

GLOBAL
EDITION



Linear Algebra and Its Applications

SIXTH EDITION

David C. Lay • Steven R. Lay • Judi J. McDonald



A large, light blue, semi-transparent graphic featuring a complex, organic pattern of white lines forming a mesh of triangles and polygons, resembling a molecular or crystal structure.

SIXTH EDITION

Linear Algebra and Its Applications

GLOBAL EDITION

David C. Lay

University of Maryland–College Park

Steven R. Lay

Lee University

Judi J. McDonald

Washington State University



Pearson

Pearson Education Limited

KAO Two
KAO Park
Hockham Way
Harlow
Essex
CM17 9SR
United Kingdom

and Associated Companies throughout the world

Visit us on the World Wide Web at: www.pearsonglobaleditions.com

© Pearson Education Limited, 2022

The rights of David C. Lay, Steven R. Lay, and Judi J. McDonald to be identified as the authors of this work have been asserted by them in accordance with the Copyright, Designs and Patents Act 1988.

Authorized adaptation from the United States edition, entitled Linear Algebra and Its Applications, 6th Edition, ISBN 978-0-13-585125-8 by David C. Lay, Steven R. Lay, and Judi J. McDonald, published by Pearson Education © 2021.

All rights reserved. No part of this publication may be reproduced, stored in a retrieval system, or transmitted in any form or by any means, electronic, mechanical, photocopying, recording or otherwise, without either the prior written permission of the publisher or a license permitting restricted copying in the United Kingdom issued by the Copyright Licensing Agency Ltd, Saffron House, 6–10 Kirby Street, London EC1N 8TS.

All trademarks used herein are the property of their respective owners. The use of any trademark in this text does not vest in the author or publisher any trademark ownership rights in such trademarks, nor does the use of such trademarks imply any affiliation with or endorsement of this book by such owners. For information regarding permissions, request forms, and the appropriate contacts within the Pearson Education Global Rights and Permissions department, please visit www.pearsoned.com/permissions.

This eBook is a standalone product and may or may not include all assets that were part of the print version. It also does not provide access to other Pearson digital products like MyLab and Mastering. The publisher reserves the right to remove any material in this eBook at any time.

British Library Cataloguing-in-Publication Data

A catalogue record for this book is available from the British Library

ISBN 10: 1-292-35121-7

ISBN 13: 978-1-292-35121-6

eBook ISBN 13: 978-1-292-35122-3

*To my wife, Lillian, and our children,
Christina, Deborah, and Melissa, whose
support, encouragement, and faithful
prayers made this book possible.*

David C. Lay

About the Authors

David C. Lay

As a founding member of the NSF-sponsored Linear Algebra Curriculum Study Group (LACSG), David Lay was a leader in the movement to modernize the linear algebra curriculum and shared those ideas with students and faculty through his authorship of the first four editions of this textbook. David C. Lay earned a B.A. from Aurora University (Illinois), and an M.A. and Ph.D. from the University of California at Los Angeles. David Lay was an educator and research mathematician for more than 40 years, mostly at the University of Maryland, College Park. He also served as a visiting professor at the University of Amsterdam, the Free University in Amsterdam, and the University of Kaiserslautern, Germany. He published more than 30 research articles on functional analysis and linear algebra. Lay was also a coauthor of several mathematics texts, including *Introduction to Functional Analysis* with Angus E. Taylor, *Calculus and Its Applications*, with L. J. Goldstein and D. I. Schneider, and *Linear Algebra Gems—Assets for Undergraduate Mathematics*, with D. Carlson, C. R. Johnson, and A. D. Porter.

David Lay received four university awards for teaching excellence, including, in 1996, the title of Distinguished Scholar-Teacher of the University of Maryland. In 1994, he was given one of the Mathematical Association of America's Awards for Distinguished College or University Teaching of Mathematics. He was elected by the university students to membership in Alpha Lambda Delta National Scholastic Honor Society and Golden Key National Honor Society. In 1989, Aurora University conferred on him the Outstanding Alumnus award. David Lay was a member of the American Mathematical Society, the Canadian Mathematical Society, the International Linear Algebra Society, the Mathematical Association of America, Sigma Xi, and the Society for Industrial and Applied Mathematics. He also served several terms on the national board of the Association of Christians in the Mathematical Sciences.

In October 2018, David Lay passed away, but his legacy continues to benefit students of linear algebra as they study the subject in this widely acclaimed text.

Steven R. Lay

Steven R. Lay began his teaching career at Aurora University (Illinois) in 1971, after earning an M.A. and a Ph.D. in mathematics from the University of California at Los Angeles. His career in mathematics was interrupted for eight years while serving as a missionary in Japan. Upon his return to the States in 1998, he joined the mathematics faculty at Lee University (Tennessee) and has been there ever since. Since then he has supported his brother David in refining and expanding the scope of this popular linear algebra text, including writing most of Chapters 8 and 9. Steven is also the author of three college-level mathematics texts: *Convex Sets and Their Applications*, *Analysis with an Introduction to Proof*, and *Principles of Algebra*.

In 1985, Steven received the Excellence in Teaching Award at Aurora University. He and David, and their father, Dr. L. Clark Lay, are all distinguished mathematicians, and in 1989, they jointly received the Outstanding Alumnus award from their alma mater, Aurora University. In 2006, Steven was honored to receive the Excellence in Scholarship Award at Lee University. He is a member of the American Mathematical Society, the Mathematics Association of America, and the Association of Christians in the Mathematical Sciences.

Judi J. McDonald

Judi J. McDonald became a co-author on the fifth edition, having worked closely with David on the fourth edition. She holds a B.Sc. in Mathematics from the University of Alberta, and an M.A. and Ph.D. from the University of Wisconsin. As a professor of Mathematics, she has more than 40 publications in linear algebra research journals and more than 20 students have completed graduate degrees in linear algebra under her supervision. She is an associate dean of the Graduate School at Washington State University and a former chair of the Faculty Senate. She has worked with the mathematics outreach project Math Central (<http://mathcentral.uregina.ca/>) and is a member of the second Linear Algebra Curriculum Study Group (LACSG 2.0).

Judi has received three teaching awards: two Inspiring Teaching awards at the University of Regina, and the Thomas Lutz College of Arts and Sciences Teaching Award at Washington State University. She also received the College of Arts and Sciences Institutional Service Award at Washington State University. Throughout her career, she has been an active member of the International Linear Algebra Society and the Association for Women in Mathematics. She has also been a member of the Canadian Mathematical Society, the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics.

Contents

About the Authors 3

Preface 12

A Note to Students 22

Chapter 1 Linear Equations in Linear Algebra 25

INTRODUCTORY EXAMPLE: Linear Models in Economics and Engineering 25

- 1.1 Systems of Linear Equations 26
- 1.2 Row Reduction and Echelon Forms 37
- 1.3 Vector Equations 50
- 1.4 The Matrix Equation $Ax = b$ 61
- 1.5 Solution Sets of Linear Systems 69
- 1.6 Applications of Linear Systems 77
- 1.7 Linear Independence 84
- 1.8 Introduction to Linear Transformations 91
- 1.9 The Matrix of a Linear Transformation 99
- 1.10 Linear Models in Business, Science, and Engineering 109
- Projects 117
- Supplementary Exercises 117

Chapter 2 Matrix Algebra 121

INTRODUCTORY EXAMPLE: Computer Models in Aircraft Design 121

- 2.1 Matrix Operations 122
- 2.2 The Inverse of a Matrix 135
- 2.3 Characterizations of Invertible Matrices 145
- 2.4 Partitioned Matrices 150
- 2.5 Matrix Factorizations 156
- 2.6 The Leontief Input–Output Model 165
- 2.7 Applications to Computer Graphics 171

2.8	Subspaces of \mathbb{R}^n	179
2.9	Dimension and Rank	186
	Projects	193
	Supplementary Exercises	193

Chapter 3 Determinants 195

INTRODUCTORY EXAMPLE: Weighing Diamonds 195		
3.1	Introduction to Determinants	196
3.2	Properties of Determinants	203
3.3	Cramer's Rule, Volume, and Linear Transformations	212
	Projects	221
	Supplementary Exercises	221

Chapter 4 Vector Spaces 225

INTRODUCTORY EXAMPLE: Discrete-Time Signals and Digital Signal Processing 225		
4.1	Vector Spaces and Subspaces	226
4.2	Null Spaces, Column Spaces, Row Spaces, and Linear Transformations	235
4.3	Linearly Independent Sets; Bases	246
4.4	Coordinate Systems	255
4.5	The Dimension of a Vector Space	265
4.6	Change of Basis	273
4.7	Digital Signal Processing	279
4.8	Applications to Difference Equations	286
	Projects	295
	Supplementary Exercises	295

Chapter 5 Eigenvalues and Eigenvectors 297

INTRODUCTORY EXAMPLE: Dynamical Systems and Spotted Owls 297		
5.1	Eigenvectors and Eigenvalues	298
5.2	The Characteristic Equation	306
5.3	Diagonalization	314
5.4	Eigenvectors and Linear Transformations	321
5.5	Complex Eigenvalues	328
5.6	Discrete Dynamical Systems	335
5.7	Applications to Differential Equations	345
5.8	Iterative Estimates for Eigenvalues	353
5.9	Applications to Markov Chains	359
	Projects	369
	Supplementary Exercises	369

Chapter 6 Orthogonality and Least Squares 373

INTRODUCTORY EXAMPLE: Artificial Intelligence and Machine Learning	373
6.1 Inner Product, Length, and Orthogonality	374
6.2 Orthogonal Sets	382
6.3 Orthogonal Projections	391
6.4 The Gram–Schmidt Process	400
6.5 Least-Squares Problems	406
6.6 Machine Learning and Linear Models	414
6.7 Inner Product Spaces	423
6.8 Applications of Inner Product Spaces Projects	431 437
Supplementary Exercises	438

Chapter 7 Symmetric Matrices and Quadratic Forms 441

INTRODUCTORY EXAMPLE: Multichannel Image Processing	441
7.1 Diagonalization of Symmetric Matrices	443
7.2 Quadratic Forms	449
7.3 Constrained Optimization	456
7.4 The Singular Value Decomposition	463
7.5 Applications to Image Processing and Statistics Projects	473 481
Supplementary Exercises	481

Chapter 8 The Geometry of Vector Spaces 483

INTRODUCTORY EXAMPLE: The Platonic Solids	483
8.1 Affine Combinations	484
8.2 Affine Independence	493
8.3 Convex Combinations	503
8.4 Hyperplanes	510
8.5 Polytopes	519
8.6 Curves and Surfaces Project	531 542
Supplementary Exercises	543

Chapter 9 Optimization 545

INTRODUCTORY EXAMPLE: The Berlin Airlift	545
9.1 Matrix Games	546
9.2 Linear Programming—Geometric Method	560
9.3 Linear Programming—Simplex Method	570
9.4 Duality Project	585 594
Supplementary Exercises	594

Chapter 10 Finite-State Markov Chains C-1 (Available Online)

INTRODUCTORY EXAMPLE: Googling Markov Chains	C-1
10.1 Introduction and Examples	C-2
10.2 The Steady-State Vector and Google's PageRank	C-13
10.3 Communication Classes	C-25
10.4 Classification of States and Periodicity	C-33
10.5 The Fundamental Matrix	C-42
10.6 Markov Chains and Baseball Statistics	C-54

Appendices

A	Uniqueness of the Reduced Echelon Form	597
B	Complex Numbers	599

Credits **604**

Glossary **605**

Answers to Odd-Numbered Exercises **A-1**

Index **I-1**

Applications Index

Biology and Ecology

Estimating systolic blood pressure, 422
Laboratory animal trials, 367
Molecular modeling, 173–174
Net primary production of nutrients, 418–419
Nutrition problems, 109–111, 115
Predator-prey system, 336–337, 343
Spotted owls and stage-matrix models, 297–298, 341–343

Business and Economics

Accelerator-multiplier model, 293
Average cost curve, 418–419
Car rental fleet, 116, 368
Cost vectors, 57
Equilibrium prices, 77–79, 82
Exchange table, 82
Feasible set, 460, 562
Gross domestic product, 170
Indifference curves, 460–461
Intermediate demand, 165
Investment, 294
Leontief exchange model, 25, 77–79
Leontief input–output model, 25, 165–171
Linear programming, 26, 111–112, 153, 484, 519, 522, 560–566
Loan amortization schedule, 293
Manufacturing operations, 57, 96
Marginal propensity to consume, 293
Markov chains, 311, 359–368, C-1–C-63
Maximizing utility subject to a budget constraint, 460–461
Population movement, 113, 115–116, 311, 361
Price equation, 170
Total cost curve, 419
Value added vector, 170
Variable cost model, 421

Computers and Computer Science

Bézier curves and surfaces, 509, 531–532
CAD, 537, 541

Color monitors, 178

Computer graphics, 122, 171–177, 498–500

Cray supercomputer, 153

Data storage, 66, 163

Error-detecting and error-correcting codes, 447, 471

Game theory, 519

High-end computer graphics boards, 176

Homogeneous coordinates, 172–173, 174

Parallel processing, 25, 132

Perspective projections, 175–176

Vector pipeline architecture, 153

Virtual reality, 174

VLSI microchips, 150

Wire-frame models, 121, 171

Control Theory

Controllable system, 296

Control systems engineering, 155

Decoupled system, 340, 346, 349

Deep space probe, 155

State-space model, 296, 335

Steady-state response, 335

Transfer function (matrix), 155

Electrical Engineering

Branch and loop currents, 111–112

Circuit design, 26, 160

Current flow in networks, 111–112, 115–116

Discrete-time signals, 228, 279–280

Inductance-capacitance circuit, 242

Kirchhoff's laws, 161

Ladder network, 161, 163–164

Laplace transforms, 155, 213

Linear filters, 287–288

Low-pass filter, 289, 413

Minimal realization, 162

Ohm's law, 111–113, 161

RC circuit, 346–347

RLC circuit, 254

Series and shunt circuits, 161

Transfer matrix, 161–162, 163

Engineering

Aircraft performance, 422, 437

Boeing Blended Wing Body, 122

Cantilevered beam, 293

CFD and aircraft design, 121–122

Deflection of an elastic beam, 137, 144

Deformation of a material, 482

Equilibrium temperatures, 36, 116–117, 193

Feedback controls, 519

Flexibility and stiffness matrices, 137, 144

Heat conduction, 164

Image processing, 441–442, 473–474, 479

LU factorization and airflow, 122

Moving average filter, 293

Superposition principle, 95, 98, 112

Mathematics

Area and volume, 195–196, 215–217

Attractors/repellers in a dynamical system, 338, 341, 343, 347, 351

Bessel's inequality, 438

Best approximation in function spaces, 426–427

Cauchy-Schwarz inequality, 427

Conic sections and quadratic surfaces, 481

Differential equations, 242, 345–347

Fourier series, 434–436

Hermite polynomials, 272

Hypercube, 527–529

Interpolating polynomials, 49, 194

Isomorphism, 188, 260–261

Jacobian matrix, 338

Laguerre polynomials, 272

Laplace transforms, 155, 213

Legendre polynomials, 430

Linear transformations in calculus, 241, 324–325
Simplex, 525–527
Splines, 531–534, 540–541
Triangle inequality, 427
Trigonometric polynomials, 434

Numerical Linear Algebra

Band matrix, 164
Block diagonal matrix, 153, 334
Cholesky factorization, 454–455, 481
Companion matrix, 371
Condition number, 147, 149, 211, 439, 469
Effective rank, 190, 271, 465
Floating point arithmetic, 33, 45, 221
Fundamental subspaces, 379, 439, 469–470
Givens rotation, 119
Gram matrix, 482
Gram–Schmidt process, 405
Hilbert matrix, 149
Householder reflection, 194
Ill-conditioned matrix (problem), 147
Inverse power method, 356–357
Iterative methods, 353–359
Jacobi’s method for eigenvalues, 312
LAPACK, 132, 153
Large-scale problems, 119, 153
LU factorization, 157–158, 162–163, 164
Operation counts, 142, 158, 160, 206
Outer products, 133, 152
Parallel processing, 25
Partial pivoting, 42, 163
Polar decomposition, 482
Power method, 353–356
Powers of a matrix, 129
Pseudoinverse, 470, 482

QR algorithm, 312–313, 357
QR factorization, 403–404, 405, 413, 438
Rank-revealing factorization, 163, 296, 481
Rayleigh quotient, 358, 439
Relative error, 439
Schur complement, 154
Schur factorization, 439
Singular value decomposition, 163, 463–473
Sparse matrix, 121, 168, 206
Spectral decomposition, 446–447
Spectral factorization, 163
Tridiagonal matrix, 164
Vandermonde matrix, 194, 371
Vector pipeline architecture, 153

Physical Sciences

Cantilevered beam, 293
Center of gravity, 60
Chemical reactions, 79, 83
Crystal lattice, 257, 263
Decomposing a force, 386
Gaussian elimination, 37
Hooke’s law, 137
Interpolating polynomial, 49, 194
Kepler’s first law, 422
Landsat image, 441–442
Linear models in geology and geography, 419–420
Mass estimates for radioactive substances, 421
Mass-spring system, 233, 254
Model for glacial cirques, 419
Model for soil pH, 419
Pauli spin matrices, 194
Periodic motion, 328
Quadratic forms in physics, 449–454

Radar data, 155
Seismic data, 25
Space probe, 155
Steady-state heat flow, 36, 164
Superposition principle, 95, 98, 112
Three-moment equation, 293
Traffic flow, 80
Trend surface, 419
Weather, 367
Wind tunnel experiment, 49

Statistics

Analysis of variance, 408, 422
Covariance, 474–476, 477, 478, 479
Full rank, 465
Least-squares error, 409
Least-squares line, 413, 414–416
Linear model in statistics, 414–420
Markov chains, 359–360
Mean-deviation form for data, 417, 475
Moore–Penrose inverse, 471
Multichannel image processing, 441–442, 473–479
Multiple regression, 419–420
Orthogonal polynomials, 427
Orthogonal regression, 480–481
Powers of a matrix, 129
Principal component analysis, 441–442, 476–477
Quadratic forms in statistics, 449
Regression coefficients, 415
Sums of squares (in regression), 422, 431–432
Trend analysis, 433–434
Variance, 422, 475–476
Weighted least-squares, 424, 431–432

This page is intentionally left blank

Preface

The response of students and teachers to the first five editions of *Linear Algebra and Its Applications* has been most gratifying. This *Sixth Edition* provides substantial support both for teaching and for using technology in the course. As before, the text provides a modern elementary introduction to linear algebra and a broad selection of interesting classical and leading-edge applications. The material is accessible to students with the maturity that should come from successful completion of two semesters of college-level mathematics, usually calculus.

The main goal of the text is to help students master the basic concepts and skills they will use later in their careers. The topics here follow the recommendations of the original Linear Algebra Curriculum Study Group (LACSG), which were based on a careful investigation of the real needs of the students and a consensus among professionals in many disciplines that use linear algebra. Ideas being discussed by the second Linear Algebra Curriculum Study Group (LACSG 2.0) have also been included. We hope this course will be one of the most useful and interesting mathematics classes taken by undergraduates.

What's New in This Edition

The *Sixth Edition* has exciting new material, examples, and online resources. After talking with high-tech industry researchers and colleagues in applied areas, we added new topics, vignettes, and applications with the intention of highlighting for students and faculty the linear algebraic foundational material for machine learning, artificial intelligence, data science, and digital signal processing.

Content Changes

- Since matrix multiplication is a highly useful skill, we added new examples in Chapter 2 to show how matrix multiplication is used to identify patterns and scrub data. Corresponding exercises have been created to allow students to explore using matrix multiplication in various ways.
- In our conversations with colleagues in industry and electrical engineering, we heard repeatedly how important understanding abstract vector spaces is to their work. After reading the reviewers' comments for Chapter 4, we reorganized the chapter, condensing some of the material on column, row, and null spaces; moving Markov chains to the end of Chapter 5; and creating a new section on signal processing. We view signals

as an infinite dimensional vector space and illustrate the usefulness of linear transformations to filter out unwanted “vectors” (a.k.a. noise), analyze data, and enhance signals.

