Integral Calculus and Differential Equations

MAT120

Lecture Notes

Preface and Acknowledgements

This lecture note has been prepared for aiding students who are taking the course MAT120 (Integral Calculus & Differential Equations) offered by BRAC University. These notes are a compilation of parts taken from two other books (listed below) that has been shortened down and altered so that it is adequate for students taking this course. These notes were created under the strict supervision of eminent mathematician, Dr. Syed Hasibul Hasan Chowdhury. The main goal of this compilation is to help keep things organized for the students and ensure continuity of the course content among every section. Since this is the first version of this note, there may be some errors and typos. Any suggestions for the improvements of this note are highly appreciated. If any mistakes are found please report them to **ahmed.rakin@bracu.ac.bd**.

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Reference Books:

- *Calculus of One Variable* by Keith E. Hirst
- *Schaum's Outline of Differential Equations*, 3^{rd} Edition"

Contents

Chapter 1

Integration

In MAT120, one learns very basic integration theory and elementary concepts of ordinary differential equations along with their versatile applications in engineering and related fields. The first six chapters of this set of lecture notes concern integral calculus while the last two chapters are dedicated to the study of ordinary differential equations. The contents of the Integral Calculus part are taken from the textbook titled "Calculus of one variable" by Keith E. Hirst while the ones for ordinary differential equations are chosen from the textbook titled "Elementary differential equations and boundary value problems" by William E. Boyce and Richard C. DiPrima.

1.1 Area under the graph of a function

The goal in this section will be to interpret integration as a means to compute the area under the graph of a given function. The area under the graph of a function signifies the area of a plane region bounded by the graph of a non-negative function $y = f(x)$, the $x-$axis, and the lines $x = a$ and $x = b$. We approximate this area using rectangles from below and from above. The idea is illustrated in Figure 1.1. For functions with continuous graphs (called continuous functions), it turns out that the approximations from below and from above can be made arbitrarily close to each other. As these approximations get better and better they converge to a definite numerical value which corresponds to the area of the region. This limiting process is a little more involved than the one that one encounters while defining derivative of a function.

We generate the approximations by subdividing the interval $a \leq x \leq b$ by means of an increasing sequence of points along the x-axis given by

$$a = x_1 < x_2 < ... < x_{n-1} < x_n = b.$$

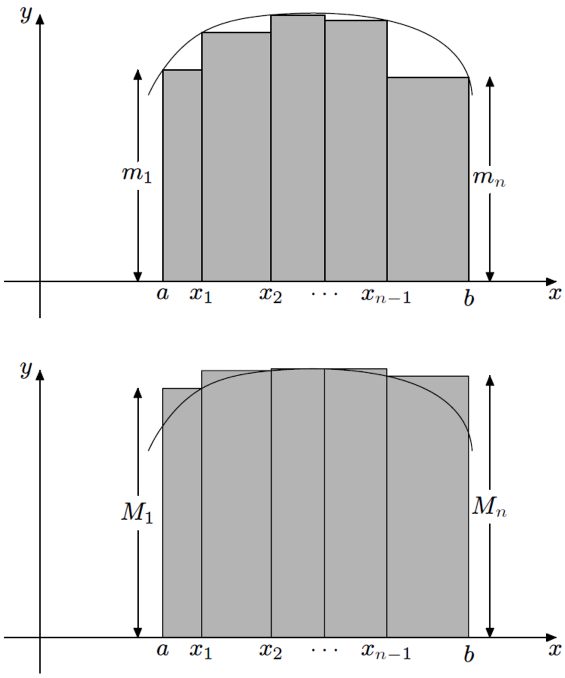

Figure 1.1: *The integral as a sum*

The sum of the areas of the rectangles lying below the graph (as in the top diagram) is called the lower sum corresponding to the subdivision and the sum of the areas of the rectangles enclosing the area underneath the graph (as in the bottom diagram) is called the upper sum corresponding to the subdivision. The lower and upper sum denoted by s and S, respectively, are given by

$$s = \sum_{i=1}^{n} m_i(x_i - x_{i-1}); \quad S = \sum_{i=1}^{n} M_i(x_i - x_{i-1}).$$

Each of the summands, in both the expressions above, represents the area of a rectangle where $(x_i - x_{i-1})$ represents the width and m_i or M_i represents the height of the rectangle in the first and in the second expression, respectively.

The fundamental idea here is that as the lengths of the intervals of the subdivision all tend to zero so that the number of intervals tends to infinity, both the upper and lower sum converge to a common limiting value which is precisely the area under the curve.

The theoretical underpinning of this idea is quite extensive and beyond the scope of this course. In fact, there are no hard and fast rules for the choice of the height of the rectangles. In other words, we can replace M_i or m_i with any number in-between, usually of the form $f(c_i)$, where $x_{i-1} \leq c_i \leq x_i$. This leads to the sum

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}),$$

bounded above and below by the upper and lower sum, respectively, i.e.

$$s \leq \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \leq S. \tag{1.1}$$

The theory confirms that all such sums (s, S or the one given by the expression (1.1)) converge to a common limiting value as the number of intervals in the subdivision increases and their lengths all tend to zero. This common limiting value is called the definite integral of the underlying function f over the interval $[a, b]$, denoted by

$$\int_a^b f(x)dx,$$

and a function for which this common limit exists is said to be integrable over the interval $a \leq x \leq b$. The expression $f(x)$ for the function f which we are integrating is referred to as the integrand. We shall exploit this idea of the integral as the limit of a sum in Chapter 5 for obtaining geometrical quantities such as area and volume.

Fundamental theorem of Calculus relates the concept of integration to that of differentiation. Its statement, under appropriate conditions, is as follows

$$\int_a^b f'(x)dx = f(b) - f(a).$$

1.2 Area function and fundamental theorem of Calculus

The Fundamental Theorem of Calculus is appropriately named because it establishes a connection between the two branches of calculus: differential calculus and integral calculus. Differential calculus arose from the tangent problem, whereas integral calculus arose from a seemingly unrelated problem, the area problem. Newton's mentor at Cambridge, Isaac Barrow (1630–1677), discovered that these two problems are actually closely related. In fact, he realized that differentiation and integration are inverse processes. The Fundamental Theorem of Calculus gives the precise inverse relationship between the derivative and the integral. It was Newton and Leibniz who exploited this relationship and used it to develop calculus into a systematic mathematical method. In particular, they saw that the Fundamental Theorem enabled them to compute areas and integrals very easily without having to compute them as limits of sums.

The first part of the Fundamental Theorem deals with functions defined by an equation of the form

$$g(x) = \int_a^x f(t)\, dt$$

where f is a continuous function on $[a, b]$ and x varies between a and b. Observe that g depends only on x, which appears as the variable upper limit in the integral. If x is a fixed number, then the integral $\int_a^x f(t)dt$ is a definite number. If we then let x vary, the number $\int_a^x f(t)dt$ also varies and defines a function of x denoted by $g(x)$.

If f happens to be a positive function, then $g(x)$ can be interpreted as the area under the graph of f from a to x, where x can vary from a to b. (Think of g as the "area so far" function; see Figure 1.2.)

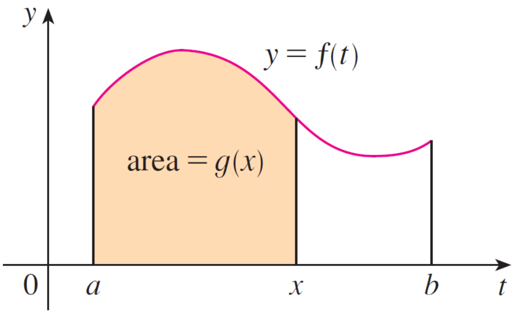

Figure 1.2: *Graph of* $g(x) = \int_a^x f(t)\ dt$

We consider any continuous function f with $f(x) \geq 0$. Then $g(x) = \int_a^x f(t)\ dt$ can be interpreted as the area under the graph of f from a to x, as in Figure 1.2.

In order to compute $g'(x)$ from the definition of derivative we first observe that, for $h > 0$, $g(x+h) - g(x)$ is obtained by subtracting areas, so it is the area under the graph of f from x to $x + h$. For small h you can see from the figure that this area is approximately equal to the area of the rectangle with height and width h:

$$g(x + h) - g(x) \approx hf(x)$$
$$\frac{g(x + h) - g(x)}{h} \approx f(x)$$

so Intuitively, we therefore expect that

$$g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = f(x)$$

The fact that this is true, even when f is not necessarily positive, is the first part of the Fundamental Theorem of Calculus.

Example 1.2.1. *Find the derivative of the function* $g(x) = \int_0^x \sqrt{1 + t^2} dt$.

Since $f(t) = \sqrt{1 + t^2}$ is continuous, Part 1 of the Fundamental Theorem of Calculus gives

$$g'(x) = \sqrt{1 + t^2}$$

Example 1.2.2. *Find* $\frac{d}{dx}\int_1^{x^4} \sec t\ dt$

Here we have to be careful to use the Chain Rule in conjunction with FTC1. Let $u = x^4$. Then

$$\begin{aligned}
\frac{d}{dx}\int_1^{x^4} \sec t\ dt &= \frac{d}{dx}\int_1^{u} \sec t\ dt \\
&= \frac{d}{du}\left(\int_1^{u} \sec t dt\right)\frac{du}{dx} \quad \text{(by the Chain Rule)} \\
&= \sec u \frac{du}{dx} \quad \text{(by FTC1)} \\
&= \sec\left(x^4\right)\cdot 4x^3
\end{aligned}$$

The fundamental theorem of Calculus, Part 2 If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$.

Example 1.2.3. *Evaluate the integral* $\int_{-2}^{1} x^3 dx$

The function $f(x) = x^3$ is continuous on [-2, 1] and we know that an antiderivative is $F(x) = \frac{1}{4}x^4$, so Part 2 of the Fundamental Theorem gives

$$\int_{-2}^{1} x^3 dx = F(1) - F(-2) = \frac{1}{4}(1)^4 - \frac{1}{4}(-2)^4 = -\frac{15}{4}$$

Notice that FTC2 says we can use *any* antiderivative F of f. So we may as well use the simplest one, namely $F(x) = \frac{1}{4}x^4$, instead of $\frac{1}{4}x^4 + 7$ or $\frac{1}{4}x^4 + C$

Example 1.2.4. *Find the area under the parabola $y = x^2$ from 0 to 1.*

An antiderivative of $f(x) = x^2$ is $F(x) = \frac{1}{3}x^3$. The required area A is found using Part 2 of the Fundamental Theorem:

$$A = \int_0^1 x^2 dx = \frac{x^3}{3}\bigg]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

Example 1.2.5. *What is wrong with the following calculation?*

$$\int_{-1}^3 \frac{1}{x^2}dx = \frac{x^{-1}}{-1}\bigg]_{-1}^3 = -\frac{1}{3} - 1 = -\frac{4}{3}$$

To start, we notice that this calculation must be wrong because the answer is negative but $f(x) = 1/x^2 \geq 0$ and Property 6 of integrals says that $\int_a^b f(x)dx \geq 0$ when $f \geq 0$. The Fundamental Theorem of Calculus applies to continuous functions. It can't be applied here because $f(x) = 1/x^2$ is not continuous on $[-1, 3]$. In fact, f has an infinite discontinuity at $x = 0$, so

$$\int_{-1}^3 \frac{1}{x^2}dx \quad \text{does not exist}$$

We end this section by bringing together the two parts of the Fundamental Theorem. Suppose f is continuous on $[a, b]$.

1. If $g(x) = \int_a^x f(t)dt$, then $g'(x) = f(x)$
2. $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f, that is, $F' = f$.

We noted that Part 1 can be rewritten as

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

which says that if f is integrated and then the result is differentiated, we arrive back at the original function f. Since $F'(x) = f(x)$., Part 2 can be rewritten as

$$\int_a^b F'(x)dx = F(b) - F(a)$$

This version says that if we take a function F, first differentiate it, and then integrate the result, we arrive back at the original function F, but in the form $F(b) - F(a)$. Taken together, the two parts of the Fundamental Theorem of Calculus say that differentiation and integration are inverse processes. Each undoes what the other does.

The Fundamental Theorem of Calculus is unquestionably the most important theorem in calculus and, indeed, it ranks as one of the great accomplishments of the human mind. Before it was discovered, from the time of Eudoxus and Archimedes to the time of Galileo and Fermat, problems of finding areas, volumes, and lengths of curves were so difficult that only a genius could meet the challenge. But now, armed with the systematic method that Newton and Leibniz fashioned out of the Fundamental Theorem, we will see in the chapters to come that these challenging problems are accessible to all of us.

1.3 Integrals of some known functions

In the remainder of this chapter, and in Chapters 8, 9 and 10, we shall consider integration as the reverse of differentiation. Given a function $f(x)$, the problem is to find another function $F(x)$ whose derivative is $f(x)$. Such a function $F(x)$ is called an **indefinite integral**, denoted by $\int f(x)dx$. In contrast, the integral $\int_a^b f(x)dx$ discussed above is called a **definite integral**.

We first note that if $\frac{d}{dx}F(x) = f(x)$ then $\frac{d}{dx}(F(x) + C) = f(x)$ for any real number C.

C is known as the constant of integration, and strictly speaking should be included whenever we evaluate an indefinite integral.

Basic integration is normally first encountered in school mathematics, and it is assumed that readers are familiar with a small number of indefinite integrals, as in the following table, where we have omitted the constant of integration, as we shall do throughout this book.

$f(x)$	$\int f(x)dx$
$x^{\alpha}(\alpha \neq -1)$	$\frac{x^{\alpha+1}}{\alpha+1}$
x^{-1}	$\ln\lvert x\rvert$
$cosx$	$sinx$
$sinx$	$-cosx$
sec^2x	$tanx$
e^{kx}	$\frac{e^{kx}}{k}$

Various rules and identities can be used to reduce many integrals to the basic ones above. There are some standard methods for doing this which are applicable to various classes of functions, and we shall consider these in later chapters. The examples in this section illustrate this idea with some elementary integrals. The algebraic rules of integration used in this chapter are as follows

$$\int (Cf_1(x) + Df_2(x))dx = C\int f_1(x)dx + D\int f_2(x)dx,$$

where C and D are constants (the addition rule), and

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

Example 1.3.1. *Evaluate* $\int \cos 2x dx$.

The table above suggests that the answer will involve $\sin 2x$, and we can check by differentiation. Now $\frac{d}{dx}\sin 2x = 2\cos 2x$, and so we can see that we have to compensate for the factor of 2. We can therefore write $\frac{d}{dx}\frac{\sin 2x}{2} = \cos 2x$, and so,

$$\int \cos 2x\, dx = \frac{\sin 2x}{2}$$

Example 1.3.2. *Evaluate* $\int a^x dx (a > 0)$.

We know that $a^x = e^{x \ln a}$. Using this, and realising that $\ln a$ is simply a constant, we can write

$$\int a^x dx = \int e^{x \ln a} dx = \frac{e^{x \ln a}}{\ln a} = \frac{a^x}{\ln a}$$

Example 1.3.3. *Evaluate* $\int_{-3}^{3} \ln(1 + x^2) \sin(x) dx$.

The function looks complicated to integrate as an indefinite integral, so we have to look at it another way, in this case geometrically. We notice that the function is an odd function, and the interval of integration is symmetric about the origin. The answer is therefore zero. This is clear from Figure 1.3, with the interpretation explained above, that areas below the x-axis correspond to a negative answer for the integral. We have implicitly used the rule of integration telling us that,

$$\int_{-3}^{3} \ln(1 + x^2) \sin(x) dx = \int_{-3}^{0} \ln(1 + x^2) \sin(x) dx + \int_{0}^{3} \ln(1 + x^2) \sin(x) dx$$

The value of the first integral on the right-hand side is then minus that of the second integral, so they cancel to zero.

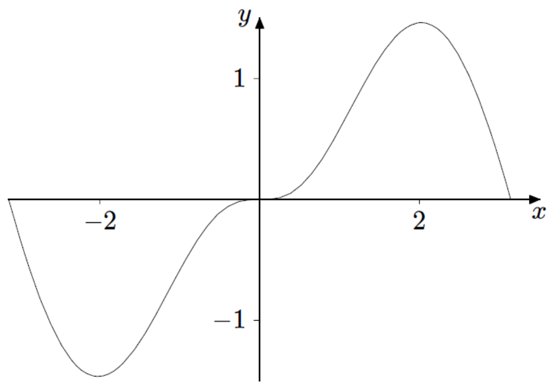

Figure 1.3: *Integral of an odd function*

1.4 The Logarithmic Integral

One of the items in the table of integrals at the beginning of Section 1.2 states that $\int \frac{1}{x} dx = \ln|x|$. Some textbooks say that $\int \frac{1}{x} dx = \ln x + C$

. Neither of these is strictly correct, and in this section we shall discuss this integral further. The problem is caused by the fact that neither $1/x$ nor $\ln|x|$ is defined for $x = 0$. We approach the problem through graphs.

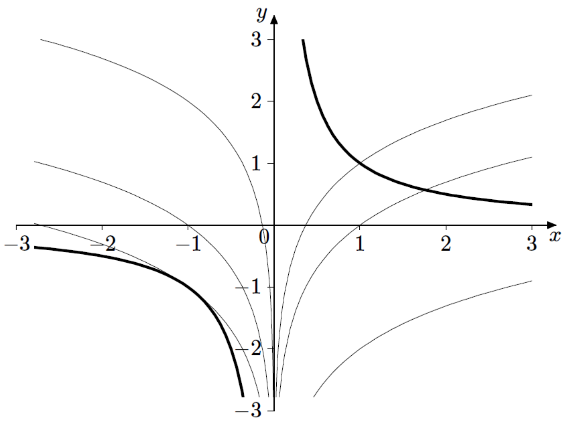

Figure 1.4: *The logarithmic integral*

In Figure 1.4 the thicker graph is that of $y = \frac{1}{x}$, and the other graphs are of $\ln x + C$ for $x > 0$, and $\ln(-x) + D$ for $x < 0$. All these logarithmic graphs have derivative $1/x$. For $x > 0$ this is a standard result. Now $1/x$ is an odd function, and so its gradient at $-x$ will be the negative of the gradient at x. This means that for $x < 0$, $1/x$ is the gradient for the logarithmic function with x replaced by $-x$, i.e., $\ln(-x)$. For $x < 0$ we can also verify this result using the chain rule.

$$\frac{d}{dx}(\ln(-x) + D) = \frac{1}{-x}\frac{d}{dx}(-x) = \frac{1}{-x}.(-1) = \frac{1}{x}$$

Finally we have to consider the constant of integration. We can see that we could choose an arbitrary function of the form $\ln x + C$ for $x > 0$, and an arbitrary function of the form $\ln(-x) + D$ for $x < 0$. There is no reason why C should be the same as D, since the two "halves" do not join together, because of discontinuity at $x = 0$. So the most complete description of the set of functions whose derivative is $1/x$ is,

$$F(x) = \begin{cases} \ln(x) + C, & (x > 0), \\ \ln(-x) + D, & (x < 0). \end{cases}$$

To write $\ln|x| + C$ is a convenient abbreviation, but it conceals the fact that C and D can be different.

1.5 Integrals with Variable Limits

So far when we have evaluated a definite integral of the form $\int_a^b f(x)dx$, the limits of integration a and b have been constants. There is no reason however why they should not involve a variable.

Example 1.5.1. *Evaluate the integral* $\displaystyle\int_{t-2}^{\sin t}(x^2-2x+3)dx$.

$$\int_{t-2}^{\sin t}(x^2-2x+3)dx = \left[\frac{x^3}{3}-x^2+3x\right]_{t-2}^{\sin t}$$
$$= \frac{\sin^3(t)}{3}-\sin^2 t+3\sin t-\frac{(t-2)^3}{3}+(t-2)^2-3(t-2)$$
$$= \frac{\sin^3(t)}{3}-\sin^2 t+3\sin t-\frac{t^3}{3}+3t^2-11t+\frac{38}{3}$$

This example shows that when such an integral is evaluated the answer involves the variable which is present in the limits of integration. Now if we want to find the derivative of this expression we can evaluate the integral, as we have done in the above example, and then differentiate the answer. However we can find the derivative without integrating first, as follows. Suppose that we know the indefinite integral of $f(x$, i.e., that we know a function $F(x)$ satisfying $F'(x)=f(x)$. We then have,

$$G(t)=\int_{a(t)}^{b(t)}f(x)dx=\int_{a(t)}^{b(t)}F'(x)dx=F(b(t))-F(a(t))$$

We can therefore differentiate using chain rule to obtain,

$$G'(t)=F'(b(t))b'(t)-F'(a(t))a'(t)=f(b(t))b'(t)-f(a(t))a'(t)$$

Example 1.5.2. *Find the derivative with respect to t of the function defined by* $G(t)=\displaystyle\int_{t^2-3t}^{t^3+4t^2}\cos(x^2)dx$

In this example we cannot find the indefinite integral, but we can still use the formula for the derivative. This tells us that the derivative is,

$$G'(t)=(3t^2+8t)\cos((t^3+4t^2)^2)-(2t-3)cos((t^2-3t)^2)$$

1.6 Infinite Integrals

So far we have considered definite integrals of the form $\int_a^b f(x)dx$, where a and b are real numbers, and for a non-negative function the integral corresponds to the area between the graph and the $x-axis$. In effect we are considering a function whose domain is limited to lie between a and b. But many functions have as their domain the set of all real numbers, or the set of all positive real numbers, or some other unbounded set. In this section we shall consider how to interpret the idea of the area of the region between such a graph and the $x - axis$. This leads to the idea of an infinite integral. Suppose we have a function $f(x)$ whose domain is the set of all real x satisfying $x \geq a$. We want to give ameaning to $\int_a^\infty f(x)dx$.

Imagine that we are going to paint such a region and that we want to know whether we can do it with a finite amount of paint. What we can do is to start at $x = a$ and paint the region up as far as $x = t$. We can then measure the amount of paint used. This will depend on t, and so can be expressed as a function $F(t)$. Using the ideas of limits from Chapter 2 we can then investigate $\lim\limits_{y\to\infty} F(t)$. If this is finite it would appear that we can paint the complete region using a finite amount of paint. This somewhat far-fetched analogy motivates the definition.

Definition 1.6.1. *If $f(x)$ is a function continuous for $x \geq a$ we define the infinite integral of f by*

$$\int_a^\infty f(x)dx = \lim_{t\to\infty} \int_a^t f(x)dx.$$

If this limit exists and is finite we say that the infinite integral ***converges****. Otherwise the infinite integral* ***diverges****. We can similarly define*

$$\int_{-\infty}^b f(x)dx = \lim_{t\to-\infty} \int_t^b f(x)dx.$$

Example 1.6.1. *Investigate the convergence or otherwise of the infinite integral* $\int_1^\infty x^\alpha dx$.

We will begin with two numerical cases by way of illustration. Firstly, let $\alpha = -2$. We then have

$$\int_1^t \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_1^t = 1 - \frac{1}{t} \to 1 \text{ as } t \to \infty$$

The integral therefore converges and we can write

$$\int_1^\infty \frac{1}{x^2}dx = 1$$

Now consider the case $\alpha = -1$. We now have,

$$\int_1^t \frac{1}{x}dx = [\ln x]_1^t = \ln t \to \infty \text{ as } t \to \infty$$

Therefore the integral in this case diverges. In general suppose that $\alpha \neq -1$. Then,

$$\int_1^t x^\alpha = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_1^t = \frac{x^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha+1}$$

Now if $\alpha + 1 > 0$ the right-hand expression tends to infinity, and so the corresponding infinite integral diverges. If $\alpha + 1 < 0$ then,

$$\lim_{t\to\infty} \frac{x^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha+1} = -\frac{1}{\alpha+1}$$

Therefore the integral converges, and we can write

$$\int_1^\infty x^\alpha = -\frac{1}{\alpha+1}$$

This is verified with the case $\alpha = -2$ we considered above.

Example 1.6.2. *Show that the infinite integral* $\displaystyle\int_0^\infty e^{-x}$ *converges and find its value.*

Evaluating the integral over the finite interval $0 \leq x \leq t$ gives,

$$\int_0^t e^{-x} = [-e^{-x}]_0^t = 1 - e^{-t} \to 1 \text{as} t \to \infty$$

Therefore the integral converges and its value is 1.

Example 1.6.3. *Show that the integral* $\int_1^\infty e^{-x^2}$ *converges.*

In this case, unlike Example 1.6.2, we cannot evaluate the indefinite integral explicitly and so we cannot investigate the required limit directly. What we do is to compare this integral with one which we already know to converge. To return to the area interpretation, if we can show that the region between the graph of e^{-x^2} and the $x - axis$ is smaller than one for which we already know that the area is finite then it too must enclose a finite area. The argument proceeds as follows in this case. For all $x \geq 1$ we know that $0 \leq e^{-x^2} \leq e^{-x}$. We therefore have, using the result of Example 1.6.2,

$$\int_0^t e^{-x^2} dx \leq \int_0^t e^{-x} dx \leq 1$$

The value of the left-hand integral increases as t increases, because the integrand is positive for all x, and so it tends to a finite limit as $t \to \infty$. Therefore the integral converges. Because we cannot evaluate the indefinite integral we do not know the value of the infinite integral. What we have showed is that

$$\int_0^\infty e^{-x^2} dx \leq 1$$

The procedure used in this example generalises, as in the following theorem.

