Linear Algebra
MATH 1201

Roy Lee

1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

Linear equation: An equation of the form $a_1x_1+a_2x_2+\cdots+a_nx_n = b$ where $a_1, a_2, \ldots, a_n, b$ are constants and $x_1, x_2, \ldots, x_n$ are variables.

Solution set: All possible solutions to a system of linear equations.

If we have a system of linear equations that looks like:

$$\begin{aligned} x_1 - 2x_2 \quad &+x_3 = 0 \\ x_2 \quad &+2x_3 = 3 \\ 3x_1 + x_2 \quad &+3x_3 = 3 \end{aligned}$$

Coefficient matrix: Rewrite system of linear equations as just coefficients

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 3 \end{bmatrix}$$

Augmented matrix: Rewrite system as matrix with coefficients and constants

$$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 1 & 2 & | & 3 \\ 3 & 1 & 3 & | & 3 \end{bmatrix}$$

Row equivalent: Two matrices are row equivalent if one can be obtained from the other by a sequence of elementary row operations.

Consistent: A system of linear equations is consistent if it has one or infinitely many solutions.

Key Concept

Inconsistent system: A system of linear equations is inconsistent if the RREF of its augmented matrix has a row with $\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & | & b \end{bmatrix}$ where $b \neq 0$.

1.2 Row Reduction and Echelon Forms

Definition

Properties of a matrix in echelon form (or reduced echelon form)

- All nonzero rows are above any rows of all zeros.
- The leading entry of each nonzero row occurs to the right of the leading entry of the previous row.
- The leading entry in any nonzero row is 1.
- All entries in the column below the leading 1 are zeros.

Example:

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Definition

Properties of a matrix in RREF (row-reduced echelon form)

- Echelon form
- The leading entry in each row is a 1.
- Each leading 1 is the only nonzero entry in its column.

Example:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Note

The RREF of a matrix is unique.

Pivot positions: The positions of the leading 1s in a matrix in RREF. Obviously, **pivot columns** and **pivot rows** are the corresponding rows/columns to pivot positions.

Free variables: Variables that are not pivot variables.

For example, in this augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 3 \end{array}\right]$$

x_1 and x_2 are **pivot variables**, and x_3 is a **free variable**.

Note

If a system of linear equations has **any** free variables, it has infinitely many solutions.

Parametric descriptions of solution sets: The standard is to use free variables as our parametric variables. If our RREF form looks like:

$$\left[\begin{array}{ccc|c} 1 & -3 & -4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

then x_2 and x_3 are free variables, which we'll parameterize and write our solution set as:

$$\begin{aligned} x_1 &= 3x_2 + 4x_3 \\ x_2 &= x_2 \\ x_3 &= x_3 \end{aligned}$$

1.3 Vector Equations

Column Vector: A matrix with only one column. These are expressed as

$$\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \textit{ or } (3, -1)$$

The geometric visualization of a vector such as $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is a vector in the plane from (0,0) to (3,1).

Note

Parallelogram Rule for Addition The sum **u** +**v** of two column vectors **u** and **v** is the fouth point of the parallelogram with sides given by **(0,0)**, **u**, and **v**.

Key Concept

A vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

has the same solution set as the system whose augmented matrix is:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n & | & b \end{bmatrix}$$

Span: The set of all possible linear combinations of a set of vectors. Thus, asking whether a vector **b** is in the span of a system is the same as asking if the system with b has a solution.

1.4 The Matrix Equation Ax = b

This section is very obvious. If there is an m x n matrix A , then we can write Ax=b, which has the same solution set as the the system whose augmented matrix is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & | & b_m \end{bmatrix}$$

Definition

The following are logically equivalent

- For each b in $\mathbb{R}^m$, the equation Ax=b has a solution.
- Each b in $\mathbb{R}^m$ is a linear combination of the columns of A.
- The columns of A span $\mathbb{R}^m$.
- A has a pivot position in every row.

1.5 Solution Sets of Linear Systems

Homogeneous Linear Systems: A system is homogeneous if it can be written in the form *Ax=0*, where 0 is the zero vector in $\mathbb{R}^m$.

