

Numerical Methods for Chemical Engineers

Study guide for 6E5X0

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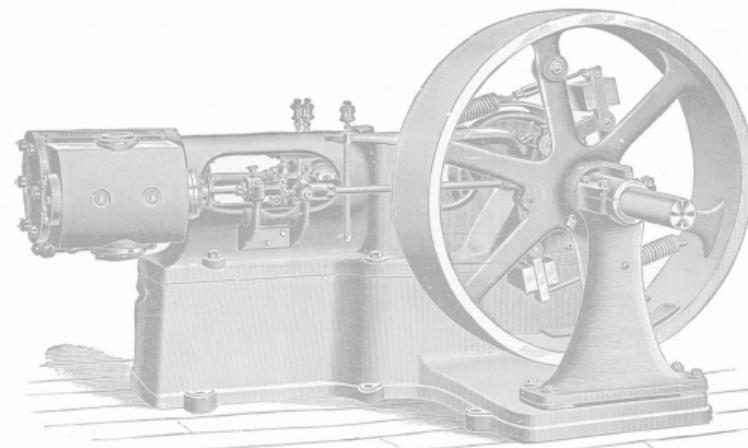
Chemical Process Intensification group
Eindhoven University of Technology

Numerical Methods (6E5X0), 2020-2021

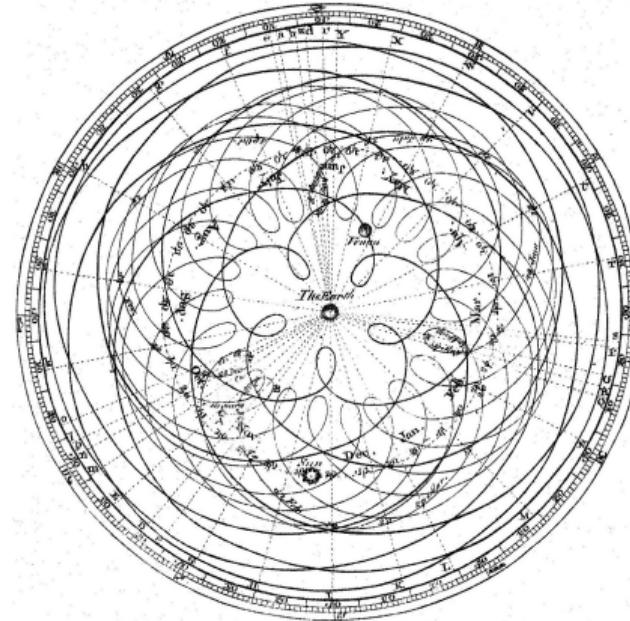
Numerical Methods

“Simulation and mathematical modeling will power the twenty-first century the way steam powered the nineteenth.”

— W.H. Press*



Ptolemy and the almagest



~150 AD. Development of numerical approximations to describe the motions of the heavenly bodies with accuracy matching reality sufficiently.

Numerical Methods

- Numerical analysis is concerned with obtaining approximate solutions to problems while maintaining reasonable bounds of error...
- ...because it is often impossible to obtain exact answers ...
- Numerical analysis makes use of algorithms to approximate solutions

Used in many fields... Including Chemical Engineering

- Description of reactors and separators (dynamic and steady state)
- Computational fluid dynamics
- Thermodynamic equations of state
- Optimizing process performance
- Design and synthesis of processes
- Regression of data, e.g. isotherms, kinetics, ...

Course Schedule

Lecture	Topic	Teacher
1	Programming and algorithms (1)	IR
2	Programming and algorithms (2)	IR
3	Numerical errors	IR
4	Linear eqns: direct methods	IR
5	Linear eqns: iterative methods	IR
6	Interpolation + integration	IR
7	Non-linear equations (1)	MSA
8	Non-linear equations (2)	MSA
9	ODEs (1)	MSA
10	ODEs (2)	MSA
11	PDEs	MSA
12	Regression and Optimization	IR

Course Objectives

- Gain experience with programming basics and algorithm design
- Acquire knowledge of and experience with various techniques for the numerical solution of systems of linear and non-linear algebraic and differential equations, as well as data analysis and optimization.
- Being able to solve various numerical problems using Matlab or Excel.

Prerequisites

The following courses should give you enough background knowledge to follow this course comfortably:

- Calculus
- Linear Algebra
- Some basic MATLAB experience
 - We will shortly cover some aspects on MATLAB programming in the first lectures. Detailed documents and courses are provided on Canvas, for your own reference.

You will definitely need a laptop with Matlab and Excel installed!

Course Materials

- Lecture slides (+ lecture recordings 2020)
- MATLAB scripts
- Additional articles
- There are some useful books for those seeking more in-depth knowledge and alternative methods, not mandatory:
 - Numerical recipes, W.H. Press et al.
 - Numerical methods for chemical engineering, K.J. Beers
 - Numerical methods for chemical engineers, A. Constantinides
 - Essential matlab-for engineers, B.D. Hahn
 - Introduction to Numerical Methods and Matlab Programming for Engineers, T. Young and M.J. Mohlenkamp

Look on Canvas for the slides, exercises, scripts, assignments and additional documentation on MATLAB.

Assessment

4 assignments

- Each 20% of the final result
- Done in groups of 2 persons, see Canvas→People
- Short report (template provided, Overleaf and Canvas)
- Questions on the assignments should be asked in-class, not through private message/mail/etc.

Final exam

- Practical and theoretical questions, covering *all topics*
- Exam taken on your own computer (ANS with proctorio+Matlab), if possible on-campus
- You can use the slides and Matlab documentation
- Sample exam will be released before Christmas
- Grade of the final exam needs to be at least a 5.0

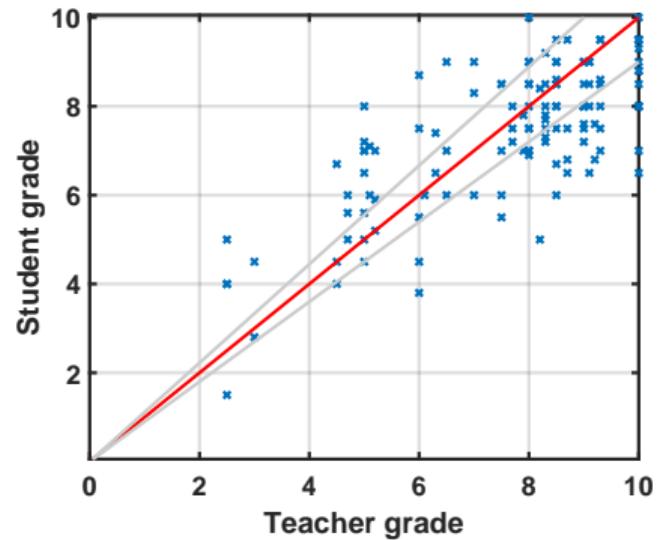
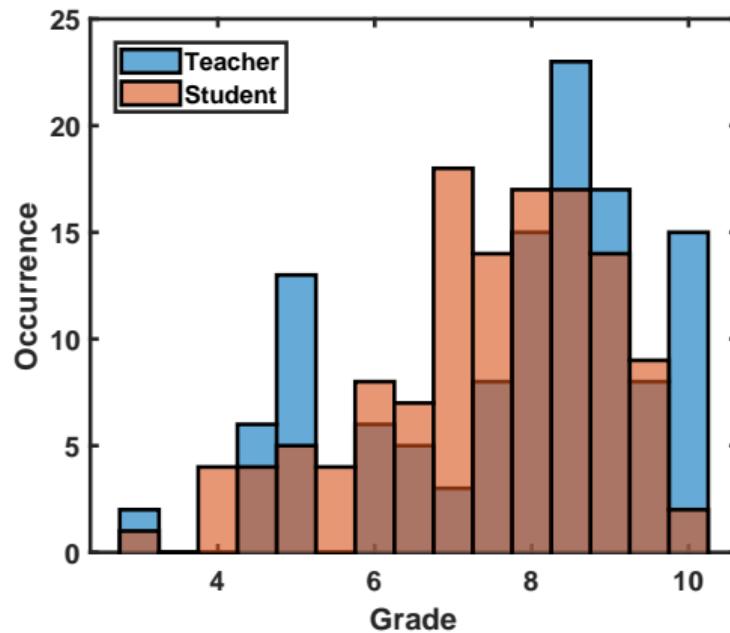
Assignment grading: peer assessment & feedback (1)

- Assignments are graded through supervised peer-assessment. After the deadline, each person will grade 2 other assignments. Rubrics are available to maintain a consistent assessment among different groups. Criteria are:
 - Functionality
 - Code style
 - Visualisation
 - Analysis
- The assessment should be done within 3 days using rubrics, additional feedback should be supplied to establish its validity. We will assess the quality of the feedback, and grade it by a multiplier (0.8-1.2).
- You can challenge one or more reviews by submitting a rebuttal;
- The final grade will be the averaged grade from the remaining peer-assessments (group), multiplied by the peer-review quality (individual), with a max. of 10.
- When statistics are poor (≤ 2 reviews), the assignment will be graded by the instructor, which discards all remaining peer-reviews.

Assignment grading: peer assessment & feedback (2)

- Along with the rubrics, you will give each other specific comments for improvement: What are you impressed with; why did you score a certain criteria low; how to improve the code or visuals, etc. Give at least 3 tips and 3 tops.
- Grades for an assignment are released only when proper assessment and feedback have been given.
- Rebuttals are turned in through an additional assignment. A rebuttal should convince us and provide evidence and in-depth argumentation why a particular review is flawed. We will evaluate the rebuttals and discard a peer-review if it is indeed disproportionate.
- We are getting help from student assistants to make the process go smoothly. I will show a possible solution after the deadline.
- Grading with rubrics: don't be afraid to use the full spectrum.

Peer assessment in the past



Peer assessment summary

Complete document can be read through Canvas, here's the summary:

- The assignment and report template are released along with the grading rubrics
- Canvas automatically performs a plagiarism check
- The lecturer will give a short overview of how the assignment could have been solved (point of reference)
- Students have 3 days for double-blind peer assessment
- Students have the opportunity to challenge their reviewers (rebuttal)
- Lecturers and TAs will check the review quality, as well as the reports that have very low or very high marks or large deviations among reviews coarsely. I will provide a full correction when no suitable peer-reviews have been done.
- If a student fails to produce a good, timely peer-review, their grade for the assignment will be marked NA.

Assignment handout and deadlines

Hand-in your assignments via Canvas

- Deadlines are given on Canvas as well
- Deliver the report in PDF format
- Send along the scripts + necessities in a .zip
- Use your student ID instead of your name for identification purposes
- Be aware that a .docx stores the original author name as metadata.

Course Philosophy

- This is a hard course! It will take a lot of hours, especially if you have no coding experience.
- We are here to help you. To have nice discussions, to show alternative ways, etc. Clarify language subtleties, or suggestions. Not to give away the answers.
- We encourage research and independent learning. It comes down to paying attention, repetition of the concepts, and practice practice practice!
- We try to make the lectures interactive, working on examples and creating scripts as we go. It is advised that you work along with us to get the most out of this course!
- It is ok to discuss the general approach to solving the problem, or to get a hint, or several hints, if you get stuck while solving a problem, but work out the details of the solution in your own group.
- It is **not ok** to take someone else's solution and simply copy their scripts, answers, etc.
- Take regular breaks! Better 6 times per hour a short break than 1 time per hour a long break. Use keyboard shortcuts, or a mouse if you must. Set up screen brightness to a pleasant value. Stretch.

Contact information

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Some Acknowledgements



Some Real Acknowledgements

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- To Roel Verstappen of Groningen University
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- To Edwin Zondervan of University of Twente
- To edX: MITx course "Introduction to Computer Science and Programming Using Python"

Matlab and Programming 1

Programming basics and algorithms

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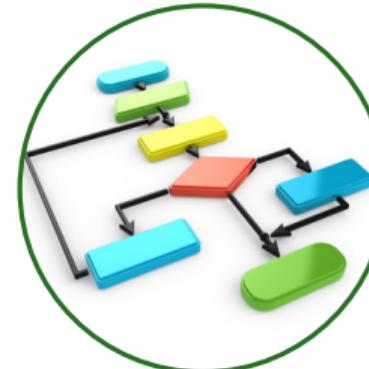
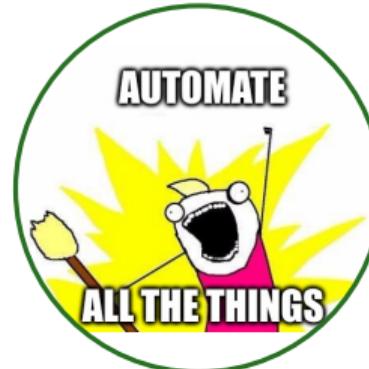
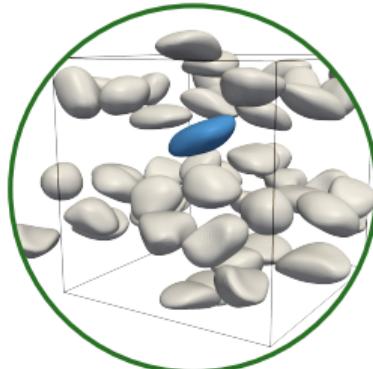
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- Introduction
- Data structures
- Plotting
- Creating algorithms
- Functions
- Conclusions
- Exercises

Why should you learn something about programming?

- Scientific techniques depend in an increasing fashion upon computer programs and simulation methods
- Knowledge of programming allows you to automate routine tasks
- Ability to understand algorithms by inspection of the code
- Learn to think by dissecting a problem into smaller bits



Introduction to programming

What is a program?

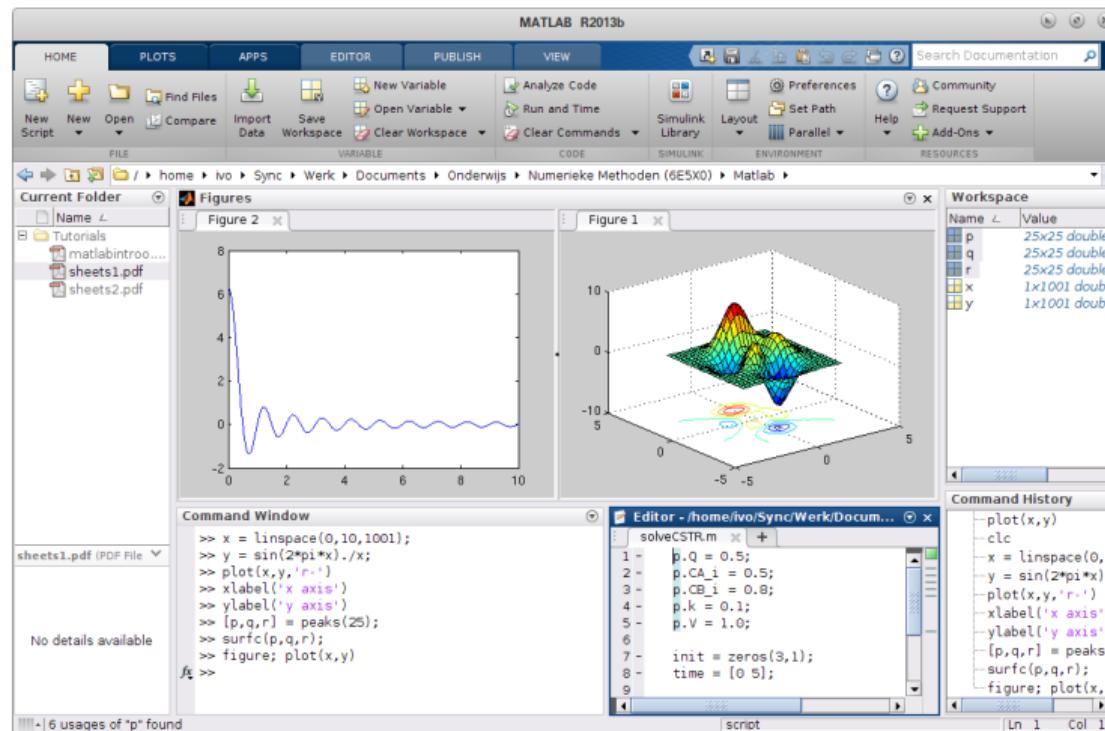
A program is a sequence of instructions that is written to perform a certain task on a computer.

- The computation might be something mathematical, a symbolic operation, image analysis, etc.

Program layout

- ① Input (Get the radius of a circle)
- ② Operations (Compute and store the area of the circle)
- ③ Output (Print the area to the screen)

Versatility of Matlab

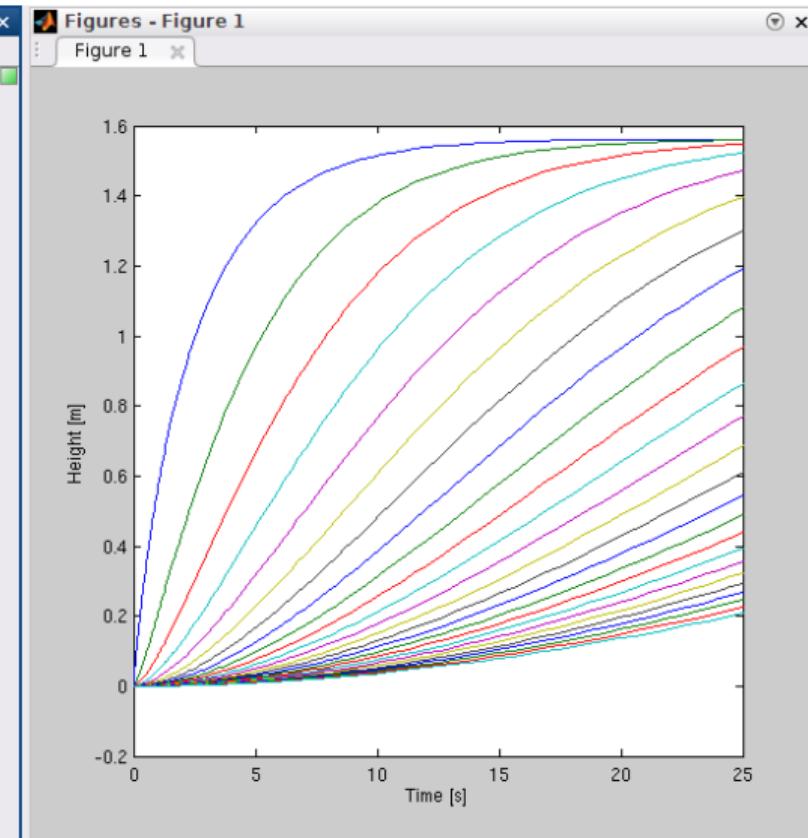


Versatility of Matlab: ODE solver

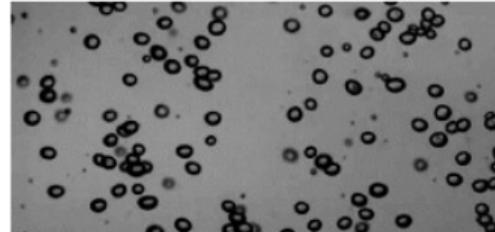
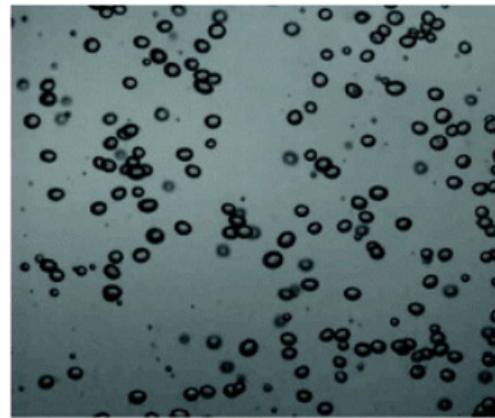
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solveTankSeries.m x tankSeries.m x +

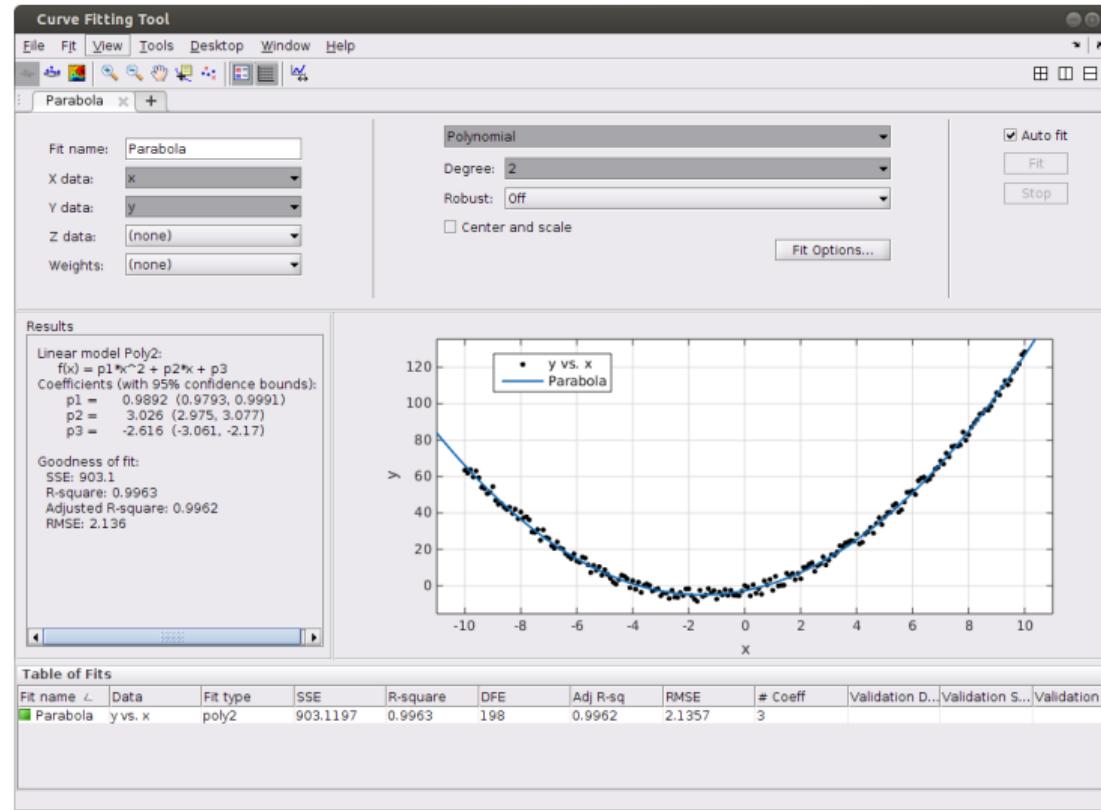
```
1 - p.F0 = 1.0;
2 - p.K = 0.8;
3 - p.N = 25;
4 - p.A = 1.0;
5
6 - init = zeros(p.N,1);
7 - time = [0 25];
8
9 - [t,h] = ode45(@tankSeries,time,init,[],p);
10
11 - plot(t,h);
12 - xlabel('Time [s]');
13 - ylabel('Height [m]');
```



Versatility of Matlab: Image analysis



Versatility of Matlab: Curve fitting



Getting started

Start Matlab, and enter the following commands on the command line. Evaluate the output.

```
>> 2 + 3          % Some simple calculations
>> 2*3
>> 2*3^2         % Powers are done with ^
>> a = 2          % Storing values into the workspace
>> b = 3
>> c = (2*3)^2   % Parentheses set priority
>> 8/a-b
>> sin(a)         % Mathematical functions can be used
>> sin(0.5*pi)   % pi is an internal Matlab variable
>> 1/0            % Infinity is a thing ...
>> sqrt(-1)       % ... as are imaginary numbers
```

Printing and formatting results

You can control the display format of the output of a command using the `format` statement. This only involves the way the numbers are printed on the screen, behind the scenes the accuracy is the same (some 15 digits).

```
>> format short % This is the default setting
>> a = 19/4
>> b = a^(-6)
>> format long
>> c = sqrt(21)
>> d = exp(-c)
>> format long e
>> d = exp(-c)
>> format short compact
>> d
```

A semi-colon at the end of a line suppresses output entirely:

```
>> f = pi/4;
```

A few helpful things

- Using the \uparrow and \downarrow keys, you can cycle through recent commands
- Typing part of a command and pressing Tab completes the command and lists the possibilities
- If a computation takes too long, you can press $\text{Ctrl} + \text{C}$ to stop the program and return to the command line. Note that you may end up with incomplete results in the workspace.
- Sequences of commands (programs, scripts) are contained as m-files, plain text files with the `.m` extension.
- Such m-files must be in the *current working directory* or in the Matlab *path*, the locations where Matlab searches for a command. If you try to run a script that is not in the path, Matlab will suggest to add that directory to the *path*, or to switch to that directory. Type `path` to view the current path.
- Anything following a `%` symbol is regarded as a comment
- In a script, pressing $\text{Ctrl} + \text{R}$ comments the current line (or selection), $\text{Ctrl} + \text{T}$ does the opposite.

Matlab help, documentation, resources

- You can look for a command with a particular purpose using the `lookfor` command:

```
>> lookfor inverse
```

- Refer to the Matlab documentation using `doc` (pops up a new window) or `help` function
 - Try for instance: `help inv` or `help help`.
- We supplied a number of basic Canvas modules: Matlab Crash Course, including small exercises.
- Matlab Academy: <https://matlabacademy.mathworks.com>
- Introduction to Numerical Methods and Matlab Programming for Engineers. T. Young and M.J. Mohlenkamp (2015). GNU-licensed document, online
- Search the web, Reddit, YouTube, etc.

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Terminology

Variable Piece of data stored in the computer memory, to be referenced and/or manipulated

Function Piece of code that performs a certain operation/sequence of operations on given input

Operators Mathematical operators (e.g. + - * or /), relational (e.g. < >or ==, and logical operators (&&, ||)

Script Piece of code that performs a certain sequence of operations without specified input/output

Expression A command that combines variables, functions, operators and/or values to produce a result.

Variables in Matlab

- Matlab stores variables in the *workspace*
- You should recognize the difference between the *identifier* of a variable (its name, e.g. `x`, `setpoint_p`), and the data that it actually stores (e.g. 0.5)
- Matlab also defines a number of variables by default, e.g. `eps`, `pi` or `i`.
- You can assign a variable by the = sign:

```
>> x = 4*3
x =
    12
```

- If you don't assign a variable, it will be stored in `ans`
- Clearing the workspace is done with `clear`.

Datatypes and variables

Matlab uses different types of variables:

Datatype	Example
string	'Wednesday'
integer	15
float	0.15
vector	[0.0; 0.1; 0.2]
matrix	[0.0 0.1 0.2; 0.3 0.4 0.5]
struct	sct.name = 'MyDataName' sct.number = 13
logical	0 (false) 1 (true)

About variables

- Matlab variables can change their type as the program proceeds (this is not common for other programming languages!):

```
>> s = 'This is a string'  
s =  
This is a string  
>> s = 10  
s =  
10
```

- Vectors and matrices are essentially *arrays* of another data type. A vector of `struct` is therefore possible.
- Variables are *local* to a function (more on this later).

Vectors in Matlab (1)

A row vector:

```
>> v = [0 1 2 3]
```

A column vector by separating elements with semi-colons:

```
>> u = [9; 10; 11; 12; 13; 14; 15]
```

Access (i.e. read) an entry in a vector:

```
>> u(2)
```

Manipulate the value of that entry:

```
>> u(2)=47
```

Get a slice of a vector:

```
>> u([2 3 4]) % With colon operator: u(2:4)
```

Transposing vectors:

```
>> w = v'
```

Vectors in Matlab (2)

Manual definition may be cumbersome. A colon (:) generates a list:

```
>> a = 1:10      % Default stride is 1
>> x = -1:.1:1    % start:stride:stop specifies list
```

Or, when you prefer to set the *number of elements* instead of the step size:

```
>> y = linspace(0,10,11)
>> p = logspace(2,6,5)
```

Manipulating multiple components:

```
>> y([1 4:7]) = 1
```

Or (by supplying a vector instead of a scalar):

```
>> y([1 4:7]) = 16:20 % equivalent to y([1 4 5 6 7]) = [16 17 18 19 20]
```

Practice

Given a vector

$$x = [2 \ 4 \ 6 \ 8 \ 10 \ 12 \ 14 \ 16 \ 18 \ 20 \ 30 \ 40 \ 50 \ 60 \ 70 \ 80]$$

- Find a way to define the vector without typing all individual elements
- Investigate the meaning of the following commands:

```
>> x(3)
>> x(1:5)
>> x(1:end-1)
>> y = x(5:end)
>> y(4)
>> y(4) = []
>> sum(x)
>> mean(x)
>> std(x)
>> max(x)
>> fliplr(x)
>> x(end:-1:1)
>> diff(x)
```

Practice

Given a vector

$$x = [2 \ 4 \ 6 \ 8 \ 10 \ 12 \ 14 \ 16 \ 18 \ 20 \ 30 \ 40 \ 50 \ 60 \ 70 \ 80]$$

- Find a way to define the vector without typing all individual elements
- Investigate the meaning of the following commands:

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>> y(4)  
>> y(4) = []  
>> sum(x)  
>> mean(x)  
>> std(x)  
>> max(x)  
>> fliplr(x)  
>> diff(x)
```

Operations on vectors (1)

```
>> e = 1:5
>> f = 2*e
>> g = 4*f + 20
>> h = e^2
```

... wait ... what's that?

```
Error using ^
Inputs must be a scalar and a square matrix.
To compute elementwise POWER, use POWER (.^) instead.
```

Matlab uses matrix operations by default, we should use a dot operator to make operations element-wise for *, / and ^.

```
>> e.^2
```

Operations on vectors (2)

To demonstrate the matrix product:

```
>> p = [1; 1; 1]
>> q = [1 2 3]
>> p*q    % which is not equal to q*p
```

All kinds of mathematical functions on vectors typically operate on elements:

```
>> x = linspace(0,2*pi,100);
>> s = sin(x)
>> e = exp(x)
```

Matrices in Matlab

Matrix A is defined as:

$$A = \begin{bmatrix} 8 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & 9 & 2 \end{bmatrix}$$

In Matlab:

```
>> A = [ 8 1 6; 3 5 7; 4 9 2]
```

Elements can be accessed/manipulated by the following syntax:

```
>> A(3,1) % Third row, first column, also A(3)
>> A(3,:) = [2 4 8] % Set entire third row
>> A(:,3) % Print third column
>> A(A>5) = 2 % Set elements by condition
```

There are a few functions that help creating matrices:

```
>> A = zeros(4) % A 4x4 matrix with zeros
>> A = ones(4,1) % A 4-element vector with ones
>> A = eye(3) % Identity matrix of 3x3
>> A = rand(3,4) % A 3x4 matrix with random numbers
```

Practice

- Find a *short* Matlab expression to create the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 9 & 7 & 5 & 3 & 1 & -1 & -3 \\ 4 & 8 & 16 & 32 & 64 & 128 & 256 \end{bmatrix}$$

- Investigate the command `max(A)`. What does it give?

Building blocks: Mathematics and number manipulation

Programming languages usually support the use of various mathematical functions (sometimes via a specialized library). Some examples of the most elementary functions in Matlab:

Command	Explanation
<code>cos(x)</code> , <code>sin(x)</code> , <code>tan(x)</code>	Cosine, sine or tangens of x
<code>mean(x)</code> , <code>std(x)</code>	Mean, st. deviation of vector x
<code>exp(x)</code>	Value of the exponential function e^x
<code>log10(x)</code> , <code>log(x)</code>	Base-10/Natural logarithm of x
<code>floor(x)</code>	Largest integer smaller than x
<code>ceil(x)</code>	Smallest integer that exceeds x
<code>abs(x)</code>	Absolute value of x
<code>size(x)</code>	Size of a vector x
<code>length(x)</code>	Number of elements in a vector x
<code>rem(x,y)</code>	Remainder of division of x by y

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Simple plotting

Let's make a plot of the following table

T (°C)	5	20	30	50	55
μ (Pa·s)	0.08	0.015	0.009	0.006	0.0055

```
>> T = [ 5 20 30 50 55 ]  
>> mu = [ 0.08 0.015 0.009 0.006 0.0055]
```

```
>> plot(T,mu)
```

```
>> plot(T,mu,'*')
```

```
>> plot(T,mu,'r--')
```

```
>> plot(T,mu,'ko-', 'LineWidth',2)
```

```
>> xlabel('Temperature ^\circC')  
>> ylabel('Viscosity [Pa s]')  
>> title('Experiment 1')
```

Practice

Create plots of the following functions in a single figure for $x \in \{0, 2\pi\}$:

$$y_1 = \cos x$$

$$y_2 = \arctan x$$

$$y_3 = \frac{\sin x}{x}$$

Strategies to draw multiple graphs in 1 figure:

```
>> plot(x,y1,x,y2,x,y3)
```

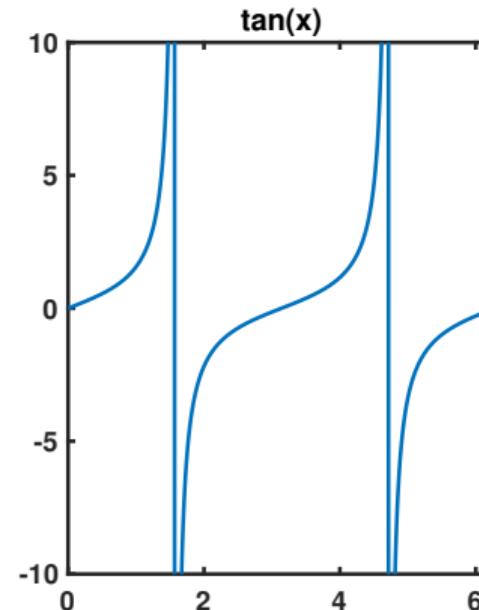
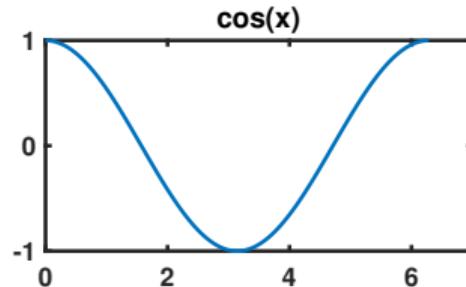
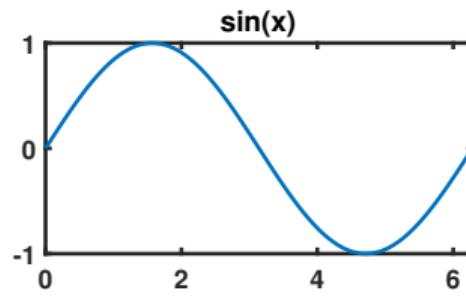
```
>> plot(x,y1)
>> hold on; % Maintain drawn plots in current figure
>> plot(x,y2)
>> plot(x,y3) % The 'hold-property' was already set
```

Multiple graph plotting

- A new figure window can be created using the `figure` command.
- The command `subplot(m,n,p)` divides the figure window into blocks using `m` rows and `n` columns. The number or vector `p` indicates which block (from left to right, top to bottom) or blocks are used for the axis.
- Graph operators (e.g. axis labels, title, hold on, etc) act on the active subplot

```
x = linspace(0,2*pi,1000)
y1 = sin(x);
y2 = cos(x);
y3 = tan(x);
subplot(2,2,1); plot(x,y); title('sin(x)');
subplot(2,2,1); plot(x,y1); title('sin(x)');
subplot(2,2,3); plot(x,y2); title('cos(x)')
subplot(2,2,[2 4]); plot(x,y3); title('tan(x)'); axis([0 2*pi -10 10])
```

Multiple graph plotting



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Building blocks: conditional statements

if-statement: Check whether a (set of) condition(s) is met, based on relational operations.