Theorem 1.6.1. *Comparison Test for Infinite Integral*

Suppose that $f(x)$ and $g(x)$ are continuous, and that $0 \leq g(x) \leq f(x)$, for all $x \geq a$. Then if the infinite integral $\int_a^\infty f(x)dx$ converges, so does the infinite integral $\int_a^\infty g(x)dx$, and

$$\int_a^\infty g(x)dx \leq \int_a^\infty f(x)dx$$

Proof. An analytical proof is outside the scope of this book. We shall give an intuitive geometrical explanation, which generalises the argument in Example 1.6.3. □

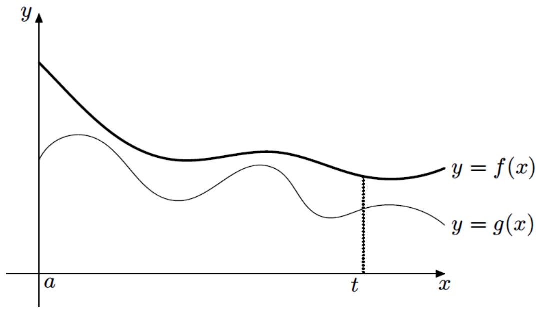

Figure 1.5: *Integral comparison test*

We can see from Figure 1.5 that the area between the graph of $f(x)$ and the $x-axis$ is greater than the area between the graph of $g(x)$ and the $x - axis$. Therefore

$$\int_a^t g(x)dx \leq \int_a^t f(x)dx$$

From the figure we can also see that these areas increase as t increases, and because the functions are both non-negative it follows that the corresponding integrals also increase as t increases. If we denote the value of the convergent integral $\int_a^\infty f(x)dx$ by K we deduce that

$$\int_a^t g(x)dx \leq K$$

for all $t \geq a$. So $\int_a^t g(x)dx$ is an increasing function of t which is bounded above. It therefore has a limit $H \leq K$. (This last result would be proved in a course on Real Analysis, for example in Howie Chapter 3).

So far we have restricted attention to non-negative functions. Definition 1.6.1 does not require this, and in the next example we consider a function taking both signs.

Example 1.6.4. *Show that the integral* $\int_{2\pi}^{\infty} \frac{\sin x}{x^2}dx$ *converges*

From Example 1.6.1 (with $\alpha = -2$) we know that the area between the graph of $1/x^2$ and the $x - axis$ is finite. We also know that

$$-\frac{1}{x^2} \leq \frac{\sin x}{x^2} \leq \frac{1}{x^2}$$

We can therefore see, in Figure 1.6 , that the areas contained by those parts of the graph of $\frac{\sin x}{x^2}$ above the $x - axis$ will be finite in total. The same will be true below the $x - axis$, so that the total area between the graph of $\frac{\sin x}{x^2}$ and the $x - axis$ will be finite. Now the integral is found by subtracting the total area below the axis from the total area above the axis, and this will therefore be finite. In other words the integral $\int_{2\pi}^{\infty} \frac{\sin x}{x^2}dx$ will converge.

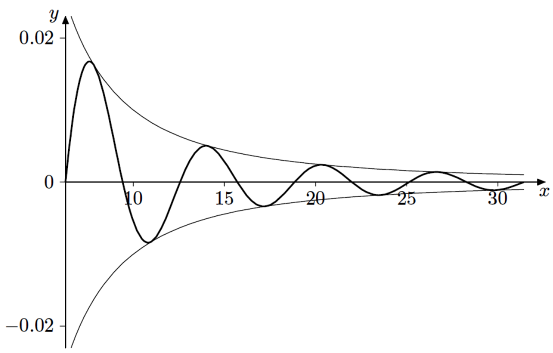

Figure 1.6: *Graph of example 1.6.4*

This is an example of a generalisation of Theorem 1.6.1 which we state here without proof.

Theorem 1.6.2. *Suppose that $f(x)$ and $g(x)$ are continuous, and that $|g(x)| \leq f(x)$, for all $x \geq a$. Then if the infinite integral $\int_a^\infty f(x)dx$ converges, so does the infinite integral $\int_a^\infty g(x)dx$, and*

$$\left|\int_a^\infty g(x)dx\right| \leq \int_a^\infty |g(x)|dx \leq \int_a^\infty f(x)dx.$$

1.7 Improper Integrals

In this section we consider integrals of the form $\int_a^b f(x)dx$, where the function f is undefined, or has a discontinuity, or is unbounded, at some point of the interval $a \leq x \leq b$. If this occurs inside the interval at some point c we can split the integral into two by using the rule of integration

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

This means that we can restrict attention to situations where the point of discontinuity etc. occurs at an end-point of the interval. We shall in fact discuss only cases where this happens at the lower end-point, the case of the upper endpoint being exactly similar. So by way of example we can consider the integral $\int_0^1 \frac{1}{x^2}dx$, where the integrand is undefined at the end-point $x = 0$. Such an integral is called an **improper integral**, and we need to investigate questions of convergence. As with infinite integrals we are asking whether we can sensibly define the area of the region between the graph and the x-axis when the region is unbounded.

Definition 1.7.1. *Let $f(x)$ be a function continuous for $a \leq x \leq b$. Then the improper integral $\int_a^b f(x)dx$ is said to converge if the integral $\int_c^b f(x)dx$ has a limit as $c \to a^+$. We then define*

$$\int_a^b f(x)dx = \lim_{c \to a^+} \int_c^b f(x)dx$$

Example 1.7.1. *Investigate convergence of the improper integral* $\int_0^1 \frac{1}{x^2}dx$

Using Definition 1.7.1, we have

$$\int_c^1 \frac{1}{x^2}dx = \left[-\frac{1}{x}\right]_c^1 = \frac{1}{c} - 1 \to \infty \text{as} c \to 0^+.$$

Therefore the integral $\int_0^1 \frac{1}{x^2}dx$ does not converge.

Chapter 2

Integration by Parts

2.1 The Basic Technique

We are considering integration as the reverse of differentiation, and we should therefore expect that rules of differentiation should relate to techniques of integration. The technique of integration by parts discussed in this chapter is a consequence of the product rule for differentiation. That rule tells us that

$$\frac{d}{dx}(U(x)V(x)) = U(x)\frac{dV}{dx} + V(x)\frac{dU}{dx}$$

If we integrate both sides with respect to x we obtain

$$U(x)V(x) = \int U(x)\frac{dV}{dx} + \int V(x)\frac{dU}{dx}$$

which can be rearranged in the form

$$\int U(x)\frac{dV}{dx} = U(x)V(x) - \int V(x)\frac{dU}{dx}$$

This equation therefore gives us a procedure for evaluating an integral of the form $\displaystyle\int f(x)g(x)dx$. We have to decide which of f and g to identify with U. If we choose $f = U$ this is usually because $\dfrac{dU}{dx}$ is a simpler expression than U. We then have to identify g with $\frac{dV}{dx}$, and we have to be able to find V, i.e., we have to be able to integrate $g(x)$.

Example 2.1.1. *Evaluate* $\int x \cos x \; dx$.

The integrand is a product, and so we have an expression for which integration by parts is a possible technique. So we have to decide which of the two components (x or $\cos x$) to identify with U. It is clear that if we choose $U = x$ then $\frac{dU}{dx} = 1$, which is simpler, whereas if we choose $\frac{dV}{dx} = x$ then $v = \frac{x^2}{2}$, which will give a more complicated integral. So we choose $\frac{dV}{dx} = \cos x$ gives $V = \sin x$. The formula for integration by parts therefore gives

$$\int x \cos x \; dx = x \sin x - \int \sin x.1.dx = x \sin x + \cos x$$

which can be checked by differentiation.

Example 2.1.2. *Evaluate* $\int e^x \sin 2x \; dx$.

In this example it is not immediately clear which of the two components of the product to choose for U. In each case the derivative is no simpler, and we can integrate either expression. In fact we could choose either $U = e^x$ or $U = \sin 2x$ in this example. We shall choose the former. We will find that integrating by parts once does not leave us with a straightforward final integral, so we repeat the calculation. We shall find that we end up with the integral we started with, but in fact this then gives us an equation which we can solve to find I.

$$I = \int e^x \sin 2x \; dx = e^x \left(-\frac{\cos 2x}{2}\right) + \int e^x \frac{\cos 2x}{2} dx$$

2.2 Reduction Formulae

When we evaluate $\int x^2 \sinh x \, dx$, we have to integrating by parts twice. If we were faced with $\int x^6 \sinh x \, dx$ we would have to integrate by parts six times.

Example 2.2.1. *Find reduction formulae for* $I_n = \int x^n e^x \, dx$ *and* $J_n = \int_0^1 x^n e^x \, dx$.

The notation I_n indicates that the integral involves the integer parameter n. The answer depends upon the value of n, as for example with

$$\int_0^1 x^n = \left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1} \quad (n \neq -1),$$

where we can see explicitly that the answer involves n.

In this example we have, in the case of the indefinite integral,

$$I_n = \int x^n e^x \, dx = x^n e^x - \int n x^{n-1} e^x \, dx$$

We can see that the final integral is obtained from I_n by replacing n by $n-1$ and multiplying by n. This is expressed in the reduction formula

$$I_n = \int x^n e^x \, dx = x^n e^x - nI_{n-1}$$

In the case of the definite integral the calculations are the same, and so we obtain

$$J_n = \int_0^1 x^n e^x \, dx = [x^n e^x]_o^1 - \int_0^1 n x^{n-1} e^x \, dx$$

which gives the reduction formula

$$J_n = e - nJ_{n-1}$$

We can use this formula to work out, for example, the value of $J_7 = \int_0^1 x^7 e^x \, dx$

We first apply the reduction formula with $n = 7$, giving $J_7 = e - J_6$. We now apply the same reduction formula, but with $n = 6$, and so

$$J_7 = e - J_6 = e - 7(e - 6J_5) = -6e + 7.6J_5 = -6e + 42J_5.$$

This process continues until we reach J_0, which we can evaluate explicitly, because

$$J_0 = \int_0^1 x^0 e^x \, dx = e - 1$$

The complete process using the reduction formula for values of n from 7 down to 1 is as follows.

$$\begin{aligned}
J_7 &= e - J_6 \\
&= e - 7(e - 6J_5) = -6e + 42J_5 \\
&= -6e + 42(e - 5J_4) = 36e - 210J_4 \\
&= 36e - 210(e - 4J_3) = -174e + 840J_3 \\
&= -174e + 840(e - 3J_2) = 666e - 2520J_2 \\
&= 666e - 2520(e - 2J_1) = -1854e + 5040J_1 \\
&= -1854e + 5040(e - 1.J_0) = -1854e + 5040(e - (e - 1)) \\
&= -1854e + 5040.
\end{aligned}$$

Example 2.2.2. *Find a reduction formula for* $I_n = \int_0^\pi x^n \sin x \, dx$.

We shall see that we need to integrate by parts twice in this case, in order to return to an integral containing $\sin x$.

$$\begin{aligned}
I_n = \int_0^\pi x^n \sin x \, dx &= [x^n(-\cos x)]_0^\pi + \int_0^\pi nx^{n-1} \cos x \, dx \\
&= \pi^n + \int_0^\pi nx^{n-1} \cos x \, dx \\
&= \pi^n + [nx^{n-1} \sin x]_0^\pi + \int_0^\pi n(n-1)x^{n-2} \sin x \, dx \\
&= \pi^n - n(n-1)I_{n-2}
\end{aligned}$$

So the reduction formula is

$$I_n = \pi^n - n(n-1)I_{n-2}$$

In Example 2.2.1 the integer parameter n was reduced by 1 at each stage. In this example n is reduced by 2 each time. So if we begin with an odd value of n we shall finish by needing to calculate I_1, and if we start with an even value of n we shall need the value of I_0. In both cases these are integrals which are easy to evaluate explicitly.

2.3 The Gamma Function

In this section we consider some elementary properties of the Gamma function. This is a function encountered in applied mathematics and in statistics. It brings together integration by parts and infinite and improper integrals discussed in Sections 1.5 and 1.6.

Definition 2.3.1. *The Gamma function is a function of the variable x defined by the integral*

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt \tag{2.1}$$

This is an infinite integral, and so we need to discuss convergence, as in Section 1.5.

If $x-1 < 0$ then the integrand contains a negative power of t, which is undefined at $t = 0$. In this case we therefore have an improper integral, and convergence at $t = 0$ must be investigated, as in Section 1.6.

We shall investigate these two cases of convergence separately by splitting the integral at $t = 1$, letting

$$I_1 = \int_0^1 t^{x-1}e^{-t}dt, \int_1^\infty t^{x-1}e^{-t}dt.$$

In both cases we use comparison to discuss convergence.

Consider I_1. Since $0 \le t \le 1$ we have $e^{-1} \le e^{-t} \le 1$, so

$$t^{x-1}e^{-1} \leq t^{x-1}e^{-t} \leq t^{x-1}$$

If $x > 0$ then for $0 < h < 1$ we have

$$\int_h^1 t^{x-1}dt = \left[\frac{t^x}{x}\right]_h^1 = \frac{1}{x}(1-h^x) \to \frac{1}{x} \text{ as } h \to 0.$$

Hence $\int_0^1 t^{x-1}dt$ converges, and so $\int_0^1 t^{x-1}e^{-1}dt$ converges by comparison. Now if $x < 0$ then for $0 < h < 1$ we have

$$\int_h^1 e^{-1}t^{x-1}dt = e^{-1}\left[\frac{t^x}{x}\right]_h^1 = \frac{e^{-1}}{x}(1-h^x) \to \infty \text{ as } h \to 0.$$

Hence $\int_0^1 e^{-1}t^{x-1}dt$ converges, and so $\int_0^1 t^{x-1}e^{-1}dt$ converges by comparison. Now if $x < 0$ then for $0 < h < 1$ we have

Now we consider $I2$. Let n denote the first integer greater than x. We therefore have,

$$t^{x-1}e^{-t} < t^{n-1}e^{-1} < t^{n-1}\frac{(n+1)!}{t^{n+1}} = \frac{(n+1)!}{t^2}$$

In Example 1.6.1 we saw that $\int_1^\infty \frac{1}{t^2}dt$ converges. Therefore by comparison (noting that $(n+1)!$ is a constant), $\int_1^\infty t^{x-1}e{-}tdt$ converges.

We would like to integrate by parts, but we have an infinite (and possibly improper) integral, so this needs careful consideration. Suppose that $x > 1$, so that the integral is not improper at $x = 0$. We then integrate by parts as follows.

$$[-t^x e^{-t}]_0^k + \int_0^k xt^{x-1}e{-}tdt$$

lrtting $k \to \infty$ then gives

$$\int_0^k t^x e^{-t}dt = x\int_0^\infty t^{x-1}e^{-1}dt, \text{ i.e., } \Gamma(x+1) = x\Gamma(x)$$

This argument will generalise to show that we can integrate convergent infinite integrals by parts. We can deal with the case $0 < x \leq 1$, when the integral is improper, in a similar fashion, using the limit definition of an improper integral as we did for an infinite integral above. So we can say that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$.

This looks like a reduction formula (see Section 2.2), and if we let $x = n$, a positive integer, we can see that

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n) = n(n-1)\Gamma(n-1)\\ &= n(n-1)(n-2)\Gamma(n-2)\\ &= n(n-1)(n-2)...2.1.\Gamma(1).\end{aligned}$$

Now $\Gamma(1) = \int_1^\infty t^0 e^{-t} dt = [-e^{-t}]_0^\infty = 1$, so $\Gamma(n+1) = n!$ The Gamma function can therefore be seen as a generalisation of the factorial function for non-integer values.

Example 2.3.1. ***A Strange Example***

We integrate $\tan x$ by parts, using $U = \dfrac{1}{\cos x}, \dfrac{dV}{dx} = \sin x$.

$$\begin{aligned}\int \tan x\, dx &= \int \frac{\sin x}{\cos x} dx\\ &= \frac{-\cos x}{\cos x} + \int -\cos x . \frac{-\sin x}{\cos^2 x} dx\\ &= -1 + \int \tan x\, dx.\end{aligned}$$

Cancelling $\int \tan x\, dx$ therefore gives $0 = -1$.

So how do we explain this apparent paradox? Is it because of the constant of integration? Well if we were to include that we would get

$$\int \tan x\, dx + C = -1 + \int \tan x\, dx + C,$$

suggesting that $C = -1 + C$, which again is wrong. If the constant of integration is arbitrary perhaps we should not have the same constant C on both sides. We would then obtain

$$\int \tan x\, dx + C = -1 + \int \tan x\, dx + D.$$

This would imply that $C = -1 + D$, but if C and D are arbitrary constants why should they be related? What we need to do is to interpret C and D as representing the set of all possible constants, so that if C and D represent all possible real numbers then the set of all numbers of the form $-1 + D$ is also the set of all real numbers, the same as C. From another perspective it means that we have to think carefully about what an indefinite integral is. This example suggests that an indefinite integral is not a function, but a set of functions, and so the two sets of functions on either side of the equation

$$\int \tan x\, dx + C = -1 + \int \tan x\, dx + C$$

are identical, and there is no question of cancelling $\int \tan x\, dx$.

When we introduced indefinite integrals in Section 1.2 we used the indefinite article, and described a function $F(x)$ whose derivative is $f(x)$ as an indefinite integral (and not the indefinite integral). This is normal usage, and it would be very complicated to develop the procedures of integration in terms of the language of sets of functions. We shall not therefore change our approach, but simply be aware that occasionally we may need to think more precisely about the definition of an indefinite integral if we encounter an apparent paradox.

Chapter 3

Integration by Substitution

The theoretical basis for integration by substitution is the chain rule for differentiation, which says that

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$$

Regarding integration as the reverse of differentiation leads us to integrate both sides of this equation to write

$$\int f'(g(x))g'(x)\,dx = f(g(x))$$

Given an integral of this form we can transform it by means of the substitution $u = g(x)$. We then have $\dfrac{du}{dx} = g'(x)$ and so

$$\int f'(g(x))g'(x)\,dx = \int f'(u)\frac{du}{dx}\,dx = \int \frac{d}{dx}(f(u))\,dx = f(u) = f(g(x))$$

where we have used the chain rule with the intermediate variable u, to recognise that $f'(u)\dfrac{du}{dx} = \dfrac{d}{dx}(f(u))$.

In fact we implement this process symbolically by rewriting $\dfrac{du}{dx} = g'(x)$ in the form $du = g'(x)\,dx$. The integration procedure then appears in the form

$$\int f'(g(x))g'(x)\,dx = \int f'(u)du = f(u) = f(g(u)) \tag{3.1}$$

The underlying idea is that the substitution gives rise to a simpler integral involving the variable u. After having evaluated this integral we then replace u in the answer by $g(x)$, so as to present the answer in terms of the original variable x.

This can all be made analytically rigorous. The details are beyond the scope of this book. In the remainder of this chapter we shall concentrate therefore on applying this technique of integration in a variety of circumstances.

3.1 Some Simple Substitutions

Example 3.1.1. *Evaluate the indefinite integral* $\int \frac{\cos x}{(1+\sin x)^3}\,dx$

The theory above looks straightforward, but if we are presented with an integral like $\int \frac{\cos x}{(1+\sin x)^3}\,dx$, how are we to find an appropriate substitution which will transform the integral into a simpler one that we can recognise? We need to be able to discern what should play the role of f and what should play the role of g. We note that the general integral expression $\int f'(g(x))g'(x)\,dx$ involves both g and its derivative, so what we need to look for is one part of the integrand which is the derivative of another part of the integrand. In this example we can see that the integrand involves $\sin x$ and also $\cos x$, which is the derivative of $\sin x$. This suggests using the substitution $u = \sin x$. We then have $du = \cos x\,dx$, and so the integral transforms as

$$\int \frac{\cos x}{(1+\sin x)^3}\,dx = \int \frac{1}{(1+u)^3}\,du$$

The latter is an integral we should know how to do, but if not we can simplify it still further with a linear transformation $w = 1 + u$, giving $dw = du$, and therefore

$$\int \frac{\cos x}{(1+\sin x)^3}\,dx = \int \frac{1}{(1+u)^3}\,du = \int \frac{1}{w^3}\,dw = -\frac{1}{2w^2}$$

The answer has to be given in terms of the original variable x, so we have

$$\int \frac{\cos x}{(1+\sin x)^3}\,dx = -\frac{1}{2w^2} = -\frac{1}{2(1+u)^2} = -\frac{1}{2(1+\sin x)^2}$$

Example 3.1.2. *Evaluate the indefinite integral* $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$

We first note that $e^{2x} = (e^x)^2$, so that the numerator is the derivative of part of the denominator. This suggests the substitution $u = e^x$, giving $du = e^x\,dx$. We therefore have

$$\int \frac{e^x}{1+e^{2x}}\,dx = \int \frac{1}{1+u^2}\,du = \tan^{-1} u = \tan^{-1}(e^x)$$

3.2 Inverse Substitutions

In the previous section the substitutions used replaced part of the integrand by a single variable, as in Example 3.1.1 where we replaced $\sin x$ by u. In this section we consider substitutions which work in the reverse direction, so that x itself is replaced by an expression in another variable, for example we may substitute $x = \sin u$. We can regard this as equivalent to $u = \sin^{-1} x$, hence the term **inverse substitution**.

Example 3.2.1. *Evaluate the indefinite integral* $I = \displaystyle\int \frac{x^2}{\sqrt{x}+2}\,dx$

It is the square root term which makes this integral slightly awkward, so we choose an inverse substitution to remove it, the obvious one being $x = u^2$. We then have $dx = 2u\,du$, and so

$$
\begin{aligned}
I &= \int \frac{u^4}{u+2} \cdot 2u \, du = \int \frac{2u^5}{u+2} \, du \\
&= \int \left(2u^4 - 4u^3 + 8u^2 - 16u + 32 - \frac{64}{u+2} \right) du \\
&\quad \text{(using polynomial division)} \\
&= \frac{2u^5}{5} - u^4 + \frac{8u^3}{3} - 8u^2 + 32u - 64 \ln|u+2| \\
&= \frac{2x^{\frac{5}{2}}}{5} - x^2 + \frac{8x^{\frac{3}{2}}}{3} - 8x + 32\sqrt{x} - 64 \ln|\sqrt{x}+2|
\end{aligned}
$$

Because $x = u^2$ is a simple inverse transformation it has a straightforward direct equivalent, namely $u = \sqrt{x}$. Applying this direct substitution, however, is slightly more awkward, because we obtain $du = \dfrac{dx}{\sqrt{x}}$, which still involves the square root, and for which the algebraic manipulation needed is slightly more involved.

Example 3.2.2. *Evaluate the indefinite integral* $I = \displaystyle\int \frac{1}{\sqrt{x}(\sqrt[3]{x}+2)} \, dx$

Here we need to look for a substitution which will eliminate both the square and the cube roots, and $x = u^6$ will achieve this. So we have $dx = 6u^5 \, du$ and therefore

$$
\begin{aligned}
I &= \int \frac{1}{\sqrt{x}(\sqrt[3]{x}+2)} \, dx = \int \frac{6u^5}{u^3(u^2+2)} \, du \\
&= 6 \int \frac{u^2}{u^2+2} \, du = 6 \int \left(1 - \frac{2}{u^2+2} \right) du \\
&= 6 \left(u - \sqrt{2} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) \right) = 6 \left(\sqrt[6]{x} - \sqrt{2} \tan^{-1} \left(\frac{\sqrt[6]{x}}{\sqrt{2}} \right) \right)
\end{aligned}
$$

In the remainder of this chapter we shall concentrate on some inverse substitutions which work for classes of integrals. They will be illustrated through particular examples, but discussed in a way which emphasises their general applicability.

3.3 Square Roots of Quadratics

These can generally be simplified by means of trigonometric or hyperbolic substitutions. A knowledge of trigonometric identities, and of the algebraic technique of completing the square, is a necessity therefore. The examples are chosen to involve only a single square root term, but the techniques are equally applicable to integrals of algebraic fractions where the numerator or the de- nominator is the square root of a quadratic.

Example 3.3.1. *Evaluate the indefinite integral* $I = \int \sqrt{4x^2 - 16x + 52}\, dx$

We shall break this example down into a number of steps, each of which is important in evaluating integrals of this type.

STEP 1 We make the coefficient of x^2 equal to 1, so

$$I = 2\int \sqrt{x^2 - 4x + 13}\, dx$$

(Note that if the coefficient of x^2 is negative then we make its coefficient equal to -1 and then proceed with the following steps.)

