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241

Examen

1. $X \sim \begin{pmatrix} 1 & 7 \\ 0.15 & 0.85 \end{pmatrix}$ $Y \sim \begin{pmatrix} 5 & 7 \\ p_1 & p_2 \end{pmatrix}$

a) $P(X=1, Y=7) = 0.03$

$E[X \mid Y=7] = 4$

X\Y	5	7	Σ
1	0.12	0.03	0.15
7	$p_1 - 0.12$	$p_2 - 0.03$	0.85
Σ	p_1	p_2	

$X \mid Y=7 \sim \begin{pmatrix} 1 & 7 \\ P(X=1, Y=7) & P(X=7, Y=7) \end{pmatrix} \sim \begin{pmatrix} 1 & 7 \\ \frac{0.03}{p_2} & \frac{p_2 - 0.03}{p_2} \end{pmatrix}$

$1 \cdot \frac{0.03}{p_2} + 7 \cdot \frac{p_2 - 0.03}{p_2} = 4 \quad | \cdot p_2$

$0.03 + 7p_2 - 7 \cdot 0.03 = 4p_2$

$3p_2 = 7 \cdot 0.03 - 0.03$

$3p_2 = 6 \cdot 0.03$

$p_2 = 2 \cdot 0.03 = 0.06$

$p_1 = 1 - p_2 = 0.94$

$X \sim \begin{pmatrix} 1 & 7 \\ 0.15 & 0.85 \end{pmatrix}$ $Y \sim \begin{pmatrix} 5 & 7 \\ 0.94 & 0.06 \end{pmatrix}$

$P\{X+Y=6\} = P(\{X=1\} \cap \{Y=5\}) = P\{X=1\} \cdot P\{Y=5\}$
$= 0.15 \cdot 0.94 = 0.141$

$P\{X+Y=8\} = P(\{X=1\} \cap \{Y=7\}) = P\{X=1\} \cdot P\{Y=7\}$
$= 0.15 \cdot 0.06 = 0.009$

$P\{X+Y=12\} = P(\{X=7\} \cap \{Y=5\}) = P\{X=7\} \cdot P\{Y=5\}$
$= 0.85 \cdot 0.94 = 0.799$

$P\{X+Y=14\} = P(\{X=7\} \cap \{Y=7\}) = P\{X=7\} \cdot P\{Y=7\}$
$= 0.85 \cdot 0.06 = \cancel{0.561}$
0.051

$X+Y \sim \begin{pmatrix} 6 & 8 & 12 & 14 \\ 0.141 & 0.009 & 0.799 & 0.051 \end{pmatrix}$

$P\{X-Y=-4\} = P(\{X=1\} \cap \{Y=5\}) = 0.141$

$P\{X-Y=-6\} = P(\{X=1\} \cap \{Y=7\}) = 0.009$

$P\{X-Y=2\} = P(\{X=7\} \cap \{Y=5\}) = 0.799$

$P\{X-Y=0\} = P(\{X=7\} \cap \{Y=7\}) = \cancel{0.561}$
0.051

$X-Y \sim \begin{pmatrix} -6 & -4 & 0 & 2 \\ 0.009 & 0.141 & 0.051 & 0.799 \end{pmatrix}$

$$X^2 \sim \begin{pmatrix} 1 & 49 \\ 0.15 & 0.85 \end{pmatrix} \qquad Y^2 \sim \begin{pmatrix} 25 & 49 \\ 0.94 & 0.06 \end{pmatrix}$$

$$5X^2 \sim \begin{pmatrix} 5 & 245 \\ 0.15 & 0.85 \end{pmatrix} \qquad 2Y^2 \sim \begin{pmatrix} 50 & 98 \\ 0.94 & 0.06 \end{pmatrix}$$

$$P\{5X^2 + 2Y^2 = 55\} = 0.15 \cdot 0.94 = 0.141$$

$$P\{5X^2 + 2Y^2 = 103\} = 0.15 \cdot 0.06 = 0.009$$

$$P\{5X^2 + 2Y^2 = 295\} = 0.85 \cdot 0.94 = 0.799$$

$$P\{5X^2 + 2Y^2 = 343\} = 0.85 \cdot 0.06 = 0.051$$

$$5X^2 + 2Y^2 \sim \begin{pmatrix} 55 & 103 & 295 & 343 \\ 0.141 & 0.009 & 0.799 & 0.051 \end{pmatrix}$$