```
num = floor (10 * rand + 1);
guess = input ('Your guess please : ');
if ( guess ~= num )
    disp (['Wrong, it was ',num2str(num),'. Kbye.']);
else
    disp ('Correct !');
end
```

Other relational operators

<code>==</code>	is equal to
<code>~=</code>	is not equal to
<code><=</code>	is less than or equal to
<code>>=</code>	is greater than or equal to
<code><</code>	is less than
<code>></code>	is greater than

Combining conditional statements

<code>&</code>	and
<code> </code>	or
<code>~</code>	negation (switches true/false)
<code>xor</code>	exclusive or

Building blocks: conditional statements

You can use nested `if`-statements and combinations of expressions:

```
a = 3;
b = 9;

if (a < b) & (b <= 10)
    disp('a is smaller than 10 (by deduction)')
elseif ((a >= b) & (b > 10))
    disp('a is larger than 10 (by deduction)')
else
    disp('We cannot deduce if a is smaller than 10 or not, unless we perform a direct
          comparison')
    if (a < 10)
        disp('a is smaller than 10')
    elseif (a > 10)
        disp('a is larger than 10')
    else
        disp('a is equal to 10')
    end
end
```

Building blocks: logical indexing

Relational operators return a type "logical", i.e. true (1) or false (0). You can use this to select subsets of existing vectors.

Simple demonstration:

```
>> x = linspace(-3,3,7) % Create a base
    vector
x =
    -3      -2      -1      0      1      2      3
>> x>0 % Illustrate logical result
ans =
  1x7 logical array
  0     0     0     0     1     1     1
>> x2 = x(x>0) % Create a subset
x2 =
    1      2      3
```

You can use the same logical index for different vectors (in script-format):

```
% Create a base vector
x = linspace(-4*pi,4*pi,1000);
y = (sin(x)+0.2*x)+1;
% Select only values where y>0 and plot
xs = x(y>0); ys = y(y>0);
plot(x,y,xs,ys)
```

Practice relational operations and logical indexing

① Let $x = [1 \ 5 \ 2 \ 8 \ 9 \ 0 \ 1]$ and $y = [5 \ 2 \ 2 \ 6 \ 0 \ 0 \ 2]$. Execute and explain the results of the following commands:

- $x > y$
- $x \leq y$
- $x \& (\sim y)$
- $y < x$
- $y \geq x$
- $(x > y) \mid (y < x)$
- $x == y$
- $x \mid y$
- $(x > y) \& (y < x)$

② Clear the workspace. Let $x = 1:10$ and $y = [3 \ 5 \ 6 \ 1 \ 8 \ 2 \ 9 \ 4 \ 0 \ 7]$. The exercises here show the techniques of logical-indexing. Execute and interpret the results of the following commands:

- $(x > 3) \& (x < 8)$
- $x(x > 5)$
- $y(x \leq 4)$
- $x((x < 2) \mid (x \geq 8))$
- $y((x < 2) \mid (x \geq 8))$
- $x(y < 0)$

Building blocks: case selection

switch-statement: Selects and runs a block of code.

```
[dnum,dnam] = weekday(now);
switch dnum
    case {1,7}
        disp('Yay! It is weekend!');
    case 6
        disp('Hooray! It is Friday!');
    case {2,3,4,5}
        disp(['Today is ', dnam]);
    otherwise
        disp('Today is not a good day...');

end
```

Building blocks: loops

for-loop: Performs a block of code a certain number of times.

```
>> p(1) = 1;
>> p(2) = 1;
>> for i = 2:10
p(i+1) = p(i)+p(i-1);
end
>> p
p =
    1     1     2     3     5     8    13    21    34    55    89
```

Building blocks: indeterminate repetition

while-loop: Performs and repeats a block of code until a certain condition.

```
num = floor (10* rand +1) ;
guess = input ('Your guess please : ');

while ( guess ~= num )
    guess = input ('That is wrong. Try again ... ');
end

if (isempty(guess))
    disp('No number supplied - exit');
else
    disp ('Correct!');
end
```

Example algorithm

Compute the factorial of N : $N! = N \cdot (N-1) \cdot (N-2) \cdots 2 \cdot 1$

How to deal with this?

Naive approach

```
Z = 1;  
Z = Z*2;  
Z = Z*3;  
Z = Z*4;  
... etc ...
```

For-loop

```
Z = 1;  
for i = 1:N  
    Z = Z*i;  
end
```

While-loop

```
Z = 1;  
i = 1;  
while (i<=N)  
    Z = Z*i;  
    i = i+1;  
end
```

Note: `N` must be set beforehand!

Note: Pay attention to the relational operators!

Input and output

Many programs require some input (data) to function correctly. A combination of the following is common:

- Input may be given in a parameters file (“hard-coded”)
- Input may be entered via the keyboard

```
>> a = input('Please enter the number ');
```

- Input may be read from a file, e.g.

```
>> data = getfield(importdata('myData.txt', ' ', 4), 'data');  
>> numdata = xlsread('myExcelDataFile.xls');
```

Importdata also has a GUI equivalent (right-click on file, select *Import Data*)

- There are many more advanced functions, e.g. `fread`, `fgets`, ...

Input and output

Output of results to screen, storing arrays to a file or exporting a graphic are the most common ways of getting data out of Matlab:

- Results of each expression are automatically shown on screen as long as the line is not ended with a semi-colon;
- Output may be stored via the GUI:
 - Use the 'Export Setup' function
 - Save figure (use .fig, .eps or .png, **not** .jpg or .pcx)
 - Save variables (right click, save as)
- Save variables automatically (scripted):

```
>> savefile = 'test.mat';
>> p = rand(1,10);
>> q = ones(10);
>> save(savefile, 'p', 'q')
```

- More advanced functions can be found in e.g. `fwrite`, `fprintf`, ...

Today's outline

- Introduction
- Data structures
- Plotting
- Creating algorithms
- Functions
- Conclusions
- Exercises

Functions - general

A function in a programming language is a program fragment that performs a certain task. Creating functions keeps your code clean, re-usable and structured.

- You can use functions supplied by the programming language, and define functions yourself
- Functions take one or more input parameters (*arguments*), and *return* an output (*result*).
- In Matlab, functions are defined as follows (2 output and 3 input arguments):

```
function [out1, out2] = myFunction(in1, in2, in3)
```

- In the function body, you can perform operations on the input parameters (`in1`, `in2` and `in3`).
- Inside a function, other values are not available (local workspace). Analogously, any variables created are not available after the function returns, except for those explicitly defined as output arguments.
- The name of the function and the name of the file should be identical and (just like variables) be unique.

Functions - Exercise

Example: write a function that takes 3 variables, and returns the average:

Approach 1

```
function res = avg1(a,b,c)
    mySum = a + b + c;
    res = mySum / 3;
end
```

Approach 2

```
function res = avg2(a,b,c)
    data = [a; b; c];
    res = mean(data);
end
```

Functions - definition overview

the first word must be "function"

function name

output argument

input argument

function avr = average (x)

comments {

%AVERAGE computes the average value of a vector
%SYNTAX: avr=average(x)

%Note| to self: this is a boring function

function body {

n = length(x);
avr = sum(x)/n;
return;

comments following a blank line are NOT shown if you type:
`>> help average`

Functions - stack

Calling functions from each other creates a 'stack', like a stack of cards. The stack works with first in, last out (FILO) principle.

```
function [] = a()
disp('Start of a()');
b();
disp('End of a()');
end

function [] = b()
disp('Start of b()');
c();
disp('End of b()');
end

function [] = c()
disp('Start of c()');
disp('End of c()');
end
```

```
>> stackcheck
Start of a()
Start of b()
Start of c()
End of c()
End of b()
End of a()
```

Exercise: create a function

Compute $N! = N \cdot (N-1) \cdot (N-2) \cdots 2 \cdot 1$

Create a function of our while-loop approach with N the argument:

Original script

```
Z = 1;  
i = 1;  
while (i<=N)  
    Z = Z*i;  
    i = i+1;  
end
```

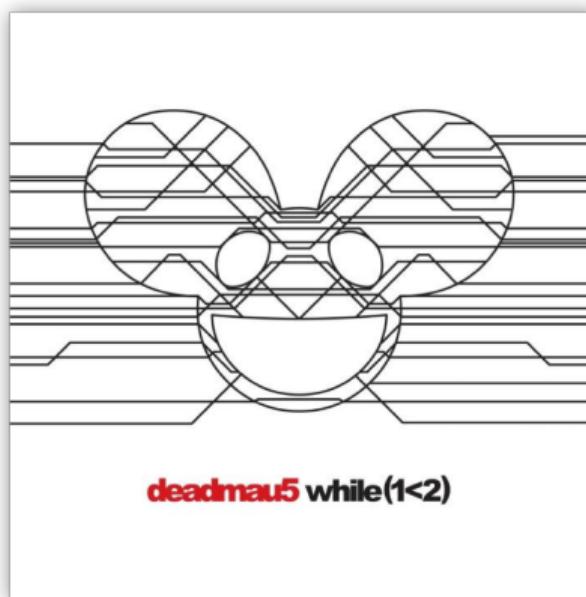
Function

```
function Z = fact_while(N)  
  
Z = 1;  
i = 1;  
while (i<=N)  
    Z = Z*i;  
    i = i+1;  
end  
  
end
```

Functions - checking input

The function we created computes the factorial correctly!

- When the supplied argument is positive and
- When the supplied argument is a natural number...



- In this case, we should check the user input to prevent an infinite loop:

```
if (fix(N) ~= N) | (N<0)
    disp 'Provide a positive integer number!'
    return;
end
```

- If no check can be done before a while-loop, you may want to stop after x loops

Functions - checking input

The whole factorial function, including comments:

```
function Z = fact_while(N)
%% This function computes a factorial of input value N
% Usage : fact_while(N)
% N      : value of which the factorial is computed
% returns: factorial of N

% Catch non-integer case
if (fix(N) ~= N) | (N<0)
    disp 'Provide a positive integer number!'
    return;
end

Z = 1;
i = 1;
while (i<=N)
    Z = Z*i;
    i = i+1;
end

end
```

Recursion

- In order to understand recursion, one must first understand recursion
- A recursive function includes a call to itself (a function within a function)
 - This could lead to infinite calls;
 - A base case is required so that recursion is stopped;
 - Base case does not call itself, simply returns.



Recursion: example

```
function out = mystery(a,b)
if (b == 1)
    % Base case
    out = a;
else
    % Recursive function call
    out = a + mystery(a,b-1);
end
```

- What does this function do?
- Can you spot the error?
- How deep can you go? Which values of b don't work anymore?

Recursion: exercise

Create a function computing the factorial of N , based on recursion.

```
function res = fact_recursive(x)

% Catch non-integer case
if (fix(x) ~= x) | (x<0)
    disp 'You should provide a positive integer number only'
    return;
end

if (x > 1)
    res = x*fact_recursive(x-1);
else
    res = 1;
end

end
```

Today's outline

- Introduction
- Data structures
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In conclusion...

- Matlab: A versatile development environment, with excellent vector and matrix computations
- Programming basics: variables, operators and functions, locality of variables, recursive operations
- For now: exercises on slide deck and Matlab modules
- Preparation for next lecture: familiarize with the concepts, use Canvas course or Matlab Academy.

Practice vectors and matrices

① Create a vector x with the elements:

- $[2, 4, 6, 8, \dots, 16]$
- $[0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{99}{100}]$

② Create a vector x with the elements: $x_n = \frac{(-1)^n}{2n-1}$ for $n = 1, 2, 3, \dots, 200$. Find the sum of the first 50 elements x_1, \dots, x_{50} .

③ Let $x = 20:10:200$. Create a vector y of the same length as x such that:

- $y_i = x_i - 3$
- $y_i = x_i$ for every even index i and $y_i = x_i + 11$ for every odd index i .

④ Let $T = [3 \ 4; \ 1 \ 8; \ -4 \ 3]$ and $A = [\text{diag}(-1:2:3), T; \ -4 \ 4 \ 1 \ 2 \ 1]$. Perform the following operations on A :

- Retrieve a vector consisting of the 2nd and 4th elements of the 3rd row.
- Find the minimum of the 3rd column.
- Find the maximum of the 2nd row.
- Compute the sum of the 2nd column
- Compute the mean of the row 1 and the mean of row 4

Practice plotting

- ① Plot the functions $f(x) = x$, $g(x) = x^3$, $h(x) = e^x$ and $z(x) = e^{x^2}$ over the interval $[0, 4]$ on the normal scale and on the log-log scale. Use an appropriate sampling to get smooth curves. Describe your plots by using the functions: `xlabel`, `ylabel`, `title` and `legend`.
- ② Make a plot of the functions: $f(x) = \sin(1/x)$ and $g(x) = \cos(1/x)$ over the interval $[0.01, 0.1]$. How do you create `x` so that the plots look sufficiently smooth?

Practice control flow and loops (1)

① Write a function that uses two logical input arguments with the following behaviour:

$$f(\text{true}, \text{true}) \mapsto \text{false}$$

$$f(\text{false}, \text{true}) \mapsto \text{true}$$

$$f(\text{true}, \text{false}) \mapsto \text{true}$$

$$f(\text{false}, \text{false}) \mapsto \text{false}$$

② Write a function that computes the factorial of x :

$$f(x) = x! = 1 \times 2 \times 3 \times 4 \times \dots \times x$$

- Using a loop-construction
- Using recursion

Practice control flow and loops (2)

- ① Write a function that computes the exponential function using the Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

until the last term is smaller than 10^{-6} .

- ② Use a script to compute the result of the following series:

$$f_n = \sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2}$$

This should give you an indication of the fraction this series converges to.

- Now plot in two vertically aligned subplots i) The result as a function of n , and ii) the difference with the earlier mentioned fraction as a function of n . For the latter, consider carefully the axis scale!

Practice logical indexing

- ① Let $x = \text{linspace}(-4, 4, 1000)$, $y_1 = 3x^2 - 4x - 6$ and $y_2 = 1.5x - 1$. Use logical indexing to determine function $y_3 = \max(\max(y_1, y_2), 0)$. Plot the function.
- ② Consider these data concerning the age (in years), length (in cm) and weight (in kg) of twelve adult men: $A = [41 \ 25 \ 33 \ 29 \ 64 \ 34 \ 47 \ 38 \ 49 \ 32 \ 26 \ 26]$; $H = [165 \ 186 \ 177 \ 190 \ 156 \ 174 \ 164 \ 205 \ 184 \ 190 \ 165 \ 171]$; $W = [75 \ 90 \ 97 \ 60 \ 74 \ 65 \ 101 \ 85 \ 91 \ 75 \ 87 \ 70]$;
 - Calculate the average of all vectors (age, weight and length).
 - Combine the command `length` with logical indexing to determine how many men in the group are taller than 182 cm.
 - What is the average age of men with a body-mass index ($B \equiv \frac{W}{L^2}$ with W in kg and L in m) larger than 25? And for men with a $B < 25$?
 - How many men are older than the average and at the same time have a BMI below 25?

Practice algorithm: Fourier series for heat equation

The unsteady 1D heat equation in 1D in a slab of material is given as:

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

We can express the temperature profile $T(x, t)$ in the slab using a Fourier sine series. For an initial profile $T(x, 0) = 20$ and fixed boundary values $T(0, t) = T(L, t) = 0$, the solution is given as:

$$T(x, t) = \sum_{n=1}^{n=\infty} \frac{40(1 - (-1)^n)}{n\pi} \sin\left(\frac{n\pi x}{L}\right) \exp\left(-kt \frac{n\pi^2}{L}\right)$$

- Create a script to solve this equation using loops and/or conditional statements

Matlab and Programming 2

Programming workflow and advanced features

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Numerical Methods (6E5X0), 2020-2021

Today's outline

● Coding style

● Debugging and profiling

● Visualisation

● Functions: revisited

● Concluding remarks

Make a habit of the following adage

MAKE IT WORK

MAKE IT RIGHT

MAKE IT FAST

Make it work

Use the building blocks of previous lecture to create an algorithm:

- ① *Problem analysis*
Contextual understanding of the nature of the problem to be solved
 - ② *Problem statement*
Develop a detailed statement of the mathematical problem to be solved with the program
 - ③ *Processing scheme*
Define the inputs and outputs of the program
 - ④ *Algorithm*
A step-by-step procedure of all actions to be taken by the program (*pseudo-code*)
 - ⑤ *Program the algorithm*
Convert the algorithm into a computer language, and debug until it runs
 - ⑥ *Evaluation*
Test all of the options and conduct a validation study

Now it's time to make it right.

Interpret the following code

```
s=checksc();
if(s==true)
a=cb();
b=cfrrsp();
if(a<5)
if(b>5)
a=gtbs();
end
if(a>b)
ubx();
end
end
else
brn();
gtbs();
end
```

WAT



Let's change that a bit... Indentation

Shown here with 2 spaces of indentation, Matlab uses 4 by default!

```
s=checksc();  
if(s==true)  
a=cb();  
b=cfrsp();  
if(a<5)  
if(b>5)  
a=gtbs();  
end  
if(a>b)  
ubx();  
end  
end  
else  
brn();  
gtbs();  
end
```

```
s = checksc();  
if (s == true)  
    a = cb();  
    b = cfrsp();  
    if (a < 5)  
        if (b > 5)  
            a = gtbs();  
        end  
        if (a > b)  
            ubx();  
        end  
    end  
else  
    brn();  
    gtbs();  
end
```

Readable variables and function names

```
s = checksc();
if (s == true)
    a = cb();
    b = cfrsp();
    if (a < 5)
        if (b > 5)
            a = gtbs();
        end
        if (a > b)
            ubx();
        end
    end
else
    brn();
    gtbs();
end
```

```
IAmFree = checkSchedule();
if (IAmFree == true)
    books = countBooks();
    shelfSize = countFreeSpaceShelf();
    if (books < 5)
        if (shelfSize > 5)
            books = goToBookStore();
        end
        if (books > shelfSize)
            useBox();
        end
    end
else
    burnBooks();
    goToBookStore();
end
```

Get rid of magic numbers in the code

```
IAmFree = checkSchedule();
if (IAmFree == true)
    books = countBooks();
    shelfSize = countFreeSpaceShelf();
    if (books < 5)
        if (shelfSize > 5)
            books = goToBookStore();
        end
        if (books > shelfSize)
            useBox();
        end
    end
else
    burnBooks();
    goToBookStore();
end
```

```
maxShelfSize = 5;
minBooksNeeded = 5;

IAmFree = checkSchedule();
if (IAmFree == true)
    books = countBooks();
    shelfSize = countFreeSpaceShelf();
    if (books < maxShelfSize)
        if (shelfSize > minBooksNeeded)
            books = goToBookStore();
        end
        if (books > shelfSize)
            useBox();
        end
    end
else
    burnBooks();
    goToBookStore();
end
```

That's more like it!

```
s=checksc();
if(s==true)
a=cb();
b=cfrsp();
if(a<5)
if(b>5)
a=gtbs();
end
if(a>b)
ubx();
end
end
else
brn();
gtbs();
end
```

```
maxShelfSize = 5;
minBooksNeeded = 5;

IAmFree = checkSchedule();
if (IAmFree == true)
    books = countBooks();
    shelfSize = countFreeSpaceShelf();
    if (books < maxShelfSize)
        if (shelfSize > minBooksNeeded)
            books = goToBookStore();
        end
        if (books > shelfSize)
            useBox();
        end
    end
else
    burnBooks();
    goToBookStore();
end
```

Writing readable code

Good code reads like a book.

- When it doesn't, make sure to use comments. In Matlab, everything following `% is a comment`
- Prevent "smart constructions" in the code
- Re-use working code (i.e. create functions for well-defined tasks).
- Documentation is also useful, but hard to maintain.
- Matlab comes with a function that generates reports from comments

How not to comment

- Useless:

```
% Start program
```

- Obvious:

```
if (a > 5)    % Check if a is greater than 5
    ...
end
```

- Too much about the life:

```
% Well... I do not know how to explain what is going on
% in the snippet below. I tried to code in the night
% with some booze and it worked then, but now I have a
% strong hangover and some parameters still need to be
% worked out...
```

- ...

```
% You may think that this function is obsolete, and doesn't seem to
% do anything. And you would be correct. But when we remove this
% function for some reason the whole program crashes and we can't
% figure out why, so here it will stay.
```

Adding comments to our program

Use comments to document design and purpose
(functionality), not mechanics (implementation).

```
IAmFree = checkSchedule();
if (IAmFree == true)
% Count books and amount of free space on a shelf.
% If minimum number of books I need is less than a
% shelf capacity, go shopping and buy additional
% literature. If the amount of books after the
% shopping is too big, use boxes to store them.
books = countBooks();
shelfSize = countFreeSpaceShelf();

...
else
burnBooks();
goToBookStore();
end
```

What else makes a good program?

- Portability (guaranteed in Matlab)
- Readability
- Efficiency
- Structural
- Flexibility
- Generality
- Documentation

Funny thing is: This list does not mention that the program should be actually working for its intended purposes!

Portability

It should work on your neighbors laptop. Well, that is guaranteed anyway. There is a little caveat though, what if your neighbor is stupid and has a polluted workspace?

```
% trust no one! nuke all evidence! no witnesses!
clear all    % destroy all variables
close all    % close all figures
```

Solution: clear all variables before your program starts

Readability

Don't use meaningless variable or function names. Rule of thumb: use verbs for functions and nouns for variables.

```
% stupid names
x = 5;
xx = myfunction(x);

% proper names
number_dams = 6;
beaver_workforce = allocate_beavers();
dams = build_dams(beaver_workforce, number_dams);
```

Efficiency

This one is difficult. Not much you can do without truly understanding how Matlab is utilizing your processor and memory. A couple of guidelines though:

- Avoid loops
- Especially avoid nested loops
- Use inherent matrix operations when possible
- Reduce IO (i.e. reading / writing to and from files)
- Don't run scripts from network disks
- Pre-allocate your matrices, that means, making it as large as the maximum required size for your particular problem
- Use tic/toc to test the execution times

```
x = linspace(0,10,1000001);
tic;
for cr = 1:length(x)
    y(cr) = sin(2*pi*x(cr));
end;
toc
```

```
x = linspace(0,10,1000001);
tic;

y = sin(2*pi*x);

toc
```

Structural

- Compartimentalize your code.
- Write functions whenever possible.
 - If you have > 15 lines of code, you can probably replace it by one or more functions.
 - In principle, it should not even matter how function works, as long as it gives the expected output.



- Put critical variables at the beginning of your program.

Structural

Write code as if it are paragraphs of a story.

```
% Step 0: Define variables
n_steps = 10000;      % number of steps
n_walks = 1000;       % number of random walk samples

% Step 1: Generate n_walks random_walks
de = zeros(n_walks, 2);
for i=1:n_walks
    angles = get_random_angles(n_steps);
    coord = transform_angles_to_coordinates(angles);
    de(i,1) = calculate_de(coord);  % store de
    de(i,2) = de(i,1)^2;           % store de^2
end

% Step 2: Plot the histogram
histogram(de(:,1), 'Normalization', 'pdf')
[D,P] = calculate_pdf(n_steps, 1000);
hold on;
plot(D,P);
```

Flexibility

If you want to add a feature or change something inside the program, it should not require rewriting the whole program. (jargon: non-linear propagation of change).

Solution: Encapsulate your code and “Don’t Repeat Yourself”

- Use functions for specific tasks (can you verbalize it? Then it is probably a function)
- Use variables, even for constants (it sounds like a oxymoron, but it's not! In fact, constant variables are a real thing)
- Use abstraction whenever possible

Generalization

If your code is working for one problem, it should also work for a similar problem, in another company, on another planet.

Pro-tip: Separate data from algorithms.

To be honest: I rarely see students making this mistake.

```
% stupid code
A = [1 2 3; 4 5 6; 7 8 9]           % hardcode data in the program

% smart code
load('some_random_dataset.dat') % or the arguably better fopen functions
```

Documentation

Properly document your code. Write your comments in a clear and concise fashion. Help your future self: Write clear and concise documentation.

```
function c = f(a,b)
switch nargin
    case 2
        c = a + b;
    case 1
        c = a + a;
    otherwise
        c = 0;
end
```



Documentation

Properly document your code. Write your comments in a clear and concise fashion. Help your future self: Write clear and concise documentation.

```
function c = f(a,b)
    % ADDME  Add two values together.
    %     C = ADDME(A) adds A to itself.
    %     C = ADDME(A,B) adds A and B together.
    %
    %     See also SUM, PLUS.
switch nargin % number of arguments
    case 2      % sum two different numbers
        c = a + b;
    case 1      % double single number
        c = a + a;
    otherwise   % shit in, shit out
        c = 0;
end
```



Make a habit of the following adage

MAKE IT WORK
MAKE IT RIGHT
MAKE IT FAST

Make a habit of the following adage

① *Make it work*

Create an algorithm that does the intended job. Make sure it works, and works repeatedly. Test and verify frequently. Add *todo* comments when you're not sure about a certain decision.

② *Make it right*

Refactor the code to improve the code design. Insert functions, comments, compartmentalize it. Get rid of magic numbers, use sensible variable names. Check input. Test and verify. Align with the team!

③ *Make it fast*

Measure and tune the performance of your code (profiling tool). In Matlab, vectorized calculations are much (!) faster than for-loops. Use sensible numerical techniques (e.g. higher-order integration).

Program by iterating over these aspects multiple times, starting at the fine-grained level, working your way up.

Today's outline

- Coding style
- Debugging and profiling
- Visualisation
- Functions: revisited
- Concluding remarks

Errors in computer programs

The following symptoms can be distinguished:

- Unable to execute the program
- Program crashes, warnings or error messages
- Never-ending loops
- Wrong (unexpected) result

Three error categories:

Syntax errors You did not obey the language rules. These errors prevent running or compilation of the program.

Runtime errors Something goes wrong during the execution of the program resulting in an error message (problem with input, division by zero, loading of non-existent files, memory problems, etc.)

Semantic errors The program does not do what you expect, but does what have told it to do.

Validation

- Testcases: run the program with parameters such that a known result is (should be) produced.
- Testcases: what happens when unforeseen input is encountered?
 - More or fewer arguments than anticipated? (Matlab uses `varargin` and `nargin` to create a varying number of input arguments, and to check the number of given input arguments)
 - Other data types than anticipated? How does the program handle this? Warnings, error messages (crash), NaN or worse: a program that silently continues?
- For physical modeling, we typically look for analytical solutions
 - Sometimes somewhat stylized cases
 - Possible solutions include Fourier-series
 - Experimental data

But: validation can only tell you *if* something is wrong, not *where* it went wrong.

The debugger (1)

- No-one can write a 1000-line code without making errors
 - If you can, please come work for us
- One of the most important skills you will acquire is debugging.
- Although it can be frustrating, debugging is one of the most intellectually rich, challenging, and interesting parts of programming.
- In some ways, debugging is like detective work. You are confronted with clues, and you have to infer the processes and events that led to the results you see.
- Actually, you are the detective, the murderer and the victim at the same time.

"When you have eliminated the impossible, whatever remains, however improbable, must be the truth."

— A. Conan Doyle, The Sign of Four

The debugger (2)

The debugger can help you to:

- Pause a program at a certain line: set a *breakpoint*
- Check the values of variables during the program
- Controlled execution of the program:
 - One line at a time
 - Run until a certain line
 - Run until a certain condition is met (conditional breakpoint)
 - Run until the current function exits
- Note: You may end up in the source code of Matlab functions!
- Check Canvas (Matlab Crash Course section) for a demonstration of the debugger.

Recursive Fibonacci

- Create a program that computes the n -th Fibonacci number using recursion:

$$F_n = F_{n-1} + F_{n-2} \text{ with } F_1 = 1 \text{ and } F_2 = 1$$

```

1 function out = fibonacci_recursive(N)
2 %FIBONACCI_RECURSIVE Prints out the Nth Fibonacci number to the screen
3 %SYNTAX: fibonacci_recursive(N)
4
5 if (N>2)
6     Nminus1 = fibonacci_recursive(N-1);
7     Nminus2 = fibonacci_recursive(N-2);
8     out = Nminus1 + Nminus2;
9 elseif (N==1) || (N==2)
10    out = 1;
11 else
12    error('Input argument was invalid')
13 end

```

- Place a breakpoint line 5 (click on dash or press **F12**), run `fibonacci_recursive(5)`
- Explore the function of step **F10**, step into **F11**, and how the local workspace changes
- Stop the debugger (red stop button on top, or **Shift** + **F5**)
- Right-click the breakpoint, select *Set/modify condition*, enter `N==2`, run again.

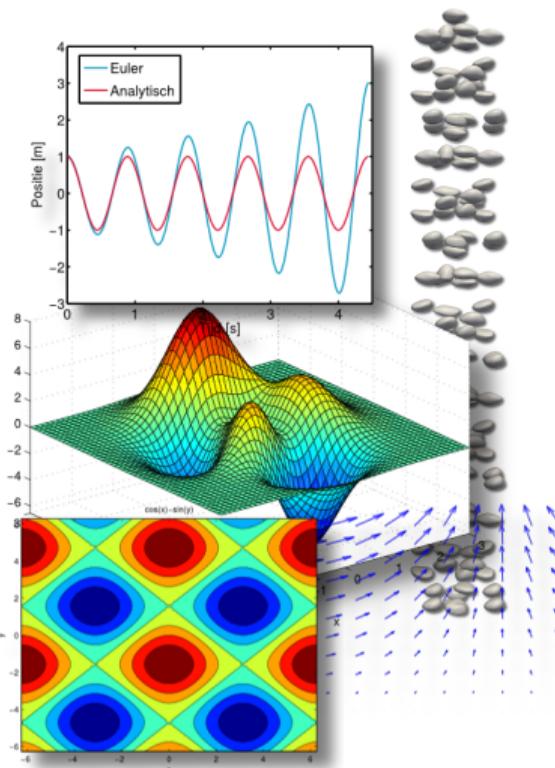
Today's outline

- Coding style
- Debugging and profiling
- Visualisation
- Functions: revisited
- Concluding remarks

Data visualisation

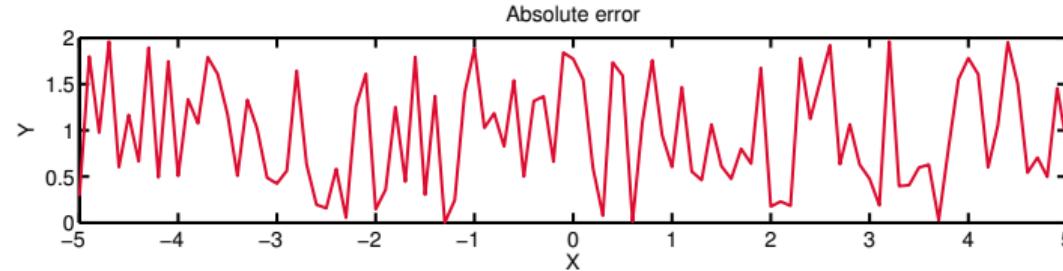
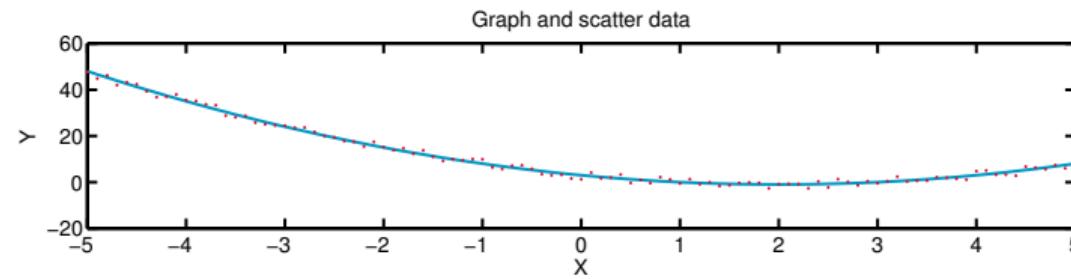
Modeling can lead to very large data sets, that require appropriate visualisation to convey your results.

- 1D, 2D, 3D visualisation
- Multiple variables at the same time (temperature, concentration, direction of flow)
- Use of colors, contour lines
- Use of stream lines or vector plots
- Animations



Plotting

```
x = -5:0.1:5;
y = x.^2-4*x+3;
y2 = y + (2-4*rand(size(y)));
subplot(2,1,1); plot(x,y,'-',x,y2,'r.');
xlabel('X'); ylabel('Y'); title('Graph and Scatter');
subplot(2,1,2); plot(x,abs(y-y2),'r-');
xlabel('X'); ylabel('Y'); title('Absolute error');
```

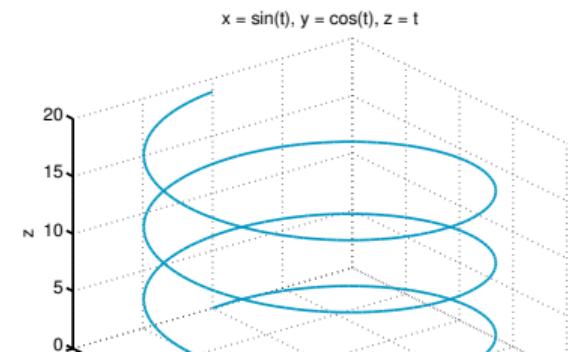
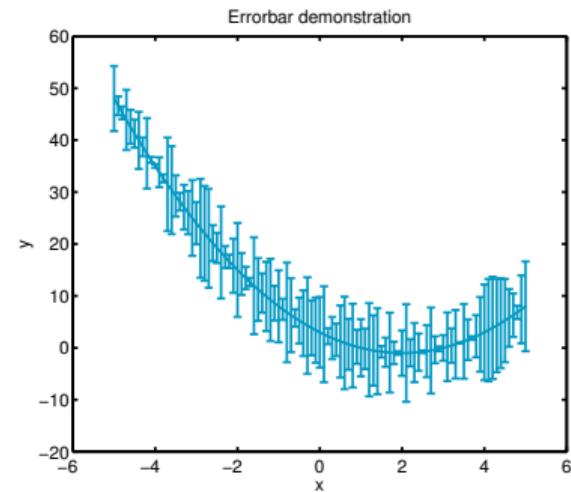


Animating plots

The `drawnow` command holds the execution of your program until the graph is updated. It allows to have a live view of a simulation result.

