Differential Equations

containing the derivative of one or more dependent variables with respect to one or more independent variables.

For example,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$y\frac{dy}{dx} = 2 \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$x\frac{dy}{dx} = y - 1 \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 2\frac{\partial u}{\partial t}$$

Classification

Differential Equations are classified by :

- Type
- Order
- Linearity

Classification by Type:

Ordinary Differential Equation

involves ordinary derivatives of one or more dependent variables with respect to a single independent variable.

For example,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \qquad \frac{dy}{dx} = 2xy \qquad x\frac{dy}{dx} = y - 1$$

Partial Differential Equation

involves partial derivatives of one or more dependent variables with respect to more than one independent variable.

For example,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 2\frac{\partial u}{\partial t}$$

Classification by Order:

- The order of the differential equation (either ODE or PDE) is the order of the highest derivative in the equation.
- The highest power of the highest ordered derivative is the degree of the differential equation (ODE or PDE).

$$\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = 0$$ Order = 3, Degree = 1

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0$$ Order = 2, Degree = 1

$$x\left(\frac{dy}{dx}\right)^2 = y - 1$$ Order = 1, Degree = 2

Classification by Linearity:

Linear Ordinary Differential Equation

- dependent variable y and its various derivatives occur to the first degree only.
- no products of y and/or any of its derivatives are present.
- no transcendental functions of y and/or its derivative occur.

For example,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = 0 \qquad\qquad x\frac{dy}{dx} = y - 1$$

Nonlinear Ordinary Differential Equation

ordinary differential equation which is not linear

For example,

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + y^2 = 0 \qquad \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y = 0 \qquad \frac{d^2y}{dx^2} + y\frac{dy}{dx} + y = 0$$

Initial Value Problem

We are often interested in problems in which we seek a solution $y(x)$ of a differential equation so that $y(x)$ satisfies prescribed side conditions—that is, conditions imposed on the unknown $y(x)$ or its derivatives. On some interval I containing x_0 the problem

$$\begin{aligned} &\textit{Solve:} && \frac{d^ny}{dx^n} = f\left(x, y, y', \ldots, y^{(n-1)}\right) \\ &\textit{Subject to:} && y(x_0) = y_0,\ y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}, \end{aligned} \tag{1}$$

where y_0, $y_1, \ldots, y_{n-1}$ are arbitrarily specified real constants, is called an **initial-value problem (IVP).** The values of $y(x)$ and its first $n - 1$ derivatives at a single point x_0, $y(x_0) = y_0$, $y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$, are called **initial conditions.**

FIRST- AND SECOND-ORDER IVPS The problem given in (1) is also called an ***n*th-order initial-value problem.** For example,

$$\begin{aligned} &\textit{Solve:} && \frac{dy}{dx} = f(x, y) \\ &\textit{Subject to:} && y(x_0) = y_0 \end{aligned} \tag{2}$$

and

$$\begin{aligned} &\textit{Solve:} && \frac{d^2y}{dx^2} = f(x, y, y') \\ &\textit{Subject to:} && y(x_0) = y_0,\ y'(x_0) = y_1 \end{aligned} \tag{3}$$

Separable Differential Equations:

A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be **separable** or to have **separable variables.**

Solving a Separable Differential Equation

Example 1

Solve $(1 + x)\,dy - y\,dx = 0$.

SOLUTION Dividing by $(1 + x)y$, we can write $dy/y = dx/(1 + x)$, from which it follows that

$$\int \frac{dy}{y} = \int \frac{dx}{1 + x}$$

$$\ln|y| = \ln|1 + x| + c_1$$

$$y = e^{\ln|1+x|+c_1} = e^{\ln|1+x|} \cdot e^{c_1}$$

$$= |1 + x|\,e^{c_1}$$

$$= \pm e^{c_1}(1 + x).$$

Relabeling $\pm e^{c_1}$ as c then gives $y = c(1 + x)$.

(Ans)

Example 2

Solve $(e^{2y} - y)\cos x \dfrac{dy}{dx} = e^y \sin 2x, \quad y(0) = 0.$

Solution

Dividing the equation by $e^y \cos x$ gives

$$\frac{e^{2y} - y}{e^y}\, dy = \frac{\sin 2x}{\cos x}\, dx.$$

Before integrating, we use termwise division on the left-hand side and the trigonometric identity $\sin 2x = 2\sin x \cos x$ on the right-hand side. Then

$$\int (e^y - ye^{-y})\, dy = 2\int \sin x\, dx$$

$$e^y + ye^{-y} + e^{-y} = -2\cos x + c.$$

[Integrating by Parts]

The initial condition $y = 0$ when $x = 0$ implies $c = 4$. Thus a solution of the initial-value problem is

$$e^y + ye^{-y} + e^{-y} = 4 - 2\cos x.$$

(Ans)

Linear Differential Equations

A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

is said to be a **linear equation** in the dependent variable y.

When $g(x) = 0$, the linear equation (1) is said to be **homogeneous;** otherwise, it is **nonhomogeneous.**

STANDARD FORM By dividing both sides of (1) by the lead coefficient $a_1(x)$, we obtain a more useful form, the **standard form,** of a linear equation:

$$\frac{dy}{dx} + P(x)y = f(x). \tag{2}$$

Solving a First Order Linear Differential Equation

Steps:

(*i*) Put a linear equation of form (1) into the standard form (2).

(*ii*) From the standard form identify $P(x)$ and then find the integrating factor $e^{\int P(x)dx}$.

(*iii*) Multiply the standard form of the equation by the integrating factor. The left-hand side of the resulting equation is automatically the derivative of the integrating factor and y:

$$\frac{d}{dx}\left[e^{\int P(x)dx}y\right] = e^{\int P(x)dx}f(x).$$

(*iv*) Integrate both sides of this last equation.

Example 1:

Solve $\frac{dy}{dx} - 3y = 0$.

SOLUTION This linear equation can be solved by separation of variables. Alternatively, since the equation is already in the standard form (2), we see that $P(x) = -3$, and so the integrating factor is $e^{\int(-3)dx} = e^{-3x}$. We multiply the equation by this factor and recognize that

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 0 \qquad \text{is the same as} \qquad \frac{d}{dx}[e^{-3x}y] = 0.$$

Integrating both sides of the last equation gives $e^{-3x}y = c$. Solving for y gives us the explicit solution $y = ce^{3x}$, $-\infty < x < \infty$. ■

Example 2:

Solve $\frac{dy}{dx} - 3y = 6$.

SOLUTION The associated homogeneous equation for this DE was solved in Example 1. Again the equation is already in the standard form (2), and the integrating factor is still $e^{\int(-3)dx} = e^{-3x}$. This time multiplying the given equation by this factor gives

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = 6e^{-3x}, \qquad \text{which is the same as} \qquad \frac{d}{dx}[e^{-3x}y] = 6e^{-3x}.$$

Integrating both sides of the last equation gives $e^{-3x}y = -2e^{-3x} + c$ or $y = -2 + ce^{3x}$, $-\infty < x < \infty$. ■