Exact Differential Equations

Introduction

Although the simple first-order equation

$$y\,dx + x\,dy = 0$$

is separable, we can solve the equation in an alternative manner by recognizing that the expression on the left-hand side of the equality is the differential of the function $f(x, y) = xy$, that is,

$$d(xy) = y\,dx + x\,dy$$

In this section we examine first-order equations in differential form

$M(x, y)dx + N(x, y)dy = 0$ by applying a simple test to M and N.

Differential of a Function of Two Variables

If $z = f(x, y)$ is a function of two variables with continuous first partial derivative in a region R of the xy −plane, then its differential is

$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \qquad (1)$$

In the special case when $f(x, y) = c$, where c is a constant, then (1) implies,

$$\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \;= 0 \qquad (2)$$

Exact Equation

A differential expression $M(x, y)\,dx + N(x, y)dy$ is an exact differential in a region R of the xy −plane if it corresponds to the differential of some function $f(x, y)$ defined in R. A first-order differential equation of the form

$$M(x, y)\,dx + N(x, y)dy = 0$$

is said to be an exact equation if the expression on the left-hand side is an exact differential.

Criterion for an Exact Differential

Let, $M(x,y)$ and $N(x,y)$ be continuous and have continuous first partial derivatives in a rectangular region R defined by $a < x < b, c < y < d$. Then a necessary and sufficient condition that $M(x,y)\,dx + N(x,y)dy$ be an exact differential is,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Solving an Exact Differential

Example 1:

Solve $2xy\,dx + (x^2 - 1)\,dy = 0$.

Solution:

With $M(x, y) = 2xy$ and $N(x, y) = x^2 - 1$ we have

$$\frac{\partial M}{\partial y} = 2x = \frac{\partial N}{\partial x}.$$

Thus the equation is exact. So there exists a function $f(x,y)$ such that,

$$\frac{\partial f}{\partial x} = 2xy \qquad \text{and} \qquad \frac{\partial f}{\partial y} = x^2 - 1.$$

From the first of these equations we obtain, after integrating,

$$f(x, y) = x^2 y + g(y).$$

Taking the partial derivative of the last expression with respect to y and setting the result equal to $N(x, y)$ gives

$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1. \qquad \leftarrow N(x, y)$$

It follows that $g'(y) = -1$ and $g(y) = -y$. Hence $f(x, y) = x^2y - y$, so the solution of the equation in implicit form is $x^2y - y = c$. The explicit form of the solution is easily seen to be $y = c/(1 - x^2)$ and is defined on any interval not containing either $x = 1$ or $x = -1$. ■

Example 2:

Solve $(e^{2y} - y \cos xy)\, dx + (2xe^{2y} - x \cos xy + 2y)\, dy = 0$.

Solution:

The equation is exact because

$$\frac{\partial M}{\partial y} = 2e^{2y} + xy \sin xy - \cos xy = \frac{\partial N}{\partial x}.$$

Hence a function $f(x, y)$ exists for which

$$M(x, y) = \frac{\partial f}{\partial x} \quad \text{and} \quad N(x, y) = \frac{\partial f}{\partial y}.$$

Now for variety we shall start with the assumption that $\partial f/\partial y = N(x, y)$; that is,

$$\frac{\partial f}{\partial y} = 2xe^{2y} - x \cos xy + 2y$$

$$f(x, y) = 2x \int e^{2y}\, dy - x \int \cos xy\, dy + 2 \int y\, dy.$$

After integrating it follows that,

$$f(x, y) = xe^{2y} - \sin xy + y^2 + h(x)$$

$$\frac{\partial f}{\partial x} = e^{2y} - y \cos xy + h'(x) = e^{2y} - y \cos xy, \quad \leftarrow M(x, y)$$

and so $h'(x) = 0$ or $h(x) = c$. Hence a family of solutions is

$$xe^{2y} - \sin xy + y^2 + c = 0.$$