STEP 2 Complete the square, giving

$$I = 2\int \sqrt{(x-2)^2 + 9}\, dx$$

STEP 3 Write the constant as a square number.

$$I = 2\int \sqrt{(x-2)^2 + 3^3}\, dx$$

STEP 4 Use a linear substitution to replace the variable square term with a single square, using $u = x - 2$ in this case.

$$I = 2\int \sqrt{u^2 + 3^2}\, du$$

This succession of steps will always transform the square root of a quadratic into a sum of squares, as in this example, or a difference of squares. The following steps are therefore applicable to integrals involving a term of the form $\sqrt{u^2+a^2}$

STEP 5 Make an appropriate trigonometric substitution. We need to find a substitution which makes use of a trigonometric identity reducing a sum of two squares to a single square term, which will then enable us to remove the square root. In this case the appropriate identity is $\tan^2 t + 1 = \sec^2 t$. However we have $\sqrt{u^2+3^2}$, and so we need to take the 3^2 into account, using $u = 3\tan t$. We then have $du = 3\sec^2 t\ dt$, and so the substitution gives

$$I = 2\int \sqrt{3^2\tan^2 t + 3^2}\cdot 3\sec^2 t\,dt = 2\cdot 3\cdot 3\int sec^3 t\,dt$$

STEP 6 We now have to evaluate the trigonometric integral. This involves integration by parts, and one of the exercises in Chapter 2 describes how to do this. We find that

$$I = 18\left(\frac{1}{2}\sec t\tan t + \frac{1}{2}\ln|\sec t + \tan t|\right)$$

STEP 7 We now have to express the result in terms of x, firstly by finding the trigonometric functions in terms of u. The substitution gives us one of them, because $\tan t = \frac{u}{3}$. To find $\sec t$ (or any other trigonometric function which might arise is integrals of this type) a helpful technique is to draw a right-angled triangle in which $\tan t = \frac{u}{3}$. Using Pythagoras' Theorem gives the third side of the triangle, and we can then read off any trigonometric ratio that we need.

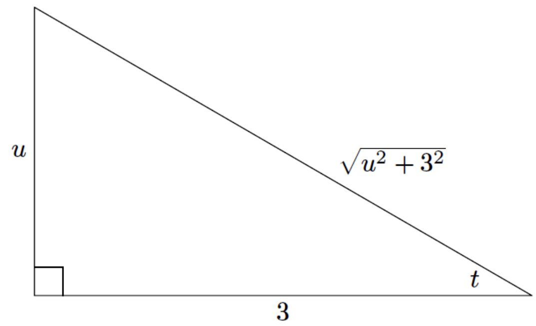

Figure 3.1: *Trigonometric substitutions*

We can now see from the diagram that $\sec t = \dfrac{\sqrt{u^2+3^2}}{3}$. So we can express in terms of u as

$$I = 9\frac{\sqrt{u^2+3^2}}{3}\frac{u}{3} + 9\ln\left|\frac{\sqrt{u^2+3^2}}{3} + \frac{u}{3}\right|$$

STEP 8 We replace u, using $u = x-2$ from *STEP 4*, and undertake some algebraic simplification, to give the answer

$$I = (x-2)\sqrt{(x-2)^2+3^2} + 9\ln\left|\sqrt{(x-2)^2+3^2} + (x-2)\right| - 9\ln 3$$

The $-9\ln 3$ term can be absorbed into the constant of integration, giving finally

$$I = (x-2)\sqrt{(x-2)^2+3^2} + 9\ln\left|\sqrt{(x-2)^2+3^2} + (x-2)\right|$$

3.4 Rational Functions of cos & sin

A rational function, is a quotient of polynomials, where the numerator and denominator involve only terms containing non-negative integer powers of the variable x. By a rational function of cos and sin we mean a quotient where the numerator and denominator involve only terms containing non-negative integer powers of cos and sin, for example

$$\frac{\cos^3 t \sin t + \sin^2 t + 3 - 2\cos^2 t}{3\sin t - 4 + 5\sin^2 t \cos^5 t - 2\cos t}$$

To integrate such a function we use the so-called **half-angle substitution** $x = \tan\left(\frac{t}{2}\right)$. This will always lead to a rational function of x. In Chapter 4 we shall consider methods of integrating rational functions in general, so the example in this section will be a straightforward one.

Because such expressions often involve both $\cos t$ and $\sin t$ we need to use identities which express these in terms of x. We can do this using a right-angled triangle together with basic trigonometric identities.

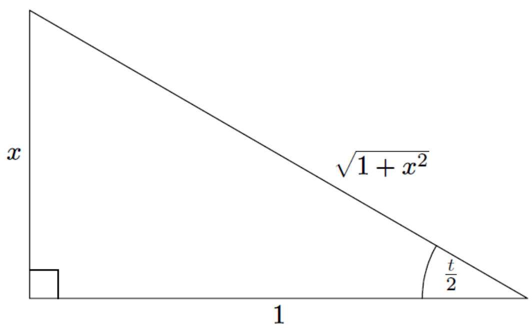

Figure 3.2: *Half-angle substitution*

In the right-angled triangle we have $x = \tan\left(\frac{t}{2}\right)$. So $dx = \frac{1}{2}\sec^2\left(\frac{t}{2}\right) dt$, which we can rearrange in the form

$$dt = 2\cos^2\left(\frac{t}{2}\right) dx = \frac{2}{1+x^2}dx$$

where we obtain the value of $\cos^2\left(\frac{t}{2}\right)$ from the triangle. We can also use the triangle to see that

$$\sin\left(\frac{t}{2}\right) = \frac{x}{\sqrt{1+x^2}}$$

We therefore have

$$\cos t = 2\cos^2\left(\frac{t}{2}\right) - 1 = \frac{2}{1+x^2} - 1 = \frac{1-x^2}{1+x^2}$$
$$\sin t = 2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right) = 2\frac{x}{\sqrt{1+x^2}}\frac{1}{\sqrt{1+x^2}} = \frac{2x}{1+x^2}$$

We can therefore see that each term in the integral of a rational function of $\cos t$ and $\sin t$ (including dt) will be transformed into a rational fraction of x, and so we shall be left with a rational function of x to integrate.

Example 3.4.1. *Evaluate the indefinite integral* $I = \int \frac{1}{2+\sin t}\, dt$

Using the substitution $x = \tan\left(\frac{x}{2}\right)$ with the identities above gives

$$\begin{aligned}
I &= \int \frac{1}{2+\sin t}\, dt = \int \frac{1}{2+\frac{2x}{1+x^2}}\frac{2}{1+x^2}\, dx \\
&= \int \frac{1}{x^2+x+1}\, dx \quad \text{(after simplification)} \\
&= \int \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}} \quad \text{(completing the square)} \\
&= \frac{1}{\sqrt{\frac{3}{4}}}\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right) = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{3}\right) \\
&= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2\tan(\frac{t}{2})+1}{\sqrt{3}}\right)
\end{aligned}$$

Chapter 4

Integration of Rational Functions

4.1 Introduction

A rational function is one of the form $R(x) = \frac{P(x)}{Q(x)}$,where are both $P(x)$ and $Q(x)$ are polynomials in the variable x, for example

$$\frac{2x^3 + 3x^2 - 4x + 1}{x^2 - 3x + 2}$$

In this chapter we shall explain the steps involved in a procedure which will enable us to integrate any rational function, provided the algebra is not too horrible! One of the algebraic tools needed is the decomposition of rational functions into partial fractions, and we discuss this in the next section. In Section 4.3 we describe the process of integrating a rational function, split into a sequence of steps. We explain what happens at each step, using different examples at each stage to illustrate some degree of generality.

Apart from partial fractions, the main algebraic prerequisite is polynomial division.

4.2 Partial Fractions

The partial fraction decomposition expresses a rational function $\frac{P(x)}{Q(x)}$ as a sum of simpler algebraic fractions. The denominators of these fractions are determined by the factorisation of the denominator $Q(x)$. When a real polynomial is completely factorised into real factors, the factors will either be linear or quadratic, and some factors may occur more than once.

Dealing with partial fractions where the denominator factorises into linear factors only, none of which are repeated, is a familiar area of school mathematics. A common method used is that of equating coefficients, as in the following example.

Example 4.2.1. *Find the partial fraction decomposition of*

$$\frac{x^3+2x^2-x+4}{(x-1)(x+2)(x-3)(x+1)}$$

The decomposition is of the form

$$\frac{x^3+2x^2-x+4}{(x-1)(x+2)(x-3)(x+1)} = \frac{A}{x-1}+\frac{B}{x+2}+\frac{C}{x-3}+\frac{D}{x+1}$$

If we now put the right hand side over the common denominator

$$(x-1)(x+2)(x-3)(x+1)$$

the numerator will be

$$\begin{aligned} &A(x+2)(x-3)(x+1)+B(x-1)(x-3)(x+1)\\ &\quad +C(x-1)(x+2)(x+1)+D(x-1)(x+2)(x-3)\end{aligned}$$

So equating the numerators gives

$$\begin{aligned} x^3+2x^2-x+4 \equiv A(x+2)(x-3)(x+1)+B(x-1)(x-3)(x+1)\\ +C(x-1)(x+2)(x+1)\\ +D(x-1)(x+2)(x-3)\end{aligned}$$

where the use of the identity symbol $\equiv$ emphasises that this is true for all values of x. One method of finding A, B, C, D is to multiply out the right-hand side, giving

$$\begin{aligned} x^3+2x^2-x+4 \equiv (A+B+C+D)x^3+(-3B+2C-2D)x^2\\ +(-7A-B-C-5D)x\\ +(-6A+3B-2C+6D)\end{aligned}$$

Because this is an identity the coefficient of each power of x on both sides must be the same. This gives the following system of four equations to solve for A, B, C, D.

$$\begin{aligned} A + B + C + D &= 1, \\ -3B + 2C - 2D &= 2, \\ -7A - B - C - 5D &= -1, \\ -6A + 3B - 2C + 6D &= 4 \end{aligned}$$

An alternative method is to substitute specific values of x in the identity, chosen so as to make one of the linear factors zero. This has the effect of making all but one of the terms zero on the right hand side of the first identity above. This enables us to determine the unknown coefficients one at a time. The results are given in the following table.

$x = 1$	$6 = -12A$	$A = -1/2$
$x = -2$	$6 = -15B$	$B = -2/5$
$x = 3$	$46 = 40C$	$C = 23/20$
$x = -1$	$6 = 8D$	$D = 3/4$

We therefore have

$$\frac{x^3 + 2x^2 - x + 4}{(x-1)(x+2)(x-3)(x+1)} = \frac{1}{2(x-1)} - \frac{2}{5(x+2)} + \frac{23}{20(x-3)} + \frac{3}{4(x+1)}$$

Example 4.2.2. *Decompose* $\dfrac{x^2 - 2x + 3}{(x+1)^3}$ *into partial fractions.*

The decomposition will be of the form

$$\frac{x^2 - 2x + 3}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

We have to find A, B, C and as usual we arrange the right-hand side as a single fraction over the common denominator, $(x+1)^3$. This gives rise to the numerator $A(x+1)^2 + B(x+1) + C$, which must be equal to $x^2 - 2x + 3$. We could find A, B, C, by comparing coefficients, as in Example 10.1, or by compensation. Here we shall demonstrate another method, involving differentiation, reminiscent of the discussion on Taylor Polynomials.

So in the equation $A(x+1)^2 + B(x+1) + C = x^2 - 2x + 3$, we substitute $x = -1$ to give $C = 6$. We then differentiate both sides of the equation to give $2A(x+1) + B = 2x - 2$. Putting $x = -1$ now gives $B = -4$. Differentiating once more gives $2A = 2$, and so $A = 1$. Therefore the decomposition is

$$\frac{x^2 - 2x + 3}{(x+1)^3} = \frac{1}{x+1} - \frac{4}{(x+1)^2} + \frac{6}{(x+1)^3}$$

Example 4.2.3. *Decompose* $\dfrac{2x^2 - x + 4}{(x^2+x+1)^2}$ *into partial fractions.*

The decomposition will be of the form $\dfrac{Ax+B}{(x^2+x+1)} + \dfrac{Cx+D}{(x^2+x+1)^2}$

Arranging this as a single fraction over a common denominator gives the numerator $(Ax+B)(x^2+x+1) + Cx + D$, which must be equal to $2x^2 - x + 4$.

We can't use the differentiation method in the same way as in Example 4.5, because there is a quadratic involved, and indeed there is no real value of x for which it is zero. We can adapt the differentiation method however, using $x = 0$ in each case to simplify the calculations. So we have to solve the identity $(Ax+B)(x^2+x+1) + Cx + D \equiv 2x^2 - x + 4$ for A, B, C. We can see immediately that $A = 0$ because there is no x^3 term on the right hand side. So the identity simplifies to $B(x^2+x+1) + Cx + D \equiv 2x^2 - x + 4$. Putting $x = 0$ gives $B + D = 4$.

Differentiating gives $2Bx + B + C \equiv 4x - 1$, and putting $x = 0$ gives $B + C = -1$.

Differentiating again gives $2B = 4$ and so $B = 2$. We can now deduce that $C = -3$ and $D = 2$. So the decomposition is

$$\frac{2x^2 - x + 4}{(x^2+x+1)^2} = \frac{2}{(x^2+x+1)} + \frac{-3x+2}{(x^2+x+1)^2}$$

Notice that although we expect each numerator to be linear, it is possible that some of the coefficients may be zero, as has happened with the first fraction in this case.

In the examples above we have used a variety of methods, chosen to keep the algebraic manipulation as straightforward as we can.

4.3 The Integration Process

Integrating a rational function $\frac{P(x)}{Q(x)}$ is a process which can be broken down into a well-defined sequence of steps. In this section we shall describe the procedure, providing illustrative examples. In the subsequent section we work through some examples in detail.

STEP 1 Polynomial Division

Is the degree of $P(x)$ greater than or equal to the degree of $Q(x)$? If the answer is YES then divide $Q(x)$ into $P(x)$ to obtain

$$\frac{P(x)}{Q(x)} = A(x) + \frac{B(x)}{Q(x)},$$

where $A(x)$ and $B(x)$ are polynomials and deg $B(x) <$ deg $Q(x)$.

STEP 2 Polynomial Division

Following polynomial division we need to factorise the denominator $Q(x)$. The problem with this step is that there is no general algorithm which will factorise all polynomials. So in practice this step can only be carried out if the polynomial $Q(x)$ is relatively straightforward.

STEP 3 Polynomial Division

Decompose $\frac{B(x)}{Q(x)}$ using partial fractions. This was discussed in detail in Section 4.2.

STEP 4 Polynomial Division

We can now integrate each term in the partial fraction decomposition separately. Each is a rational function, but only a few different types of expression occur, as we have seen in Section 4.2, and we discuss each of them. The first two types involve a linear factor in the denominator, which may be repeated, and a constant numerator. From the basic integrals described in Section 1.2 we have the following two results.

$$\int \frac{A}{x+k}dx = A\ln|x+k|,$$
$$\int \frac{A}{(x+k)^n}dx = A\frac{(x+k)^{-n+1}}{-n+1} \qquad (n \neq 1)$$

We now need to consider quadratic denominators. Where the quadratic is not a repeated factor the integral will be of the form $\int \frac{\text{linear}}{\text{quadratic}}$

We can re-write the integrand in the form $\frac{\text{linear}}{q(x)} = c\frac{q'(x)}{q(x)} + \frac{d}{q(x)}$, where c and d are constants, and integrate each term separately.

Example 4.3.1. *Evaluate the integral* $\int \frac{x+3}{x^2-x+4}dx$

We rewrite the integrand as explained above to give

$$\int \frac{x+3}{x^2-x+4}dx = \int \frac{\frac{1}{2}(2x-1)}{x^2-x+4}dx + \int \frac{\frac{7}{2}}{x^2-x+4}dx$$

Using the general result

$$\int \frac{q'(x)}{q(x)}dx = \ln|q(x)|$$

we can deal with the first integral as follows.

$$\int \frac{\frac{1}{2}(2x-1)}{x^2-x+4}dx = \frac{1}{2}\ln|x^2-x+4|$$

We deal with the second integral by completing the square of the denominator, which doesn't have real roots.

$$\int \frac{\frac{7}{2}}{x^2-x+4}dx = \frac{7}{2}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{15}{4}} = \frac{7}{2}\sqrt{\frac{4}{15}}\tan^{-1}\left(\sqrt{\frac{4}{15}}\left(x-\frac{1}{2}\right)\right)$$

Finally we must deal with integrals of the form $\int \frac{\text{linear}}{(\text{quadratic})^n}$ $(n > 1)$

We can write the integrand as $\frac{\text{linear}}{(q(x))^n} = c\frac{q'(x)}{(q(x))^n} + \frac{d}{(q(x))^n}$, where c and d are constants.

The following example illustrates the general process for evaluating such integrals.

Example 4.3.2. *Evaluate the integral*

$$\int \frac{2x-1}{(x^2-2x+5)^2}dx$$

Splitting the integral as explained above gives

$$\int \frac{2x-1}{(x^2-2x+5)^2}dx = \int \frac{2x-2}{(x^2-2x+5)^2}dx + \int \frac{1}{(x^2-2x+5)^2}dx$$

Evaluating the first integral gives

$$\int \frac{2x-2}{(x^2-2x+5)^2}dx = -\frac{1}{x^2-2x+5}$$

The second integral can be evaluated as follows, using the trigonometric substitution $u = 2\tan t, du = 2\sec^2 t dt$.

$$\begin{aligned}
\int \frac{1}{(x^2-2x+5)^2}dx &= \int \frac{1}{((x-1)^2+4)^2}dx = \int \frac{1}{(u^2+4)^2}du \\
&= \int \frac{2\sec^2 t}{(4\tan^2 t+4)^2}dt = \int \frac{2\sec^2 t}{(4\sec^2 t)^2}dt \\
&= \frac{1}{8}\int \cos^2 t dt = \frac{1}{16}\int (1+\cos 2t)dt \\
&= \frac{1}{16}\left(t + \frac{\sin 2t}{2}\right) = \frac{1}{16}(t+\sin t\cos t) \\
&= \frac{1}{16}\left(\tan^{-1}\left(\frac{u}{2}\right) + \frac{u}{\sqrt{u^2+4}}\frac{2}{\sqrt{u^2+4}}\right) \\
&= \frac{1}{16}\left(\tan^{-1}\left(\frac{x-1}{2}\right) + \frac{2(x-1)}{x^2-2x+5}\right)
\end{aligned}$$

where in the penultimate line we have used the right-angled triangle method.

Adding the two results together then gives

$$\int \frac{2x-1}{(x^2-2x+5)^2}dx = \frac{1}{16}\left(\tan^{-1}\left(\frac{x-1}{2}\right) + \frac{2(x-9)}{x^2-2x+5}\right)$$

Chapter 5

Geometrical Applications of Integration

In Section 1.1 we discussed integration as summation, and we use that interpretation of the integral in this chapter to construct integral formulae for some geometrical quantities. We shall consider length, area and volume, and the notions of centroid and centre of mass. The most important thing in this chapter is not to remember particular formulae, but to understand the principles underpinning their construction, so that analogous formulae can be constructed in other areas of application.

5.1 Arc Length

In this section we derive formulae for the length of a curve. Not all curves can be given as the graph of a function, for example a circle, and so we shall deal with the more general situation where a curve is described parametrically by means of the equations

$$x = x(t),\ y = y(t),\ a \leq t \leq b.$$

We shall assume that we have a smooth curve, for which the functions $x(t)$ and $y(t)$ have continuous derivatives for $a \leq t \leq b$. On the graph represented by these parametric equations, we divide the curve into small pieces by means of a sequence of points $P_0, P_1, P_2, \ldots, P_n$, specified by the sequence of values of the parameter, given by

$$a = t_0 < t_1 < t_2 < \cdots < t_n = b.$$

This is shown in the left-hand diagram in Figure 5.1. In the right-hand diagram we have isolated one small piece of the graph lying between two successive points. The length of that small piece of arc is denoted conventionally by ds.

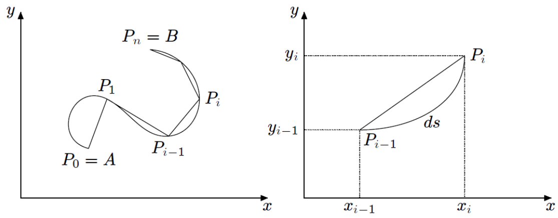

Figure 5.1: *Arc length*

In the right-hand diagram, if the piece of arc is very small then the gradient will not change much along the arc because the parametric functions are assumed to have continuous derivatives. So the length ds will be approximately equal to that of the line segment $P_{i-1}P_i$. Using Pythagoras' Theorem gives

$$\begin{aligned}(P_{i-1}P_i)^2 &= (x_i - x_{i-1})^2 + (y_i - y_{i-1})^2 \\ &= (x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2 \\ &= (x'(c_i)(t_i - t_{i-1}))^2 + (y'(c_i)(t_i - t_{i-1}))^2\end{aligned}$$

using the Mean Value Theorem, we now deduce that,

$$ds^2 \approx (P_{i-1}P_i)^2 = (x'(c_i)^2 + y'(d_i)^2)(t_i - t_{i-1})^2$$

Taking square roots gives

$$ds \approx \sqrt{(x'(c_i)^2 + y'(d_i)^2)}(t_i - t_{i-1})$$

The total arc length is therefore given by

$$L \approx \sum_{i=1}^{n} \sqrt{(x'(c_i)^2 + y'(d_i)^2)}(t_i - t_{i-1})$$

This is an approximating sum to an integral, as outlined in Section 1.1, and so we finally we have the formula

$$L = \int_a^b \sqrt{(x'(t)^2 + y'(t)^2)}dt$$

As a special case, suppose that the curve is the graph of the function specified by $y = f(x), p \leq x \leq q$. This can be expressed in the parametric form $x = t, y = f(t), p \leq t \leq q$. We then have

$$x'(t) = 1,\ y'(t) = f'(t) = f'(x) = \frac{dy}{dx}.$$

So the formula becomes

$$L = \int_p^q \sqrt{(1 + f'(x)^2)}dx = \int_p^q \sqrt{1 + \left(\frac{dy}{dx}\right)^2}dx$$

Example 5.1.1. *Find the length of the curve given by* $x = t^2, y = 2t^3, -1 \leq t \leq 1$, *shown in Figure 5.2.*

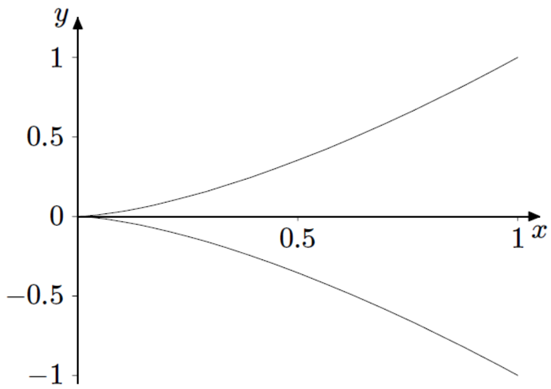

Figure 5.2: *Graph of* $x = t^2, y = 2t^3, -1 \leq t \leq 1$

Using the formula derived above gives

$$L = \int_{-1}^{1} \sqrt{4t^2 + 36t^4}dt = \int_{-1}^{1} 2t\sqrt{1 + 9t^2}dt = \left[\frac{2}{3}\frac{1}{9}(1 + 9t^2)^{\frac{3}{2}}\right]_{-1}^{1} = 0$$

This is clearly wrong. The curve does not have zero length! The problem is that we were not sufficiently careful with the square root when we factored out t^2 from under the square root in the first integral. It is not true in general that $\sqrt{t^2} = t$, especially in this case where the integrand must be positive for all values of t because it represents a length. A correct version of the calculations is as follows.

$$
\begin{aligned}
L &= \int_{-1}^{1} \sqrt{4t^2 + 36t^4}dt = \int_{-1}^{1} 2|t|\sqrt{1+9t^2}dt \\
&= 2\int_{0}^{1} 2t\sqrt{1+9t^2}dt = 2\left[\frac{2}{3}\frac{1}{9}(1+9t^2)^{\frac{3}{2}}\right]_0^1 \\
&= \frac{4}{27}\left(10^{\frac{3}{2}} - 1\right).
\end{aligned}
$$

Example 5.1.2. *Find the length of the curve given by* $y = x^2 - \frac{\ln x}{8}(1 \leq x \leq 2)$.

In this example we need to use the cartesian formula for arc length.

$$
\begin{aligned}
L &= \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \int_1^2 \sqrt{1 + \left(2x - \frac{1}{8x}\right)^2} dx \\
&= \int_1^2 \sqrt{1 + 4x^2 + \frac{1}{64x^2} - \frac{1}{2}}dx = \int_1^2 \sqrt{4x^2 + \frac{1}{64x^2}0\frac{1}{2}}dx \\
&= \int_1^2 \sqrt{\left(2x + \frac{1}{8x}\right)^2}dx = \int_1^2 \left(2x + \frac{1}{8x}\right) dx \\
&= \left[x^2 + \frac{\ln x}{8}\right]_1^2 = 3 + \frac{\ln 2}{8}
\end{aligned}
$$

In passing from line 2 to line 3 we had to be able to spot that $4x^2 + \frac{1}{64x^2} + \frac{1}{2}$ is a perfect square.

