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Gamma, Beta Functions, Differentiation Under the Integral Sign

21.1 Gamma Function

$$\int_0^\infty e^{-x} x^{n-1}\, dx \qquad (n > 0)$$

is called gamma function of n. It is also written as $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$.

Example 1. *Prove that* $\Gamma 1 = 1$

Solution. $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$

Put $n = 1$, $\Gamma 1 = \int_0^\infty e^{-x}\, dx = \left[\dfrac{e^{-x}}{-1}\right]_0^\infty = 1$ **Proved**

Example 2. *Prove that*

(*i*) $\overline{\Gamma n+1} = n\,\overline{\Gamma n}$ (*ii*) $\overline{\Gamma n+1} = |\underline{n}$ (*Reduction formula*)

Solution.

(*i*) $\overline{\Gamma n} = \int_0^\infty x^{n-1} e^{-x}\, dx$ ...(1)

Integrating by parts, we have

$$= \left[x^{n-1}\frac{e^{-x}}{-1}\right]_0^\infty - (n-1)\int_0^\infty x^{n-2}\frac{e^{-x}}{-1}\, dx$$

$$= \left[\lim_{x\to 0}\frac{x^{n-1}}{e^x} = \lim_{x\to 0} 1 + \frac{x}{|\underline{1}} + \frac{x^2}{|\underline{2}} + \dots + \frac{x^n}{|\underline{n}} + \dots + x^{n-1}\right] = 0$$

$$= (n-1)\int_0^\infty x^{n-2} e^{-x}\, dx$$

$\therefore$ $\Gamma n = (n-1)\,\overline{\Gamma n-1}$ ...(2)

$\overline{\Gamma n+1} = n\Gamma n$ Replacing n by $(n+1)$ **Proved**

(*ii*) Replace n by $n-1$ in (2), we get

$$\overline{\Gamma n-1} = (n-2)\,\overline{\Gamma n-2}$$

Putting the value $\Gamma n-1$ in (2), we get

$$\Gamma n = (n-1)(n-2)\overline{\Gamma n-2}$$

Similarly $\Gamma n = (n-1)(n-2)\dots 3.2.1\,\Gamma 1$ ...(3)

Putting the value of $\Gamma 1$ in (3), we have

$$\Gamma n = (n-1)(n-2)\dots 3.2.1.1$$

$$\Gamma n = |\underline{n-1}$$

Replacing n by $n+1$, we have

$$\overline{\Gamma n+1} = |\underline{n}$$ **Proved**

Example 3. *Evaluate* $\int_0^\infty \sqrt[4]{x}\, e^{-\sqrt{x}}\, dx$

Solution. Let $I = \int_0^\infty x^{1/4} e^{-\sqrt{x}}\, dx$ ...(1)

Putting $\sqrt{x} = t$ or $x = t^2$ or $dx = 2t\, dt$ in (1), we get

$$I = \int_0^\infty t^{1/2} e^{-t}\, 2t\, dt = 2\int_0^\infty t^{3/2} e^{-t}\, dt$$

$$= 2\Gamma\frac{5}{2} \quad \text{By definition}$$

$$= 2\cdot\frac{3}{2}\Gamma\frac{3}{2} = 2\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2} = \frac{3}{2}\sqrt{\pi}$$ **Ans.**

Example 4. *Evaluate* $\int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx$.

Solution. Let $I = \int_0^\infty \sqrt{x}\, e^{-\sqrt[3]{x}}\, dx$ ...(1)

Putting $\sqrt[3]{x} = t$ or $x = t^3$ or $dx = 3t^2\, dt$ in (1) we get

$$I = \int_0^\infty t^{3/2} e^{-t}\, 3t^2\, dt = 3\int_0^\infty t^{7/2} e^{-t}\, dt = 3\Gamma\frac{9}{2} = 3\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2} = \frac{315}{16}\sqrt{\pi}$$ **Ans.**

Example 5. *Evaluate* $\int_0^\infty x^{n-1} e^{-h^2x^2}\, dx$.

Solution. Let $I = \int_0^\infty x^{n-1} e^{-h^2x^2}\, dx$ ...(1)

Putting $t = h^2x^2$ or $x = \dfrac{\sqrt{t}}{h}$ or $dx = \dfrac{dt}{2h\sqrt{t}}$,

(1) becomes $I = \int_0^\infty \left(\dfrac{\sqrt{t}}{h}\right)^{n-1} e^{-t}\dfrac{dt}{2h\sqrt{t}}$

$$= \frac{1}{2h^n}\int_0^\infty t^{\frac{n-1}{2}} e^{-t}\frac{dt}{\sqrt{t}} = \frac{1}{2h^n}\int_0^\infty t^{\frac{n-2}{2}} e^{-t}\, dt$$

$$= \frac{1}{2h^n}\Gamma\frac{n}{2}$$ **Ans.**

Example 6. *Evaluate* $\int_0^\infty \frac{x^a}{a^x}dx.$ $(a > 1)$

Solution: $I = \int_0^\infty \frac{x^a}{a^x}dx$ ...(1)

Putting $a^x = e^t$ or $x\log a = t$, $x = \frac{t}{\log a}$, $dx = \frac{dt}{\log a}$ in (1), we have

$$I = \int_0^\infty \left(\frac{t}{\log a}\right)^a e^{-t}\frac{dt}{\log a} = \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t}t^a\,dt$$

$$= \frac{1}{(\log a)^{a+1}}\Gamma(a+1)$$ **Ans.**

Example 7. *Evaluate* $\int_0^1 x^{n-1}\cdot\left[\log_e\cdot\left(\frac{1}{x}\right)\right]^{m-1}\cdot dx$