```
x = -5:0.1:5;
y = x.^2-4*x+3;
y2 = y + (2-4*rand(size(y)));
for i = 1:length(x)
    subplot(2,1,1);
    plot(x(1:i),y(1:i),'-',x(1:i),y2(1:i),'r.');
    xlabel('X'); ylabel('Y'); title('Graph and Scatter');
    axis([min(x) max(x) min(y) max(y)])
    
    subplot(2,1,2);
    plot(x(1:i),abs(y(1:i)-y2(1:i)), 'r-');
    xlabel('X'); ylabel('Y'); title('Absolute error');
    axis([min(x) max(x) 0 2])
    
    drawnow
end
```

Other plotting tools



- Errorbars: `errorbar(x,y,err)`

3D plots: `plot3(x,y,z)`

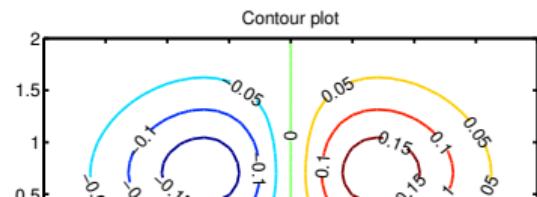
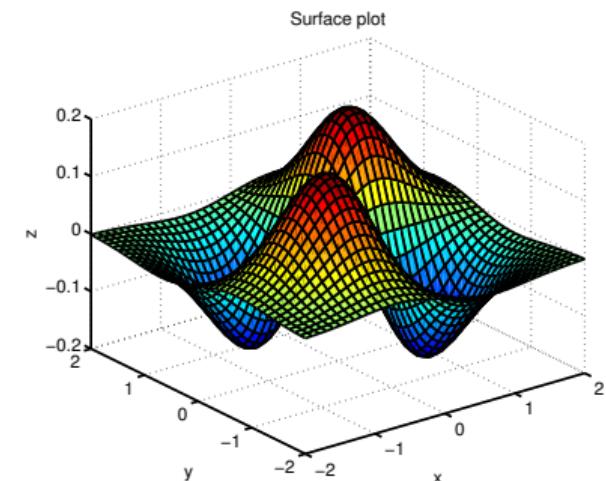
Histograms: `histogram(x,20)`

Multi-dimensional data

Matlab typically requires the definition of rectangular grid coordinates using [meshgrid](#):

```
[x y] = meshgrid(-2:0.1:2, -2:0.1:2);  
z = x .* y .* exp(-x.^2 - y.^2);
```

- Surface plot
- Contour plot
- Waterfall
- Ribbons



Vector data

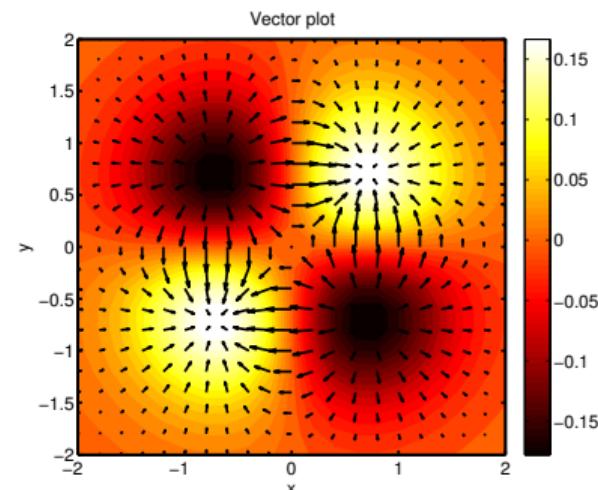
The gradient operator, as expected, is used to obtain the gradient of a scalar field. Colors can be used in the background to simultaneously plot field data:

```
[x y] = meshgrid(-2:0.2:2, -2:0.2:2);
z = x .* y .* exp(-x.^2 - y.^2)
[dx dy] = gradient(z,8,8)

% Background
contourf(x,y,z,30,'LineColor','none');
colormap(hot); colorbar;

axis tight; hold on;

% Vectors
quiver(x,y,dx,dy,'k');
```



Today's outline

- Coding style
- Debugging and profiling
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Functions: revisited

In MATLAB you can define your own functions to re-use certain functionalities. We now define the mathematical function $f = x^2 + e^x$:

```
function y = f(x)
y = x.^2 + exp(x);
```

Note:

- The first line of the file has to contain the `function` keyword
- The variables used are *local*. They will not be available in your Workspace
- The file needs to be saved with the same name as the function, i.e. “f.m”
- The semi-colon prevents that at each function evaluation output appears on the screen
- If `x` is an array, then `y` becomes an array of function values.

Anonymous functions

If you do not want to create a file, you can create an *anonymous function*:

```
>> f = @(x) (x.^2+exp(x))
```

- f: the name of the function
- @: the function handle
- x: the input argument
- x.^2+exp(x): the actual function

```
>> f(0:0.1:1)
```

Using function handles

A function handle points to a function. It behaves as a variable

```
>> myFunctionHandle = @exp  
>> myFunctionHandle(1)
```

Used a.o. for passing a function to another function, for instance for optimization functions.

$$f(x) = x^3 - x^2 - 3 \arctan x + 1$$

Matlab offers a function `fzero` that can find the roots of a function in a certain range:

```
>> f = @(x) x.^3 - x.^2 - 3*atan(x) + 1;  
>> fzero(f, [-2 2])  
>> ezplot(f)  
>> f(ans)
```

Practice function handles

Consider the function

$$f(x) = -x^2 - 3x + 3 + e^{x^2}$$

The built-in Matlab function `fminbnd` allows to find the minimum of a function in a certain range. Find the minimum of $f(x)$ on $-2 \leq x \leq 2$. Example usage:

```
x = fminbnd(fun,x1,x2)
```

Answer using an anonymous function:

```
>> f = @(x) -x.^2 - 3*x + 3 + exp(x.^2)
>> ezplot(f,[-2 2])
>> fminbnd(f,-2,2)
>> f(ans)
```

Various related functions are `fzero`, `feval`, `fsolve`, `fminsearch`. They will be discussed later in the course.

Today's outline

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Advanced concepts

- Object oriented programming: classes and objects
- Memory management: some programming languages require you to allocate computer memory yourself (e.g. for arrays)
- External libraries: in many cases, someone already built the general functionality you are looking for
- Compiling and scripting (“interpreted”); compiling means converting a program to computer-language before execution. Interpreted languages do this on the fly.
- Parallelization: Distributing expensive calculations over multiple processors or GPUs.

Make a habit of the following adage

MAKE IT WORK
MAKE IT RIGHT
MAKE IT FAST

Make a habit of the following adage

① *Make it work*

Create an algorithm that does the intended job. Make sure it works, and works repeatedly. Test and verify frequently. Add *todo* comments when you're not sure about a certain decision.

② *Make it right*

Refactor the code to improve the code design. Insert functions, comments, compartmentalize it. Get rid of magic numbers, use sensible variable names. Check input. Test and verify. Align with the team!

③ *Make it fast*

Measure and tune the performance of your code (profiling tool). In Matlab, vectorized calculations are much (!) faster than for-loops. Use sensible numerical techniques (e.g. higher-order integration).

Program by iterating over these aspects multiple times, starting at the fine-grained level, working your way up.

Exercise: finding the roots of a parabola

We are writing a program that finds for us the roots of a parabola. We use the form

$$y = ax^2 + bx + c$$

What is our program in pseudo-code?

- ① Input data (a , b and c)
- ② Identify special cases ($a = b = c = 0$, $a = 0$)

$a = b = c = 0$ Solution indeterminate

$a = 0$ Solution: $x = -\frac{c}{b}$

- ③ Find $D = b^2 - 4ac$

- ④ Decide, based on D :

$D < 0$ Display message: complex roots

$D = 0$ Display 1 root value

$D > 0$ Display 2 root values

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Example: finding the roots of a parabola

```
>> roots([1 -4 -3])
ans =
    4.6458
   -0.6458
```

Numerical errors in computer simulations

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Chemical Process Intensification group
Eindhoven University of Technology

Numerical Methods (6E5X0), 2020-2021

Today's outline

- Introduction
- Roundoff and truncation errors
- Break errors
- Loss of digits
- (Un)stable methods
- Symbolic math
- Summary

Example 1

Start your spreadsheet program (Excel, ...)

Enter:

Cell	Value
A1	0.1
A2	= (A1*10)-0.9
A3	= (A2*10)-0.9
A4	= (A3*10)-0.9

(repeat until A30)

What's happening?

Enter:

Cell	Value
A1	2
A2	= (A1*10)-18
A3	= (A2*10)-18
A4	= (A3*10)-18

(repeat until A30)

Example 2

Start Matlab

Investigate the result of `sin(1e40 * pi)`

Create a vector `v` containing the powers of 10, e.g. from 10^0 up to 10^{40} and solve `sin(v * pi)`:

```
v = logspace(0,40,41);
y = sin(v*pi);
loglog(v,abs(y));      % Double log plot, values on y-axis must be positive.
```

Errors in computer simulations

In this lecture I will outline different numerical errors that can appear in computer simulations, and how these errors can affect the simulation results.

- Errors in the mathematical model (physics)
- Errors in the program (implementation)
- Errors in the entered parameters
- Roundoff- and truncation errors
- Break errors

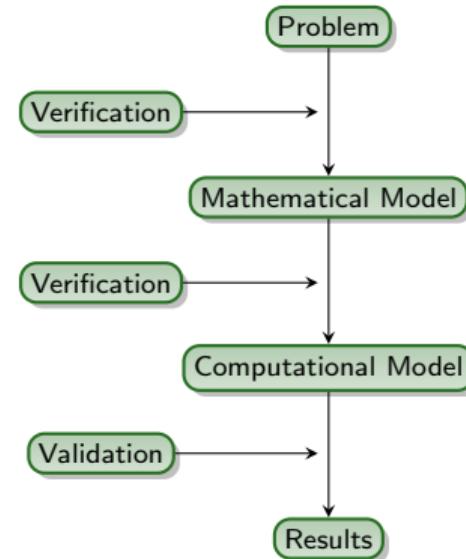
Verification and validation

Verification

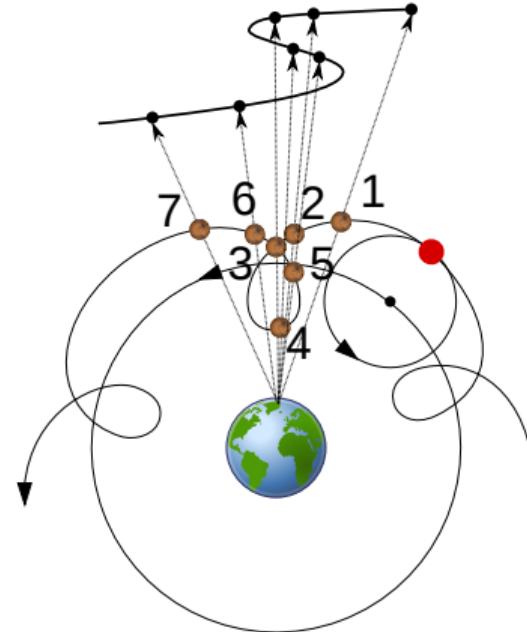
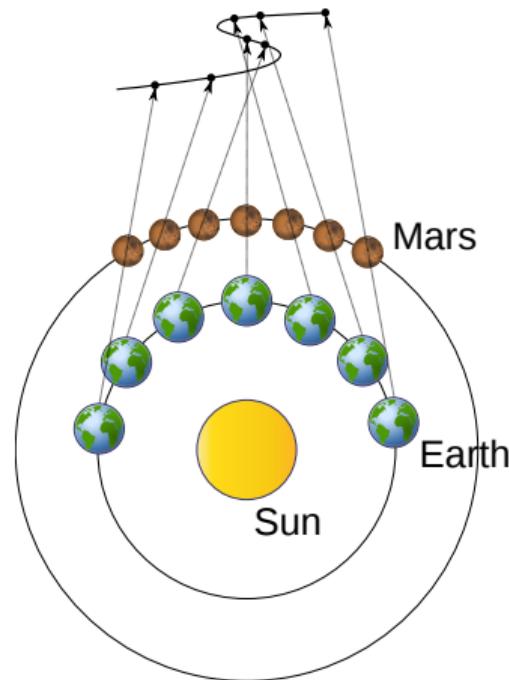
Verification is the process of mathematically and computationally assuring that the model computes the equations you intended to implement.

Validation

Validation is the process of determining the degree to which a model is an accurate representation of the real world from the perspective of the intended uses of the model



Verification of the physical model



- The perceived orbit of Mars from Earth shows a zig-zag (in contrast to the Sun, Mercury, Venus)
- Even though they were not 'right', Earth-centered models (Ptolemy) were still valid

Be aware of your uncertainties

Aleatory uncertainty

Uncertainty that arises due to inherent randomness of the system, features that are too complex to measure and take into account

Epistemic uncertainty

Uncertainty that arises due to lack of knowledge of the system, but could in principle be known

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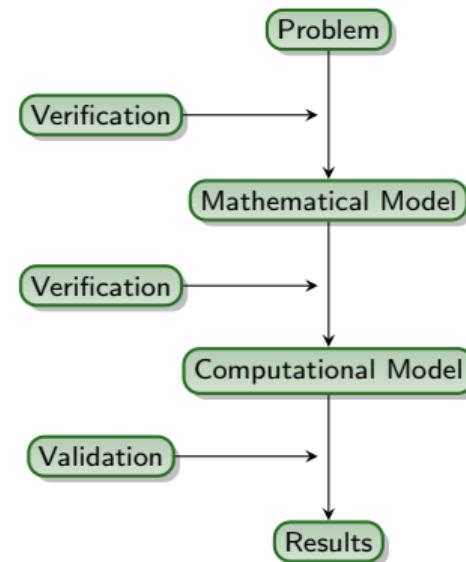
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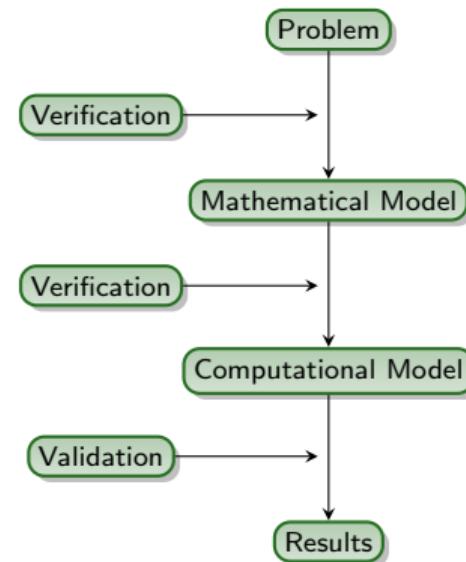
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- Roundoff- and truncation errors
- Break errors

Significant digits

A numerical result \tilde{x} is an approximation of the real value x .

- Absolute error

$$\delta = |\tilde{x} - x|, x \neq 0$$

- Relative error

$$\frac{\delta}{\tilde{x}} = \left| \frac{\tilde{x} - x}{\tilde{x}} \right|$$

- Error margin

$$\tilde{x} - \delta \leq x \leq \tilde{x} + \delta$$

$$x = \tilde{x} \pm \delta$$

Significant digits

- \tilde{x} has m significant digits if the absolute error in x is smaller or equal to 5 at the $(m+1)$ -th position:

$$10^{q-1} \leq |\tilde{x}| \leq 10^q$$

$$|x - \tilde{x}| = 0.5 \times 10^{q-m}$$

- For example:

$$x = \frac{1}{3}, \tilde{x} = 0.333 \Rightarrow \delta = 0.0003333\dots$$

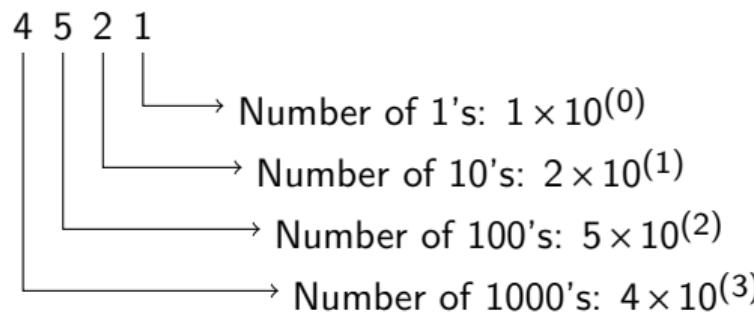
3 significant digits

Today's outline

- Introduction
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- (Un)stable methods
- Symbolic math
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Representation of numbers

- Computers represent a number with a finite number of digits: each number is therefore an approximation due to roundoff and truncation errors.
- In the decimal system, a digit c at position n has a value of $c \times 10^{n-1}$



$$(4521)_{10} = 4 \times 10^3 + 5 \times 10^2 + 2 \times 10^1 + 1 \times 10^0$$

Representation of numbers

- You could use another basis, computers often use the basis 2:

$$\begin{aligned}(4521)_{10} &= 1 \times 2^{12} + 0 \times 2^{11} + 0 \times 2^{10} + 0 \times 2^9 + 1 \times 2^8 + \dots \\ &\quad \dots 1 \times 2^7 + 0 \times 2^6 + 1 \times 2^5 + 0 \times 2^4 + \dots \\ &\quad \dots 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\ &= (1000110101001)_2\end{aligned}$$

- In general:

$$(c_m \dots c_1 c_0)_q = c_0 q^0 + c_1 q^1 + \dots + c_m q^m, c \in \{0, 1, 2, \dots, q-1\}$$

Representation of numbers

- Numbers are stored in binary in the memory of a computer, in segments of a specific length (called a *word*).
- We distinguish multiple types of numbers:
 - Integers: $-301, -1, 0, 1, 96, 2293, \dots$
 - Floating points: $-301.01, 0.01, 3.14159265, 14498.2$
- A binary integer representation looks like the following bit sequence:

$$z = \sigma (c_0 2^0 + c_1 2^1 + \dots + c_{\lambda-1} 2^{\lambda-1})$$

σ is the sign of z (+ or -), and λ is the length of the word

- Endianness: the order of bits stored by a computer

Excercise

- Convert the following decimal number to base-2: 214

$$214_{10} = 11010110_2$$

- Excel:
 - Decimal: =DEC2BIN(214)
 - Octal: =DEC2OCT(214)
 - Hexadecimal: =DEC2HEX(214)
- Matlab:
 - Decimal: dec2bin(214)
 - Other base: dec2base(214,<base>)

Arithmetic operations with binary numbers

$$0 + 0 = 0$$

$$0 + 1 = 1$$

$$1 + 0 = 1$$

$$1 + 1 = 0 \text{ (carry one)}$$

$$\begin{array}{r} & 1 & 4 & 5 \\ + & 2 & 3 \\ \hline 1 & 6 & 8 \end{array}$$

$$\begin{array}{r} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ + & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \end{array}$$

Addition:

$$0 - 0 = 0$$

$$1 - 0 = 1$$

$$1 - 1 = 0$$

$$0 - 1 = 1 \text{ (borrow one)}$$

$$\begin{array}{r} & 1 & 4 & 5 \\ - & 2 & 3 \\ \hline 1 & 2 & 2 \end{array}$$

$$\begin{array}{r} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ - & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ \hline 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 \end{array}$$

Subtraction:

- Multiplication and division are more expensive, and more elaborate

Excercise

Try the following commands in Matlab:

Command	Result
intmin	-2147483648
intmax	2147483647
i = int16(intmax)	i = 32767
whos i	int16 information
i = i + 100	i = 32767
realmax	1.7977e+308
f = 0.1	
whos f	double information
format long e	
realmax	1.797693134862316e+308
f	
fprintf("%0.16f",f)	0.1000000000000000
fprintf("%0.20f",f)	0.1000000000000000555

Representation of integer numbers

- In Matlab, integers of the type `int32` are represented by 32-bit words ($\lambda = 31$).
- The set of numbers that an `int32` z can represent is:

$$-2^{31} \leq z \leq 2^{31} - 1 \approx 2 \times 10^9$$

- If, during a calculation, an integer number becomes larger than $2^\lambda - 1$, the computer reports an **overflow** with most programming languages. Matlab does not perform actual integer overflows, it just stops at the maximum.
- How can a computer identify an overflow?

Representation of real (floating point) numbers

- Formally, a real number is represented by the following bit sequence

$$x = \sigma \left(2^{-1} + c_2 2^{-2} + \dots + c_m 2^{-m} \right) 2^{e-1023}$$

Here, σ is the sign of x and e is an integer value.

- A floating point number hence contains sections that contain the sign, the exponent and the mantissa

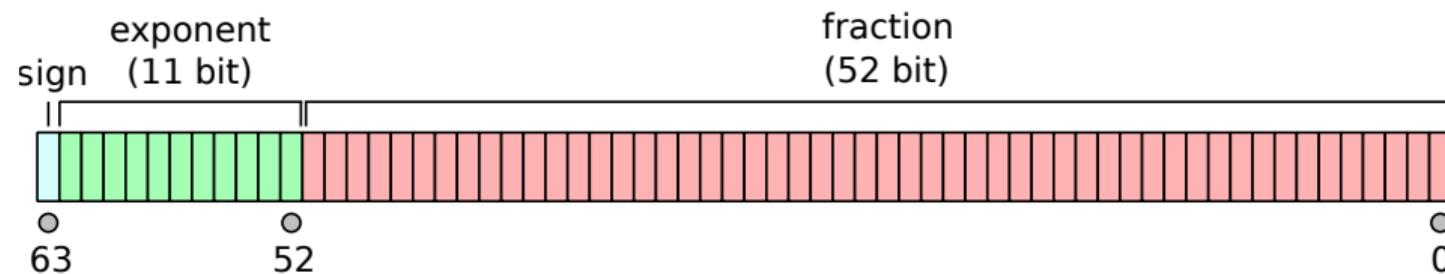


Image: Wikimedia Commons CC by-SA

Representation of real (floating point) numbers

- Example: $\lambda = 3, m = 2, x = \frac{2}{3}$

$$x = \pm(2^{-1} + c_2 2^{-2}) 2^e$$

- $c_0 \in \{0, 1\}$
- $e = \pm a_0 2^0$
- $a_0 \in \{0, 1\}$
- Truncation: $fl(x) = 2^{-1} = 0.5$
- Round off: $fl(x) = 2^{-1} + 2^{-2} = 0.75$

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Trigonometric, Logarithmic, and Exponential computations

- Processors can do logic and arithmetic instructions
- Trigonometric, logarithmic and exponential calculations are “higher-level” functions: \exp , \sin , \cos , \tan , \sec , \arcsin , \arccos , \arctan , \log , \ln , ...
- Such functions can be performed using these “low level” instructions, for instance using a Taylor series:

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Trigonometric, Logarithmic, and Exponential computations

- These operations involve many multiplications and additions, and are therefore *expensive*
- Computations can only take finite time, for infinite series, calculations are interrupted at N

$$\sin(x) = \sum_{n=0}^N \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots + \frac{(-1)^N}{(2N+1)!} x^{2N+1}$$

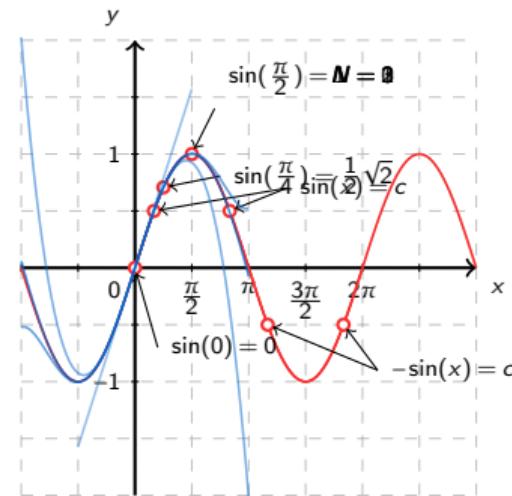
$$e^x = \sum_{n=0}^N \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots + \frac{x^N}{N!}$$

- This results in a *break error*

Algorithm for sine-computation

A computer may use a clever algorithm to limit the number of operations required to perform a higher-level function. A (fictional!) example for the computation of $\sin(x)$:

- ① Use periodicity so that $0 \leq x \leq 2\pi$
- ② Use symmetry ($0 \leq x \leq \frac{\pi}{2}$)
- ③ Use lookup tables for known values
- ④ Perform taylor expansion



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Loss of digits

- During operations such as $+$, $-$, \times , \div , an error can add up
- Consider the summation of x and y

$$\tilde{x} - \delta \leq x \leq \tilde{x} + \delta \quad \text{and} \quad \tilde{y} - \varepsilon \leq y \leq \tilde{y} + \varepsilon$$

$$(\tilde{x} + \tilde{y}) - (\delta + \varepsilon) \leq x + y \leq (\tilde{x} + \tilde{y}) + (\delta + \varepsilon)$$

Loss of digits: Example 1

$$\left. \begin{array}{l} x = \pi, \tilde{x} = 3.1416 \\ y = 22/7, \tilde{y} = 3.1429 \end{array} \right\} \Rightarrow \left. \begin{array}{l} \delta = \tilde{x} - x = 7.35 \times 10^{-6} \\ \varepsilon = \tilde{y} - y = 4.29 \times 10^{-5} \end{array} \right\}$$

$$x + y = \tilde{x} + \tilde{y} \pm (\delta + \varepsilon) \approx 6.2845 - 5.025 \times 10^{-5}$$

$$x - y = \tilde{x} - \tilde{y} \pm (\delta + \varepsilon) \approx -0.0013 + 3.55 \times 10^{-5}$$

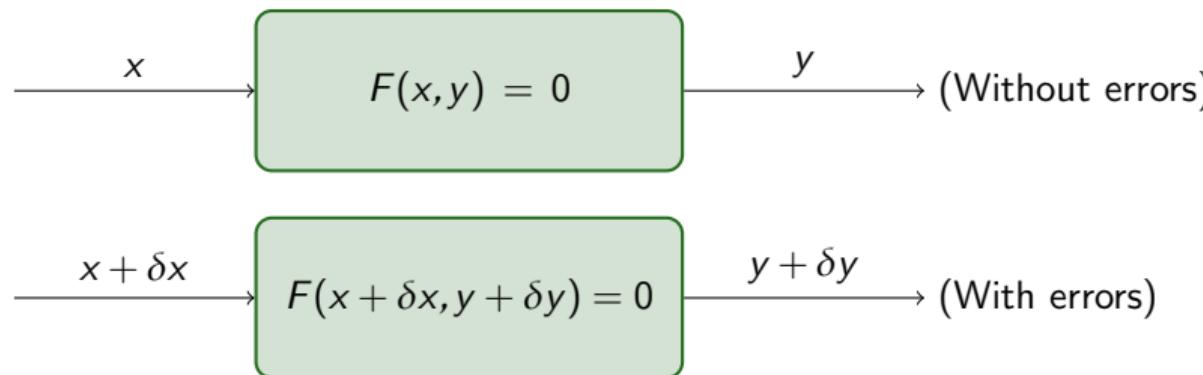
- The absolute error is small ($\approx 10^{-5}$), but the relative error is much bigger (0.028).
- Adding up the errors results in a loss of significant digits!

Loss of digits: Example 2

- Calculate e^{-5}
 - Use the Taylor series
 - Calculate the first 26 terms ($N = 26$)
- Now repeat the calculation, but use for each calculation only 4 digits. What do you find?
Use: `str2double(sprintf('%.4g', term))`
- Without errors you would find: $e^{-5} = 0.006738$
- If you only use 4 digits in the calculations, you'll find 0.00998

Badly (ill) conditioned problems

We consider a system $F(x, y)$ that computes a solution from input data. The input data may have errors:



$$y(x + \delta x) - y(x) \approx y'(x)\delta x$$

Propagated error on the basis of Taylor expansion

$$C = \max_{\delta x} \left(\left| \frac{\delta y/y}{\delta x/x} \right| \right)$$

Condition criterion, $C < 10$ error development small

Badly (ill) conditioned problems: Example

Solve the following linear system in Matlab using double and single precision:

$$A = \begin{bmatrix} 1.2969 & 0.8648 \\ 0.2161 & 0.1441 \end{bmatrix}, \quad x = \begin{bmatrix} 0.8642 \\ 0.1440 \end{bmatrix}, \quad y = \begin{bmatrix} 2.0 \\ -2.0 \end{bmatrix}$$

Double precision

```
>> clear;clc;format long e;
>> A = [[1.2969 0.8648]; [0.2161 0.1441]];
>> x = [0.8642; 0.1440];
>> y = A\x
y =
    2.00000002400302e+00
   -2.00000003599621e+00
```

Single precision

```
>> clear;clc;format long e;
>> A = single(
    [[1.2969 0.8648];
     [0.2161 0.1441]] );
>> x = single(
    [0.8642; 0.1440] );
>> y = A\x
y =
    1.3331791e+00
   -1.0000000e+00
```

Badly (ill) conditioned problems: Example

- Matlab already warned us about the bad condition number:

Warning: Matrix is `close` to singular or badly scaled. Results may be inaccurate. `RCOND = 1.148983e-08.`

- The `RCOND` is the reciprocal condition number
- A small error in x results in a big error in y . This is called an ill conditioned problem.

Today's outline

- Introduction
- Roundoff and truncation errors
- Break errors
- Loss of digits
- (Un)stable methods
- Symbolic math
- Summary

(Un)stable methods

- The condition criterion does not tell you anything about the quality of a numerical solution method!
- It is very well possible that a certain solution method is more sensitive for one problem than another
- If the method propagates the error, we call it an *unstable method*. Let's look at an example.

The Golden mean

- Let's evaluate the following recurrent relationship:

$$y_{n+1} = y_{n-1} - y_n$$

$$y_0 = 1, \quad y_1 = \frac{2}{1 + \sqrt{5}}$$

- You can prove (by substitution) that:

$$y_n = x^{-n}, \quad n = 0, 1, 2, \dots, \quad x = \frac{1 + \sqrt{5}}{2}$$

The Golden mean

Recurrent version

```
% initialise
y(1) = 1;
y(2) = 2 / (1 + sqrt(5));

% Perform recurrent approach
for n = 2:39
    y(n+1) = y(n-1)-y(n);
end
```

Powerlaw version

```
% initialise
x = (1 + sqrt(5))/2;
y2(1) = x^0; % n = 1

% Perform powerlaw approach
for n = 0:39
    y2(n+1) = x^-n
end
```

The Golden mean

n	Recurrent	Powerlaw
1	1.0000	1.0000
1	0.6180	0.6180
2	0.3820	0.3820
3	0.2361	0.2361
...
37	$3.080 \cdot 10^{-8}$	$2.995 \cdot 10^{-8}$
38	$1.714 \cdot 10^{-8}$	$1.851 \cdot 10^{-8}$
39	$1.366 \cdot 10^{-8}$	$1.144 \cdot 10^{-8}$
40	$3.485 \cdot 10^{-8}$	$7.071 \cdot 10^{-9}$

- The recurrent approach enlarges errors from earlier calculations!

Example 1: Explanation

Recall example 1, where the errors blew up our computation of 0.1, whereas they did not for 2. Why did we see these results?

- The number 0.1 is not exactly represented in binary
 - A tiny error can accumulate up to catastrophic proportions!
- The number 2 does have an exact binary representation

Example 2 (large sine series)

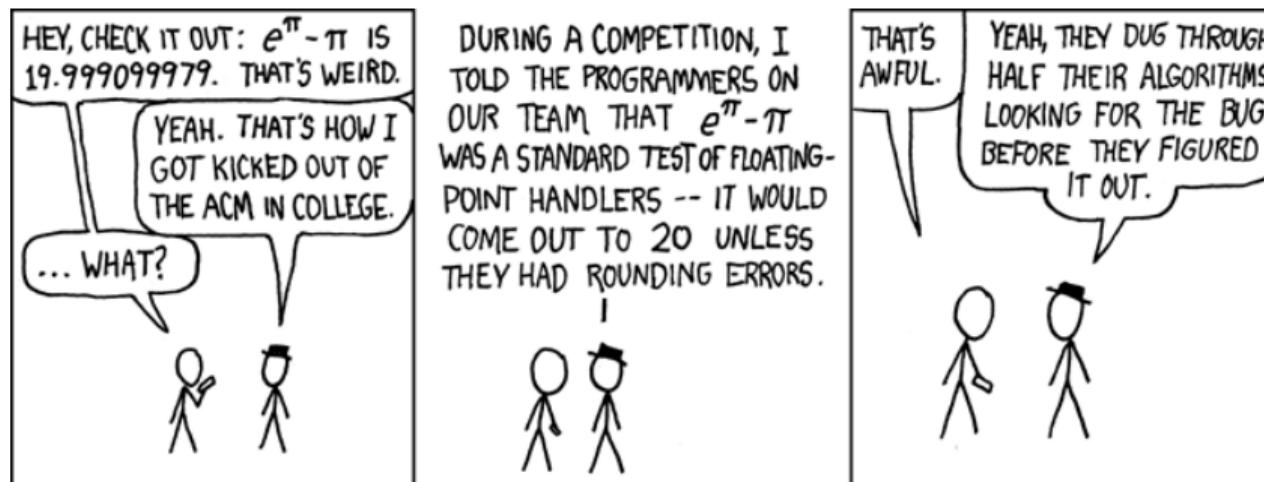
The `sin(1e40*pi)` result gives poor results, because $1e40$ has an error of `eps`, about 1×10^{-14} . In Matlab, the number of 2π cycles is still much larger than $10^{40} \times 10^{-14}$. Also, π is not stored with enough digits.

Example 3

Start your calculation program of choice (Excel, Matlab, ...)

Calculate the result of y :

$$y = e^\pi - \pi = 19.999099979 \neq 20$$



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Symbolic math packages

Definition

The use of computers to manipulate mathematical equations and expressions in symbolic form, as opposed to manipulating the numerical quantities represented by those symbols.

- Symbolic integration or differentiation, substitution of one expression into another
- Simplification of an expression, change of subject etc.
- Packages and toolboxes:

Symbolic math packages

Mathematica Well known software package, license available via [TU/e](#)

Maple Well known, license available via [TU/e](#)

Wolfram|Alpha Web-based interface by Mathematica developer. Less powerful in mathematical respect, but more accessible and has a broad application range (unit conversion, semantic commands).

Sage Open-source alternative to Maple, Mathematica, Magma, and MATLAB.

Matlab Symbolic math toolbox

Symbolic math: simplify

$$f(x) = (x - 1)(x + 1)(x^2 + 1) + 1$$

```
>> syms x
>> f = (x - 1)*(x + 1)*(x^2 + 1) + 1
f =
(x^2 + 1)*(x - 1)*(x + 1) + 1
>> f2 = simplify(f)
f2 =
x^4
```

Symbolic math: integration and differentiation

$$f(x) = \frac{1}{x^3 + 1}$$

```
>> syms x
>> f = 1/(x^3+1);
>> my_f_int = int(f)

my_f_int = log(x + 1)/3 - log((x - 1/2)^2 + 3/4)/6 + (3^(1/2)*atan((2*3^(1/2)*(x - 1/2))/3))/3

>> my_f_diff = diff(my_f_int)

my_f_diff = 1/(3*(x + 1)) + 2/(3*((4*(x - 1/2)^2)/3 + 1)) - (2*x - 1)/(6*((x - 1/2)^2 + 3/4))

>> simplify(my_f_diff)

ans = 1/(x^3 + 1)
```

Symbolic math: exercises

Exercise 1

Simplify the following expression:

$$f(x) = \frac{2 \tan x}{(1 + \tan^2 x)} = \sin 2x$$

```
>> simplify(2*tan(x)/(1 + tan(x)^2))
```

Exercise 2

Calculate the *value* of p :

$$p = \int_0^{10} \frac{e^x - e^{-x}}{\sinh x} dx$$

```
>> f = ((exp(x)- exp(-x))/sinh(x));
>> p = int(f,0,10)
p = 20
```

Symbolic math: root finding

A root finding method searches for the values where a function reaches zero. We will cover the numerical methods later, here we show how to use root finding with symbolic math in Matlab.

Symbolic math function

$$f(x) = \frac{3}{x^2 + 3x} - 2$$

```
>> syms x
>> f = 3 / (x^2 + 3*x) - 2;
>> solve(f)
ans =
 15^(1/2)/2 - 3/2
 - 15^(1/2)/2 - 3/2
```

Symbolic math toolbox: variable precision arithmetic

Variable precision can be used to specify the number of significant digits.