- By moving Markov chains to the end of Chapter 5, we can now discuss the steady state vector as an eigenvector. We also reorganized some of the summary material on determinants and change of basis to be more specific to the way they are used in this chapter.
- In Chapter 6, we present pattern recognition as an application of orthogonality, and the section on linear models now illustrates how machine learning relates to curve fitting.
- Chapter 9 on optimization was previously available only as an online file. It has now been moved into the regular textbook where it is more readily available to faculty and students. After an opening section on finding optimal strategies to two-person zero-sum games, the rest of the chapter presents an introduction to linear programming—from two-dimensional problems that can be solved geometrically to higher dimensional problems that are solved using the Simplex Method.

Other Changes

- In the high-tech industry, where most computations are done on computers, judging the validity of information and computations is an important step in preparing and analyzing data. In this edition, students are encouraged to learn to analyze their own computations to see if they are consistent with the data at hand and the questions being asked. For this reason, we have added “Reasonable Answers” advice and exercises to guide students.
- We have added a list of projects to the end of each chapter (available online and in MyLab Math). Some of these projects were previously available online and have a wide range of themes from using linear transformations to create art to exploring additional ideas in mathematics. They can be used for group work or to enhance the learning of individual students.
- PowerPoint lecture slides have been updated to cover all sections of the text and cover them more thoroughly.

Distinctive Features

Early Introduction of Key Concepts

Many fundamental ideas of linear algebra are introduced within the first seven lectures, in the concrete setting of \mathbb{R}^n , and then gradually examined from different points of view. Later generalizations of these concepts appear as natural extensions of familiar ideas, visualized through the geometric intuition developed in Chapter 1. A major achievement of this text is that the level of difficulty is fairly even throughout the course.

A Modern View of Matrix Multiplication

Good notation is crucial, and the text reflects the way scientists and engineers actually use linear algebra in practice. The definitions and proofs focus on the columns of a matrix rather than on the matrix entries. A central theme is to view a matrix–vector product $A\mathbf{x}$ as a linear combination of the columns of A . This modern approach simplifies many arguments, and it ties vector space ideas into the study of linear systems.

Linear Transformations

Linear transformations form a “thread” that is woven into the fabric of the text. Their use enhances the geometric flavor of the text. In Chapter 1, for instance, linear transformations provide a dynamic and graphical view of matrix–vector multiplication.

Eigenvalues and Dynamical Systems

Eigenvalues appear fairly early in the text, in Chapters 5 and 7. Because this material is spread over several weeks, students have more time than usual to absorb and review these critical concepts. Eigenvalues are motivated by and applied to discrete and continuous dynamical systems, which appear in Sections 1.10, 4.8, and 5.9, and in five sections of Chapter 5. Some courses reach Chapter 5 after about five weeks by covering Sections 2.8 and 2.9 instead of Chapter 4. These two optional sections present all the vector space concepts from Chapter 4 needed for Chapter 5.

Orthogonality and Least-Squares Problems

These topics receive a more comprehensive treatment than is commonly found in beginning texts. The original Linear Algebra Curriculum Study Group has emphasized the need for a substantial unit on orthogonality and least-squares problems, because orthogonality plays such an important role in computer calculations and numerical linear algebra and because inconsistent linear systems arise so often in practical work.

Pedagogical Features

Applications

A broad selection of applications illustrates the power of linear algebra to explain fundamental principles and simplify calculations in engineering, computer science, mathematics, physics, biology, economics, and statistics. Some applications appear in separate sections; others are treated in examples and exercises. In addition, each chapter opens with an introductory vignette that sets the stage for some application of linear algebra and provides a motivation for developing the mathematics that follows.

A Strong Geometric Emphasis

Every major concept in the course is given a geometric interpretation, because many students learn better when they can visualize an idea. There are substantially more drawings here than usual, and some of the figures have never before appeared in a linear algebra text. Interactive versions of many of these figures appear in MyLab Math.

Examples

This text devotes a larger proportion of its expository material to examples than do most linear algebra texts. There are more examples than an instructor would ordinarily present in class. But because the examples are written carefully, with lots of detail, students can read them on their own.

Theorems and Proofs

Important results are stated as theorems. Other useful facts are displayed in tinted boxes, for easy reference. Most of the theorems have formal proofs, written with the beginner student in mind. In a few cases, the essential calculations of a proof are exhibited in a carefully chosen example. Some routine verifications are saved for exercises, when they will benefit students.

Practice Problems

A few carefully selected Practice Problems appear just before each exercise set. Complete solutions follow the exercise set. These problems either focus on potential trouble spots in the exercise set or provide a “warm-up” for the exercises, and the solutions often contain helpful hints or warnings about the homework.

Exercises

The abundant supply of exercises ranges from routine computations to conceptual questions that require more thought. A good number of innovative questions pinpoint conceptual difficulties that we have found on student papers over the years. Each exercise set is carefully arranged in the same general order as the text; homework assignments are readily available when only part of a section is discussed. A notable feature of the exercises is their numerical simplicity. Problems “unfold” quickly, so students spend little time on numerical calculations. The exercises concentrate on teaching understanding rather than mechanical calculations. The exercises in the *Sixth Edition* maintain the integrity of the exercises from previous editions, while providing fresh problems for students and instructors.

Exercises marked with the symbol **T** are designed to be worked with the aid of a “matrix program” (a computer program, such as MATLAB, Maple, Mathematica, MathCad, or Derive, or a programmable calculator with matrix capabilities, such as those manufactured by Texas Instruments).

True/False Questions

To encourage students to read all of the text and to think critically, we have developed over 300 simple true/false questions that appear throughout the text, just after the computational problems. They can be answered directly from the text, and they prepare students for the conceptual problems that follow. Students appreciate these questions—after they get used to the importance of reading the text carefully. Based on class testing and discussions with students, we decided not to put the answers in the text. (The *Study Guide*, in MyLab Math, tells the students where to find the answers to the odd-numbered questions.) An additional 150 true/false questions (mostly at the ends of chapters) test understanding of the material. The text does provide simple T/F answers to most of these supplementary exercises, but it omits the justifications for the answers (which usually require some thought).

Writing Exercises

An ability to write coherent mathematical statements in English is essential for all students of linear algebra, not just those who may go to graduate school in mathematics.

The text includes many exercises for which a written justification is part of the answer. Conceptual exercises that require a short proof usually contain hints that help a student get started. For all odd-numbered writing exercises, either a solution is included at the back of the text or a hint is provided and the solution is given in the *Study Guide*.

Projects

A list of projects (available online) have been identified at the end of each chapter. They can be used by individual students or in groups. These projects provide the opportunity for students to explore fundamental concepts and applications in more detail. Two of the projects even encourage students to engage their creative side and use linear transformations to build artwork.

Reasonable Answers

Many of our students will enter a workforce where important decisions are being made based on answers provided by computers and other machines. The Reasonable Answers boxes and exercises help students develop an awareness of the need to analyze their answers for correctness and accuracy.

Computational Topics

The text stresses the impact of the computer on both the development and practice of linear algebra in science and engineering. Frequent Numerical Notes draw attention to issues in computing and distinguish between theoretical concepts, such as matrix inversion, and computer implementations, such as LU factorizations.

Acknowledgments

David Lay was grateful to many people who helped him over the years with various aspects of this book. He was particularly grateful to Israel Gohberg and Robert Ellis for more than fifteen years of research collaboration, which greatly shaped his view of linear algebra. And he was privileged to be a member of the Linear Algebra Curriculum Study Group along with David Carlson, Charles Johnson, and Duane Porter. Their creative ideas about teaching linear algebra have influenced this text in significant ways. He often spoke fondly of three good friends who guided the development of the book nearly from the beginning—giving wise counsel and encouragement—Greg Tobin, publisher; Laurie Rosatone, former editor; and William Hoffman, former editor.

Judi and Steven have been privileged to work on recent editions of Professor David Lay’s linear algebra book. In making this revision, we have attempted to maintain the basic approach and the clarity of style that has made earlier editions popular with students and faculty. We thank Eric Schulz for sharing his considerable technological and pedagogical expertise in the creation of the electronic textbook. His help and encouragement were essential in bringing the figures and examples to life in the Wolfram Cloud version of this textbook.

Mathew Hudelson has been a valuable colleague in preparing the *Sixth Edition*; he is always willing to brainstorm about concepts or ideas and test out new writing and exercises. He contributed the idea for new vignette for Chapter 3 and the accompanying

project. He has helped with new exercises throughout the text. Harley Weston has provided Judi with many years of good conversations about how, why, and who we appeal to when we present mathematical material in different ways. Katerina Tsatsomeros' artistic side has been a definite asset when we needed artwork to transform (the fish and the sheep), improved writing in the new introductory vignettes, or information from the perspective of college-age students.

We appreciate the encouragement and shared expertise from Nella Ludlow, Thomas Fischer, Amy Johnston, Cassandra Seubert, and Mike Manzano. They provided information about important applications of linear algebra and ideas for new examples and exercises. In particular, the new vignettes and material in Chapters 4 and 6 were inspired by conversations with these individuals.

We are energized by Sepideh Stewart and the other new Linear Algebra Curriculum Study Group (LACSG 2.0) members: Sheldon Axler, Rob Beezer, Eugene Boman, Minerva Catral, Guershon Harel, David Strong, and Megan Wawro. Initial meetings of this group have provided valuable guidance in revising the *Sixth Edition*.

We sincerely thank the following reviewers for their careful analyses and constructive suggestions:

Maila C. Brucal-Hallare, *Norfolk State University*
 Kristen Campbell, *Elgin Community College*
 Charles Conrad, *Volunteer State Community College*
 R. Darrell Finney, *Wilkes Community College*
 Xiaofeng Gu, *University of West Georgia*
 Jeong Mi-Yoon, *University of Houston–Downtown*
 Michael T. Muzheve, *Texas A&M U.–Kingsville*
 Jason Rusodimos, *Perimeter C. at Georgia State U.*
 Rebecca Swanson, *Colorado School of Mines*
 Casey Wynn, *Kenyon College*
 Taoye Zhang, *Penn State U.–Worthington Scranton*

Steven Burrow, *Central Texas College*
 J. S. Chahal, *Brigham Young University*
 Kevin Farrell, *Lyndon State College*
 Chris Fuller, *Cumberland University*
 Jeffrey Jauregui, *Union College*
 Christopher Murphy, *Guildford Tech. C.C.*
 Charles I. Odion, *Houston Community College*
 Desmond Stephens, *Florida Ag. and Mech. U.*
 Jiyuan Tao, *Loyola University–Maryland*
 Amy Yielding, *Eastern Oregon University*
 Houlong Zhuang, *Arizona State University*

We appreciate the proofreading and suggestions provided by John Samons and Jennifer Blue. Their careful eye has helped to minimize errors in this edition.

We thank Kristina Evans, Phil Oslin, and Jean Choe for their work in setting up and maintaining the online homework to accompany the text in MyLab Math, and for continuing to work with us to improve it. The reviews of the online homework done by Joan Saniuk, Robert Pierce, Doron Lubinsky and Adriana Corinaldesi were greatly appreciated. We also thank the faculty at University of California Santa Barbara, University of Alberta, Washington State University and the Georgia Institute of Technology for their feedback on the MyLab Math course. Joe Vetere has provided much appreciated technical help with the *Study Guide* and *Instructor's Solutions Manual*.

We thank Jeff Weidenaar, our content manager, for his continued careful, well-thought-out advice. Project Manager Ron Hampton has been a tremendous help guiding us through the production process. We are also grateful to Stacey Sveum and Rosemary Morton, our marketers, and Jon Krebs, our editorial associate, who have also contributed to the success of this edition.

Steven R. Lay and Judi J. McDonald

Acknowledgments for the Global Edition

Pearson would like to acknowledge and thank the following for their work on the Global Edition.

Contributors

José Luis Zuleta Estrugo, *École Polytechnique Fédérale de Lausanne*
Mohamad Rafi Segi Rahmat, *University of Nottingham Malaysia*

Reviewers

Sibel Doğru Akgöl, *Attilim University*
Hossam M. Hassan, *Cairo University*
Kwa Kiam Heong, *University of Malaya*
Yanghong Huang, *University of Manchester*
Natanael Karjanto, *Sungkyunkwan University*
Somitra Sanadhya, *Indraprastha Institute of Information Technology*
Veronique Van Lierde, *Al Akhawayn University in Ifrane*

Get the most out of MyLab Math



Pearson
MyLab

MyLab Math for *Linear Algebra and Its Applications* Lay, Lay, McDonald

MyLab Math features hundreds of assignable algorithmic exercises that mirror those in the text. It is tightly integrated with each author team's style, offering a range of author-created resources, so your students have a consistent experience.

eText with Interactive Figures

The eText includes **Interactive Figures** that bring the geometry of linear algebra to life. Students can manipulate figures and experiment with matrices to provide a deeper geometric understanding of key concepts and examples.

PEARSON Linear Algebra and Its Applications Lay, Lay, McDonald Section 7.2 Content ▾

▲ Classifying Quadratic Forms

When A is an $n \times n$ matrix, the quadratic form $Q(\mathbf{x}) = \mathbf{x}^T A \mathbf{x}$ is a real-valued function with domain \mathbb{R}^n . The sliders in Figure 4 can be adjusted to display several graphs of quadratic forms with domain \mathbb{R}^2 . For each point $\mathbf{x} = (x_1, x_2)$ in the domain of a quadratic form Q , the graph displays the point (x_1, x_2, z) where $z = Q(\mathbf{x})$. Notice that except at $\mathbf{x} = 0$, the values of $Q(\mathbf{x})$ are all positive when λ_1 and λ_2 are chosen to be positive. If they are both chosen to be negative, then except at $\mathbf{x} = 0$, the values of $Q(\mathbf{x})$ are all negative. In both these cases, the horizontal cross-sections of the graphs are ellipses. When λ_1 and λ_2 have opposite signs, then $Q(\mathbf{x})$ takes on both positive and negative values, and the cross-sections of graphs are hyperbolas.

this quadratic form is positive definite

λ_1 4
 λ_2 4

Examples: [positive definite](#) [negative definite](#) [indefinite](#)

Graphs of quadratic forms.

FIGURE 4 ▾

Teaching with Interactive Figures

Interactive Figure files are available within MyLab Math to use as a teaching tool for classroom demonstrations. Instructors can illustrate concepts that are difficult for students to visualize, leading to greater conceptual understanding.

Supporting Instruction

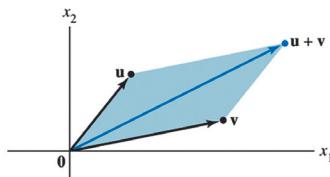
MyLab Math provides resources to help you assess and improve student results and unparalleled flexibility to create a course tailored to you and your students.

PowerPoint® Lecture Slides

Fully editable PowerPoint slides are available for all sections of the text. The slides include definitions, theorems, examples and solutions. When used in the classroom, these slides allow the instructor to focus on teaching, rather than writing on the board. PowerPoint slides are available to students (within the Video and Resource Library in MyLab Math) so that they can follow along.

PARALLELOGRAM RULE FOR ADDITION

- If \mathbf{u} and \mathbf{v} in \mathbb{R}^2 are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are \mathbf{u} , $\mathbf{0}$, and \mathbf{v} . See Fig. 3 below.



© 2016 Pearson Education, Inc.

Slide 1.3- 6

Copy and Assign Sample Assignments

Start Select Assignments

Course name Lay Linear Algebra 5e
Book Lay: Linear Algebra and Its Applications, 5e

Select the assignments you wish to copy.

Show All Homework Quizzes Tests

4. Vector Spaces Go

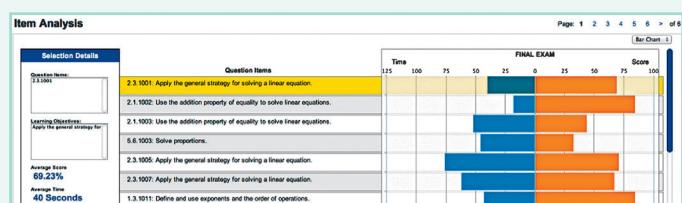
Copy	Assign	Ch.	Assignment Name	New Assignment Name
<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>	4	Section 4.1 Homework	Section 4.1 Homework
<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>	4	Section 4.2 Homework	Section 4.2 Homework
<input checked="" type="checkbox"/>	<input checked="" type="checkbox"/>	4	Section 4.3 Homework	Section 4.3 Homework

Sample Assignments

Sample Assignments are crafted to maximize student performance in the course. They make course set-up easier by giving instructors a starting point for each section.

Comprehensive Gradebook

The gradebook includes enhanced reporting functionality, such as item analysis and a reporting dashboard to enable you to efficiently manage your course. Student performance data are presented at the class, section, and program levels in an accessible, visual manner so you'll have the information you need to keep your students on track.



Resources for Success



Pearson
MyLab

Instructor Resources

Online resources can be downloaded from MyLab Math or from www.pearsonglobaleditions.com.

Instructor's Solution Manual

Includes fully worked solutions to all exercises in the text and teaching notes for many sections.

PowerPoint® Lecture Slides

These fully editable lecture slides are available for all sections of the text.

Instructor's Technology Manuals

Each manual provides detailed guidance for integrating technology throughout the course, written by faculty who teach with the software and this text. Available For MATLAB, Maple, Mathematica, and Texas Instruments graphing calculators.

TestGen®

TestGen (www.pearsoned.com/testgen) enables instructors to build, edit, print, and administer tests using a computerized bank of questions developed to cover all the objectives of the text.

Student Resources

Additional resources to enhance student success. All resources can be downloaded from MyLab Math.

Study Guide

Provides detailed worked-out solutions to every third odd-numbered exercise. Also, a complete explanation is provided whenever an odd-numbered writing exercise has a Hint in the answers. Special subsections of the *Study Guide* suggest how to master key concepts of the course. Frequent "Warnings" identify common errors and show how to prevent them. MATLAB boxes introduce commands as they are needed. Appendixes in the *Study Guide* provide comparable information about Maple, Mathematica, and TI graphing calculators. Available within MyLab math.

Getting Started with Technology

A quick-start guide for students to the technology they may use in this course. Available for MATLAB, Maple, Mathematica, or Texas Instrument graphing calculators. Downloadable from MyLab Math.

Projects

Exploratory projects, written by experienced faculty members, invite students to discover applications of linear algebra.



A Note to Students

This course is potentially the most interesting and worthwhile undergraduate mathematics course you will complete. In fact, some students have written or spoken to us after graduation and said that they still use this text occasionally as a reference in their careers at major corporations and engineering graduate schools. The following remarks offer some practical advice and information to help you master the material and enjoy the course.

In linear algebra, the *concepts* are as important as the *computations*. The simple numerical exercises that begin each exercise set only help you check your understanding of basic procedures. Later in your career, computers will do the calculations, but you will have to choose the calculations, know how to interpret the results, analyze whether the results are reasonable, then explain the results to other people. For this reason, many exercises in the text ask you to explain or justify your calculations. A written explanation is often required as part of the answer. If you are working on questions in MyLab Math, keep a notebook for calculations and notes on what you are learning. For odd-numbered exercises in the textbook, you will find either the desired explanation or at least a good hint. You must avoid the temptation to look at such answers before you have tried to write out the solution yourself. Otherwise, you are likely to think you understand something when in fact you do not.

To master the concepts of linear algebra, you will have to read and reread the text carefully. New terms are in boldface type, sometimes enclosed in a definition box. A glossary of terms is included at the end of the text. Important facts are stated as theorems or are enclosed in tinted boxes, for easy reference. We encourage you to read the Preface to learn more about the structure of this text. This will give you a framework for understanding how the course may proceed.

In a practical sense, linear algebra is a language. You must learn this language the same way you would a foreign language—with daily work. Material presented in one section is not easily understood unless you have thoroughly studied the text and worked the exercises for the preceding sections. Keeping up with the course will save you lots of time and distress!

Numerical Notes

We hope you read the Numerical Notes in the text, even if you are not using a computer or graphing calculator with the text. In real life, most applications of linear algebra involve numerical computations that are subject to some numerical error, even though that error may be extremely small. The Numerical Notes will warn you of potential difficulties in

using linear algebra later in your career, and if you study the notes now, you are more likely to remember them later.