Solution: Put $\log_e\frac{1}{x} = t$ or $x = e^{-t}$ $\quad\therefore dx = -e^{-t}dt$

$$\int_0^1 x^{n-1}\left[\log_e\left(\frac{1}{x}\right)\right]^{m-1}dx = \int_\infty^0 (e^{-t})^{n-1}[t]^{m-1}(-e^{-t}dt) = \int_0^\infty e^{-nt}t^{m-1}dt$$

Put $nt = u$ or $t = \frac{u}{n}$ $\quad\therefore\ dt = \frac{du}{n}$

$$= \int_0^\infty e^{-u}\left(\frac{u}{n}\right)^{m-1}\frac{du}{n} = \frac{1}{n^m}\int_0^\infty e^{-u}u^{m-1}du = \frac{1}{n^m}\Gamma m$$ **Ans.**

21.2 Transformation of Gama Function

Prove that (1) $\int_0^\infty e^{-ky}y^{n-1}dy = \frac{\Gamma n}{k^n}$ (2) $\Gamma\frac{1}{2} = \sqrt{\pi}$ (3) $\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy = \Gamma n$

Solution: We know that $\Gamma n = \int_0^\infty x^{n-1}e^{-x}dx$ ...(1)

(i) Replace x by $k\,y$, so that $dx = kdy$; then

(1) becomes $\Gamma n = \int_0^\infty (ky)^{n-1}e^{-ky}k\,dy.$

$$\Gamma n = k^n\int_0^\infty e^{-ky}y^{n-1}dy$$

$$\therefore \quad \int_0^\infty e^{-ky}y^{n-1}dy = \frac{\Gamma n}{k^n}$$ ...(2) **Proved**

(*ii*) Replace x^n by y. $nx^{n-1}dx = dy$ in (1), then

$$\Gamma n = \int_0^\infty y^{\frac{n-1}{n}}e^{-y^{1/n}}\frac{dy}{nx^{n-1}}$$

$$= \int_0^\infty y^{\frac{n-1}{n}}e^{-y^{1/n}}\frac{dy}{ny^{\frac{n-1}{n}}} = \frac{1}{n}\int_0^\infty e^{-y^{1/n}}dy$$

When $n = \frac{1}{2}$, $\quad \Gamma\frac{1}{2} = \frac{1}{\frac{1}{2}}\int_0^\infty e^{-y^2}dy = 2\left[\frac{1}{2}\sqrt{\pi}\right]$

$$\Gamma\frac{1}{2} = \sqrt{\pi}$$

(*iii*) Substitute e^{-x} by y, $-e^{-x}dx = dy$

$-x = \log y$, $x = \log\frac{1}{y}$, Then (1) becomes

$$\Gamma n = -\int_1^0\left(\log\frac{1}{y}\right)^{n-1} y\cdot\frac{dy}{e^{-x}}$$

$$= \int_0^1\left(\log\frac{1}{y}\right)^{n-1} y\cdot\frac{dy}{y} = \int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy.$$

Exercise 21.1

Evaluate :

1. (*i*) $\Gamma -\frac{1}{2}$ (*ii*) $\Gamma\frac{-3}{2}$ (*iii*) $\Gamma\frac{-15}{2}$ (*iv*) $\Gamma\frac{7}{2}$ (*v*) $\Gamma 0$

Ans. (*i*) $-2\sqrt{\pi}$ (*ii*) $\frac{4}{3}\sqrt{\pi}$ (*iii*) $\frac{2^8\sqrt{\pi}}{15\times 13\times 11\times 9\times 7\times 5\times 3}$ (*iv*) $\frac{15\sqrt{\pi}}{8}$ (*v*) ∞

2. $\int_0^\infty \sqrt{x}\,e^{-x}dx$ **Ans.** $\sqrt{\frac{3}{2}}$ 3. $\int_0^\infty x^4e^{-x^2}dx$ **Ans.** $\frac{3\sqrt{\pi}}{8}$.

4. $\int_0^\infty e^{-h^2x^2}dx$ **Ans.** $\frac{\sqrt{\pi}}{2h}$

5. $\int_0^\infty\int_0^\infty e^{-(ax^2+by^2)}x^{2m-1}y^{2n-1}dx\,dy,\ a, b, m, n > 0$ **Ans.** $\frac{\Gamma m\Gamma n}{4a^mb^n}$

6. $\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy,\ n > 0$ **Ans.** Γn 7. $\int_0^1\frac{dx}{\sqrt{-\log x}}$ **Ans.** $\sqrt{\pi}$

8. $\int_0^1 (x\log x)^3dx$ **Ans.** $-\frac{3}{128}$ 9. $\int_0^1\frac{dx}{\sqrt{x\log\frac{1}{x}}}$ **Ans.** $\sqrt{2\pi}$