$E[X] = 1 \cdot 0.15 + 7 \cdot 0.85 = 6.1$

$E[Y] = 5 \cdot 0.94 + 7 \cdot 0.06 = 5.12$

$E[X^2] = 1 \cdot 0.15 + 49 \cdot 0.85 = 41.8$

$E[Y^2] = 25 \cdot 0.94 + 49 \cdot 0.06 = 26.44$

$Var(X) = E[X^2] - E[X]^2 = 41.8 - 37.21 = 4.59$

$Var(Y) = E[Y^2] - E[Y]^2 = 26.44 - 26.21 = 0.23$

$$Var(2X-4Y+15) = Var(2X-4Y)$$
$$= Var(2X) + Var(-4Y)$$
$$= 4\,Var(X) + 16\,Var(Y)$$
$$= 4\cdot 4.59 + 16\cdot 0.23 = 22.04$$

$$\rho(X,Y) = \frac{COV(X,Y)}{\sqrt{Var(X)}\cdot\sqrt{Var\,Y}}$$

$$COV(X,Y) = E[XY] - E[X]\cdot E[Y]$$

$$XY \sim \begin{pmatrix} 25 & 49 & 175 & 343 \\ 0.141 & 0.009 & 0.799 & 0.051 \end{pmatrix}$$

$$E[XY] = 25\cdot 0.141 + 49\cdot 0.009 + 175\cdot 0.799 + 343\cdot 0.051$$

$$= 161.284$$

$$COV(X,Y) = 161.284 - 6.1\cdot 5.12 = 161.284 - 31.232$$
$$= 130.052$$

$$\rho(X,Y) = \frac{130.052}{\sqrt{4.59}\cdot\sqrt{0.23}} = \frac{130.052}{1.027} = 126.632$$

$= 200 - \left(5\frac{\sqrt{2\pi}}{2}\right) = 200 - 12.53 = 1$

2. $E[\log(x)] \geq \log(E[x])$

$E[x] \; ? \; \sqrt{E[x]}$ – depinde de intervalul radicalului

$E[\sin^2(x)] + E[\cos^2(x)] = 1$ → media de constantă e constantă

$P(x>c) \; ? \; \frac{E[x^3]}{c^3}$ – depinde de constanta

$P(x \le y) = P(x \ge y)$

$P(x+y>10) \; ? \; P(x>5$ sau $y>5)$

$E[\min(x,y)] < \min E[x], E[y]$

$E\left[\frac{x}{y}\right] \geq \frac{E[x]}{E[y]}$ → din proprietăți

$E[x^2(x^2+1)] \; ? \; E[x^2(y^2+1)]$

$E\left[\frac{1}{x}\right] > \frac{1}{E[x]}$

$\frac{E(x)}{E(y)} = \frac{P(x)\cdot x}{P(y)\cdot y}$

3. 5 telefoane ; X - nr teste pt identificarea primului telefon defect
Y - nr teste pt identificarea celui de-al doilea telefon defect

X \ Y	0	1	2	3
1	0	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$
2	0	$\frac{1}{10}$	$\frac{2}{10}$	0
3	$\frac{1}{10}$	$\frac{2}{10}$	0	0

$X+Y \leq 4$

$P(X=i, Y=j) = P(X=i) \cdot P(Y=j)$

$P(X=1, Y=1) = \frac{2}{5} \cdot \frac{1}{4} = \frac{1}{10}$

$P(X=1, Y=2) = \frac{2}{5} \cdot \frac{\cancel{3}}{4} \cdot \frac{1}{\cancel{3}} = \frac{1}{10}$