```
>> p = vpa(1/3,16)
p = 0.3333333333333333
>> p = vpa(1/3,4)
p = 0.3333
>> a = vpa(0.1, 30)
a = 0.1
>> b = vpa(0.1, 5);
b = 0.1
>> a-b

ans = 0.000000000000056843418860808014869689938467514
```

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Summary

- Numerical errors may arise due to truncation, roundoff and break errors, which may seriously affect the accuracy of your solution
- Errors may propagate and accumulate, leading to smaller accuracy
- Ill-conditioned problems and unstable methods have to be identified so that proper measures can be taken
- Symbolic math computations may be performed to solve certain equations algebraically, bypassing numerical errors, but this is not always possible.

Numerical interpolation

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Chemical Process Intensification group
Eindhoven University of Technology

Numerical Methods (6E5X0), 2020-2021

Today's outline

- Introduction

- Piecewise constant

- Linear

- Polynomial

- Splines

- Tutorials

Interpolation problem

Definition

Given a set of points x_k , $k = 0, \dots, n$, $x_i \neq x_j$ with associated function values f_k , $k = 0, \dots, n$, or simply: $\{x_k, f_k\}_{k=0}^n$. The interpolation problem is defined as: find a polynomial p_n such that this interpolates the values of f_k on the points x_k :

$$p_n(x_k) = f_k, \quad k = 0, \dots, n$$

Theorem

The interpolation problem for $\{x_k, f_k\}_{k=0}^n$ has a unique solution when $x_i \neq x_j$ for $i \neq j$. Note that we cannot allow multiple function values f_k for the same value of x_k .

What is interpolation?

Interpolation means constructing additional data points within the range of, and using, a discrete set of known data points.

It is typically performed on a uniformly spread data set, but this is not strictly necessary for all methods

Is interpolation the same as curve fitting?

NO

- Curve-fitting requires additionally some way of computing the error between function (curve) and data
- Curve-fitting does not strictly enforce the function to match the data exactly
- Curve-fitting may be done on multiple datapoints at one position
- Curve-fitting is much more expensive to do, requires optimisation

Why do chemical engineers need interpolation?

- Comparison of two data sets which are given at different positions
 - An experimental data set may have been recorded at a constant rate, but the numerical solution is computed at irregular intervals
- Reconstruction of field values distant of computing nodes
 - A CFD simulation on a regular grid containing structures that are not grid-conformant requires interpolation to the structures
- Calculation of a physical property at a condition between those of a lookup table
 - The viscosity of a substance may have been measured at 20°C and 30°C, but not at the desired 28.5°C

General

Several important numerical interpolation methods are discussed today:

- Piecewise constant interpolation
- Linear interpolation
 - Bilinear interpolation
- Polynomial interpolation (Newton's method)
- Spline interpolation

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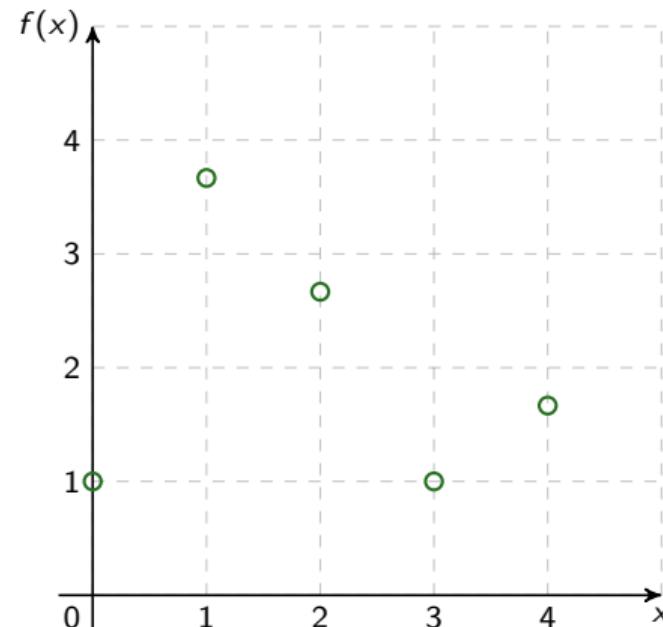
Today's data set

Data set $f_n(x_n)$ represented by ○ at discrete intervals
 $x_n \in \{0, 5\}$

Download the datafile `interpolation-dataset.mat`,
which contains multiple data sets.

We start with x_1 and y_1 :

x_k	f_k
0	1.00
1	$\frac{11}{3} = 3.67$
2	$\frac{8}{3} = 2.67$
3	1.00
4	$\frac{5}{3} = 1.67$
5	$\frac{23}{3} = 7.67$



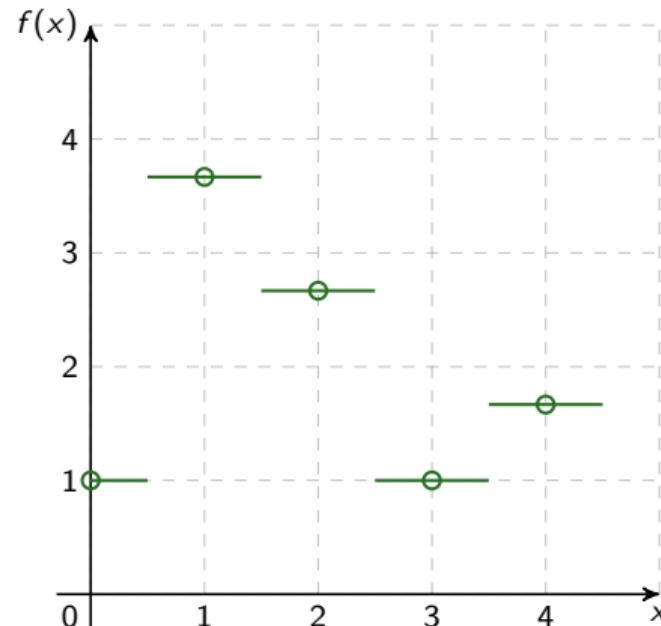
Piecewise constant interpolation

Data set $f_n(x_n)$ represented by ○ at discrete intervals
 $x_n \in \{0, 5\}$

- Nearest-neighbor interpolation in the continuous range $x \in [0, 5]$
- How to treat the point halfway (e.g. at $x = 2.5$)?

$$\begin{array}{ll} x \in [0, 0.5] & \rightarrow f(x) = f(0) \\ x \in]0.5, 1.5] & \rightarrow f(x) = f(1) \\ x \in]1.5, 2.5] & \rightarrow f(x) = f(2) \\ x \in]2.5, 3.5] & \rightarrow f(x) = f(3) \\ x \in]3.5, 4.5] & \rightarrow f(x) = f(4) \end{array}$$

- Not often used for simple problems, but e.g. for 2D (Voronoi)



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Linear interpolation

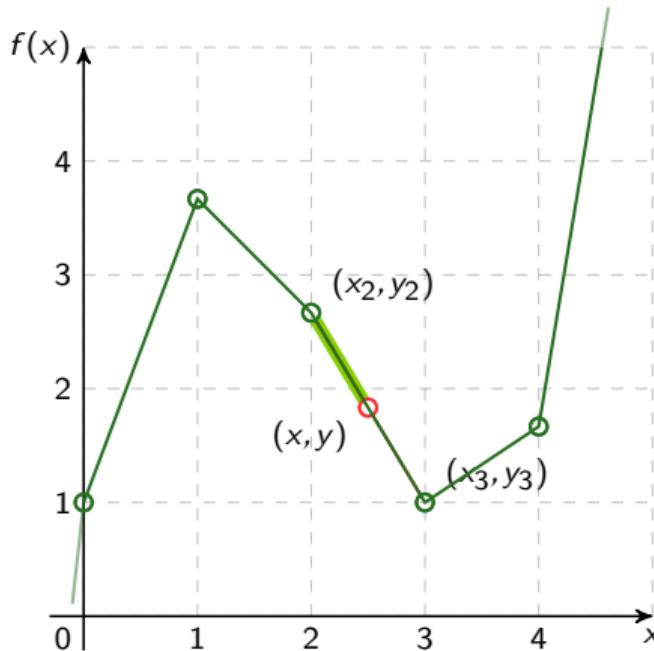
Data set $f_n(x_n)$ represented by ○ at discrete intervals
 $x_n \in \{0, 5\}$

- Linear interpolation to (x, y) between 2 data points (x_2, y_2) and (x_3, y_3) :

$$\frac{y - y_2}{x - x_2} = \frac{y_3 - y_2}{x_3 - x_2}$$

- Reordered, and more formally:

$$y = y_n + (y_{n+1} - y_n) \frac{x - x_n}{x_{n+1} - x_n}$$

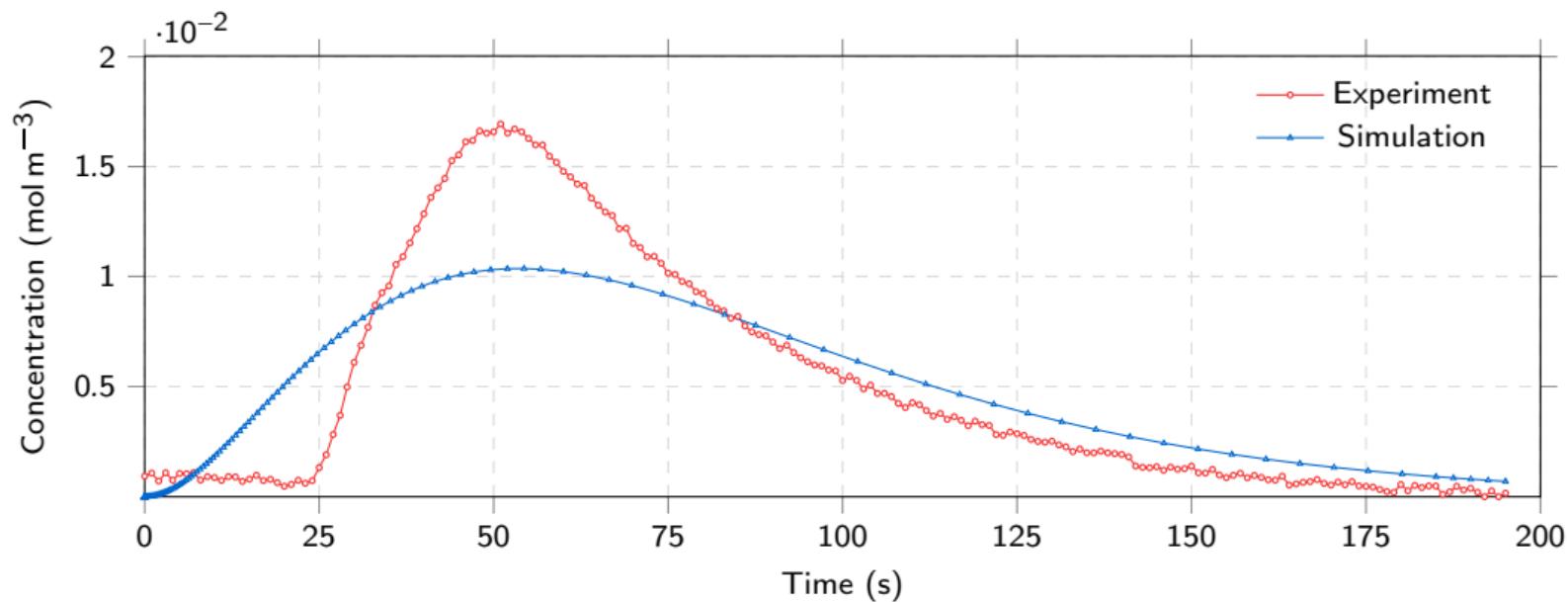


Linear interpolation

- While linear interpolation is fast, and relatively easy to program, it is not very accurate
- At the nodes, the derivatives are discontinuous i.e. not differentiable
- Error is proportional to the square of the distance between nodes

Example: Linear interpolation in Matlab

Consider the data set in `sim_exp_dataset.mat`, containing a normalized concentration and time vector for an experiment and a simulation. The simulation was performed with adaptive node distance to save computation time, thus the concentration is not known at the same times. We are not able to compare yet.



Example: Linear interpolation in Matlab

Consider the data set in `sim_exp_dataset.mat`, containing a normalized concentration and time vector for an experiment and a simulation. The simulation was performed with adaptive node distance to save computation time, thus the concentration is not known at the same times. We are not able to compare yet.

```
% Linear interpolation
c_sim_new = interp1(t_sim,c_sim,t_exp,'linear');
diff = abs(c_exp-c_sim_new);
% Plot the solution
subplot(2,1,1);
plot(t_exp,c_exp,'b-x',t_exp,c_sim_new,'r-o');
subplot(2,1,2);
stem(t_exp,diff);
% Compute the L2-norm
norm(diff)
```

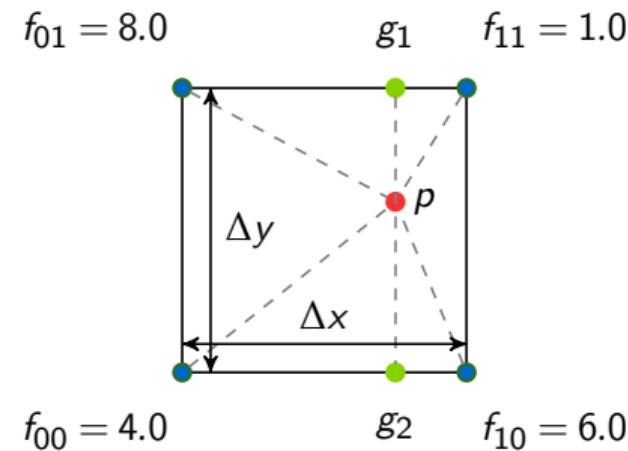
Bi-linear interpolation

When a 2D field of some quantity is known, we can interpolate the solution to an arbitrary position in the 2D domain $p(x, y)$ using 4 field values f_{00} , f_{10} , f_{01} and f_{11} .

$$\begin{aligned}g_1 &= f_{01} \frac{x_1 - x}{x_1 - x_0} + f_{11} \frac{x - x_0}{x_1 - x_0} \\&= f_{01} \frac{x_1 - x}{\Delta x} + f_{11} \frac{x - x_0}{\Delta x}\end{aligned}$$

$$g_2 = f_{00} \frac{x_1 - x}{\Delta x} + f_{10} \frac{x - x_0}{\Delta x}$$

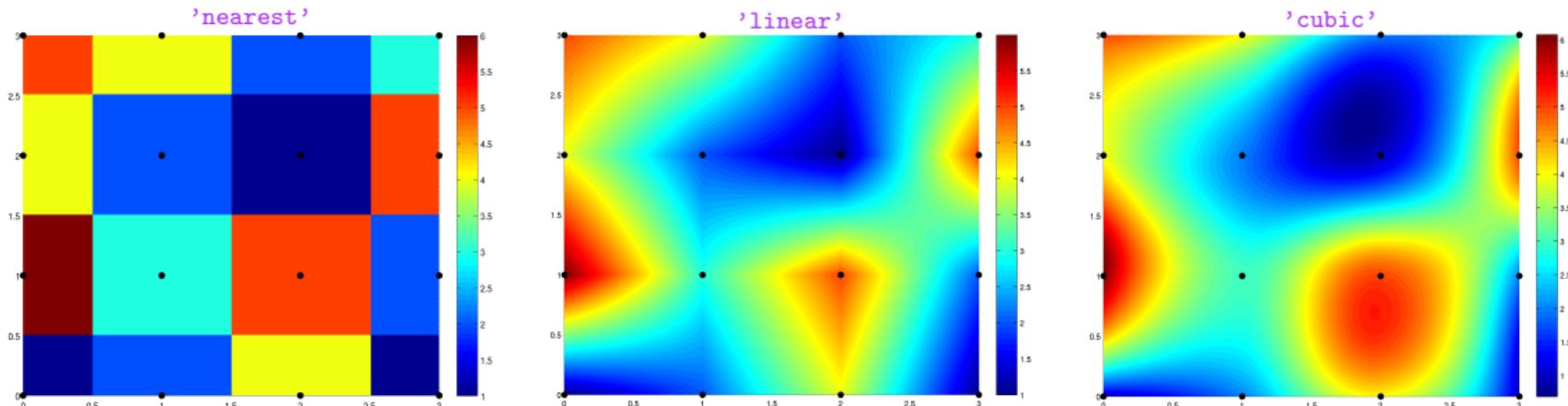
$$p = g_2 \frac{y_1 - y}{\Delta y} + g_1 \frac{y - y_0}{\Delta y}$$



- The order of interpolation (x or y direction first) does not matter; the results are equal

Higher-dimensional field interpolation in Matlab

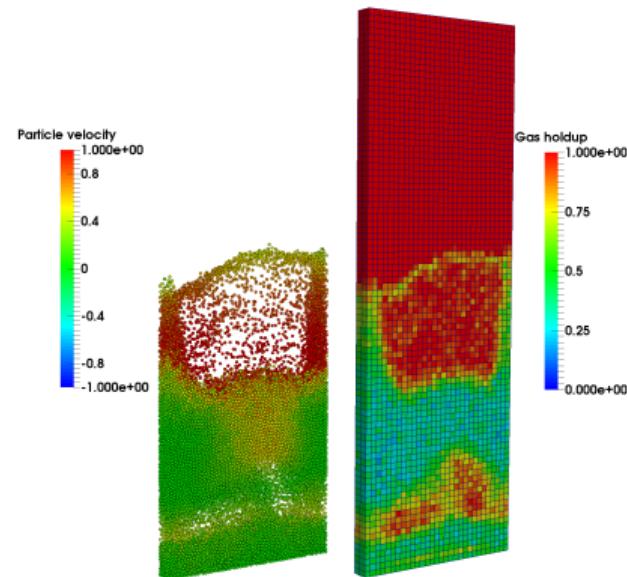
2D or higher-dimensional fields of data can be interpolated in Matlab using the `interp2`, `interp3` or even `interpN` functions, the method can be adjusted:



- Similar to 1D linear interpolation, the derivatives are discontinuous on the grid nodes
- Also consider tri-linear interpolation (for 3D fields), or bicubic interpolation (2D, but third order)

A practical example

Field interpolation is used in e.g. CFD simulations, e.g. a fluidized bed simulation using a *discrete particle model*, where particles are found in between the grid nodes used for velocity computation.



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Polynomial interpolation

The examples that we have seen, are simplified forms of *Newton polynomials*. We can interpolate our data with a polynomial of degree n :

$$p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$$

Polynomial interpolation via Vandermonde matrix

Consider the data points $(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)$, the Vandermonde matrix V , coefficient vector a and function value vector y :

$$V_{m,n} = \begin{pmatrix} x_1^0 & x_1^1 & x_1^2 & \cdots & x_1^{n-1} \\ x_2^0 & x_2^1 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_m^0 & x_m^1 & x_m^2 & \cdots & x_m^{n-1} \end{pmatrix} \quad a = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

The coefficients of a polynomial through the data points can be obtained by solving the linear system $Va = y$.

```
>> x = [0 1 2];
>> y = [1.0000; 3.6667; 2.6667];
>> V = vander(x);
>> a = V\y;
a =
    -1.8333
     4.5000
    1.0000
```

So we found the equation:

$$p_2(x) = -1.8333x^2 + 4.5x - 1$$

These Vandermonde-systems are often *ill-conditioned*,
so we need another, more stable, method!

Construction of Newton polynomials

Formally, the polynomials $p_n(x)$ are described using prefactors $f[x_0, \dots, x_k]$ and polynomial terms $w_m(x)$:

$$p_n(x) = \sum_{k=0}^n f[x_0, \dots, x_k] w_k(x)$$

The polynomial terms are computed via:

$$\begin{aligned}w_0(x) &= 1, \quad w_1(x) = (x - x_0), \quad w_2(x) = (x - x_0) \cdot (x - x_1), \\w_m(x) &= (x - x_0) \cdot (x - x_1) \cdots (x - x_{m-1}) = w_{m-1} \cdot (x - x_{m-1})\end{aligned}$$

$$w_m(x) = \prod_{j=0}^{m-1} (x - x_j), \quad m = 0, \dots, n$$

The prefactors are *forward divided differences*, which can be computed as:

$$f[x_{r-k}, \dots, x_r] \equiv \frac{f[x_{r-k+1}, \dots, x_r] - f[x_{r-k}, \dots, x_{r-1}]}{x_r - x_{r-k}}$$

Construction of Newton polynomials: example

Sample data

x_k	f_k
0	1.00
1	$\frac{11}{3} = 3.67$
2	$\frac{8}{3} = 2.67$

$$p_n(x) = \sum_{k=0}^n f[x_0, \dots, x_k] w_k(x)$$

$$f[x_{r-k}, \dots, x_r] \equiv \frac{f[x_{r-k+1}, \dots, x_r] - f[x_{r-k}, \dots, x_{r-1}]}{x_r - x_{r-k}}$$

$$w_m(x) = \prod_{j=0}^{m-1} (x - x_j)$$

x_k	f_k
x_0	$f[x_0] = f_0$
x_1	$f[x_1] = f_1$
x_2	$f[x_2] = f_2$
	$f[x_0, x_1] = \frac{f_1 - f_0}{x_1 - x_0}$
	$f[x_1, x_2] = \frac{f_2 - f_1}{x_2 - x_1}$
	$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}$

x_k	f_k
0	1
1	3.67
2	2.67
	$\frac{\frac{11}{3} - 1}{1 - 0} = \frac{8}{3}$
	$\frac{\frac{8}{3} - \frac{11}{3}}{2 - 1} = \frac{-1}{1} = -1$
	$\frac{(-1) - \frac{8}{3}}{2 - 0} = -\frac{11}{6}$

Construction of Newton polynomials: example

Sample data

x_k	f_k
0	1.00
1	$\frac{11}{3} = 3.67$
2	$\frac{8}{3} = 2.67$

$$p_n(x) = \sum_{k=0}^n f[x_0, \dots, x_k] w_k(x)$$

$$f[x_{x-k}, \dots, x_r] \equiv \frac{f[x_{r-k+1}, \dots, x_r] - f[x_{r-k}, \dots, x_{r-1}]}{x_r - x_{r-k}}$$

$$w_m(x) = \prod_{j=0}^{m-1} (x - x_j)$$

x_k	f_k	
0	1	
1	3.67	$\frac{\frac{11}{3} - 1}{1 - 0} = \frac{8}{3}$
2	2.67	$\frac{\frac{8}{3} - \frac{11}{3}}{2 - 1} = \frac{-1}{1} = -1 \quad \frac{(-1) - \frac{8}{3}}{2 - 0} = -\frac{11}{6}$

$$\begin{aligned}
 p_2(x) &= 1 \cdot w_m(0) + \frac{8}{3} \cdot w_m(1) + \left(-\frac{11}{6}\right) \cdot w_m(2) \\
 &= 1 \cdot 1 + \frac{8}{3} \cdot (x - 0) + \left(-\frac{11}{6}\right) \cdot (x - 0)(x - 1) = -\frac{11}{6}x^2 + 4\frac{1}{2}x + 1
 \end{aligned}$$

Construction of Newton polynomials: example

For each three points, a new polynomial interpolant can be derived:

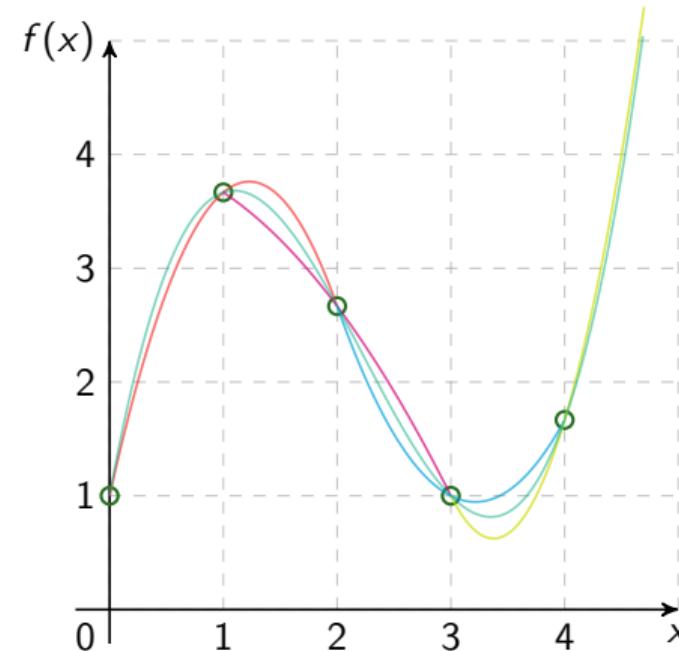
$$p_2(x) = -\frac{11}{6}x^2 + 4\frac{1}{2}x + 1$$

$$p_2(x) = 4 - \frac{x^2}{3}$$

$$p_2(x) = \frac{7x^2}{6} - 7\frac{1}{2}x + 13$$

$$p_2(x) = \frac{8}{3}x^2 - 18x + 31$$

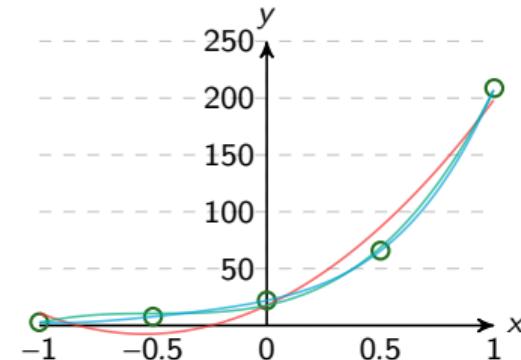
$$f(x) = \frac{x^3}{2} - \frac{10x^2}{3} + \frac{11x}{2} + 1$$



Polynomial fitting in Matlab: example

Develop the $p_2(x)$, $p_3(x)$ and $p_4(x)$ from the following data set (example data x_2 and y_2):

x_k	y_k
-1.0	2.8677
-0.5	7.7530
0.0	22.0000
0.5	65.7863
1.0	208.6744



We use the built-in `polyfit(x,y,n)` and `polyval(p,x)` functions:

```
x_cont = linspace(-1,1,1001);
p2 = polyfit(x2,y2,2);
p3 = polyfit(x2,y2,3);
p4 = polyfit(x2,y2,4);

y_cont2 = polyval(p2,x_cont);
y_cont3 = polyval(p3,x_cont);
y_cont4 = polyval(p4,x_cont);
plot(x2,y2,'o',x_cont,y_cont2,x_cont,y_cont3,x_cont,y_cont4)
```

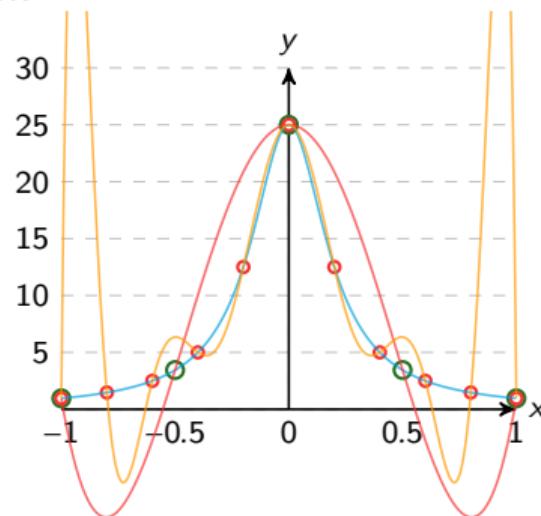
Exercise

Develop the $p_4(x)$ and $p_{10}(x)$ interpolants from the following data sets:

$$f(x) = \frac{1}{x^2 + \frac{1}{25}} \quad x \in [-1, 1]$$

```
% Generate the data
x3a = linspace(-1, 1, 5);
x3b = linspace(-1, 1, 11);
y3a = 1 ./ (x3a.^2 + (1/25));
y3b = 1 ./ (x3b.^2 + (1/25));
```

