Linear equations

Linear algebra basics

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Today's outline

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Overview

Goals

- Different ways of looking at a system of linear equations
- Determination of the inverse, determinant and the rank of a matrix
- The existence of a solution to a set of linear equations

Different views of linear systems

- Separate equations:

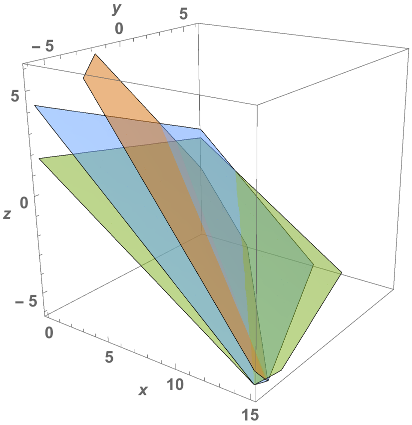
$$\begin{aligned} x + y + z &= 4 \\ 2x + y + 3z &= 7 \\ 3x + y + 6z &= 5 \end{aligned}$$

- Matrix mapping $Mx = b$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix}$$

- Linear combination:

$$x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix}$$



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Inverse of a matrix

- The inverse M^{-1} is defined such that:

$$MM^{-1} = I \quad \text{and} \quad M^{-1}M = I$$

- Use the inverse to solve a set of linear equations:

$$\begin{aligned} M\boldsymbol{x} &= \boldsymbol{b} \\ M^{-1}M\boldsymbol{x} &= M^{-1}\boldsymbol{b} \\ I\boldsymbol{x} &= M^{-1}\boldsymbol{b} \\ \boldsymbol{x} &= M^{-1}\boldsymbol{b} \end{aligned}$$

How to calculate the inverse?

- The inverse of an $N \times N$ matrix can be calculated using the co-factors of each element of the matrix:

$$M^{-1} = \frac{1}{\det\left|M\right|} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

- $\det\left|M\right|$ is the *determinant* of matrix M.
- C_{ij} is the *co-factor* of the ij^{th} element in M.

Computing the co-factors

Consider the following example matrix: $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{bmatrix}$

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$$\begin{bmatrix} 1 & \times & \times \\ \times & 1 & 3 \\ \times & 1 & 6 \end{bmatrix}$$

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$$\begin{bmatrix} 1 & \times & \times \\ \times & 1 & 3 \\ \times & 1 & 6 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad C_{11} = +1 \cdot \det \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix}$$

$$= 6 \times 1 - 3 \times 1 = 3$$

Computing the co-factors

Back to our example:

$$M^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{bmatrix}^{-1} = \frac{1}{\det\left|M\right|} \begin{bmatrix} 3 & -3 & -1 \\ -5 & 3 & 2 \\ 2 & -1 & -1 \end{bmatrix}^{T}$$

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- The determinant is very important
- If $\det \left| M \right| = 0$, the inverse does not exist (singular matrix)

Calculating the determinant

Compute the determinant by multiplication of each element on a row (or column) by its cofactor and adding the results:

$$\det\left|\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{bmatrix}\right| = +\det\left|\begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix}\right| - \det\left|\begin{bmatrix} 2 & 3 \\ 3 & 6 \end{bmatrix}\right| + \det\left|\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\right| = -1$$

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$$\det\left|\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{bmatrix}\right| = +\det\left|\begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}\right| - 3\det\left|\begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}\right| + 6\det\left|\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\right| = -1$$

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Solving a linear system

- Our example:

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- The solution is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = M^{-1}b = \frac{1}{-1} \begin{bmatrix} 3 & -5 & 2 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} -13 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 13 \\ -4 \\ -5 \end{bmatrix}$$

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- The inverse exists, because $\det \left| M \right| = -1$.

Solving a linear system in Matlab using the inverse

- Create the matrix:

```
>> A = [1 1 1; 2 1 3; 3 1 6];
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- Matlab's internal direct solver:

```
>> x = A\b
```

- These are black boxes! We are going over some methods later!

Exercise: performance of inverse computation

Create a script that generates matrices with random elements of various sizes $N \times N$. Compute the inverse of each matrix, and use `tic` and `toc` to see the computing time for each inversion. Plot the time as a function of the matrix size N.

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s = [10:10:90 100:100:1000 2000:1000:5000 10000]
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for n = 1:length(s)
    s(n)
    A = rand(s(n));
    tic;
    Ainv = inv(A);
    t_inv(n) = toc;
end
loglog(s,t_inv)
xlabel('N')
ylabel('Time [s]')
```
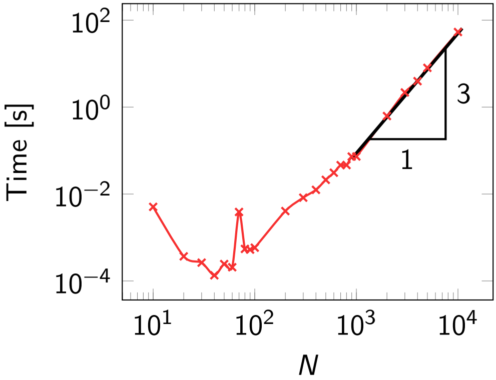

Exercise: sample results

Each computer produces slightly different results because of background tasks, different matrices, etc. This is especially noticable for small systems.