$P(X=1, Y=3) = \frac{2}{5} \cdot \frac{\cancel{3}}{4} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{2}{\cancel{2}} = \frac{4}{20} = \frac{2}{10}$

$P(X=2, Y=1) = \frac{\cancel{3}}{5} \cdot \frac{2}{4} \cdot \frac{1}{\cancel{3}} = \frac{1}{10}$

$P(X=2, Y=2) = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{2}{2} = \frac{2}{10}$

$P(X=3, Y=1) = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} \cdot \frac{2}{2} = \frac{2}{10}$

$P(X=3, Y=0) = \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} = \frac{1}{10}$ → am găsit 3 telefoane defecte => știm că ultimele 2 sunt defecte

$P_X(x) \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{4}{10} & \frac{3}{10} & \frac{3}{10} \end{pmatrix}$ $P_Y(y) \sim \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{10} & \frac{4}{10} & \frac{3}{10} & \frac{2}{10} \end{pmatrix}$

b) $E[X] = 1 \cdot \frac{4}{10} + 2 \cdot \frac{3}{10} + 3 \cdot \frac{3}{10} = \frac{4+6+9}{10} = \frac{19}{10}$

$E[Y] = 0 \cdot 1/10 + 1 \cdot \frac{4}{10} + 2 \cdot \frac{3}{10} + 3 \cdot \frac{2}{10} = \frac{4+6+6}{10} = \frac{16}{10}$

$X^2 \sim \begin{pmatrix} 1 & 4 & 9 \\ \frac{4}{10} & \frac{3}{10} & \frac{3}{10} \end{pmatrix}$ $Y^2 \sim \begin{pmatrix} 0 & 1 & 4 & 9 \\ \frac{1}{10} & \frac{4}{10} & \frac{3}{10} & \frac{2}{10} \end{pmatrix}$

$E[X^2] = 1 \cdot \frac{4}{10} + 4 \cdot \frac{3}{10} + 9 \cdot \frac{3}{10} = \frac{43}{10}$

$E[Y^2] = 0 \cdot \frac{1}{10} + 1 \cdot \frac{4}{10} + 4 \cdot \frac{3}{10} + 9 \cdot \frac{2}{10} = \frac{34}{10}$

$$Var(x) = E[x^2] - (E[x])^2 = \frac{43}{10} - \left(\frac{19}{10}\right)^2 = 0.69$$

$$Var(y) = E[y^2] - (E[y])^2 = \frac{34}{10} - \left(\frac{16}{10}\right)^2 = 0.84$$

$$\rho(x,y) = \frac{cov(x,y)}{\sqrt{var(x)} \cdot \sqrt{var(y)}}$$

$$cov(x,y) = E[xy] - E[x] \cdot E[y]$$

$$xy \sim \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 6 & 9 \\ 0.1 & 0.16 & 0.24 & 0.2 & 0.09 & 0.15 & 0.06 \end{pmatrix}$$

$$E[xy] = 0 \cdot 0.1 + 1 \cdot 0.16 + 2 \cdot 0.24 + 3 \cdot 0.2 + 4 \cdot 0.09 + 6 \cdot 0.15 + 9 \cdot 0.06$$

$$= 3.04$$

$$cov(x,y) = 3.04 - 1.9 \cdot 1.6 = 0 \Rightarrow \rho(x,y) = 0$$

c) $E[x|y=2] = 1 \cdot \frac{1}{3} + 2 \cdot \frac{2}{3} + 3 \cdot 0 = \frac{5}{3}$

$$Var(x|y=2) = E[(x|y=2)^2] - (E[x|y=2])^2$$

$$(x|y=2)^2 \sim \begin{pmatrix} 1 & 4 & 9 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$$

$$E[(x|y=2)^2] = 1 \cdot \frac{1}{3} + 4 \cdot \frac{2}{3} + 9 \cdot 0 = \frac{9}{3} = 3$$

$$Var(x|y=2) = 3 - \left(\frac{5}{3}\right)^2 = \frac{2}{9}$$

4. $f(x) = \frac{x}{100} e^{-\frac{x^2}{200}} \quad 1\{x \geq 0\}$

$F(x) = \int_0^x \frac{t}{100} e^{-\frac{t^2}{200}} dt = -e^{-\frac{t^2}{200}} \Big|_0^x = 1 - e^{-\frac{x^2}{200}}$