```
x_cont = linspace(-1, 1, 1001);
p4 = polyfit(x3a, y3a, 4);
p10 = polyfit(x3b, y3b, 10);
y_cont4 = polyval(p4, x_cont);
y_cont10 = polyval(p10, x_cont);
ezplot('1./(x.^2+(1/25))', [-1 1]); hold on;
plot(x3a, y3a, 'o', x3b, y3b, 'x', x_cont, y_cont4, x_cont, y_cont10);
```



Final thoughts on polynomial interpolation

- An polynomial interpolant of order n requires $n+1$ data points
 - More data points: interpolant does *not always* cross the points
 - Fewer data points: interpolant is not unique
- Higher-degree polynomials at equidistant points may cause strong oscillatory behaviour (Runge's phenomenon)
 - Mitigation of the problem on Chebyshev (i.e. non uniform grid)...
 - ... or by performing piecewise interpolation (next topic)
- Matlab functions `polyfit(x,y,n)` and `polyval(p,x_new)` were demonstrated.

Today's outline

- Introduction
- Piecewise constant
- Linear
- Polynomial
- Splines
- Tutorials

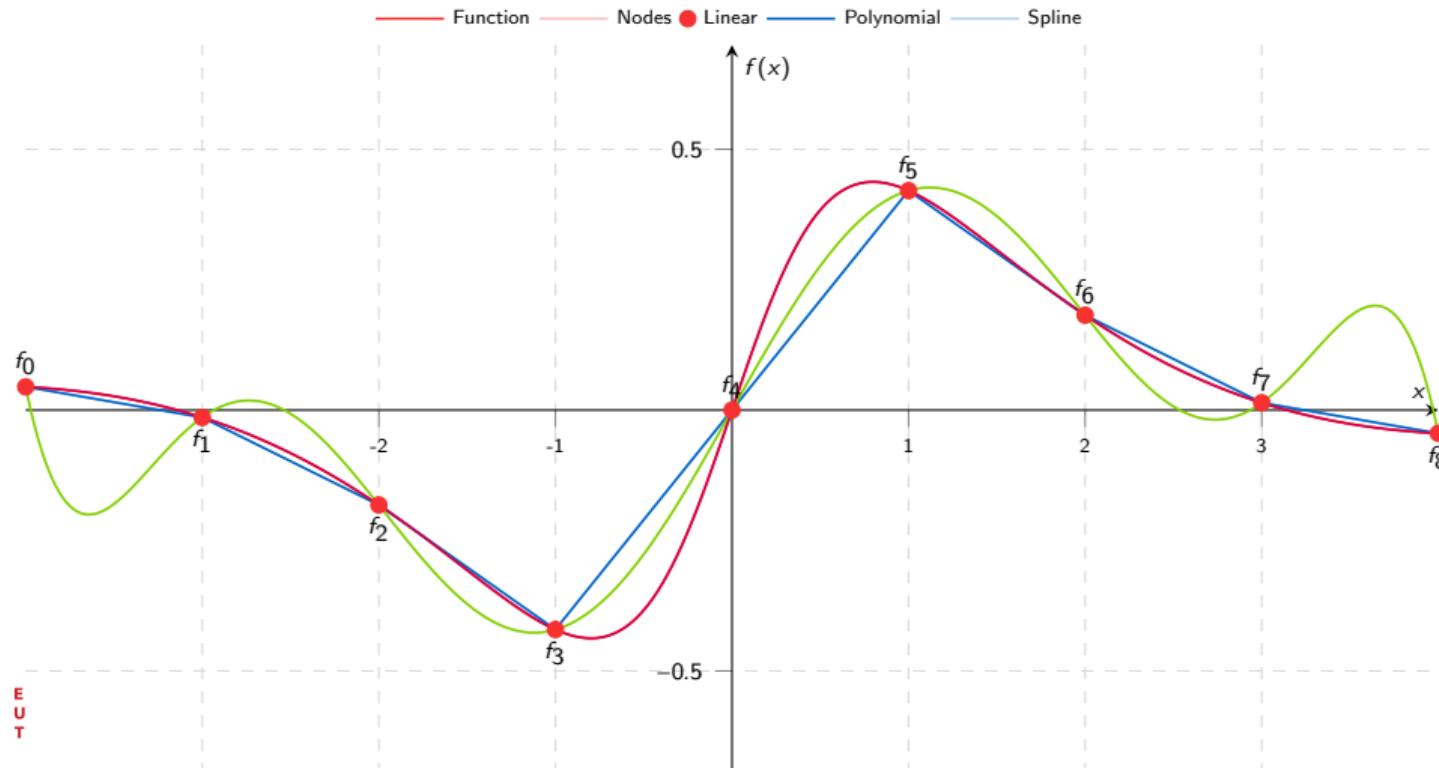
Spline interpolation

A spline is a numerical function that represents a smooth, higher order, piecewise polynomial interpolants of a data set.

- Smooth: the interpolant is continuous in the first and second derivatives
- Higher order: The most common type of splines uses third-order polynomials (cubic splines)
- Piecewise polynomial: The interpolant is constructed between each two consecutive tabulated points

Splines: comparison to other interpolation techniques

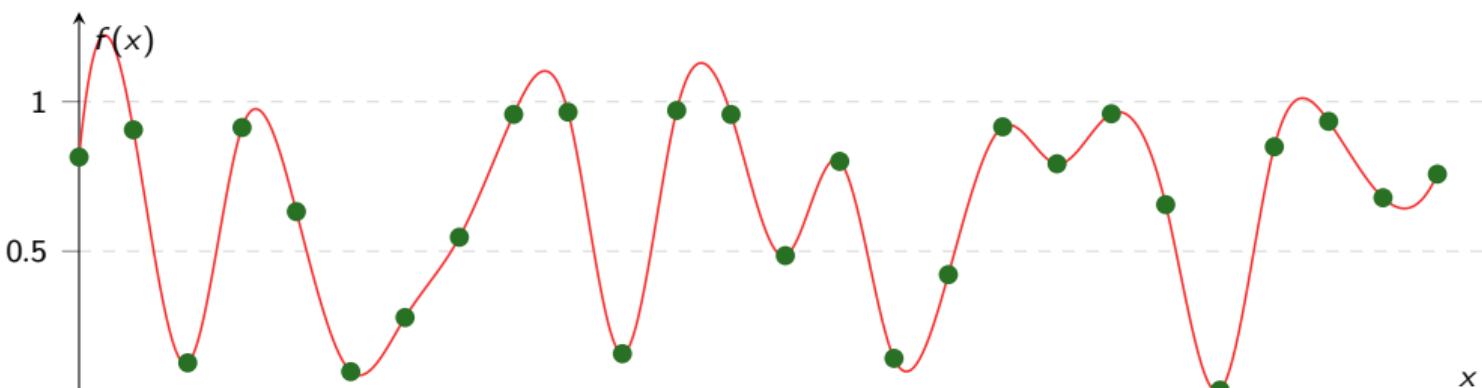
Interpolation of $f(x) = \frac{\sin x}{1+x^2}$



Spline interpolation in Matlab

We can generate a random data set, and interpolate using `interp1`:

```
% Generate random data set
x=0:25;
y = rand(size(x));
% Interpolant on a fine mesh
xc = linspace(0,25,1001);
yc = interp1(x,y,xc,'spline');
plot(x,y,'o',xc,yc,'-r')
```



Summary

- Interpolation is used to obtain data between existing data points
 - (Bi-)Linear, polynomial and spline interpolation methods
 - Construction of Newton polynomials
 - Oscillations of high-order polynomials
- Legendre polynomials: alternative way of performing the polynomial interpolation (not discussed here)

Interpolation tutorials

- ① In Matlab, generate the data:

```
x = -4:1:5;
y = [ 0 0 0 1 1 1 0 0 0 0];
```

Interpolate the data using polynomial interpolation (which order do you use?) and a spline. Plot the results together with the original data in a graph.

- ② Do the same exercise for the following data. Can you explain your observations?

```
t = [ 0 .1 .499 .5 .6 1.0 1.4 1.5 1.899 1.9 2.0];
y = [ 0 .06 .17 .19 .21 .26 .29 .29 .30 .31 .31];
```

Numerical integration

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Numerical Methods (6E5X0), 2020-2021

Today's outline

- Introduction
- Riemann integrals
- Trapezoid rule
- Simpson's rule
- Conclusion
- Tutorials

What is numerical integration?

To determine the integral $I(x)$ of an integrand $f(x)$, which can be used to compute the area underneath the integrand between $x = a$ and $x = b$.

$$I(x) = \int_a^b f(x)dx$$

Today we will outline different numerical integration methods.

- Riemann integrals
- Trapezoidal rule
- Simpson's rule

Why do chemical engineers need integration?

- Obtaining the cumulative particle size distribution from a particle size distribution
- The concentration outflow over time may be integrated to yield the residence time distribution
- Integration of a varying product outflow yields the total product outflow
- Quantitative analysis of mixture components via e.g. GC/MS
- Not all function have an explicit antiderivative, e.g. $\int e^{x^2} dx$ or $\int \frac{1}{\ln x} dx$

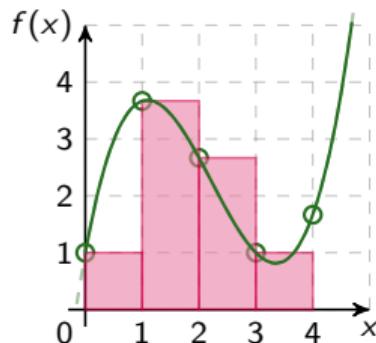
Today's outline

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Riemann integrals

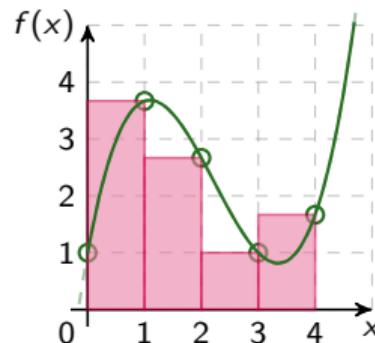
Basic idea: Subdivide the interval $[a, b]$ into n subintervals of equal length $\Delta x = \frac{b-a}{n}$ and use the sum of area to approximate the integral.

Left endpoint rule



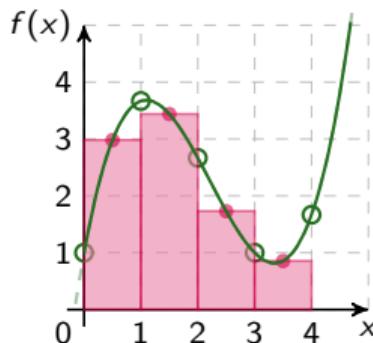
$$L_n = \sum_{i=1}^n f(x_{i-1})\Delta x_i$$

Right endpoint rule



$$R_n = \sum_{i=1}^n f(x_i)\Delta x_i$$

Midpoint rule



$$M_n = \sum_{i=1}^n f(\bar{x}_i)\Delta x_i$$

$$\text{with } \bar{x}_i = \frac{x_{i-1}+x_i}{2}$$

Errors in Riemann integrals

We define the exact integral as $I = \int_a^b f(x)dx$, and L_n , R_n and M_n represent the left, right and midpoint rule approximations of I based on n intervals.

Writing $f_{\max}^{(k)}$ for the maximum value of the k -th derivative, the upper-bounds of the errors by Riemann integrals are:

- $|I - L_n| \leq \frac{f_{\max}^{(1)}(b-a)^2}{2n}$
- $|I - R_n| \leq \frac{f_{\max}^{(1)}(b-a)^2}{2n}$
- $|I - M_n| \leq \frac{f_{\max}^{(2)}(b-a)^3}{24n^2}$

Note that while $|I - L_n|$ and $|I - R_n|$ give the same *upper-bounds* of the error, this does not mean the same error. Rather, the error is of opposite sign!

Today's outline

- Introduction
- Riemann integrals
- Trapezoid rule
- Simpson's rule
- Conclusion
- Tutorials

Trapezoid rule

Since the sign of the approximation error of the left and right endpoint rules is opposite, we can take the average of these approximations:

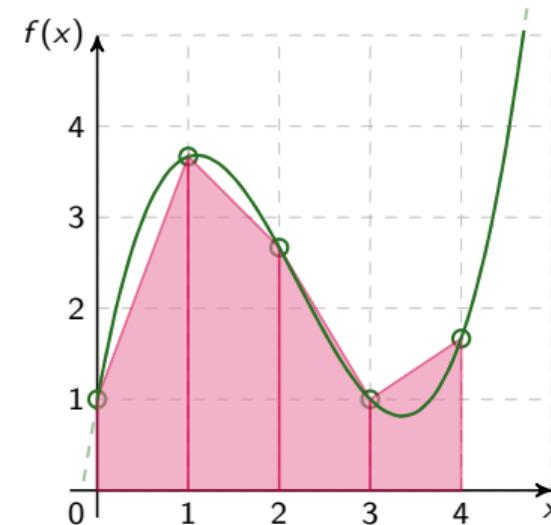
$$T_n = \frac{L_n + R_n}{2}$$

The total area is obtained by geometric reconstruction of trapezoids:

$$T_n = \sum_{i=1}^n \frac{f(x_{i+1}) + f(x_i)}{2} \Delta x_i$$

Note that this can be rewritten for equidistant intervals:

$$T_n = \frac{b-a}{2n} (f(x_0) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(x_n))$$



Error in trapezoid integration

The trapezoid rule result over n intervals T_n approximates the exact integral $I = \int_a^b f(x)dx$.
The upper-bounds of the error is given as:

$$|I - T_n| \leq \frac{f_{\max}^{(2)}(b-a)^3}{12n^2}$$

Recall that the midpoint rule approximates with an upper-bound error of

$$|I - M_n| \leq \frac{f_{\max}^{(2)}(b-a)^3}{24n^2}$$

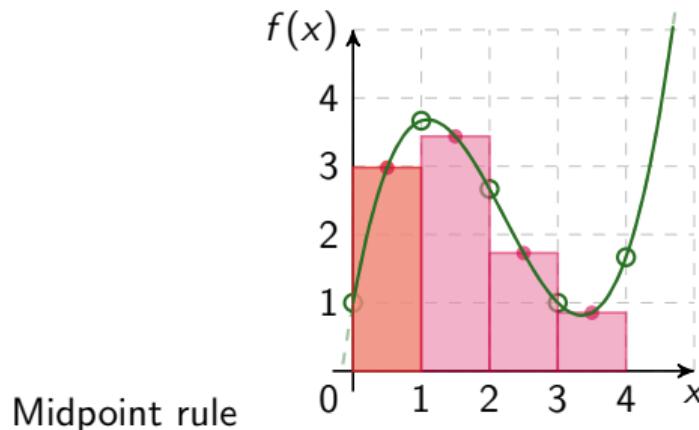
The midpoint rule approximation has lower error bounds than the trapezoid rule. A linear function is, however, better approximated by the trapezoid rule.

Today's outline

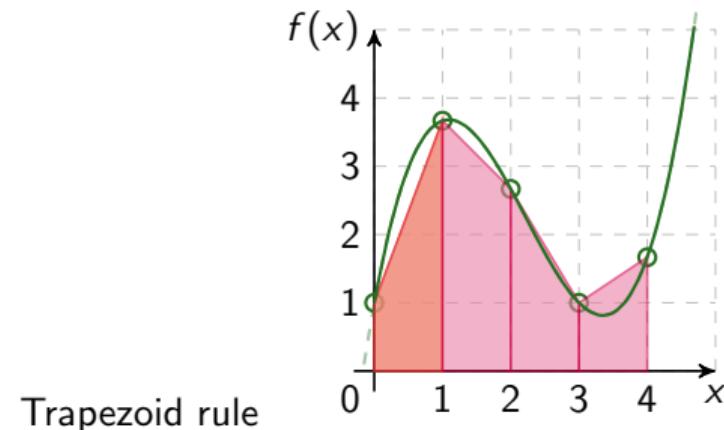
- Introduction
- Riemann integrals
- Trapezoid rule
- Simpson's rule
- Conclusion
- Tutorials

Towards higher-order integration

Compare how the midpoint and trapezoid functions behave on convex and concave parts of a graph.



Midpoint rule



Trapezoid rule

In convex parts (bending down), the midpoint rule tends to overestimate the integral (trapezoid underestimates). In concave parts (bending up), the midpoint rule tends to underestimate the integral (trapezoid overestimates).

Towards higher-order integration

The errors of the midpoint rule and trapezoid rule behave in a similar way, but have opposite signs.

- Midpoint: $|I - M_n| \leq \frac{f_{\max}^{(2)}(b-a)^3}{24n^2}$
- Trapezoid: $|I - T_n| \leq \frac{f_{\max}^{(2)}(b-a)^3}{12n^2}$

For a quadratic function, the errors relate as:

$$|I - M_n| = \frac{1}{2}|I - T_n|$$

Taking the weighted average of these two yields the Simpson's rule:

$$S_{2n} = \frac{2}{3}M_n + \frac{1}{3}T_n$$

The $2n$ means we have $2n$ subintervals: the n trapezoid intervals are subdivided by the

Simpson's rule

Consider the interval $i \in [x_0, x_2]$, subdivided in three equidistant interpolation points: x_0, x_1, x_2 .

- Midpoint: $M_i = f\left(\frac{x_0 + x_2}{2}\right)2\Delta x = f(x_1)2\Delta x$
- Trapezoid: $T_i = \frac{f(x_0) + f(x_2)}{2}2\Delta x$
- Simpson: $S_i = \frac{2}{3}M_i + \frac{1}{3}T_i$

Note that M_i and T_i were computed on interval $x_2 - x_0 = 2\Delta x$.

Now we have:

$$\begin{aligned} S_i &= \frac{2}{3}[f(x_1)2\Delta x] + \frac{1}{3}\left[\frac{f(x_0) + f(x_2)}{2}2\Delta x\right] \\ &= \frac{4\Delta x}{3}f(x_1) + \frac{\Delta x}{3}f(x_0) + f(x_2) = \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + f(x_2)) \end{aligned}$$

Simpson's rule

We write $f(x_k) = f_k$. The integral of an interval $i \in [x_0, x_2]$ is approximated as:

$$S_i = \frac{\Delta x}{3} (f_0 + 4f_1 + f_2)$$

The next interval, S_j with $j \in [x_2, x_4]$ with midpoint $x_3 = \frac{x_2+x_4}{2}$ is approximated as:

$$S_j = \frac{\Delta x}{3} (f_2 + 4f_3 + f_4)$$

If we sum these two intervals we obtain:

$$\begin{aligned} I \approx S_i + S_j &= \left[\frac{\Delta x}{3} (f_0 + 4f_1 + f_2) \right] + \left[\frac{\Delta x}{3} (f_2 + 4f_3 + f_4) \right] \\ &= \frac{\Delta x}{3} (f_0 + 4f_1 + 2f_2 + 4f_3 + f_4) \end{aligned}$$

Simpson's rule

In general, Simpson's rule can be written as:

$$\begin{aligned} \int_a^b f(x)dx &\approx \sum_{\substack{k=2 \\ k \text{ even}}}^n \frac{\Delta x}{3} (f_{k-2} + 4f_{k-1} + f_k) \\ &= \frac{\Delta x}{3} (f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + \dots + 2f_{n-2} + 4f_{n-1} + f_n) \end{aligned}$$

The error is given by:

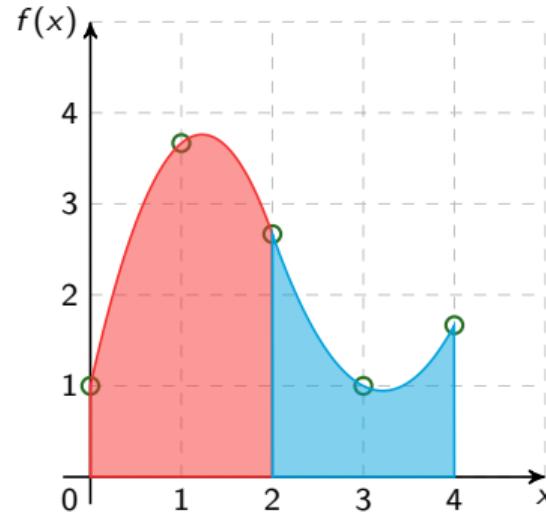
$$|I - S_n| \leq \frac{f_{\max}^{(4)}(b-a)^5}{180n^4}$$

if integrand f is differentiable on $[a, b]$.

Simpson's rule: example

Recall our example data, described by $f(x) = \frac{x^3}{2} - \frac{10x^2}{3} + \frac{11x}{2} + 1$
 $I = \int_0^4 \frac{x^3}{2} - \frac{10x^2}{3} + \frac{11x}{2} + 1 = \frac{80}{9} \approx 8.888\dots$

- Interpolating x_0, x_1 and x_2 :
 $p_{2a}(x) = -\frac{11}{6}x^2 + 4\frac{1}{2}x + 1$
 $\int_0^2 p_{2a} = \frac{55}{9} \approx 6.1111$
- Interpolating x_2, x_3 and x_4 :
 $p_{2b}(x) = \frac{7x^2}{6} - 7\frac{1}{2}x + 13$
 $\int_2^4 p_{2b} = \frac{25}{9} \approx 2.777\dots$
- Adding the separate integrals:
 $\int_0^2 p_{2a} + \int_2^4 p_{2b} = \frac{80}{9}$



Using Simp-

$$\text{son's rule: } I \approx \frac{\Delta x}{3} (f_0 + 4f_1 + 2f_2 + 4f_3 + f_4) = \frac{1}{3} (1 + 4 \cdot 3.6667 + 2 \cdot 2.6667 + 4 \cdot 1.0000 + 1.6667) = 8.88888 = \frac{80}{9}$$

Simpson's method is of fourth order, and it gives exact approximations of third order polynomials!

Integration in Matlab

Integration can be done numerically in Matlab.

- `trapz(x,y)` uses the trapezoid rule to integrate the data. Make sure you use the `x` variable if your data is not spaced with $\Delta x = 1$. Can handle non-equidistant data.

```
>> x = linspace(-2,2,2001);  
>> y = 1./(x.^2+1);  
>> I = trapz(x,y)  
I =  
    2.2143
```

- Integration of functions can be done using the `integral(fun,xmin,xmax)` function:

```
>> fun = @(x) exp(-x.^2);  
>> I = integral(fun,0,10)  
I =  
    0.886226925452758
```

Today's outline

- Introduction
- Riemann integrals
- Trapezoid rule
- Simpson's rule
- Conclusion
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What hasn't been discussed?

This course is by no means complete, and further reading is possible.

- Gaussian quadrature: A third-order integration method that requires only two base points (in contrast to the third order Simpson's method, which requires three points)
- Adaptive techniques: Parts of a function that are relatively steady (no wild oscillations) and differentiable can be integrated with much larger step sizes than other parts of the function.
- Simpson's 3/8-rule: Yet another integration technique, requiring an additional data point

Summary

- Several techniques for numerical integration were discussed:
 - Riemann sums, trapezoid rule, Simpson's rule
 - Upper-bound errors were given for each technique
 - Built-in Matlab functions were illustrated
- Continue with characterization of convergence of the integration methods in the tutorials!

Integration tutorials

- ① Implement a function to integrate a mathematical function for a specific number of integration intervals. Implement it as a function, which can be called with arguments:

- Function (handle) to integrate
- Integration boundaries (as separate arguments or as 2×1 vector)
- Number of integration intervals

For instance: `function result = leftrule(func, x0, x1, N).`

- ② Set up a function to integrate:

```
function [f] = myfunction(x)
f = x.^2 - 4*x + 6 + sin(5*x);
end
```

- ③ Integrate the function, e.g. `int_left = leftrule(@myfunction, 0, 10, 25)`
- ④ Assess how the number of intervals affects the deviation from the true integral value.
- ⑤ Create a log-log plot of the deviation vs. number of intervals used.
- ⑥ Do this for all methods discussed³ and compare their performance in a graph

Linear equations 1

Linear algebra basics

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Numerical Methods (6E5X0), 2020-2021

Today's outline

● Introduction

● Matrix inversion

● Solving a linear system

● Towards larger systems

● Summary

Overview

Goals

- Different ways of looking at a system of linear equations
- Determination of the inverse, determinant and the rank of a matrix
- The existence of a solution to a set of linear equations

Different views of linear systems

- Separate equations:

$$x + y + z = 4$$

$$2x + y + 3z = 7$$

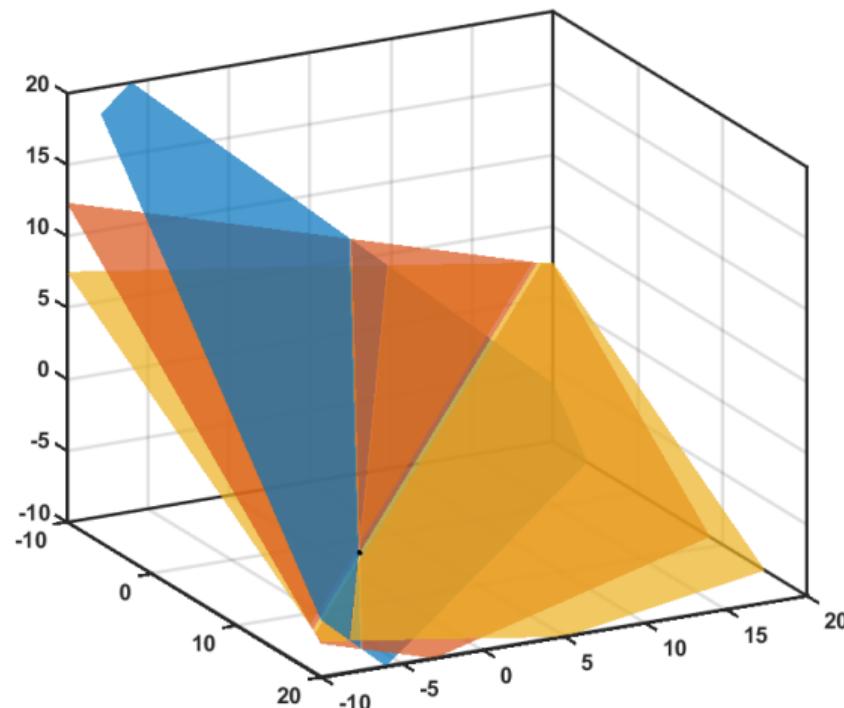
$$3x + y + 6z = 5$$

- Matrix mapping $Mx = b$:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix}$$

- Linear combination:

$$x \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + z \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix}$$



Today's outline

- Introduction

- Matrix inversion

- Solving a linear system

- Towards larger systems

- Summary

Inverse of a matrix

- The inverse M^{-1} is defined such that:

$$MM^{-1} = I \quad \text{and} \quad M^{-1}M = I$$

- Use the inverse to solve a set of linear equations:

$$Mx = \mathbf{b}$$

$$M^{-1}Mx = M^{-1}\mathbf{b}$$

$$Ix = M^{-1}\mathbf{b}$$

$$x = M^{-1}\mathbf{b}$$

How to calculate the inverse?

- The inverse of an $N \times N$ matrix can be calculated using the co-factors of each element of the matrix:

$$M^{-1} = \frac{1}{\det|M|} \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

- $\det|M|$ is the *determinant* of matrix M .
- C_{ij} is the *co-factor* of the ij^{th} element in M .

Computing the co-factors

Consider the following example matrix: $M = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{bmatrix}$

A co-factor (e.g. C_{11}) is the determinant of the elements left over when you cover up the row and column of the element in question, multiplied by ± 1 , depending on the position.

$$\begin{bmatrix} 1 & \times & \times \\ \times & 1 & 3 \\ \times & 1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$C_{11} = +1 \cdot \det \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} \\ = 6 \times 1 - 3 \times 1 = 3$$

Computing the co-factors

Back to our example:

$$M^{-1} = \frac{1}{\det|M|} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{bmatrix}^{-1} = \frac{1}{\det|M|} \begin{bmatrix} 3 & -3 & -1 \\ -5 & 3 & 2 \\ 2 & -1 & -1 \end{bmatrix}^T$$

- The determinant is very important
- If $\det|M| = 0$, the inverse does not exist (singular matrix)

Calculating the determinant

Compute the determinant by multiplication of each element on a row (or column) by its cofactor and adding the results:

$$\det \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{vmatrix} = +\det \begin{vmatrix} 1 & 3 \\ 1 & 6 \end{vmatrix} - \det \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} + \det \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1$$

$$\det \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{vmatrix} = +\det \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} - 3\det \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} + 6\det \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1$$

Today's outline

- Introduction
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Solving a linear system

- Our example:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix}$$

- The solution is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = M^{-1}b = \frac{1}{-1} \begin{bmatrix} 3 & -5 & 2 \\ -3 & 3 & -1 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix} = \frac{1}{-1} \begin{bmatrix} -13 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 13 \\ -4 \\ -5 \end{bmatrix}$$

- The inverse exists, because $\det|M| = -1$.

Solving a linear system in Matlab using the inverse

- Create the matrix:

```
>> A = [1 1 1; 2 1 3; 3 1 6];
```

- Create solution vector:

```
>> b = [4; 7; 5];
```

- Get the matrix inverse:

```
>> Ainv = inv(A);
```

- Compute the solution:

```
>> x = Ainv * b
```

- Matlab's internal direct solver:

```
>> x = A\b
```

Exercise: performance of inverse computation

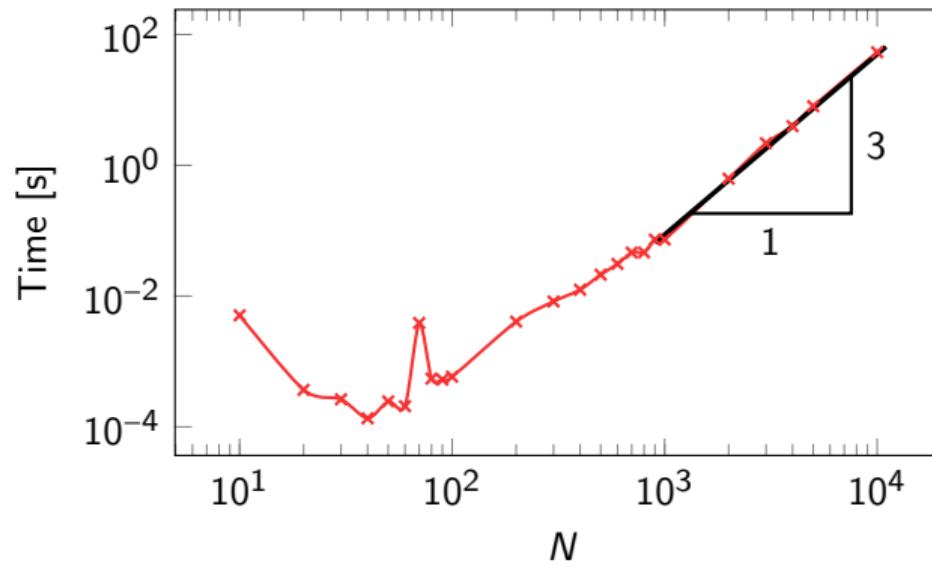
Create a script that generates matrices with random elements of various sizes $N \times N$. Compute the inverse of each matrix, and use `tic` and `toc` to see the computing time for each inversion. Plot the time as a function of the matrix size N .

Hints:

- Create an array that contains the sizes of the systems n
 - Loop over the array elements to:
 - Create a random matrix of size $n \times n$
 - Perform the matrix inversion
 - Record the time required
 - Plot the time required for inversion vs size of the system on a double-log scale
-

Exercise: sample results

Each computer produces slightly different results because of background tasks, different matrices, etc. This is especially noticeable for small systems.



The time increases by 3 orders of magnitude, for every magnitude in N . The *computational complexity* of matrix inversion scales with $\mathcal{O}(N^3)$!

Solving a linear system in Excel using the inverse

$$Ax = b \quad \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 3 & 1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 5 \end{bmatrix}$$

- Create matrix A in 3×3 cells
- Create right hand side vector b in 3 vertical cells
- Compute the inverse I :
 - Select an empty area of 3×3 cells
 - Type `=MINVERSE(B2:D4)` (In Dutch Excel: `INVERSEMAT`)
 - Close with `Ctrl+Shift+Enter`
- Solution:
 - Select 3 vertical cells
 - Type `=MMULT(H2:J4 ; B6:B8)` (In Dutch Excel: `PRODUCTMAT`. The semicolon may be a comma.)
 - Close with `Ctrl+Shift+Enter`

Today's outline

- Introduction
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Towards larger systems

Computation of determinants and inverses of large matrices in this way is too difficult (slow), so we need other methods to solve large linear systems!

Towards larger systems

- Determinant of upper triangular matrix:

$$\det|M_{\text{tri}}| = \prod_{i=1}^n a_{ii} \quad M = \begin{bmatrix} 5 & 3 & 2 \\ 0 & 9 & 1 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \det|M| = 5 \times 9 \times 1 = 45$$

- Matrix multiplication:

$$\det|AM| = \det|A| \times \det|M|$$

- When A is an identity matrix ($\det|A| = 1$):

$$\det|AM| = \det|A| \times \det|M| = 1 \times \det|M|$$

- With rules like this, we can use row-operations so that we can compute the determinant more cheaply.

Solutions of linear systems

Rank of a matrix: the number of linearly independent columns (columns that can not be expressed as a linear combination of the other columns) of a matrix.

$$M = \begin{bmatrix} 5 & 3 & 2 \\ 0 & 9 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- 3 independent columns
- In Matlab:

```
>> rank(M)
```

- col 2 = 2 × col 1
- col 4 = col 3 - col 1
- 2 independent columns: rank = 2

Solutions of linear systems

The solution of a system of linear equations may or may not exist, and it may or may not be unique. Existence of solutions can be determined by comparing the rank of the Matrix M with the rank of the augmented matrix M_a :