5.2 Surface Area of Revolution

If we take a curve and rotate it about a line we will obtain a curved surface of revolution. For example if we rotate the semicircle in Figure 5.3 about its base through a complete rotation of 2π we will generate a sphere. In general of course we consider not just a semicircle but an arbitrary smooth curve, which we represent in parametric form as $x = x(t), y = y(t), a \leq t \leq b$. To obtain a formula for such a surface area we divide the curve into small pieces as we did in Section 1.1.

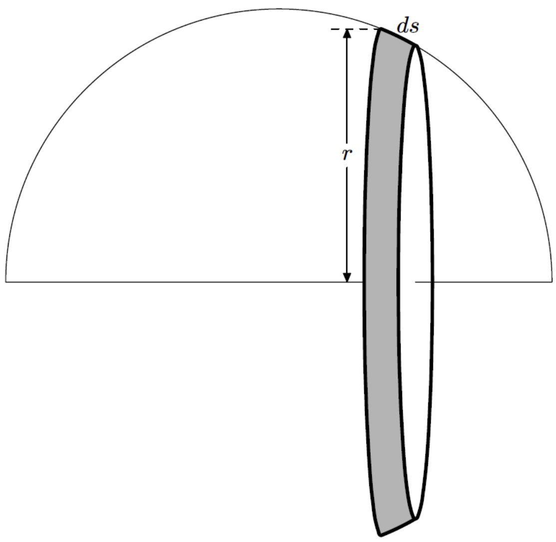

Figure 5.3: *Surface of revolution*

In Figure 5.3 we have shown the effect of rotating one such piece of arc. To find the area generated we imagine cutting and unwrapping the section of surface shown. This will give us a piece of "ribbon" approximately rectangular in shape. Its length will be the circumference, $2\pi r$, of the circle generated by rotating a point on the small piece of arc. Its width will be the length ds of the piece of arc, for which we found an approximate formula in Section 5.1. So, using the same notation as in Section 5.1, the small piece of surface area will be approximately

$$2\pi r.ds \approx 2\pi|y(t)|\sqrt{(x'(t_i)^2) + y'(t_i)^2)}(t_i - t_{i-1}).$$

Note that we have used $|y(t)|$ as the value of r to allow for the fact that $y(t)$ might be negative for some values of t. So the total surface area will be given by

$$S \approx \sum_{i=1}^{n} 2\pi |y(t)| \sqrt{(x'(t_i)^2) + y'(t_i)^2)} (t_i - t_{i-1}).$$

This is an approximating sum for an integral, and so the formula for the surface area of rotation is

$$S = 2\pi \int_a^b |y(t)| \sqrt{(x'(t_i)^2) + y'(t_i)^2)} (t_i - t_{i-1}) dt.$$

If the curve is the graph of a function, $y = f(x), p \leq x \leq q$, then it can be expressed in the parametric form $x = t, y = f(t), p \leq t \leq q$, as in Section 5.1. The argument in that section shows that in this case

$$S = 2\pi \int_p^q |f(x)| \sqrt{1 + f'(x)^2)} dx = 2\pi \int_p^q |f(x)| \sqrt{1 + \left(\frac{dy}{dx}\right)^2}) dx.$$

Example 5.2.1. *Find the area of the surface obtained by rotating the curve $y = x^3, 0 \leq x \leq 1$ about the $x-axis$.*

We use the cartesian formula to obtain

$$S = 2\pi \int_0^1 x^3 \sqrt{1 + (3x^2)^2} dx = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4} dx$$
$$= 2\pi \left[\frac{1}{54} (1 + 9x^4)^{\frac{3}{2}} \right]_0^1 = \frac{\pi}{27} \left(10^{\frac{3}{2}} - 1 \right).$$

Note that because $x^3 \geq 0$ in the interval we do not need the modulus signs.

Example 5.2.2. *Verify the formula for the surface of a sphere of radius a.*

In this example we shall use the parametric formula. The sphere can be obtained by rotating the semicircle given by $x^2 + y^2 = a^2,\ y \geq 0$, about the $x - axis$. We parameterise the semicircle by $x = a\cos t,\ y = a\sin t, 0 \leq t \leq \pi$. Since $y(t) \geq 0$ for all t in the interval we do not need the modulus signs in the formula, and so we have

$$\begin{aligned} S &= 2\pi \int_0^\pi a\sin t\sqrt{a^2\sin^2 t + a^2\cos^2 t}dt \\ &= 2\pi a^2 \int_0^\pi \sin t dt = 2\pi a^2[-\cos t]_0^\pi = 4\pi a^2. \end{aligned}$$

5.3 Volumes by Slicing

Consider a prism, whose uniform cross section can be any shape, for example triangular, rectangular, or circular (a circular prism is of course a cylinder). Its volume is equal to the cross-sectional area multiplied by its length. We can extend this idea to solids where the cross section is not uniform. So suppose we have a solid, contained between planes $x = a$ and $x = b$, and that we can calculate the area $A(c)$ of the cross section made by the plane $x = c$. A good way to imagine this is to think of a sliced loaf of bread, where the cross section will vary from one end to the other. We now slice up the solid by means of a sequence of planes

$$x = x_0(= a),\ x = x_1,\ x = x_2, \ldots,\ x = x_n(= b).$$

Assuming that the slices are sufficiently thin, and that $A(x)$ is a continuous function of x, the volume of the slice contained between $x = x_{i-1}$ and $x = x_i$ will be approximately $A(x_i)(x_i - x_{i-1})$. The total volume will therefore be approximately the sum of these slice volumes, so

$$V \approx \sum_{i=1}^{n} A(x_i)(x_i - x_{i-1}).$$

This is an approximating sum for an integral, as in Section 1.1, and so we have

$$V = \int_a^b A(x)dx.$$

Naturally for this to be useful we have to be able to find $A(x)$ for the solid we are concerned with, and we do this is the next example.

Example 5.3.1. *A tetrahedron is formed by cutting a corner from a cube by means of a plane. Find its volume.*

We place the corner of the cube O at the origin, with the sides of the cube meeting at that corner along the coordinate axes. Let the cutting plane meet the coordinate axes at $x = a$, $y = b$, $z = c$ respectively. Figure 5.4 shows the tetrahedron with a triangular cross section PQR made by a plane $x = p$. We therefore need to calculate the area of this triangle.

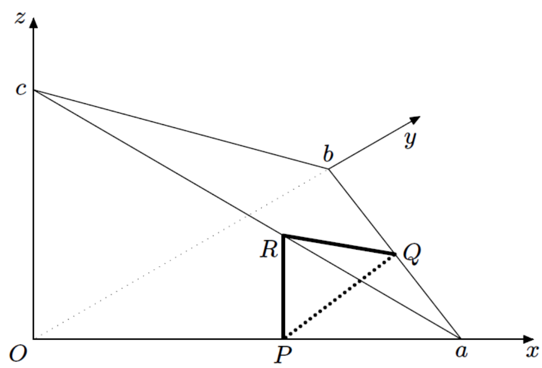

Figure 5.4: *Tetrahedron for Example 5.3.1*

Some elementary calculation with similar triangles tells us that

$$PR = \frac{c(a-p)}{a}, \quad PQ = \frac{b(a-p)}{a},$$

so the area of triangle PQR is given by

$$A(p) = \frac{PR.PQ}{2} = \frac{bc(a-p)^2}{2a^2}.$$

Therefore the volume of the tetrahedron is given by

$$V = \int_0^a A(x)dx = \int_0^a \frac{bc(a-x)^2}{2a^2}dx = \frac{bc}{2a^2}\left[-\frac{(a-x)^3}{3}\right]_0^a = \frac{abc}{6}$$

5.4 Volumes of Revolution

If we imagine rotating a plane region about a line in that plane then a solid will be generated with circular cross sections. For example if we rotate a rectangular region about one of its edges we obtain a cylinder. A right-angled triangle rotated about one of its shorter edges will generate a cone. A semicircular region rotated about its diameter will generate a sphere. In this section we shall investigate two methods of calculating the volume of a solid of revolution.

5.4.1 The Disc Method

If we rotate a rectangular region about a line parallel to one of its edges which does not intersect the rectangle then we will generate a solid cylinder with a cylindrical hole through its centre. This is a simple example of a general type of solid obtained by rotating the region specified by

$$f(x) \le y \le g(x), a \le x \le b$$

about the line $y = c$, shown in Figure 5.5.

As before we calculate the cross sectional area of the solid and use the integral formula derived above. We subdivide the interval $a \le x \le b$ as in Section 1.1, and use this to divide the region into strips parallel to the $y - axis$. We have shown one of these in Figure 5.5, together with the solid obtained by rotating this strip about the line $y = c$. It looks like a "washer", i.e., a disc with a smaller disc removed from its centre. The cross-sectional area is therefore

$$\pi R^2 - \pi r^2 = \pi\left((g(x)-c)^2 - (f(x)-c)^2\right).$$

The integral formula tells us that the total volume is given by

$$V = \pi \int_a^b \left((g(x) - c)^2 - (f(x) - c)^2\right) dx$$

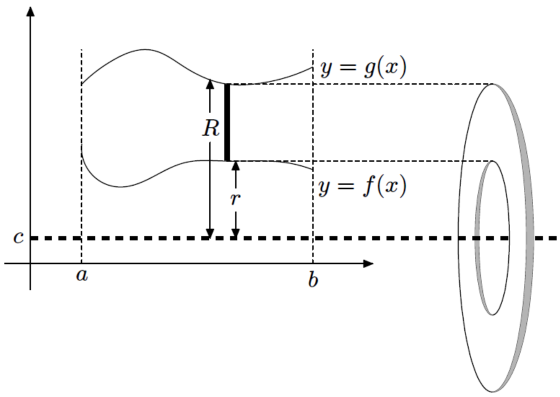

Figure 5.5: *Volume of revolution; the disc method*

Example 5.4.1. *Find the volume of the solid obtained by rotating the region $0 \leq y \leq \sin x$, $0 \leq x \leq \pi$, about (a) the $x-$axis & (b) the line $y = -1$*

(a) In this case we have $c = 0$, $f(x) = 0$, $g(x) = \sin x$, and so the volume is given by

$$V = \pi \int_0^\pi \sin^2 x\, dx = \frac{\pi}{2}\left[x - \frac{\sin 2x}{x}\right]_0^\pi = \frac{\pi^2}{2}$$

(b) Here we have $c = -1$ and so the volume is given by

$$\begin{aligned} V &= \pi \int_0^\pi \left((\sin^2 x + 1)^2 - (-1)^2\right) dx \\ &= \pi \int_0^\pi \sin^2 x\, dx + \pi \int_0^\pi 2 \sin x\, dx = \frac{\pi^2}{2} + 4\pi \end{aligned}$$

5.4.2 The Cylindrical Method

This method applies to the same region

$$f(x) \leq y \leq g(x), a \leq x \leq b.$$

for which we developed the disc method, but this time we rotate the region about the line $x = d$, parallel to the $y - axis$, as shown in Figure 5.6. In this case the strips produced by subdividing the interval $a \leq x \leq b$ generate cylindrical shells rather than a cross section of the solid. One has to imagine each of these shells fitting inside the previous one to form the solid, like some childrens' toys where plastic beakers fit inside each other, or like the separate parts of Russian dolls.

In Figure 5.6, suppose that the strip is specified by one of the intervals of the subdivision, i.e.,

$$f(x) \leq y \leq g(x), xi - 1 \leq x \leq xi.$$

The distance of this strip from the axis of rotation is shown as R, where we have $R = x - d$. (In the diagram d is negative, and so $R = x - d > x$, as the figure suggests.) On the left of the figure is shown the cylindrical shell generated by rotating the strip, and we need to find its volume. If the cylinder is made of some flexible material, we can cut it parallel to the axis of rotation. We can then open it out and we will obtain an approximately rectangular slab. Its height will be the height of the strip, namely $g(x) - f(x)$, its width will be

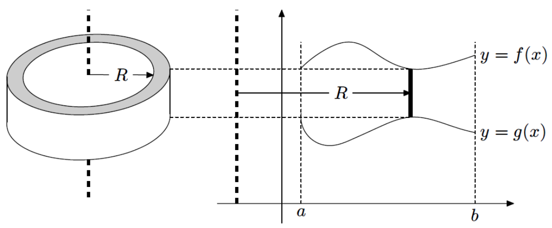

Figure 5.6: *Volume of revolution; the shell method*

the circumference of the cylinder, namely $2\pi R$, and its thickness will be the width of the strip, namely $x_i - x_{i-1}$. The volume is therefore approximately

$$2\pi R(g(x) - f(x))(x_i - x_{i-1}) = 2\pi(x - d)(g(x) - f(x))(x_i - x_{i-1}).$$

The total volume is therefore approximately

$$\sum_{i=1}^{n} 2\pi(x - d)(g(x) - f(x))(x_i - x_{i-1}),$$

which is an approximating sum for an integral, and hence

$$V = 2\pi \int_a^b (x - d)(g(x) - f(x))\, dx.$$

Example 5.4.2. *Rotate the region in Example 5.4.1 about the line $x = -\pi$, and find the volume of the solid obtained.*

The region is $0 \leq y \leq \sin x$, $0 \leq x \leq \pi$, and so using the formula obtained above we have

$$f(x) = 0,\ g(x) = \sin x,\ a = 0,\ b = \pi,\ d = -\pi.$$

We therefore have

$$V = 2\pi \int_0^\pi (x + \pi) \sin x\, dx = 2\pi^2 \int_0^\pi \sin x\, dx + 2\pi \int_0^\pi x \sin x\, dx = 6\pi^2$$

Both integrals are straightforward. The second is done by parts, and the first is a basic integral. This integral is encountered frequently, so it is worth learning that its value, which corresponds to the area of the region in this example, is equal to 2.

Example 5.4.3. *Let R denote the region contained between the two graphs $y = x^2$, $y = \sqrt{x}$, $0 \leq x \leq 1$.Find the volume obtained by rotating this region about (a) the $x - axis$, (b) the $y - axis$.*

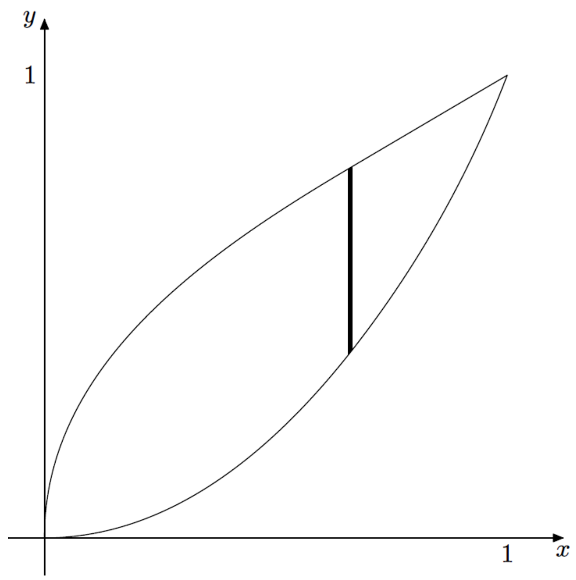

Figure 5.7: *Diagram for Example 5.4.3*

The region is shown in Figure 5.7, and we have drawn a strip parallel to the $y - axis$. If we rotate the strip round the $x - axis$ this will generate a washer, and so we can use the disc method. If we rotate it round the $y - axis$ we will generate a cylinder, so we can use the shell method.

(a) The volume obtained by rotating round the $x - axis$ is given by

$$V = \pi \int_0^1 \left(\sqrt{x}^2 - (x^2)^2 \right) \, dx = \pi \int_0^1 (x - x^4) \, dx = \frac{3\pi}{10}.$$

(b) The volume obtained by rotating round the $y - axis$ is given by

$$V = 2\pi \int_0^\pi x(\sqrt{x} - x^2) \, dx = 2\pi \int_0^\pi \left(x^{\frac{3}{2}} - x^3 \right) \, dx = \frac{3\pi}{10}.$$

Because of the symmetry of the region we can see that the two solids will in fact be identical in shape and size.

Chapter 6

Multiple Integrals

In calculus and elementary physics, you have seen a number of uses for integration such as finding area, volume, mass, moment of inertia, and so on. In this chapter we want to consider these and other applications of both single and multiple integrals. We shall discuss both how to set up integrals to represent physical quantities and methods of evaluating them. In later chapters we will need to use both single and multiple integrals.

6.1 Double and Triple Integrals

Recall from calculus that $\int_a^b y dx = \int_a^b f(x)\, dx$ gives the area "under the curve" in Figure 6.1. Recall also the definition of the integral as the limit of a sum: We approximate the area by a sum of rectangles as in Figure 6.1; a representative rectangle (shaded) has width Δx. The geometry indicates that if we increase the number of rectangles and let all the widths $\Delta x \to 0$, the sum of the areas of the rectangles will tend to the area under the curve. We define $\int_a^b f(x)dx$ as the limit of the sum of the areas of the rectangles; then we evaluate the integral as an anti-derivative, and use $\int_a^b f(x)\, dx$ to calculate the area under the curve.

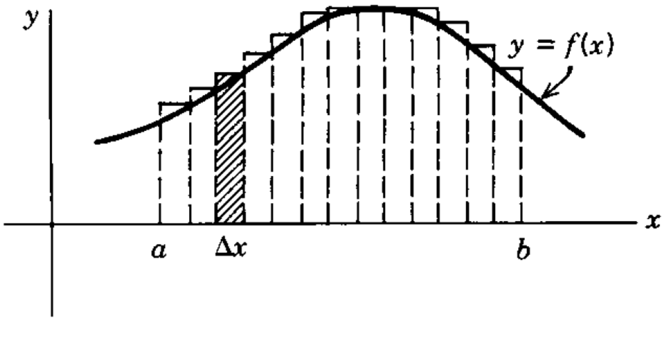

Figure 6.1: *The area under a curve*

We are going to do something very similar in order to find the volume of the cylinder in Figure 6.2 under the surface $z = f(x, y)$. We cut the (x, y) plane into little rectangles of area $\Delta A = (\Delta x)(\Delta y)$ as shown in Figure 6.2; above each $\Delta x \Delta y$ is a tall slender box reaching up to the surface. We can approximate the desired volume by a sum of these boxes just as we approximated the area in Figure 6.1 by a sum of rectangles. As the number of boxes increases and all Δx and $\Delta y \to 0$, the geometry indicates that the sum of the volumes of the boxes will tend to the desired volume. We define the double integral of $f(x, y)$ over the area A in the (x, y) plane (Figure 6.2) as the limit of this sum, and we write it as $\iint_A f(x, y)\, dx\, dy$. Before we can use the double integral to compute volumes, however, we need to see how double integrals are evaluated.

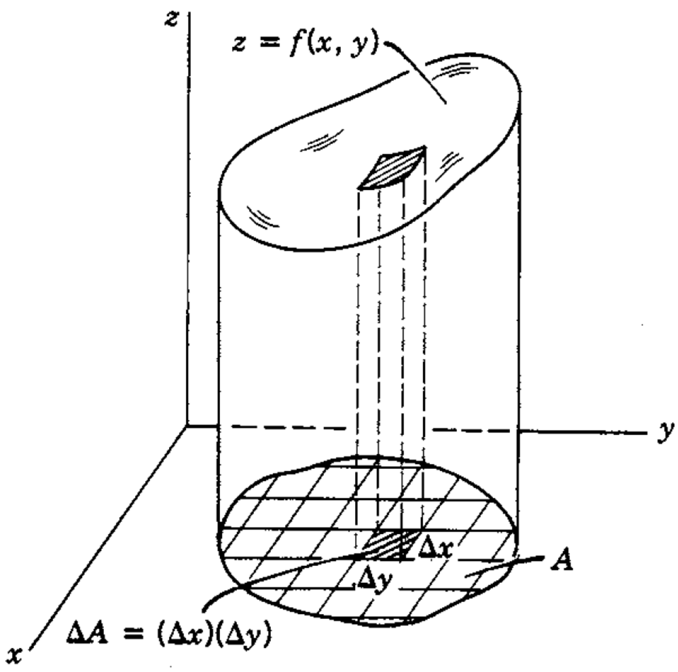

Figure 6.2: *A Cylinder*

Iterated Integrals We now show by some examples the details of evaluating double integrals.

Example 6.1.1. *Find the mass of a rectangular plate bounded by $x = 0$, $x = 2$, $y = 0$, $y = 1$, if its density (mass per unit area) is $f(x, y) = xy$.*

The mass of a tiny rectangle $\Delta A = \Delta x \Delta y$ is approximately $f(x, y) = \Delta x\ \Delta y$, where $f(x, y)$ is evaluated at some point in ΔA. We want to add up the masses of all the ΔA's; this is what we find by evaluating the double integral of $dM = xy\,dx\,dy$. We call dM the mass element and think of adding up all the dM's to get M.

$$M = \iint Axy\,dx\,dy = \int_{x=0}^{2}\int_{y=0}^{1} xy\,dx\,dy$$
$$= \left(\int_0^2 x\,dx\right)\left(\int_0^1 y\,dy\right) = 2 \cdot \frac{1}{2} = 1$$

A triple integral of $f(x, y, z)$ over a volume V , written $\iiint V f(x, y, z)$ $dx\,dy\,dz$, is also defined as the limit of a sum and is evaluated by an iterated integral. If the integral is over a **box**, that is, all limits are constants, then we can do the x, y, z integrations in any order. If the volume is complicated, then we have to consider the geometry.

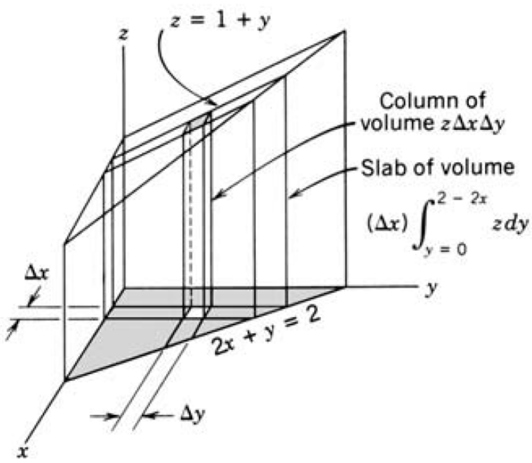

Figure 6.3: *Solid from Example 6.1.2*

Example 6.1.2. *Find the mass of the solid in Figure 6.3 if the density (mass per unit volume) is $x + z$.*

The mass element is $dM = (x+z)\,dx\,dy\,dz$. We add up elements of mass just as we add up elements of volume.

$$M = \int_{x=0}^{1} \int_{y=0}^{2-2x} \int_{z=0}^{1+y} (x+z)\,dz\,dy\,dx = 2$$

where we evaluate the integrals. (Check the result by hand and by computer.)

6.2 Applications of Integration; Single and Multiple Integrals

Many different physical quantities are given by integrals; let us do some problems to illustrate setting up and evaluating these integrals. The basic idea which we use in setting up the integrals in these problems is that an integral is the "limit of a sum." Thus we imagine the physical object (whose volume, moment of inertia, etc., we are trying to find) cut into a large number of small pieces called elements. We write an approximate formula for the function given, of a certain element and then sum over all elements of the object.

Our job would mainly be setting up integrals. It is imperative that when evaluating integral we keep in mind utilizing appropriate limits, deciding the order of integration, detecting and correcting errors, making useful changes of variables, and understanding the meaning of the symbols used, it is imperative that practise and quite a lot of care is applied when going through this section.

Example 6.2.1. *Given the curve $y = x^2$ from $x = 0$ to $x = 1$, find*

a) *the area under the curve (that is, the area bounded by the curve, the x axis, and the line $x = 1$; see Figure 3.1).*

b) *the mass of a plane sheet of material cut in the shape of this area if its density (mass per unit area) is xy.*

c) *the arc length of the curve.*

a) The area is,

$$A = \int_{x=0}^{1} y\,dx = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}$$

We could also find the area as a double integral of $dA = dx\,dy$ (see Figure 6.4) . We then have,

$$A = \int_{x=0}^{1} \int_{y=0}^{x^2} dy\,dx = \int_0^1 x^2\,dx$$

as before. Although the double integral is entirely unnecessary in t=finding the areas on this problem, we shall need to use a double integral to find the mass in part (b).

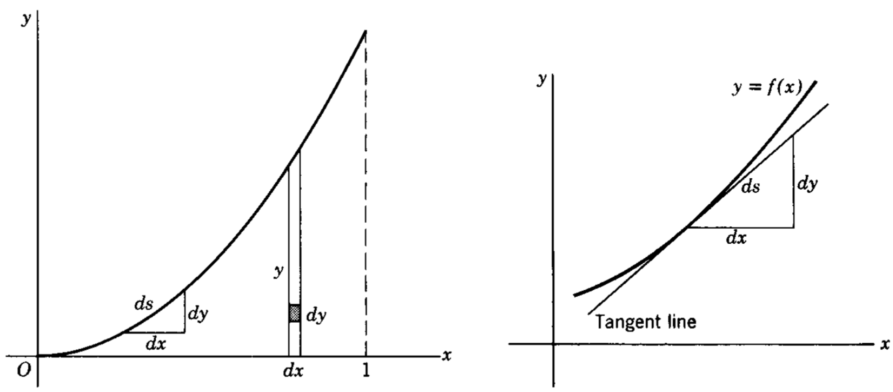

Figure 6.4: *Figures for Example 6.2.1*

b) The element of area, as in the double integral method in (a), is $dA = dy\,dx$. Since the density is $\rho = xy$, the element of mass is $dM = xy\,dy\,dx$ and the total mass is

$$M = \int_{x=0}^{1}\int_{y=0}^{x^2} xy\,dy\,dx = \int_0^1 x\,dx\left[\frac{y^2}{2}\right]_0^{x^2} = \int_0^1 \frac{x^5}{2}\,dx = \frac{1}{12}$$

Observe that we could not do this problem as a single integral because the density depends on both x and y.

