Week 3

Beta Function

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

$m>0,\ n>0$

$\sqrt{1-x^3}$

$= (1-x^3)^{1/2}$

simplify

Exercise:

2 iii) $\displaystyle\int_0^1 \frac{1}{\sqrt{1-x^3}}\, dx$

$= \displaystyle\int_0^1 (1-x^3)^{-1/2}\, dx$

$= \displaystyle\int_0^1 \frac{1}{3} z^{-2/3} (1-z)^{-1/2}\, dz$

$-2/3 \to m-1$, $-1/2 \to n-1$

$= \displaystyle\frac{1}{3}\int_0^1 z^{\frac{1}{3}-1} (1-z)^{\frac{1}{2}-1}\, dz$

$= \frac{1}{3}\beta\left(\frac{1}{3}, \frac{1}{2}\right)$

$x^3 = z$

$3x^2\, dx = dz$

$dx = \dfrac{1}{3x^2}\, dz$

$= \dfrac{1}{3z^{2/3}}\, dz$

$x^3 = z$

$x = z^{1/3}$

$\therefore x^2 = z^{2/3}$

limits

$x = 0 \to z = 0$

$x = 1 \to z = 1$

$\to m-1 = -2/3$

$m = 1 - 2/3 = 1/3$

$\to n-1 = -1/2$

$n = \frac{1}{2}$

$$\sqrt{a^4(1-y^4)} = a^2\sqrt{1-y^4}$$

Let

2vi) $\displaystyle\int_0^a y^7 \sqrt{a^4 - y^4}\, dy$

a^4y^4 factor out a^4

$$= \int_0^1 a^7 z^7 \sqrt{a^4 - a^4 z^4}\; a\,dz$$

$$= a^8 \int z^7 a^2 \sqrt{1-z^4}\, dz$$

$$= a^{10} \int_0^1 z^7 (1-z^4)^{1/2}\, dz$$

$$= a^{10} \int_0^1 z^4 \cdot \underline{z^3} (1-z^4)^{1/2}\, \underline{dz}$$

$$= a^{10} \int_0^1 x\, \frac{1}{4} (1-x)^{1/2}\, dx$$

$$= \frac{a^{10}}{4} \int_0^1 x^1 (1-x)^{\overset{\to\, n-1}{1/2}}\, dx$$

$$= \frac{a^{10}}{4} \int_0^1 x^{\overset{m-1}{2-1}} (1-x)^{\overset{n-1}{3/2-1}}\, dx$$

$$= \frac{a^{10}}{4} \beta\left(2, 3/2\right)$$

$y^4 = a^4 z^4$

let

$y = az$

$dy = a\,dz$

limits

$y = 0 \Rightarrow az = 0$

$z = 0$

$y = a \Rightarrow az = a$

$z = 1$

let

$z^4 = x$

$4z^3 dz = dx$

$z^3 dz = \frac{1}{4} dx$

limits

$z = 0 \to x = 0$

$z = 1 \to x = 1$

$n - 1 = \frac{1}{2}$

$n = \frac{1}{2} + 1 = \frac{3}{2}$

Gamma Function

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \; ; \; n>0$$

$$\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \; ; \; \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

$m>0, n>0$

1 i) $\int_0^4 x^{3/2} (4 - x)^{5/2}\, dx$

$\underset{1}{4}$ $\quad \downarrow 4x$

$x = 4y$

$dx = 4dy$

limits

$x=0 \to y=0$

$x=4 \to y=1$

$$= \int_0^1 4^{3/2} y^{3/2} (4-4y)^{5/2}\, 4dy$$

$[4(1-y)]^{5/2}$

$$= 4^{3/2+1} \int_0^1 y^{3/2}\, 4^{5/2} (1-y)^{5/2}\, dy$$

$$= 4^{\frac{3}{2}+1+\frac{5}{2}} \int_0^1 y^{\overset{m-1}{3/2}} (1-y)^{\overset{n-1}{5/2}}\, dy$$