10. Prove that $1.3.5.....(2n-1) = \frac{2^n\Gamma\left(n+\frac{1}{2}\right)}{\sqrt{\pi}}$

11. $\int_0^\infty e^{-y^{1/m}}dy = m\Gamma m.$

21.3 Beta Function

$$\int_0^1 x^{l-1}(1-x)^{m-1}dx$$ $(l > 0, m >$

s called the Beta function of l, m. It is also written as

$$\beta(l,m)=\int_0^1 x^{l-1}(1-x)^{m-1}\,dx.$$

1.4 Evaluation of Beta Function

$$\beta(l,m)=\frac{\Gamma l\,\Gamma m}{\Gamma(l+m)}$$

Solution. We have $\beta(l,m)=\int_0^1 x^{l-1}(1-x)^{m-1}\,dx=\int_0^1 (1-x)^{m-1}x^{l-1}\,dx$

Integrating by parts, we have

$$=\left[(1-x)^{m-1}\frac{x^l}{l}\right]_0^1+(m-1)\int_0^1(1-x)^{m-2}\left(\frac{x^l}{l}\right)dx$$

$$=\frac{(m-1)}{l}\int_0^1(1-x)^{m-2}x^l\,dx$$

Again integrating by parts

$$=\frac{(m-1)(m-2)}{l(l+1)}\int_0^1(1-x)^{m-3}x^{l+1}\,dx$$

$$=\frac{(m-1)(m-2)\ldots 2.1}{l(l+1)\ldots\ldots(l+m-2)}\int_0^1 x^{l+m-2}\,dx$$

$$=\frac{(m-1)(m-2)\ldots 2.1}{l(l+1)\ldots\ldots(l+m-2)}\left[\frac{x^{l+m-1}}{l+m-1}\right]_0^1$$

$$=\frac{(m-1)(m-2)\ldots 2.1}{l(l+1)\ldots(l+m-2)(l+m-1)}$$

$$=\frac{\lfloor m-1}{l(l+1)\ldots(l+m-2)(l+m-1)}\times\frac{(l-1)(l-2)\ldots 1}{(l-1)(l-2\ldots 1}$$

$$=\frac{\lfloor m-1\,\lfloor l-1}{1.2\ldots(l-2)(l-1)\,.\,l(l+1)\ldots(l+m-2)(l+m-1)}$$

$$=\frac{\lfloor l-1\ \lfloor m-1}{\lfloor l+m-1}$$

$$=\frac{\Gamma l\,\Gamma m}{\Gamma(l+m)}$$

And if only l is positive integer and not m then

$$\beta(l,m)=\frac{\lfloor l-1}{m(m+1)\ldots(m+l-1)}$$ **Ans.**

1.5 A property of Beta Function

$$\beta(l,m)=\beta(m,l)$$

Solution. We have

$$\beta(l,m)=\int_0^1 x^{l-1}(1-x)^{m-1}\,dx\qquad\left[\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx\right]$$

$$=\int_0^1(1-x)^{l-1}[1-(1-x)]^{m-1}\,dx$$

$$=\int_0^1(1-x)^{l-1}x^{m-1}\,dx$$

$$=\int_0^1 x^{m-1}(1-x)^{l-1}\,dx=\beta(m,l)$$ l and m are interchanged. **Proved**

Example 8. *Evaluate* $\int_0^1 x^4(1-\sqrt{x})^5\,dx$

Solution. Let $\sqrt{x}=t$ or $x=t^2$ or $dx=2t\,dt$

$$\int_0^1 x^4(1-\sqrt{x})^5\,dx=\int_0^1(t^2)^4(1-t)^5(2t\,dt)$$

$$=2\int_0^1 t^9(1-t)^5\,dt=2\beta(10,6)=2\frac{\Gamma 10\,\Gamma 6}{\Gamma 16}=2\frac{\lfloor 9\,\lfloor 5}{\lfloor 15}$$

$$=2.\frac{\lfloor 5}{10\times 11\times 12\times 13\times 14\times 15}=\frac{2\times1\times2\times3\times4\times5}{10\times 11\times 12\times 13\times 14\times 15}$$

$$=\frac{1}{11\times 13\times 7\times 15}=\frac{1}{15015}$$ **Ans.**

Example 9. *Evaluate* $\int_0^1(1-x^3)^{-\frac{1}{2}}\,dx$

Solution. Let $x^3=y$ or $x=y^{1/3}$ or $dx=\frac{1}{3}y^{-\frac{2}{3}}dy$

$$\int_0^1(1-x^3)^{-\frac{1}{2}}\,dx=\int_0^1(1-y)^{-\frac{1}{2}}\left(\frac{1}{3}y^{-\frac{2}{3}}dy\right)$$

$$=\frac{1}{3}\int_0^1 y^{-\frac{2}{3}}(1-y)^{-\frac{1}{2}}dy=\frac{1}{3}\beta\left(\frac{1}{3},\frac{1}{2}\right)=\frac{1}{3}\frac{\Gamma\frac{1}{3}\,\Gamma\frac{1}{2}}{\Gamma\frac{5}{6}}$$ **Ans.**

21.6 Transformation of Beta Function

We know that

$$\beta(l,m)=\int_0^1 x^{l-1}(1-x)^{m-1}\,dx$$

Putting $x=\frac{1}{1+y}$ so that $dx=-\frac{1}{(1+y)^2}dy$ and $1-x=\frac{y}{1+y}$.