```
>> rank(A)
>> rank([A b])
```

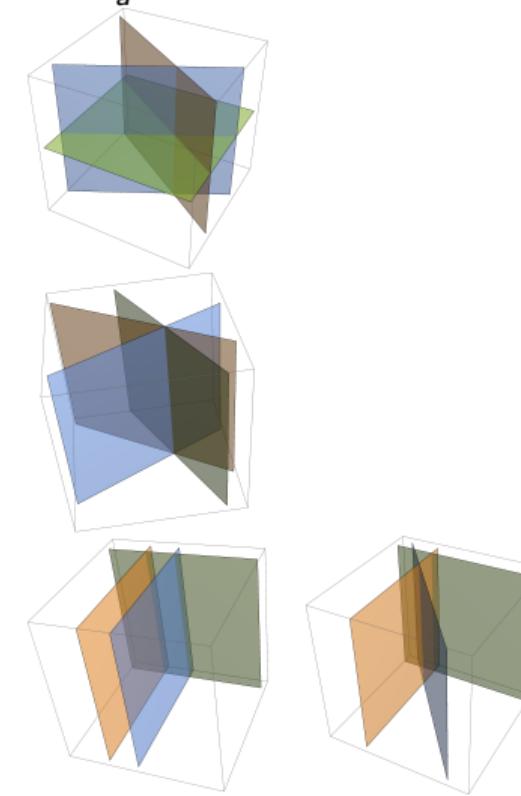
Our system: $Mx = b$

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow M_a = \begin{bmatrix} M_{11} & M_{12} & M_{13} & b_1 \\ M_{21} & M_{22} & M_{23} & b_2 \\ M_{31} & M_{32} & M_{33} & b_3 \end{bmatrix}$$

Existence of solutions for linear systems

For a matrix M of size $n \times n$, and augmented matrix M_a :

- $\text{Rank}(M) = n$:
Unique solution
- $\text{Rank}(M) = \text{Rank}(M_a) < n$:
Infinite number of solutions
- $\text{Rank}(M) < n$, $\text{Rank}(M) < \text{Rank}(M_a)$:
No solutions



Two examples

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad b = \begin{bmatrix} 17 \\ 11 \\ 4 \end{bmatrix} \Rightarrow M_a = \begin{bmatrix} 1 & 1 & 2 & 17 \\ 0 & 3 & 1 & 11 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

$\text{rank}(M) = 3 = n \Rightarrow$ Unique solution

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 17 \\ 11 \\ 0 \end{bmatrix} \Rightarrow M_a = \begin{bmatrix} 1 & 1 & 2 & 17 \\ 0 & 3 & 1 & 11 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(M) = \text{rank}(M_a) = 2 < n \Rightarrow$ Infinite number of solutions

Today's outline

● Introduction

● Matrix inversion

● Solving a linear system

● Towards larger systems

● Summary

Summary

- Linear equations can be written as matrices
- Using the inverse, the solution can be determined
 - Inverse via cofactors
 - Inverse and solution in Matlab
 - Inverse and solution in Excel
- Introduced the concept of computational complexity: matrix inversion scales with N^3
- A solution depends on the rank of a matrix

Linear equations 2

Direct methods

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Eindhoven University of Technology

Numerical Methods (6E5X0), 2020-2021

Today's outline

● Introduction

● Gauss elimination

● Partial Pivoting

● LU decomposition

● Summary

Overview

Goals

Today we are going to write a program, which can solve a set of linear equations

- The first method is called Gaussian elimination
- We will encounter some problems with Gaussian elimination
- Then LU decomposition will be introduced

Today's outline

- Introduction
- Gauss elimination
- Partial Pivoting
- LU decomposition
- Summary

Define the linear system

Consider the system:

$$Ax = b$$

In general:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Desired solution:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_2 \\ b'_3 \end{bmatrix}$$

Using row operations

- Use row operations to simplify the system. Eliminate element A_{21} by subtracting $A_{21}/A_{11} = d_{21}$ times row 1 from row 2.
- In this case, Row 1 is the pivot row, and A_{11} is the pivot element.

$$\left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ A_{21} & A_{22} & A_{23} & b_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ 0 & A'_{22} & A'_{23} & b'_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{array} \right]$$

Using row operations

Eliminate element A_{21} using $d_{21} = A_{21}/A_{11}$.

$$\left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ A_{21} & A_{22} & A_{23} & b_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ 0 & A'_{22} & A'_{23} & b'_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{array} \right]$$

- $d_{21} \rightarrow A_{21}/A_{11}$
- $A_{21} \rightarrow A_{21} - A_{11}d_{21}$
- $A_{22} \rightarrow A_{22} - A_{12}d_{21}$
- $A_{23} \rightarrow A_{23} - A_{13}d_{21}$
- $b_2 \rightarrow b_2 - b_1 d_{21}$

```
d21 = A(2,1)/A(1,1);
A(2,1) = A(2,1) - A(1,1)*d21;
A(2,2) = A(2,2) - A(1,2)*d21;
A(2,3) = A(2,3) - A(1,3)*d21;
b(2) = b(2) - b(1)*d21;
```

Using row operations

Eliminate element A_{31} using $d_{31} = A_{31}/A_{11}$.

$$\left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ 0 & A'_{22} & A'_{23} & b'_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ 0 & A'_{22} & A'_{23} & b'_2 \\ 0 & A'_{32} & A'_{33} & b'_3 \end{array} \right]$$

- $d_{31} \rightarrow A_{31}/A_{11}$
- $A_{31} \rightarrow A_{31} - A_{11}d_{31}$
- $A_{32} \rightarrow A_{32} - A_{12}d_{31}$
- $A_{33} \rightarrow A_{33} - A_{13}d_{31}$
- $b_3 \rightarrow b_3 - b_1 d_{31}$

```
d31 = A(3,1)/A(1,1);
A(3,1) = A(3,1) - A(1,1)*d31;
A(3,2) = A(3,2) - A(1,2)*d31;
A(3,3) = A(3,3) - A(1,3)*d31;
b(3) = b(3) - b(1)*d31;
```

Using row operations

Eliminate element A_{32} using $d_{32} = A_{32}/A'_{22}$. Note that now the second row has become the pivot row.

$$\left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ 0 & A'_{22} & A'_{23} & b'_2 \\ 0 & A_{32} & A_{33} & b_3 \end{array} \right] \longrightarrow \left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ 0 & A'_{22} & A'_{23} & b'_2 \\ 0 & 0 & A''_{33} & b''_3 \end{array} \right]$$

- $d_{32} \rightarrow A_{32}/A'_{22}$
- $A_{32} \rightarrow A_{32} - A'_{22}d_{32}$
- $A_{33} \rightarrow A_{33} - A'_{23}d_{32}$
- $b_3 \rightarrow b_3 - b'_2 d_{32}$

```
d32 = A(3,2)/A(2,2);  
A(3,2) = A(3,1) - A(2,2)*d32;  
A(3,3) = A(3,2) - A(2,3)*d32;  
b(3) = b(3) - b(2)*d32;
```

The matrix is now a triangular matrix — the solution can be obtained by back-substitution.

Backsubstitution

The system now reads:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A'_{22} & A'_{23} \\ 0 & 0 & A''_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \end{bmatrix}$$

Start at the last row N , and work upward until row 1.

$$x_3 = b''_3 / A''_{33}$$

$$x_2 = (b'_2 - A'_{23}x_3) / A'_{22}$$

$$x_1 = (b_1 - A_{12}x_2 - A_{13}x_3) / A_{11}$$

```
x(3) = b(3) / A(3,3)
x(2) = (b(2) - A(2,3)*x(3)) / A(2,2)
x(1) = (b(1) - A(1,2)*x(2) - A(1,3)*x(3)) / A(1,1)
```

In general:

$$x_N = \frac{b_N}{A_{NN}} \quad x_i = \frac{b_i - \sum_{j=i+1}^N A_{ij}x_j}{A_{ii}}$$

Writing the program

- Create a function that provides the framework: take matrix A and vector b as an input, and return the solution x as output:

```
function [x,A,b] = GaussianEliminate(A,b)
```

- We will use *for-loops* instead of typing out each command line.
- Useful Matlab shortcuts:
 - $A(1,:) = [A_{11}, A_{12}, A_{13}]$
 - $A(:,2) = [A_{12}, A_{22}, A_{32}]^T$
 - $A(1,2:end) = [A_{12}, A_{13}]$
- A row operation could look like:

```
A(i,:) = A(i,:) - d*A(1,:)
```

The program: elimination

```
function [x,A,b] = GaussianEliminate(A,b)

% Perform elimination to obtain an upper triangular matrix
N = length(b);
for column=1:(N-1) % Select pivot
    for row=(column+1):N % Loop over subsequent rows (below pivot)
        d=A(row,column)/A(column,column);
        A(row,:)=A(row,:)-d*A(column,:);
        b(row)= b(row)-d*b(column);
    end
end
```

The program: Backsubstitution

```
% Assign b to x
x=b;

% Perform backsubstitution
for row=N:-1:1
    x(row) = b(row);
    for i =(row+1):N
        x(row)=x(row)-A(row,i)*x(i);
    end
    x(row)=x(row)/A(row,row);
end
```

$$x_N = \frac{b_N}{A_{NN}} \quad x_i = \frac{b_i - \sum_{j=i+1}^N A_{ij}x_j}{A_{ii}}$$

Exercise: Gaussian Elimination

- The function we just made can be found on Canvas
- Use `help GaussianEliminate` to find out how it works
- Solve the following system of equations:

$$\begin{bmatrix} 9 & 9 & 5 & 2 \\ 6 & 7 & 1 & 3 \\ 6 & 4 & 3 & 5 \\ 2 & 6 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7 \\ 4 \\ 10 \\ 1 \end{bmatrix}$$

- Compare your solution with `A\b`

Today's outline

- Introduction
- Gauss elimination
- Partial Pivoting
- LU decomposition
- Summary

Partial pivoting

- Now try to run the algorithm with the following system:

$$\begin{bmatrix} 0 & 2 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 10 \end{bmatrix}$$

- It does not work! Division by zero, due to $A_{11} = 0$.
- Solution: Swap rows to move largest element to the diagonal.

Partial pivoting: implementing row swaps

- Find maximum element row below pivot in current column
- Store current row
- Swap pivot row and desired row in A
- Do the same for b: store and swap

```
[dummy, index] = max(abs(A(column:end, column)));
Index = index+column-1;
```

```
temp = A(column,:);
```

```
A(column,:) = A(index,:);
A(index,:) = temp;
```

```
temp = b(column);
b(column) = b(index);
b(index) = temp;
```

Improve the program by using re-usable functions

```
function [x] = GaussianEliminate(A,b)
% GaussianEliminate(A,b): solves x in Ax=b
N = length(b);
for c=1:(N-1)
    [dummy , index]=max(abs(A(c:end ,c)));
    index=index+c-1;
    A = SWAP(A,c,index); % Created swap function
    b = SWAP(b,c,index);
    for row=(column+1):N
        d=A(row,column)/A(column,column);
        A(row,:)=A(row,:)-d*A(column,:);
        b(row)= b(row)-d*b(column);
    end
end
x = backwardSub(A,b); % Created BS function
return
```

This function is also provided (named GaussianEliminate_v2 and GaussianEliminate_v3 on Canvas).

Alternatives to this program

- MATLAB can compute the solution to $Ax=b$ with its own solvers (more efficient) $\backslash\backslash b$
- Too many loops. Loops make MATLAB slow.
- There are fundamental problems with Gaussian elimination
 - You can add a counter to the algorithm to see how many subtraction and multiplication operations it performs for a given size of matrix A.
 - The number of operations to perform Gaussian elimination is $\mathcal{O}(2N^3)$ (where N is the number of equations)
 - Exercise: verify this for our script
 - LU decomposition takes $\mathcal{O}(2N^3/3)$ flops, 3 times less!
 - Forward and backward substitution each take $\mathcal{O}(N^2)$ flops (both cases)

Today's outline

- Introduction
- Gauss elimination
- Partial Pivoting
- LU decomposition
- Summary

LU Decomposition

Suppose we want to solve the previous set of equations, but with several right hand sides:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} \vdots & \vdots & \vdots \\ x_1 & x_2 & x_3 \\ \vdots & \vdots & \vdots \end{bmatrix} = \begin{bmatrix} \vdots & \vdots & \vdots \\ b_1 & b_2 & b_3 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

Factor the matrix A into two matrices, L and U, such that $A = LU$:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \times & 1 & 0 \\ \times & \times & 1 \end{bmatrix} \begin{bmatrix} \times & \times & \times \\ 0 & \times & \times \\ 0 & 0 & \times \end{bmatrix}$$

Now we can solve for each right hand side, using only a forward followed by a backward substitution!

Substitutions

- Define a lower and upper matrix L and U so that $A = LU$
- Therefore $LUX = b$
- Define a new vector $y = UX$ so that $Ly = b$
- Solve for y , use L and forward substitution
- Then we have y , solve for x , use $UX = y$
- Solve for x , use U and backward substitution
- But how to get L and U?

Decomposing the matrix (1)

When we eliminate the element A_{21} we can keep multiplying by a matrix that undoes this row operations, so that the product remains equal to A .

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ d_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} - d_{21}A_{12} & A_{23} - d_{21}A_{13} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

Decomposing the matrix (2)

When we eliminate the element A_{31} we can keep multiplying by a matrix that undoes this row operations, so that the product remains equal to A .

$$A = \begin{bmatrix} 1 & 0 & 0 \\ d_{21} & 1 & 0 \\ d_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A'_{22} = A_{22} - d_{21}A_{12} & A'_{23} = A_{23} - d_{21}A_{13} \\ 0 & A'_{32} = A_{32} - d_{31}A_{12} & A'_{33} = A_{33} - d_{31}A_{12} \end{bmatrix}$$

Decomposing the matrix (3)

When we eliminate the element A_{32} we can keep multiplying by a matrix that undoes this row operations, so that the product remains equal to A .

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ d_{21} & 1 & 0 \\ d_{31} & d_{32} & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A'_{22} & A'_{23} \\ 0 & A'_{32} & A''_{33} = A'_{33} - d_{32}A'_{23} \end{bmatrix}$$

This finishes the LU decomposition!

Pivoting during decomposition

Suppose we have arrived at the situation below, where $A'_{32} > A'_{22}$:

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ d_{21} & 1 & 0 \\ d_{31} & 0 & 1 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A'_{22} & A'_{23} \\ 0 & A'_{32} & A'_{33} \end{bmatrix}$$

Exchange rows 2 and 3 to get the largest value on the main diagonal. Use a permutation matrix to store the swapped rows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ d_{31} & 0 & 1 \\ d_{21} & 1 & 0 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A'_{22} & A'_{23} \\ 0 & A'_{32} & A'_{33} \end{bmatrix}$$

Multiplying with a permutation matrix will swap the rows of a matrix. The permutation matrix is just an identity matrix, whose rows have been interchanged.

Recipe for LU decomposition

When decomposing matrix A into $A = LU$, it may be beneficial to swap rows to get the largest values on the diagonal of U (pivoting). A permutation matrix P is used to store row swapping such that:

$$PA = LU$$

- Write down a permutation matrix and the linear system
- Promote the largest value in the column diagonal
- Eliminate all elements below diagonal
- Move on to the next column and move largest elements to diagonal
- Eliminate elements below diagonal
- Repeat 5 and 6
- Write down L,U and P

LU decomposition example (1)

Write down a permutation matrix, which starts as the identity matrix, and the linear system:

$$PA = LU$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

Promote the largest value into the diagonal of column 1 — swap row 1 and 2:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$$

LU decomposition example (2)

Eliminate all elements below the diagonal — row 2 already contains a zero in column 1, row 3 = row 3 - 0.5 row 1. Record the multiplier 0.5 in L :

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1.5 & -0.5 \end{bmatrix}$$

Elimination of column 1 is done. Step to the next column, and move the largest value below/on the diagonal to the diagonal (swap rows 2 and 3). Adjust P and lower triangle of L accordingly:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1.5 & -0.5 \\ 0 & 1 & 1 \end{bmatrix}$$

LU decomposition example (3)

Eliminate all elements below the diagonal —

row 3 = row 3 - $\frac{2}{3}$ row 2. Record the multiplier $\frac{2}{3}$ in L:

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1.5 & -0.5 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

We have obtained the matrices from $PA = LU$:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1.5 & -0.5 \\ 0 & 0 & \frac{4}{3} \end{bmatrix}$$

Proceed with solving for x.

Substitutions

$$Ax = b \Rightarrow PAx = Pb \equiv d$$

$$PA = LU \Rightarrow LUx = d$$

- Define a new vector $y = Ux$
 - $Ly = b \Rightarrow Ly = d$
 - Solve for y , forward substitution:

$$y_1 = \frac{d_1}{L_{11}}$$

$$y_i = \frac{d_i - \sum_{j=1}^{i-1} L_{ij}y_j}{L_{ii}}$$

- Then solve $Ux = y$:
 - Solve for x , backward substitution:

$$x_N = \frac{y_N}{U_{NN}}$$

$$x_i = \frac{y_i - \sum_{j=i+1}^{N-1} U_{ij}x_j}{U_{ii}}$$

How to use the solver in Matlab

```
A = rand(5,5); % Get random matrix
[L, U, P] = lu(A); % Get L, U and P
b = rand(5,1); % Random b vector
d = P*b; % Permute b vector
y = forwardSub(L,d); % Can also do y=L\d
x = backwardSub(U,y); % Can also do x=U\y
rnorm = norm(A*x - b); % Residual

% Compare results to internal Matlab solver
x = A\b
```

- Use this as a basis to create a function that takes A and b , and returns x .
- Use the function to check the performance for various matrix sizes and inspect the performance.

Today's outline

- Introduction
- Gauss elimination
- Partial Pivoting
- LU decomposition
- Summary

Summary

- This lecture covered direct methods using elimination techniques.
- Gaussian elimination can be slow ($\mathcal{O}(N^3)$)
- Back substitution is often faster ($\mathcal{O}(N^2)$)
- LU decomposition means that we don't have to do Gaussian elimination every time (saves time and effort), but the matrix has to stay the same.
- Matlab has build in routines for solving linear equations (backslash operator \) and LU decomposition ([lu](#)).
- Advanced techniques such as (preconditioned) conjugate gradient or biconjugate gradient solvers are also available.
- Next part covers iterative approaches

Linear equations 3

Iterative methods

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Numerical Methods (6E5X0), 2020-2021

Today's outline

- Introduction
 - Sparse matrices
 - Laplace's equation
 - Creating a sparse system
 - Iterative methods
 - Summary

Sparse matrices

- In many engineering cases, we deal with sparse matrices (as opposed to dense matrices)
 - A matrix is sparse when it mostly consists of zeros
 - Linear systems where equations depend on a limited number of variables (e.g. spatial discretization)
 - Storing zeros is not very efficient:

```
>> A = eye(10000);  
>> whos A  
>> S = sparse(A);  
>> whos S
```

- Can you think of a way to achieve this?
 - Sparse matrix formats: Yale, CRS, CCS

Sparse matrix storage format

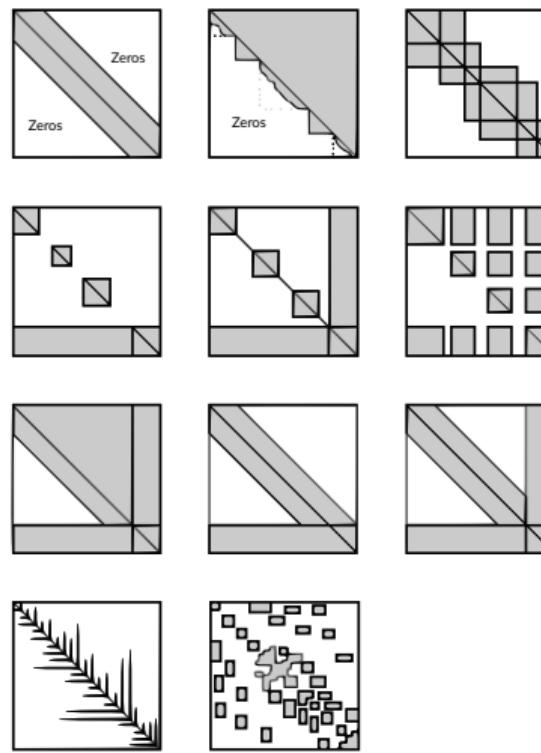
- Example: Yale storage format, storing 3 vectors

- $A = [5 \ 8 \ 3 \ 6]$
 - $IA = [0 \ 1 \ 2 \ 3 \ 4]$
 - $JA = [0 \ 1 \ 2 \ 1]$

- A stores the non-zero values
 - IA stores the index in A of the first non-zero in row i
 - JA stores the column index
 - Note: zero-based indices are used here!

$$A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix}$$

Sparse matrix layout examples



Today's outline

- Introduction
 - Sparse matrices
 - Laplace's equation
 - Creating a sparse system
 - Iterative methods
 - Summary

Laplace's equation

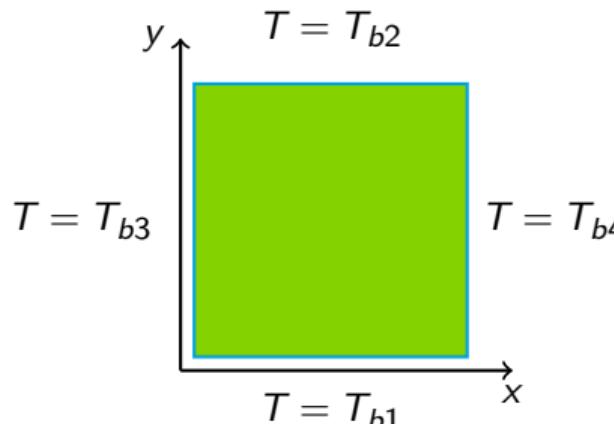
$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

T = Temperature

α = Thermal diffusivity

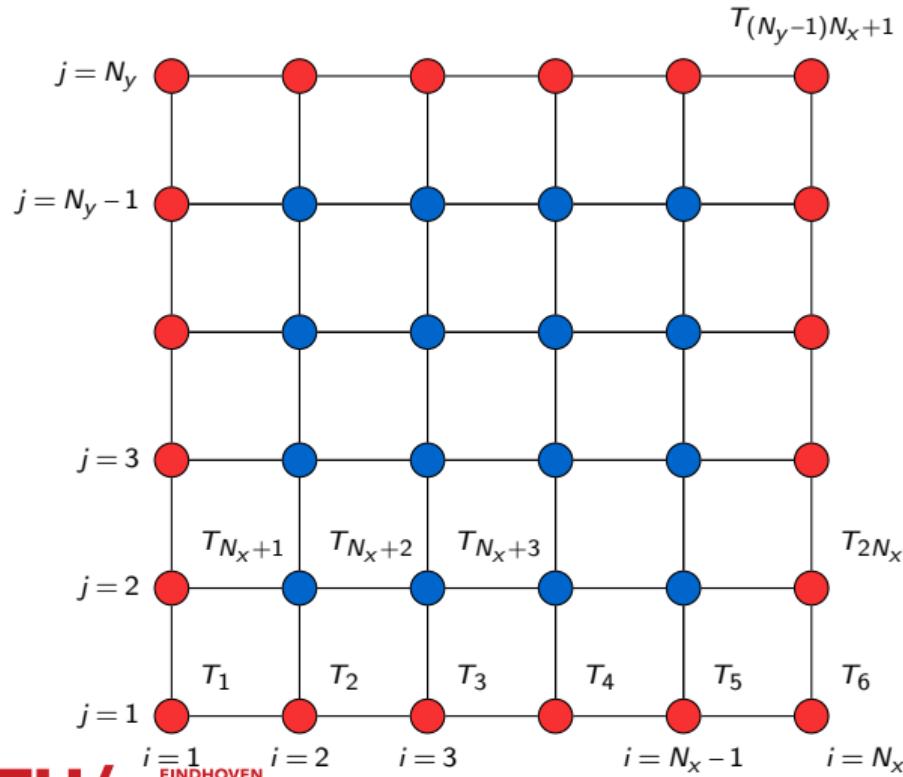
In steady state:

$$\nabla^2 T = 0$$



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

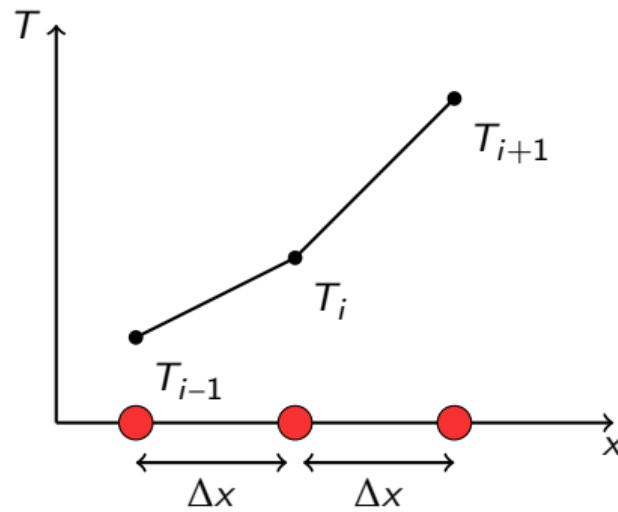
Discretization of Laplace's equation (I)



- Define a grid of points in x and y
- Index of the grid points using 2D coordinates i and j
- Set up the equations using a 1D index system:
$$T_{i,j} = T_{i+N_x(j-1)}$$

Discretization of Laplace's equation (II)

Estimate the second-order differentials: assume a piece-wise linear profile in the temperature:



$$\frac{\partial^2 T}{\partial x^2} \approx \frac{\frac{\partial T}{\partial x}\Big|_{i+\frac{1}{2}} - \frac{\partial T}{\partial x}\Big|_{i-\frac{1}{2}}}{\Delta x}$$

$$\approx \frac{\frac{(T_{i+1,j} - T_{i,j})}{\Delta x} - \frac{(T_{i,j} - T_{i-1,j})}{\Delta x}}{\Delta x}$$

$$= \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2}$$

Discretization of Laplace's equation (III)

The y -direction is derived analogously, so that the 2D Laplace's equation is discretized as:

$$\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2} + \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2} = 0$$

Use a single index counter $k = i + N_x(j-1)$, so that the equation becomes:

$$\frac{T_{k+1} - 2T_k + T_{k-1}}{(\Delta x)^2} + \frac{T_{k+N_x} - 2T_k + T_{k-N_x}}{(\Delta y)^2} = 0$$

For an equal spaced grid $\Delta x = \Delta y = 1$:

$$T_{k-N_x} + T_{k-1} - 4T_k + T_{k+1} + T_{k+N_x} = 0$$

$$\Rightarrow AT = b$$

Today's outline

- Introduction
 - Sparse matrices
 - Laplace's equation
 - Creating a sparse system
 - Iterative methods
 - Summary

Creating the linear system

$$T_{k-N_x} + T_{k-1} - 4T_k + T_{k+1} + T_{k+N_x} = 0$$

Create a *banded* matrix A : the main diagonal k contains -4, whereas the bands at $k-1$, $k+1$, $k-N_x$ and $k+N_x$ contain a 1. Boundary cells just contain a 1 on the main diagonal so that the temperature is equal to T_b (e.g. $T_1 = 1T_b$).

$$\left[\begin{array}{ccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \cdots & 1 & \cdots & 1 & -4 & 1 & \cdots & 1 & \ddots & 0 \\ 0 & \cdots & 1 & \cdots & 1 & -4 & 1 & \cdots & 1 & \vdots \\ \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right] \left[\begin{array}{c} T_1 \\ T_2 \\ \vdots \\ T_k \\ T_{k+1} \\ \vdots \\ T_{(N_y-1)N_x} \\ T_{(N_y-1)N_x+1} \end{array} \right] = \left[\begin{array}{c} T_b \\ T_b \\ \vdots \\ 0 \\ 0 \\ \vdots \\ T_b \\ T_b \end{array} \right]$$

Creating the linear system

$$T_{k-N_x} + T_{k-1} - 4T_k + T_{k+1} + T_{k+N_x} = 0$$

Create a *banded* matrix A in Matlab, by setting the coefficients for the internal cells:

```

Nx=5; %number of points along x direction
Ny=5; %number of points in the y direction
Nc=Nx*Ny; % Total number of points

e = ones(Nc,1);
A = spdiags([e,e,-4*e,e,e],[-Nx,-1,0,1,Nx],Nc,Nc);
b = zeros(Nc,1);

```

The function `spdiags` uses the following arguments:

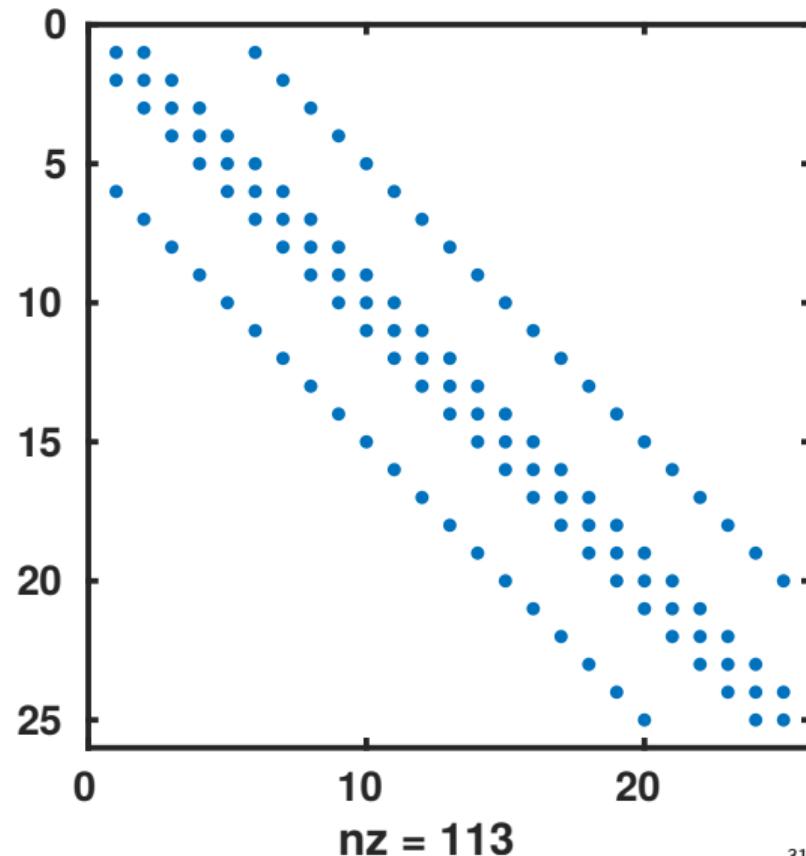
- The coefficients that have to be put on the diagonals arranged as columns in a matrix
 - The position of the bands with respect to the main diagonal
 - Size of the resulting matrix (in our case square $N_x N_y \times N_x N_y$)

Matrix sparsity

- Let's check the matrix layout:

```
>> spy(A)
```

- This command shows the non-zero values of a matrix
 - Apart from the main diagonal, there are offset bands!



About boundary conditions

- For the nodes on the boundary, we have a simple equation:

$T_{k,\text{boundary}}$ = Some fixed value

- However, we have set all nodes to be a function of their neighbors...
 - Find the boundary node indices using $k = i + Nx(j - 1)$
 - $i = 1, j = 1:Ny$
 - $i = Nx, j = 1:Ny$
 - $j = 1, i = 1:Nx$
 - $j = Ny, i = 1:Nx$
 - Reset the row in A to zeros, set $A_{kk} = 1$
 - Set value in rhs: $b_k = T_{k,\text{boundary}}$
 - Boundary conditions are often more elaborate to implement! See `setBoundaryConditions.m`.

Partial implementation of the boundary conditions

See `setBoundaryConditions.m`.

```

function [A,b] = setBoundaryConditions(A,b,Tb,Nx,Ny)

% Set boundary conditions over x-direction
for i=1:Nx
    j = 1;
    ind = i + Nx * (j-1);
    A(ind,:) = 0; % Reset matrix for boundary cells
    A(ind,ind) = 1; % Add a 1 on the diagonal
    b(ind) = Tb(1);
    j = Ny;
    ind = i + Nx * (j-1);
    A(ind,:) = 0; % Reset matrix for boundary cells
    A(ind,ind) = 1; % Add a 1 on the diagonal
    b(ind) = Tb(2);
end

%% Repeat for v-direction

```

How applying boundary conditions affects the linear system

```
function [A, b] = setBoundaryConditions(A, b, Tb, Nx, Ny)
```

- Make sure that matrix A and right hand side vector b are in your workspace, as well as N_x and N_y
- Create a vector that holds the temperature at each boundary:

```
>> T = [10 20 30 40];
```

- Call the function, store A and b in new variables:

```
>> [A2, b2] = setBoundaryConditions(A, b, T, Nx, Ny);
```

- Check the new structure of the matrix and the right hand side:

```
>> subplot(1,2,1); spy(A2);
>> subplot(1,2,2); spy(b2);
```

A full program, including solver

The program and auxiliary functions are on Canvas (`solveLaplaceEq.m`)

```
function [x,y,T,A] = solveLaplaceEq(Nx,Ny)
% Solves the steady-state Laplace equation

Tb = [10 20 30 40]; % Fixed boundary temperatures

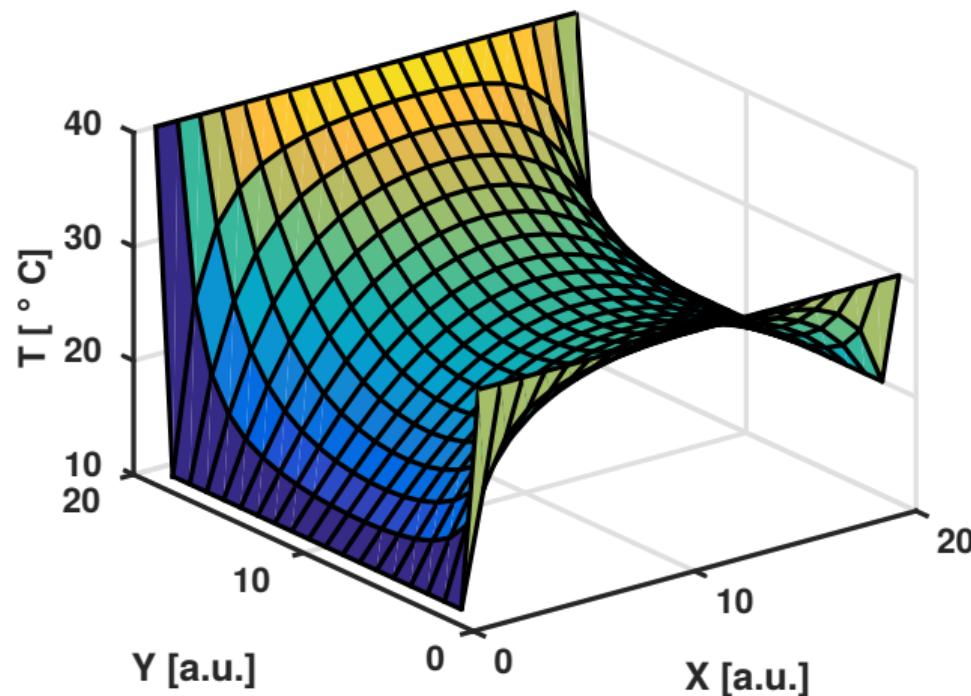
% Fill sparse matrix with [1 1 -4 1 1]
e = ones(Nx*Ny,1);
A = spdiags([e,e,-4*e,e,e],[-Nx,-1,0,1,Nx],Nx*Ny,Nx*Ny);
b = zeros(Nx*Ny,1);

[A,b] = setBoundaryConditions(A,b,Tb,Nx,Ny);

T = A\b; % Solve matrix
Tc = reshape(T,[Nx,Ny]); % Reshape x-vec to mat Nx,Ny
[xc yc] = meshgrid(1:Nx,1:Ny); % Get position arrays
surf(xc,yc,Tc); % Surface plot
```

Sample results

Solved for a 20×20 system with $T_b = [10 \ 20 \ 30 \ 40]$.



Exercise: Verify the numerical solution using Fourier-series

A Fourier-series expansion for the steady-state heat conduction in a flat plate is given for a domain: $x, y \in [0, 1]$, with fixed-temperature boundaries $T|_{x=0} = T|_{x=1} = T|_{y=0} = 0$ and $T|_{y=1} = 1$:

$$T = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin(m\pi x)\sinh(m\pi y)}{m\sinh(m\pi)} \quad \text{with} \quad m = 2n - 1$$

Compute and plot the exact temperature profile in the 2D plate, and compare it with the numerical solution:

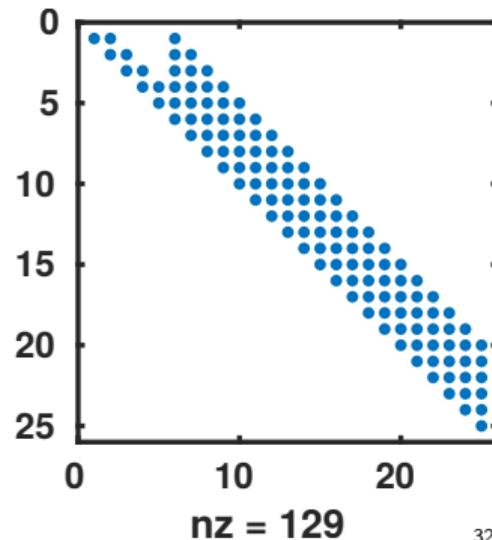
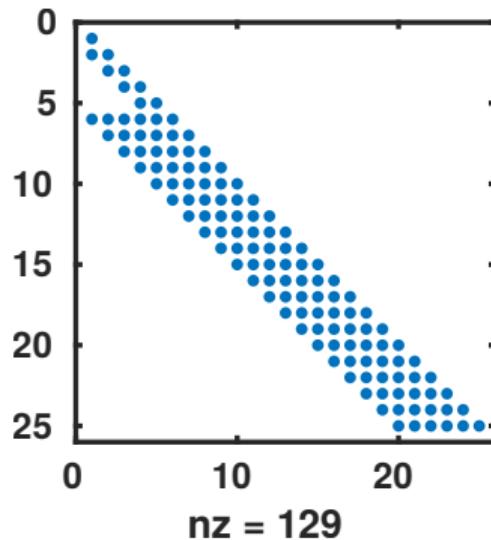
Hints:

- Use meshgrid to create a mesh in x and y
- Compute the temperature using the Fourier series, use vectorised computations over x and y so that only 1 loop (over n) is required.
- Solve the numerics for the same problem (note the boundary conditions)
- Compare the numerical and exact solutions (e.g. a plot).

LU decomposition of a sparse matrix

- With LU decomposition we produce matrices that are less sparse than the original matrix.
- Sparse storage often required, and also numerical techniques that fully utilizes this!