Note

The homogeneous equation *Ax=0* has a **nontrivial solution** if and only if the equation has **at least one free variable**

EXAMPLE: Determine whether the following system is consistent. If so, describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 \quad -4_3 &= 0 \\ -3x_1 - 2x_2 \quad +4x_3 &= 0 \\ 6x_1 + x_2 \quad -8x_3 &= 0 \end{aligned}$$

Reduce it part way to get to this echelon form:

$$\begin{bmatrix} 3 & 5 & -4 & | & 0 \\ 0 & 3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Notice x_3 is free, so this homogeneous system of equations has a free variable at x_3, which means there is a nontrivial solution.
Reduce it further to get

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Now, you can rewrite this as the general solution, which is shown below

Key Concept

General solution to a system of linear equations (example)

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{4}{3}x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

We'll factor out the x_3 to get the general solution, and this will extend later on to all the other variables, as well as constants, which you'll see later.

Nonhomogeneous Systems: These are systems that are not of the form $Ax = 0$, but of the form $Ax = b$ where $b \neq 0$ If we had an example of a nonhomogeneous system, the general solution might be writtten in the form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + \frac{4}{3}x_3 \\ 2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix}$$

Note

The solution set of $Ax = b$ is the set of all vectors of the form $w = p + v_h$ where v_h is the solution of the homogeneous equation $Ax = 0$.

1.6 Applications of Systems of Linear Equations

TLDR: we use systems of linear equations for a lot of irl things.

1.7 Linear Independence

Definition

Linear Independence

- A set of vectors is **linearly independent** if the only solution to the equation $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$ is $c_1 = c_2 = \cdots = c_k = 0$.
- The set $v_1, v_2, \cdot, v_p$ is said to be **linearly dependent** if there exists weights $c_1, c_2, \cdots, c_p$ **not all zero** such that $c_1v_1 + c_2v_2 + \cdots + c_pv_p = 0$.

EXAMPLE: Determine whether the following set of vectors is linearly independent

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} v_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} v_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Solution:

We must determine if there is a nontrivial solution. Row operations on the associated augmented matrix show that

$$\begin{bmatrix} 1 & 4 & 2 & | & 0 \\ 2 & 5 & 1 & | & 0 \\ 3 & 6 & 0 & | & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Key Concept

x_3 is free, meaning there are infinite solutions and there exists a nontrivial solution. **Thus, the set of vectors is linearly dependent.**

EXAMPLE: For the above example, find a linear dependence relation among v_1, v_2, and v_3

Solution:

To find a linear dependence relation, completely row reduce and write the new system.

$$\begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} = \begin{matrix} x_1 & & -2x_3 = 0 \\ & x_2 & +x_3 = 0 \\ & & 0 = 0 \end{matrix}$$

Key Concept

Linear dependence relation

Thus, $x_1 = 2x_3$ and $x_2 = -x_3$. We can write this as one of infinitely many linear dependence relation:

$$10v_1 - 5v_2 + 5v_3 = 0$$

Note that linear dependence relations are always of the form:

$$c_1v_1 + c_2v_2 + \cdot + c_pv_p = 0$$

Key Concept

Linear independence of matrix columns: Suppose we begin with a matrix

$$A = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

The columns of matrix A are **linearly independent** if and only if the equation $Ax = 0$ has *only* the trivial solution.

OR

Linear independence means there are **no free variables** in $Ax = 0$

EXAMPLE: Determine if the columns of the matrix $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent

Solution:

Row reduce it a bit

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}$$

Observe there are **no free variables**, so the equation $Ax = 0$ **has only the trivial solution**, and the columns of A are linearly independent.

Linear independence of a single vector: A set containing only one vector v is linearly independent if and only if $v \neq 0$.

EXAMPLE: Determine if the following set of vectors is linearly independent:

$$v_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} v_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

Solution:

Notice that v_2 is a multiple of v_1 and thus, the set is linearly dependent.

Alternatively, notice that if we wrote this as an augmented matrix of the form $Ax = 0$, we'd have $\begin{bmatrix} 3 & 1 & | & 0 \\ 6 & 2 & | & 0 \end{bmatrix}$

which would simplify to $\begin{bmatrix} 1 & \frac{1}{3} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$ giving us the free variable x_2, thus linear dependence.

Note

If a set $S = v_1, v_2, ..., v_p$ in $\mathbb{R}^m$ contains the zero vector, then the set is linearly dependent.

If a set contains more vectors than there are entires in each vector, then the set is linearly independent.