Logică Propoziţională Clasică – demonstraţii sintactice

SEMINAR DE LOGICĂ MATEMATICĂ ŞI COMPUTAŢIONALĂ

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Def.: $n \in \mathbb{N}^*$, $\gamma_1, \ldots, \gamma_n \in E$.

Se definește $\bigwedge_{i=1}^{n} \gamma_i := \begin{cases} \gamma_1, & \text{dc. } n=1, \\ \left(\bigwedge_{i=1}^{n-1} \gamma_i\right) \wedge \gamma_n, & \text{dc. } n \geq 2. \end{cases}$

Exerc.: $n \in \mathbb{N}^*$, $\gamma_1, \ldots, \gamma_n, \varphi \in E$.

Să se dem. că: $\vdash \left(\bigwedge_{i=1}^{n} \gamma_i\right) \rightarrow \varphi \Leftrightarrow$

$\Leftrightarrow \vdash \gamma_1 \rightarrow (\gamma_2 \rightarrow (\ldots \rightarrow (\gamma_n \rightarrow \varphi)\ldots))$.

REZOLVARE:

Inducție matem. după $n \in \mathbb{N}^*$.

$n=1$: $\vdash \gamma_1 \rightarrow \varphi \Leftrightarrow \vdash \gamma_1 \rightarrow \varphi$.

$n=2$: $\vdash (\gamma_1 \wedge \gamma_2) \rightarrow \varphi \underset{4}{\Leftrightarrow} \vdash \gamma_1 \rightarrow (\gamma_2 \rightarrow \varphi)$.

$\vdash \gamma_1 \rightarrow (\gamma_2 \rightarrow \varphi) \overset{(TD)}{\Longleftrightarrow} \{\gamma_1\} \vdash \gamma_2 \rightarrow \varphi \overset{(TD)}{\Longleftrightarrow}$

$\Leftrightarrow \{\gamma_1, \gamma_2\} \vdash \varphi$ (I)

Pe de altă parte: $\vdash (\gamma_1 \wedge \gamma_2) \to \varphi$

$\overset{(TD)}{\Leftrightarrow} \{\gamma_1 \wedge \gamma_2\} \vdash \varphi$. (II)

Dacă $\{\gamma_1 \wedge \gamma_2\} \vdash \varphi$, atunci:

$\vdash (\gamma_1 \wedge \gamma_2) \to \varphi$, $\Rightarrow \{\gamma_1, \gamma_2\} \vdash (\gamma_1 \wedge \gamma_2) \to \varphi$

$\left.\begin{array}{l} \{\gamma_1, \gamma_2\} \vdash \gamma_1 \\ \{\gamma_1, \gamma_2\} \vdash \gamma_2 \end{array}\right\} \overset{\text{prop.}}{\Longrightarrow} \{\gamma_1, \gamma_2\} \vdash \gamma_1 \wedge \gamma_2$

$\overset{(mp)}{\Longrightarrow} \{\gamma_1, \gamma_2\} \vdash \varphi$. (III)

Dacă $\{\gamma_1, \gamma_2\} \vdash \varphi$, atunci:

$\vdash \gamma_1 \to (\gamma_2 \to \varphi)$ (cf. (I)) $\Rightarrow$

$\Rightarrow \{\gamma_1 \wedge \gamma_2\} \vdash \gamma_1 \to (\gamma_2 \to \varphi)$.

prop. $\Rightarrow \vdash (\gamma_1 \wedge \gamma_2) \to \gamma_1 \overset{(TD)}{\Leftrightarrow} \{\gamma_1 \wedge \gamma_2\} \vdash \gamma_1$ $\Rightarrow$

$\overset{(mp)}{\Longrightarrow} \{\gamma_1 \wedge \gamma_2\} \vdash \gamma_2 \to \varphi$.

prop. $\Rightarrow \vdash (\gamma_1 \wedge \gamma_2) \to \gamma_2 \overset{(TD)}{\Leftrightarrow} \{\gamma_1 \wedge \gamma_2\} \vdash \gamma_2$. $\Rightarrow$

$\overset{(mp)}{\Longrightarrow} \{\gamma_1 \wedge \gamma_2\} \vdash \varphi$. (IV)

(I), (II), (III), (IV) $\Rightarrow$ [$\vdash (\gamma_1 \wedge \gamma_2) \to \varphi \Leftrightarrow$

$\Leftrightarrow \vdash \gamma_1 \to (\gamma_2 \to \varphi)$].