$$\beta(l,m)=\int_\infty^0\left(\frac{1}{1+y}\right)^{l-1}\left(\frac{y}{1+y}\right)^{m-1}\left[-\frac{1}{(1+y)^2}dy\right]$$

$$=\int_0^\infty\frac{y^{m-1}}{(1+y)^{l+m}}dy$$

Since l, m can be interchanged in $\beta(l,m)$,

$$\beta(l,m)=\int_0^\infty\frac{y^{l-1}}{(1+y)^{m+l}}dy\quad\text{or}\quad\beta(l,m)=\int_0^\infty\frac{x^{l-1}}{(1+x)^{m+l}}dx$$

Example 10. *Evaluate* $\int_0^1 \frac{x^{m-l}+x^{n-l}}{(1+x)^{m+n}}\,dx$ (Hamirpur, 1995)

Solution. We know that

$$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \quad \text{...(1)}$$

Consider $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$ $\left(\text{Put } x = \frac{1}{t}\right)$

$$= \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}}\left(-\frac{1}{t^2}\,dt\right) = \int_0^1 \frac{\left(\frac{1}{t}\right)^{m-1}\frac{1}{t^2}}{\left(\frac{1}{t}\right)^{m+n}(t+1)^{m+n}}\,dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

Putting the value of $\int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$ in (1) we get

$$\beta(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+m}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx \qquad \textbf{Ans.}$$

21.7 Relation between Beta and Gamma Functions

We know that

$$\Gamma l = \int_0^\infty e^{-x} x^{l-1}\,dx, \quad \frac{\Gamma l}{z^l} = \int_0^\infty e^{-zx} x^{l-1}\,dx$$

$$\Gamma l = \int_0^\infty z^l e^{-zx} x^{l-1}\,dx$$

Multiplying both sides by $e^{-z} z^{m-1}$, we have

$$\Gamma l \cdot e^{-z} \cdot z^{m-1} = \int_0^\infty e^{-z} \cdot z^{m-1} \cdot z^l \cdot e^{-zx} x^{l-1}\,dx$$

$$\Gamma l \cdot e^{-z} \cdot z^{m-1} = \int_0^\infty e^{-(1+x)z} z^{l+m-1} x^{l-1}\,dx$$

Integrating both sides w.r.t. 'x' we get

$$\int_0^\infty \Gamma l\, e^{-z} z^{m-1}\,dz = \int_0^\infty \int_0^\infty e^{-(1+x)z} z^{l+m-1} x^{l-1}\,dx\,dz$$

$$\Gamma l\, \Gamma m = \int_0^\infty x^{l-1}\,dx \int_0^\infty e^{-(1+x)z} z^{l+m-1}\,dz$$

$$= \int_0^\infty x^{l-1}\,dx \cdot \frac{\Gamma(l+m)}{(1+x)^{l+m}}$$

$$\Gamma l\, \Gamma m = \Gamma(l+m) \int_0^\infty \frac{x^{l-1}}{(1+x)^{l+m}}\,dx = \Gamma(l+m) \cdot \beta(l,m)$$

$$\therefore \quad \beta(l,m) = \frac{\Gamma l\, \Gamma m}{\Gamma(l+m)}$$

This is the required relation.

Example 11. *Show that*

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta\,d\theta = \frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{P+q+2}{2}\right)}$$

Solution. We know that

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad \text{...(1)}$$

Putting $x = \sin^2\theta, \; dx = 2\sin\theta\cos\theta\,d\theta$

and $1-x = 1-\sin^2\theta = \cos^2\theta$

Then (1) becomes

$$\beta(m,n) = \int_0^{\frac{\pi}{2}} \sin^{2m-2}\theta \cos^{2n-2}\theta\, 2\sin\theta\cos\theta\,d\theta$$

or $$\frac{\Gamma m\, \Gamma n}{\Gamma(m+n)} = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$$

Putting $2m-1 = p$, *i.e.* $m = \frac{p+1}{2}$

and $2n-1 = q$, *i.e.* $n = \frac{q+1}{2}$

$$\frac{\Gamma\frac{p+1}{2}\,\Gamma\frac{q+1}{2}}{\Gamma\frac{P+q+2}{2}} = 2\int_0^{\frac{\pi}{2}} \sin^p\theta \cdot \cos^q\theta\,d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \cdot \cos^q\theta\,d\theta = \frac{\Gamma\frac{P+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{P+q+2}{2}} \qquad \textbf{Proved}$$

Example 12. *Find the value of* $\Gamma\frac{1}{2}$.

Solution. We know that

$$\int_0^{\frac{\pi}{2}} \sin^p\theta \cos^q\theta\,d\theta = \frac{\Gamma\frac{P+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{P+q+2}{2}}$$

Putting $P = q = 0$ $$\int_0^{\frac{\pi}{2}} d\theta = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{2\,\Gamma 1}$$

or $$[\theta]_0^{\pi/2} = \frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2 \quad \text{or} \quad \frac{\pi}{2} = \frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2$$

or $$\left(\Gamma\frac{1}{2}\right)^2 = \pi \quad \text{or} \quad \Gamma\frac{1}{2} = \sqrt{\pi}$$ **Ans.**

Example 13. *Show that*

$$\int_0^{\frac{\pi}{2}} \sqrt{(\cot\theta)}\, d\theta = \frac{1}{2}\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$$

Solution. We know that

$$\int_0^{\frac{\pi}{2}} \sin^P x \cos^q x\, dx = \frac{\Gamma\frac{P+1}{2}\,\Gamma\frac{q+1}{2}}{2\,\Gamma\frac{P+q+2}{2}} \qquad \ldots(1)$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\cos^{1/2}\theta}{\sin^{1/2}\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^{-1/2}\theta \cos^{1/2}\theta\, d\theta$$

