Exercise (2.6) in Polchinski

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I. PROBLEM STATEMENT

Given the transformation law for the metric:

$$\delta g_{ab} = -\partial_a v_b - \partial_b v_a \qquad (1)$$

determine the most general $v^a(\sigma)$ that leaves flat d-dimensional Euclidian metric δ_{ab} invariant up to a local rescaling.

II. SOLUTION

The case of $d = 2$ is covered extensively in String Theory textbooks, so we will concentrate our efforts on $d > 2$ here.

A. Notation

Everywhere below a, b, and c are three different indices: $a \neq b$, $b \neq c$, $a \neq c$. Note that it means that most of the derivation does not apply to $d = 2$. This the reason why $d = 2$ is special.

m and n are possibly equal, but both non-negative integers: $m \geq 0$, $n \geq 0$.

B. Derivation

The local rescaling invariance imposes following constraints:

$$\partial_a v_a = \partial_b v_b, \qquad (2)$$
$$\partial_b v_a = -\partial_a v_b. \qquad (3)$$

For $d > 2$ we can derive an additional constraint:

$$\partial_c\left(\partial_b v_a\right) = -\partial_c\left(\partial_a v_b\right) = -\partial_a\left(\partial_c v_b\right) =$$
$$\partial_a \partial_b v_c = \partial_b\left(\partial_a v_c\right), \qquad (4)$$

which is equivalent to:

$$\partial_b\left(\partial_c v_a - \partial_a v_c\right) = 2\partial_b\partial_c v_a = 0. \qquad (5)$$

This immediately implies that v_a can't have terms with three different coordinate variables:

$$v_a = \sum_{\substack{m \geq 0, n \geq 0 \\ b \neq a}} c_{mn}^{(ab)}(\sigma_a)^m(\sigma_b)^n, \qquad (6)$$

$$c_{m0}^{(ab)} = c_{m0}^{(ac)}. \qquad (7)$$

In fact, the space of solutions is even smaller. The term

$$v_a = (\sigma_a)^{m+2}(\sigma_b)^{n+1} + \cdots \qquad (8)$$

is incompatible with constraints:

$$\partial_c v_c = \partial_a v_a = (m+2)(\sigma_a)^{m+1}(\sigma_b)^{n+1}, \qquad (9)$$

but v_c can't have a term proportional to $(\sigma_c)(\sigma_a)^{m+1}(\sigma_b)^{n+1}$ due to (5). Therefore

$$c_{(m+2)(n+1)}^{(ab)} = 0. \qquad (10)$$

As we shall see only finitely few of them are non-zero, and even less are independent of each other.

The first constraint (2) can be translated from differential equation

$$\partial_a v_a = \sum m c_{mn}^{(ab)}(\sigma_a)^{m-1}(\sigma_b)^n \qquad (11)$$

$$= \partial_b v_b = \sum n c_{nm}^{(ba)}(\sigma_b)^{n-1}(\sigma_a)^m \qquad (12)$$

to a relation between coefficients

$$(m+1)c_{(m+1)n}^{(ab)} = (n+1)c_{(n+1)m}^{(ba)}. \qquad (13)$$

Similarly (3) translates to

$$(n+1)c_{m(n+1)}^{(ab)} = -(m+1)c_{n(m+1)}^{(ba)}. \qquad (14)$$

Combining (13) and (14) together we see that

$$c_{(m+2)n}^{(ab)} = -\frac{(n+2)(n+1)}{(m+2)(m+1)}c_{m(n+2)}^{(ab)}, \qquad (15)$$

which together with (10) implies that

$$c_{m(n+3)}^{(ab)} \simeq c_{(m+2)(n+1)}^{(ab)} = 0, \qquad (16)$$

$$c_{(m+4)n}^{(ab)} \simeq c_{(m+2)(n+2)}^{(ab)} = 0. \qquad (17)$$

Therefore the only coefficients that can be non-zero are

$$c^{(a)} \equiv c_{00}^{(ab)}, \; c_1^{(a)} \equiv c_{10}^{(ab)}, \; c_{01}^{(ab)}, \qquad (18)$$
$$c_{11}^{(ab)}, \; c_{12}^{(ab)}, \; c_3^{(a)} \equiv c_{30}^{(ab)}. \qquad (19)$$

We can also work out relations between them. Using (13) and (14):

$$c_1^{(a)} = c_{10}^{(ab)} = c_{10}^{(ba)} = c_1^{(b)}, \qquad (20)$$

$$c_{01}^{(ab)} = -c_{01}^{(ba)}, \qquad (21)$$

$$c_{11}^{(ab)} = -2c_{02}^{(ba)} = 2c_{20}^{(ba)} = 2c_2^{(b)}, \qquad (22)$$

$$c_3^{(a)} = c_{30}^{(ab)} = \frac{1}{3}c_{12}^{(ba)} = -\frac{1}{3}c_{12}^{(ab)} = -c_{30}^{(ba)} = -c_3^{(b)}, \qquad (23)$$

but the (23) has to be zero

$$c_3^{(a)} = -c_3^{(b)} = c_3^{(c)} = -c_3^{(a)} = 0. \qquad (24)$$

Before writing out the most general form of v^a, let us count the number of independent solutions

$$\# \left\{ c^{(a)} \right\} = d, \qquad (25)$$

$$\# \left\{ c_1^{(a)} = c_1^{(b)} \right\} = 1, \qquad (26)$$

$$\# \left\{ c_{01}^{(ab)} = -c_{01}^{(ba)} \right\} = \frac{d(d-1)}{2}, \qquad (27)$$

$$\# \left\{ c_2^{(a)} \right\} = d, \qquad (28)$$

and the total

$$d + 1 + \frac{d(d-1)}{2} + d \qquad (29)$$

$$= \frac{2d + 2 + d^2 - d + 2d}{2} \qquad (30)$$

$$= \frac{d^2 + 3d + 2}{2} \qquad (31)$$

$$= \frac{(d+1)(d+2)}{2}. \qquad (32)$$

C. Result

The most general form of solution is:

$$\begin{aligned} v^a =& c_0 + c_1(\sigma_a) + c_2(\sigma_a)^2 \\ &+ \sum_b \left[c_3(\sigma_a)(\sigma_b) + -2c_4(\sigma_b)^2 \right] \\ &+ \sum_b c_5 \left[(\sigma_a)^3 - \frac{1}{3}(\sigma_a)(\sigma_b)^2 \right]. \end{aligned} \qquad (33)$$