$n \geq 2 \rightarrow n+1$:

Prin ip. de ind., $\vdash \left(\bigwedge_{i=1}^{n} \gamma_i\right) \to \varphi \Leftrightarrow$

$$\Leftrightarrow \vdash \gamma_1 \to (\gamma_2 \to (\ldots \to (\gamma_n \to \varphi)\ldots)).$$

$$\vdash \left(\bigwedge_{i=1}^{n+1} \gamma_i\right) \to \varphi \overset{(def.)}{\Leftrightarrow}$$

$$\Leftrightarrow \vdash \left(\left(\bigwedge_{i=1}^{n} \gamma_i\right) \wedge \gamma_{n+1}\right) \to \varphi \overset{(\text{cazul } n=2)}{\Longleftrightarrow}$$

$$\Leftrightarrow \vdash \left(\bigwedge_{i=1}^{n} \gamma_i\right) \to (\gamma_{n+1} \to \varphi) \overset{(\text{ip. de ind.})}{\Longleftrightarrow}$$

$$\Leftrightarrow \vdash \gamma_1 \to (\gamma_2 \to (\ldots \to (\gamma_n \to (\gamma_{n+1} \to \varphi))\ldots))).$$

Exerc.: $\Sigma \subseteq E$; $\varphi \in E$.

At.: $\Sigma \vdash \varphi \Leftrightarrow (\exists n \in \mathbb{N}^*)(\exists \gamma_1, \ldots, \gamma_n \in \Sigma)\left(\vdash \left(\bigwedge_{i=1}^{n} \gamma_i\right) \to \varphi\right)$.

REZOLVARE:

$\Sigma \vdash \varphi \overset{(prop.)}{\Leftrightarrow} (\exists \Gamma \subseteq \Sigma)(|\Gamma| < \infty$ și $\Gamma \vdash \varphi)$.

$\tau \in \Sigma$, arbitrar, dc. $\Sigma \neq \emptyset$

dc. nițel e 0, i.e. $\Gamma = \emptyset$, at. luăm (măcar) $\Gamma = \{$[illegible]$\}$, $n=1$

Fie $|\Gamma| = n \in \mathbb{N}^*$, $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$.

dc. $\Sigma = \emptyset$: fără id.: $n=0$.

Așadar, $\Sigma \vdash \varphi \Leftrightarrow (\exists n \in \mathbb{N}^*)$
$(\exists \gamma_1, \ldots, \gamma_n \in \Sigma)(\{\gamma_1, \ldots, \gamma_n\} \vdash \varphi)$.

Dar: $\{\gamma_1, \ldots, \gamma_n\} \vdash \varphi \overset{(TD)}{\Leftrightarrow}$
$\Leftrightarrow \{\gamma_1, \ldots, \gamma_{n-1}\} \vdash \gamma_n \to \varphi \overset{(TD)}{\Leftrightarrow}$
$\Leftrightarrow \{\gamma_1, \ldots, \gamma_{n-2}\} \vdash \gamma_{n-1} \to (\gamma_n \to \varphi) \overset{(TD)}{\Leftrightarrow}$
$\Leftrightarrow \ldots \overset{(TD)}{\Leftrightarrow} \vdash \gamma_1 \to (\gamma_2 \to (\ldots \to (\gamma_n \to \varphi) \ldots)) \underset{\text{(exerc. anter.)}}{\Leftrightarrow}$
$\vdash \left(\bigwedge_{i=1}^{n} \gamma_i\right) \to \varphi$.

Prin urmare: $\Sigma \vdash \varphi \Leftrightarrow$
$\Leftrightarrow (\exists n \in \mathbb{N}^*)(\exists \gamma_1, \ldots, \gamma_n \in \Sigma)$
$(\vdash \gamma_1 \to (\gamma_2 \to (\ldots \to (\gamma_n \to \varphi) \ldots))) \Leftrightarrow$
$\Leftrightarrow (\exists n \in \mathbb{N}^*)(\exists \gamma_1, \ldots, \gamma_n \in \Sigma)$
$\left(\vdash \left(\bigwedge_{i=1}^{n} \gamma_i\right) \to \varphi\right)$.

Obs.: Din rezolvarea de mai sus rezultă și: $(\forall n \in \mathbb{N}^*)(\forall \gamma_1, \ldots, \gamma_n \in E)$ (au loc: $\{\gamma_1, \ldots, \gamma_n\} \vdash \varphi \Leftrightarrow \vdash \left(\bigwedge_{i=1}^{n} \gamma_i\right) \to \varphi$
$\Leftrightarrow \vdash \gamma_1 \to (\gamma_2 \to (\ldots \to (\gamma_n \to \varphi) \ldots)) \Leftrightarrow$
$\Leftrightarrow \{\bigwedge_{i=1}^{n} \gamma_i\} \vdash \varphi$.