On applying formula (1), we have

$$= \frac{\Gamma\frac{-1/2+1}{2}\,\Gamma\frac{1/2+1}{2}}{2\,\Gamma(-1/2+1/2)} = \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{2} = \frac{1}{2}\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$$ **Proved**

Example 14. *Evaluate* $\int_{-1}^{+1} (1+x)^{P-1}(1-x)^{q-1}\, dx.$

Solution. Put $x = \cos 2\theta$, then $dx = -2\sin 2\theta\, d\theta$

$$\int_{-1}^{+1} (1+x)^{P-1}(1-x)^{q-1}\, dx = \int_{\frac{\pi}{2}}^{0} (1+\cos 2\theta)^{P-1}(1-\cos 2\theta)^{q-1}(-2\sin 2\theta\, d\theta)$$

$$= \int_{\frac{\pi}{2}}^{0} (1+2\cos^2\theta - 1)^{P-1}(1-1+2\sin^2\theta)^{q-1}(-4\sin\theta\cos\theta\, d\theta)$$

$$= 4\int_0^{\frac{\pi}{2}} 2^{P-1}\cos^{2P-2}\theta \,.\, 2^{q-1}\sin^{2q-2}\theta \,.\, \sin\theta\cos\theta\, d\theta$$

$$= 2^{P+q}\int_0^{\pi} \sin^{2q-1}\theta\cos^{2P-1}\theta\, d\theta$$

$$= 2^{P+q}\frac{\Gamma\frac{2q}{2}\,\Gamma\frac{2P}{2}}{2\,\Gamma\frac{2P+2q}{2}} = 2^{P+q-1}\frac{\Gamma P\,\Gamma q}{\Gamma P+q}$$ **Ans.**

Example 15. *Show that* $\Gamma n\,\Gamma 1-n = \frac{\pi}{\sin n\pi}$ $\quad (0 < n < 1)$

Solution. We know that

$$\beta(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$\frac{\Gamma m\,\Gamma n}{\Gamma m+n} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

Putting $m + n = 1$ or $m = 1 - n$

$$\frac{\Gamma 1-n\,\Gamma n}{\Gamma 1} = \int_0^\infty \frac{x^{n-1}}{(1+x)^1}\, dx$$

$$\Gamma 1-n\,\Gamma n = \int_0^\infty \frac{x^{n-1}}{1+x}\, dx \qquad \left[\int_0^\infty \frac{x^{n-1}}{1+x}\, dx = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{\pi}{\sin n\pi}$$ **Proved**

Example 16. *Evaluate* $\int_0^1 \frac{dx}{(1-x^n)^{1/n}}.$

Solution. Let $x^n = \sin^2\theta$ or $x = \sin^{2/n}\theta$

So that $$dx = \frac{2}{n}\sin^{2/n-1}\theta\cos\theta\, d\theta$$

$$\int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \int_0^{\frac{\pi}{2}} \frac{\frac{2}{n}\sin^{2/n-1}\theta\cos\theta\, d\theta}{(1-\sin^2\theta)^{1/n}} = \frac{2}{n}\int_0^{\frac{\pi}{2}} \frac{\sin^{2/n-1}\theta\cos\theta\, d\theta}{(\cos^2\theta)^{1/n}}$$

$$= \frac{2}{n}\int_0^{\frac{\pi}{2}} \sin^{2/n-1}\theta\cos^{1-2/n}\theta\, d\theta$$

$$= \frac{2}{n}\,\frac{\Gamma\frac{\frac{2}{n}-1+1}{2}\,\Gamma\frac{1-\frac{2}{n}+1}{2}}{2\,\Gamma\frac{\frac{2}{n}-1+1+2-\frac{2}{n}}{2}}$$

$$= \frac{2}{n}\,\frac{\Gamma\frac{1}{n}\,\Gamma\frac{n-1}{n}}{\Gamma 1} \qquad \left(\Gamma\frac{1}{n}\,\Gamma 1-\frac{1}{n} = \frac{\pi}{\sin\frac{\pi}{n}}\right)$$

$$= \frac{2\pi}{n \sin \frac{\pi}{n}}$$ **Ans.**

Example 17. Show that $\int_0^{\frac{\pi}{2}} \tan^P \theta \, d\theta = \frac{\pi}{2} \sec \frac{P\pi}{2}$ *and indicate the restriction on the values of P.*

Solution. $\int_0^{\frac{\pi}{2}} \tan^P \theta \, d\theta = \int_0^{\frac{\pi}{2}} \sin^P \theta \cos^{-P} \theta \, d\theta$

$$= \frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{-P+1}{2}\right)}{2\,\Gamma\left(\frac{P+1-P+1}{2}\right)} \qquad \left[\begin{array}{r} 1-P>0 \\ 1>P \end{array}\right]$$

$$= \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{-p+1}{2}\right)}{2\,\Gamma 1} \qquad \left[\begin{array}{r} 1+P>0 \\ P>1 \end{array}\right]$$

$$= \frac{1}{2}\Gamma\left(\frac{1+p}{2}\right)\Gamma\left(\frac{-p+1}{2}\right) \qquad \therefore 1>P>-1$$

$$= \frac{1}{2}\,\frac{\pi}{\sin \frac{p+1}{2}\pi} = \frac{1}{2}\,\frac{\pi}{\cos \frac{p\pi}{2}} = \frac{\pi}{2} \sec \frac{p\pi}{2}$$ **Proved**

