Exercise (2.6) in Polchinski

indutny

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I. PROBLEM STATEMENT

Given the transformation law for the metric:

$$\delta g_{ab} = -\partial_a v_b - \partial_b v_a \tag{1}$$

determine the most general $v^a(\sigma)$ that leaves flat d-dimensional Euclidian metric δ_{ab} invariant up to a local rescaling.

II. SOLUTION

The case of $d = 2$ is covered extensively in String Theory textbooks, so we will concentrate our efforts on $d > 2$ here.

A. Notation

Everywhere below a, b, and c are three different indices: $a \neq b$, $b \neq c$, $a \neq c$. Note that it means that most of the derivation does not apply to $d = 2$. This the reason why $d = 2$ is special.

m and n are possibly equal, but both non-negative integers: $m \geq 0$, $n \geq 0$.

B. Derivation

The local rescaling invariance imposes following constraints:

$$\partial_a v_a = \partial_b v_b, \tag{2}$$
$$\partial_b v_a = -\partial_a v_b. \tag{3}$$

For $d > 2$ we can derive an additional constraint:

$$\partial_c (\partial_b v_a) = -\partial_c (\partial_a v_b) = -\partial_a (\partial_c v_b) = \tag{4}$$
$$\partial_a \partial_b v_c = \partial_b (\partial_a v_c), \tag{5}$$

which is equivalent to:

$$\partial_b (\partial_c v_a - \partial_a v_c) = 2\partial_b \partial_c v_a = 0. \tag{6}$$

This immediately implies that v_a can't have terms with three different coordinate variables:

$$v_a = \sum_{\substack{m\geq 0, n\geq 0 \\ b\neq a}} c^{(ab)}_{mn} (\sigma_a)^m (\sigma_b)^n, \tag{7}$$
$$c^{(ab)}_{m0} = c^{(ac)}_{m0}. \tag{8}$$

In fact, the space of solutions is even smaller. The term

$$v_a = (\sigma_a)^{m+2} (\sigma_b)^{n+1} + \cdots \tag{9}$$

is incompatible with constraints:

$$\partial_c v_c = \partial_a v_a = (m+2)(\sigma_a)^{m+1}(\sigma_b)^{n+1}, \tag{10}$$

but v_c can't have a term proportional to $(\sigma_c)(\sigma_a)^{m+1}(\sigma_b)^{n+1}$ due to (6). Therefore

$$c^{(ab)}_{(m+2)(n+1)} = 0. \tag{11}$$

As we shall see only finitely few of them are non-zero, and even less are independent of each other.

The first constraint (2) can be translated from differential equation

$$\partial_a v_a = \sum m c^{(ab)}_{mn} (\sigma_a)^{m-1} (\sigma_b)^n, \tag{12}$$
$$\partial_b v_b = \sum n c^{(ba)}_{nm} (\sigma_b)^{n-1} (\sigma_a)^m, \partial_a v_a = \partial_b v_b \tag{13}$$

to a relation between coefficients

$$(m+1) c^{(ab)}_{(m+1)n} = (n+1) c^{(ba)}_{(n+1)m}. \tag{14}$$

Similarly (3) translates to

$$(n+1) c^{(ab)}_{m(n+1)} = -(m+1) c^{(ba)}_{n(m+1)}. \tag{15}$$

Combining (14) and (15) together we see that

$$c^{(ab)}_{(m+2)n} = -\frac{(n+2)(n+1)}{(m+2)(m+1)} c^{(ab)}_{m(n+2)}, \tag{16}$$

which together with (11) implies that

$$c^{(ab)}_{m(n+3)} \simeq c^{(ab)}_{(m+2)(n+1)} = 0, \tag{17}$$
$$c^{(ab)}_{(m+4)n} \simeq c^{(ab)}_{(m+2)(n+2)} = 0. \tag{18}$$

Therefore the only coefficients that can be non-zero are

$$c^{(a)} \equiv c^{(ab)}_{00},\ c^{(a)}_1 \equiv c^{(ab)}_{10},\ c^{(ab)}_{01}, \tag{19}$$
$$c^{(ab)}_{11},\ c^{(ab)}_{12},\ c^{(a)}_3 \equiv c^{(ab)}_{30}. \tag{20}$$

We can also work out relations between them. Using (14) and (15):

$$c^{(a)}_1 = c^{(ab)}_{10} = c^{(ba)}_{10} = c^{(b)}_1, \tag{21}$$
$$c^{(ab)}_{01} = -c^{(ba)}_{01}, \tag{22}$$
$$c^{(ab)}_{11} = -2c^{(ba)}_{02} = 2c^{(ba)}_{20} = 2c^{(b)}_2, \tag{23}$$
$$c^{(a)}_3 = c^{(ab)}_{30} = \frac{1}{3} c^{(ba)}_{12} = -\frac{1}{3} c^{(ab)}_{12} = -c^{(ba)}_{30} = -c^{(b)}_3, \tag{24}$$

but the (24) has to be zero

$$c_3^{(a)} = -c_3^{(b)} = c_3^{(c)} = -c_3^{(a)} = 0. \tag{25}$$

Before writing out the most general form of v^a, let us count the number of independent solutions

$$\# \left\{ c^{(a)} \right\} = d, \tag{26}$$

$$\# \left\{ c_1^{(a)} = c_1^{(b)} \right\} = 1, \tag{27}$$

$$\# \left\{ c_{01}^{(ab)} = -c_{01}^{(ba)} \right\} = \frac{d(d-1)}{2}, \tag{28}$$

$$\# \left\{ c_2^{(a)} \right\} = d, \tag{29}$$

and the total

$$d + 1 + \frac{d(d-1)}{2} + d \tag{30}$$

$$= \frac{2d + 2 + d^2 - d + 2d}{2} \tag{31}$$

$$= \frac{d^2 + 3d + 2}{2} \tag{32}$$

$$= \frac{(d+1)(d+2)}{2}. \tag{33}$$

C. Result

The most general form of solution is:

$$\begin{aligned} v^a =& c_0 + c_1(\sigma_a) + c_2(\sigma_a)^2 \\ &+ \sum_b \left[c_3(\sigma_a)(\sigma_b) + -2c_4(\sigma_b)^2 \right] \\ &+ \sum_b c_5 \left[(\sigma_a)^3 - \frac{1}{3}(\sigma_a)(\sigma_b)^2 \right]. \end{aligned} \tag{34}$$