Linear Algebra
MATH 1201

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1 Linear Equations in Linear Algebra

1.1 Systems of Linear Equations

Linear equation: An equation of the form $a_1x_1+a_2x_2+\cdots+a_nx_n = b$ where $a_1, a_2, \ldots, a_n, b$ are constants and $x_1, x_2, \ldots, x_n$ are variables.

Solution set: All possible solutions to a system of linear equations.

If we have a system of linear equations that looks like:

$$\begin{aligned} x_1 - 2x_2 \quad &+x_3 = 0 \\ x_2 \quad &+2x_3 = 3 \\ 3x_1 + x_2 \quad &+3x_3 = 3 \end{aligned}$$

Coefficient matrix: Rewrite system of linear equations as just coefficients

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 3 & 1 & 3 \end{bmatrix}$$

Augmented matrix: Rewrite system as matrix with coefficients and constants

$$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 1 & 2 & | & 3 \\ 3 & 1 & 3 & | & 3 \end{bmatrix}$$

Row equivalent: Two matrices are row equivalent if one can be obtained from the other by a sequence of elementary row operations.

Consistent: A system of linear equations is consistent if it has one or infinitely many solutions.

Key Concept

Inconsistent system: A system of linear equations is inconsistent if the RREF of its augmented matrix has a row with $\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & | & b \end{bmatrix}$ where $b \neq 0$.

1.2 Row Reduction and Echelon Forms

Definition

Properties of a matrix in echelon form (or reduced echelon form)

- All nonzero rows are above any rows of all zeros.
- The leading entry of each nonzero row occurs to the right of the leading entry of the previous row.
- The leading entry in any nonzero row is 1.
- All entries in the column below the leading 1 are zeros.

Example:

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Definition

Properties of a matrix in RREF (row-reduced echelon form)

- Echelon form
- The leading entry in each row is a 1.
- Each leading 1 is the only nonzero entry in its column.

Example:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

Note

The RREF of a matrix is unique.

Pivot positions: The positions of the leading 1s in a matrix in RREF. Obviously, **pivot columns** and **pivot rows** are the corresponding rows/columns to pivot positions.

Free variables: Variables that are not pivot variables.

For example, in this augmented matrix:

$$\begin{bmatrix} 1 & 0 & 0 & | & 3 \\ 0 & 1 & 2 & | & 3 \end{bmatrix}$$

x_1 and x_2 are **pivot variables**, and x_3 is a **free variable**.

Note

If a system of linear equations has **any** free variables, it has infinitely many solutions.

Parametric descriptions of solution sets: The standard is to use free variables as our parametric variables. If our RREF form looks like:

$$\begin{bmatrix} 1 & -3 & -4 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

then x_2 and x_3 are free variables, which we'll parameterize and write our solution set as:

$$x_1 = 3x_2 + 4x_3$$
$$x_2 = x_2$$
$$x_3 = x_3$$

1.3 Vector Equations

Column Vector: A matrix with only one column. These are expressed as

$$\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \textit{ or } (3, -1)$$

The geometric visualization of a vector such as $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is a vector in the plane from (0,0) to (3,1).

Note

Parallelogram Rule for Addition The sum **u** +**v** of two column vectors **u** and **v** is the fouth point of the parallelogram with sides given by **(0,0)**, **u**, and **v**.

Key Concept

A vector equation

$$x_1a_1 + x_2a_2 + \cdots + x_na_n = b$$

has the same solution set as the system whose augmented matrix is:

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_n & | & b \end{bmatrix}$$

Span: The set of all possible linear combinations of a set of vectors. Thus, asking whether a vector **b** is in the span of a system is the same as asking if the system with b has a solution.

1.4 The Matrix Equation Ax = b

This section is very obvious. If there is an m x n matrix A , then we can write Ax=b, which has the same solution set as the the system whose augmented matrix is

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & | & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & | & b_2 \\ \vdots & \vdots & \ddots & \vdots & | & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & | & b_m \end{bmatrix}$$

Definition

The following are logically equivalent

- For each b in $\mathbb{R}^m$, the equation Ax=b has a solution.
- Each b in $\mathbb{R}^m$ is a linear combination of the columns of A.
- The columns of A span $\mathbb{R}^m$.
- A has a pivot position in every row.