Example 18. *Prove Duplication Formula*

$$\Gamma m\, \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma 2m.$$

Hence show that $\beta(m, m) = 2^{1-2m}\beta\left(m, \frac{1}{2}\right)$ *(U.P., II Semeter, Summer 2001)*

Solution. We know that

$$\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)} = \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta \, d\theta$$

Putting $q = p$ we get

$$\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{2\,\Gamma(p+1)} = \int_0^{\frac{\pi}{2}} \sin^P \theta \cos^P \theta \, d\theta = \int_0^{\frac{\pi}{2}} (\sin\theta\cos\theta)^P \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2^P}(2\sin\theta\cos\theta)^P \, d\theta = \frac{1}{2^P}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^P \, d\theta$$

Putting $2\theta = t$, we have

$$= \frac{1}{2^P}\int_0^{\pi} \sin^P t \, \frac{dt}{2}$$

$$= \frac{1}{2^P}\cdot\frac{1}{2}\cdot 2\int_0^{\frac{\pi}{2}} \sin^P t \, dt = \frac{1}{2^P}\int_0^{\frac{\pi}{2}} \sin^P t \cos^0 t \, dt$$

$$= \frac{1}{2^P}\,\frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{P+2}{2}\right)}$$

or $$\frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{P+1}{2}\right)}{2\,\Gamma(P+1)} = \frac{1}{2^P}\,\frac{\Gamma\left(\frac{P+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(\frac{P+2}{2}\right)}$$

$\therefore$ or $$\frac{\Gamma\left(\frac{P+1}{2}\right)}{\Gamma(P+1)} = \frac{1}{2^P}\,\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{P+2}{2}\right)}$$

$\therefore$ or $$\frac{\Gamma\left(\frac{P+1}{2}\right)}{\Gamma(P+1)} = \frac{1}{2^P}\,\frac{\sqrt{\pi}}{\Gamma\left(\frac{P+2}{2}\right)}$$

Take $\frac{P+1}{2} = m$ or $P = 2m-1$

or $$\frac{\Gamma m}{\Gamma 2m} = \frac{1}{2^{2m-1}}\,\frac{\sqrt{\pi}}{\Gamma\left(\frac{2m+1}{2}\right)} \qquad \ldots(1)$$

$$\Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma 2m$$ **Proved**

Multiplying both sides of (1) by Γm, we have

$$\frac{\Gamma m\,\Gamma m}{\Gamma 2m} = 2^{1-2m}\,\frac{\Gamma\left(\frac{1}{2}\right)\Gamma m}{\Gamma\left(m+\frac{1}{2}\right)}$$ **Proved**

$$\beta(m, m) = 2^{1-2m}\beta\left(m, \frac{1}{2}\right)$$

Example 19. *Evaluate* $\iint_A \frac{dx\,dy}{\sqrt{xy}}$, *using the substitutions*

$$x = \frac{u}{1+v^2}, \quad y = \frac{uv}{1+v^2}$$

where A is bounded by $x^2+y^2-x=0,\ y=0,\ y>0.$

Solution. Here $\sqrt{xy} = \sqrt{\left(\frac{u}{1+v^2}\right)\left(\frac{uv}{(1+v^2)}\right)} = \frac{u\sqrt{v}}{1+v^2}$

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv = \begin{vmatrix} \frac{1}{1+v^2} & -\frac{2uv}{(1+v^2)^2} \\ \frac{v}{1+v^2} & \frac{u(1-v^2)}{(1+v^2)^2} \end{vmatrix} du\,dv$$

$$= \left[\frac{u(1-v^2)}{(1+v^2)^3} + \frac{2uv^2}{(1+v^2)^3}\right] du\,dv = \left[\frac{u-uv^2+2uv^2}{(1+v^2)^3}\right] du\,dv$$

$$= \frac{u(1+v^2)}{(1+v^2)^3}\,du\,dv = \frac{u}{(1+v^2)^2}\,du\,dv$$

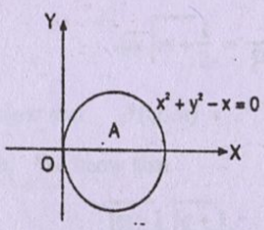
Also the circle $x^2+y^2-x=0$ is transformed into

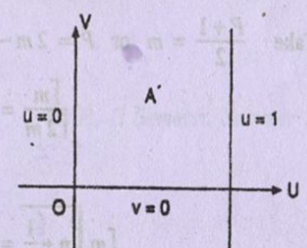
$$\frac{u^2}{(1+v^2)^2} + \frac{u^2v^2}{(1+v^2)^2} - \frac{u}{1+v^2} = 0 \quad \text{or} \quad \frac{u^2(1+v^2)}{(1+v^2)^2} - \frac{u}{1+v^2} = 0$$

$$\frac{u^2}{1+v^2} - \frac{u}{1+v^2} = 0 \quad \text{or} \quad u^2-u=0 \quad \text{or} \quad u(u-1)=0 \Rightarrow u=0,\ u=1$$

Further $\qquad y=0 \Rightarrow \frac{uv}{1+v^2} = 0 \Rightarrow u=0,\ v=0$

and $y>0 \Rightarrow uv>0$ either both u and v are positive or both negative.