The time increases by 3 orders of magnitude, for every magnitude in N. A matrix inversion scales with $\mathcal{O}(N^3)$!

Solving a linear system in Excel using the inverse

$$Ax = b \qquad \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix}$$

- Create matrix A in 3×3 cells
- Create right hand side vector b in 3 vertical cells

[1] In Dutch Excel: `INVERSEMAT`

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- Create matrix A in 3×3 cells
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- Compute the inverse I :
  - Select an empty area of 3×3 cells
  - Type `=MINVERSE(B2:D4)` [1]
  - Close with Ctrl+Shift+Enter

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  - Select an empty area of 3×3 cells
  - Type `=MINVERSE(B2:D4)` [1]
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- Solution:
  - Select 3 vertical cells
  - Type `=MMULT(H2:J4;B6:B8)` [2]
  - Close with Ctrl+Shift+Enter

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Today's outline

Towards larger systems

Computation of determinants and inverses of large matrices in this way is too difficult (slow), so we need other methods to solve large linear systems!

Towards larger systems

- Determinant of upper triangular matrix:

$$\det \left| M_{\text{tri}} \right| = \prod_{i=1}^{n} a_{ii} \qquad M = \begin{bmatrix} 5 & 3 & 2 \\ 0 & 9 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \det \left| M \right| = 5 \times 9 \times 1 = 45$$

- Matrix multiplication:

$$\det \left| AM \right| = \det \left| A \right| \times \det \left| M \right|$$

- When A is an identity matrix ($\det \left| A \right| = 1$):

$$\det \left| AM \right| = \det \left| A \right| \times \det \left| M \right| = 1 \times \det \left| M \right|$$

- With rules like this, we can use row-operations so that we can compute the determinant more cheaply.

Solutions of linear systems

Rank of a matrix: the number of linearly independent columns (columns that can not be expressed as a linear combination of the other columns) of a matrix.

$$M = \begin{bmatrix} 5 & 3 & 2 \\ 0 & 9 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- 3 independent columns
- In Matlab:

```
>> rank(M)
```

$$M = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- col 2 $= 2\times$ col 1
- col 4 $=$ col 3 $-$ col 1
- 2 independent columns: rank $= 2$

Solutions of linear systems

The solution of a system of linear equations may or may not exist, and it may or may not be unique. Existence of solutions can be determined by comparing the rank of the Matrix M with the rank of the augmented matrix M_a:

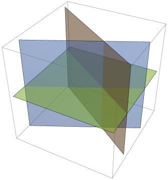
```
>> rank(A)
>> rank([A b])
```

Our system: $Mx = b$

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow M_a = \begin{bmatrix} M_{11} & M_{12} & M_{13} & b_1 \\ M_{21} & M_{22} & M_{23} & b_2 \\ M_{31} & M_{32} & M_{33} & b_3 \end{bmatrix}$$

Existence of solutions for linear systems

For a matrix M of size $n \times n$, and augmented matrix M_a:



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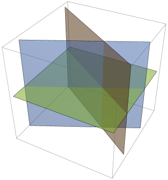
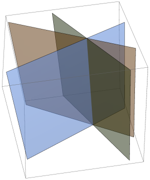


- $\text{Rank}(M) = \text{Rank}(M_a) < n$:
  Infinite number of solutions



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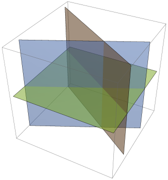

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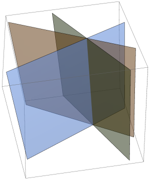
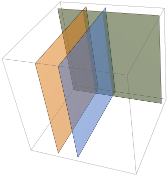
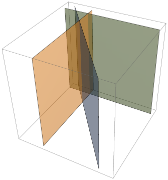


- $\text{Rank}(M) < n$,
  $\text{Rank}(M) < \text{Rank}(M_a)$:
  No solutions





Two examples

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 17 \\ 11 \\ 4 \end{bmatrix} \Rightarrow M_a = \begin{bmatrix} 1 & 1 & 2 & 17 \\ 0 & 3 & 1 & 11 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$\mathrm{rank}(M) = 3 = n \Rightarrow$ Unique solution

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$\text{rank}(M) = \text{rank}(M_a) = 2 < n \Rightarrow$ Infinite number of solutions

Today's outline

Summary

- Linear equations can be written as matrices
- Using the inverse, the solution can be determined
  - Inverse via cofactors
  - Inverse and solution in Matlab
  - Inverse and solution in Excel
- Inversion scales with N^3
- A solution depends on the rank of a matrix