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x^2}{200}}, & x > 0 \end{cases}$$

$F^{-1}(x) = y \mid oF$

$x = F(y)$

$x = 1 - e^{-\frac{y^2}{200}}$

$e^{-\frac{y^2}{200}} = 1 - x$

$e^{-\frac{y^2}{200}} = 1 - x$

$-\frac{y^2}{200} = \ln(1-x)$

$y^2 = 200 \ln \frac{1}{1-x}$

$y = 10\sqrt{2}\sqrt{\ln \frac{1}{1-x}}$

$F^{-1}(0.75) = 10\sqrt{2} \sqrt{\ln \frac{1}{0{,}25}} = 10\sqrt{2} \cdot \sqrt{\ln 4}$

$F^{-1}(0.25) = 10\sqrt{2} \cdot \sqrt{\ln \frac{1}{0.75}} = 10\sqrt{2} \sqrt{\ln \frac{100}{75}}$

$F^{-1}(0.75) - F^{-1}(0.25) = 10\sqrt{2}\left(\sqrt{\ln 4} - \sqrt{\ln \frac{20}{15}}\right)$

$Var(X) = E[X^2] - E[X]^2$

$E[X] = \int_0^\infty x f(x)\, dx = \int_0^\infty \frac{x^2}{100} e^{-\frac{x^2}{200}} dx$

$= -\frac{x}{2} e^{-\frac{x^2}{200}} \Big|_0^\infty + \frac{1}{2}\int_0^\infty e^{-\frac{x^2}{200}} dx$

$$E[X] = \frac{1}{2}\int_0 e^{-\frac{x^2}{200}}\,dx \overset{y=\frac{x}{10}}{=} 5\int_0^\infty e^{-\frac{y^2}{2}}\,dy = 5\frac{\sqrt{2\pi}}{2}$$

$$E[X^2] = \int_0^\infty x^2 f(x) = \int_0^\infty \frac{x^3}{100} e^{-\frac{x^2}{200}}\,dx = 200$$

$$\Rightarrow Var(X) = 200 - \left(5\frac{\sqrt{2\pi}}{2}\right)^2 = 200 - 12.53 = 187.47$$

2. $E[\log(X)] \geq \log(E[X])$

$E[X]$? $\sqrt{E[X]}$ – depinde de intervalul radicalului

$E[\sin^2(X)] + E[\cos^2(X)] = 1$ → medie de constantă e constantă

$P(X>c)$? $\frac{E[X^3]}{c^3}$ – depinde de constanta

$P(X \le Y) = P(X \geq Y)$

$P(X+Y>10)$? $P(X>5$ sau $Y>5)$

$E[\min(X,Y)] \le \min E[X], E[Y]$

$E\left[\frac{X}{Y}\right] \geq \frac{E[X]}{E[Y]}$ → din proprietăți

5. X - votează Cîțu $P(X)=0.39$

Y - votează Orban $P(Y)=0.61$

$X \sim P(\lambda)$

$X \sim P(604)$

$P(X=k)=e^{-\lambda}\cdot\frac{\lambda^k}{k!}=e^{-604}\cdot\frac{604^k}{k!}$

$E[X]=604=\lambda$

$Var(X)=604=\lambda$

6. $p=0.11$

X - nr succes

~~$\Omega = \{t, \dots\}^2 = \{\dots\}$~~

~~$\Omega=\{s,e\}=\{s,es,ees,eees,eeee,$~~

s - succes, e - eșec

1 eșec 4 succes

2 eșec 3 succes

3 eșec 2 succes

$\Omega=\{s,e\}=$

s - succes, e - eșec

funcția de repartiție $\to$ $F:\mathbb{R}\to[0,1]$

$F(x)=P(X\le x), \forall x\in\mathbb{R}$