```
>> [L,U,P] = lu(A)
>> subplot(1,2,1)
>> spy(L)
>> subplot(1,2,2)
>> spy(U)
```



LU decomposition

- LU decomposition and Gaussian elimination on a matrix like A requires more memory (with 3D problems, the offset in the diagonal would even be bigger!)
 - In general extra memory allocation will not be a problem for MATLAB
 - MATLAB is clever, in that sense that it attempts to reorder equations, to move elements closer to the diagonal)

Alternatives for elimination methods

- Use iterative methods when systems are large and sparse.
 - Often such systems are encountered when we want to solve PDE's of higher dimensions

Today's outline

- Introduction
- Sparse matrices
- Laplace's equation
- Creating a sparse system
- Iterative methods
- Summary

Examples of iterative methods

- Jacobi method
 - Gauss-Seidel method
 - Successive over relaxation
 - `bicg` — Bi-conjugate gradient method
 - `pcg` — preconditioned conjugate gradient method
 - `gmres` — generalized minimum residuals method
 - `bicgstab` — Bi-conjugate gradient method

The Jacobi method

- In our example we derived the following equation

$$T_{k-N_x} + T_{k-1} - 4T_k + T_{k+1} + T_{k+N_x} = 0$$

- Rearranging gives:

$$T_k = \frac{T_{k-N_x} + T_{k-1} + T_{k+1} + T_{k+N_x}}{4}$$

- In the Jacobi scheme the iteration proceeds as follows:

- ① Start with an initial guess for the values of T at each node.
 - ② Compute updated values and store a new vector:

$$T_k^{\text{new}} = \frac{T_{k-N_x}^{\text{old}} + T_{k-1}^{\text{old}} + T_{k+1}^{\text{old}} + T_{k+N_x}^{\text{old}}}{4}$$

- ③ Do this for all nodes
 - ④ Repeat the procedure until converged

Jacobi method for Laplace's equation

See `laplace_jacobi.m` (from Canvas)

```

% Grid size
nx = 40; ny = 40;
% The temperature field + boundaries at old and new times
T = zeros(nx,ny);
T(1,:) = 40; % Left
T(nx,:) = 60; % Right
T(:,1) = 20; % Bottom
T(:,ny) = 30; % Top
Tnew = T;
% For plotting
[x y] = meshgrid(1:nx, 1:ny);
for iter = 1:1000
    for i = 2:nx-1
        for j = 2:ny-1
            Tnew(i,j) = (T(i-1,j)+T(i+1,j)+T(i,j-1)+T(i,j+1))/4.0;
        end
    end
    surf(x,y,Tnew);
    title(['Iteration: ' num2str(iter)]);
    drawnow
    T = Tnew; % Update T
end

```

About the straightforward implementation

- The method as implemented works fine for a simple Laplace equation
 - For generic systems of linear equations, the implementation cannot be used.

We will now introduce the Jacobi method so it can be used for generic systems of linear equations.

The Jacobi method with matrices

We can split our (banded) matrix A into a diagonal matrix D and a remainder R :

$$A = D + R$$

Jacobi method: solving a system

- We can solve $AT = b$, now written generally as $Ax = b$, by:

$$Ax = b$$

$$(D + R)x = b$$

$$Dx \equiv b - Rx$$

$$Dx^{\text{new}} = h - Rx^{\text{old}}$$

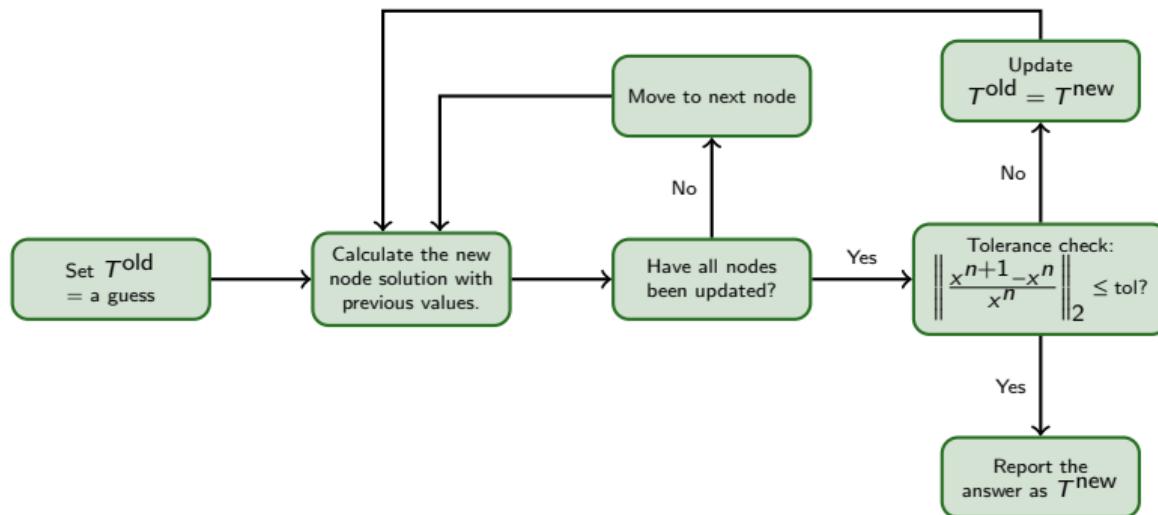
$$x^{\text{new}} = D^{-1}(b - Rx^{\text{old}})$$

- Using the n and $n+1$ notation for old and new time steps, we find in general:

$$x^{n+1} = D^{-1}(b - Rx^n)$$

$$x_i^{n+1} = \frac{1}{A_{ii}} \left(b_i - \sum_{j \neq i} A_{ij} x_j^n \right)$$

Diagram of the Jacobi method



The core of the solver

The full file is on Canvas, `solveJacobi.m`.

```

1 while ( xDiff > tol && it_jac < 1000 )
2     x_old = x;
3     for i=1:N
4         s = 0;
5         for j = 1:N
6             if (j ~= i)
7                 s = s+A(i,j)*x_old(j);
8             end
9         end
10        x(i) = (b(i)-s)/A(i,i);
11    end
12    it_jac = it_jac+1;
13    xDiff = norm((x-x_old)./x,2);
14 end
15 it_jac

```

Try to call it from the `solveLaplaceEq.m` file, instead of using \.

A few details on this algorithm

- The while loop holds two aspects
 - A convergence criterion (`norm((x-x_old)./x,2) > tol`). Some considerations are:
 - L_1 -norm (sum)
 - L_2 -norm (Euclidian distance)
 - L_∞ -norm (max)
 - Protection against infinite loops (no convergence)
 - Reset the sum for each row, before summing for the new unknown node
 - Start vector x is not shown in the example, but should be there!
 - It can have huge impact on performance!
 - The for-loops also have a large performance penalty!

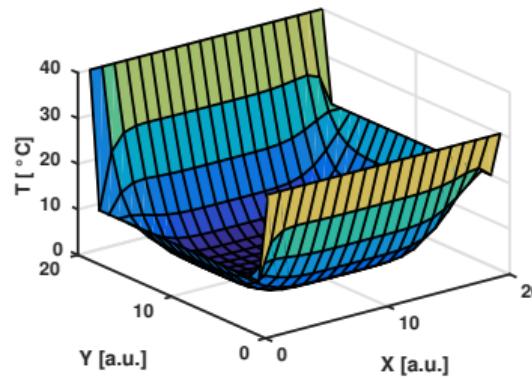
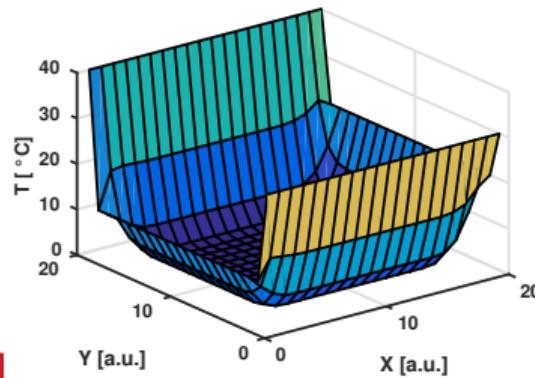
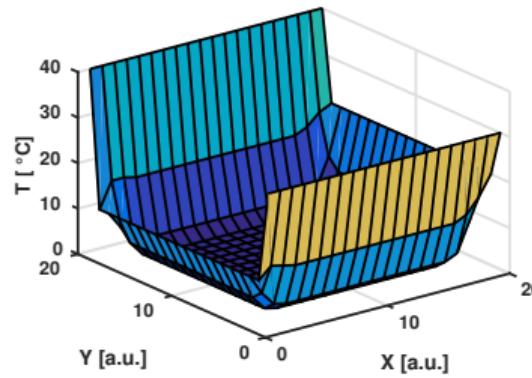
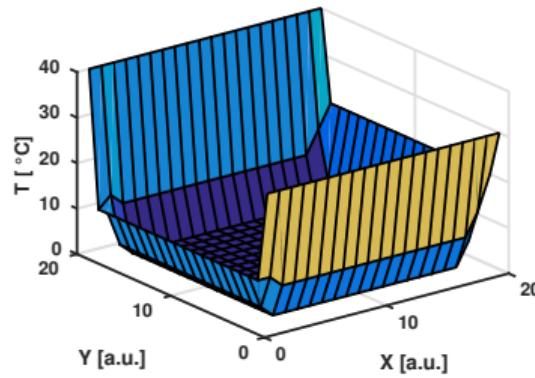
The solver using array indices

Make a copy of the Jacobian solver, and replace the for-loop by a vector-operation:

```
% While not converged or max_it not reached
while ( xDiff > tol && it_jac < 1000 )
    x_old = x;
    for i=1:N
        % Sum off-diagonal*x_old
        offDiagonalIndex = [1:(i-1) (i+1):N];
        Aij_Xj = A(i,offDiagonalIndex)*x_old(offDiagonalIndex);

        % Compute new x value
        x(i) = (b(i)-Aij_Xj)/A(i,i);
    end
    it_jac = it_jac+1;
    xDiff = norm((x-x_old)./x,2);
end
```

Iterations 1, 2, 3 and 10



Gauss-Seidel method

The Gauss-Seidel method is quite similar to Jacobi method

- The only difference is that the new estimate x^{new} is returned to the solution x^{old} as soon as it is completed
- For following nodes, the updated solution is used immediately
- Our straightforward script (from the Jacobi method) is therefore changed easily:
 - Do not create a `Tnew` array (save memory!)
 - Do not store the solution in `Tnew`, but simply in `T`
 - Do not perform the update step `T=Tnew`
 - See `laplace_gaussseidel.m` for the algorithm.
- The straightforward script works well for the current Laplace equation, but we define the generic Gauss-Seidel algorithm on the following slides.

Gauss-Seidel method

- Define a lower and strictly upper triangular matrix, such that $A = L + U$
 - Now we can solve $AT = b$ by:

$$(L + U)T = b$$

$$LT = b - UT$$

$$LT^{\text{new}} = b - UT^{\text{old}}$$

$$T^{\text{new}} = L^{-1}(b - UT^{\text{old}})$$

- Using the n and $n+1$ notation for old and new time steps, we find in for the general Gauss-Seidel method:

$$x^{n+1} = L^{-1}(b - Ux^n)$$

$$x_i^{n+1} = \frac{1}{A_{ii}} \left(b_i - \sum_{j < i} A_{ij} x_j^{n+1} - \sum_{j > i} A_{ij} x_j^n \right)$$

Today's outline

Summary

- Partial differential equations can be discretized into sparse systems of linear equations
 - Sparse matrices can be stored in memory efficiently using specialised formats (e.g. compressed row storage)
 - The Jacobi and Gauss–Seidel methods were introduced as iterative methods; other methods are based on the same principle (successive over-relaxation method, for example)
 - Various implementation issues were discussed, e.g. vectorised computing, convergence tolerances

Direct methods vs. Iterative methods

- Iterative methods converge *gradually* to a solution while direct methods (possibly with partial pivoting) factorise a (set of) matrix(es) which allow to compute the solution by *substitution*.
 - Direct methods generally use more memory, since they need to store also the result matrices.
 - A strictly (or irreducibly) diagonally dominant matrix is a prerequisite for convergence of the Jacobi and Gauss-Seidel method.
 - For real-life situations; 1D problems are generally solved with direct methods (LU decomposition). If you have systems of more than 1 dimension, a direct method still can be used, if there are no memory issues, otherwise an iterative method would be more attractive.

Curve fitting, regression and optimization

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Numerical Methods (6E5X0), 2020-2021

Today's outline

● Introduction

- Curve fitting
- Regression
- Fitting numerical models
- Optimization
- Linear programming
- Summary

Overview

- We are going to fit measurements to models today
- You will also learn what R^2 actually means
- We get introduced to constrained and unconstrained optimization.
- We will use the simplex method to solve linear programming problems (LP)

Today's outline

- Introduction
- Curve fitting
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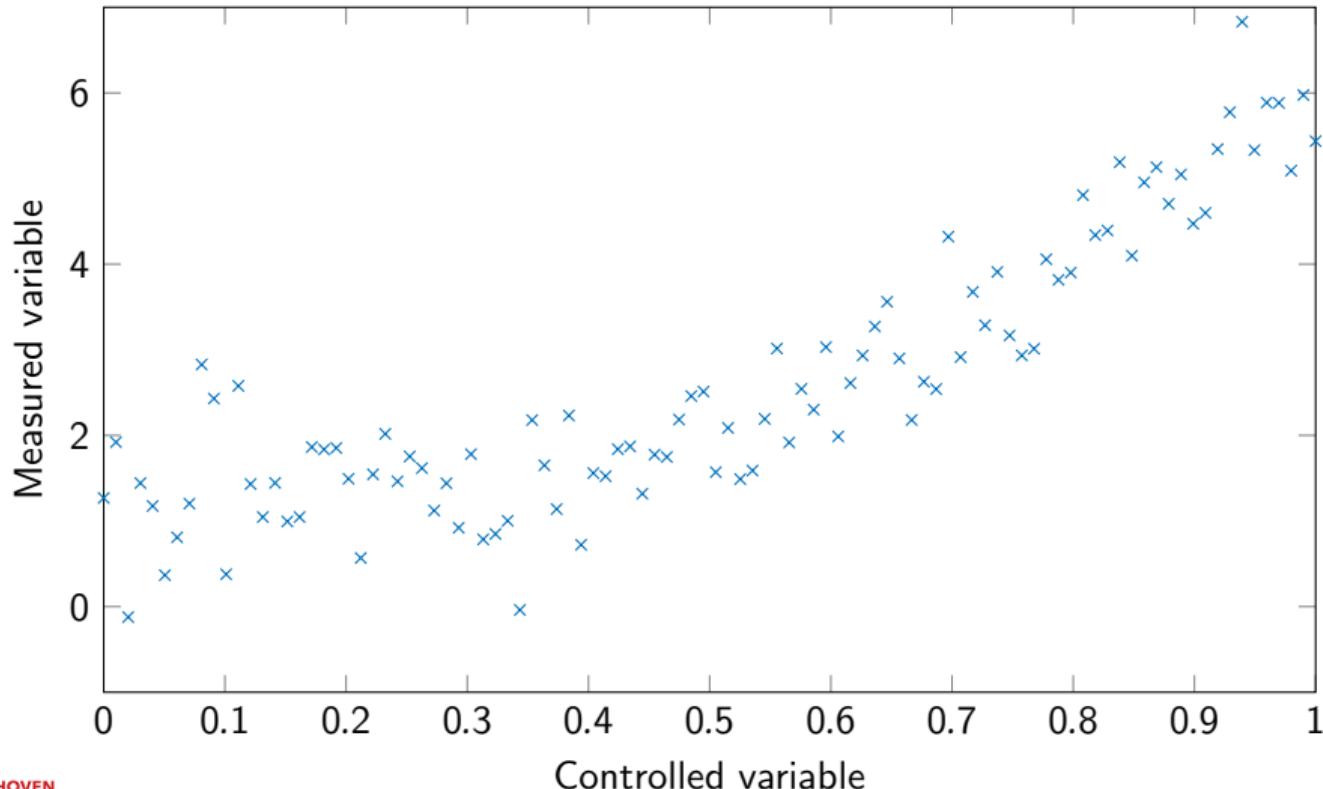
Let's do an 'experiment' to gather data

```
%% Generate linear space of control points
N = 100; % Number of data points
x = linspace(0,1,N); % Points (independent variable)
A = [1 1/3 1.5 3.5]; % Coefficients of polynomial

%% Generate 'measurement values' with errors following a normal distribution
% Initialize randomizer
pd = makedist('normal',0,0.5);
% Add scatter data to the polynomial
y = A(4)*x.^3 + A(3)*x.^2 + A(2).*x + A(1) + random(pd,1,N);

%% Plot the generated data
plot(x,y,'x');
```

Fitting models to data



How to fit a model to the data?

We would like to fit the following model to the data:

$$\hat{y} = a_1 + a_2x + a_3x^2 + a_4x^3$$

First step: If we have N data points, we could write the model as the product of a matrix and a vector:

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \hat{y}_3 \\ \vdots \\ \hat{y}_N \end{bmatrix} = \begin{bmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_N & x_N^2 & x_N^3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$$

$$\hat{y} = Xa$$

X is called the design matrix
and a are the fit parameters.

Residuals

Second step: work out the residuals for each data point:

$$d_i = (y_i - \hat{y}_i)$$

Third step: work out the sum of squares of the residuals:

$$\text{SSE} = \sum_i d_i^2 = \sum_i (y_i - \hat{y}_i)^2$$

This can be written using the dot-product operation:

$$\text{SSE} = \sum_i d_i^2 = d \cdot d = d^T \cdot d = (y_i - \hat{y}_i)^T \cdot (y_i - \hat{y}_i)$$

Minimizing the sum of squares

Choose the parameter vector such that the sum of squares of the residuals is minimized; the partial derivative with respect to each parameter should be set to zero:

$$\frac{\partial}{\partial a_i} \left[(y - (Xa)^T)(y - Xa) \right]$$

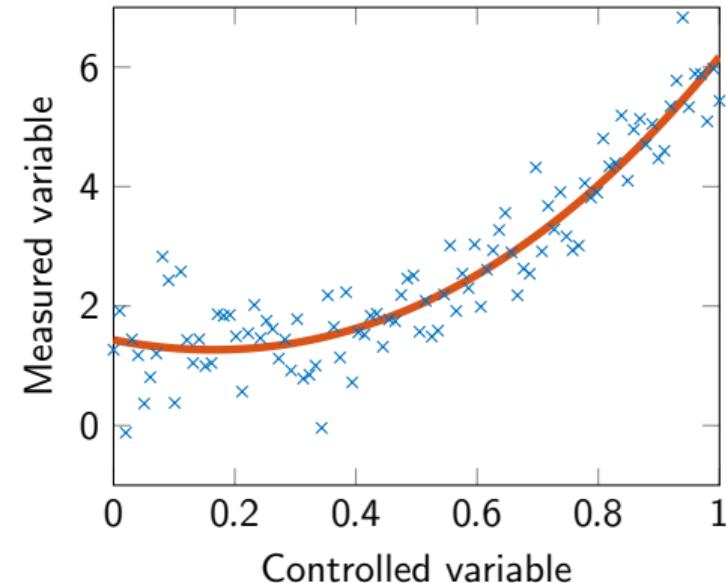
In Matlab, we can solve our linear system $\hat{Y} = Xa$ simply by running `a = x\y`.

- If there are more data points ($N > 4$), we can write an analogue, but maybe a consistent solution does not exist (the system is over specified).
- However, matlab will find values for the vector a which minimize $\|y - aX\|^2$, so i.e. a least squares fit.

Fitting our problem: Matlab solver

As a follow-up of the script provided in slide 346

```
N=length(x);  
X(:,1) = ones(N,1);  
X(:,2) = x;  
X(:,3) = x.^2;  
X(:,4) = x.^3;  
A = X\y';
```



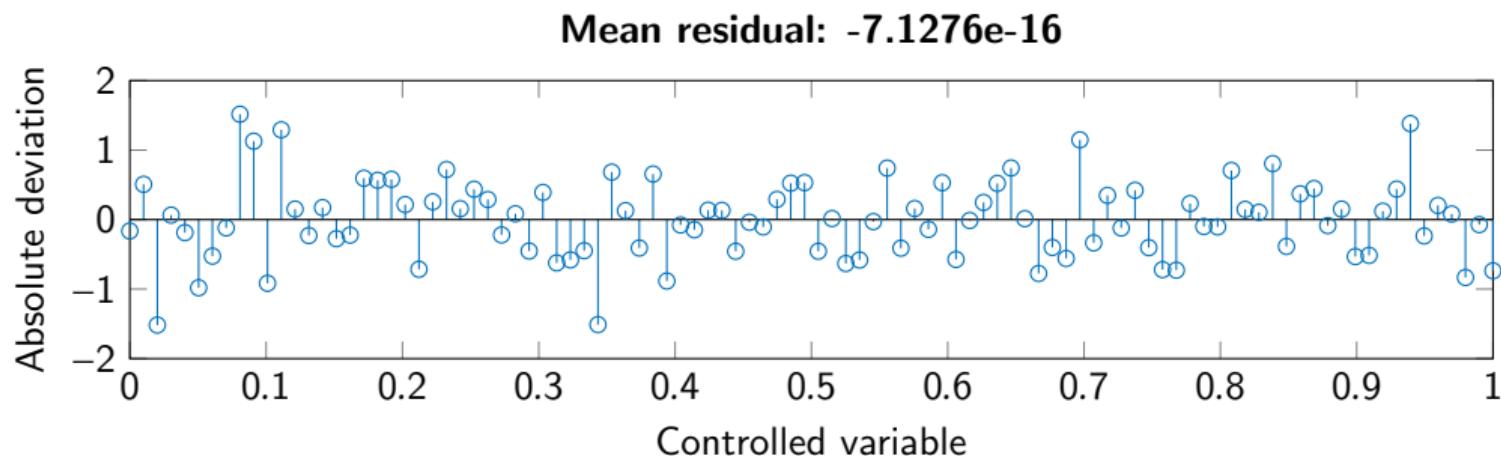
Fitting our problem: Matlab curve-fitting toolbox

- Start the toolbox: `cftool`
- Choose the dataset (x and y data)
- Choose the interpolant type (polynomial, exponential, ..., custom)
- Get the coefficients (save to workspace or write them down)

Fitting our problem: Excel

- Create a column with the independent and dependent data series (x and y)
- Create a column that computes \hat{y} , keeping the coefficients as separate cells
- Compute the sum of squares of the residuals (another column, sum the results)
- Use the solver to minimize this sum, modifying the coefficient cells
- Note: regression in Excel + display equation is dangerous if you choose the 'line' plot (use scatter if you can)

How good is the model?



- For a model to make sense the data points should be scattered randomly around the model predictions, the mean of the residuals d should be zero: $d_i = (y_i - \hat{y}_i)$
- It's always good to check if the residuals are not correlated with the measured values, if that is the case, it can indicate that your model is wrong.

Today's outline

- Introduction
- Curve fitting
- Regression
- Fitting numerical models
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- Summary

Regression coefficients

- Variance measured in the data (y) is:

$$\sigma_y^2 = \frac{1}{N} \sum_i (y_i - \bar{y})^2$$

- Variance of the residuals is:

$$\sigma_{\text{error}}^2 = \frac{1}{N} \sum_i (d_i)^2$$

- Variance in the model is:

$$\sigma_{\text{model}}^2 = \frac{1}{N} \sum_i (\hat{y}_i - \bar{\hat{y}})^2$$

Regression coefficients

Given that the error is uncorrelated we can state that:

$$\sigma_y^2 = \sigma_{\text{error}}^2 + \sigma_{\text{model}}^2$$

$$R^2 = \frac{\sigma_{\text{model}}^2}{\sigma_y^2} = 1 - \frac{\sigma_{\text{error}}^2}{\sigma_y^2}$$

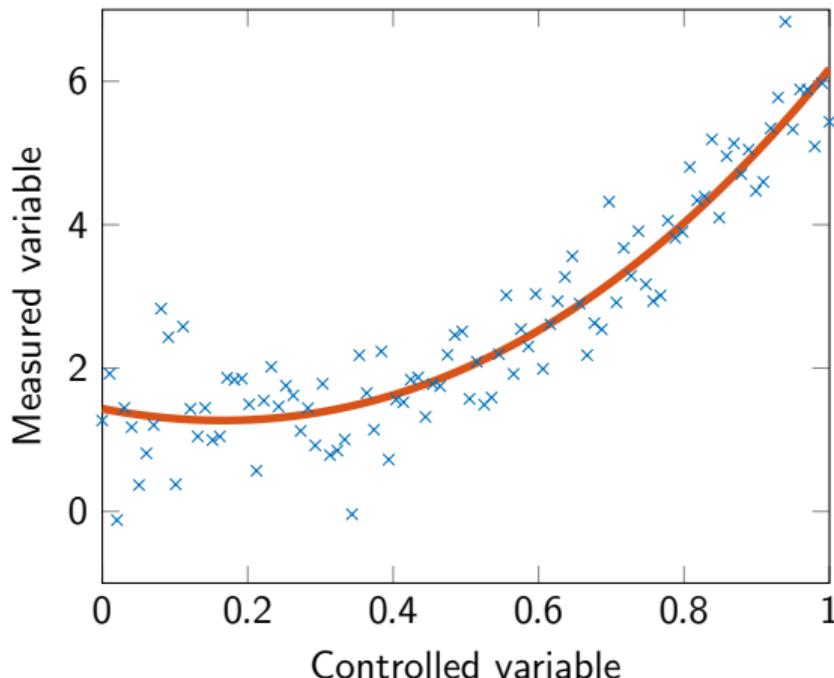
$$R^2 = 1 - \frac{\text{SSE}}{\text{SST}}$$

- SSE: Sum of errors (residuals) squared (difference between data and model)
- SST: Total sum of squares (variance of the data)
- SSR: Sum of squares (model)

Back to the example

The statistics:

	Value
N	100
SSE	32.042
SST	896.907
SSR	928.950
R^2	0.964



Today's outline

- Introduction
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Curve fitting from commandline: lsqcurvefit

Matlab offers various non-linear parameter and curve fitting tools that can be run from the commandline. The function `lsqcurvefit` allows to fit a model to a given dataset. Again, based on the data generated in slide 346:

```
% Initial guess of coefficients
a0 = [1 2 1 3];

% Perform fitting, store resulting coeffs in a_fit
a_fit = lsqcurvefit(@curve_fit_model, a0, x, y)

% Run the model once more, with fitted coefficients
y_model = curve_fit_model(a_fit, x);

plot(x,y_model, '-r')
```

We just need a function that captures the model that we want to fit. This can be an inline function or a separate file:

```
function [y] = curve_fit_model(a,xdata)
y = a(1)*xdata.^3+a(2)*xdata.^2 + a(3)*xdata + a(4);
end
```

Dynamic fitting of non-linear equations: lsqnonlin

You may encounter situations where the model data is slightly more complicated to obtain (e.g. a numerical model based on ODEs where coefficients are unknown), or you want to perform fitting of multiple functions/coefficients, or just want to automate things via scripts. Matlab's Optimization toolbox gives access to a powerful function `lsqnonlin`, least-squares non-linear optimization.

General use of lsqnonlin

```
k = lsqnonlin(fun,k0,lb,ub,options)
```

- `fun` is a function handle to the fit criterium (e.g. `@myFitCrit`). The fit criterium function `myFitCrit` should return the residuals vector, e.g. $d_i = (y_i - \hat{y}_i)$. Here, y_i would again be the measurement data and \hat{y} the solution computed by a model.
- `k0` is the initial guess for the fitting coefficient (or: array of initial guesses when fitting multiple coefficients)
- `lb` and `ub` are the lower and upper boundaries for `k0`. These should both be the size of the `k0`-array.
- `options` are some fitting options, for more fine-grained control on the fit procedure. Use e.g. `options = optimset('TolX',1.0E-6,'MaxFunEvals',1000);` to create an `options` object, or leave it empty (`options = []`).

Example use of lsqnonlin

We have experimental data stored in the file `tudataset1.mat`, containing T and U data. We want to fit a model with coefficients k_1 and k_2 with the following structure:

$$\frac{du}{dt} = -k_1 u + k_2$$

- First, we need to create a function that contains the ODE:

```
function dudt = simpleode(t,u,k);
dudt = -k(1)*u + k(2);
```

Note that we supply a vector `k`, containing both coefficients for fitting

- We create a fit criterium function:

