Higher Order Differential Equations

The general form of nth order linear ordinary differential equation is,

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

Homogenous Equations

A linear nth order differential equation of the form,

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{2}$$

is said to be homogenous.

Non- homogenous Equations

A linear nth order differential equation of the form,

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{3}$$

with $g(x)$ is not identically zero is said to be non-homogenous.

General Solution

Homogenous Equation

$$y = c_1 y_1 + c_2 y_2 + \cdots \dots \dots \dots \dots \dots + c_n y_n$$

Non-homogenous Equation

$$\begin{aligned} y &= y_c + y_p \\ &= c_1y_1 + c_2y_2 + \cdots \dots \dots \dots \dots \dots + c_ny_n + y_p \\ &= complementary\ function + any\ particular\ solution \end{aligned}$$

Homogenous Linear Equations with Constant Coefficients

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2y'' + a_1y' + a_0y = 0, \tag{4}$$

where the coefficients a_i, $i = 0, 1, \ldots, n$ are real constants and $a_n \neq 0$.

Auxiliary Equation

We begin by the special case of second order equation,

$$ay'' + by' + cy = 0, \tag{5}$$

where a, b, and c are constants. If we try to find a solution of the form $y = e^{mx}$, then after substitution of $y' = me^{mx}$ and $y'' = m^2e^{mx}$, equation (2) becomes

$$am^2e^{mx} + bme^{mx} + ce^{mx} = 0 \quad \text{or} \quad e^{mx}(am^2 + bm + c) = 0.$$

Since, $e^{mx} \neq 0$, therefore,

$$am^2 + bm + c = 0. \tag{6}$$

Eqn, (6) is called auxiliary eqn of the DE (5). Since, (6) is a quadratic equation then it has two roots of the form,

$$m_1 = (-b + \sqrt{b^2 - 4ac})/2a$$

$$m_2 = (-b - \sqrt{b^2 - 4ac})/2a$$

So there will be three forms of the general solution of the differential equation (5) corresponding to the three cases.

- m_1 and m_2 real and distinct ($b^2 - 4ac > 0$),
- m_1 and m_2 real and equal ($b^2 - 4ac = 0$), and
- m_1 and m_2 conjugate complex numbers ($b^2 - 4ac < 0$).

Case I: Distinct Real Roots

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case II: Repeated Real Roots

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}.$$

Case III: Conjugate Complex Roots

If, m_1 and m_2 are complex, then we can write, $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$. Where, $i = \sqrt{-1}$. Then the general solution is of the form,

$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x).$$

Example 1

$$2y'' - 5y' - 3y = 0$$

Solve:

Solution: The corresponding auxiliary equation is,

$$2m^2 - 5m - 3 = (2m + 1)(m - 3) = 0, \quad m_1 = -\tfrac{1}{2}, m_2 = 3$$

The roots are distinct and real, therefore the general solution is,

$$y = c_1 e^{-x/2} + c_2 e^{3x}. \quad \text{(Ans)}$$

Example 2

$$y'' - 10y' + 25y = 0$$

Solve:

Solution: The corresponding auxiliary equation is,

$$m^2 - 10m + 25 = (m-5)^2 = 0, \quad m_1 = m_2 = 5$$

The roots are repeated and real, therefore the general solution is,

$$y = c_1 e^{5x} + c_2 x e^{5x}. \quad \text{(Ans)}$$

Example 3

$$y'' + 4y' + 7y = 0$$

Solve:

Solution: The corresponding auxiliary equation is,

$$m^2 + 4m + 7 = 0, \; m_1 = -2 + \sqrt{3}i, \quad m_2 = -2 - \sqrt{3}i$$

The roots are conjugate complex, therefore the general solution is,

with $\alpha = -2, \beta = \sqrt{3}, y = e^{-2x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$. (Ans)

Method of Undetermined Coefficients

- The underlying idea behind this method is the conjecture about the form y_p.
- $g(x)$ is a constant k, a polynomial function, an exponential function $e^{\alpha x}$, a sine or cosine function $\sin \beta x$ or $\cos \beta x$, or finite sums and products of these functions.

Example 1

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6.$ (1)

Solution:

Step1.We first solve the assosiated homogeneous equation. The roots of the auxiliary eqn,

$m^2 + 4m - 2 = 0$ are $m_1 = -2 - \sqrt{6}$ and $m_2 = -2 + \sqrt{6}$.

Hence, the complementary function is,

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}.$$

Step 2. Now, because the function $g(x)$ is a quadratic polynomial, let us assume a particular solution that is also in the form of a quadratic polynomial:

$$y_p = Ax^2 + Bx + C.$$

Therefore,

$$y_p' = 2Ax + B \qquad \text{and} \qquad y_p'' = 2A$$

Substituing in the differential equation (1) we get,

$$y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6.$$

Now, we will equate the coefficients.

equal

$$\boxed{-2A}\,x^2 + \boxed{8A - 2B}\,x + \boxed{2A + 4B - 2C} = 2x^2 - 3x + 6$$

That is, $\quad -2A = 2, \quad 8A - 2B = -3, \quad 2A + 4B - 2C = 6.$

Solving this system of equations leads to the values $A = -1$, $B = -\frac{5}{2}$, and $C = -9$. Thus a particular solution is

$$y_p = -x^2 - \frac{5}{2}x - 9.$$

Step 3. The general solution of the given equation is

$$y = y_c + y_p = c_1 e^{-(2+\sqrt{6})x} + c_1 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9.$$

(Ans)

Example 2:

Find a particular solution of $y'' - 2y' + y = e^x$.

Solution:

The complementary function is $y_c = c_1 e^x + c_2 x e^x$.

the assumption $y_p = Ae^x$ will fail, since it is apparent from y_c that e^x is a solution of the associated homogeneous equation $y'' - 2y' + y = 0$. Moreover, we will not be able to find a particular solution of the form $y_p = Axe^x$, since the term xe^x is also duplicated in y_c. We next try

$$y_p = Ax^2 e^x.$$

Substituting into the given differential equation yields $2Ae^x = e^x$, so $A = \frac{1}{2}$. Thus a particular solution is $y_p = \frac{1}{2}x^2 e^x$. ■

Example 3

Solve $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$.

Solution:

The complementary function is $y_c = c_1e^{3x} + c_2xe^{3x}$.
the usual assumption for a particular solution would be

$$y_p = \underbrace{Ax^2 + Bx + C}_{y_{p_1}} + \underbrace{Ee^{3x}}_{y_{p_2}}.$$

Inspection of these functions shows that the one term in y_{p_2} is duplicated in y_c. If we multiply y_{p_2} by x, we note that the term xe^{3x} is still part of y_c. But multiplying y_{p_2} by x^2 eliminates all duplications. Thus the operative form of a particular solution is

$$y_p = Ax^2 + Bx + C + Ex^2e^{3x}.$$

Differentiating this last form, substituting into the differential equation, and collecting like terms gives

$$y_p'' - 6y_p' + 9y_p$$

$$= 9Ax^2 + (-12A + 9B)x + 2A - 6B + 9C + 2Ee^{3x} = 6x^2 + 2 - 12e^{3x}.$$

It follows from this identity that $A = \frac{2}{3}$, $B = \frac{8}{9}$, $C = \frac{2}{3}$, and $E = -6$. Hence the general solution $y = y_c + y_p$ is $y = c_1e^{3x} + c_2xe^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2e^{3x}$. ■