If you enjoy reading the Numerical Notes, you may want to take a course later in numerical linear algebra. Because of the high demand for increased computing power, computer scientists and mathematicians work in numerical linear algebra to develop faster and more reliable algorithms for computations, and electrical engineers design faster and smaller computers to run the algorithms. This is an exciting field, and your first course in linear algebra will help you prepare for it.

Study Guide

To help you succeed in this course, we suggest that you use the *Study Guide* available in MyLab Math. Not only will it help you learn linear algebra, it also will show you how to study mathematics. At strategic points in your textbook, marginal notes will remind you to check that section of the *Study Guide* for special subsections entitled “Mastering Linear Algebra Concepts.” There you will find suggestions for constructing effective review sheets of key concepts. The act of preparing the sheets is one of the secrets to success in the course, because you will construct *links between ideas*. These links are the “glue” that enables you to build a solid foundation for learning and *remembering* the main concepts in the course.

The *Study Guide* contains a detailed solution to more than a third of the odd-numbered exercises, plus solutions to all odd-numbered writing exercises for which only a hint is given in the Answers section of this book. The *Guide* is separate from the text because you must learn to write solutions by yourself, without much help. (We know from years of experience that easy access to solutions in the back of the text slows the mathematical development of most students.) The *Guide* also provides warnings of common errors and helpful hints that call attention to key exercises and potential exam questions.

If you have access to technology—MATLAB, Octave, Maple, Mathematica, or a TI graphing calculator—you can save many hours of homework time. The *Study Guide* is your “lab manual” that explains how to use each of these matrix utilities. It introduces new commands when they are needed. You will also find that most software commands you might use are easily found using an online search engine. Special matrix commands will perform the computations for you!

What you do in your first few weeks of studying this course will set your pattern for the term and determine how well you finish the course. Please read “How to Study Linear Algebra” in the *Study Guide* as soon as possible. Many students have found the strategies there very helpful, and we hope you will, too.

This page is intentionally left blank

1

Linear Equations in Linear Algebra



Introductory Example

LINEAR MODELS IN ECONOMICS AND ENGINEERING

It was late summer in 1949. Harvard Professor Wassily Leontief was carefully feeding the last of his punched cards into the university's Mark II computer. The cards contained information about the U.S. economy and represented a summary of more than 250,000 pieces of information produced by the U.S. Bureau of Labor Statistics after two years of intensive work. Leontief had divided the U.S. economy into 500 "sectors," such as the coal industry, the automotive industry, communications, and so on. For each sector, he had written a linear equation that described how the sector distributed its output to the other sectors of the economy. Because the Mark II, one of the largest computers of its day, could not handle the resulting system of 500 equations in 500 unknowns, Leontief had distilled the problem into a system of 42 equations in 42 unknowns.

Programming the Mark II computer for Leontief's 42 equations had required several months of effort, and he was anxious to see how long the computer would take to solve the problem. The Mark II hummed and blinked for 56 hours before finally producing a solution. We will discuss the nature of this solution in Sections 1.6 and 2.6.

Leontief, who was awarded the 1973 Nobel Prize in Economic Science, opened the door to a new era in mathematical modeling in economics. His efforts at Harvard in 1949 marked one of the first significant uses of computers to analyze what was then a large-scale

mathematical model. Since that time, researchers in many other fields have employed computers to analyze mathematical models. Because of the massive amounts of data involved, the models are usually *linear*; that is, they are described by *systems of linear equations*.

The importance of linear algebra for applications has risen in direct proportion to the increase in computing power, with each new generation of hardware and software triggering a demand for even greater capabilities. Computer science is thus intricately linked with linear algebra through the explosive growth of parallel processing and large-scale computations.

Scientists and engineers now work on problems far more complex than even dreamed possible a few decades ago. Today, linear algebra has more potential value for students in many scientific and business fields than any other undergraduate mathematics subject! The material in this text provides the foundation for further work in many interesting areas. Here are a few possibilities; others will be described later.

- *Oil exploration.* When a ship searches for offshore oil deposits, its computers solve thousands of separate systems of linear equations *every day*. The seismic data for the equations are obtained from underwater shock waves created by explosions from air guns. The waves bounce off subsurface

rocks and are measured by geophones attached to mile-long cables behind the ship.

- *Linear programming.* Many important management decisions today are made on the basis of linear programming models that use hundreds of variables. The airline industry, for instance, employs linear programs that schedule flight crews, monitor the locations of aircraft, or plan the varied schedules of support services such as maintenance and terminal operations.
- *Electrical networks.* Engineers use simulation software to design electrical circuits and microchips involving millions of transistors. Such software

relies on linear algebra techniques and systems of linear equations.

- *Artificial intelligence.* Linear algebra plays a key role in everything from scrubbing data to facial recognition.
- *Signals and signal processing.* From a digital photograph to the daily price of a stock, important information is recorded as a signal and processed using linear transformations.
- *Machine learning.* Machines (specifically computers) use linear algebra to learn about anything from online shopping preferences to speech recognition.

Systems of linear equations lie at the heart of linear algebra, and this chapter uses them to introduce some of the central concepts of linear algebra in a simple and concrete setting. Sections 1.1 and 1.2 present a systematic method for solving systems of linear equations. This algorithm will be used for computations throughout the text. Sections 1.3 and 1.4 show how a system of linear equations is equivalent to a *vector equation* and to a *matrix equation*. This equivalence will reduce problems involving linear combinations of vectors to questions about systems of linear equations. The fundamental concepts of spanning, linear independence, and linear transformations, studied in the second half of the chapter, will play an essential role throughout the text as we explore the beauty and power of linear algebra.

1.1 Systems of Linear Equations

A **linear equation** in the variables x_1, \dots, x_n is an equation that can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \quad (1)$$

where b and the **coefficients** a_1, \dots, a_n are real or complex numbers, usually known in advance. The subscript n may be any positive integer. In textbook examples and exercises, n is normally between 2 and 5. In real-life problems, n might be 50 or 5000, or even larger.

The equations

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2(\sqrt{6} - x_1) + x_3$$

are both linear because they can be rearranged algebraically as in equation (1):

$$3x_1 - 5x_2 = -2 \quad \text{and} \quad 2x_1 + x_2 - x_3 = 2\sqrt{6}$$

The equations

$$4x_1 - 5x_2 = x_1x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 6$$

are not linear because of the presence of x_1x_2 in the first equation and $\sqrt{x_1}$ in the second.

A **system of linear equations** (or a **linear system**) is a collection of one or more linear equations involving the same variables—say, x_1, \dots, x_n . An example is

$$\begin{aligned} 2x_1 - x_2 + 1.5x_3 &= 8 \\ x_1 - 4x_3 &= -7 \end{aligned} \tag{2}$$

A **solution** of the system is a list (s_1, s_2, \dots, s_n) of numbers that makes each equation a true statement when the values s_1, \dots, s_n are substituted for x_1, \dots, x_n , respectively. For instance, $(5, 6.5, 3)$ is a solution of system (2) because, when these values are substituted in (2) for x_1, x_2, x_3 , respectively, the equations simplify to $8 = 8$ and $-7 = -7$.

The set of all possible solutions is called the **solution set** of the linear system. Two linear systems are called **equivalent** if they have the same solution set. That is, each solution of the first system is a solution of the second system, and each solution of the second system is a solution of the first.

Finding the solution set of a system of two linear equations in two variables is easy because it amounts to finding the intersection of two lines. A typical problem is

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned}$$

The graphs of these equations are lines, which we denote by ℓ_1 and ℓ_2 . A pair of numbers (x_1, x_2) satisfies *both* equations in the system if and only if the point (x_1, x_2) lies on both ℓ_1 and ℓ_2 . In the system above, the solution is the single point $(3, 2)$, as you can easily verify. See Figure 1.

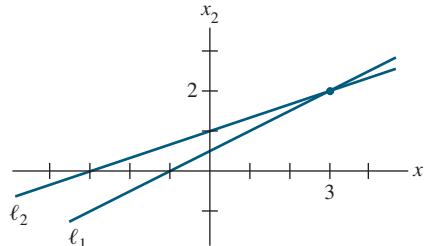


FIGURE 1 Exactly one solution.

Of course, two lines need not intersect in a single point—they could be parallel, or they could coincide and hence “intersect” at every point on the line. Figure 2 shows the graphs that correspond to the following systems:

$$\begin{array}{ll} \text{(a)} & x_1 - 2x_2 = -1 \\ & -x_1 + 2x_2 = 3 \\ \text{(b)} & x_1 - 2x_2 = -1 \\ & -x_1 + 2x_2 = 1 \end{array}$$

Figures 1 and 2 illustrate the following general fact about linear systems, to be verified in Section 1.2.

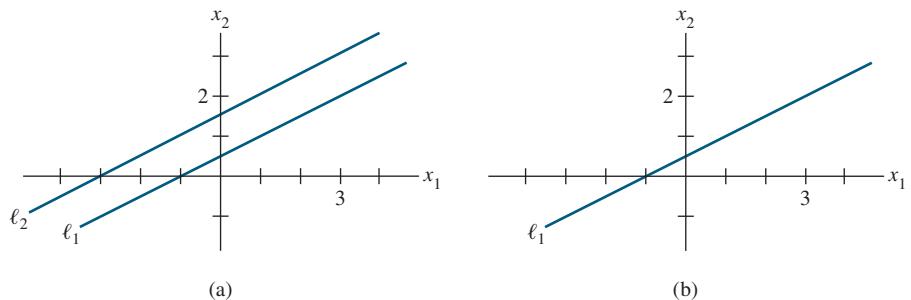


FIGURE 2 (a) No solution. (b) Infinitely many solutions.

A system of linear equations has

1. no solution, or
2. exactly one solution, or
3. infinitely many solutions.

A system of linear equations is said to be **consistent** if it has either one solution or infinitely many solutions; a system is **inconsistent** if it has no solution.

Matrix Notation

The essential information of a linear system can be recorded compactly in a rectangular array called a **matrix**. Given the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 &\quad - 5x_3 = 10 \end{aligned} \tag{3}$$

with the coefficients of each variable aligned in columns, the matrix

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix}$$

is called the **coefficient matrix** (or **matrix of coefficients**) of the system (3), and the matrix

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{bmatrix} \tag{4}$$

is called the **augmented matrix** of the system. (The second row here contains a zero because the second equation could be written as $0 \cdot x_1 + 2x_2 - 8x_3 = 8$.) An augmented matrix of a system consists of the coefficient matrix with an added column containing the constants from the respective right sides of the equations.

The **size** of a matrix tells how many rows and columns it has. The augmented matrix (4) above has 3 rows and 4 columns and is called a 3×4 (read “3 by 4”) matrix. If m and n are positive integers, an **$m \times n$ matrix** is a rectangular array of numbers with m rows and n columns. (The number of rows always comes first.) Matrix notation will simplify the calculations in the examples that follow.

Solving a Linear System

This section and the next describe an algorithm, or a systematic procedure, for solving linear systems. The basic strategy is *to replace one system with an equivalent system (that is one with the same solution set) that is easier to solve*.

Roughly speaking, use the x_1 term in the first equation of a system to eliminate the x_1 terms in the other equations. Then use the x_2 term in the second equation to eliminate the x_2 terms in the other equations, and so on, until you finally obtain a very simple equivalent system of equations.

Three basic operations are used to simplify a linear system: Replace one equation by the sum of itself and a multiple of another equation, interchange two equations, and multiply all the terms in an equation by a nonzero constant. After the first example, you will see why these three operations do not change the solution set of the system.

EXAMPLE 1 Solve system (3).

SOLUTION The elimination procedure is shown here with and without matrix notation, and the results are placed side by side for comparison:

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 = 0 & \left[\begin{array}{rrr} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{array} \right] \\ 2x_2 - 8x_3 = 8 & \\ 5x_1 - 5x_3 = 10 & \end{array}$$

Keep x_1 in the first equation and eliminate it from the other equations. To do so, add -5 times equation 1 to equation 3. After some practice, this type of calculation is usually performed mentally:

$$\begin{array}{rcl} -5 \cdot [\text{equation 1}] & -5x_1 + 10x_2 - 5x_3 = 0 \\ + [\text{equation 3}] & 5x_1 - 5x_3 = 10 \\ \hline [\text{new equation 3}] & 10x_2 - 10x_3 = 10 \end{array}$$

The result of this calculation is written in place of the original third equation:

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 = 0 & \left[\begin{array}{rrr} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 0 & 10 & -10 \end{array} \right] \\ 2x_2 - 8x_3 = 8 & \\ 10x_2 - 10x_3 = 10 & \end{array}$$

Now, multiply equation 2 by $\frac{1}{2}$ in order to obtain 1 as the coefficient for x_2 . (This calculation will simplify the arithmetic in the next step.)

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 = 0 & \left[\begin{array}{rrr} 1 & -2 & 1 \\ 0 & 1 & -4 \\ 0 & 10 & -10 \end{array} \right] \\ x_2 - 4x_3 = 4 & \\ 10x_2 - 10x_3 = 10 & \end{array}$$

Use the x_2 in equation 2 to eliminate the $10x_2$ in equation 3. The “mental” computation is

$$\begin{array}{rcl} -10 \cdot [\text{equation 2}] & -10x_2 + 40x_3 = -40 \\ + [\text{equation 3}] & 10x_2 - 10x_3 = 10 \\ \hline [\text{new equation 3}] & 30x_3 = -30 \end{array}$$

The result of this calculation is written in place of the previous third equation (row):

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 = 0 & \left[\begin{array}{rrr} 1 & -2 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 30 \end{array} \right] \\ x_2 - 4x_3 = 4 & \\ 30x_3 = -30 & \end{array}$$

Now, multiply equation 3 by $\frac{1}{30}$ in order to obtain 1 as the coefficient for x_3 . (This calculation will simplify the arithmetic in the next step.)

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 = 0 & \left[\begin{array}{rrr} 1 & -2 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{array} \right] \\ x_2 - 4x_3 = 4 & \\ x_3 = -1 & \end{array}$$

The new system has a *triangular* form (the intuitive term *triangular* will be replaced by a precise term in the next section):

$$\begin{array}{l} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = -1 \end{array} \quad \left[\begin{array}{cccc} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Eventually, you want to eliminate the $-2x_2$ term from equation 1, but it is more efficient to use the x_3 in equation 3 first, to eliminate the $-4x_3$ and $+x_3$ terms in equations 2 and 1. The two “mental” calculations are

$$\begin{array}{rcl} 4 \cdot [\text{equation 3}] & 4x_3 = -4 & -1 \cdot [\text{equation 3}] \\ + [\text{equation 2}] & x_2 - 4x_3 = 4 & + [\text{equation 1}] \\ \hline [\text{new equation 2}] & x_2 = 0 & [\text{new equation 1}] \\ & & \end{array} \quad \begin{array}{rcl} & & -x_3 = 1 \\ & & x_1 - 2x_2 + x_3 = 0 \\ & & x_1 - 2x_2 = 1 \end{array}$$

It is convenient to combine the results of these two operations:

$$\begin{array}{l} x_1 - 2x_2 = 1 \\ x_2 = 0 \\ x_3 = -1 \end{array} \quad \left[\begin{array}{cccc} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Now, having cleaned out the column above the x_3 in equation 3, move back to the x_2 in equation 2 and use it to eliminate the $-2x_2$ above it. Because of the previous work with x_3 , there is now no arithmetic involving x_3 terms. Add 2 times equation 2 to equation 1 and obtain the system:

$$\begin{array}{l} x_1 = 1 \\ x_2 = 0 \\ x_3 = -1 \end{array} \quad \left[\begin{array}{cccc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

The work is essentially done. It shows that the only solution of the original system is $(1, 0, -1)$. However, since there are so many calculations involved, it is a good practice to check the work. To verify that $(1, 0, -1)$ is a solution, substitute these values into the left side of the original system, and compute:

$$\begin{aligned} 1(1) - 2(0) + 1(-1) &= 1 - 0 - 1 = 0 \\ 2(0) - 8(-1) &= 0 + 8 = 8 \\ 5(1) - 5(-1) &= 5 + 5 = 10 \end{aligned}$$

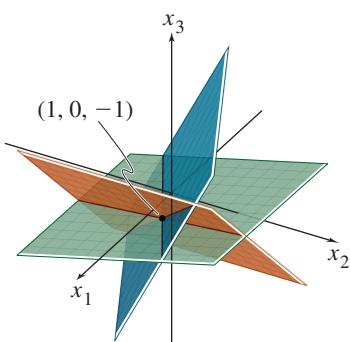
The results agree with the right side of the original system, so $(1, 0, -1)$ is a solution of the system. ■

Example 1 illustrates how operations on equations in a linear system correspond to operations on the appropriate rows of the augmented matrix. The three basic operations listed earlier correspond to the following operations on the augmented matrix.

ELEMENTARY ROW OPERATIONS

1. (Replacement) Replace one row by the sum of itself and a multiple of another row.¹
2. (Interchange) Interchange two rows.
3. (Scaling) Multiply all entries in a row by a nonzero constant.

¹A common paraphrase of row replacement is “Add to one row a multiple of another row.”



Each of the original equations determines a plane in three-dimensional space. The point $(1, 0, -1)$ lies in all three planes.

Row operations can be applied to any matrix, not merely to one that arises as the augmented matrix of a linear system. Two matrices are called **row equivalent** if there is a sequence of elementary row operations that transforms one matrix into the other.

It is important to note that row operations are *reversible*. If two rows are interchanged, they can be returned to their original positions by another interchange. If a row is scaled by a nonzero constant c , then multiplying the new row by $1/c$ produces the original row. Finally, consider a replacement operation involving two rows—say, rows 1 and 2—and suppose that c times row 1 is added to row 2 to produce a new row 2. To “reverse” this operation, add $-c$ times row 1 to (new) row 2 and obtain the original row 2. See Exercises 39–42 at the end of this section.

At the moment, we are interested in row operations on the augmented matrix of a system of linear equations. Suppose a system is changed to a new one via row operations. By considering each type of row operation, you can see that any solution of the original system remains a solution of the new system. Conversely, since the original system can be produced via row operations on the new system, each solution of the new system is also a solution of the original system. This discussion justifies the following statement.

If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

Though Example 1 is lengthy, you will find that after some practice, the calculations go quickly. Row operations in the text and exercises will usually be extremely easy to perform, allowing you to focus on the underlying concepts. Still, you must learn to perform row operations accurately because they will be used throughout the text.

The rest of this section shows how to use row operations to determine the size of a solution set, without completely solving the linear system.

Existence and Uniqueness Questions

Section 1.2 will show why a solution set for a linear system contains either no solutions, one solution, or infinitely many solutions. Answers to the following two questions will determine the nature of the solution set for a linear system.

To determine which possibility is true for a particular system, we ask two questions.

TWO FUNDAMENTAL QUESTIONS ABOUT A LINEAR SYSTEM

1. Is the system consistent; that is, does at least one solution *exist*?
2. If a solution exists, is it the *only* one; that is, is the solution *unique*?

These two questions will appear throughout the text, in many different guises. This section and the next will show how to answer these questions via row operations on the augmented matrix.

EXAMPLE 2 Determine if the following system is consistent:

$$\begin{aligned}x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 - 5x_3 &= 10\end{aligned}$$

SOLUTION This is the system from Example 1. Suppose that we have performed the row operations necessary to obtain the triangular form

$$\begin{array}{l} x_1 - 2x_2 + x_3 = 0 \\ x_2 - 4x_3 = 4 \\ x_3 = -1 \end{array} \quad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

At this point, we know x_3 . Were we to substitute the value of x_3 into equation 2, we could compute x_2 and hence could determine x_1 from equation 1. So a solution exists; the system is consistent. (In fact, x_2 is uniquely determined by equation 2 since x_3 has only one possible value, and x_1 is therefore uniquely determined by equation 1. So the solution is unique.) ■

EXAMPLE 3 Determine if the following system is consistent:

$$\begin{array}{l} x_2 - 4x_3 = 8 \\ 2x_1 - 3x_2 + 2x_3 = 1 \\ 4x_1 - 8x_2 + 12x_3 = 1 \end{array} \quad (5)$$

SOLUTION The augmented matrix is

$$\left[\begin{array}{ccc|c} 0 & 1 & -4 & 8 \\ 2 & -3 & 2 & 1 \\ 4 & -8 & 12 & 1 \end{array} \right]$$

To obtain an x_1 in the first equation, interchange rows 1 and 2:

$$\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 4 & -8 & 12 & 1 \end{array} \right]$$

To eliminate the $4x_1$ term in the third equation, add -2 times row 1 to row 3:

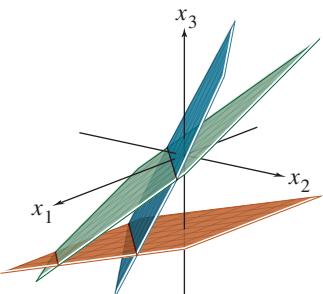
$$\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & -2 & 8 & -1 \end{array} \right] \quad (6)$$

Next, use the x_2 term in the second equation to eliminate the $-2x_2$ term from the third equation. Add 2 times row 2 to row 3:

$$\left[\begin{array}{ccc|c} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 15 \end{array} \right] \quad (7)$$

The augmented matrix is now in triangular form. To interpret it correctly, go back to equation notation:

$$\begin{array}{l} 2x_1 - 3x_2 + 2x_3 = 1 \\ x_2 - 4x_3 = 8 \\ 0 = 15 \end{array} \quad (8)$$



The system is inconsistent because there is no point that lies on all three planes.