Example 6.2.2. *Rotate the area from Example 6.2.1 about the $x-$axis to form a volume and surface of revolution, and find*

a) the volume,

b) the area of the curved surface.

a) We want to find the given volume.

The easiest way to find a volume of revolution is to take as volume element a thin slab of the solid as shown in Figure 6.5. The slab has a circular cross section of radius y and thickness dx; thus the volume element is $\pi y^2\,dx$.

Then the volume in our example is,

$$v = \int_0^1 \pi y^2\,dx = \int_0^1 \pi x^4\,dx = \frac{\pi}{5}$$

We have really avoided part of the integration here because we knew the formula for the area of a circle. In finding volumes of solids which are not solids of revolution, we may have to use double or triple integrals. Even for a solid of revolution we might need multiple integrals to find the mass if the density is variable.

To illustrate setting up such integrals, let us do the above problem using triple integrals. For this we need the equation of the surface which is,

$$y^2 + z^2 = x^4, x > 0.$$

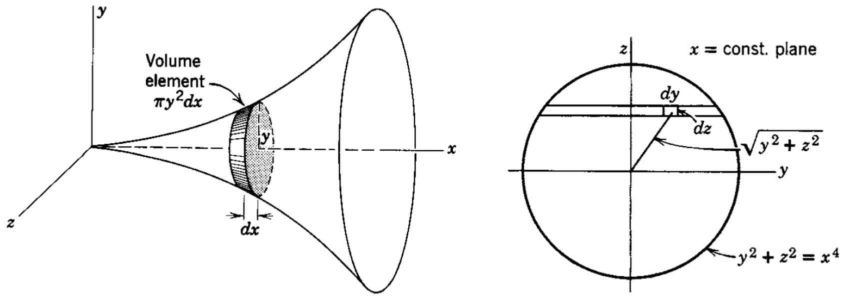

Figure 6.5: *Figures for Example 6.2.2*

To set up a multiple integral for the volume of a solid, we cut the solid into slabs as in Figure 6.5 ***(left)*** (not necessarily circular slabs, although they are in our example) and then as in Figure 6.5 ***(right)*** we cut each slab into strips and each strip into tiny boxes of volume $dx\,dy\,dz$. The volume is

$$V = \iiint dx\,dy\,dz$$

The only problem is to find the limits! To do this, we start by adding up tiny boxes to get a strip; as we have drawn Figure 6.5 ***(right)***, this means to integrate with respect to y from one side of the circle $y^2 + z^2 = x^4$ to the other, that is, from

$$y = -\sqrt{x^4 - z^2} \quad \text{to} \quad y = +\sqrt{x^4 - z^2}.$$

Next we add all the strips in a slab. This means that, in Figure 6.5 ***(right)***, we integrate with respect to z from the bottom to the top of the circle $y^2 + z^2 = x^4$; thus the z limits are $z = \pm radiusofcircle = \pm x^2$. And finally we add all the slabs to obtain the solid. This means to integrate in Figure 6.5 ***(left)*** from $x = 0$ to $x = 1$; this is just what we did in our first simple method. The final integral is then

$$V = \int_{x=0}^{1} \int_{z=-x^2}^{x^2} \int_{y=-\sqrt{x^4-z^2}}^{\sqrt{x^4-z^2}} dy\, dz\, dx$$

Although the triple integral is an unnecessarily complicated way of finding a volume of revolution, this simple problem illustrates the general method of setting up an integral for any kind of volume. Once we have the volume as a triple integral, it is easy to write the integrals for the mass with a given variable density, for the coordinates of the centroid, for the moments of inertia, and so on. The limits of integration are the same as for the volume; we need only insert the proper expressions (density, etc.) in the integrand to get the mass, centroid, and so on.

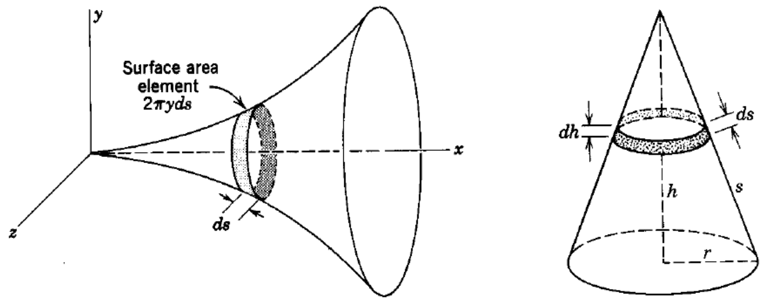

Figure 6.6: *Figures for Example 6.2.2*

b) We find the area of the surface of revolution by using as element the curved surface of a thin slab as in Figure 6.6 ***(left)***. This is a strip of circumference $2\pi y$ and width ds. To see this clearly and to understand why we use ds here but dx in the **??**, think of the slab as a thin section of a cone (Figure 6.6 ***(right)***) between planes perpendicular to the axis of the cone. If you wanted to find the total volume $V = \frac{1}{3}\pi r 2h$ of the cone, youwould use the height h perpendicular to the base, but in finding the total curved surface area $S = \frac{1}{2} 2\pi r s$, you would use the slant height s. The same ideas hold in finding the volume and surface elements. The approximate volume of the thin slab is the area of a face of the slab times its thickness dh in (Figure 6.6 ***(right)***), dx in Figure 6.5 ***(left)***). But if you think of a narrow strip of paper just covering the curved surface of the thin slab, the width of the strip of paper is ds, and its length is the circumference of the thin slab.

The element of surface area (in Figure 6.6 ***(left)***) is then

$$dA = 2\pi y ds.$$

The total area is [using ds from Example 6.2.1 (c)]

$$A = \int_{x=0}^{1} 2\pi y \, ds = \int_{0}^{1} 2\pi x^2 \sqrt{1+4x^2} \, dx.$$

6.3 Change of Variables in Double Integrals

Let us begin by examining the change of variables in a double integral. Suppose that we require to change an integral

$$I = \iint_R f(x,y) \, dx \, dy$$

in terms of coordinates x and y, into one expressed in new coordinates u and v, given in terms of x and y by differentiable equations $u = u(x,y)$ and $v = v(x,y)$ with inverses $x = x(u,v)$ and $y = y(u,v)$. The region R in the xy-plane and the curve C that bounds it will become a new region R' and a new boundary C' in the uv-plane, and so we must change the limits of integration accordingly. Also, the function $f(x,y)$ becomes a new function $g(u,v)$ of the new coordinates.

Now the part of the integral that requires most consideration is the area element. In the $xy - plane$ the element is the rectangular area $dA_{xy} = dx\,dy$ generated by constructing a grid of straight lines parallel to the x & y $axes$ respectively. Our task is to determine the corresponding area element in the uv coordinate system. In general the corresponding element dA_{uv} will not be the same shape as dA_{xy}, but this does not matter since all elements are infinitesimally small and the value of the integrand is considered constant over them. Since the sides of the area element are infinitesimal, dA_{uv} will in general have the shape of a parallelogram. We can find the connection between dA_{xy} and dA_{uv} by considering the grid formed by the family of curves $u = constant$ and $v = constant$, as shown in Figure 6.7. Since v is constant along the line element KL, the latter has components $(\frac{\partial x}{\partial u})du$ and $(\frac{\partial y}{\partial u})du$ in the directions of the x & y $axes$ respectively. Similarly, since u is constant along the line element KN, the latter has corresponding components $(\frac{\partial x}{\partial v})dv$ and $(\frac{\partial y}{\partial v})dv$. Using the result for the area of a parallelogram given, we find that the area of the parallelogram $KLMN$ is given by,

$$\begin{aligned} dA_{uv} &= \left|\frac{\partial x}{\partial u}\,du\,\frac{\partial y}{\partial v}\,dv\, - \frac{\partial x}{\partial v}\,dv\,\frac{\partial y}{\partial u}\,du\right| \\ &= \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|\; du\,dv \end{aligned}$$

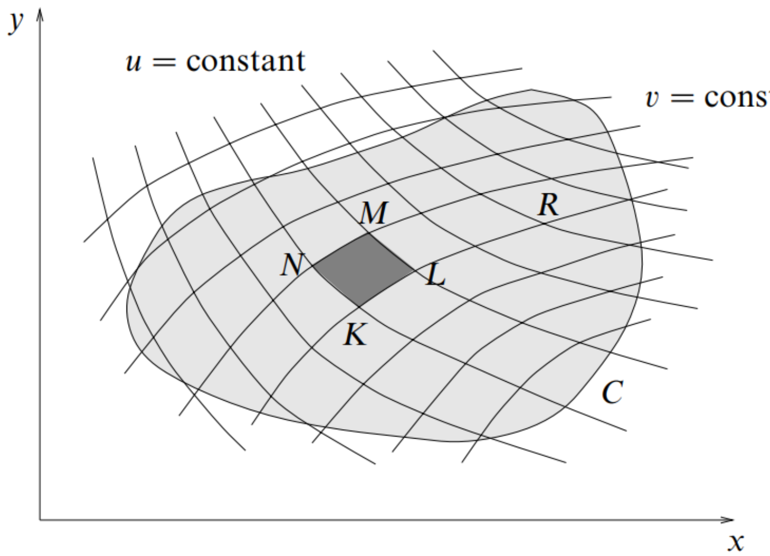

Figure 6.7: *A region of integration R overlaid with a grid formed by the family of curves $u = constant$ and$v = constant$. The parallelogram $KLMN$ defines the area element dA_{uv}*

Defining the Jacobian of x, y with respect to u, v as

$$J = \frac{\partial(x,y)}{\partial(u,v)} \equiv \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

We have,

$$dA_{uv} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

The reader acquainted with determinants will notice that the Jacobian can also be written as the 2×2 determinant

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

So, in summary, the relationship between the size of the area element generated by dx, dy and the size of the corresponding area element generated by du, dv is

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

This equality should be taken as meaning that when transforming from coordinates x, y to coordinates u, v, the area element $dx\,dy$ should be replaced by the expression on the **RHS** of the above equality. Of course, the Jacobian can, and, in general will, vary over the region of integration. We may express the double integral in either coordinate system as

$$I = \iint_R f(x, y)dx\,dy = \iint_{R'} g(u, v) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \tag{6.1}$$

When evaluating the integral in the new coordinate system, it is usually advisable to sketch the region of integration R' in the uv-plane.

Example 6.3.1. *Evaluate the double integral*

$$I = \iint_R \left(a + \sqrt{x^2 + y^2}\right)$$

where R is the region bounded by the circle $x^2 + y^2 = a^2$.

In Cartesian coordinates, the integral may be written

$$I = \int_{-a}^{a} dx \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \left(a + \sqrt{x^2 + y^2}\right)$$

and can be calculated directly. However, because of the circular boundary of the integration region, a change of variables to plane polar coordinates ρ, ϕ is indicated. The relationship between Cartesian and plane polar coordinates is given by $x = \rho\cos\phi$ and $y = \rho\sin\phi$. Using (6.1) we can therefore write

$$I = \iint_{R'} (a + \rho) \left|\tfrac{\partial(x,y)}{\partial(\rho,\phi)}\right| d\rho\, d\phi$$

where R' is the rectangular region in the $\rho\phi$-plane whose sides are $\rho = 0$, $\rho = a$, $\phi = 0$ and $\phi = 2\pi$. The Jacobian is easily calculated, and we obtain

$$J = \frac{\partial(x,y)}{\partial(\rho,\phi)} = \begin{vmatrix} \cos\phi & \sin\phi \\ -\rho\sin(\phi) & \rho\cos\phi \end{vmatrix} = \rho(\cos^2\phi + \sin^2\phi) = \rho$$

So the relationship between the area elements in Cartesian and in plane polar coordinates is

$$dx\, dy = \rho\, d\rho\, d\phi$$

Therefore, when expressed in plane polar coordinates, the integral is given by

$$\begin{aligned} I &= \iint_{R'} (a + \rho)\rho\, d\rho\, d\phi \\ &= \int_0^{2\pi} d\,\phi \int_0^a d\rho\, (a + \rho)\rho = 2\pi \left[\frac{a\rho^2}{2} + \frac{\rho^3}{3}\right]_0^a = \frac{5\pi a^3}{3} \end{aligned}$$

6.4 Change of Variables in Triple Integrals

A change of variable in a triple integral follows the same general lines as that for a double integral. Suppose we wish to change variables from x, y, z to u, v, w. In the x, y, z coordinates the element of volume is a cuboid of sides dx, dy, dz and volume $dV_{xyz} = dx\,dy\,dz$. If, however, we divide up the total volume into infinitesimal elements by constructing a grid formed from the coordinate surfaces $u = constant$, $v = constant$ and $w = constant$, then the element of volume dV_{uvw} in the new coordinates will have the shape of a ***parallelepiped*** whose faces are the coordinate surfaces and whose edges are the curves formed by the intersections of these surfaces (see Figure 6.8). Along the line element PQ the coordinates v and w are constant, and so PQ has components $(\frac{\partial x}{\partial u})du$, $(\frac{\partial y}{\partial u})du$ and $(\frac{\partial z}{\partial u})du$ in the directions of the $x-$, $y-$ and $z-axes$ respectively. The components of the line elements PS and ST are found by replacing u by v and w respectively.

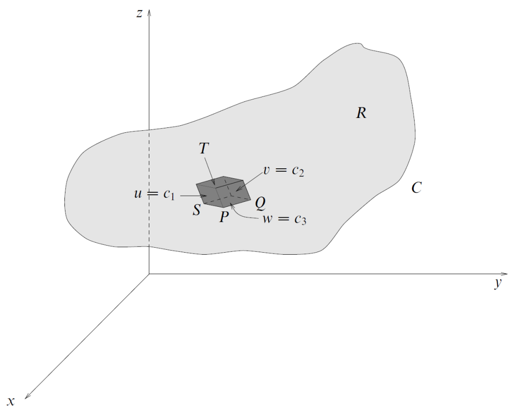

Figure 6.8: *A three-dimensional region of integration R, showing an element of volume in u, v, w coordinates formed by the coordinate surfaces $u = constant$, $v = constant$, $w = constant$.*

The volume of a parallelepiped is the scalar triple product. Using this, we find that the element of volume in u, v,w coordinates is given by

$$dV_{uvw} = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

where the Jacobian of x, y, z with respect to u, v, w is a short-hand for a 3×3 determinant:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \equiv \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w} \end{vmatrix}$$

So, in summary, the relationship between the elemental volumes in multiple integrals formulated in the two coordinate systems is given in Jacobian form by

$$\partial x\,\partial y\,\partial z = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw.$$

and we can write a triple integral in either set of coordinates as

$$I = \iiint_R f(x,y,z)dx\,dy\,dz = \iiint_{R'} g(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Example 6.4.1. *Find an expression for a volume element in spherical polar coordinates, and hence calculate the moment of inertia about a diameter of a uniform sphere of radius a and mass M.*

Spherical polar coordinates r, θ, ϕ are defined by

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$$

The required Jacobian is therefore

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{vmatrix}$$

The determinant is most easily evaluated by expanding it with respect to the last column, which gives

$$\begin{aligned} J &= \cos\theta(r^2\sin\theta\cos\theta) + r\sin\theta(r\sin^2\theta) \\ &= r^2\sin\theta(\cos^2\theta + \sin^2\theta) = r^2\sin\theta \end{aligned}$$

Therefore the volume element in spherical polar coordinates is given by

$$dV = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}\, dr\, d\theta\, d\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

If we place the sphere with its centre at the origin of an x, y, z coordinate system then its moment of inertia about the $z-axis$ (which is, of course, a diameter of the sphere) is

$$I = \int (x^2+y^2)\, dM = \rho \int (x^2+y^2)\, dV$$

where the integral is taken over the sphere, and ρ is the density. Using spherical polar coordinates, we can write this as

$$\begin{aligned} I &= \rho \iiint_V (r^2) r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= \rho \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin^2\theta \int_0^a dr r^4 \\ &= \rho \times 2\pi \times \frac{4}{3} \times \frac{1}{5} a^5 \\ &= \frac{8}{15}\pi a^5 \rho. \end{aligned}$$

Since the mass of the sphere is $M = \frac{4}{3}\pi a^3 \rho$, the moment of inertia can also be written as $I = \frac{2}{5} M a^2$.

6.5 Surface Integrals

In the preceding sections we found surface areas, moments of them, etc., for surfaces of revolution. We now want to consider a way of computing surface integrals in general whether the surface is a surface of revolution or not. Consider a part of a surface as in Figure 6.9 and its projection in the $(x,\ y)$ *plane*. We assume that any line parallel to the $z - axis$ intersects the surface only once. If this is not true, we must work with part of the surface at a time, or project the surface into a different plane. For example, if the surface is closed, we could find the areas of the upper and lower parts separately. For a cylinder with axis parallel to the $z - axis$ we could project the front and back parts separately into the $(y,\ z)$ *plane*.

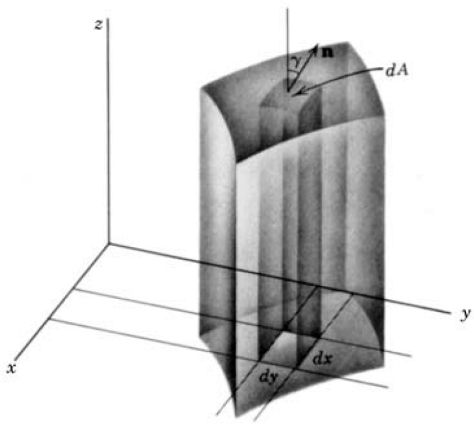

Figure 6.9: *Element dA*

Let dA 6.9 be an element of surface area which projects onto $dx\ dy$ in the $(x,\ y)$ *plane* and let be the acute angle between dA (that is, the tangent plane at dA) and the $(x,\ y)$ *plane*. Then we have

$$dx\ dy = dA\ \cos\gamma \qquad \text{or} \qquad dA = \sec\gamma\ dx\ dy. \tag{6.2}$$

The surface area is then

$$\iint dA = \iint \sec\gamma\ dx\ dy \tag{6.3}$$

where the limits on x and y must be such that we integrate over the projected area in the (x, y) plane.

Now we must find $\sec\gamma$. The (acute) angle between two planes is the same as the (acute) angle between the normals to the planes. If n is a unit vector normal to the surface at dA (Figure 5.6), then γ is the (acute) angle between n and the $z-axis$, that is, between the vectors n and k, so $\cos\gamma = |n \cdot k|$. Let the equation of the surface be $\phi(x, y, z) = const$.The vector

$$grad\,\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} \tag{6.4}$$

is normal to the surface $\phi(x, y, z) = const$. Then n is a unit vector in the direction of grad ϕ, so

$$n = \frac{(grad\,\phi)}{|grad\,\phi|} \tag{6.5}$$

From equations 6.4 and 6.5 we find,

$$n \cdot k = \frac{k \cdot grad\,\phi}{|grad\,\phi|} = \frac{\frac{\partial\phi}{\partial z}}{grad\,\phi}$$
$$\sec\gamma = \frac{1}{\cos\gamma} = \frac{1}{|n \cdot k|}$$

so

$$\sec\gamma = \frac{|grad\,\phi|}{|\frac{\partial\phi}{\partial z}|} = \frac{\sqrt{(\frac{\partial\phi}{\partial x})^2 + (\frac{\partial\phi}{\partial y})^2 + (\frac{\partial\phi}{\partial z})^2}}{|\frac{\partial\phi}{\partial z}|} \tag{6.6}$$

Often the equation of a surface is given in the form $z = f(x, y)$. In this case $\phi(x, y, z) = z - f(x, y)$, so $\frac{\partial\phi}{\partial z} = 1$, and (6.6) simplifies to

$$\sec\gamma =, \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1} \tag{6.7}$$

We then substitute (6.6) or (6.7) into (6.3) and integrate to find the area. To find centroids, moments of inertia, etc., we insert the proper factor into (6.3).

Example 6.5.1. *Find the area cut from the upper half of the sphere* $x^2 + y^2 + z^2 = 1$ *by the cylinder* $x^2 + y^2 - y = 0$.

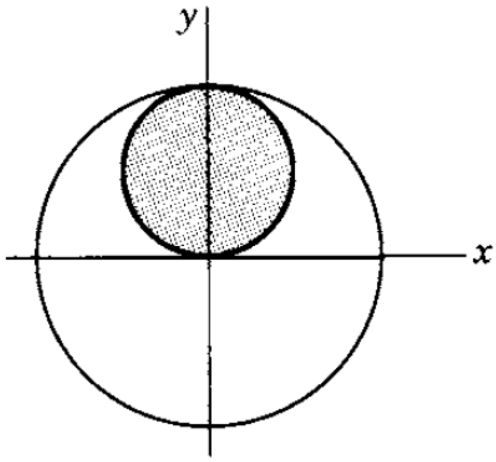

Figure 6.10: *Figure for Example 6.5.1*

This is the same as the area on the sphere which projects onto the disk $x2+y2-y0$ in the (x, y) *plane*. Thus we want to integrate (6.3) over the area of this disk. Figure 6.10 shows the disk of integration (shaded) and the equatorial circle of the sphere (large circle). We compute sec from the equation of the sphere; we could use (6.7), but it is easier in this problem to use (6.6):

$$\phi = x^2 + y^2 + z^2,$$

$$\sec\gamma = \frac{|grad\,\phi|}{|\frac{\partial\phi}{\partial z}|} = \frac{1}{2z}\sqrt{(2x)^2 + (2y)^2 + (2z)^2} = \frac{1}{z} = \frac{1}{\sqrt{1 - x2 - y2}}$$

We find the limits of integration from the equation of the shaded disk, $x^2 + y^2 - y \leq 0$. Because of the symmetry we can integrate over the first-quadrant part of the shaded area and double our result. Then the limits are:

$$x \text{ from } 0 \text{ to } \sqrt{y - y^2}, \qquad y \text{ from } 0 \text{ to } 1.$$

The desired area is

$$A = 2\int_{y=0}^{1}\int_{x=0}^{\sqrt{y-y^2}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}} \tag{6.8}$$

This integral is simpler in polar coordinates. The equation of the cylinder is then $r = \sin\theta$, so the limits are: r from 0 to $\sin\theta$, and θ from 0 to $\frac{\pi}{2}$. Thus (6.9) becomes

$$A = 2\int_{\theta=0}^{\frac{\pi}{2}}\int_{r=0}^{\sin\theta} \frac{r\,dr\,d\theta}{\sqrt{1-r^2}} \tag{6.9}$$

This is still simpler if we make the change of variable $z = \sqrt{1-r^2}$. Then $dz = \frac{-r\,dr}{\sqrt{1-r^2}}$, and the limits $r = 0$ to $\sin\theta$ become $z = 1$ to $\cos\theta$. Thus (6.10) becomes

$$A = -2\int_{\theta=0}^{\frac{\pi}{2}}\int_{z=1}^{\cos\theta} dx\,d\theta = \pi - 2 \tag{6.10}$$

Chapter 7

First Order Differential Equations

7.1 Basic Concepts

DIFFERENTIAL EQUATIONS

A *differential equation* is an equation involving an unknown function and it's derivatives.

The following are differential equations involving the unknown function y,

$$\frac{dy}{dx} = 5x + 3 \tag{7.1}$$

$$e^y \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 1 \tag{7.2}$$

$$4\frac{d^3y}{dx^3} + (\sin x)\frac{d^2y}{dx^2} + 5xy = 0 \tag{7.3}$$

$$\left(\frac{d^2y}{dx^2}\right)^3 + 3y\left(\frac{dy}{dx}\right)^7 + y^3\left(\frac{dy}{dx}\right)^2 = 5x \tag{7.4}$$

$$\frac{\partial^2 y}{\partial t^2} - 4\frac{\partial^2 y}{\partial x^2} = 0 \tag{7.5}$$

A differential equation is an *ordinary differential equation* (ODE) if the unknown function depends on only <u>one</u> independent variable. If the unknown function depends on two or more independent variables. the differential equation is a *partial differential equation* (PDE). *With the exceptions of Chapters 31 and 34, the primary focus of this hook mil he ordinary differential equations.*

Equations (7.1) through (7.4) are examples, of ordinary differential equations, since the unknown function Y depends solely on the variable x Equation (7.5) is a partial differential equation, since Y depends on both the independent variables t and x.

The *order* of a differential equation is the order of the highest derivative appearing in the equation.

Equation (7.1) is a first-order differential equation - (7.2), (7.4), and (7.5) are second-order differential equations. [Note in (1.5) that the order of the highest derivative appearing in the equation is two.] Equation (7.3) is a third order differential equation.

