The Gamma Function

- The Gamma function is the function of variable x defined by the integral,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

- It brings together integration by parts and improper and infinite integrals.
- $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx = (n-1)!$
- $\Gamma(n+1) = n!$

Example: Evaluate $\int_0^\infty \sqrt{x}\, e^{\sqrt[6]{x}} dx$

- Let, $\sqrt[6]{x} = t; x = t^6;\ \sqrt{x} = \sqrt{t^6} = t^3$
- Therefore, $dx = 6t^5 dt$
- Now substituting we get,

$$\int_0^\infty t^3\, e^{-t} 6t^5 dt = 6 \int_0^\infty t^{9-1}\, e^{-t} dt$$

- Comparing with the definition of gamma function, we get,

$$6\Gamma(9) = 6\Gamma(8+1) = 6.8! = 241920$$

* $\Gamma\frac{1}{2} = \sqrt{\pi}$ * $\Gamma 1 = 1$ * $\Gamma(n+1) = n\Gamma n$

---

* Find a) $\Gamma(3/2)$ b) $\Gamma(-3/2)$

Solⁿ a) $\Gamma(3/2) = \Gamma(1+1/2) = \frac{1}{2}\Gamma\frac{1}{2}$

$$= \frac{1}{2}\sqrt{\pi} \quad (\text{Ans})$$

b) $\Gamma(n+1) = n\Gamma n$

$$\Rightarrow \Gamma n = \frac{\Gamma(n+1)}{n}$$

$$\Gamma(-3/2) = \frac{\Gamma(-3/2+1)}{-3/2} = -\frac{2}{3}\Gamma(-1/2)$$

$$= \left(-\frac{2}{3}\right)\frac{\Gamma\left(-\frac{1}{2}+1\right)}{-1/2}$$

$$= \left(-\frac{2}{3}\right)\left(-\frac{2}{1}\right)\Gamma\frac{1}{2}$$

$$= \frac{4}{3}\sqrt{\pi} \quad (\text{Ans})$$

Beta Function

- Also known as Euler's Integral of First Kind.

$$\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$$

Symmetry Property

- We have to show, $\beta(x,y) = \beta(y,x)$.
- By definition, we know $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$

$$\boxed{\int_0^a f(x)dx = \int_0^a f(x-a)dx}$$

- Replacing t by $1-t$ in beta function we get,

$$\beta(x,y) = \int_0^1 (1-t)^{x-1}(1-1+t)^{y-1}dt$$

$$=\int_0^1 (1-t)^{x-1}t^{y-1}dt$$
$$=\int_0^1 t^{y-1}(1-t)^{x-1}dt$$
$$= \beta(y,x) \quad \text{[proved]}$$

Relationship between Gamma and Beta Function

- We have to show that, $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.
- By the definition of gamma function we can write that,

$$\Gamma(m) = \int_0^\infty x^{m-1}e^{-x}\,dx$$
$$\Gamma(n) = \int_0^\infty y^{n-1}e^{-y}\,dy$$

- Therefore,

$$\Gamma(m)\Gamma(n) = \int_0^\infty \int_0^\infty x^{m-1} y^{n-1} e^{-x-y}\, dxdy$$

- Substituting, $x = vt \quad and \quad y = v(1-t)$ we get,

$$\Gamma(m)\Gamma(n) = \int_0^1 t^{m-1}(1-t)^{n-1} dt \int_0^\infty v^{m+n-1} e^{-v} dv$$

$$= \beta(m,n).\Gamma(m+n)$$

Therefore, $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ [proved]

Trigonometric representation of Beta Function

- We have to show that,

$$\beta(x,y) = \int_0^{\pi/2} 2\, sin^{2x-1}(t) cos^{2y-1}(t) dt$$

- By definition of beta function we know that,

$$\beta(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1} du$$

- Substituting, $u = sin^2(t), du = 2 sint cost dt$
- Limits of integration will change into,

u	0	1
t	0	$\pi/2$

- Therefore, from the definition of beta function we can write,

$$\beta(x,y) = \int_0^{\pi/2} (sin^2 t)^{x-1}(1 - sin^2 t)^{y-1}. 2 sint cost\, dt$$

$$= \int_0^{\pi/2} sin^{2x-2}(t) cos^{2y-2}(t). 2 sint cost\, dt$$

$$= \int_0^{\pi/2} 2 sin^{2x-2+1}(t) cos^{2y-2+1}(t)\, dt$$

$$= \int_0^{\pi/2} 2\, sin^{2x-1}(t) cos^{2y-1}(t) dt \quad \text{[proved]}$$

Example on Beta Function

- Evaluate $\int_0^3 x^{1/2}(27-x^3)^{-1/2}dx$.

  Solution:

$$\int_0^3 x^{1/2}(27-x^3)^{-1/2}dx$$

$$=\int_0^3 x^{1/2}27^{-1/2}(1-\left(\frac{x}{3}\right)^3)^{-1/2}dx$$

- Substituting, $u=\frac{x}{3}, x=3u, dx=3du$
- Limits of the integration will change into,

x	0	3
u	0	1

- Therefore,

$$=\int_0^3 x^{1/2}27^{-1/2}(1-\left(\frac{x}{3}\right)^3)^{-1/2}dx$$

$$=27^{-1/2}\int_0^3 (3u)^{1/2}(1-u^3)^{-1/2}.3du$$

$$=3.27^{-1/2}.3^{1/2}\int_0^1 u^{1/2}(1-u^3)^{-1/2}du$$

$$=\int_0^1 u^{1/2}(1-u^3)^{-1/2}du$$

- Again substituting, $u^3=t, u=t^{1/3}, du=\frac{1}{3}t^{-2/3}$
- Limits of the integration will change into,

u	0	1
t	0	1

Therefore,

$$\int_0^1 u^{1/2}(1-u^3)^{-1/2}du$$
$$= \int_0^1 (t^{1/3})^{1/2}(1-t)^{-1/2}\frac{1}{3}t^{-2/3}dt$$
$$= \frac{1}{3}\int_0^1 t^{\frac{1}{6}-\frac{2}{3}}(1-t)^{-1/2}dt$$
$$= \frac{1}{3}\int_0^1 t^{-\frac{1}{2}}(1-t)^{-1/2}dt$$
$$= \frac{1}{3}\int_0^1 t^{\frac{1}{2}-1}(1-t)^{\frac{1}{2}-1}dt$$
$$= \frac{1}{3}\beta(\frac{1}{2},\frac{1}{2})$$
$$= \frac{1}{3}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2})}$$
$$= \frac{1}{3}\frac{\sqrt{\pi}\sqrt{\pi}}{\Gamma(1)} \quad \text{[Since, } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\text{]}$$
$$= \frac{1}{3}\pi \quad \text{[Since, } \Gamma(1) = 1\text{]} \quad \text{(ans)}$$