The equation $0 = 15$ is a short form of $0x_1 + 0x_2 + 0x_3 = 15$. This system in triangular form obviously has a built-in contradiction. There are no values of x_1, x_2, x_3 that satisfy (8) because the equation $0 = 15$ is never true. Since (8) and (5) have the same solution set, the original system is inconsistent (it has no solution). ■

Pay close attention to the augmented matrix in (7). Its last row is typical of an inconsistent system in triangular form.

Reasonable Answers

Once you have one or more solutions to a system of equations, remember to check your answer by substituting the solution you found back into the original equation. For example, if you found $(2, 1, -1)$ was a solution to the system of equations

$$\begin{array}{rcl} x_1 - 2x_2 + x_3 & = & 2 \\ x_1 & - 2x_3 & = -2 \\ x_2 + x_3 & = & 3 \end{array}$$

you could substitute your solution into the original equations to get

$$\begin{array}{rcl} 2 - 2(1) + (-1) & = & -1 \neq 2 \\ 2 & - 2(-1) & = 4 \neq -2 \\ 1 + (-1) & = & 0 \neq 3 \end{array}$$

It is now clear that there must have been an error in your original calculations. If upon rechecking your arithmetic, you find the answer $(2, 1, 2)$, you can see that

$$\begin{array}{rcl} 2 - 2(1) + (2) & = & 2 = 2 \\ 2 & - 2(2) & = -2 = -2 \\ 1 + 2 & = & 3 = 3 \end{array}$$

and you can now be confident you have a correct solution to the given system of equations.

Numerical Note

In real-world problems, systems of linear equations are solved by a computer. For a square coefficient matrix, computer programs nearly always use the elimination algorithm given here and in Section 1.2, modified slightly for improved accuracy.

The vast majority of linear algebra problems in business and industry are solved with programs that use *floating point arithmetic*. Numbers are represented as decimals $\pm.d_1 \cdots d_p \times 10^r$, where r is an integer and the number p of digits to the right of the decimal point is usually between 8 and 16. Arithmetic with such numbers typically is inexact, because the result must be rounded (or truncated) to the number of digits stored. “Roundoff error” is also introduced when a number such as $1/3$ is entered into the computer, since its decimal representation must be approximated by a finite number of digits. Fortunately, inaccuracies in floating point arithmetic seldom cause problems. The numerical notes in this book will occasionally warn of issues that you may need to consider later in your career.

Practice Problems

Throughout the text, practice problems should be attempted before working the exercises. Solutions appear after each exercise set.

- State in words the next elementary row operation that should be performed on the system in order to solve it. [More than one answer is possible in (a).]

Practice Problems (Continued)

$$\begin{array}{l} \text{a. } x_1 + 4x_2 - 2x_3 + 8x_4 = 12 \\ \quad x_2 - 7x_3 + 2x_4 = -4 \\ \quad 5x_3 - x_4 = 7 \\ \quad x_3 + 3x_4 = -5 \\ \text{b. } x_1 - 3x_2 + 5x_3 - 2x_4 = 0 \\ \quad x_2 + 8x_3 = -4 \\ \quad 2x_3 = 3 \\ \quad x_4 = 1 \end{array}$$

2. The augmented matrix of a linear system has been transformed by row operations into the form below. Determine if the system is consistent.

$$\left[\begin{array}{cccc} 1 & 5 & 2 & -6 \\ 0 & 4 & -7 & 2 \\ 0 & 0 & 5 & 0 \end{array} \right]$$

3. Is $(3, 4, -2)$ a solution of the following system?

$$\begin{array}{l} 5x_1 - x_2 + 2x_3 = 7 \\ -2x_1 + 6x_2 + 9x_3 = 0 \\ -7x_1 + 5x_2 - 3x_3 = -7 \end{array}$$

4. For what values of h and k is the following system consistent?

$$\begin{array}{l} 2x_1 - x_2 = h \\ -6x_1 + 3x_2 = k \end{array}$$

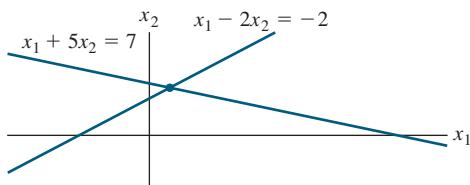
1.1 Exercises

Solve each system in Exercises 1–4 by using elementary row operations on the equations or on the augmented matrix. Follow the systematic elimination procedure described in this section.

1. $x_1 + 5x_2 = 7$
 $-2x_1 - 7x_2 = -5$

2. $2x_1 + 4x_2 = -4$
 $5x_1 + 7x_2 = 11$

3. Find the point (x_1, x_2) that lies on the line $x_1 + 5x_2 = 7$ and on the line $x_1 - 2x_2 = -2$. See the figure.



4. Find the point of intersection of the lines $x_1 - 5x_2 = 1$ and $3x_1 - 7x_2 = 5$.

Consider each matrix in Exercises 5 and 6 as the augmented matrix of a linear system. State in words the next two elementary row operations that should be performed in the process of solving the system.

5. $\left[\begin{array}{ccccc} 1 & 3 & -4 & 0 & 9 \\ 1 & 1 & 5 & 0 & -8 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 & -6 \end{array} \right]$

6. $\left[\begin{array}{ccccc} 1 & -6 & 4 & 0 & -1 \\ 0 & 2 & -7 & 0 & 4 \\ 0 & 0 & 1 & 2 & -3 \\ 0 & 0 & 3 & 1 & 6 \end{array} \right]$

In Exercises 7–10, the augmented matrix of a linear system has been reduced by row operations to the form shown. In each case, continue the appropriate row operations and describe the solution set of the original system.

7. $\left[\begin{array}{ccccc} 1 & 7 & 3 & -4 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{array} \right]$

8. $\left[\begin{array}{ccccc} 1 & 1 & 5 & 0 \\ 0 & 1 & 9 & 0 \\ 0 & 0 & 7 & -7 \end{array} \right]$

9. $\left[\begin{array}{ccccc} 1 & -1 & 0 & 0 & -4 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 & 4 \end{array} \right]$

10.
$$\begin{bmatrix} 1 & -2 & 0 & 3 & 0 \\ 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Solve the systems in Exercises 11–14.

11. $x_2 + 4x_3 = -4$

$x_1 + 3x_2 + 3x_3 = -2$

$3x_1 + 7x_2 + 5x_3 = 6$

12. $x_1 - 3x_2 + 4x_3 = -4$

$3x_1 - 7x_2 + 7x_3 = -8$

$-4x_1 + 6x_2 + 2x_3 = 4$

13. $x_1 - 3x_3 = 8$

$2x_1 + 2x_2 + 9x_3 = 7$

$x_2 + 5x_3 = -2$

14. $x_1 - 3x_2 = 5$

$-x_1 + x_2 + 5x_3 = 2$

$x_2 + x_3 = 0$

15. Verify that the solution you found to Exercise 11 is correct by substituting the values you obtained back into the original equations.

16. Verify that the solution you found to Exercise 12 is correct by substituting the values you obtained back into the original equations.

17. Verify that the solution you found to Exercise 13 is correct by substituting the values you obtained back into the original equations.

18. Verify that the solution you found to Exercise 14 is correct by substituting the values you obtained back into the original equations.

Determine if the systems in Exercises 19 and 20 are consistent. Do not completely solve the systems.

19. $x_1 + 3x_3 = 2$

$x_2 - 3x_4 = 3$

$-2x_2 + 3x_3 + 2x_4 = 1$

$3x_1 + 7x_4 = -5$

20. $x_1 - 2x_4 = -3$

$2x_2 + 2x_3 = 0$

$x_3 + 3x_4 = 1$

$-2x_1 + 3x_2 + 2x_3 + x_4 = 5$

21. Do the three lines $x_1 - 4x_2 = 1$, $2x_1 - x_2 = -3$, and $-x_1 - 3x_2 = 4$ have a common point of intersection? Explain.

22. Do the three planes $x_1 + 2x_2 + x_3 = 4$, $x_2 - x_3 = 1$, and $x_1 + 3x_2 = 0$ have at least one common point of intersection? Explain.

In Exercises 23–26, determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

23.
$$\begin{bmatrix} 1 & h & 4 \\ 3 & 6 & 8 \end{bmatrix}$$

24.
$$\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$$

25.
$$\begin{bmatrix} 1 & 3 & -2 \\ -4 & h & 8 \end{bmatrix}$$

26.
$$\begin{bmatrix} 3 & -4 & h \\ -6 & 8 & 9 \end{bmatrix}$$

In Exercises 27–34, key statements from this section are either quoted directly, restated slightly (but still true), or altered in some way that makes them false in some cases. Mark each statement True or False, and *justify* your answer. (If true, give the approximate location where a similar statement appears, or refer to a definition or theorem. If false, give the location of a statement that has been quoted or used incorrectly, or cite an example that shows the statement is not true in all cases.) Similar true/false questions will appear in many sections of the text and will be flagged with a (T/F) at the beginning of the question.

27. (T/F) Every elementary row operation is reversible.

28. (T/F) Elementary row operations on an augmented matrix never change the solution set of the associated linear system.

29. (T/F) A 5×6 matrix has six rows.

30. (T/F) Two matrices are row equivalent if they have the same number of rows.

31. (T/F) The solution set of a linear system involving variables x_1, \dots, x_n is a list of numbers (s_1, \dots, s_n) that makes each equation in the system a true statement when the values s_1, \dots, s_n are substituted for x_1, \dots, x_n , respectively.

32. (T/F) An inconsistent system has more than one solution.

33. (T/F) Two fundamental questions about a linear system involve existence and uniqueness.

34. (T/F) Two linear systems are equivalent if they have the same solution set.

35. Find an equation involving g , h , and k that makes this augmented matrix correspond to a consistent system:

$$\left[\begin{array}{cccc|c} 1 & -3 & 5 & g \\ 0 & 2 & -3 & h \\ -3 & 5 & -9 & k \end{array} \right]$$

36. Construct three different augmented matrices for linear systems whose solution set is $x_1 = -2$, $x_2 = 1$, $x_3 = 0$.

37. Suppose the system below is consistent for all possible values of f and g . What can you say about the coefficients c and d ? Justify your answer.

$x_1 + 5x_2 = f$

$cx_1 + dx_2 = g$

38. Suppose a , b , c , and d are constants such that a is not zero and the system below is consistent for all possible values of f and g . What can you say about the numbers a , b , c , and d ? Justify your answer.

$$ax_1 + bx_2 = f$$

$$cx_1 + dx_2 = g$$

In Exercises 39–42, find the elementary row operation that transforms the first matrix into the second, and then find the reverse row operation that transforms the second matrix into the first.

39. $\begin{bmatrix} 0 & -2 & 5 \\ 1 & 4 & -7 \\ 3 & -1 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 4 & -7 \\ 0 & -2 & 5 \\ 3 & -1 & 6 \end{bmatrix}$

40. $\begin{bmatrix} 1 & 3 & -4 \\ 0 & -2 & 6 \\ 0 & -5 & 9 \end{bmatrix}, \begin{bmatrix} 1 & 3 & -4 \\ 0 & 1 & -3 \\ 0 & -5 & 9 \end{bmatrix}$

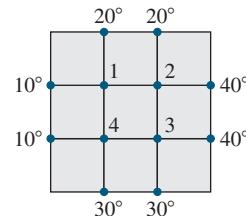
41. $\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 4 & -5 & 6 \\ 5 & -7 & 8 & -9 \end{bmatrix}, \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 4 & -5 & 6 \\ 0 & 8 & -2 & -9 \end{bmatrix}$

42. $\begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & -3 & 9 & 5 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -3 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

An important concern in the study of heat transfer is to determine the steady-state temperature distribution of a thin plate when the

temperature around the boundary is known. Assume the plate shown in the figure represents a cross section of a metal beam, with negligible heat flow in the direction perpendicular to the plate. Let T_1, \dots, T_4 denote the temperatures at the four interior nodes of the mesh in the figure. The temperature at a node is approximately equal to the average of the four nearest nodes—to the left, above, to the right, and below.² For instance,

$$T_1 = (10 + 20 + T_2 + T_4)/4, \quad \text{or} \quad 4T_1 - T_2 - T_4 = 30$$



43. Write a system of four equations whose solution gives estimates for the temperatures T_1, \dots, T_4 .
44. Solve the system of equations from Exercise 43. [Hint: To speed up the calculations, interchange rows 1 and 4 before starting “replace” operations.]

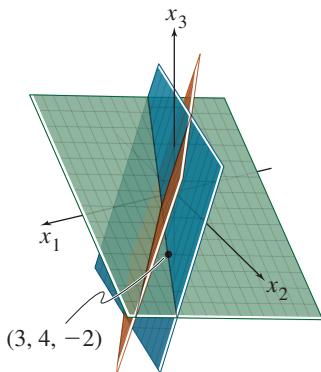
² See Frank M. White, *Heat and Mass Transfer* (Reading, MA: Addison-Wesley Publishing, 1991), pp. 145–149.

Solutions to Practice Problems

1. a. For “hand computation,” the best choice is to interchange equations 3 and 4. Another possibility is to multiply equation 3 by 1/5. Or, replace equation 4 by its sum with $-1/5$ times row 3. (In any case, do not use the x_2 in equation 2 to eliminate the $4x_2$ in equation 1. Wait until a triangular form has been reached and the x_3 terms and x_4 terms have been eliminated from the first two equations.)
- b. The system is in triangular form. Further simplification begins with the x_4 in the fourth equation. Use the x_4 to eliminate all x_4 terms above it. The appropriate step now is to add 2 times equation 4 to equation 1. (After that, move to equation 3, multiply it by 1/2, and then use the equation to eliminate the x_3 terms above it.)
2. The system corresponding to the augmented matrix is

$$\begin{aligned} x_1 + 5x_2 + 2x_3 &= -6 \\ 4x_2 - 7x_3 &= 2 \\ 5x_3 &= 0 \end{aligned}$$

The third equation makes $x_3 = 0$, which is certainly an allowable value for x_3 . After eliminating the x_3 terms in equations 1 and 2, you could go on to solve for unique values for x_2 and x_1 . Hence a solution exists, and it is unique. Contrast this situation with that in Example 3.



Since $(3, 4, -2)$ satisfies the first two equations, it is on the line of the intersection of the first two planes. Since $(3, 4, -2)$ does not satisfy all three equations, it does not lie on all three planes.

3. It is easy to check if a specific list of numbers is a solution. Set $x_1 = 3$, $x_2 = 4$, and $x_3 = -2$, and find that

$$\begin{aligned} 5(3) - (4) + 2(-2) &= 15 - 4 - 4 = 7 \\ -2(3) + 6(4) + 9(-2) &= -6 + 24 - 18 = 0 \\ -7(3) + 5(4) - 3(-2) &= -21 + 20 + 6 = 5 \end{aligned}$$

Although the first two equations are satisfied, the third is not, so $(3, 4, -2)$ is not a solution of the system. Notice the use of parentheses when making the substitutions. They are strongly recommended as a guard against arithmetic errors.

4. When the second equation is replaced by its sum with 3 times the first equation, the system becomes

$$\begin{aligned} 2x_1 - x_2 &= h \\ 0 &= k + 3h \end{aligned}$$

If $k + 3h$ is nonzero, the system has no solution. The system is consistent for any values of h and k that make $k + 3h = 0$.

1.2 Row Reduction and Echelon Forms

This section refines the method of Section 1.1 into a row reduction algorithm that will enable us to analyze any system of linear equations.¹ By using only the first part of the algorithm, we will be able to answer the fundamental existence and uniqueness questions posed in Section 1.1.

The algorithm applies to any matrix, whether or not the matrix is viewed as an augmented matrix for a linear system. So the first part of this section concerns an arbitrary rectangular matrix and begins by introducing two important classes of matrices that include the “triangular” matrices of Section 1.1. In the definitions that follow, a *nonzero* row or column in a matrix means a row or column that contains at least one nonzero entry; a **leading entry** of a row refers to the leftmost nonzero entry (in a nonzero row).

DEFINITION

A rectangular matrix is in **echelon form** (or **row echelon form**) if it has the following three properties:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matrix in echelon form satisfies the following additional conditions, then it is in **reduced echelon form** (or **reduced row echelon form**):

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

¹ The algorithm here is a variant of what is commonly called *Gaussian elimination*. A similar elimination method for linear systems was used by Chinese mathematicians in about 250 B.C. The process was unknown in Western culture until the nineteenth century, when a famous German mathematician, Carl Friedrich Gauss, discovered it. A German engineer, Wilhelm Jordan, popularized the algorithm in an 1888 text on geodesy.

An **echelon matrix** (respectively, **reduced echelon matrix**) is one that is in echelon form (respectively, reduced echelon form). Property 2 says that the leading entries form an *echelon* (“steplike”) pattern that moves down and to the right through the matrix. Property 3 is a simple consequence of property 2, but we include it for emphasis.

The “triangular” matrices of Section 1.1, such as

$$\begin{bmatrix} 2 & -3 & 2 & 1 \\ 0 & 1 & -4 & 8 \\ 0 & 0 & 0 & 5/2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

are in echelon form. In fact, the second matrix is in reduced echelon form. Here are additional examples.

EXAMPLE 1 The following matrices are in echelon form. The leading entries (\blacksquare) may have any nonzero value; the starred entries (*) may have any value (including zero).

$$\begin{bmatrix} \blacksquare & * & * & * \\ 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & \blacksquare & * & * & * & * & * & * & * \\ 0 & 0 & 0 & \blacksquare & * & * & * & * & * \\ 0 & 0 & 0 & 0 & \blacksquare & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & \blacksquare & * & * \end{bmatrix}$$

The following matrices are in reduced echelon form because the leading entries are 1's, and there are 0's below *and above* each leading 1.

$$\begin{bmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & * & 0 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 1 & 0 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & * \end{bmatrix}$$

Any nonzero matrix may be **row reduced** (that is, transformed by elementary row operations) into more than one matrix in echelon form, using different sequences of row operations. However, the reduced echelon form one obtains from a matrix is unique. The following theorem is proved in Appendix A at the end of the text.

THEOREM I

Uniqueness of the Reduced Echelon Form

Each matrix is row equivalent to one and only one reduced echelon matrix.

If a matrix A is row equivalent to an echelon matrix U , we call U **an echelon form** (or **row echelon form**) **of A** ; if U is in reduced echelon form, we call U **the reduced echelon form of A** . [Most matrix programs and calculators with matrix capabilities use the abbreviation RREF for reduced (row) echelon form. Some use REF for (row) echelon form.]

Pivot Positions

When row operations on a matrix produce an echelon form, further row operations to obtain the reduced echelon form do not change the positions of the leading entries. Since the reduced echelon form is unique, *the leading entries are always in the same positions*

in any echelon form obtained from a given matrix. These leading entries correspond to leading 1's in the reduced echelon form.

DEFINITION

A **pivot position** in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A . A **pivot column** is a column of A that contains a pivot position.

In Example 1, the squares (■) identify the pivot positions. Many fundamental concepts in the first four chapters will be connected in one way or another with pivot positions in a matrix.