The area A, *i.e.*, $x^2+y^2-x=0$ is transformed into A' bounded by $u=0,\ v=0$ and $u=1$ *and* $v=\infty$.

$$\iint \frac{dx\,dy}{\sqrt{x}} = \int_0^1\int_0^\infty \frac{\frac{u}{(1+v^2)^2}\,du\,dv}{\frac{u\sqrt{v}}{1+v^2}} = \int_1^0\int_0^\infty \frac{1}{\sqrt{v}\,(1+v^2)}\,dv\,du$$

On putting $v=\tan\theta$, $dv = \sec^2\theta\,d\theta$

$$= \int_0^1\int_0^{\frac{\pi}{2}} \frac{\sec^2\theta\,d\theta\,du}{\sqrt{\tan\theta}\,(1+\tan^2\theta)} = \int_0^1 du\int_0^{\frac{\pi}{2}} \sqrt{\frac{\cos\theta}{\sin\theta}}\,d\theta = \int_0^1 du\int_0^{\frac{\pi}{2}} \sin\theta^{-\frac{1}{2}}\cos\theta^{\frac{1}{2}}\,d\theta$$

duplication formula $\Gamma m\ \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma 2m$

$$= \int_0^1 du\,\frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{\Gamma 2}\,\frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{\Gamma 2} = \frac{1}{2}\int_0^1 du\,\Gamma\frac{1}{4}\,\Gamma\frac{3}{4} = \frac{1}{2}\int_0^1 du\left[\frac{\sqrt{\pi}}{2^{\frac{-1}{2}}}\,\Gamma\frac{1}{2}\right]$$

$$= \frac{1}{2}\int_0^1 du\,\sqrt{2}\,\sqrt{\pi}\cdot\sqrt{\pi} = \frac{\pi}{\sqrt{2}}\left[u\right]_0^1 = \frac{\pi}{\sqrt{2}}$$ **Ans.**

Example 20. *Prove that*

$$\iint_D x^{l-1}y^{m-1}\,dx\,dy = \frac{\Gamma l\,\Gamma m}{\Gamma(l+m+1)}\,h^{l+m}$$

where D is the domain $x\ge 0,\ y\ge 0$ *and* $x+y\le h$.

Solution. Putting $x = Xh$ and $y = Yh$, $dx\,dy = h^2\,dX\,dY$

$$\iint_D x^{l-1}y^{m-1}\,dx\,dy = \iint_{D'} (Xh)^{l-1}(Yh)^{m-1}h^2\,dX\,dY$$

where D' is the domain

$X\ge 0,\ Y\ge 0,\ X+Y\le 1$

$$= h^{l+m}\int_0^1\int_0^{1-X} X^{l-1}Y^{m-1}\,dX\,dY = h^{l+m}\int_0^1 X^{l-1}\,dX\int_0^{1-X} Y^{m-1}\,dY$$

$$= h^{l+m}\int_0^1 X^{l-1}\,dX\left[\frac{Y^m}{m}\right]_0^{1-X} = \frac{h^{l+m}}{m}\int_0^1 X^{l-1}(1-X)^m\,dX$$

$$= \frac{h^{l+m}}{m}\,\beta(l,\ m+1) = \frac{h^{l+m}}{m}\,\frac{\Gamma l\,\Gamma(m+1)}{\Gamma(l+m+1)}$$

$$= \frac{h^{l+m}}{m}\,\frac{m\,\Gamma l\,\Gamma m}{\Gamma(l+m+1)} = h^{l+m}\,\frac{\Gamma l\,\Gamma m}{\Gamma(l+m+1)}.$$ **Proved.**

Example 21. *Establish Dirichlet's integral*

$$\iiint_V x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$$

where V is the region $x\ge 0,\ y\ge 0,\ z\ge 0$ *and* $x+y+z\le 1$.

Solution. Putting $y+z\le 1-x = h$. Then $z\le h-y$

$$\iiint_V x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \int_0^1 x^{l-1}\,dx\int_0^{1-x} y^{m-1}\,dy\int_0^{1-x-y} z^{n-1}\,dz$$

$$= \int_0^1 x^{l-1}\,dx\left[\int_0^h\int_0^{h-y} y^{m-1}z^{n-1}\,dy\,dz\right]$$

$$= \int_0^1 x^{l-1}\,dx\left[\frac{\Gamma m\,\Gamma n}{\Gamma(m+n+1)}\,h^{m+n}\right]$$

$$= \frac{\Gamma m\,\Gamma n}{\Gamma(m+n+1)}\int_0^1 x^{l-1}(1-x)^{m+n}\,dx$$

$$= \frac{\Gamma m\,\Gamma n}{\Gamma(m+n+1)}\,\beta(l,\ m+n+1)$$

$$= \frac{\Gamma m\,\Gamma n}{\Gamma(m+n+1)} \cdot \frac{\Gamma l\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)}$$

$$= \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$$ **Proved.**

Note. $\iiint_V x^{l-1}\, y^{m-1}\, z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}\, h^{l+m+n}$

where V is the domain, $x \ge 0,\ y \ge 0,\ z \ge 0$ and $x+y+z \le h$.

Example 22. *Find the mass of an octant of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$, *the density at any point being* $\rho = k\,x\,y\,z$.

