Deriving (2.8.25a) in Polchinski

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I. TARGET FORMULA

$$\alpha_n = -\frac{in}{(2\alpha')^{1/2}} X_{-n} - i\left(\frac{\alpha'}{2}\right)^{1/2} \frac{\partial}{\partial X_n} \tag{2.8.25a}$$

II. DERIVATION

Start by noting that:

$$X = X_i + X_0 + \sum_{n=1}^{\infty} \left(z^n X_n + \bar{z}^n X_{-n}\right), \tag{1}$$

$$S = \frac{1}{2\pi\alpha'} \int_{|z|<1} d^2z\, \partial X \bar{\partial} X. \tag{2}$$

Computing a partial derivative of the action with respect to the boundary field mode we get:

$$\begin{aligned}
\frac{\partial S}{\partial X_n} &= \frac{1}{2\pi\alpha'} \int_{|z|<1} d^2z\, \partial(z^n) \bar{\partial} X && (3)\\
&= \frac{1}{2\pi\alpha'} \int_{|z|<1} d^2z\, \partial(z^n \bar{\partial} X) && (4)\\
&= i\frac{1}{2\pi\alpha'} \oint_{|z|=1} d\bar{z}\, z^n \bar{\partial} X && (5)\\
&= i\frac{1}{2\pi\alpha'} \oint_{|z|=1} d\bar{z}\, \bar{z}^{-n} \bar{\partial} X && (6)\\
&= \frac{1}{\alpha'} n X_{-n}. && (7)
\end{aligned}$$

Which means that the wavefunctional is an eigenstate of such operator:

$$\begin{aligned}
\frac{\partial}{\partial X_n} \Psi_1[X_b] &= \frac{\partial}{\partial X_n} \int [dX_i]_{X_b=0} \exp(-S) && (8)\\
&= -\frac{n}{\alpha'} X_{-n} \left(\int [dX_i]_{X_b=0} \exp(-S)\right) && (9)\\
&= -\frac{n}{\alpha'} X_{-n} \Psi_1[X_b] && (10)
\end{aligned}$$

or, using the factors from 2.8.25a we see that the first and the second terms of 2.8.25a cancel each other when applied to Ψ_1

$$-i\left(\frac{\alpha'}{2}\right)^{1/2} \frac{\partial}{\partial X_n} \Psi_1[X_b] = -\frac{in}{(2\alpha')^{1/2}} X_n \Psi_1[X_b]. \tag{11}$$

Now using the definition of α_m we can compute it's value using the boundary contour

$$\alpha_m = i\left(\frac{2}{\alpha'}\right)^{1/2} \oint_z \frac{dz}{2\pi i} z^m \partial X_b(z) = -in\left(\frac{2}{\alpha'}\right)^{1/2} X_{-n} \tag{12}$$

which is precisely twice the first term of 2.8.25a.

I have proven nothing.