NOTATION

The expressions $y', y'', y''', y^{(4)}, ..., y^{(n)}$ are often used to represent, respectively, the first, second, third, fourth,..., nth derivatives of y with respect to the independent variable under consideration. Thus, y'' represents $\dfrac{d^2y}{dx^2}$ if the independent variable is x, but represents $\dfrac{d^2y}{dp^2}$ if the independent variable is p. Observe that parentheses are used in $y^{(n)}$ to distinguish it from the nth power, $y^{(n)}$ If the independent variable is time, usually denoted by t, primes are often replaced by dots. Thus, $\dot{y}, \ddot{y}$, and $\dddot{y}$ represent $\dfrac{dy}{dt}, \dfrac{d^2y}{dt^2}$, and $, \dfrac{d^3y}{dt^3}$ respectively.

SOLUTIONS

A solution of a differential equation in the unknown function y and the independent variable x on the interval $\mathscr{I}$, is a function $y(x)$ that satisfies the differential equation identically for all x in $\mathscr{I}$.

Example 7.1.1. *Is $y(x) = c_1 \sin 2x + c_2 \cos 2x$, where c_1 and c_2 are arbitrary constants, a solution of $y'' + 4y = 0$?*

Differentiating y, we find

$$y' = 2c_1 \cos 2x - 2c_2 \sin 2x \quad \text{and} \quad y'' = -4c_1 \sin 2x - 4c_2 \cos 2x$$

Hence,

$$\begin{aligned} y'' + 4y &= (-4c_1 \sin 2x - 4c_2 \cos 2x) + 4(2c_1 \cos 2x - 2c_2 \sin 2x) \\ &= (-4c_1 + 4c_1) \sin 2x + (-4c_2 + 4c_2) \cos 2x \\ &= 0 \end{aligned}$$

Thus, $y(x) = c_1 \sin 2x + c_2 \cos 2x$ satisfies the differential equation for all values of x and is a solution on the interval $(-\infty, \infty)$

Example 7.1.2. *Determine whether $y = x^2 - 1$ is a solution of $(y')^4 + y^2 = -1$.*

Note that the left side of the differential equation must be nonnegative for every real function $y(x)$ and any x, since it is the sum of terms raised to the second and fourth powers, while the right side of the equation is negative. Since no function $y(x)$ will satisfy this equation, the given differential equation has no solution.

We see that some differential equations have infinitely many solutions (Example 7.1.1), whereas other differential equations have no solutions (Example 7.1.2). It is also possible that a differential equation has exactly one solution. Consider $(y')^4 + y^2 = 0$, which for reasons identical to those given in Example 7.1.2 has only one solution $y \equiv 0$.

A *particular solution* of a differential equation is any one solution. The *general solution* of a differential equation is the set of all solutions.

Example 7.1.3. *The general solution to the differential equation in Example 7.1.1 can be shown to be (see Chapters 8 and 9) $y(x) = c_1 \sin 2x + c_2 \cos 2x$. That is, every particular solution of the differential equation has this general form. A few particular solutions are: (a)$y = 5 \sin 2x - 3 \cos 2x$ (choose $c_1 = 5$ and $c_2 = -3$), (b)$y = \sin 2x$ (choose $c_1 = 1$ and $c_2 = 0$), and (c)$y \equiv 0$ (choose $C_1 = c_2, = 0$).*

The general solution of a differential equation cannot always be expressed by a single formula. As an example consider the differential equation $y' + y^2 = 0$, which has two particular solutions $y = 1/x$ and $y \equiv 0$.

INITIAL-VALUE AND BOUNDARY-VALUE PROBLEMS

A differential equation along with subsidiary conditions on the unknown function and its derivatives, all given at the same value of the independent variable, constitutes an *initial-value problem*. The subsidiary conditions are *initial conditions*. If the subsidiary conditions are given at more than one value of the independent variable, the problem is a *boundary-value problem* and the conditions are *boundary conditions*.

A solution to an initial-value or boundary-value problem is a function $y(x)$ that both solves the differential equation and satisfies all given subsidiary conditions.

7.2 Classifications of First Order Differential Equations

MATHEMATICAL MODELS

Mathematical models can he thought of as equations. In this chapter, and in other parts of this book (sec Chapter 7. Chapter 14 and Chapter 31. for example), we will consider equations which model certain real-world situations.

For example, when considering a simple direct current (DC) electrical circuit, the equation $V = RI$ models the voltage drop (measured in volts) across a resistor (measured in ohms), here I is the current (measured in amperes). This equation is called Ohm's Law. named in honor of G. S. Ohm (1787-1854), a German physicist,

Once constructed, some models can he used to predict main physical situations. for example, weather forecasting, the growth of a tumor, or the outcome of a roulette wheel, can all be connected with some form of mathematical modeling.

In this chapter, we consider variables that are continuous and how *differential equations* can he used in modeling. Chapter 34 introduces the idea of *difference equations.* These are equations in which we consider *discrete* variables; that is, variables which can lake on only certain values, such as whole numbers. With few modifications, everything presented about modeling with differential equations also holds true with regard to modeling with difference equations.

THE "MODELING CYCLE"

Suppose we have a *real-life* situation (we want to find the amount of radio-active material in some element). Research may be able to model this situation (in the form of a "very difficult" differential equation). Technology may be used to help us solve the equation (computer programs give us an answer). The technological answers are then *interpreted.*

QUALITATIVE METHODS

To build a model can be a long and arduous process; it may take many years of research. Once they are formulated, models may be virtually impossible to solve analytically. Then the researcher has two options:

- Simplify, or "tweak", the model so that it can be dealt with in a more manageable way. This is a valid approach, provided the simplification does not overly compromise the "real-world" connection, and therefore, its usefulness
- Retain the model as is and use other techniques, such as numerical or graphical methods (see Chapter 18, Chapter 19, and Chapter 20). This represents a *qualitative* approach. While we do not possess an exact, analytical solution, we do obtain *some* information which can shed *some* light on the model and its application. Technological tools can be extremely helpful with this approach (see Appendix B).

7.3 Separable First Order Differential Equations

GENERAL SOLUTION

The solution to the first-order separable differential equation (see Chapter 3)

$$A(x)dx + B(y)dy = 0 \tag{7.6}$$

is

$$\int A(x)dx + \int B(y)dy = c \tag{7.7}$$

where c represents an arbitrary constant.

The integrals obtained in Eq.(7.7) may be. for all practical purposes, impossible lo evaluate. In such eases. numerical techniques (see Chapters 18, 14. 20) are used to obtain an approximate solution. Even if the indicated integrations in (7.7) can be performed, it may not be algebraically possible lo solve for y explicitly in terms of x. In that case, the solution is left in implicit form.

The solution to the initial-value problem

$$A(x)dx + B(y)dy = 0; \quad y(x_0) = y_0 \tag{7.8}$$

can be obtained, as usual. by first using Eq.(7.7) to solve the differential equation and then applying The initial condition directly lo evaluate r.

Alternatively, the solution lo Eq.(7.8) can be obtained from

$$\int_{x_0}^{x} A(x)dx + \int_{y_0}^{y} B(y)dy = 0 \tag{7.9}$$

Equation (7.9). however, may not determine the solution of (7.8) *uniquely*; that is. (7.9) may have many solutions, of which only one will satisfy the initial-value problem.

REDUCTION OF HOMOGENEOUS EQUATIONS

The homogeneous differential equation

$$\frac{dy}{dx} = f(x, y) \tag{7.10}$$

having the property that $f(tx, ty) = f(x, y)$ (see Chapter 3) can be transformed into a separable equation by making the substitution

$$y = xv \tag{7.11}$$

along with its corresponding derivative

$$\frac{dy}{dx} = v + x\frac{dv}{dx} \tag{7.12}$$

The resulting equation in the variables v and x is solved as a separable differential equation; the required solution to Eq.(7.10) is obtained by back substitution.

Alternatively, the solution to (7.10) can be obtained by rewriting the differential equation as

$$\frac{dx}{dy} = \frac{1}{f(x, y)} \tag{7.13}$$

and then substituting

$$x = yu \tag{7.14}$$

and the corresponding derivative

$$\frac{dx}{dy} = u + y\frac{du}{dy} \tag{7.15}$$

into Eq.(7.13). After simplifying, the resulting differential equation will be one with variables (this time, u and y) separable.

Ordinarily, it is immaterial which method of solution is used. Occasionally, however, one of the substitutions (7.11) or (7.14) is definitely superior to the other one. In such cases, the better substitution is usually apparent from the form of the differential equation itself.

Example 7.3.1. *Solve* $y' = \dfrac{x+1}{y^4+1}$

This equation, in differential form, is $(x+1)dx + (-y^4 - 1)dy = 0$ which is separable. Its solution is

$$\int (x+1)dx + \int (-y^4 - 1)dy = c$$

or, by evaluating,

$$\frac{x^2}{2} + x - \frac{y^5}{5} - y + c$$

Since it is impossible algebraically to solve this equation explicitly for y, the solution must be left in its present implicit form.

Example 7.3.2. *Solve* $y' = \dfrac{2xy}{x^2 - y^2}$

Phis differential equation is not separable. Instead it has the form $y' = f(x, y)$, with

$$f(x, y) = \frac{2xy}{x^2 - y^2}$$

where

$$f(tx, ty) = \frac{2(tx)(ty)}{(tx)^2 - (ty)^2} = \frac{t^2(2xy)}{t^2(x^2 - y^2)} = \frac{2xy}{x^2 - y^2} = f(x, y)$$

so it is homogenous. Substituting Eqs.(7.11) and (7.12) into the differential equation as originally given, we obtain

$$v + x\frac{dv}{dx} = \frac{2x(xy)}{x^2 - (xy)^2}$$

which can be algebraically simplified to

$$x\frac{dv}{dx} = -\frac{v(v^2+1)}{v^2-1}$$

or

$$\frac{1}{x}dx + \frac{v^2-1}{v(v^2+1)}dv = 0 \tag{7.16}$$

Using partial fractions, we can expand (7.16) to

$$\frac{1}{x}dx + \left(-\frac{1}{v} + \frac{2v}{v^2+1}\right)dv = 0 \tag{7.17}$$

The solution to this separable equation is found by integrating both sides of (7.17). Doing so, we obtain $\ln|x| - \ln|v| + \ln(v^2+1) = c$, which can be simplified to

$$x(v^2+1) = kv \quad (c = \ln|k|) \tag{7.18}$$

Substituting $v = y/x$ into (7.18), we find the solution of the given differential equation is $x^2 + y^2 = ky$.

7.4 Exact First Order Differential Equations

DEFINING PROPERTIES

A differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{7.19}$$

is *exact* if there exists a function $g(x,y)$ such that

$$dg(x,y) = M(x,y)dx + N(x,y)dy \tag{7.20}$$

Test for exactness: If $M(x,y)$ and $N(x,y)$ are continuous functions and have continuous first partial derivatives on some rectangle of the xy-plane. then (7.19) is exact if and only if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \tag{7.21}$$

METHOD OF SOLUTION

To solve Eq.(7.19). assuming that it is exact, first solve the equations

$$\frac{\partial g(x,y)}{\partial x} = M(x,y) \tag{7.22}$$

$$\frac{\partial g(x,y)}{\partial y} = N(x,y) \tag{7.23}$$

for $g(x,y)$. The solution to (7.19) is then given implicitly In

$$g(x,y) = c \tag{7.24}$$

where c represents an arbitrary constant.

Equation (7.24) is immediate from Eqs. (7.19) and (7.20). If (7.20) is substituted into If (7.19) we obtain $dg(x,y(x)) = 0$. Integrating this equation (note that we can write 0 as 0 dx), we have $\int dg(x, y(x)) = \int 0\ dx$, which, in turn, implies (7.24).

INTEGRATING FACTORS

In general, Eq. (7.19) is not exact. Occasionally, it is possible to transform (7.19) into an exact differential equation by a judicious multiplication. A function $I(x, y)$ is an integrating factor for (7.19) if the equation

$$I(x, y)[M(x, y) + N(x, y)] = 0 \tag{7.25}$$

is exact. A solution to (7.19) is obtained by solving the exact differential equation defined by (7.25). Some of the more common integrating factors are displayed in Table 7.4 and the conditions that follow:

if $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) \equiv g(x)$, a function of x alone, then

$$I(x, y) = e^{\int g(x)dx} \tag{7.26}$$

if $\dfrac{1}{M}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right) \equiv g(x)$, a function of y alone, then

$$I(x, y) = e^{-\int h(y)dy} \tag{7.27}$$

Group of terms	Integrating factor $I(x,y)$	Exact differential $dg(x,y)$
$y\,dx - x\,dy$	$-\frac{1}{x^2}$	$\frac{x\,dy - y\,dx}{x^2} = d(\frac{y}{x})$
$y\,dx - x\,dy$	$\frac{1}{y}$	$\frac{y\,dx - x\,dy}{y^2} = d(\frac{x}{y})$
$y\,dx - x\,dy$	$-\frac{1}{xy}$	$\frac{x\,dy - y\,dx}{xy} = d(\ln\frac{y}{x})$
$y\,dx - x\,dy$	$-\frac{1}{x^2+y^2}$	$\frac{x\,dy - y\,dx}{x^2+y^2} = d(arctan\,\frac{y}{x})$
$y\,dx + x\,dy$	$\frac{1}{xy}$	$\frac{y\,dx + x\,dy}{xy} = d(\ln xy)$
$y\,dx + x\,dy$	$\frac{1}{(xy)^n}, n > 1$	$\frac{y\,dx + x\,dy}{(xy)^n} = d\left[\frac{-1}{(n-1)(xy)^{n-1}}\right]$
$y\,dx + x\,dy$	$\frac{1}{x^2+y^2}$	$\frac{y\,dy + x\,dx}{x^2+y^2} = d\left[\frac{1}{2}\ln x^2 + y^2\right]$
$y\,dx + x\,dy$	$\frac{1}{(x^2+y^2)^n}, n > 1$	$\frac{y\,dy + x\,dx}{(x^2+y^2)^n} = d\left[\frac{-1}{2(n-1)(x^2+y^2)^{n-1}}\right]$
$ay\,dx + bx\,dy$	$x^{a-1}y^{b-1}$	$x^{a-1}y^{b-1}(ay\,dx + bx\,dy) = d(x^a y^b)$

Table 7.1: Common integrating factors

if $M = yf(xy)$ and $n = xg(xy)$, then

$$I(x,y) = \frac{1}{xM - yN} \tag{7.28}$$

In general, integrating factors are difficult to uncover. If a differential equation does not have one of the forms given above, then a search for an integrating factor likely will not be successful, and other methods of solution are recommended

Example 7.4.1. *Determine whether the differential equation* $2xydx + (1 + x^2)dy = 0$ *is exact. and solve it*

This equation has the form of Eq. (7.19) with $M(x, y) = 2xy$ and $N(x, y) = 1 + x^2$. Since $\partial M/\partial y = \partial N/\partial x = 2x$, the differential equation is exact.

This equation was shown to be exact. We now determine a function $g(x, y)$ that satisfies Eqs. (7.22) and (7.23). Substituting $M(x, y) = 2xy$

into (7.22), we obtain $\partial g/\partial x = 2xy$. Integrating both sides of this equation with respect to x, we find

$$\int \frac{\partial g}{\partial x}\, dx = \int 2xy\, dx$$

or,

$$g(x, y) = x^2 y + h(y) \tag{7.29}$$

Note that when integrating with respect to x, the constant (*with respect to* x) of integration can depend on y.

We now determine $h(y)$. Differentiating (7.29) with respect to y, we obtain $\partial g/\partial y = x^2 + h'(y)$. Substituting this equation along with $N(x, y) = 1 + x^2$ into (7.23), we have

$$x^2 + h'(y) = 1 + x^2 \quad \text{or} \quad h'(y) = 1$$

Integrating this last equation with respect to y, we obtain $h(y) = y + c_1$ (c_1 = constant). Substituting this expression into (7.29) yields

$$g(x, y) = x^2 y + y + c_1$$

The solution to the differential equation, which is given implicitly by (7.24) as $g(x, y) = c$, is

$$x^2 y + y = c_2 \quad (c_2 = c - c_1)$$

Solving for y explicitly, we obtain the solution as $y = c_2/(x^2 + 1)$.

Example 7.4.2. *Solve* $y' = \dfrac{2 + ye^{xy}}{2y - xe^{xy}}$

Rewriting this equation in differential form, we obtain

$$(2 + ye^{xy})dx + (xe^{xy} - 2y)dy = 0$$

Here, $M(x, y) = 2 + ye^{xy}$ and $N(x, y) = xe^{xy} - 2y$ and, since $\partial M/\partial y = \partial N/\partial x = e^{xy} + xye^{xy}$ the differential equation is exact. Substituting $M(x, y)$ into (7.22), we find $\partial g/\partial x = 2+ye^{xy}$; then integrating with respect to x, we obtain

$$\int \frac{\partial g}{\partial x} dx = \int [2 + ye^{xy}] dx$$

or

$$g(x, y) = 2x + e^{xy} + h(y) \tag{7.30}$$

To find $h(y)$, first differentiate (7.30) with respect to y, obtaining $\partial g/\partial y = xe^{xy} + h'(y)$; then substitute this result along with $N(x, y)$ into (7.23) to obtain

$$xe^{xy} + h'(y) = xe^{xy} - 2y \quad or \quad h'(y) = -2y$$

It follows that $h(y) = -y^2 + c_1$. Substituting this $h(y)$ into (7.30), we obtain

$$g(x, y) = 2x + e^{xy} - y^2 + c_1$$

The solution to the differential equation is given implicitly by (7.24) as

$$2x + e^{xy} - y^2 = c_2 \quad (c_2 = c - c_1)$$

7.5 Linear First Order Differential Equations

METHOD OF SOLUTION

A first-order linear differential equation has the form (see Chapter 3)

$$y' + p(x)y = q(x) \tag{7.31}$$

An integrating factor for Eq. (7.31) is

$$I(x) = e^{\int p(x)dx} \tag{7.32}$$

which depends only on y and is independent of v. When both sides of (7.31) are multiplied by $I(x)$, the resulting equation

$$I(x)y' + p(x)I(x)y = I(x)q(x) \tag{7.33}$$

is *exact*. This equation can he solved by the method described in Chapter 5. A simpler procedure is to rewrite (7.33) as

$$\frac{d(yI)}{dx} = Iq(x) \tag{7.34}$$

integrate both sides of this last equation with respect to x, and then solve the resulting equation for Y.

REDUCTION OF BERNOULLI EQUATIONS

A Bernoulli differential equation has the form

$$y' + p(x)y = q(x)y^n \tag{7.35}$$

where n is a real number. The substitution

$$z = y^{1-n} \tag{7.36}$$

transforms (7.35) into a linear differential equation in ihe unknown function $z(x)$.

Example 7.5.1. *Find an integrating factor for $y' - 3y = 6$.*

The differential equation has the form of Eq. 7.31), with $p(x) = -3$ and $q(x) = 6$, and is linear. Here

$$\int p(x)dx = \int -3dx = -3x$$

so (7.32) becomes

$$I(x) = e^{\int p(x)dx} = e^{-3x}$$

Example 7.5.2. *Solve* $\dfrac{dz}{dx} - xz = -x$

This is a linear differential equation for the unknown function $z(x)$. It has the form of Eq. (7.31) with y replaced by z and $p(x) = q(x) = -x$. The integrating factor is

$$I(x) = e^{\int(-x)dx} = e^{-x^2/2}$$

Multiplying the differential equation by $I(x)$, we obtain

$$e^{-x^2/2}\frac{dz}{dx} - xe^{-x^2/2}z = -xe^{-x^2/2}$$

or

$$\frac{dz}{dx}(ze^{-x^2/2}) = -xe^{-x^2/2}$$

Upon integrating both sides of this last equation, we have

$$ze^{-x^2/2} = -e^{-x^2/2} + c$$

whereupon

$$z(x) = ce^{x^2/2} + 1$$

7.6 Applications of First Order Differential Equations

GROWTH AND DECAY PROBLEMS

Let $N(t)$ denote the amount of substance (or population) that is either growing or decaying. It' we assume that dN/dt. the lime rate of change of this amount of substance, is proportional to the amount of substance present. Then $dN/dt = kN$. or

$$\frac{dN}{dt} - kN = 0 \tag{7.37}$$

where k is the constant of proportionality.

We are assuming that $N(t)$ is a differentiable, hence continuous, function of time. For population problems, where $N(t)$ is actually discrete and integer-valued, this assumption is incorrect. Nonetheless, (7.37) still provides a good approximation to the physical laws governing such a system.

TEMPERATURE PROBLEMS

Newton's law of cooling, which is equally applicable lo healing, states that *the time rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium.* Let T denote the temperature of the body and let T_m denote the temperature of the surrounding medium. Then the time rate of change of Ihe temperature of the body is dT/dt, and Newton's law of cooling can be formulated as $dT/dt = -k(T - T_m)$. or as

$$\frac{dT}{dt} + kT = kT_m \tag{7.38}$$

where k is a positive constant of proportionality Once k is chosen positive, the minus sign is required in Newton's law to make dT/dt negative in a cooling process, when T is greater than T_m. and positive in a heating process, when TMs less than T_m

FALLING BODY PROBLEMS

Consider a vertically falling body of mass m that is being influenced only by gravity g and an air resistance that is proportional to the velocity of the body. Assume that both gravity and mass remain constant and, for convenience, choose the downward direction as the positive direction.

Newton's second law of motion: *The net force acting on a body is equal to the time rate of change of the momentum of the body; or, for constant mass,*

$$F = m\frac{dv}{dt} \tag{7.39}$$

where F is the net force on the body and v is the velocity of the body, both at time t.

For the problem at hand, there are two forces acting on the body: (1) the force due to gravity given by the weight w of the body, which equals mg, and (2) the force due to air resistance given by $-kv$, where $k > 0$ is a constant of proportionality. The minus sign is required because this force opposes the velocity; that is, it acts in the upward, or negative, direction. The net force F on the body is, therefore, $F = mg - kv$. Substituting this result into (7.39), we obtain

$$mg - kv = m\frac{dv}{dt}$$

or

$$\frac{dv}{dt} + \frac{k}{m}v = g \tag{7.40}$$

as the equation of motion for the body.

If air resistance is negligible or nonexistent, then $k = 0$ and (7.40) simplifies to

$$\frac{dv}{dt} = g \tag{7.41}$$

When $k > 0$, the limiting velocity v_l is defined by

$$v_l = \frac{mg}{k} \tag{7.42}$$

Caution: Equations (7.40), (7.41), and (7.42), are valid only if the given conditions are satisfied. These equations are not valid if, for example, air resistance is not proportional to velocity but to the velocity squared, or if the upward direction is taken to be the positive direction.

Consider a tank which initially holds V_0 gal of brine that contains a lb of salt. Another brine solution, containing b lb of salt per gallon, is poured into the tank at the rate of e gal/min while, simultaneously, the well-stirred solution leaves the tank at the rate of f gal/min . The problem is to find the amount of salt in the tank at any time t.

Let Q denote the amount (in pounds) of salt in the tank at any time t. The time rate of change of Q, dQ/dt, equals the rate at which salt enters the tank minus the rate at which salt leaves the tank. Salt enters the tank at the rate of be lb/min. To determine the rate at which salt leaves the tank, we first calculate the volume of brine in the tank at any time t, which is the initial volume V_0 plus the volume of brine added et minus the volume of brine removed ft . Thus, the volume of brine at any time is

$$V_0 + et - ft \tag{7.43}$$

The concentration of salt in the tank at any time is $Q/(V_0 + et - ft)$, from which it follows that salt leaves the tank at the rate of

$$f\left(\frac{Q}{V_0 + et - ft}\right) \text{ lb/min}$$

Thus,

$$\frac{dQ}{dt} = be - f\left(\frac{Q}{V_0 + et - ft}\right)$$

or

$$\frac{dQ}{dt} + f\left(\frac{Q}{V_0 + et - ft}\right) = be \tag{7.44}$$

ELECTRICAL CIRCUITS

The basic equation governing the amount of current I (in amperes) in a simple RL circuit consisting of a resistance R (in ohms), an inductor L (in henries), and an electromotive force (abbreviated emf) E (in volts) is

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{E}{L} \tag{7.45}$$

For an RC circuit consisting of a resistance, a capacitance C (in farads), an emf, and no inductance, the equation governing the amount of electrical charge q (in coulombs) on the capacitor is

$$\frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{r} \tag{7.46}$$

The relationship between q and I is

$$i = \frac{dq}{dt} \tag{7.47}$$

ORTHOGONAL TRAJECTORIES

Consider a one-parameter family of curves in the xy-plane defined by

$$F(x, y, c) = 0 \tag{7.48}$$

where c denotes the parameter. The problem is to find another one-parameter family of curves, called the *orthogonal trajectories* of the family (7.48) and given analytically by

$$G(x, y, k) = 0 \tag{7.49}$$

such that every curve in this new family (7.49) intersects at right angles every curve in the original family (7.48).