$m-1 = 3/2$

$m = 3/2 + 1 = 5/2$

$n-1 = 5/2$

$n = 5/2 + 1 = 7/2$

$$= 4^{\frac{10}{2}} \int_0^1 y^{5/2 - 1} (1-y)^{7/2 - 1}\, dy$$

$$= 4^5 \; \beta\left(\frac{5}{2}, 7/2\right)$$

$$= 1024 \frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(7/2\right)}{\Gamma\left(\frac{5}{2}+\frac{7}{2}\right)} = 1024 \frac{\Gamma\left(5/2\right)\Gamma\left(7/2\right)}{\Gamma(6)}.$$

1(ii) $\int_0^b y^5 \sqrt{b^2-y^2}\, dy$

let $y = bz$

$dy = b\,dz$

limits

$y=0 \rightarrow z=0$

$y=b \rightarrow z=1$

$$= \int_0^1 b^5 z^5 \sqrt{b^2-b^2z^2}\, b\, dz$$

$$= b^6 \int_0^1 z^5\, b\sqrt{1-z^2}\, dz$$

$$= b^7 \int_0^1 z^5 (1-z^2)^{1/2}\, dz$$

$z^2 = x \rightarrow z^4 = x^2$

$2z\,dz = dx$

$z\,dz = \frac{1}{2}dx$

$$= b^7 \int_0^1 z \cdot z^4 (1-z^2)^{1/2}\, dz$$

$$= b^7 \int_0^1 \frac{1}{2} \cdot x^2 (1-x)^{1/2}\, dx$$

limits:

$z=0 \rightarrow x=0$

$z=1 \rightarrow x=1$

$$= \frac{b^7}{2} \int_0^1 x^{3-1} (1-x)^{3/2-1}\, dx$$

$$= \frac{b^7}{2} \beta\left(3, 3/2\right)$$

$$= \frac{b^7}{2} \frac{\Gamma(3)\,\Gamma(3/2)}{\Gamma(3+3/2)}$$

$$= \frac{b^7}{2} \frac{\Gamma(3)\,\Gamma(3/2)}{\Gamma(9/2)}$$

1 vii) $\int_0^\infty x^6 e^{-3x} dx$

$3x = z \longrightarrow \frac{z}{3} = x$

$3dx = dz$

$dx = \frac{1}{3} dz$

limits:

$x = 0 \rightarrow z = 0$

$x = \infty \rightarrow z = \infty$

$= \int_0^\infty \left(\frac{z}{3}\right)^6 e^{-z} \frac{1}{3} dz$

$= \frac{1}{3^7} \int_0^\infty e^{-z} z^{7-1} dz$

$= \frac{1}{3^7} \Gamma(7)$

1 ix) $\int_0^\infty \sqrt{x}\, e^{-x^2} dx$

$x^2 = z \longrightarrow x = \sqrt{z}$

$\therefore \sqrt{x} = z^{1/4}$

$2x dx = dz$

$dx = \frac{1}{2x} dz$

$= \frac{1}{2\sqrt{z}} dz$

$= \int_0^\infty z^{1/4} e^{-z} \frac{1}{2\sqrt{z}} dz$

$= \frac{1}{2} \int_0^\infty z^{\frac{1}{4} - \frac{1}{2}} e^{-z} dz$

limits

$x = 0 \rightarrow z = 0$

$x = \infty \rightarrow z = \infty$

$= \frac{1}{2} \int_0^\infty z^{-1/4} e^{-z} dz$ $\quad -1/4 \rightarrow n-1$

$= \frac{1}{2} \int_0^\infty e^{-z} z^{\overbrace{\frac{3}{4}-1}^{n-1}} dz$

$n - 1 = -\frac{1}{4}$

$n = -\frac{1}{4} + 1 = \frac{3}{4}$

$= \frac{1}{2} \Gamma(3/4)$

$$\to \frac{1}{\sqrt{x}\sqrt{\ln\frac{1}{x}}} = \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{\ln(\frac{1}{x})}} = x^{-1/2} \frac{1}{\sqrt{\ln\frac{1}{x}}}$$

