models (\varphi \vee \psi) \leftrightarrow (\chi \to \neg\psi) & (1) \\ \text{și} \\ \Sigma_1 \cup \Sigma_3 \cup \{\varphi \leftrightarrow \chi\} \models \psi & (2). \end{cases}$$

Dem. că: $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \models \chi \to \psi$.
Fie $h : V \to \mathcal{L}_2$, a.î. $h \models \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \Leftrightarrow \begin{cases} h \models \Sigma_1 \cup \Sigma_2 & (*) \\ \text{și} \\ h \models \Sigma_1 \cup \Sigma_3 & (**) \end{cases}$

<u>Caz 1:</u> dc. $\tilde{h}(\psi) = 1 \Rightarrow$

$\Rightarrow \tilde{h}(\chi \to \psi) = \tilde{h}(\chi) \to \tilde{h}(\psi) =$

$= \tilde{h}(\chi) \to 1 = 1.$

Caz 2: dacă $\tilde{h}(\psi) = 0$, atunci:

Pp. prin absurd că $\tilde{h}(\varphi \leftrightarrow \chi) =$

$= 1 \overset{(**)}{\Longrightarrow} h \vDash \Sigma_1 \cup \Sigma_3 \cup \{\varphi \leftrightarrow \chi\} \Rightarrow$

$\overset{(2)}{\Longrightarrow} \tilde{h}(\psi) = 1 \Rightarrow 0 = 1$ în $\mathcal{L}_2$; $\Rightarrow$

$\Rightarrow \tilde{h}(\varphi \leftrightarrow \chi) = 0 \Leftrightarrow \tilde{h}(\varphi) \leftrightarrow \tilde{h}(\chi) =$

$\neq 1 \Leftrightarrow \tilde{h}(\varphi) \neq \tilde{h}(\chi).$ (I)

$(*), (1) \Rightarrow \tilde{h}((\varphi \vee \psi) \leftrightarrow (\chi \to \neg\psi)) =$

$= 1 \Leftrightarrow (\tilde{h}(\varphi) \vee \tilde{h}(\psi)) \leftrightarrow (\tilde{h}(\chi) \to$

$\to \overline{\tilde{h}(\psi)}) = 1 \Leftrightarrow \tilde{h}(\varphi) \vee \tilde{h}(\psi) =$

$= \tilde{h}(\chi) \to \overline{\tilde{h}(\psi)} \xLeftrightarrow{\tilde{h}(\psi) = 0}$

$\Leftrightarrow \tilde{h}(\varphi) \vee 0 = \tilde{h}(\chi) \to \overline{0} \Leftrightarrow$

$\Leftrightarrow \tilde{h}(\varphi) = \tilde{h}(\chi) \to 1 \Leftrightarrow \tilde{h}(\varphi) = 1$

$\overset{(I)}{\Longrightarrow} \tilde{h}(\chi) = 0 \Rightarrow \tilde{h}(\chi \to \psi) =$

$= \tilde{h}(\chi) \to \tilde{h}(\psi) = 0 \to 0 = 1.$

$\Rightarrow \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vDash \chi \to \psi. \overset{(TCT)}{\Longleftrightarrow}$

$\overset{(TCT)}{\Longleftrightarrow} \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vdash \chi \to \psi.$

ALTĂ REZOLVARE:

Fie $\Sigma_1, \Sigma_2, \Sigma_3 \subseteq E$ și $\varphi, \psi, \chi \in E$, a.î.:

$$\begin{cases} \Sigma_1 \cup \Sigma_2 \vdash (\varphi \vee \psi) \leftrightarrow (\chi \rightarrow \neg\psi) \\ \text{și} \\ \Sigma_1 \cup \Sigma_3 \cup \{\varphi \leftrightarrow \chi\} \vdash \psi. \end{cases}$$

$$\overset{\text{TCT}}{\Leftrightarrow} \begin{cases} \Sigma_1 \cup \Sigma_2 \vDash (\varphi \vee \psi) \leftrightarrow (\chi \rightarrow \neg\psi) \quad (*) \\ \text{și} \\ \Sigma_1 \cup \Sigma_3 \cup \{\varphi \leftrightarrow \chi\} \vDash \psi \quad (**) \end{cases}$$

Fie $h : V \rightarrow L_2 = \{0,1\}$, a.î. $h \vDash \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \Rightarrow \begin{cases} h \vDash \Sigma_1 \cup \Sigma_3 \\ \text{și} \\ h \vDash \Sigma_1 \cup \Sigma_2 \end{cases} \Rightarrow$

$$\overset{(*)}{\Rightarrow} 1 = \tilde{h}((\varphi \vee \psi) \leftrightarrow (\chi \rightarrow \neg\psi)) =$$

$$= (\tilde{h}(\varphi) \vee \tilde{h}(\psi)) \leftrightarrow (\tilde{h}(\chi) \rightarrow \overline{\tilde{h}(\psi)}) =$$

$$= (\tilde{h}(\varphi) \vee \tilde{h}(\psi)) \leftrightarrow (\overline{\tilde{h}(\chi)} \vee \overline{\tilde{h}(\psi)}). \quad (***)$$

<u>Caz 1:</u> dacă $\tilde{h}(\varphi \leftrightarrow \chi) = 1$, atunci, cum $h \vDash \Sigma_1 \cup \Sigma_3 \Rightarrow$

$\Rightarrow h \vDash \Sigma_1 \cup \Sigma_3 \cup \{\varphi \leftrightarrow \chi\} \overset{(**)}{\Rightarrow}$

$$\xRightarrow{(**)} \tilde{h}(\psi) = 1 \Rightarrow \tilde{h}(\chi \to \psi) = \tilde{h}(\chi) \to \tilde{h}(\psi) = \tilde{h}(\chi) \to 1 = 1 \quad (1)$$

Caz 2: dacă $\tilde{h}(\varphi \leftrightarrow \chi) = 0$,

at. $\tilde{h}(\varphi) \leftrightarrow \tilde{h}(\chi) = 0 \neq 1 \Rightarrow$

$\Rightarrow \tilde{h}(\varphi) \neq \tilde{h}(\chi)$.