EXAMPLE 2 Row reduce the matrix A below to echelon form, and locate the pivot columns of A .

$$A = \begin{bmatrix} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{bmatrix}$$

SOLUTION Use the same basic strategy as in Section 1.1. The top of the leftmost nonzero column is the first pivot position. A nonzero entry, or *pivot*, must be placed in this position. A good choice is to interchange rows 1 and 4 (because the mental computations in the next step will not involve fractions).

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

↑ Pivot column

Create zeros below the pivot, 1, by adding multiples of the first row to the rows below, and obtain matrix (1) below. The pivot position in the second row must be as far left as possible—namely in the second column. Choose the 2 in this position as the next pivot.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 5 & 10 & -15 & -15 \\ 0 & -3 & -6 & 4 & 9 \end{bmatrix}$$

↑ Next pivot column

(1)

Add $-5/2$ times row 2 to row 3, and add $3/2$ times row 2 to row 4.

$$\begin{bmatrix} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 \end{bmatrix}$$

(2)

The matrix in (2) is different from any encountered in Section 1.1. There is no way to create a leading entry in column 3! (We can't use row 1 or 2 because doing so would

destroy the echelon arrangement of the leading entries already produced.) However, if we interchange rows 3 and 4, we can produce a leading entry in column 4.

$$\left[\begin{array}{rrrrr} 1 & 4 & 5 & -9 & -7 \\ 0 & 2 & 4 & -6 & -6 \\ 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \text{Pivot} \quad \text{General form:} \quad \left[\begin{array}{rrrrr} \blacksquare & * & * & * & * \\ 0 & \blacksquare & * & * & * \\ 0 & 0 & 0 & \blacksquare & * \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

↑↑ Pivot columns

The matrix is in echelon form and thus reveals that columns 1, 2, and 4 of A are pivot columns.

$$A = \left[\begin{array}{rrrrr} 0 & -3 & -6 & 4 & 9 \\ -1 & -2 & -1 & 3 & 1 \\ -2 & -3 & 0 & 3 & -1 \\ 1 & 4 & 5 & -9 & -7 \end{array} \right] \quad \text{Pivot positions}$$

↑↑ Pivot columns

■
(3)

A **pivot**, as illustrated in Example 2, is a nonzero number in a pivot position that is used as needed to create zeros via row operations. The pivots in Example 2 were 1, 2, and -5 . Notice that these numbers are not the same as the actual elements of A in the highlighted pivot positions shown in (3).

With Example 2 as a guide, we are ready to describe an efficient procedure for transforming a matrix into an echelon or reduced echelon matrix. Careful study and mastery of this procedure now will pay rich dividends later in the course.

The Row Reduction Algorithm

The algorithm that follows consists of four steps, and it produces a matrix in echelon form. A fifth step produces a matrix in reduced echelon form. We illustrate the algorithm by an example.

EXAMPLE 3 Apply elementary row operations to transform the following matrix first into echelon form and then into reduced echelon form:

$$\left[\begin{array}{cccccc} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array} \right]$$

SOLUTION

Step 1

Begin with the leftmost nonzero column. This is a pivot column. The pivot position is at the top.

$$\left[\begin{array}{cccccc} 0 & 3 & -6 & 6 & 4 & -5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{array} \right]$$

↑ Pivot column

Step 2

Select a nonzero entry in the pivot column as a pivot. If necessary, interchange rows to move this entry into the pivot position.

Interchange rows 1 and 3. (We could have interchanged rows 1 and 2 instead.)

$$\left[\begin{array}{cccccc} 3 & -9 & 12 & -9 & 6 & 15 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array} \right]$$

Pivot

Step 3

Use row replacement operations to create zeros in all positions below the pivot.

As a preliminary step, we could divide the top row by the pivot, 3. But with two 3's in column 1, it is just as easy to add -1 times row 1 to row 2.

$$\left[\begin{array}{cccccc} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array} \right]$$

Pivot

Step 4

Cover (or ignore) the row containing the pivot position and cover all rows, if any, above it. Apply steps 1–3 to the submatrix that remains. Repeat the process until there are no more nonzero rows to modify.

With row 1 covered, step 1 shows that column 2 is the next pivot column; for step 2, select as a pivot the “top” entry in that column.

$$\left[\begin{array}{cccccc} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 3 & -6 & 6 & 4 & -5 \end{array} \right]$$

Pivot

New pivot column

For step 3, we could insert an optional step of dividing the “top” row of the submatrix by the pivot, 2. Instead, we add $-3/2$ times the “top” row to the row below. This produces

$$\left[\begin{array}{cccccc} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right]$$

When we cover the row containing the second pivot position for step 4, we are left with a new submatrix having only one row:

$$\left[\begin{array}{cccccc} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & \downarrow 4 \end{array} \right] \quad \text{Pivot}$$

Steps 1–3 require no work for this submatrix, and we have reached an echelon form of the full matrix. If we want the reduced echelon form, we perform one more step.

Step 5

Beginning with the rightmost pivot and working upward and to the left, create zeros above each pivot. If a pivot is not 1, make it 1 by a scaling operation.

The rightmost pivot is in row 3. Create zeros above it, adding suitable multiples of row 3 to rows 2 and 1.

$$\left[\begin{array}{cccccc} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 2 & -4 & 4 & 0 & -14 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right] \quad \begin{array}{l} \leftarrow \text{Row 1} + (-6) \cdot \text{row 3} \\ \leftarrow \text{Row 2} + (-2) \cdot \text{row 3} \end{array}$$

The next pivot is in row 2. Scale this row, dividing by the pivot.

$$\left[\begin{array}{cccccc} 3 & -9 & 12 & -9 & 0 & -9 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right] \quad \leftarrow \text{Row scaled by } \frac{1}{2}$$

Create a zero in column 2 by adding 9 times row 2 to row 1.

$$\left[\begin{array}{cccccc} 3 & 0 & -6 & 9 & 0 & -72 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right] \quad \leftarrow \text{Row 1} + (9) \cdot \text{row 2}$$

Finally, scale row 1, dividing by the pivot, 3.

$$\left[\begin{array}{cccccc} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right] \quad \leftarrow \text{Row scaled by } \frac{1}{3}$$

This is the reduced echelon form of the original matrix. ■

The combination of steps 1–4 is called the **forward phase** of the row reduction algorithm. Step 5, which produces the unique reduced echelon form, is called the **backward phase**.

Numerical Note

In step 2 on page 41, a computer program usually selects as a pivot the entry in a column having the largest absolute value. This strategy, called **partial pivoting**, is used because it reduces roundoff errors in the calculations.

Solutions of Linear Systems

The row reduction algorithm leads directly to an explicit description of the solution set of a linear system when the algorithm is applied to the augmented matrix of the system.

Suppose, for example, that the augmented matrix of a linear system has been changed into the equivalent *reduced* echelon form

$$\begin{bmatrix} 1 & 0 & -5 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are three variables because the augmented matrix has four columns. The associated system of equations is

$$\begin{aligned} x_1 - 5x_3 &= 1 \\ x_2 + x_3 &= 4 \\ 0 &= 0 \end{aligned} \tag{4}$$

The variables x_1 and x_2 corresponding to pivot columns in the matrix are called **basic variables**.² The other variable, x_3 , is called a **free variable**.

Whenever a system is consistent, as in (4), the solution set can be described explicitly by solving the *reduced* system of equations for the basic variables in terms of the free variables. This operation is possible because the reduced echelon form places each basic variable in one and only one equation. In (4), solve the first equation for x_1 and the second for x_2 . (Ignore the third equation; it offers no restriction on the variables.)

$$\begin{cases} x_1 = 1 + 5x_3 \\ x_2 = 4 - x_3 \\ x_3 \text{ is free} \end{cases} \tag{5}$$

The statement “ x_3 is free” means that you are free to choose any value for x_3 . Once that is done, the formulas in (5) determine the values for x_1 and x_2 . For instance, when $x_3 = 0$, the solution is $(1, 4, 0)$; when $x_3 = 1$, the solution is $(6, 3, 1)$. *Each different choice of x_3 determines a (different) solution of the system, and every solution of the system is determined by a choice of x_3 .*

EXAMPLE 4 Find the general solution of the linear system whose augmented matrix has been reduced to

$$\begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}$$

SOLUTION The matrix is in echelon form, but we want the reduced echelon form before solving for the basic variables. The row reduction is completed next. The symbol \sim before a matrix indicates that the matrix is row equivalent to the preceding matrix.

$$\begin{aligned} \begin{bmatrix} 1 & 6 & 2 & -5 & -2 & -4 \\ 0 & 0 & 2 & -8 & -1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} &\sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 2 & -8 & 0 & 10 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \\ \sim \begin{bmatrix} 1 & 6 & 2 & -5 & 0 & 10 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} &\sim \begin{bmatrix} 1 & 6 & 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix} \end{aligned}$$

² Some texts use the term *leading variables* because they correspond to the columns containing leading entries.

There are five variables because the augmented matrix has six columns. The associated system now is

$$\begin{aligned} x_1 + 6x_2 + 3x_4 &= 0 \\ x_3 - 4x_4 &= 5 \\ x_5 &= 7 \end{aligned} \tag{6}$$

The pivot columns of the matrix are 1, 3, and 5, so the basic variables are x_1 , x_3 , and x_5 . The remaining variables, x_2 and x_4 , must be free. Solve for the basic variables to obtain the general solution:

$$\left\{ \begin{array}{l} x_1 = -6x_2 - 3x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 4x_4 \\ x_4 \text{ is free} \\ x_5 = 7 \end{array} \right. \tag{7}$$

Note that the value of x_5 is already fixed by the third equation in system (6). ■

Parametric Descriptions of Solution Sets

The descriptions in (5) and (7) are *parametric descriptions* of solution sets in which the free variables act as parameters. *Solving a system* amounts to finding a parametric description of the solution set or determining that the solution set is empty.

Whenever a system is consistent and has free variables, the solution set has many parametric descriptions. For instance, in system (4), we may add 5 times equation 2 to equation 1 and obtain the equivalent system

$$\begin{aligned} x_1 + 5x_2 &= 21 \\ x_2 + x_3 &= 4 \end{aligned}$$

We could treat x_2 as a parameter and solve for x_1 and x_3 in terms of x_2 , and we would have an accurate description of the solution set. However, to be consistent, we make the (arbitrary) convention of always using the free variables as the parameters for describing a solution set. (The answer section at the end of the text also reflects this convention.)

Whenever a system is inconsistent, the solution set is empty, even when the system has free variables. In this case, the solution set has *no* parametric representation.

Back-Substitution

Consider the following system, whose augmented matrix is in echelon form but is *not* in reduced echelon form:

$$\begin{aligned} x_1 - 7x_2 + 2x_3 - 5x_4 + 8x_5 &= 10 \\ x_2 - 3x_3 + 3x_4 + x_5 &= -5 \\ x_4 - x_5 &= 4 \end{aligned}$$

A computer program would solve this system by back-substitution, rather than by computing the reduced echelon form. That is, the program would solve equation 3 for x_4 in terms of x_5 and substitute the expression for x_4 into equation 2, solve equation 2 for x_2 , and then substitute the expressions for x_2 and x_4 into equation 1 and solve for x_1 .

Our matrix format for the backward phase of row reduction, which produces the reduced echelon form, has the same number of arithmetic operations as back-substitution. But the discipline of the matrix format substantially reduces the likelihood of errors

during hand computations. The best strategy is to use only the *reduced* echelon form to solve a system! The *Study Guide* that accompanies this text offers several helpful suggestions for performing row operations accurately and rapidly.

Numerical Note

In general, the forward phase of row reduction takes much longer than the backward phase. An algorithm for solving a system is usually measured in flops (or floating point operations). A **flop** is one arithmetic operation (+, −, *, /) on two real floating point numbers.³ For an $n \times (n+1)$ matrix, the reduction to echelon form can take $2n^3/3 + n^2/2 - 7n/6$ flops (which is approximately $2n^3/3$ flops when n is moderately large—say, $n \geq 30$). In contrast, further reduction to reduced echelon form needs at most n^2 flops.

Existence and Uniqueness Questions

Although a nonreduced echelon form is a poor tool for solving a system, this form is just the right device for answering two fundamental questions posed in Section 1.1.

EXAMPLE 5 Determine the existence and uniqueness of the solutions to the system

$$\begin{aligned} 3x_2 - 6x_3 + 6x_4 + 4x_5 &= -5 \\ 3x_1 - 7x_2 + 8x_3 - 5x_4 + 8x_5 &= 9 \\ 3x_1 - 9x_2 + 12x_3 - 9x_4 + 6x_5 &= 15 \end{aligned}$$

SOLUTION The augmented matrix of this system was row reduced in Example 3 to

$$\left[\begin{array}{cccccc} 3 & -9 & 12 & -9 & 6 & 15 \\ 0 & 2 & -4 & 4 & 2 & -6 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array} \right] \quad (8)$$

The basic variables are x_1 , x_2 , and x_5 ; the free variables are x_3 and x_4 . There is no equation such as $0 = 1$ that would indicate an inconsistent system, so we could use back-substitution to find a solution. But the *existence* of a solution is already clear in (8). Also, the solution is *not unique* because there are free variables. Each different choice of x_3 and x_4 determines a different solution. Thus the system has infinitely many solutions. ■

When a system is in echelon form and contains no equation of the form $0 = b$, with b nonzero, every nonzero equation contains a basic variable with a nonzero coefficient. Either the basic variables are completely determined (with no free variables) or at least one of the basic variables may be expressed in terms of one or more free variables. In the former case, there is a unique solution; in the latter case, there are infinitely many solutions (one for each choice of values for the free variables).

These remarks justify the following theorem.

³ Traditionally, a *flop* was only a multiplication or division because addition and subtraction took much less time and could be ignored. The definition of *flop* given here is preferred now, as a result of advances in computer architecture. See Golub and Van Loan, *Matrix Computations*, 2nd ed. (Baltimore: The Johns Hopkins Press, 1989), pp. 19–20.

THEOREM 2**Existence and Uniqueness Theorem**

A linear system is consistent if and only if the rightmost column of the augmented matrix is *not* a pivot column—that is, if and only if an echelon form of the augmented matrix has *no* row of the form

$$\begin{bmatrix} 0 & \cdots & 0 & b \end{bmatrix} \quad \text{with } b \text{ nonzero}$$

If a linear system is consistent, then the solution set contains either (i) a unique solution, when there are no free variables, or (ii) infinitely many solutions, when there is at least one free variable.

The following procedure outlines how to find and describe all solutions of a linear system.

USING ROW REDUCTION TO SOLVE A LINEAR SYSTEM

1. Write the augmented matrix of the system.
2. Use the row reduction algorithm to obtain an equivalent augmented matrix in echelon form. Decide whether the system is consistent. If there is no solution, stop; otherwise, go to the next step.
3. Continue row reduction to obtain the reduced echelon form.
4. Write the system of equations corresponding to the matrix obtained in step 3.
5. Rewrite each nonzero equation from step 4 so that its one basic variable is expressed in terms of any free variables appearing in the equation.

Reasonable Answers

Remember that each augmented matrix corresponds to a system of equations. If

you row reduce the augmented matrix $\begin{bmatrix} 1 & -2 & 1 & 2 \\ 1 & -1 & 2 & 5 \\ 0 & 1 & 1 & 3 \end{bmatrix}$ to get the matrix

$\begin{bmatrix} 1 & 0 & 3 & 8 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, the solution set is

$$\begin{cases} x_1 = 8 - 3x_3 \\ x_2 = 3 - x_3 \\ x_3 \text{ is free} \end{cases}$$

The system of equations corresponding to the original augmented matrix is

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 2 \\ x_1 - x_2 + 2x_3 &= 5 \\ x_2 + x_3 &= 3 \end{aligned}$$

You can now check whether your solution is correct by substituting it into the original equations. Notice that you can just leave the free variables in the solution.

$$\begin{aligned}
 (8 - 3x_3) - 2(3 - x_3) + (x_3) &= 8 - 3x_3 - 6 + 2x_3 + x_3 = 2 \\
 (8 - 3x_3) - (3 - x_3) + 2(x_3) &= 8 - 3x_3 - 3 + x_3 + 2x_3 = 5 \\
 (3 - x_3) + (x_3) &= 3 - x_3 + x_3 = 3
 \end{aligned}$$

You can now be confident you have a correct solution to the system of equations represented by the augmented matrix.

Practice Problems

1. Find the general solution of the linear system whose augmented matrix is

$$\left[\begin{array}{cccc} 1 & -3 & -5 & 0 \\ 0 & 1 & -1 & -1 \end{array} \right]$$

2. Find the general solution of the system

$$\begin{aligned}
 x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\
 -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\
 3x_1 - 6x_2 - 6x_3 + 8x_4 &= 2
 \end{aligned}$$

3. Suppose a 4×7 coefficient matrix for a system of equations has 4 pivots. Is the system consistent? If the system is consistent, how many solutions are there?

1.2 Exercises

In Exercises 1 and 2, determine which matrices are in reduced echelon form and which others are only in echelon form.

1. a. $\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$ b. $\left[\begin{array}{ccccc} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right]$

c. $\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$ d. $\left[\begin{array}{ccccc} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{array} \right]$

2. a. $\left[\begin{array}{cccc} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array} \right]$ b. $\left[\begin{array}{ccccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$

c. $\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array} \right]$ d. $\left[\begin{array}{ccccc} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$

Row reduce the matrices in Exercises 3 and 4 to reduced echelon form. Circle the pivot positions in the final matrix and in the original matrix, and list the pivot columns.

3. $\left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{array} \right]$ 4. $\left[\begin{array}{ccccc} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{array} \right]$

5. Describe the possible echelon forms of a nonzero 2×2 matrix. Use the symbols ■, *, and 0, as in the first part of Example 1.

6. Repeat Exercise 5 for a nonzero 3×2 matrix.

Find the general solutions of the systems whose augmented matrices are given in Exercises 7–14.

7. $\left[\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 4 & 8 & 9 & 4 \end{array} \right]$ 8. $\left[\begin{array}{ccccc} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 11 \end{array} \right]$

9. $\left[\begin{array}{cccc} 0 & 1 & -6 & 5 \\ 1 & -2 & 7 & -4 \end{array} \right]$ 10. $\left[\begin{array}{ccccc} 1 & -2 & -1 & 3 \\ 3 & -6 & -2 & 2 \end{array} \right]$

11. $\left[\begin{array}{ccccc} 3 & -4 & 2 & 0 \\ -9 & 12 & -6 & 0 \\ -6 & 8 & -4 & 0 \end{array} \right]$ 12. $\left[\begin{array}{ccccc} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{array} \right]$

13. $\left[\begin{array}{cccccc} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$

14.
$$\left[\begin{array}{cccccc} 1 & 2 & -5 & -4 & 0 & -5 \\ 0 & 1 & -6 & -4 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

You may find it helpful to review the information in the Reasonable Answers box from this section before answering Exercises 15–18.

15. Write down the equations corresponding to the augmented matrix in Exercise 9 and verify your answer to Exercise 9 is correct by substituting the solutions you obtained back into the original equations.
16. Write down the equations corresponding to the augmented matrix in Exercise 10 and verify your answer to Exercise 10 is correct by substituting the solutions you obtained back into the original equations.
17. Write down the equations corresponding to the augmented matrix in Exercise 11 and verify your answer to Exercise 11 is correct by substituting the solutions you obtained back into the original equations.
18. Write down the equations corresponding to the augmented matrix in Exercise 12 and verify your answer to Exercise 12 is correct by substituting the solutions you obtained back into the original equations.

Exercises 19 and 20 use the notation of Example 1 for matrices in echelon form. Suppose each matrix represents the augmented matrix for a system of linear equations. In each case, determine if the system is consistent. If the system is consistent, determine if the solution is unique.

19. a.
$$\left[\begin{array}{ccccc} \blacksquare & * & * & * & \\ 0 & \blacksquare & * & * & \\ 0 & 0 & \blacksquare & 0 & \end{array} \right]$$

b.
$$\left[\begin{array}{ccccc} 0 & \blacksquare & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & 0 & \blacksquare \end{array} \right]$$

20. a.
$$\left[\begin{array}{ccc} \blacksquare & * & * \\ 0 & \blacksquare & * \\ 0 & 0 & 0 \end{array} \right]$$

b.
$$\left[\begin{array}{ccccc} \blacksquare & * & * & * & * \\ 0 & 0 & \blacksquare & * & * \\ 0 & 0 & 0 & \blacksquare & * \end{array} \right]$$

In Exercises 21 and 22, determine the value(s) of h such that the matrix is the augmented matrix of a consistent linear system.