We first implicitly differentiate (7.48) with respect to x, then eliminate c between this derived equation and (7.48). This gives an equation connecting x, y, and y', which we solve for y' to obtain a differential equation of the form

$$\frac{dy}{dx} = f(x, y) \tag{7.50}$$

The orthogonal trajectories of (7.48) are the solutions of

$$\frac{dx}{dy}x = \frac{1}{f(x, y)} \tag{7.51}$$

For many families of curves, one cannot explicitly solve for dy/dx and obtain a differential equation of the form (7.50). We do not consider such curves in this book.

Example 7.6.1. *What constant interest rate is required if an initial deposit placed into an account that accrues interest compounded continuously is to double its value in six years?*

The balance $N(t)$ in the account at any time t is governed by (7.37)

$$\frac{dN}{dt} - kN = 0$$

which has as its solution

$$N(t) = ce^{kt} \tag{7.52}$$

We are not given an amount for the initial deposit, so we denote it as N_0. At $t = 0$, $N(0) = N_0$, which when substituted into (7.52) yields

$$N_0 = ce^{k(0)} = c$$

and (7.52) becomes

$$N(t) = N_0e^{kt} \tag{7.53}$$

We seek the value of k for which $N = 2N_0$ when $t = 6$. Substituting these values into (7.53) and solving for k, we find

$$2N_0 = N_0 e^{k*6}$$
$$e^{6k} = 2$$
$$6k = \ln|2|$$
$$k = \frac{1}{6}\ln|2| = 0.1155$$

Example 7.6.2. *An RL circuit has an emf given (in volts) by* $3\sin 2t$*, a resistance of* 10 *ohms, an inductance of* 0.5 *henry, and an initial current of* 6 *amperes. Find the current in the circuit at any time* t*.*

Here, $E = 3\sin 2t$, $R = 10$, and $L = 0.5$; hence (7.45) becomes

$$\frac{dI}{dt} + 20I = 6\sin 2t$$

This equation is linear, with solution (see section 6.5)

$$\int d(Ie^{20t}) + \int 6e^{20t}\sin 2t\, dt$$

Carrying out the integrations (the second integral requires two integrations by parts), we obtain

$$I = ce^{-20t} + \frac{30}{101}\sin 2t - \frac{3}{101}\cos 2t$$

At $t = 0, 1 = 6$; hence,

$$6 = ce^{-20*0} + \frac{30}{101}\sin 2*0 - \frac{3}{101}\cos 2*0 \quad \text{or} \quad 6 = c - \frac{3}{101}$$

whence $c = 609/101$. The current at any time t is

$$I = \frac{609}{101}e^{-20t} + \frac{30}{101}\sin 2t - \frac{3}{101}\cos 2t$$

the current is the sum of a transient current, here $(609/101)e^{-20t}$, and a steady-state current,

$$\frac{30}{101}\sin 2t - \frac{3}{101}\cos 2t$$

7.7 Graphical & Numerical Methods of Solving First Order Differential Equations

QUALITATIVE METHODS

In Chapter 2, we touched upon the concept of qualitative methods regarding differential equations; that is, techniques which are used when analytical solutions are difficult or virtually impossible to obtain. In this chapter, and in the two succeeding chapters, we introduce several qualitative approaches in dealing with differential equations.

DIRECTION FIELDS

Graphical methods produce plots of solutions to first-order differential equations of the form

$$y'(x, y) \tag{7.54}$$

where the derivative appears only on the left side of the equation.

Example 7.7.1.

(a) *For the problem* $y' = -y + x + 2$*, we have* $f(x, y) = -y + x + 2$*.*

(b) *For the problem*$y' = y^2 + 1$*, we have* $f(x, y) = y^2 + 1$*.*

(c) *For the problem* $y' = 3$*, we have* $f(x, y) = 3$*. Observe that in a particular problem,* $f(x, y)$ *may be independent of* x*, of* y*, or of* x *and* y*.*

Equation (7.54) defines the slope of the solution curve $y(x)$ at any point (x, y) in the plane. A line element is a short line segment that begins at the point (x, y) and has a slope specified by (7.54); it represents an approximation to the solution curve through that point. A collection of line elements is a *direction field*. The graphs of solutions to (7.54) are generated from direction fields by drawing curves that pass through the points at which line elements are drawn and also are tangent to those line elements.

If the left side of Eq. (7.54) is set equal to a constant, the graph of the resulting equation is called an *isocline*. Different constants define different isoclines, and each isocline has the property that all line elements emanating from points on that isocline have the same slope, a slope equal to the constant that generated the isocline. When they are simple to draw, isoclines yield many line elements at once which is useful for constructing direction fields.

EULER'S METHOD

If an initial condition of the form

$$y(x_0) = y_0 \tag{7.55}$$

is also specified, then the only solution curve of Eq. (7.54) of interest is the one that passes through the initial point (X_0, y_0).

To obtain a graphical approximation to the solution curve of Eqs. (7.54) and (7.55), begin by constructing a line element at the initial point (x_0, y_0) and then continuing it for a short distance. Denote the terminal point of this line element as (x_1, y_1). Then construct a second line element at (x_1, y_1) and continue it a short distance. Denote the terminal point of this second line element as (x_2, y_2). Follow with a third line element constructed at (x_2, y_2) and continue it a short distance. The process proceeds iteratively and concludes when enough of the solution curve has been drawn to meet the needs of those concerned with the problem.

If the difference between successive x values are equal, that is, if for a specified constant h, $h = x_1 - x_0 = x_2 - x_1 = x_3 - x_2 = ...$, then the graphical method given above for a first-order initial-value problem is known as Euler's method. It satisfies the formula

$$y_{n+1} = y_n + hf(x_n, y_n) \tag{7.56}$$

for $n = 1, 2, 3, ...$ This formula is often written as

$$y_{n+1} = y_n + hy'_n \tag{7.57}$$

where

$$y_n' = f(x_n, y_n) \tag{7.58}$$

as required by Eq. (7.54).

STABILITY

The constant h in Eqs. (7.56) and (7.57) is called the step-size, and its value is arbitrary. In general, the smaller the step-size, the more accurate the approximate solution becomes at the price of more work to obtain that solution. Thus, the final choice of h may be a compromise between accuracy and effort. If h is chosen too large, then the approximate solution may not resemble the real solution at all, a condition known as *numerical instability*. To avoid numerical instability, Euler's method is repeated, each time with a step-size one half its previous value, until two successive approximations are close enough to each other to satisfy the needs of the solver.

Chapter 8

Higher Order Differential Equations

8.1 Linear Differential Equations

LINEAR DIFFERENTIAL EQUATIONS

An nth-order linear differential equation has the form

$$b_n(x)y^{(n)} + b_{n-1}(x)y^{(n-1)} + ... + b_2(x)y'' + b_1(x)y' + b_0(x)y = g(x) \tag{8.1}$$

where $g(x)$ and the coefficients $b_j(x)$ $(j = 0, 1, 2, ..., n)$ depend solely on the variable x. In other words, they do not depend on y or on any derivative of y,

If $g(x) \equiv 0$, then Eq (8.1) is *homogeneous*; if not, (8.1) is *nonhomogeneous.* A linear differential equation has constant coefficients if all the coefficients $b_j(x)$ in (8.1) are constants; if one or more of these coefficients is not constant, (8.1) has *variable coefficients.*

Theorem 8.1.1. *Consider the initial-value problem given by the linear differential equation (8.1) and the n initial conditions*

$$y(x_0) = c_0, \quad y'(x_0) = c_1, \quad y''(x_0) = c_2, ..., y^{(n-1)}(x_0) = c_{n-1} \tag{8.2}$$

If $g(x)$ and $b_j(x)$ $(j = 0, 1, 2, ..., n)$ are continuous in some interval $\mathcal{I}$ containing x_0 and if $b_n(x) \neq 0$ in $\mathcal{I}$, then the initial-value problem given by (8.1) and (8.2) has a unique (only one) solution defined throughout $\mathcal{I}$.

When the conditions on $b_n(x)$ in Theorem 8.1.1 hold, we can divide Eq (8.1) by $b_n(x)$ to get

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_2(x)y'' + a_1(x)y' + a_0(x)y = \phi(x) \quad (8.3)$$

where $a_j(x) = b_j(x)/b_n(x)$ $(j = 0, 1, 2, ..., n-1)$ and $\phi(x) = g(x)/b_n(x)$

Let us define the differential operator $\mathbf{L}(y)$ by

$$\mathbf{L}(y) \equiv y^{(n)} + a_{n-1}(x)y^{(n-1)} + ... + a_2(x)y'' + a_1(x)y' + a_0(x)y \quad (8.4)$$

where $a_i(x)$ $(i = 0, 1, 2, ..., n - 1)$ is continuous on some interval of interest. Then (8.3) can be rewritten as

$$\mathbf{L}(y) = \phi(x) \quad (8.5)$$

and, in particular, a linear *homogeneous* differential equation can he expressed as

$$\mathbf{L}(y) = 0 \quad (8.6)$$

LINEARLY INDEPENDENT SOLUTIONS

A set of functions $\{y_1(x), y_2(x), ..., y_n(x)\}$ is linearly dependent on $a \leq x \leq b$ if there exist constants c_1, c_2 , ... , c_n , *not all zero*, such that

$$c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x) \equiv 0 \quad (8.7)$$

on $a \leq x \leq b$

Example 8.1.1. *The set $\{x, 5x, 1, \sin x\}$ is linearly dependent on $[-1, 1]$ since there exist constants $c_1 = -5$, $c_2 = 1$, $c_3 = 0$, and $c_4 = 0$, not all zero, such that (8.7) is satisfied. In particular,*

$$-5 \cdot x + 1 \cdot 5x + 0 \cdot 1 + 0 \cdot \sin x \equiv 0$$

Note that $c_1 = c_2 = ... = c_n = 0$ is a set of constants that always satisfies (8.7). A set of functions is linearly dependent if there exists *another* set of constants, *not all zero*, that also satisfies (8.7). If the only solution to (8.7) is $c_1 = c_2 = ... = c_n = 0$, then the set of functions $\{y_1(x), y_2(x), ..., y_n(x)\}$ is *linearly independent* on $a \leq x \leq b$.

Theorem 8.1.2. *The nth-order linear* homogeneous *differential equation* $\mathbf{L}(y) = 0$ *always has* n *linearly independent solutions. If* $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ *represent these solutions, then the general solution of* $\mathbf{L}(y) = 0$ *is*

$$y(x) = c_1y_1(x) + c_2y_2(x) + ... + c_ny_n(x) \tag{8.8}$$

where c_1, c_2, ... , c_n *denote arbitrary constants.*

THE WRONSKIAN

The *Wronskian* of a set of functions $\{z_1(x), z_2(x), ..., z_n(x)\}$ on the interval $a \leq x \leq b$, having the property that each function possesses $n - 1$ derivatives on this interval, is the determinant

$$W(z_1, z_2, ..., z_n) = \begin{vmatrix} z_1 & z_2 & \cdots & z_n \\ z_1' & z_2' & \cdots & z_n' \\ z_1'' & z_2'' & \cdots & z_n'' \\ \vdots & \vdots & & \vdots \\ z_1^{(n-1)} & z_2^{(n-1)} & \cdots & z_n^{(n-1)} \end{vmatrix} \tag{8.9}$$

Theorem 8.1.3. *If the Wronskian of a set of* n *functions defined on the interval* $a \leq x \leq b$ *is nonzero for at least one point in this interval, then the set of functions is linearly independent there. If the Wronskian is identically zero on this interval and if each of the functions is a solution to the same linear differential equation, then the set of functions is linearly dependent.*

Caution: Theorem 8.1.3 is silent when the Wronskian is identically zero and the functions are not known to be solutions of the same linear differential equation. In this case, one must test directiy whether Eq. (8.7) is satisfied.

NON-HOMOGENEOUS EQUATIONS

Let y_p denote any *particular* solution of Eq. (8.5) and let y_h (henceforth called the *homogeneous* or *complementary solution*) represent the general solution of the associated homogeneous equation $\mathbf{L}(y) = 0$.

Theorem 8.1.4. *The general solution to* $\mathbf{L}(y) = \phi(x)$ *is*

$$y = y_h + y_p \tag{8.10}$$

Example 8.1.2. *Two solutions of* $y'' - 2y' + y = 0$ *are* e^x *and* xe^x. *Is the general solution* $y = c_1 e^x + c_2 x e^x$*?*

We have

$$W(e^x, xe^x) = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} \not\equiv 0$$

It follows, first from Theorem 8.1.3 that the two particular solutions are linearly independent and then from 8.1.2, that the general solution is

$$y = c_1 e^x + c_2 x e^x$$

Example 8.1.3. *Use the results of Example 8.1.2 to find the general solution of*

$$y'' - 2y' + y = x^2$$

if it is known that $x^2 + 4x + 6$ *is a particular solution.*

We have from Example 8.1.2 that the general solution to the associated homogeneous differential equation is

$$y_h = c_1 e^x + c_2 x e^x$$

Since we are given that $y_p = x^2 + 4x + 6$, it follows from Theorem 8.1.4 that

$$y = y_h + y_p = c_1 e^x + c_2 x e^x + x^2 + 4x + 6$$

8.2 Second Order Linear Homogeneous Differential Equations

INTRODUCTORY REMARK

Thus far we have concentrated on first-order differential equations. We will now turn our attention to the second-order case. After investigating solution techniques, we will discuss applications of these differential equations.

THE CHARACTERISTIC EQUATION

Corresponding to the differential equation

$$y'' + a_1 y' + a_0 y = 0 \tag{8.11}$$

in which a_1 and a_0 are constants, is the algebraic equation

$$\lambda^2 + a_1 \lambda + a_0 = 0 \tag{8.12}$$

which is obtained from Eq (8.11) by replacing y'', y' and y by λ^2, λ^1 and $\lambda^0 = 1$, respectively. Equation (8.12) is called the *characteristic equation* of (8.11).

Example 8.2.1. *The characteristic equation of* $y'' + 3y' - 4y = 0$ *is* $\lambda^2 + 3\lambda - 4 = 0$*; the characteristic equation of* $y'' - 2y' + y = 0$ *is* $\lambda^2 - 2\lambda + 1 = 0$.

Characteristic equations for differential equations having dependent variables other than y are obtained analogously, by replacing the jth derivative of the dependent variable by λ^j $(j = 0, 1, 2)$.

The characteristic equation can be factored into

$$(\lambda - \lambda_1)(\lambda - \lambda_2) = 0 \tag{8.13}$$

THE GENERAL SOLUTION

The general solution of (8.11) is obtained directly from the roots of (8.13). There are three cases to consider.

Case 1. $\boldsymbol{\lambda_1}$ and $\boldsymbol{\lambda_2}$ both real and distinct. Two linearly independent solutions are $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$, and the general solution is (Theorem 8.1.2)

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \tag{8.14}$$

In the special case $\lambda_2 = -\lambda_1$, the solution (8.14) can be rewritten as $y = k_1 \cosh \lambda_1 x + k_2 \sinh \lambda_1 x$

Case 2. $\boldsymbol{\lambda_1 = a + ib}$, a complex number. Since a_1 and a_0 in (8.11) and (8.12) are assumed real, the roots of (8.12) must appear in conjugate pairs; thus, the other root is $\lambda_2 = a - ib$. Two linearly independent solutions are $e^{(a+ib)x}$ and $e^{(a-ib)x}$, and the general complex solution is

$$y = d_1 e^{(a+ib)x} + d_2 e^{(a-ib)x} \tag{8.15}$$

which is algebraically equivalent to

$$y = c_1 e^{ax} \cos bx + c_2 e^{ax} \sin bx \tag{8.16}$$

Case 3. $\boldsymbol{\lambda_1 = \lambda_2}$. Two linearly independent solutions are $e^{\lambda_1 x}$ and $x e^{\lambda_1 x}$, and the general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x} \tag{8.17}$$

Warning: The above solutions *are not valid* if the differential equation is not linear or does not have constant coefficients. Consider, for example, the equation $y'' - x^2y = 0$. The roots of the characteristic equation are $\lambda_1 = x$ and $\lambda_2 = -x$, but the solution is *not*

$$y = c_1e^{(x)x} + c_2e^{(-x)x} = c_1e^{x^2} + c_2e^{-x^2}$$

Example 8.2.2. *Solve* $y'' - y' - 2y = 0$.

The characteristic equation is $\lambda^2 - \lambda - 2 = 0$, which can be factored into $(\lambda + 1)(\lambda - 2) = 0$. Since the roots $\lambda_1 = -1$ and $\lambda_2 = 2$ are real and distinct, the solution given by (8.14) as

$$y = c_1e^{-x} + c_2e^{2x}$$

Example 8.2.3. *Solve* $y'' - 8y' + 16y = 0$.

The characteristic equation is

$$\lambda^2 - 8\lambda + 16 = 0$$

which can be factored into

$$(\lambda - 4)^2 = 0$$

The roots $\lambda_1 = \lambda_2 = 4$ are real and equal, so the general solution is given by (8.17) as

$$y = c_1e^{4x} + c_2xe^{4x}$$

8.3 n^{th} order Linear Homogeneous Differential Equations

THE CHARACTERISTIC EQUATION

The characteristic equation of the differential equation

$$y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1y' + a_0y = 0 \tag{8.18}$$

with constant coefficients a_j $(j = 0, 1, ..., n-1)$ is

$$\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0 = 0 \tag{8.19}$$

The characteristic equation (8.19) is obtained from (8.18) by replacing $y^{(j)}$ by λ^j $(j = 0,\ 1,\ ...,\ n-1)$. Characteristic equations for differential equations having dependent variables other than y are obtained analogously, by replacing the jth derivative of the dependent variable by λ^j $(j = 0,\ 1,\ ...,\ n-1)$.

Example 8.3.1. *The characteristic equation of* $y^{(4)} - 3y''' + 2y'' - y = 0$ *is* $\lambda^4 - 3\lambda^3 + 2\lambda^2 - \lambda = 0$. *The characteristic equation of*

$$\frac{d^5x}{dt^5} - 3\frac{d^3x}{dt^3} + 5\frac{dx}{dt} - 7x = 0$$

is

$$\lambda^5 - 3\lambda^3 + 5\lambda - 7 = 0$$

Caution: Characteristic equations are only defined for linear homogeneous differential equations with constant coefficients.

THE GENERAL SOLUTION

The roots of the characteristic equation determine the solution of (8.18). If the roots $\lambda_1,\ \lambda_2,\ ...,\ \lambda_n$ are all real and distinct the solution is

$$y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x} + ... + c_ne^{\lambda_n x} \tag{8.20}$$

If the roots are distinct, but some are complex, then the solution is again given by (8.20). Those terms involving complex exponentials can be combined to yield terms involving sines and cosines. If λ_k is a root of multiplicity p [that is, if $(\lambda - \lambda_k)^p$ is a factor of the characteristic equation, but $(\lambda - \lambda_k)^{p+1}$ is not] then there will be p linearly independent solutions associated with λ_k given by $e^{\lambda_k x}$, $xe^{\lambda_k x}$, $x^2 e^{\lambda_k x}$, ... , $x^{p-1}e^{\lambda_k x}$. These solutions are combined in the usual way with the solutions associated with the other roots to obtain the complete solution.

In theory it is always possible to factor the characteristic equation, but in practice this can be extremely difficult, especially for differential equations of high order. In such cases, one must often use numerical techniques to approximate the solutions.

Example 8.3.2. *Solve* $y''' - 6y'' + 11y' - 6y = 0$

The characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$, which can be factored into

$$(\lambda - 1)(\lambda - 2)(\lambda - 3) = 0$$

The roots are $\lambda_1 = 1$, $\lambda_2 = 2$ and $\lambda_3 = 3$; hence the solution is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

Example 8.3.3. *Solve* $y^{(4)} + 8y''' + 24y'' + 32y' + 16y = 0$

The characteristic equation, $\lambda^4 + 8\lambda^3 + 24\lambda^2 + 32\lambda + 16 = 0$, can be factored into $(\lambda + 2)^4 = 0$. Here $\lambda_1 = -2$ is a root of multiplicity four; hence the solution is

$$y = c_1 e^{-2x} + c_2 x e^{-2x} + c_3 x^2 e^{-2x} + c_4 x^3 e^{-2x}$$

8.4 Method of Undetermined Coefficients

The general solution to the linear differential equation $\mathbf{L}(y) = \phi(x)$ is given by Theorem 8.1.4 as $y = y_h + y_p$ where y_p denotes one solution to the differential equation and y_h is Ihe general solution to the associated homogeneous equation, $\mathbf{L}(y) = 0$. Methods for obtaining y_h when the differential equation has constant coefficients are given in the previous sections. In this chapter and the next, we give methods for obtaining a particular solution y_p *once* y_h *is known.*

SIMPLE FORM OF THE METHOD

The *method of undetermined coefficients* is applicable only if $\phi(x)$ and *all* of its derivatives can be written in terms of the same *finite* set of linearly independent functions. which we denote by $\{y_1(x), y_2(x), ..., y_n(x)\}$. The method is initiated by assuming a particular solution of the form

$$y_p(x) = A_1y_1(x) + A_2y_2(x) + ... + A_ny_n(x)$$

where A_1, A_2, ... , A_n denote arbitrary multiplicative constants. These arbitrary constants are then evaluated by substituting the proposed solution into the given differential equation and equating the coefficients of like terms.

Case 1. $\boldsymbol{\phi(x) = p_n(x)}$**, an** $\boldsymbol{n}$**th degree polynomial in** $\boldsymbol{x}$**.** Assume a solution of the form

$$y_p = A_nx^n + A_{n-1}x^{n-1} + ... + A_1x + A_0 \tag{8.21}$$

where A_j $(j = 0, 1, 2, ..., n)$ is a constant to be determined.

Case 2. $\boldsymbol{\phi(x) = ke^{\alpha x}}$ **where** $\boldsymbol{k}$ **and** $\boldsymbol{\alpha}$ **are known constants.** Assume a solution of the form

$$y_p = Ae^{\alpha x} \tag{8.22}$$

where A is a constant to be determined.

Case 3. $\phi(x) = k_1 \sin\beta x + k_2 \cos\beta x$ where k_1, k_2 and β are known constants. Assume a solution of the form

$$y_p = A\sin\beta x + B\cos\beta x \tag{8.23}$$

where A and B are constants to be determined.

Note: (8.23) in its entirety is assumed even when k_1 or k_2 is zero, because the derivatives of sines or cosines involve both sines and cosines.

GENERALIZATIONS

If $\phi(x)$ is the product of terms considered in Cases 1 through 3, take y_p to be the product of the corresponding assumed solutions and algebraically combine arbitrary constants where possible. In particular, if $\phi(x) = e^{\alpha x}p_n(x)$ is the product of a polynomial with an exponential, assume

$$y_p = e^{\alpha x}(A_n x^n + A_{n-1}x^{n-1} + ... + A_1 x + A_0) \tag{8.24}$$

where A_j is as in Case 1. If, instead, $\phi(x) = e^{\alpha x}p_n(x)\sin\beta x$ is the product of a polynomial, exponential, and sine term, or if $\phi(x) = e^{\alpha x}p_n(x)\cos\beta x$ is the product of a polynomial, exponential, and cosine term, then assume

$$y_p = e^{\alpha x}\sin\beta x(A_n x^n + ... + A_1 x + A_0) + e^{\alpha x}\cos\beta x(B_n x^n + ... + B_1 x + B_0) \tag{8.25}$$

where A_j and B_j $(j = 0, 1, ..., n)$ are constants which still must be determined.

if $\phi(x)$ is the sum (or difference) of terms already considered, then we take y_p to be the sum (or difference) of the corresponding assumed solutions and algebraically combine arbitrary constants where possible.

MODIFICATIONS

If any term of the assumed solution, disregarding multiplicative constants, is also a term of y_h (the homogeneous solution), then the assumed solution must be modified by multiplying it by x^m, where m

is the smallest positive integer such that the product of x^m with the assumed solution has no terms in common with y_h.

LIMITATIONS OF THE METHOD

In general, if $\phi(x)$ is not one of the types of functions considered above, or if the differential equation *does not have constant coefficients*, then other methods apply.