Solution. $$\text{Mass} = \iiint \rho\, dv = \iiint (k\,x\,y\,z)\, dx\, dy\, dz$$

$$= k \iiint (x dx)\,(y dy)\,(z dz) \qquad \ldots (1)$$

Putting $\frac{x^2}{a^2} = u,\ \frac{y^2}{b^2} = v,\ \frac{z^2}{c^2} = w$ and $u+v+w = 1$

so that $\frac{2x dx}{a^2} = du,\quad \frac{2\,y\,dy}{b^2} = dv,\quad \frac{2\,z\,dz}{c^2} = dw$

$$\text{Mass} = k \iiint \left(\frac{a^2\,du}{2}\right)\left(\frac{b^2\,dv}{2}\right)\left(\frac{c^2\,dw}{2}\right)$$

$$= \frac{k\,a^2\,b^2\,c^2}{8} \iiint du\,dv\,dw \quad \text{where} \quad u+v+w \le 1$$

$$= \frac{k\,a^2\,b^2\,c^2}{8} \iiint u^{1-1}\, v^{1-1}\, w^{1-1}\, du\,dv\,dw$$

$$= \frac{k\,a^2\,b^2\,c^2}{8}\, \frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma(3+1)} = \frac{k\,a^2\,b^2\,c^2}{8 \times 6}$$

$$= \frac{k\,a^2\,b^2\,c^2}{48}$$ **Ans.**

Example 23. *Show that*

$$\int_0^1 \frac{x^{m-1}\,(1-x)^{n-1}}{(a+x)^{m+n}}\, dx = \frac{\beta(m,n)}{a^n\,(1+a)^m}$$

Solution: Put $$\frac{x}{a+x} = \frac{t}{a+1}$$

$$(a+1)\,x = t\,(a+x) \quad \text{or} \quad x = \frac{at}{a+1-t}$$

$$dx = \frac{(a+1-t)\,a\,dt - at\,(-dt)}{(a+1-t)^2}$$

$$= \frac{(a^2+a-at+at)}{(a+1-t)^2}\, dt = \frac{a\,(a+1)}{(a+1-t)^2}\, dt$$

$$\int_0^1 \frac{x^{m-1}\,(1-x)^{n-1}}{(a+x)^{m+n}}\, dx = \int_0^1 \frac{\left(\frac{at}{a+1-t}\right)^{m-1} \cdot \left(1-\frac{at}{a+1-t}\right)^{n-1}}{\left(a+\frac{at}{a+1-t}\right)^{m+n}}\, \frac{a\,(a+1)}{(a+1-t)^2}\, dt$$

$$= \int_0^1 \frac{(at)^{m-1}\,(a+1-t-at)^{n-1}}{(a^2+a-at+at)^{m+n}}\, a\,(a+1)\, dt$$

$$= \int_0^1 \frac{a^{m-1}\,t^{m-1}\,(a+1)^{n-1}\,(1-t)^{n-1}}{a^{m+n}\,(a+1)^{m+n}}\, a\,(a+1)\, dt$$

$$= \frac{1}{a^n\,(a+1)^m} \int_0^1 t^{m-1}\,(1-t)^{n-1}\, dt$$

$$= \frac{1}{a^n\,(a+1)^m}\, \beta(m,n)$$ **Proved**

Exercise 21.2

Prove that

1. (a) $\int_0^{\frac{\pi}{2}} \sin^2\theta \cos^4\theta\, d\theta = \frac{5\pi}{256}$ (b) $\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta = \frac{5\pi}{32}$

2. (a) $\beta(m+1, n) = \frac{m}{m+n}\, \beta(m,n)$ (b) $\beta(m, n+1) = \frac{n}{m+n}\, \beta(m,n)$

   (c) $\beta(m+1, n) + \beta(m, m+1) = \beta(m,n)$

3. $\int_0^1 \sqrt{x}\, \sqrt[3]{1-x^2}\, dx = \frac{\Gamma\frac{3}{4}\,\Gamma\frac{4}{3}}{2\,\Gamma\frac{7}{12}}$

4. $\int_0^1 (1-x^n)^{-\frac{1}{2}}\, dx = \frac{\Gamma\frac{1}{n}\,\Gamma\frac{1}{2}}{n\,\Gamma\frac{n+2}{2n}}$

5. $\int_0^1 (1-x^{1/n})^m\, dx = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$

6. $\int_1 \frac{dx}{x^{P+1}\,(x-1)^q} = \beta(p+q, 1-q)$ if $-P < q < 1$

7. $\int_0^1 x^m\,(1-x^n)^P\, dx = \frac{1}{n}\, \frac{\Gamma\frac{m+1}{2}\,\Gamma(P+1)}{\Gamma\left(\frac{m+1}{n}+P+1\right)}$

8. $\int_0^b (x-a)^m\,(b-x)^n\, dx = (b-a)^{m+n+1}\,.\,\beta(m+1, n+1)$

9. $\int_3^7 \sqrt[4]{(x-3)\,(7-x)}\, dx = \frac{2\left(\Gamma\frac{1}{4}\right)^2}{3\sqrt{\pi}}$ Put $x = 4t+3$

10. $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}} = \frac{\left(\Gamma\frac{1}{4}\right)^2}{4\sqrt{\pi}}$

11. If $\int_0^{\infty} e^{-x}\,x^{n-1}\, dx =$ In for $n >$ 0 find $\frac{I_{n+1}}{I_n}$ *(A.M.I.E., Summer 2000)* Ans. n