21.
$$\left[\begin{array}{ccc} 2 & 3 & h \\ 4 & 6 & 7 \end{array} \right]$$

22.
$$\left[\begin{array}{ccc} 1 & -4 & -3 \\ 6 & h & -9 \end{array} \right]$$

In Exercises 23 and 24, choose h and k such that the system has (a) no solution, (b) a unique solution, and (c) many solutions. Give separate answers for each part.

23. $x_1 + hx_2 = 2$
 $4x_1 + 8x_2 = k$

24. $x_1 + 4x_2 = 5$
 $2x_1 + hx_2 = k$

In Exercises 25–34, mark each statement True or False (T/F). Justify each answer.⁴

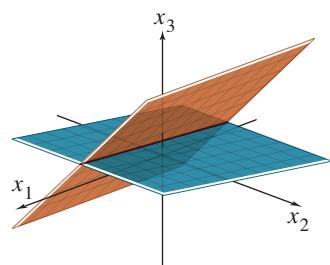
25. (T/F) In some cases, a matrix may be row reduced to more than one matrix in reduced echelon form, using different sequences of row operations.
26. (T/F) The echelon form of a matrix is unique.
27. (T/F) The row reduction algorithm applies only to augmented matrices for a linear system.
28. (T/F) The pivot positions in a matrix depend on whether row interchanges are used in the row reduction process.
29. (T/F) A basic variable in a linear system is a variable that corresponds to a pivot column in the coefficient matrix.
30. (T/F) Reducing a matrix to echelon form is called the *forward phase* of the row reduction process.
31. (T/F) Finding a parametric description of the solution set of a linear system is the same as *solving* the system.
32. (T/F) Whenever a system has free variables, the solution set contains a unique solution.
33. (T/F) If one row in an echelon form of an augmented matrix is $[0 \ 0 \ 0 \ 0 \ 5]$, then the associated linear system is inconsistent.
34. (T/F) A general solution of a system is an explicit description of all solutions of the system.
35. Suppose a 3×5 coefficient matrix for a system has three pivot columns. Is the system consistent? Why or why not?
36. Suppose a system of linear equations has a 3×5 augmented matrix whose fifth column is a pivot column. Is the system consistent? Why (or why not)?
37. Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent.
38. Suppose the coefficient matrix of a linear system of three equations in three variables has a pivot in each column. Explain why the system has a unique solution.
39. Restate the last sentence in Theorem 2 using the concept of pivot columns: “If a linear system is consistent, then the solution is unique if and only if _____.”
40. What would you have to know about the pivot columns in an augmented matrix in order to know that the linear system is consistent and has a unique solution?
41. A system of linear equations with fewer equations than unknowns is sometimes called an *underdetermined system*.

⁴ True/false questions of this type will appear in many sections. Methods for justifying your answers were described before the True or False exercises in Section 1.1.

Suppose that such a system happens to be consistent. Explain why there must be an infinite number of solutions.

42. Give an example of an inconsistent underdetermined system of two equations in three unknowns.
43. A system of linear equations with more equations than unknowns is sometimes called an *overdetermined system*. Can such a system be consistent? Illustrate your answer with a specific system of three equations in two unknowns.
44. Suppose an $n \times (n+1)$ matrix is row reduced to reduced echelon form. Approximately what fraction of the total number of operations (flops) is involved in the backward phase of the reduction when $n = 30$? when $n = 300$?

Suppose experimental data are represented by a set of points in the plane. An **interpolating polynomial** for the data is a polynomial whose graph passes through every point. In scientific work, such a polynomial can be used, for example, to estimate values between the known data points. Another use is to create curves for graphical images on a computer screen. One method for finding an interpolating polynomial is to solve a system of linear equations.



The general solution of the system of equations is the line of intersection of the two planes.

45. Find the interpolating polynomial $p(t) = a_0 + a_1t + a_2t^2$ for the data $(1, 11), (2, 16), (3, 19)$. That is, find a_0, a_1 , and a_2 such that

$$\begin{aligned}a_0 + a_1(1) + a_2(1)^2 &= 11 \\a_0 + a_1(2) + a_2(2)^2 &= 16 \\a_0 + a_1(3) + a_2(3)^2 &= 19\end{aligned}$$

46. In a wind tunnel experiment, the force on a projectile due to air resistance was measured at different velocities:

Velocity (100 ft/sec)	0	2	4	6	8	10
Force (100 lb)	0	2.90	14.8	39.6	74.3	119

Find an interpolating polynomial for these data and estimate the force on the projectile when the projectile is traveling at 750 ft/sec. Use $p(t) = a_0 + a_1t + a_2t^2 + a_3t^3 + a_4t^4 + a_5t^5$. What happens if you try to use a polynomial of degree less than 5? (Try a cubic polynomial, for instance.)⁵

⁵ Exercises marked with the symbol **T** are designed to be worked with the aid of a “Matrix program” (a computer program, such as MATLAB, Maple, Mathematica, MathCad, Octave, or Derive, or a programmable calculator with matrix capabilities, such as those manufactured by Texas Instruments or Hewlett-Packard).

Solutions to Practice Problems

1. The reduced echelon form of the augmented matrix and the corresponding system are

$$\left[\begin{array}{cccc|c} 1 & 0 & -8 & -3 \\ 0 & 1 & -1 & -1 \end{array} \right] \quad \text{and} \quad \begin{array}{l} x_1 - 8x_3 = -3 \\ x_2 - x_3 = -1 \end{array}$$

The basic variables are x_1 and x_2 , and the general solution is

$$\begin{cases} x_1 = -3 + 8x_3 \\ x_2 = -1 + x_3 \\ x_3 \text{ is free} \end{cases}$$

Note: It is essential that the general solution describe each variable, with any parameters clearly identified. The following statement does *not* describe the solution:

$$\begin{cases} x_1 = -3 + 8x_3 \\ x_2 = -1 + x_3 \\ x_3 = 1 + x_2 \end{cases} \quad \text{Incorrect solution}$$

This description implies that x_2 and x_3 are *both* free, which certainly is not the case.

2. Row reduce the system's augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & -2 & -1 & 3 & 0 \\ -2 & 4 & 5 & -5 & 3 \\ 3 & -6 & -6 & 8 & 2 \end{array} \right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & -3 & -1 & 2 \end{array} \right] \sim \left[\begin{array}{ccccc|c} 1 & -2 & -1 & 3 & 0 \\ 0 & 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{array} \right]$$

Solutions to Practice Problems (Continued)

This echelon matrix shows that the system is *inconsistent*, because its rightmost column is a pivot column; the third row corresponds to the equation $0 = 5$. There is no need to perform any more row operations. Note that the presence of the free variables in this problem is irrelevant because the system is inconsistent.

3. Since the coefficient matrix has four pivots, there is a pivot in every row of the coefficient matrix. This means that when the coefficient matrix is row reduced, it will *not* have a row of zeros, thus the corresponding row reduced augmented matrix can never have a row of the form $[0 \ 0 \ \dots \ 0 \ b]$, where b is a nonzero number. By Theorem 2, the system is consistent. Moreover, since there are seven columns in the coefficient matrix and only four pivot columns, there will be three free variables resulting in infinitely many solutions.

1.3 Vector Equations

Important properties of linear systems can be described with the concept and notation of vectors. This section connects equations involving vectors to ordinary systems of equations. The term *vector* appears in a variety of mathematical and physical contexts, which we will discuss in Chapter 4, “Vector Spaces.” Until then, *vector* will mean an *ordered list of numbers*. This simple idea enables us to get to interesting and important applications as quickly as possible.

Vectors in \mathbb{R}^2

A matrix with only one column is called a **column vector** or simply a **vector**. Examples of vectors with two entries are

$$\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} .2 \\ .3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where w_1 and w_2 are any real numbers. The set of all vectors with two entries is denoted by \mathbb{R}^2 (read “r-two”). The \mathbb{R} stands for the real numbers that appear as entries in the vectors, and the exponent 2 indicates that each vector contains two entries.¹

Two vectors in \mathbb{R}^2 are **equal** if and only if their corresponding entries are equal. Thus $\begin{bmatrix} 4 \\ 7 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 4 \end{bmatrix}$ are *not* equal, because vectors in \mathbb{R}^2 are *ordered pairs* of real numbers.

Given two vectors \mathbf{u} and \mathbf{v} in \mathbb{R}^2 , their **sum** is the vector $\mathbf{u} + \mathbf{v}$ obtained by adding corresponding entries of \mathbf{u} and \mathbf{v} . For example,

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} + \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1+2 \\ -2+5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Given a vector \mathbf{u} and a real number c , the **scalar multiple** of \mathbf{u} by c is the vector $c\mathbf{u}$ obtained by multiplying each entry in \mathbf{u} by c . For instance,

$$\text{if } \mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \text{ and } c = 5, \quad \text{then } c\mathbf{u} = 5 \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$$

¹ Most of the text concerns vectors and matrices that have only real entries. However, all definitions and theorems in Chapters 1–5, and in most of the rest of the text, remain valid if the entries are complex numbers. Complex vectors and matrices arise naturally, for example, in electrical engineering and physics.

The number c in $c\mathbf{u}$ is called a **scalar**; it is written in lightface type to distinguish it from the boldface vector \mathbf{u} .

The operations of scalar multiplication and vector addition can be combined, as in the following example.

EXAMPLE 1 Given $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, find $4\mathbf{u}$, $(-3)\mathbf{v}$, and $4\mathbf{u} + (-3)\mathbf{v}$.

SOLUTION

$$4\mathbf{u} = \begin{bmatrix} 4 \\ -8 \end{bmatrix}, \quad (-3)\mathbf{v} = \begin{bmatrix} -6 \\ 15 \end{bmatrix}$$

and

$$4\mathbf{u} + (-3)\mathbf{v} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} + \begin{bmatrix} -6 \\ 15 \end{bmatrix} = \begin{bmatrix} -2 \\ 7 \end{bmatrix} \blacksquare$$

Sometimes, for convenience (and also to save space), this text may write a column vector such as $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ in the form $(3, -1)$. In this case, the parentheses and the comma distinguish the vector $(3, -1)$ from the 1×2 row matrix $[3 \ -1]$, written with brackets and no comma. Thus

$$\begin{bmatrix} 3 \\ -1 \end{bmatrix} \neq [3 \ -1]$$

because the matrices have different shapes, even though they have the same entries.

Geometric Descriptions of \mathbb{R}^2

Consider a rectangular coordinate system in the plane. Because each point in the plane is determined by an ordered pair of numbers, we can identify a geometric point (a, b) with the column vector $\begin{bmatrix} a \\ b \end{bmatrix}$. So we may regard \mathbb{R}^2 as the set of all points in the plane.

See Figure 1.

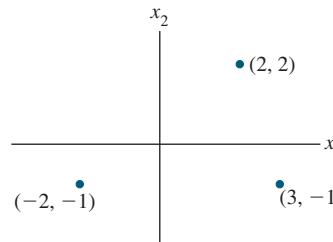


FIGURE 1 Vectors as points.

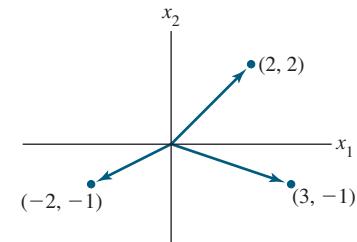


FIGURE 2 Vectors with arrows.

The geometric visualization of a vector such as $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$ is often aided by including an arrow (directed line segment) from the origin $(0, 0)$ to the point $(3, -1)$, as in Figure 2. In this case, the individual points along the arrow itself have no special significance.²

The sum of two vectors has a useful geometric representation. The following rule can be verified by analytic geometry.

²In physics, arrows can represent forces and usually are free to move about in space. This interpretation of vectors will be discussed in Section 4.1.

Parallelogram Rule for Addition

If \mathbf{u} and \mathbf{v} in \mathbb{R}^2 are represented as points in the plane, then $\mathbf{u} + \mathbf{v}$ corresponds to the fourth vertex of the parallelogram whose other vertices are \mathbf{u} , $\mathbf{0}$, and \mathbf{v} . See Figure 3.

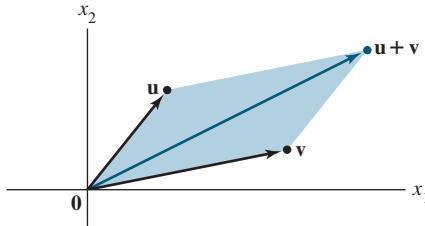


FIGURE 3 The parallelogram rule.

EXAMPLE 2 The vectors $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$, and $\mathbf{u} + \mathbf{v} = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ are displayed in Figure 4. ■

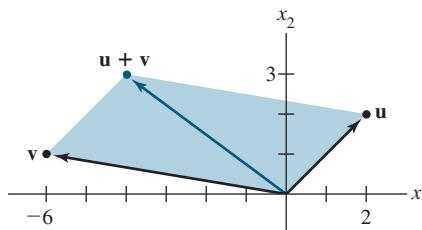
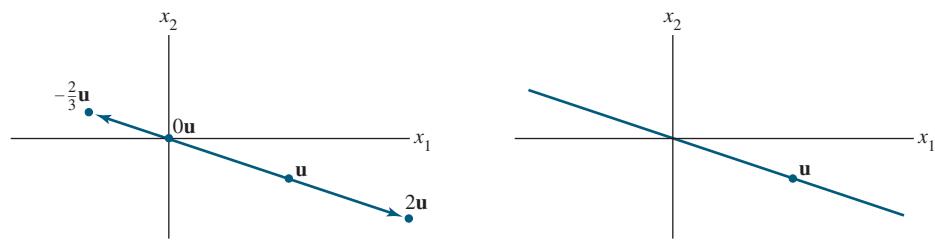


FIGURE 4

The next example illustrates the fact that the set of all scalar multiples of one fixed nonzero vector is a line through the origin, $(0, 0)$.

EXAMPLE 3 Let $\mathbf{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$. Display the vectors \mathbf{u} , $2\mathbf{u}$, and $-\frac{2}{3}\mathbf{u}$ on a graph.

SOLUTION See Figure 5, where \mathbf{u} , $2\mathbf{u} = \begin{bmatrix} 6 \\ -2 \end{bmatrix}$, and $-\frac{2}{3}\mathbf{u} = \begin{bmatrix} -2 \\ 2/3 \end{bmatrix}$ are displayed. The arrow for $2\mathbf{u}$ is twice as long as the arrow for \mathbf{u} , and the arrows point in the same direction. The arrow for $-\frac{2}{3}\mathbf{u}$ is two-thirds the length of the arrow for \mathbf{u} , and the arrows point in opposite directions. In general, the length of the arrow for $c\mathbf{u}$ is $|c|$ times the length of the arrow for \mathbf{u} . [Recall that the length of the line segment from $(0, 0)$ to (a, b) is $\sqrt{a^2 + b^2}$.]



Typical multiples of \mathbf{u}

The set of all multiples of \mathbf{u}

FIGURE 5

Vectors in \mathbb{R}^3

Vectors in \mathbb{R}^3 are 3×1 column matrices with three entries. They are represented geometrically by points in a three-dimensional coordinate space, with arrows from the origin

sometimes included for visual clarity. The vectors $\mathbf{a} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ and $2\mathbf{a}$ are displayed in

Figure 6.

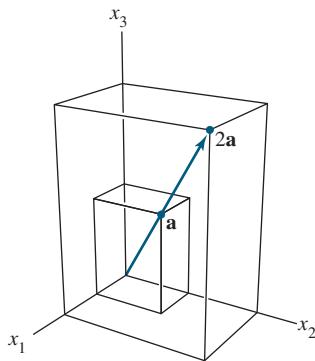


FIGURE 6
Scalar multiples.

Vectors in \mathbb{R}^n

If n is a positive integer, \mathbb{R}^n (read “r-n”) denotes the collection of all lists (or *ordered n-tuples*) of n real numbers, usually written as $n \times 1$ column matrices, such as

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$$

The vector whose entries are all zero is called the **zero vector** and is denoted by $\mathbf{0}$. (The number of entries in $\mathbf{0}$ will be clear from the context.)

Equality of vectors in \mathbb{R}^n and the operations of scalar multiplication and vector addition in \mathbb{R}^n are defined entry by entry just as in \mathbb{R}^2 . These operations on vectors have the following properties, which can be verified directly from the corresponding properties for real numbers. See Practice Problem 1 and Exercises 41 and 42 at the end of this section.

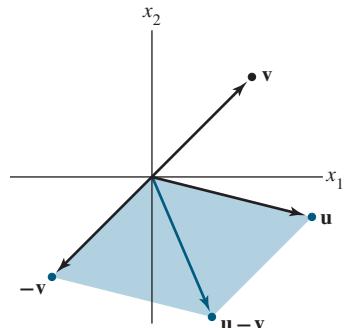


FIGURE 7
Vector subtraction.

Algebraic Properties of \mathbb{R}^n

For all $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in \mathbb{R}^n and all scalars c and d :

- | | |
|---|--|
| (i) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | (v) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ |
| (ii) $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ | (vi) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ |
| (iii) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$ | (vii) $c(d\mathbf{u}) = (cd)\mathbf{u}$ |
| (iv) $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$,
where $-\mathbf{u}$ denotes $(-1)\mathbf{u}$ | (viii) $1\mathbf{u} = \mathbf{u}$ |

For simplicity of notation, a vector such as $\mathbf{u} + (-1)\mathbf{v}$ is often written as $\mathbf{u} - \mathbf{v}$. Figure 7 shows $\mathbf{u} - \mathbf{v}$ as the sum of \mathbf{u} and $-\mathbf{v}$.

Linear Combinations

Given vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p$ in \mathbb{R}^n and given scalars c_1, c_2, \dots, c_p , the vector \mathbf{y} defined by

$$\mathbf{y} = c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \dots, \mathbf{v}_p$ with **weights** c_1, \dots, c_p . Algebraic Property (ii) above permits us to omit parentheses when forming such a linear combination. The weights in a linear combination can be any real numbers, including zero. For example, some linear combinations of vectors \mathbf{v}_1 and \mathbf{v}_2 are

$$\sqrt{3}\mathbf{v}_1 + \mathbf{v}_2, \quad \frac{1}{2}\mathbf{v}_1 (= \frac{1}{2}\mathbf{v}_1 + 0\mathbf{v}_2), \quad \text{and} \quad \mathbf{0} (= 0\mathbf{v}_1 + 0\mathbf{v}_2)$$

EXAMPLE 4 Figure 8 identifies selected linear combinations of $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. (Note that sets of parallel grid lines are drawn through integer multiples of \mathbf{v}_1 and \mathbf{v}_2 .) Estimate the linear combinations of \mathbf{v}_1 and \mathbf{v}_2 that generate the vectors \mathbf{u} and \mathbf{w} .

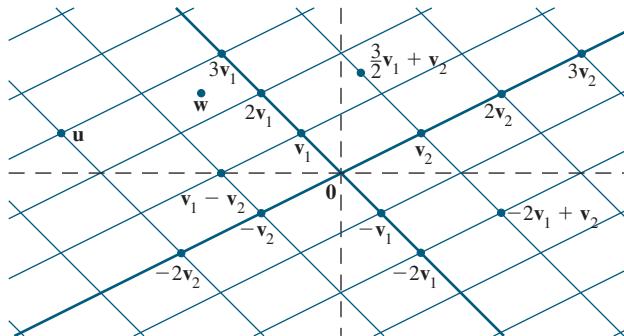


FIGURE 8 Linear combinations of \mathbf{v}_1 and \mathbf{v}_2 .