Example 8.4.1. *Solve* $y'' - y' - 2y = 4x^2$

It can be shown that $y_h = c_1e^{-x} + c_2e^{2x}$. Here $\phi(x) = 4x^2$, a second degree polynomial. Using (8.21), we assume that

$$y_p = A_2x^2 + A_1x + A_0$$

Thus, $y_p' = 2A_2x + A_1$ and $y_p'' = 2A_2$. Substituting these results into the differential equation, we have

$$2A_2 - (2A_2x + A_1) - 2(A_2x^2 + A_1x + A_0) = 4x^2$$

or, equivalently,

$$(-2A_2)x^2 + (-2A_2 - 2A_1)x + (2A_2 - A_1 - 2A_0) = 4x^2 + (0)x + 0$$

Equating the coefficients of like powers of x, we obtain

$$-2A_2 = 4 \quad -2A_2 - 2_A1 = 0 \quad 2A_2 - A_1 - 2A_0 = 0$$

Solving this system, we find that $A_2 = -2$, $A_1 = 2$ and $A_0 = -3$. Hence

$$y_p = -2x^2 + 2x - 3$$

and the general solution is

$$y = y_h + y_p = c_1e^{-x} + c_2e^{2x} - 2x^2 + 2x - 3$$

Example 8.4.2. *Solve* $y''' - 6y'' + 11y' - 6y = 2xe^x$

It can be shown that $y_h = c_1e^x + c_2e^{2x} + c_3e^{3x}$. Here $\phi(x) = e^{\alpha x}p_n(x)$, where $\alpha = -1$ and $p_n(x) = 2x$, a first degree polynomial. Using (8.24), we assume that $y_p = e^{-x}(A_1x + A_0)$, or

$$y_p = A_1xe^{-x} + A_0e^{-x}$$

Thus,

$$\begin{aligned} y_p' &= -A_1xe^{-x} + A_1e^{-x} - A_0e^{-x} \\ y_p'' &= A_1xe^{-x} - 2A_1e^{-x} + A_0e^{-x} \\ y_p''' &= -A_1xe^{-x} + 3A_1e^{-x} - A_0e^{-x} \end{aligned}$$

Substituting these results into the differential equation and simplifying, we obtain

$$-24A_1xe^{-x} + (26A_1 - 24A_0)e^{-x} = 2xe^{-x} + (0)e^{-x}$$

Equating coefficients of like terms, we have

$$-24A_1 = 2 \quad 26A_1 - 24A_0 = 0$$

from which $A_1 = -1/12$ and $A_0 = -13/144$.
So,

$$y_p = -\frac{1}{12}xe^{-x} - \frac{13}{144}e^{-x}$$

and the general solution is

$$y = c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{1}{12}xe^{-x} - \frac{13}{144}e^{-x}$$

8.5 Initial Value Problems for Linear Differential Equations

Initial-value problems are solved by applying the initial conditions to the general solution of the differential equation. It must be emphasized that the initial conditions are applied *only* to the general solution and *not* to the homogeneous solution y_h, even though it is y_h that possesses all arbitrary constants that must be evaluated. The one exception is when the general solution is the homogeneous solution; that is, when the differential equation under consideration is itself homogeneous.

Example 8.5.1. *Solve* $y'' - y' - 2y = 4x^2$*;* $y(0) = 1,\ y'(0) = 4$.

The general solution of the differential equation is

$$y = c_1e^{-x} + c_2e^{2x} - 2x^2 + 2x - 3$$

Therefore,

$$y' = -c_1e^{-x} + 2c_2e^{2x} - 4x + 2$$

Applying the first initial condition to the equation for y, we obtain

$$y(0) = c_1e^{-(0)} + c_2e^{2(0)} - 2(0)^2 + 2(0) - 3 = 1 \quad \text{or} \quad c_1 + c_2 = 4$$

Applying the second initial condition to the equation for y', we obtain

$$y'(0) = -c_1e^{-(0)} + 2c_2e^{2(0)} - 4(0) + 2 = 0 \quad \text{or} \quad -c_1 + 2c_2 = 2$$

Solving the two equations above simultaneously, we find that $c_1 = 2$ and $c_2 = 2$. Substituting these values into the general solution, we obtain the solution of the initial-value problem as

$$y = 2e^{-x} + 2e^{2x} - 2x^2 + 2x - 3$$

Example 8.5.2. *Solve* $y'' + 4y' + 8y = \sin x$*;* $y(0) = 1,\ y'(0) = 0$

Here $y_h = e^{-2x}(c_1\cos 2x + c_2\sin 2x)$, and, by the method of undetermined coefficients,

$$y_p = \frac{7}{65}\sin x - \frac{4}{65}\cos x$$

Thus, the general solution to the differential equation is

$$y = e^{-2x}(c_1 \cos 2x + c_2 \sin 2x) + \frac{7}{65}\sin x - \frac{4}{65}\cos x$$

Therefore,

$$\begin{aligned} y' = -2e^{-2x}(c_1 \cos 2x + c_2 \sin 2x) + e^{-2x}(-2c_1 \sin 2x + 2c_2 \cos 2x) \\ + \frac{7}{65}\cos x + \frac{4}{65}\sin x \end{aligned}$$

Applying the first initial condition to the general solution, we obtain

$$c_1 = \frac{69}{65}$$

Applying the second initial condition to the general solution, we obtain

$$-2c_1 + c_2 = -\frac{7}{65}$$

Solve the two equations above simultaneously, we find that $c_1 = 69/65$ and $c_2 = 131/130$. Substituting these values into the general solution, we obtain the solution of the initial-value problem as

$$y = e^{-2x}\left(\frac{69}{65}\cos 2x + \frac{131}{130}\sin 2x\right) + \frac{7}{65}\sin x - \frac{4}{65}\cos x$$

8.6 Application of Second Order Linear Differential Equations

SPRING PROBLEMS

A simple spring system consists of a mass m attached to the lower end of a spring that is itself suspended vertically from a mounting. The system is in its *equilibrium position* when it is at rest. The mass is set in motion by one or more of the following means: displacing the mass from its equilibrium position, providing it with an initial velocity, or subjecting it to an external force $F(t)$.

Hooke's law: *The restoring force F of a spring is equal and opposite to the forces applied to the spring and is proportional to the extension (contraction) l of the spring as a result of the applied force; that is, $F = -kl$, where k denotes the constant of proportionality, generally called the spring constant.*

Example 8.6.1. *A steel ball weighing 128 lb is suspended from a spring, whereupon the spring is stretched 2 ft from its natural length. The applied force responsible for the 2-ft displacement is the weight of the ball, 128 lb. Thus, F = -128 lb. Hooke's law then gives -128 = -k(2), or k = 64 lb/ft.*

For convenience, we choose the downward direction as the positive direction and take the origin to be the center of gravity of the mass in the equilibrium position. We let x denote the displacement of the mass of the spring. We assume that the mass of the spring is negligible and can be neglected and that air resistance, when present, is proportional to the velocity of the mass. Thus, at any time t, there are three forces acting on the system: (1) $F(t)$, measured in the positive direction; (2) a restoring force given by Hooke's law as $F_s = -kx$, $k > 0$; and (3) a force due to air resistance given by $F_a = -ax$, $a > 0$, where a is the constant of proportionality. Note that the restoring force F_s always acts in a direction that will tend to return the system to the equilibrium position: if the mass is below the equilibrium position, then x is positive and $-kx$ is negative; whereas if the mass is above the equilibrium position, then x

is negative and $-kx$ is positive. Also note that because $a > 0$ the force F_a due to air resistance acts in the opposite direction of the velocity and thus tends to retard, or damp, the motion of the mass.

It now follows from Newton's second law that $m\ddot{x} = -kx - a\dot{x} + F(t)$, or

$$\ddot{x} + \frac{a}{m}\dot{x} + \frac{k}{m}x = \frac{F(t)}{m} \tag{8.26}$$

If the system starts at $t = 0$ with an initial velocity v_0 and from an initial position x_0 , we also have the initial conditions

$$x(0) = x_0 \quad \dot{x}(0) = v_0 \tag{8.27}$$

The force of gravity does not explicitly appear in the first equation, but it is present nonetheless. We automatically compensated for this force by measuring distance from the equilibrium position of the spring. If one wishes to exhibit gravity explicitly, then distance must be measured from the bottom end of the *natural length* of the spring. That is, the motion of a vibrating spring can be given by

$$\ddot{x} + \frac{a}{m}\dot{x} + \frac{k}{m}x = g + \frac{F(t)}{m}$$

ELECTRICAL CIRCUIT PROBLEMS

We have a simple electrical circuit consisting of a resistor R in ohms; a capacitor C in farads, an inductor L in henries; and an electromotive force (emf) $E(t)$ in volts, usually a battery or generator, all connected in a series. The current I flowing through the circuit is measured in amperes and the charge q on the capacitor is measured in coulombs.

Kirchoff's loop law: *The algebraic sum of the voltage drops in a simple closed electric circuit is zero.*

It is known that the voltage drops across a resistor, a capacitor, and an inductor are respectively RI, $(1/C)q$, and $L(dI/dt)$ where q is the charge on the capacitor. The voltage drop across an emf is $-E(t)$. Thus, from Kirchhoff s loop law, we have

$$RI + L\frac{dI}{dt} + \frac{1}{C}q - E(t) = 0 \tag{8.28}$$

The relationship between q and I is

$$I = \frac{dq}{dt} \quad \frac{dI}{dt} = \frac{d^2q}{dt^2} \tag{8.29}$$

Substituting these values into the initial equation, we obtain

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{1}{L}E(t) \tag{8.30}$$

The initial conditions for q are

$$q(0) = q_0 \quad \left.\frac{dq}{dt}\right|_{t=0} = I(0) = I_0 \tag{8.31}$$

To obtain a differential equation for the current, we differentiate the initial equation with respect to t and then substitute Eq. (8.29) into the resulting equation to obtain

$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{1}{L}\frac{dE(t)}{dt} \tag{8.32}$$

The first initial condition is $I(0) = I_0$. The second initial condition is obtained from Eq. (8.28) by solving for dI/dt and then setting $t = 0$. Thus,

$$\left.\frac{dI}{dt}\right|_{t=0} = \frac{1}{L}E(0) - \frac{R}{L}I_0 - \frac{1}{LC}q_0 \tag{8.33}$$

An expression for the current can be gotten either by solving Eq. (8.32) directly or by solving Eq. (8.30) for the charge and then differentiating that expression.

BUOYANCY PROBLEMS

Consider a body of mass m submerged either partially or totally in a liquid of weight density ρ. Such a body experiences two forces, a downward force due to gravity and a counter force governed by:

Archimedes' principle: *A body in liquid experiences a buoyant upward force equal to the weight of the liquid displaced by that body.*

Equilibrium occurs when the buoyant force of the displaced liquid equals the force of gravity on the body. The situation for a cylinder of radius r and height H where h units of cylinder height are submerged at equilibrium, is like this: at equilibrium, the volume of water displaced by the cylinder is $\pi r^2 h$, which provides a buoyant force of $\pi r^2 h\rho$ that must equal the weight of the cylinder mg. Thus,

$$\pi r^2 h\rho = mg \tag{8.34}$$

Motion will occur when the cylinder is displaced from its equilibrium position. We arbitrarily take the upward direction to be the positive x-direction. If the cylinder is raised out of the water by $x(t)$ units, then it is no longer in equilibrium. The downward or negative force on such a body remains mg but the buoyant or positive force is reduced to $\pi r^2[h - x(t)]\rho$. It now follows from Newton's second law that

$$m\ddot{x} = \pi r^2[h - x(t)]\rho - mg \tag{8.35}$$

Substituting (8.34) into this last equation, we can simplify it to

$$m\ddot{x} = -\pi r^2 x(t)\rho$$

or

$$\ddot{x} + \frac{\pi r^2 \rho}{m}x = 0 \tag{8.36}$$

CLASSIFYING SOLUTIONS

Vibrating springs, simple electrical circuits, and floating bodies are all governed by second-order linear differential equations with constant coefficients of the form

$$\ddot{x} + a_1\dot{x} + a_0 = f(t) \tag{8.37}$$

For vibrating spring problems defined by Eq. (8.26), $a_1 = a/m$, $a_0 = k/m$, and $f(t) = F(t)/m$. For buoyancy problems defined by Eq.(8.36), $a_1 = 0$, $a_0 = \pi r^2\rho/m$, and $f(t) \equiv 0$. For electrical circuit problems, the independent variable x is replaced either by q in Eq. (8.30) or I in (8.32).

The motion or current in all of these systems is classified as *free* and *undamped* when $f(t) \equiv 0$ and $a_1 = 0$. It is classified as *free* and *damped* when $f(t)$ is identically zero but a_1 is not zero. For damped motion, there are three separate cases to consider, depending on whether the roots of the associated characteristic equation re (1) real and distinct, (2) equal, or (3) complex conjugate. These cases are respectively classified as (1) *overdamped*, (2) *critically damped*, and (3) *oscillatory damped* (or, in electrical problems, *underdamped*). If $f(t)$ is not identically zero, the motion or current is classified as *forced*.

A motion or current is *transient* if it "dies out" (that is, goes to zero) as $t \to \infty$. A *steady-state* motion or current is one that is not transient and does not become unbounded. Free damped systems always yield transient motions, while forced damped systems (assuming the external force to be sinusoidal) yield both transient and steady-state motions.

Free undamped motion defined by Eq. (8.37) with $a_1 = 0$ and $f(t) \equiv 0$ always has solutions of the form

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t \tag{8.38}$$

which defines *simple harmonic motion*. Here c_1, c_2, and ω are constants with ω often referred to as *circular frequency*. The *natural frequency* f is

$$f = \frac{\omega}{2\pi} \tag{8.39}$$

and it represents the number of complete oscillations per time unit undertaken by the solution. The *period* of the system of the time required to complete one oscillation is

$$T = \frac{1}{f} \tag{8.40}$$

Equation (8.38) has the alternate form

$$x(t) = (-1)^k A\cos(\omega t - \phi) \tag{8.41}$$

where the *amplitude* $A = \sqrt{c_1^2 + c_2^2}$, the *phase angle* $\phi = \arctan(c_2/c_1)$, and k is zero when c_1 is positive and unity when c_1 is negative.

Example 8.6.2. *A steel ball weighing* 128 *lb is suspended from a spring, whereupon the spring is stretched* 2 *ft from its natural length. The ball is started in motion with no initial velocity by displacing it* 6 *in above the equilibrium position. Assuming no air resistance, find (a) an expression for the position of the ball at any time t, and (b) the position of the ball at $t = \pi/12$ sec.*

(a) The equation of motion is governed by Eq. (8.26). There is no externally applied force, so $F(t) = 0$, and no resistance from the surrounding medium, so $a = 0$. The motion is free and undamped. Here $g = 32$ ft/sec^2, $m = 128/32 = 4$ slugs, and it follows from Example 8.6.1 that $k = 64$ lb/ft. Equation (8.26) becomes $\ddot{x} + 16x = 0$. The roots of its characteristic equation are $\lambda = \pm 4i$, so its solution is

$$x(t) = c_1\cos 4t + c_2\sin 4t$$

At $t = 0$, the position of the ball is $x_0 = -\frac{1}{2}$ ft (the minus sign is required because the ball is initially displaced *above* the equilibrium position, which is in the *negative* direction). Applying this initial condition to the equation above, we find that

$$-\frac{1}{2} = x(0) = c_1 \cos 0 + c_2 \sin 0 = c_1$$

so the initial equation becomes

$$x(t) = -\frac{1}{2}\cos 4t + c_2 \sin 4t$$

The initial velocity is given as $v_0 = 0$ ft/sec. Differentiating the equation above, we obtain

$$v(t) = \dot{x}(t) = 2\sin 4t + 4c_2 \cos 4t$$

whereupon

$$0 = v(0) = 2\sin 0 + 4c_2 \cos 0 = 4c_2$$

Thus, $c_2 = 0$, and $x(t)$'s expression simplifies to

$$x(t) = -\frac{1}{2}\cos 4t$$

as the equation of motion of the steel ball at any time t.

(b) At $t = \pi/12$,

$$x\left(\frac{\pi}{12}\right) = -\frac{1}{2}\cos\frac{4\pi}{12} = -\frac{1}{4} \text{ ft}$$

Example 8.6.3. *A mass of* $1/4$ *slug is attached to a spring, whereupon the spring is stretched* 1.28 *ft from its natural length. The mass is started in motion from the equilibrium position with an initial velocity of* 4 *ft/sec in the down-ward direction. Find the subsequent motion of the mass if the force due to air resistance is* $-2\dot{x}$ *lb.*

Here $m = 1/4$, $a = 2$, $F(t) \equiv 0$ (there is no external force), and, from Hooke's law, $k = mg/l = (1/4)(32)/1.28 = 6.25$. Equation (8.26) becomes

$$\ddot{x} + 8\dot{x} + 25x = 0$$

The roots of the associated characteristic equation are $\lambda_1 = -4 + 3i$ and $\lambda_2 = -4 - 3i$, which are complex conjugates; hence this problem is an example of oscillatory damped motion. The solution of the equation above is

$$x = e^{-4t}(c_1 \cos 3t + c_2 \sin 3t)$$

The initial conditions are $x(0) = 0$ and $\dot{x}(0) = 4$. Applying these conditions, we find that $c_1 = 0$ and $c_2 = \frac{4}{3}$; thus, $x = \frac{4}{3}e^{-4t} \sin 3t$. Since $x \to 0$ as $t \to \infty$, the motion is transient.

8.7 Reduction of Linear Differential Equation to First Order Equations

AN EXAMPLE

Consider the following second-order differential equation:

$$t^4 \frac{d^2x}{dt^2} + (\sin t)\frac{dx}{dt} - 4x = \ln t \tag{8.42}$$

We see that (8.42) implies

$$\frac{d^2x}{dt^2} = \frac{4}{t^4}x - \frac{\sin t}{t^4}\frac{dx}{dt} + \frac{\ln t}{t^4} \tag{8.43}$$

Since that derivatives can be expressed in many ways — using *primes* or *dots* are but two of them — we let $v = \dfrac{dx}{dt} = x' = \dot{x}$ and $v' = \dfrac{d^2x}{dt^2} = x'' = \ddot{x}$. Then Eq. (8.42) can be written as the following *matrix equation*:

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{4}{t^4} & \frac{-\sin t}{t^4} \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{\ln t}{t^4} \end{bmatrix} \tag{8.44}$$

because $\dot{x} = 0x + 1\dot{v}$ and $\dot{v} = \frac{4}{t^4}x - \frac{\sin t}{t^4}v + \frac{\ln t}{t^4}$. We note, finally, that Eq (8.42) can also be expressed as

$$dx(t)/dt = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) \tag{8.45}$$

Note that if $x(0) = 5$ and $\dot{x}(0) = -12$ in (8.42), then these initial conditions are written as $x(0) = 5$, $v(0) = -12$.

REDUCTION OF AN *n*th-ORDER EQUATION

As in the case of the second-order differential equation, with associated initial conditions, we can recast higher order initial-value problems into a first-order matrix system as illustrated below:

$$b_n(t)\frac{d^n x}{dt^n} + b_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + b_1(t)\dot{x} + b_0(t)x = g(t); \tag{8.46}$$

$$x(t_0) = c_0, \quad x(\dot{t_0}) = c_1, \ldots, x^{(n-1)}(t_0) = c_{n-1} \tag{8.47}$$

with $b_n(t) \neq 0$, can be reduced to the first-order matrix system.

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t) \\ \mathbf{x}(t_0) &= \mathbf{c} \end{aligned} \tag{8.48}$$

where $\mathbf{A}(t)$, $\mathbf{f}(t)$ $\mathbf{c}$, and the initial time t_0 are known. The method of reduction is as follows.

Step 1. Rewrite (8.46) so that $d^n x/dt^n$ appears by itself. Thus,

$$\frac{d^n x}{dt^n} = a_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_1(t)\dot{x} + a_0(t)x + f(t) \tag{8.49}$$

where $a_j(t) = -b_j(t)/b_n(t)$ $(j = 0, 1, \ldots, n-1)$ and $f(t) = g(t)/b_n(t)$.

Step 2. Define n new variables (the same number as the order of the original differential equation); $x_1(t)$, $x_2(t)$, ... , $x_n(t)$, by the equations

$$x_1(t) = x(t), \; x_2(t) = \frac{dx(t)}{dt}, \; x_3(t) = \frac{d^2x(t)}{dt^2}, \ldots, x_n(t) = \frac{d^{n-1}x}{dt^{n-1}} \tag{8.50}$$

These new variables are interrelated by the equations

$$
\begin{aligned}
\dot{x}_1(t) &= x_2(t)\\
\dot{x}_2(t) &= x_3(t)\\
\dot{x}_3(t) &= x_4(t)\\
&\cdots\cdots\cdots\\
\dot{x}_{n-1}(t) &= x_n(t)
\end{aligned}
\tag{8.51}
$$

Step 3. Express dx_n/dt in terms of the new variables. Proceed by first differentiating the last equation of (8.50) to obtain

$$
\dot{x}_n(t) = \frac{d}{dt}\left[\frac{d^{n-1}x(t)}{dt^{n-1}}\right] = \frac{d^n x(t)}{dt^n}
$$

Then, from Eqs. (8.49) and (8.50),

$$
\begin{aligned}
\dot{x}_n(t) &= a_{n-1}(t)\frac{d^{n-1}x(t)}{dt^{n-1}} + ... + a_1(t)\dot{x}(t) + a_0(t)x(t) + f(t)\\
&= a_{n-1}(t)x_n(t) + ... + a_1(t)x_2(t) + a_0(t)x_1(t) + f(t)
\end{aligned}
$$

For convenience, we rewrite this last equation so that $x_1(t)$, appears before $x_2(t)$, etc. Thus,

$$
\dot{x}_n(t) = a_0(t)x_1(t) + a_1(t)x_2(t) + ... + a_{n-1}(t)x_n(t) + f(t) \tag{8.52}
$$

Step 4. Equations (8.51) and (8.52) are a system of first-order linear differential equations in $x_1(t)$, $x_2(t)$, ... , $x_n(t)$. This system is equivalent to the single matrix equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$ if we define

$$
\mathbf{x}(t) \equiv \begin{bmatrix} x_1(t)\\ x_2(t)\\ \vdots\\ x_n(t) \end{bmatrix} \tag{8.53}
$$

$$\mathbf{f}(t) \equiv \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{bmatrix} \tag{8.54}$$

$$\mathbf{A}(t) \equiv \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ a_0(t) & a_1(t) & a_2(t) & a_3(t) & \dots & a_{n-1}(t) \end{bmatrix} \tag{8.55}$$

Step 5. Define

$$\mathbf{c} \equiv \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix}$$

Then the initial conditions (8.47) can be given by the matrix (vector) equation $\mathbf{x}(t_0) = \mathbf{c}$. This last equation is an immediate consequence of Eqs. (8.53), (8.54), and (8.47), since

$$\mathbf{x}(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ \vdots \\ x_n(t_0) \end{bmatrix} = \begin{bmatrix} x(t_0) \\ \dot{x}(t_0) \\ \vdots \\ x^{(n-1)}(t_0) \end{bmatrix} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{n-1} \end{bmatrix} \equiv \mathbf{c}$$

Observe that if no initial conditions are prescribed, Steps 1 through 4 by themselves reduce any linear differential Eqs. (8.46) to the matrix equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$.

REDUCTION OF A SYSTEM

A set of linear differential equations with initial conditions also can be reduced to System (8.48). The procedure is nearly identical to the method for reducing a single equation to matrix form; only Step 2 changes. With a system of equations, Step 2 is generalized so that new variables are defined for each of the unknown functions in the set.

Example 8.7.1. *Put the initial-value problem*

$$\ddot{x} + 2\dot{x} - 8x = e^t; \quad x(0) = 1, \quad \dot{x}(0) = -4$$

into the form of System (8.48).

Following Step 1, we write $\ddot{x} = -2\dot{x} + 8x + e^t$; hence, $a_1(t) = -2$, $a_0(t) = 8$, and $f(t) = e^t$. Then, defining $x_1(t) = x$ and $x_2(t) = \dot{x}$ (the differential equation is second-order, so we need two new variables), we obtain) $\dot{x}_1 = x_2$. Following Step 3, we find

$$\dot{x}_2 = \frac{d^2x}{dt^2} = -2\dot{x} + 8x + e^t = -2x_2 + 8x_1 + e^t$$

Thus,

$$\begin{aligned} \dot{x}_1 &= 0x_1 + 1x_2 + 0 \\ \dot{x}_2 &= 8x_1 - 2x_2 + e^t \end{aligned}$$

These equations are equivalent to the matrix equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$ if we define

$$\mathbf{x}(t) \equiv \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \mathbf{A}(t) \equiv \begin{bmatrix} 0 & 1 \\ 8 & -2 \end{bmatrix} \quad \mathbf{f}(t) \equiv \begin{bmatrix} 0 \\ e^t \end{bmatrix}$$

Furthermore, if we also define $\mathbf{c} \equiv \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, then the initial conditions can be given by $\mathbf{x}(t_0) = \mathbf{c}$, where $t_0 = 0$.

Example 8.7.2. *Convert the differential equation $\ddot{x} - 6\dot{x} + 9x = t$ into the matrix equation*

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$$

Here we omit Step 5, because the differential equation has no prescribed initial conditions. Following Step 1, we obtain

$$\ddot{x} = 6\dot{x} - 9x + t$$

Hence $a_1(t) = 6$, $a_0(t) = -9$, and $f(t) = t$. If we define two new variables, $x_1(t) = x$ and $x_2(t) = \dot{x}$, we have

$$\dot{x}_1 = x_2 \quad \text{and} \quad \dot{x}_2 = \ddot{x} = 6\dot{x} - 9x + t = 6x_2 - 9x_1 + t$$

Thus,

$$\begin{aligned} \dot{x}_1 &= 0x_1 + 1x_2 + 0 \\ \dot{x}_2 &= -9x_1 + 6x_2 + t \end{aligned}$$

These equations are equivalent to the matrix equation $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{f}(t)$ if we define

$$\mathbf{x}(t) \equiv \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \quad \mathbf{A}(t) \equiv \begin{bmatrix} 0 & 1 \\ -9 & 6 \end{bmatrix} \quad \mathbf{f}(t) \equiv \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Appendices

List of Figures