1 xi) $\int_0^1 \frac{1}{\sqrt{x \ln(\frac{1}{x})}} dx$

$$= \int_{\infty}^{0} \frac{-e^{-z} dz}{\sqrt{e^{-z} z}}$$

$$= -\int_{\infty}^{0} \frac{e^{-z} dz}{e^{-z/2} z^{1/2}}$$

$$= \int_0^{\infty} e^{-z-(-z/2)} z^{-1/2} dz$$

$$= \int_0^{\infty} e^{-z/2} z^{-1/2} dz$$

$$= \int_0^{\infty} e^{-y} (2y)^{-1/2} 2\, dy$$

$$= 2^{-1/2+1} \int_0^{\infty} e^{-y} y^{-1/2} dy$$

$$= \sqrt{2} \int_0^{\infty} e^{-y} y^{\frac{1}{2}-1} dy$$

$$= \sqrt{2}\, \Gamma\left(\frac{1}{2}\right)$$

$\ln\left(\frac{1}{x}\right) = z$

$\ln 1 - \ln x = z$

$0 - \ln x = z$

$\ln x = -z$

$\log_e x = -z$

$x = e^{-z}$

$dx = -e^{-z} dz$

limits:

$x = 0 \to z = \infty \to \ln(\frac{1}{0}) = \ln\infty = \infty$

$x = 1 \to z = 0 \to \ln(\frac{1}{1}) = \ln 1 = 0$

$\frac{z}{2} = y$

$z = 2y$

$dz = 2dy$

limits

$z = 0 \to y = 0$

$z = \infty \to y = \infty$

$$\int_a^b f\, dx = -\int_b^a f\, dx$$

1(xii) $\int_0^1 \left(1 - \frac{1}{x}\right)^{1/3} dx$

$$= \int_0^1 \left(\frac{x-1}{x}\right)^{1/3} dx = \int_0^1 -\left(\frac{1-x}{x}\right)^{1/3} dx$$

$$= -\int_0^1 x^{-1/3} (1-x)^{1/3} dx$$

⋮ continue

Formulas:

$\boxed{1}$ $$\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}.$$

OR

$\boxed{2}$ $$\int_0^{\pi/2} 2\sin^{2x-1}(t)\cos^{2y-1}(t)\,dt = \beta(x,y).$$

3 iii) $$\int_0^{\pi/6} \sin^2 6x \cos^4 3x \, dx$$

$\sin^2\theta$
$(\sin\theta)^2$

$$= \int_0^{\pi/6} (\sin 2\cdot 3x)^2 \cos^4 3x \, dx$$

$\sin^2 6x$
$= (\sin 6x)^2$
$= (\sin 2\cdot 3x)^2$

$$= \int_0^{\pi/6} (2\sin 3x \cdot \cos 3x)^2 \cos^4 3x \, dx$$

$$= 4\int_0^{\pi/6} \sin^2 3x \cos^2 3x \cos^4 3x \, dx$$

$$= 4\int_0^{\pi/6} \sin^2 3x \cos^6 3x \, dx$$

$3x = z$
$3\,dx = dz$
$dx = \frac{1}{3}dz$

$$= 4\cdot\frac{1}{3}\int_0^{\pi/2} \sin^{\overset{2 \to p}{}} z \cos^{\overset{6 \to q}{}} z \, dz$$

limits:

$x=0 \to z=0$

$x=\frac{\pi}{6} \to z=\pi/2$

$$= \frac{4}{3}\,\frac{\Gamma\left(\frac{2+1}{2}\right)\Gamma\left(\frac{6+1}{2}\right)}{2\Gamma\left(\frac{2+6+2}{2}\right)}$$

$$= \frac{4}{3}\,\frac{\Gamma(3/2)\Gamma(7/2)}{2\Gamma(5)} = \frac{2}{3}\,\frac{\Gamma(3/2)\Gamma(7/2)}{\Gamma(5)}$$

Alternative Method:

$$= \frac{2}{3}\int_0^{\pi/2} 2\sin^2 z\cos^6 z\,dz$$

$2x-1=2$
$2x=2+1=3$
$x=3/2$

$2y-1=6$
$2y=7$
$y=7/2$

$$= \frac{2}{3}\beta(x,y) = \frac{2}{3}\beta\left(\frac{3}{2},\frac{7}{2}\right)$$

$\boxed{7}$

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m,n)$$

$$\beta\left(\frac{3}{2},\frac{7}{2}\right) = \frac{\Gamma(3/2)\Gamma(7/2)}{\Gamma\left(\frac{3}{2}+\frac{7}{2}\right)}$$