SOLUTION The parallelogram rule shows that \mathbf{u} is the sum of $3\mathbf{v}_1$ and $-2\mathbf{v}_2$; that is,

$$\mathbf{u} = 3\mathbf{v}_1 - 2\mathbf{v}_2$$

This expression for \mathbf{u} can be interpreted as instructions for traveling from the origin to \mathbf{u} along two straight paths. First, travel 3 units in the \mathbf{v}_1 direction to $3\mathbf{v}_1$, and then travel -2 units in the \mathbf{v}_2 direction (parallel to the line through \mathbf{v}_2 and $\mathbf{0}$). Next, although the vector \mathbf{w} is not on a grid line, \mathbf{w} appears to be about halfway between two pairs of grid lines, at the vertex of a parallelogram determined by $(5/2)\mathbf{v}_1$ and $(-1/2)\mathbf{v}_2$. (See Figure 9.) Thus a reasonable estimate for \mathbf{w} is

$$\mathbf{w} = \frac{5}{2}\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2$$

The next example connects a problem about linear combinations to the fundamental existence question studied in Sections 1.1 and 1.2.

EXAMPLE 5 Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Determine whether \mathbf{b} can be generated (or written) as a linear combination of \mathbf{a}_1 and \mathbf{a}_2 . That is, determine whether weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \quad (1)$$

If vector equation (1) has a solution, find it.

SOLUTION Use the definitions of scalar multiplication and vector addition to rewrite the vector equation

$$x_1 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow$

$\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}$

which is the same as

$$\begin{bmatrix} x_1 \\ -2x_1 \\ -5x_1 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 5x_2 \\ 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 + 2x_2 \\ -2x_1 + 5x_2 \\ -5x_1 + 6x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix} \quad (2)$$

The vectors on the left and right sides of (2) are equal if and only if their corresponding entries are both equal. That is, x_1 and x_2 make the vector equation (1) true if and only if x_1 and x_2 satisfy the system

$$\begin{aligned} x_1 + 2x_2 &= 7 \\ -2x_1 + 5x_2 &= 4 \\ -5x_1 + 6x_2 &= -3 \end{aligned} \quad (3)$$

To solve this system, row reduce the augmented matrix of the system as follows:³

$$\left[\begin{array}{ccc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 7 \\ 0 & 9 & 18 \\ 0 & 16 & 32 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 7 \\ 0 & 1 & 2 \\ 0 & 16 & 32 \end{array} \right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array} \right]$$

The solution of (3) is $x_1 = 3$ and $x_2 = 2$. Hence \mathbf{b} is a linear combination of \mathbf{a}_1 and \mathbf{a}_2 , with weights $x_1 = 3$ and $x_2 = 2$. That is,

$$3 \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$$
■

Observe in Example 5 that the original vectors \mathbf{a}_1 , \mathbf{a}_2 , and \mathbf{b} are the columns of the augmented matrix that we row reduced:

$$\left[\begin{array}{ccc|c} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{array} \right]$$



 $\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{b}$

For brevity, write this matrix in a way that identifies its columns—namely

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{b}] \quad (4)$$

It is clear how to write this augmented matrix immediately from vector equation (1), without going through the intermediate steps of Example 5. Take the vectors in the order in which they appear in (1) and put them into the columns of a matrix as in (4).

The discussion above is easily modified to establish the following fundamental fact.

³The symbol \sim between matrices denotes row equivalence (Section 1.2).

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \quad (5)$$

In particular, \mathbf{b} can be generated by a linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_n$ if and only if there exists a solution to the linear system corresponding to the matrix (5).

One of the key ideas in linear algebra is to study the set of all vectors that can be generated or written as a linear combination of a fixed set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ of vectors.

DEFINITION

If $\mathbf{v}_1, \dots, \mathbf{v}_p$ are in \mathbb{R}^n , then the set of all linear combinations of $\mathbf{v}_1, \dots, \mathbf{v}_p$ is denoted by $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ and is called the **subset of \mathbb{R}^n spanned (or generated) by $\mathbf{v}_1, \dots, \mathbf{v}_p$** . That is, $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is the collection of all vectors that can be written in the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

with c_1, \dots, c_p scalars.

Asking whether a vector \mathbf{b} is in $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ amounts to asking whether the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{b}$$

has a solution, or, equivalently, asking whether the linear system with augmented matrix $[\mathbf{v}_1 \ \cdots \ \mathbf{v}_p \ \mathbf{b}]$ has a solution.

Note that $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ contains every scalar multiple of \mathbf{v}_1 (for example), since $c\mathbf{v}_1 = c\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p$. In particular, the zero vector must be in $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$.

A Geometric Description of $\text{Span}\{\mathbf{v}\}$ and $\text{Span}\{\mathbf{u}, \mathbf{v}\}$

Let \mathbf{v} be a nonzero vector in \mathbb{R}^3 . Then $\text{Span}\{\mathbf{v}\}$ is the set of all scalar multiples of \mathbf{v} , which is the set of points on the line in \mathbb{R}^3 through \mathbf{v} and $\mathbf{0}$. See Figure 10.

If \mathbf{u} and \mathbf{v} are nonzero vectors in \mathbb{R}^3 , with \mathbf{v} not a multiple of \mathbf{u} , then $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is the plane in \mathbb{R}^3 that contains \mathbf{u} , \mathbf{v} , and $\mathbf{0}$. In particular, $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ contains the line in \mathbb{R}^3 through \mathbf{u} and $\mathbf{0}$ and the line through \mathbf{v} and $\mathbf{0}$. See Figure 11.

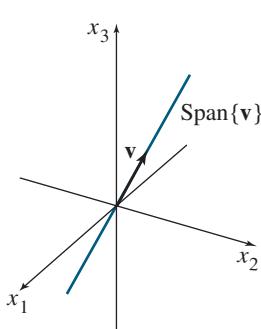


FIGURE 10 $\text{Span}\{\mathbf{v}\}$ as a line through the origin.

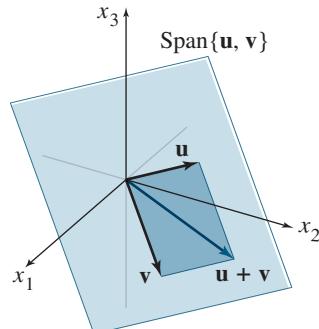


FIGURE 11 $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ as a plane through the origin.

EXAMPLE 6 Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -3 \\ 8 \\ 1 \end{bmatrix}$. Then

$\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$ is a plane through the origin in \mathbb{R}^3 . Is \mathbf{b} in that plane?

SOLUTION Does the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ have a solution? To answer this, row reduce the augmented matrix $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{b}]$:

$$\left[\begin{array}{ccc} 1 & 5 & -3 \\ -2 & -13 & 8 \\ 3 & -3 & 1 \end{array} \right] \sim \left[\begin{array}{ccc} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & -18 & 10 \end{array} \right] \sim \left[\begin{array}{ccc} 1 & 5 & -3 \\ 0 & -3 & 2 \\ 0 & 0 & -2 \end{array} \right]$$

The third equation is $0 = -2$, which shows that the system has no solution. The vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ has no solution, and so \mathbf{b} is *not* in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2\}$. ■

Linear Combinations in Applications

The final example shows how scalar multiples and linear combinations can arise when a quantity such as “cost” is broken down into several categories. The basic principle for the example concerns the cost of producing several units of an item when the cost per unit is known:

$$\left\{ \begin{array}{l} \text{number} \\ \text{of units} \end{array} \right\} \cdot \left\{ \begin{array}{l} \text{cost} \\ \text{per unit} \end{array} \right\} = \left\{ \begin{array}{l} \text{total} \\ \text{cost} \end{array} \right\}$$

EXAMPLE 7 A company manufactures two products. For \$1.00 worth of product B, the company spends \$.45 on materials, \$.25 on labor, and \$.15 on overhead. For \$1.00 worth of product C, the company spends \$.40 on materials, \$.30 on labor, and \$.15 on overhead. Let

$$\mathbf{b} = \begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix} \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} .40 \\ .30 \\ .15 \end{bmatrix}$$

Then \mathbf{b} and \mathbf{c} represent the “costs per dollar of income” for the two products.

- What economic interpretation can be given to the vector $100\mathbf{b}$?
- Suppose the company wishes to manufacture x_1 dollars worth of product B and x_2 dollars worth of product C. Give a vector that describes the various costs the company will have (for materials, labor, and overhead).

SOLUTION

- Compute

$$100\mathbf{b} = 100 \begin{bmatrix} .45 \\ .25 \\ .15 \end{bmatrix} = \begin{bmatrix} 45 \\ 25 \\ 15 \end{bmatrix}$$

The vector $100\mathbf{b}$ lists the various costs for producing \$100 worth of product B—namely \$45 for materials, \$25 for labor, and \$15 for overhead.

- The costs of manufacturing x_1 dollars worth of B are given by the vector $x_1\mathbf{b}$, and the costs of manufacturing x_2 dollars worth of C are given by $x_2\mathbf{c}$. Hence the total costs for both products are given by the vector $x_1\mathbf{b} + x_2\mathbf{c}$. ■

Practice Problems

1. Prove that $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for any \mathbf{u} and \mathbf{v} in \mathbb{R}^n .
2. For what value(s) of h will \mathbf{y} be in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

3. Let $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{u}$, and \mathbf{v} be vectors in \mathbb{R}^n . Suppose the vectors \mathbf{u} and \mathbf{v} are in $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. Show that $\mathbf{u} + \mathbf{v}$ is also in $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$. [Hint: The solution requires the use of the definition of the span of a set of vectors. It is useful to review this definition before starting this exercise.]

1.3 Exercises

In Exercises 1 and 2, compute $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - 2\mathbf{v}$.

1. $\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$

2. $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

In Exercises 3 and 4, display the following vectors using arrows on an xy -graph: \mathbf{u} , \mathbf{v} , $-\mathbf{v}$, $-2\mathbf{v}$, $\mathbf{u} + \mathbf{v}$, $\mathbf{u} - \mathbf{v}$, and $\mathbf{u} - 2\mathbf{v}$. Notice that $\mathbf{u} - \mathbf{v}$ is the vertex of a parallelogram whose other vertices are \mathbf{u} , $\mathbf{0}$, and $-\mathbf{v}$.

3. \mathbf{u} and \mathbf{v} as in Exercise 1

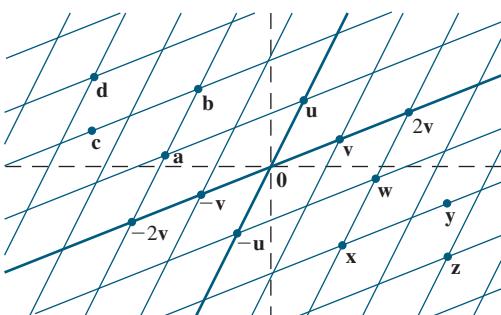
4. \mathbf{u} and \mathbf{v} as in Exercise 2

In Exercises 5 and 6, write a system of equations that is equivalent to the given vector equation.

5. $x_1 \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -8 \\ 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \\ -5 \end{bmatrix}$

6. $x_1 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 8 \\ 5 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Use the accompanying figure to write each vector listed in Exercises 7 and 8 as a linear combination of \mathbf{u} and \mathbf{v} . Is every vector in \mathbb{R}^2 a linear combination of \mathbf{u} and \mathbf{v} ?



7. Vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and \mathbf{d}

8. Vectors $\mathbf{w}, \mathbf{x}, \mathbf{y}$, and \mathbf{z}

In Exercises 9 and 10, write a vector equation that is equivalent to the given system of equations.

9. $x_2 + 5x_3 = 0$ 10. $4x_1 + x_2 + 3x_3 = 9$
 $4x_1 + 6x_2 - x_3 = 0$ $x_1 - 7x_2 - 2x_3 = 2$
 $-x_1 + 3x_2 - 8x_3 = 0$ $8x_1 + 6x_2 - 5x_3 = 15$

In Exercises 11 and 12, determine if \mathbf{b} is a linear combination of $\mathbf{a}_1, \mathbf{a}_2$, and \mathbf{a}_3 .

11. $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 5 \\ -6 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix}$

12. $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \mathbf{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \mathbf{a}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}$

In Exercises 13 and 14, determine if \mathbf{b} is a linear combination of the vectors formed from the columns of the matrix A .

13. $A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 3 & 5 \\ -2 & 8 & -4 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 3 \\ -7 \\ -3 \end{bmatrix}$

14. $A = \begin{bmatrix} 1 & -2 & -6 \\ 0 & 3 & 7 \\ 1 & -2 & 5 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$

In Exercises 15 and 16, list five vectors in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. For each vector, show the weights on \mathbf{v}_1 and \mathbf{v}_2 used to generate the vector and list the three entries of the vector. Do not make a sketch.

15. $\mathbf{v}_1 = \begin{bmatrix} 7 \\ 1 \\ -6 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}$

16. $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$

17. Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ h \end{bmatrix}$. For what value(s) of h is \mathbf{b} in the plane spanned by \mathbf{a}_1 and \mathbf{a}_2 ?

18. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -5 \\ 1 \\ 7 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} h \\ -1 \\ -5 \end{bmatrix}$. For what value(s) of h is \mathbf{y} in the plane generated by \mathbf{v}_1 and \mathbf{v}_2 ?

19. Give a geometric description of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for the vectors $\mathbf{v}_1 = \begin{bmatrix} 8 \\ 2 \\ -6 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 12 \\ 3 \\ -9 \end{bmatrix}$.

20. Give a geometric description of $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for the vectors in Exercise 16.

21. Let $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Show that $\begin{bmatrix} h \\ k \end{bmatrix}$ is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ for all h and k .

22. Construct a 3×3 matrix A , with nonzero entries, and a vector \mathbf{b} in \mathbb{R}^3 such that \mathbf{b} is *not* in the set spanned by the columns of A .

In Exercises 23–32, mark each statement True or False (T/F). Justify each answer.

23. (T/F) Another notation for the vector $\begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is $[-4 \ 3]$.

24. (T/F) Any list of five real numbers is a vector in \mathbb{R}^5 .

25. (T/F) The points in the plane corresponding to $\begin{bmatrix} -2 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ lie on a line through the origin.

26. (T/F) The vector \mathbf{u} results when a vector $\mathbf{u} - \mathbf{v}$ is added to the vector \mathbf{v} .

27. (T/F) An example of a linear combination of vectors \mathbf{v}_1 and \mathbf{v}_2 is the vector $\frac{1}{2}\mathbf{v}_1$.

28. (T/F) The weights c_1, \dots, c_p in a linear combination $c_1\mathbf{v}_1 + \dots + c_p\mathbf{v}_p$ cannot all be zero.

29. (T/F) The solution set of the linear system whose augmented matrix is $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}]$ is the same as the solution set of the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$.

30. (T/F) When \mathbf{u} and \mathbf{v} are nonzero vectors, $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ contains the line through \mathbf{u} and the origin.

31. (T/F) The set $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is always visualized as a plane through the origin.

32. (T/F) Asking whether the linear system corresponding to an augmented matrix $[\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3 \ \mathbf{b}]$ has a solution amounts to asking whether \mathbf{b} is in $\text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

33. Let $A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & -2 \\ -2 & 6 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ -4 \end{bmatrix}$. Denote the

columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and let $W = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$.

- Is \mathbf{b} in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$? How many vectors are in $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$?
- Is \mathbf{b} in W ? How many vectors are in W ?
- Show that \mathbf{a}_1 is in W . [Hint: Row operations are unnecessary.]

34. Let $A = \begin{bmatrix} 2 & 0 & 6 \\ -1 & 8 & 5 \\ 1 & -2 & 1 \end{bmatrix}$, let $\mathbf{b} = \begin{bmatrix} 10 \\ 3 \\ 3 \end{bmatrix}$, and let W be

the set of all linear combinations of the columns of A .

- Is \mathbf{b} in W ?
- Show that the third column of A is in W .

35. A mining company has two mines. One day's operation at mine 1 produces ore that contains 20 metric tons of copper and 550 kilograms of silver, while one day's operation at mine 2 produces ore that contains 30 metric tons of copper and 500 kilograms of silver. Let $\mathbf{v}_1 = \begin{bmatrix} 20 \\ 550 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 30 \\ 500 \end{bmatrix}$.

Then \mathbf{v}_1 and \mathbf{v}_2 represent the “output per day” of mine 1 and mine 2, respectively.

- What physical interpretation can be given to the vector $5\mathbf{v}_1$?
- Suppose the company operates mine 1 for x_1 days and mine 2 for x_2 days. Write a vector equation whose solution gives the number of days each mine should operate in order to produce 150 tons of copper and 2825 kilograms of silver. Do not solve the equation.
- Solve the equation in (b).

36. A steam plant burns two types of coal: anthracite (A) and bituminous (B). For each ton of A burned, the plant produces 27.6 million Btu of heat, 3100 grams (g) of sulfur dioxide, and 250 g of particulate matter (solid-particle pollutants). For each ton of B burned, the plant produces 30.2 million Btu, 6400 g of sulfur dioxide, and 360 g of particulate matter.

- How much heat does the steam plant produce when it burns x_1 tons of A and x_2 tons of B?
- Suppose the output of the steam plant is described by a vector that lists the amounts of heat, sulfur dioxide, and particulate matter. Express this output as a linear combination of two vectors, assuming that the plant burns x_1 tons of A and x_2 tons of B.
- Over a certain time period, the steam plant produced 162 million Btu of heat, 23,610 g of sulfur dioxide, and 1623 g of particulate matter. Determine how many tons of each type of coal the steam plant must have burned. Include a vector equation as part of your solution.

37. Let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be points in \mathbb{R}^3 and suppose that for $j = 1, \dots, k$ an object with mass m_j is located at point \mathbf{v}_j . Physicists call such objects *point masses*. The total mass of the system of point masses is

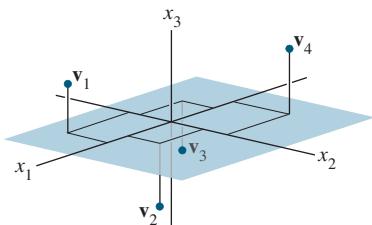
$$m = m_1 + \dots + m_k$$

The *center of mass* (or *center of gravity*) of the system is

$$\bar{\mathbf{v}} = \frac{1}{m} [m_1 \mathbf{v}_1 + \dots + m_k \mathbf{v}_k]$$

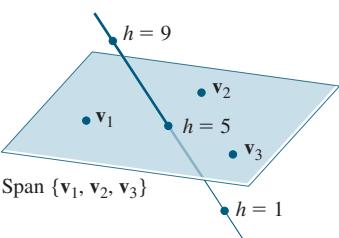
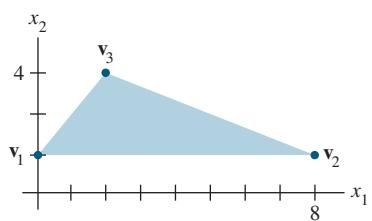
Compute the center of gravity of the system consisting of the following point masses (see the figure):

Point	Mass
$\mathbf{v}_1 = (5, -4, 3)$	2 g
$\mathbf{v}_2 = (4, 3, -2)$	5 g
$\mathbf{v}_3 = (-4, -3, -1)$	2 g
$\mathbf{v}_4 = (-9, 8, 6)$	1 g



38. Let \mathbf{v} be the center of mass of a system of point masses located at $\mathbf{v}_1, \dots, \mathbf{v}_k$ as in Exercise 37. Is \mathbf{v} in $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$? Explain.

39. A thin triangular plate of uniform density and thickness has vertices at $\mathbf{v}_1 = (0, 1)$, $\mathbf{v}_2 = (8, 1)$, and $\mathbf{v}_3 = (2, 4)$, as in the figure below, and the mass of the plate is 3 g.

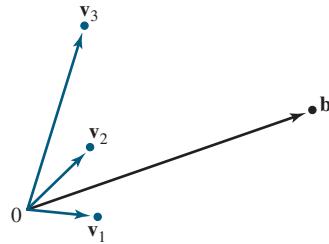


The points $\begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$ lie on a line that intersects the plane when $h = 5$.

- a. Find the (x, y) -coordinates of the center of mass of the plate. This “balance point” of the plate coincides with the center of mass of a system consisting of three 1-gram point masses located at the vertices of the plate.

- b. Determine how to distribute an additional mass of 6 g at the three vertices of the plate to move the balance point of the plate to $(2, 2)$. [Hint: Let w_1 , w_2 , and w_3 denote the masses added at the three vertices, so that $w_1 + w_2 + w_3 = 6$.]