$\tilde{h}(\varphi), \tilde{h}(\chi) \in L_2 = \{0, 1\}$.

$\Rightarrow$ ($\bar{0} = 1$ și $\bar{1} = 0$)

$$\Rightarrow \tilde{h}(\varphi) = \overline{\tilde{h}(\chi)} \xRightarrow{(***)} 1 = \left(\overline{\tilde{h}(\chi)} \vee \tilde{h}(\psi)\right) \leftrightarrow \left(\overline{\tilde{h}(\chi)} \vee \overline{\tilde{h}(\psi)}\right) \Rightarrow \overline{\tilde{h}(\chi)} \vee \tilde{h}(\psi) = \overline{\tilde{h}(\chi)} \vee \overline{\tilde{h}(\psi)}.$$

$\Rightarrow$ $\tilde{h}(\psi) \in L_2 = \{0, 1\}$; $\bar{0} = 1$; $\bar{1} = 0$.

$$\Rightarrow \overline{\tilde{h}(\chi)} \vee 0 = \overline{\tilde{h}(\chi)} \vee 1 \Leftrightarrow$$

$\Leftrightarrow \overline{\widetilde{h}(\chi)} = 1 \Leftrightarrow \widetilde{h}(\chi) = 0. \Rightarrow$

$\Rightarrow \widetilde{h}(\chi \to \psi) = \widetilde{h}(\chi) \to \widetilde{h}(\psi) =$

$= 0 \to \widetilde{h}(\psi) = 1.$ (2)

(1), (2) $\Rightarrow \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \models \chi \to \psi. \Leftrightarrow$

$\overset{(TCT)}{\Leftrightarrow} \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \vdash \chi \to \psi.$

<u>Exerc.:</u> $\alpha, \beta, \gamma, \delta, \varepsilon \in E.$

$E \supseteq \Sigma = \{\alpha \to (\gamma \to \beta), \alpha \to (\beta \to \delta),$
$\varepsilon \to \alpha, \varepsilon \to \gamma, \varepsilon, \neg \delta\}.$

Dem. că Σ e inconsistentă.

REZOLVARE:

Pp. prin absurd că Σ e consistentă $\overset{\text{(prop.)}}{\Leftrightarrow}$ Σ e satisfiabilă $\overset{\text{(def)}}{\Leftrightarrow}$

$\overset{\text{(def)}}{\Leftrightarrow} (\exists\, h : V \to \mathcal{L}_2)(h \vDash \Sigma) \Rightarrow$ (Cf. axiomei alegerii, putem alege una.)

Fie $h : V \to \mathcal{L}_2$ a.î. $h \vDash \Sigma \Leftrightarrow$

$$\overset{\text{(def)}}{\Leftrightarrow} \begin{cases} 1 = \tilde{h}(\alpha \to (\gamma \to \beta)) = \tilde{h}(\alpha) \to (\tilde{h}(\gamma) \to \tilde{h}(\beta)) \\ 1 = \tilde{h}(\alpha \to (\beta \to \delta)) = \tilde{h}(\alpha) \to (\tilde{h}(\beta) \to \tilde{h}(\delta)) \\ 1 = \tilde{h}(\varepsilon \to \alpha) = \tilde{h}(\varepsilon) \to \tilde{h}(\alpha) \\ 1 = \tilde{h}(\varepsilon \to \gamma) = \tilde{h}(\varepsilon) \to \tilde{h}(\gamma) \\ \tilde{h}(\varepsilon) = 1 \\ 1 = \tilde{h}(\neg \delta) = \overline{\tilde{h}(\delta)} \Leftrightarrow \tilde{h}(\delta) = 0. \end{cases}$$

$$\Rightarrow \begin{cases} \tilde{h}(\alpha) \geq \\ \tilde{h}(\gamma) \geq \end{cases} \tilde{h}(\varepsilon) = 1 \Rightarrow \tilde{h}(\alpha) = \tilde{h}(\gamma) = 1,$$

$$\Rightarrow \begin{cases} 1 = 1 \to (1 \to \tilde{h}(\beta)) = \overline{1} \vee \overline{1} \vee \tilde{h}(\beta) = \\ \quad = 0 \vee \tilde{h}(\beta) = \tilde{h}(\beta). \\ 1 = 1 \to (\tilde{h}(\beta) \to 0) \end{cases} \Rightarrow 1 = 1 \to (1 \to 0) = 1 \to 0 = 0 \Rightarrow$$

$\Rightarrow 0 = 1$ în $\mathcal{L}_2$. ✗. $\Rightarrow \Sigma$ e inconsistentă.