MAT 120

Beta and Gamma Functions

1. Evaluate in terms of gamma function: (WEEK 2)

$$(i)\int_0^4 x^{3/2}(4-x)^{5/2}\,dx,\quad (ii)\int_0^b y^5\sqrt{b^2-y^2}\,dy,\quad (iii)\int_0^\infty e^{-t^2}\,dt,$$

$$(iv)\int_0^\infty x^5e^{-4x}dx,\quad (v)\int_0^\infty e^{-y^2}y^5dy,\quad (vi)\int_0^\infty e^{-x^2}\,dx,\quad (vii)\int_0^\infty x^6e^{-3x}dx,$$

$$(viii)\int_0^\infty e^{-x^2}x^9dx,\quad (ix)\int_0^\infty \sqrt{x}e^{-x^2}\,dx,\quad (x)\int_0^1 \frac{x^3}{\sqrt{1-x^3}}dx,$$

$$(xi)\int_0^1 \frac{1}{\sqrt{x\ln(1/x)}}dx,\quad (xii)\int_0^1\left(1-\frac{1}{x}\right)^{1/3}dx.$$

2. Evaluate in terms of beta function: (WEEK 3)

$$(i)\int_0^1 \frac{x^2}{\sqrt{1-x}}dx,\quad (ii)\int_0^1 x^7(1-x)^3dx,\quad (iii)\int_0^1 \frac{1}{\sqrt{1-x^3}}dx,$$

$$(iv)\int_0^1 (1-x)^{1/2}x^3dx,\quad (v)\int_0^1 x^{5/2}(1-x)^{3/2}dx,$$

$$(vi)\int_0^a y^7\sqrt{a^4-y^4}\,dy,\quad (vii)\int_0^4 y^3\sqrt{64-y^3}\,dy,$$

$$(viii)\int_0^1 x^2(1-x^3)^{3/2}dx,\quad (ix)\int_0^\infty \frac{1}{1+x^4}dx.$$

3. Evaluate the following integrals: (WEEK 3)

$$(i)\int_0^\pi \sin^5\theta\cos^4\theta d\theta,\quad (ii)\int_0^\pi \sin^6\theta\cos^7\theta d\theta,$$

$$(iii)\int_0^{\pi/6}\sin^2 6x\cos^4 3xdx,\quad (iv)\int_0^{\pi/4}\sin^2 4\theta\cos^3 2\theta d\theta,$$

$$(v)\int_0^{\pi/2}\sin^4\theta\cos^2\theta d\theta,\quad (vi)\int_0^{\pi/8}\sin^2 8x\cos^4 4xdx.$$

Formula

1. $\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \quad where \quad n > 0$
2. $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx, \quad where \quad m > 0, n > 0.$
3. $\int_0^{\pi/2} \sin^p x \cos^q x dx = \frac{\Gamma(\frac{p+1}{2})\,\Gamma(\frac{q+1}{2})}{2\,\Gamma(\frac{p+q+2}{2})}$
4. $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$
5. $\Gamma(n) = (n-1)!$
6. $\Gamma(n+1) = n\Gamma(n) = n!$
7. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$
8. $\Gamma(1) = 1$
9. $\Gamma(\frac{p}{2}) = (\frac{p}{2}-1)(\frac{p}{2}-2)(\frac{p}{2}-3).....\frac{1}{2}.\Gamma(\frac{1}{2})$
10. $\Gamma(n) = (n-1)(n-2).....3.2.1$
11. $\int_0^{\pi/2} 2sin^{2x-1}(t)cos^{2y-1}(t)dt = \beta(x,y)$

Gamma Beta Distribution Solution

1. (i) $8.4.32\,\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{7}{2}\right)}{\Gamma(6)}$

   (ii) $\frac{b^7}{2}\frac{\Gamma(3)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}$

   (iii) ½ (1/2)

   (iv) $\frac{1}{4^6}(6)$

   (v) ½ (3)

   (vii) $\frac{1}{3^7}$ (7)

   (viii) ½ (5)

   (ix) ½ (3/4)

   (x) $\frac{1}{3}\frac{\Gamma\left(\frac{4}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{4}{3}+\frac{1}{2}\right)}$

   (xi) $\sqrt{2}$ (1/2)

   (xii) $-\frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma(2)}$

2. (i) β(3, ½)

   (ii) β(8, 4)

   (iii) $\frac{1}{3}$β(1/3, ½)

   (iv) β(4, 3/2)

   (v) β(7/2, 5/2)

   (vi) $\frac{a^{10}}{4}$β(2, 3/2)

   (vii) $4^4.8.\frac{1}{3}$ β(4/3, 3/2)

   (viii) $\frac{1}{3}$β(1, 5/2)

   (ix) ½ $\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{2\Gamma(1)} = \frac{1}{4}\,\beta\left(\frac{1}{4},\frac{3}{4}\right)$

3. (i) $\frac{\Gamma(3)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{11}{2}\right)}$

   (ii) $\frac{\Gamma\left(\frac{7}{2}\right)\Gamma(4)}{\Gamma\left(\frac{15}{2}\right)}$

   (iii) $\frac{4}{3}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{7}{2}\right)}{2\Gamma(5)}$

   (iv) $\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(3)}{\Gamma\left(\frac{9}{2}\right)}$

   (v) $\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\Gamma(4)}$

   (vi) $\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{7}{2}\right)}{2\Gamma(5)}$