40. Consider the vectors \mathbf{v}_1 , \mathbf{v}_2 , \mathbf{v}_3 , and \mathbf{b} in \mathbb{R}^2 , shown in the figure. Does the equation $x_1 \mathbf{v}_1 + x_2 \mathbf{v}_2 + x_3 \mathbf{v}_3 = \mathbf{b}$ have a solution? Is the solution unique? Use the figure to explain your answers.



41. Use the vectors $\mathbf{u} = (u_1, \dots, u_n)$, $\mathbf{v} = (v_1, \dots, v_n)$, and $\mathbf{w} = (w_1, \dots, w_n)$ to verify the following algebraic properties of \mathbb{R}^n .

- a. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
b. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ for each scalar c

42. Use the vector $\mathbf{u} = (u_1, \dots, u_n)$ to verify the following algebraic properties of \mathbb{R}^n .

- a. $\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = \mathbf{0}$
b. $c(d\mathbf{u}) = (cd)\mathbf{u}$ for all scalars c and d

Solutions to Practice Problems

1. Take arbitrary vectors $\mathbf{u} = (u_1, \dots, u_n)$ and $\mathbf{v} = (v_1, \dots, v_n)$ in \mathbb{R}^n , and compute

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= (u_1 + v_1, \dots, u_n + v_n) && \text{Definition of vector addition} \\ &= (v_1 + u_1, \dots, v_n + u_n) && \text{Commutativity of addition in } \mathbb{R} \\ &= \mathbf{v} + \mathbf{u} && \text{Definition of vector addition} \end{aligned}$$

2. The vector \mathbf{y} belongs to $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if there exist scalars x_1, x_2, x_3 such that

$$x_1 \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -4 \\ -7 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ h \end{bmatrix}$$

This vector equation is equivalent to a system of three linear equations in three unknowns. If you row reduce the augmented matrix for this system, you find that

$$\left[\begin{array}{cccc} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array} \right] \sim \left[\begin{array}{cccc} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{array} \right] \sim \left[\begin{array}{cccc} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{array} \right]$$

The system is consistent if and only if there is no pivot in the fourth column. That is, $h-5$ must be 0. So \mathbf{y} is in $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ if and only if $h=5$.

Remember: The presence of a free variable in a system does not guarantee that the system is consistent.

3. Since the vectors \mathbf{u} and \mathbf{v} are in $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$, there exist scalars c_1, c_2, c_3 and d_1, d_2, d_3 such that

$$\mathbf{u} = c_1 \mathbf{w}_1 + c_2 \mathbf{w}_2 + c_3 \mathbf{w}_3 \quad \text{and} \quad \mathbf{v} = d_1 \mathbf{w}_1 + d_2 \mathbf{w}_2 + d_3 \mathbf{w}_3.$$

Notice

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= c_1 \mathbf{w}_1 + c_2 \mathbf{w}_2 + c_3 \mathbf{w}_3 + d_1 \mathbf{w}_1 + d_2 \mathbf{w}_2 + d_3 \mathbf{w}_3 \\ &= (c_1 + d_1) \mathbf{w}_1 + (c_2 + d_2) \mathbf{w}_2 + (c_3 + d_3) \mathbf{w}_3 \end{aligned}$$

Since $c_1 + d_1, c_2 + d_2$, and $c_3 + d_3$ are also scalars, the vector $\mathbf{u} + \mathbf{v}$ is in $\text{Span}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

1.4 The Matrix Equation $Ax = \mathbf{b}$

A fundamental idea in linear algebra is to view a linear combination of vectors as the product of a matrix and a vector. The following definition permits us to rephrase some of the concepts of Section 1.3 in new ways.

DEFINITION

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if \mathbf{x} is in \mathbb{R}^n , then the **product of A and \mathbf{x}** , denoted by $A\mathbf{x}$, is the **linear combination of the columns of A using the corresponding entries in \mathbf{x} as weights**; that is,

$$A\mathbf{x} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \mathbf{a}_1 + x_2 \mathbf{a}_2 + \cdots + x_n \mathbf{a}_n$$

Note that $A\mathbf{x}$ is defined only if the number of columns of A equals the number of entries in \mathbf{x} .

EXAMPLE 1

$$\begin{aligned} \text{a. } \left[\begin{array}{ccc} 1 & 2 & -1 \\ 0 & -5 & 3 \end{array} \right] \left[\begin{array}{c} 4 \\ 3 \\ 7 \end{array} \right] &= 4 \left[\begin{array}{c} 1 \\ 0 \end{array} \right] + 3 \left[\begin{array}{c} 2 \\ -5 \end{array} \right] + 7 \left[\begin{array}{c} -1 \\ 3 \end{array} \right] \\ &= \left[\begin{array}{c} 4 \\ 0 \end{array} \right] + \left[\begin{array}{c} 6 \\ -15 \end{array} \right] + \left[\begin{array}{c} -7 \\ 21 \end{array} \right] = \left[\begin{array}{c} 3 \\ 6 \end{array} \right] \end{aligned}$$

$$\begin{aligned} \text{b. } \left[\begin{array}{cc} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{array} \right] \left[\begin{array}{c} 4 \\ 7 \end{array} \right] &= 4 \left[\begin{array}{c} 2 \\ 8 \\ -5 \end{array} \right] + 7 \left[\begin{array}{c} -3 \\ 0 \\ 2 \end{array} \right] = \left[\begin{array}{c} 8 \\ 32 \\ -20 \end{array} \right] + \left[\begin{array}{c} -21 \\ 0 \\ 14 \end{array} \right] = \left[\begin{array}{c} -13 \\ 32 \\ -6 \end{array} \right] \blacksquare \end{aligned}$$

EXAMPLE 2 For $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ in \mathbb{R}^m , write the linear combination $3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3$ as a matrix times a vector.

SOLUTION Place $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into the columns of a matrix A and place the weights 3, -5, and 7 into a vector \mathbf{x} . That is,

$$3\mathbf{v}_1 - 5\mathbf{v}_2 + 7\mathbf{v}_3 = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} = A\mathbf{x} \quad \blacksquare$$

Section 1.3 showed how to write a system of linear equations as a vector equation involving a linear combination of vectors. For example, the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned} \tag{1}$$

is equivalent to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \tag{2}$$

As in Example 2, the linear combination on the left side is a matrix times a vector, so that (2) becomes

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \tag{3}$$

Equation (3) has the form $A\mathbf{x} = \mathbf{b}$. Such an equation is called a **matrix equation**, to distinguish it from a vector equation such as is shown in (2).

Notice how the matrix in (3) is just the matrix of coefficients of the system (1). Similar calculations show that any system of linear equations, or any vector equation such as (2), can be written as an equivalent matrix equation in the form $A\mathbf{x} = \mathbf{b}$. This simple observation will be used repeatedly throughout the text.

Here is the formal result.

THEOREM 3

If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and if \mathbf{b} is in \mathbb{R}^m , the matrix equation

$$A\mathbf{x} = \mathbf{b} \tag{4}$$

has the same solution set as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b} \tag{5}$$

which, in turn, has the same solution set as the system of linear equations whose augmented matrix is

$$[\mathbf{a}_1 \ \mathbf{a}_2 \ \cdots \ \mathbf{a}_n \ \mathbf{b}] \tag{6}$$

Theorem 3 provides a powerful tool for gaining insight into problems in linear algebra, because a system of linear equations may now be viewed in three different but equivalent ways: as a matrix equation, as a vector equation, or as a system of linear equations. Whenever you construct a mathematical model of a problem in real life, you are free to choose whichever viewpoint is most natural. Then you may switch from one formulation of a problem to another whenever it is convenient. In any case, the matrix equation (4), the vector equation (5), and the system of equations are all solved in the same way—by row reducing the augmented matrix (6). Other methods of solution will be discussed later.

Existence of Solutions

The definition of Ax leads directly to the following useful fact.

The equation $Ax = b$ has a solution if and only if b is a linear combination of the columns of A .

Section 1.3 considered the existence question, “Is \mathbf{b} in $\text{Span}\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$?” Equivalently, “Is $Ax = b$ consistent?” A harder existence problem is to determine whether the equation $Ax = b$ is consistent for all possible \mathbf{b} .

EXAMPLE 3 Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Is the equation $Ax = b$ consistent for all possible b_1, b_2, b_3 ?

SOLUTION Row reduce the augmented matrix for $Ax = b$:

$$\begin{bmatrix} 1 & 3 & 4 & b_1 \\ -4 & 2 & -6 & b_2 \\ -3 & -2 & -7 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 7 & 5 & b_3 + 3b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 4 & b_1 \\ 0 & 14 & 10 & b_2 + 4b_1 \\ 0 & 0 & 0 & b_3 + 3b_1 - \frac{1}{2}(b_2 + 4b_1) \end{bmatrix}$$

The third entry in column 4 equals $b_1 - \frac{1}{2}b_2 + b_3$. The equation $Ax = b$ is not consistent for every \mathbf{b} because some choices of \mathbf{b} can make $b_1 - \frac{1}{2}b_2 + b_3$ nonzero. ■

The reduced matrix in Example 3 provides a description of all \mathbf{b} for which the equation $Ax = b$ is consistent: The entries in \mathbf{b} must satisfy

$$b_1 - \frac{1}{2}b_2 + b_3 = 0$$

This is the equation of a plane through the origin in \mathbb{R}^3 . The plane is the set of all linear combinations of the three columns of A . See Figure 1.

The equation $Ax = b$ in Example 3 fails to be consistent for all \mathbf{b} because the echelon form of A has a row of zeros. If A had a pivot in all three rows, we would not care about the calculations in the augmented column because in this case an echelon form of the augmented matrix could not have a row such as $[0 \ 0 \ 0 \ 1]$.

In the next theorem, the sentence “The columns of A span \mathbb{R}^m ” means that every \mathbf{b} in \mathbb{R}^m is a linear combination of the columns of A . In general, a set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in \mathbb{R}^m spans (or generates) \mathbb{R}^m if every vector in \mathbb{R}^m is a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_p$ —that is, if $\text{Span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = \mathbb{R}^m$.

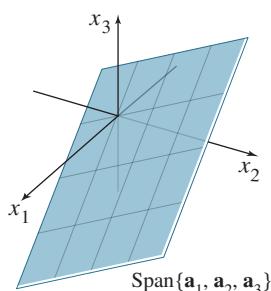


FIGURE 1

The columns of $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$ span a plane through $\mathbf{0}$.

THEOREM 4

Let A be an $m \times n$ matrix. Then the following statements are logically equivalent. That is, for a particular A , either they are all true statements or they are all false.

- a. For each \mathbf{b} in \mathbb{R}^m , the equation $Ax = b$ has a solution.
- b. Each \mathbf{b} in \mathbb{R}^m is a linear combination of the columns of A .
- c. The columns of A span \mathbb{R}^m .
- d. A has a pivot position in every row.

Theorem 4 is one of the most useful theorems in this chapter. Statements (a), (b), and (c) are equivalent because of the definition of Ax and what it means for a set of vectors to span \mathbb{R}^m . The discussion after Example 3 suggests why (a) and (d) are equivalent; a proof is given at the end of the section. The exercises will provide examples of how Theorem 4 is used.

Warning: Theorem 4 is about a *coefficient matrix*, not an augmented matrix. If an augmented matrix $[A \ b]$ has a pivot position in every row, then the equation $Ax = b$ may or may not be consistent.

Computation of Ax

The calculations in Example 1 were based on the definition of the product of a matrix A and a vector \mathbf{x} . The following simple example will lead to a more efficient method for calculating the entries in Ax when working problems by hand.

EXAMPLE 4 Compute Ax , where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

SOLUTION From the definition,

$$\begin{aligned} \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} &= x_1 \begin{bmatrix} 2 \\ -1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ -3 \\ 8 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 \\ -x_1 \\ 6x_1 \end{bmatrix} + \begin{bmatrix} 3x_2 \\ 5x_2 \\ -2x_2 \end{bmatrix} + \begin{bmatrix} 4x_3 \\ -3x_3 \\ 8x_3 \end{bmatrix} \\ &= \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \\ -x_1 + 5x_2 - 3x_3 \\ 6x_1 - 2x_2 + 8x_3 \end{bmatrix} \end{aligned} \quad (7)$$

The first entry in the product Ax is a sum of products (sometimes called a *dot product*), using the first row of A and the entries in \mathbf{x} . That is,

$$\begin{bmatrix} 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 + 3x_2 + 4x_3 \end{bmatrix}$$

This matrix shows how to compute the first entry in Ax directly, without writing down all the calculations shown in (7). Similarly, the second entry in Ax can be calculated at once by multiplying the entries in the second row of A by the corresponding entries in \mathbf{x} and then summing the resulting products:

$$\begin{bmatrix} -1 & 5 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_1 + 5x_2 - 3x_3 \end{bmatrix}$$

Likewise, the third entry in Ax can be calculated from the third row of A and the entries in \mathbf{x} . ■

Row–Vector Rule for Computing Ax

If the product Ax is defined, then the i th entry in Ax is the sum of the products of corresponding entries from row i of A and from the vector \mathbf{x} .

EXAMPLE 5

$$\text{a. } \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 3 + (-1) \cdot 7 \\ 0 \cdot 4 + (-5) \cdot 3 + 3 \cdot 7 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\text{b. } \begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \cdot 4 + (-3) \cdot 7 \\ 8 \cdot 4 + 0 \cdot 7 \\ (-5) \cdot 4 + 2 \cdot 7 \end{bmatrix} = \begin{bmatrix} -13 \\ 32 \\ -6 \end{bmatrix}$$

$$\text{c. } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \cdot r + 0 \cdot s + 0 \cdot t \\ 0 \cdot r + 1 \cdot s + 0 \cdot t \\ 0 \cdot r + 0 \cdot s + 1 \cdot t \end{bmatrix} = \begin{bmatrix} r \\ s \\ t \end{bmatrix}$$
■

By definition, the matrix in Example 5(c) with 1's on the diagonal and 0's elsewhere is called an **identity matrix** and is denoted by I . The calculation in part (c) shows that $I\mathbf{x} = \mathbf{x}$ for every \mathbf{x} in \mathbb{R}^3 . There is an analogous $n \times n$ identity matrix, sometimes written as I_n . As in part (c), $I_n\mathbf{x} = \mathbf{x}$ for every \mathbf{x} in \mathbb{R}^n .

Properties of the Matrix–Vector Product $A\mathbf{x}$

The facts in the next theorem are important and will be used throughout the text. The proof relies on the definition of $A\mathbf{x}$ and the algebraic properties of \mathbb{R}^n .

THEOREM 5

If A is an $m \times n$ matrix, \mathbf{u} and \mathbf{v} are vectors in \mathbb{R}^n , and c is a scalar, then:

- a. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$;
- b. $A(c\mathbf{u}) = c(A\mathbf{u})$.

PROOF For simplicity, take $n = 3$, $A = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$, and \mathbf{u}, \mathbf{v} in \mathbb{R}^3 . (The proof of the general case is similar.) For $i = 1, 2, 3$, let u_i and v_i be the i th entries in \mathbf{u} and \mathbf{v} , respectively. To prove statement (a), compute $A(\mathbf{u} + \mathbf{v})$ as a linear combination of the columns of A using the entries in $\mathbf{u} + \mathbf{v}$ as weights.

$$\begin{aligned} A(\mathbf{u} + \mathbf{v}) &= [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3] \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix} \\ &= (u_1 + v_1)\mathbf{a}_1 + (u_2 + v_2)\mathbf{a}_2 + (u_3 + v_3)\mathbf{a}_3 \\ &\quad \begin{array}{c} \downarrow \qquad \downarrow \qquad \downarrow \\ \text{Entries in } \mathbf{u} + \mathbf{v} \end{array} \\ &\quad \begin{array}{c} \uparrow \qquad \uparrow \qquad \uparrow \\ \text{Columns of } A \end{array} \\ &= (u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) + (v_1\mathbf{a}_1 + v_2\mathbf{a}_2 + v_3\mathbf{a}_3) \\ &= A\mathbf{u} + A\mathbf{v} \end{aligned}$$

To prove statement (b), compute $A(c\mathbf{u})$ as a linear combination of the columns of A using the entries in $c\mathbf{u}$ as weights.

$$\begin{aligned} A(c\mathbf{u}) &= [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3] \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix} = (cu_1)\mathbf{a}_1 + (cu_2)\mathbf{a}_2 + (cu_3)\mathbf{a}_3 \\ &= c(u_1\mathbf{a}_1) + c(u_2\mathbf{a}_2) + c(u_3\mathbf{a}_3) \\ &= c(u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + u_3\mathbf{a}_3) \\ &= c(A\mathbf{u}) \end{aligned}$$
■

Numerical Note

To optimize a computer algorithm to compute $A\mathbf{x}$, the sequence of calculations should involve data stored in contiguous memory locations. The most widely used professional algorithms for matrix computations are written in Fortran, a language that stores a matrix as a set of columns. Such algorithms compute $A\mathbf{x}$ as a linear combination of the columns of A . In contrast, if a program is written in the popular language C, which stores matrices by rows, $A\mathbf{x}$ should be computed via the alternative rule that uses the rows of A .

PROOF OF THEOREM 4 As was pointed out after Theorem 4, statements (a), (b), and (c) are logically equivalent. So, it suffices to show (for an arbitrary matrix A) that (a) and (d) are either both true or both false. This will tie all four statements together.

Let U be an echelon form of A . Given \mathbf{b} in \mathbb{R}^m , we can row reduce the augmented matrix $[A \ \mathbf{b}]$ to an augmented matrix $[U \ \mathbf{d}]$ for some \mathbf{d} in \mathbb{R}^m :

$$[A \ \mathbf{b}] \sim \cdots \sim [U \ \mathbf{d}]$$

If statement (d) is true, then each row of U contains a pivot position and there can be no pivot in the augmented column. So $A\mathbf{x} = \mathbf{b}$ has a solution for any \mathbf{b} , and (a) is true. If (d) is false, the last row of U is all zeros. Let \mathbf{d} be any vector with a 1 in its last entry. Then $[U \ \mathbf{d}]$ represents an *inconsistent* system. Since row operations are reversible, $[U \ \mathbf{d}]$ can be transformed into the form $[A \ \mathbf{b}]$. The new system $A\mathbf{x} = \mathbf{b}$ is also inconsistent, and (a) is false. ■

Practice Problems

1. Let $A = \begin{bmatrix} 1 & 5 & -2 & 0 \\ -3 & 1 & 9 & -5 \\ 4 & -8 & -1 & 7 \end{bmatrix}$, $\mathbf{p} = \begin{bmatrix} 3 \\ -2 \\ 0 \\ -4 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} -7 \\ 9 \\ 0 \end{bmatrix}$. It can be shown

that \mathbf{p} is a solution of $A\mathbf{x} = \mathbf{b}$. Use this fact to exhibit \mathbf{b} as a specific linear combination of the columns of A .

2. Let $A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$. Verify Theorem 5(a) in this case by computing $A(\mathbf{u} + \mathbf{v})$ and $A\mathbf{u} + A\mathbf{v}$.
3. Construct a 3×3 matrix A and vectors \mathbf{b} and \mathbf{c} in \mathbb{R}^3 so that $A\mathbf{x} = \mathbf{b}$ has a solution, but $A\mathbf{x} = \mathbf{c}$ does not.

1.4 Exercises

Compute the products in Exercises 1–4 using (a) the definition, as in Example 1, and (b) the row–vector rule for computing $A\mathbf{x}$. If a product is undefined, explain why.

1. $\begin{bmatrix} -4 & 2 \\ 1 & 6 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 7 \end{bmatrix}$

2. $\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

3. $\begin{bmatrix} 6 & 5 \\ -4 & -3 \\ 7 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

4. $\begin{bmatrix} 8 & 3 & 1 \\ 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

In Exercises 5–8, use the definition of $A\mathbf{x}$ to write the matrix equation as a vector equation, or vice versa.

5. $\begin{bmatrix} 7 & 2 & -9 & 3 \\ -4 & -5 & 7 & -2 \end{bmatrix} \begin{bmatrix} 6 \\ -9 \\ 1 \\ -8 \end{bmatrix} = \begin{bmatrix} -9 \\ 44 \end{bmatrix}$

6. $\begin{bmatrix} 7 & -3 \\ 2 & 1 \\ 9 & -6 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \\ 12 \\ -4 \end{bmatrix}$