```
function err = fitcrit(ke,T,U,U0)
[t,ue] = ode45(@simpleode,T,U0,[],ke);
err = (ue-U);
```

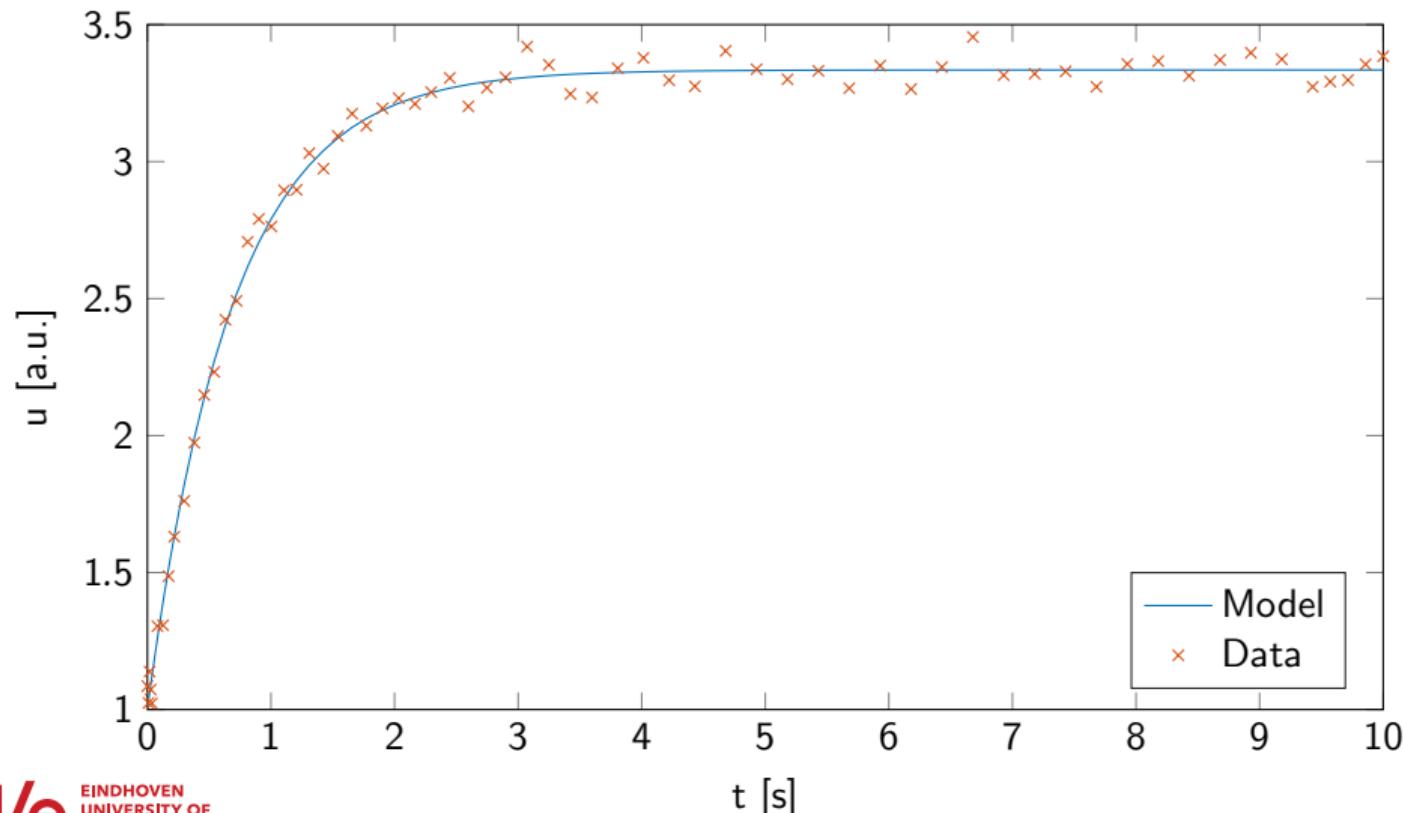
Example use of lsqnonlin

Now let's make a script that uses `lsqnonlin` to yield k-values fitted to our dataset:

```
% initial value
U0 = 1.00;
% initial guesses for model parameters
k0 = [1.00 1.00];
% lower and upper bounds for model parameters
LB = [0.00 0.00];
UB = [Inf Inf];
% Perform nonlinear least squares fit
options = optimset('TolX',1.0E-6,'MaxFunEvals',1000);
[ke,RESNORM,RESIDUAL,EXITFLAG,OUTPUT,LAMBDA,JACOBIAN] = lsqnonlin(@fitcrit,k0,LB,UB,
    options,T,U,U0);
```

Our fitted coefficients are stored in `ke`. Note that we get a lot more data back that allows to check the fitting results in more detail.

Example use of lsqnonlin



Postprocessing of results

The data returned by `lsqnonlin` can be used to obtain the 95% confidence intervals. Recall the command:

```
[ke,RESNORM,RESIDUAL,EXITFLAG,OUTPUT,LAMBDA,JACOBIAN] = lsqnonlin(@fitcrit,k0,LB,UB,  
options,T,U,U0);
```

Using the residuals and Jacobian we can use `nlpaci` to get the confidence bounds:

```
cflim = nlpaci(ke, RESIDUAL, JACOBIAN);  
  
clc  
disp('model parameters and confidence limits');  
T = table;  
T.ke = ke';  
T.LowerCI = cflim(:,1);  
T.UpperCI = cflim(:,2)
```

Second example: lsqnonlin

The model does not have to be an ODE. It can be any model that is part of the fitting goal function. Let's consider an (adapted) version of the Laplace equation solver, where we can set a single, central node to a certain temperature:

```
function T = solveLaplaceEq(Nx,Ny,Tb,Tint)
% Solves the Laplace equation for steady-state heat conduction

e = ones(Nx*Ny,1);
A = spdiags([e,e,-4*e,e,e],[-Nx,-1,0,1,Nx],Nx*Ny,Nx*Ny);
b = zeros(Nx*Ny,1); % Right hand side vector

[A,b] = setBoundaryConditions(A,b,Tb,Nx,Ny);

% Set a central node to Tint
ind = round(Nx * (Ny/2)); % Find index in center
A(ind,:) = 0; % Reset matrix for boundary cells
A(ind,ind) = 1; % Add a 1 on the diagonal
b(ind) = Tint; % Set rhs to desired T

T = A\b;
```

Second example: lsqnonlin

Now let's make a goal function based on our model and the desired setpoint (the mean temperature in the domain):

```
function [err] = fitcrit_laplace(actuate_T, Nx, Ny, boundary_T, setpoint_T)
% Compute model:
T = solveLaplaceEq(Nx, Ny, boundary_T, actuate_T);

% Compute error (deviation of mean T with desired setpoint T)
err = mean(T,'all') - setpoint_T;
end
```

Second example: lsqnonlin

- Set up system parameters
- Run the fitting procedures
- Compute and plot the final solution

```
% Set up parameters
Nx = 35; Ny = 35;
Tb = [40 10 40 10]; % Fixed boundary temperatures
T0 = 0; % Initial guess
T_set = 20; % Setpoint
UB = Inf; LB = -Inf; % Upper and lower bounds

opts = optimoptions('lsqnonlin', 'Display', 'iter');

% Run fitting
T_fit = lsqnonlin(@(T) fitcrit_laplace(T,Nx,Ny,Tb,T_set),T0, LB, UB, opts)
% Is the same as:
% T_fit = lsqnonlin(@fitcrit_laplace, T0, LB, UB, opts, Nx, Ny, Tb, T_set)

% Compute again and plot
T_model = solveLaplaceEq(Nx, Ny, Tb, T_fit);
T_plot = reshape(T_model,[Nx Ny]); % Reshape x-vec to mat Nx,Ny
[x,y] = meshgrid(1:Ny,1:Nx); % Get position arrays
surf(x,y,T_plot); % Surface plot
mean(T_model)
```

Today's outline

- Introduction
- Curve fitting
- Regression
- Fitting numerical models
- Optimization
- Linear programming
- Summary

What is optimization?

Optimization is minimization or maximization of an objective function (also called a performance index or goal function) that may be subject to certain constraints.

- $\min f(x)$: Goal function
- $g(x) = 0$: Equality constraints
- $h(x) \geq 0$: Inequality constraints

Optimization Spectrum

Problem	Method	Solvers
LP	Simplex method	Linprog (Matlab)
	Barrier methods	CPLEX (GAMS, AIMMS, AMPL, OPB)
NLP	Lagrange multiplier method	Fminsearch/fmincon (Matlab)
	Successive linear programming	MINOS (GAMS, AMPL)
	Quadratic programming	CONOPT (GAMS)
MIP	Branch and bound	
MILP	Dynamic programming	Bintprog (Matlab)
MINLP	Generalized Benders decomposition	DICOPT (GAMS)
	Outer approximation method	BARON (GAMS)
MIQP	Disjunctive programming	

Factors of concern

- Continuity of the functions
- Convexity of the functions
- Global versus local optima
- Constrained versus unconstrained optima

Today's outline

- Introduction
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- Fitting numerical models
- Optimization
- Linear programming
- Summary

Linear programming

In linear programming the objective function and the constraints are linear functions.

For example:

$$\begin{aligned} \max z &= f(x_1, x_2) = 40x_1 + 88x_2 \\ \text{s.t. (subject to)} \end{aligned}$$

$$2x_1 + 8x_2 \leq 60$$

$$5x_1 + 2x_2 \leq 60$$

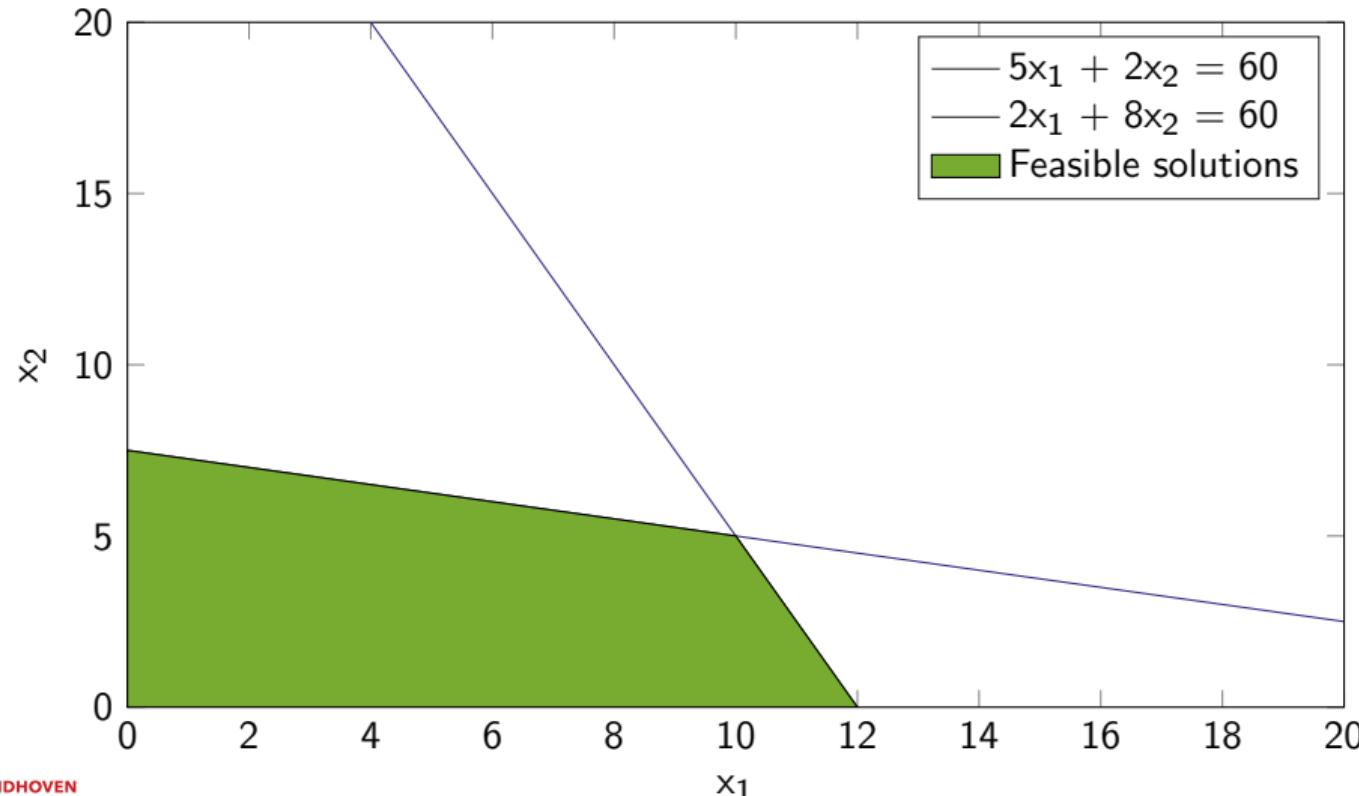
$$x_1 \geq 0$$

$$x_2 \geq 0$$

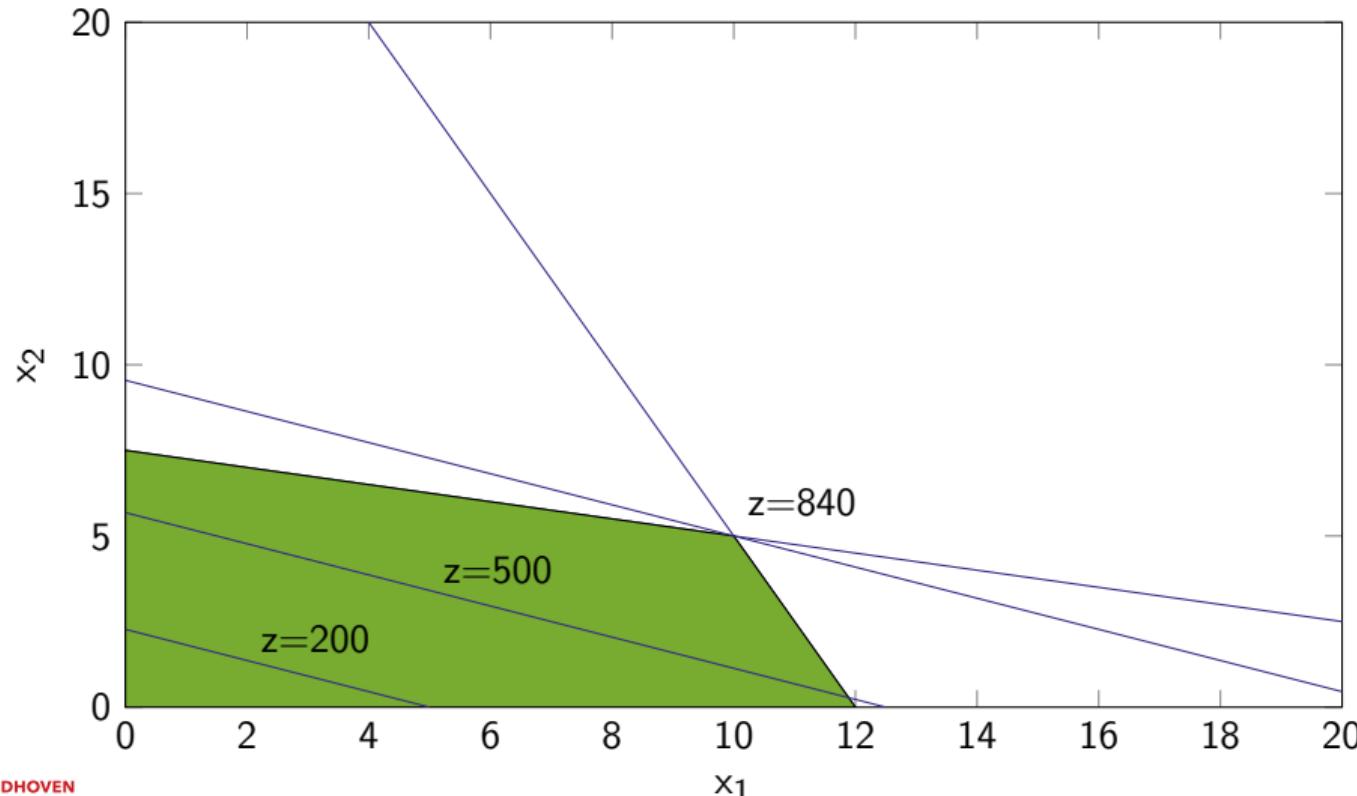
If the constraints are satisfied, but the objective function is not maximized/minimized we speak of a feasible solution.

If also the objective function is maximized/minimized, we speak of an optimal solution!

Plotting the constraints



Plotting the constraints



Normal form of an LP problem

$$\max z = f(x_1, x_2) = 40x_1 + 88x_2$$

s.t.

$$2x_1 + 8x_2 \leq 60$$

$$5x_1 + 2x_2 \leq 60$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$\max z = f(x) = 40x_1 + 88x_2$$

s.t.

$$2x_1 + 8x_2 + x_3 = 60$$

$$5x_1 + 2x_2 + x_4 = 60$$

$$x_i \geq 0 \quad i \in \{1, 2, 3, 4\}$$

x_3 and x_4 are called slack variables, they are non auxiliary variables introduced for the purpose of converting inequalities into equalities

The simplex method

We can formulate our earlier example to the normal form and consider it as the following augmented matrix with $T_0 = [z \ x_1 \ x_2 \ x_3 \ x_4 \ b]$:

$$T_0 = \begin{bmatrix} 1 & -40 & -88 & 0 & 0 & 0 \\ 0 & 2 & 8 & 1 & 0 & 60 \\ 0 & 5 & 2 & 0 & 1 & 60 \end{bmatrix}$$

This matrix is called the (initial) simplex table. Each simplex table has two kinds of variables, the basic variables (columns having only one nonzero entry) and the nonbasic variables

The simplex method

$$T_0 = \begin{bmatrix} 1 & -40 & -88 & 0 & 0 & 0 \\ 0 & 2 & 8 & 1 & 0 & 60 \\ 0 & 5 & 2 & 0 & 1 & 60 \end{bmatrix}$$

Every simplex table has a feasible solution. It can be obtained by setting the nonbasic variables to zero: $x_1 = 0$, $x_2 = 0$, $x_3 = 60/1$, $x_4 = 60/1$, $z = 0$.

The optimal solution?

- The optimal solution is now obtained stepwise by pivoting in such way that z reaches a maximum.
- The big question is, how to choose your pivot equation ...

Step 1: Selection of the pivot column

Select as the column of the pivot, the first column with a negative entry in Row 1. In our example, that's column 2 (-40)

$$T_0 = \begin{bmatrix} 1 & -40 & -88 & 0 & 0 & 0 \\ 0 & 2 & 8 & 1 & 0 & 60 \\ 0 & 5 & 2 & 0 & 1 & 60 \end{bmatrix}$$

Step 2: Selection of the pivot row

Divide the right sides by the corresponding column entries of the selected pivot column. In our example that is $60/2 = 30$ and $60/5 = 12$.

$$T_0 = \begin{bmatrix} 1 & -40 & -88 & 0 & 0 & 0 \\ 0 & 2 & 8 & 1 & 0 & 60 \\ 0 & 5 & 2 & 0 & 1 & 60 \end{bmatrix}$$

Take as the pivot equation the equation that gives the smallest quotient, so $60/5$.

Step 3: Elimination by row operations

- Row 1 = Row 1 + 8 * Row 3
- Row 2 = Row 2 - 0.4 * Row 3

$$T_1 = \begin{bmatrix} 1 & 0 & -72 & 0 & 8 & 480 \\ 0 & 0 & 7.2 & 1 & -0.4 & 36 \\ 0 & 5 & 2 & 0 & 1 & 60 \end{bmatrix}$$

The basic variables are now x_1 , x_3 and the nonbasic variables are x_2 , x_4 . Setting the nonbasic variables to zero will give a new feasible solution: $x_1 = 60/5$, $x_2 = 0$, $x_3 = 36/1$, $x_4 = 0$, $z = 480$.

The simplex method

- We moved from $z = 0$ to $z = 480$. The reason for the increase is because we eliminated a negative term from the equation, so: elimination should only be applied to negative entries in Row 1, but no others.
- Although we found a feasible solution, we did not find the optimal solution yet (the entry of -72 in our simplex table) → repeat step 1 to 3.

The simplex method

Another iteration is required:

- Step 1: Select column 3
- Step 2: $36/7.2 = 5$ and $60/2 = 30 \rightarrow$ select 7.2 as the pivot
- Elimination by row operations:
 - Row 1 = Row 1 + 10*Row 2
 - Row 3 = Row 3 - (2/7.2)*Row 2

$$T_2 = \begin{bmatrix} 1 & 0 & 0 & 10 & 4 & 840 \\ 0 & 0 & 7.2 & 1 & -0.4 & 36 \\ 0 & 5 & 0 & -1/36 & 1/0.9 & 50 \end{bmatrix}$$

- The basic feasible solution: $x_1 = 50/5$, $x_2 = 36/7.2$, $x_3 = 0$, $x_4 = 0$, $z = 840$ (no more negative entries: so this solution is also the optimal solution)

Using Matlab for LP problems

We are going to solve the following LP problem:

$$\min f(x) = -5x_1 - 4x_2 - 6x_3$$

s.t.

$$x_1 - x_2 + x_3 \leq 20$$

$$3x_1 + 2x_2 + 4x_3 \leq 42$$

$$3x_1 + 2x_2 \leq 30$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$x_3 \geq 0$$

Using the function `linprog`:

```
f = [-5; -4; -6];
A = [1 -1 1; 3 2 4; 3 2 0];
b = [20; 42; 30];
lb = zeros(3,1);
[x,fval,exitflag,output,lambda]
= linprog(f,A,b,[],[],lb);
```

Gives:

```
x = 0.00 15.00 3.00
lambda.ineqlin = 0 1.50 0.50
lambda.lower = 1.00 0 0
```

Summary

- Curve fitting: Manual procedures for polynomial fitting in Matlab
- Curve fitting: Matlab's curve-fitting toolbox
- Curve fitting: Matlab's non-linear least-squares solver `lsqnonlin`
- Optimization: An introduction to the Simplex method
- Optimization: Use of the `linprog` solver

Ordinary differential equations 1

Explicit techniques for ODEs

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Numerical Methods (6E5X0), 2020-2021

Today's outline

● Introduction

● Euler's method

- Forward Euler

● Rates of convergence

● Runge-Kutta methods

- RK2 methods
- RK4 method

● Step size control

● Solving ODEs in Matlab

Overview

Ordinary differential equations

An equation containing a function of one independent variable and its derivatives, in contrast to a *partial differential equation*, which contains derivatives with respect to more independent variables.

Main question

How to solve

$$\frac{dy}{dx} = f(y(x), x) \quad \text{with} \quad y(x=0) = y_0$$

accurately and efficiently?

What is an ODE?

- Algebraic equation:

$$f(y(x), x) = 0 \quad \text{e.g. } -\ln(K_{eq}) = (1 - \zeta)$$

- First order ODE:

$$f\left(\frac{dy}{dx}(x), y(x), x\right) = 0 \quad \text{e.g. } \frac{dc}{dt} = -kc^n$$

- Second order ODE:

$$f\left(\frac{d^2y}{dx^2}(x), \frac{dy}{dx}(x), y(x), x\right) = 0 \quad \text{e.g.} \quad \mathcal{D}\frac{d^2c}{dx^2} = -\frac{kc}{1+Kc}$$

About second order ODEs

Very often a second order ODE can be rewritten into a system of first order ODEs (whether it is handy depends on the boundary conditions!)

More general

Consider the second order ODE:

$$\frac{d^2y}{dx^2} + q(x) \frac{dy}{dx} = r(x)$$

Now define and solve using z as a new variable:

$$\frac{dy}{dx} = z(x)$$

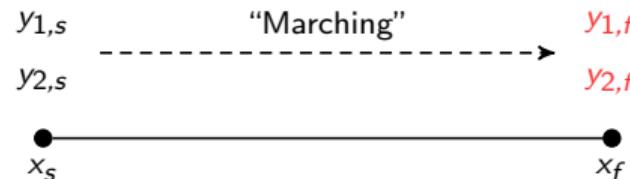
$$\frac{dz}{dx} = r(x) - q(x)z(x)$$

Importance of boundary conditions

The nature of boundary conditions determines the appropriate numerical method. Classification into 2 main categories:

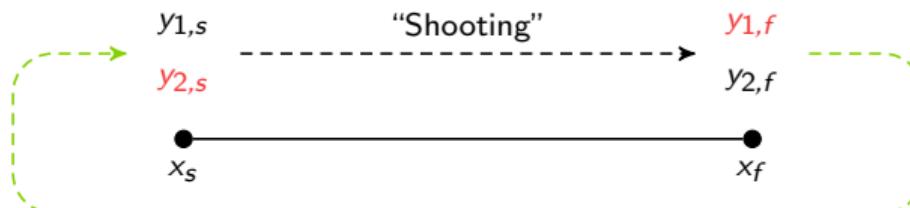
- *Initial value problems (IVP)*

We know the values of all y_i at some starting position x_s , and it is desired to find the values of y_i at some final point x_f .



- *Boundary value problems (BVP)*

Boundary conditions are specified at more than one x . Typically, some of the BC are specified at x_s and the remainder at x_f .



Overview

Initial value problems:

- Explicit methods
 - First order: forward Euler
 - Second order: improved Euler (RK2)
 - Fourth order: Runge-Kutta 4 (RK4)
 - Step size control
- Implicit methods
 - First order: backward Euler
 - Second order: midpoint rule

Boundary value problems

- Shooting method

Today's outline

- Introduction

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- RK2 methods
 - RK4 method

- Step size control

- Solving ODEs in Matlab

Euler's method

Consider the following single initial value problem:

$$\frac{dc}{dt} = f(c(t), t) \quad \text{with} \quad c(t=0) = c_0 \quad (\text{initial value problem})$$

Easiest solution algorithm: Euler's method, derived here via Taylor series expansion:

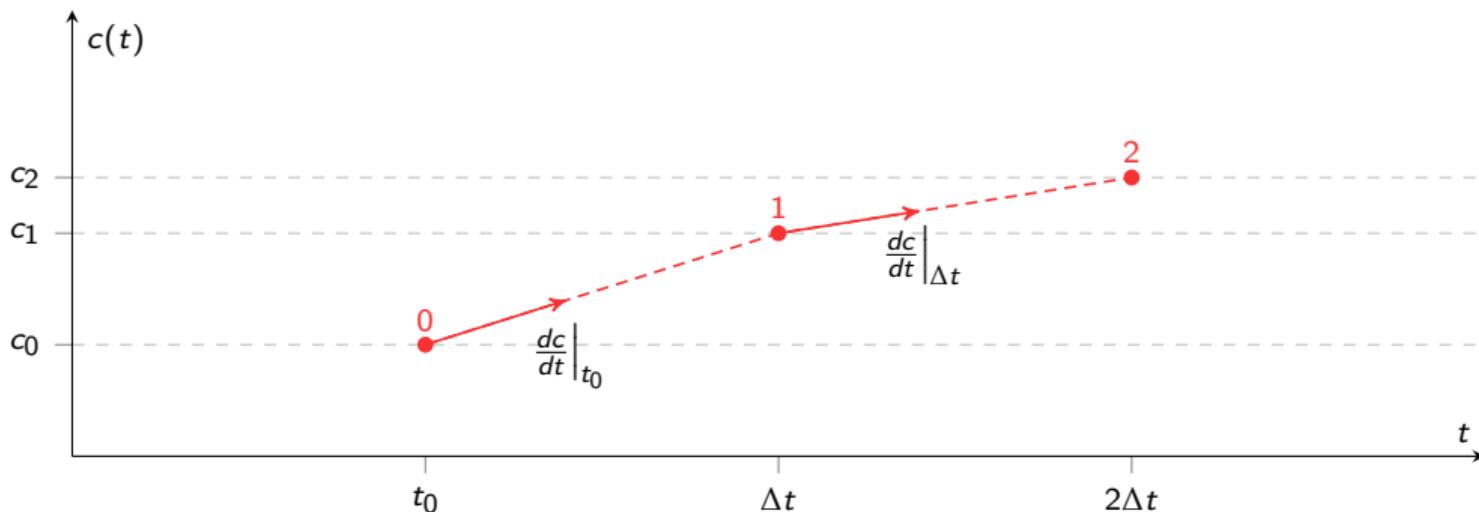
$$c(t_0 + \Delta t) \approx c(t_0) + \left. \frac{dc}{dt} \right|_{t_0} \Delta t + \frac{1}{2} \left. \frac{d^2 c}{dt^2} \right|_{t_0} (\Delta t)^2 + \mathcal{O}(\Delta t^3)$$

Neglect terms with higher order than two: $\left. \frac{dc}{dt} \right|_{t_0} = \frac{c(t_0 + \Delta t) - c(t_0)}{\Delta t}$ Substitution:

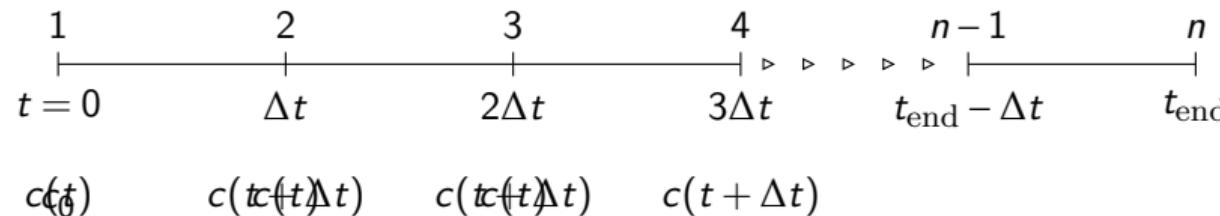
$$\frac{c(t_0 + \Delta t) - c(t_0)}{\Delta t} = f(c_0, t_0) \Rightarrow c(t_0 + \Delta t) = c(t_0) + \Delta t f(c_0, t_0)$$

Euler's method: graphical example

$$\frac{c(t_0 + \Delta t) - c(t_0)}{\Delta t} = f(c_0, t_0) \Rightarrow c(t_0 + \Delta t) = c(t_0) + \Delta t f(c_0, t_0)$$



Euler's method - solution method



Start with $t = t_0$, $c = c_0$, then calculate at discrete points in time:
 $c(t_1 = t_0 + \Delta t) = c(t_0) + \Delta t f(c_0, t_0)$.

Pseudo-code Euler's method: $\frac{dy}{dx} = f(x, y)$ and $y(x_0) = y_0$.

- ① Initialize variables, functions; set $h = \frac{x_1 - x_0}{N}$
- ② Set $x = x_0$, $y = y_0$
- ③ While $x < x_{\text{end}}$ do
 $x_{i+1} = x_i + h$; $y_{i+1} = y_i + hf(x_i, y_i)$

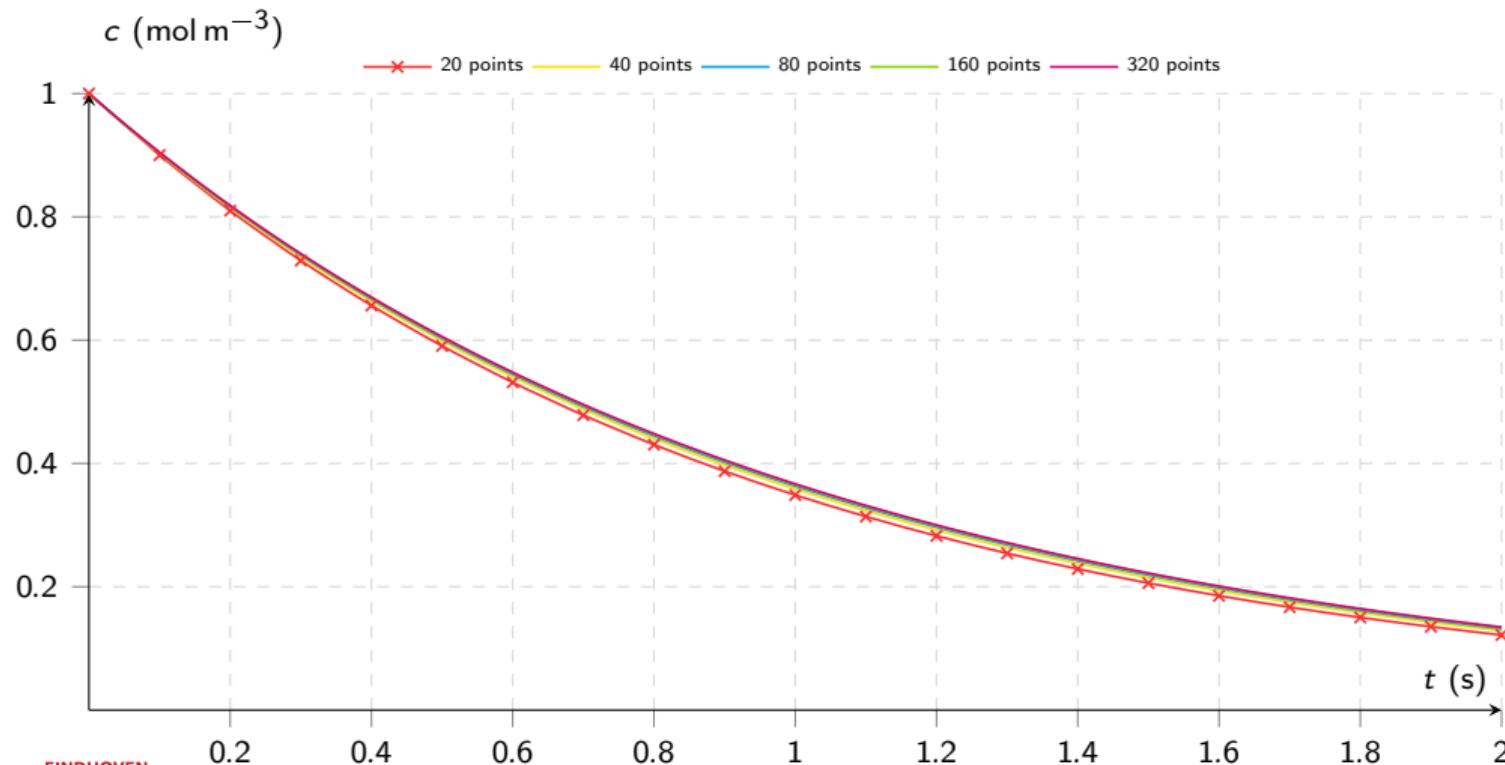
Euler's method - example

First order reaction in a batch reactor:

$$\frac{dc}{dt} = -kc \quad \text{with} \quad c(t=0) = 1 \text{ mol m}^{-3}, \quad k = 1 \text{ s}^{-1}, \quad t_{\text{end}} = 2 \text{ s}$$

Time [s]	Concentration [mol m ⁻³]
$t_0 = 0$	$c_0 = 1.00$
$t_1 = t_0 + \Delta t$ $= 0 + 0.1 = 0.1$	$c_1 = c_0 + \Delta t \cdot (-kc_0)$ $= 1 + 0.1 \cdot (-1 \cdot 1) = 0.9$
$t_2 = t_1 + \Delta t$ $= 0.1 + 0.1 = 0.2$	$c_2 = c_1 + \Delta t \cdot (-kc_1)$ $= 0.9 + 0.1 \cdot (-1 \cdot 0.9) = 0.81$
$t_3 = t_2 + \Delta t$ $= 0.2 + 0.1 = 0.3$	$c_3 = c_2 + \Delta t \cdot (-kc_2)$ $= 0.81 + 0.1 \cdot (-1 \cdot 0.81) = 0.729$
...	...
$t_{i+1} = t_i + \Delta t$	$c_{i+1} = c_i + \Delta t \cdot (-kc_i)$
...	...
$t_{20} = 2.0$	$c_{20} = c_{19} + \Delta t \cdot (-kc_{19}) = 0.121577$

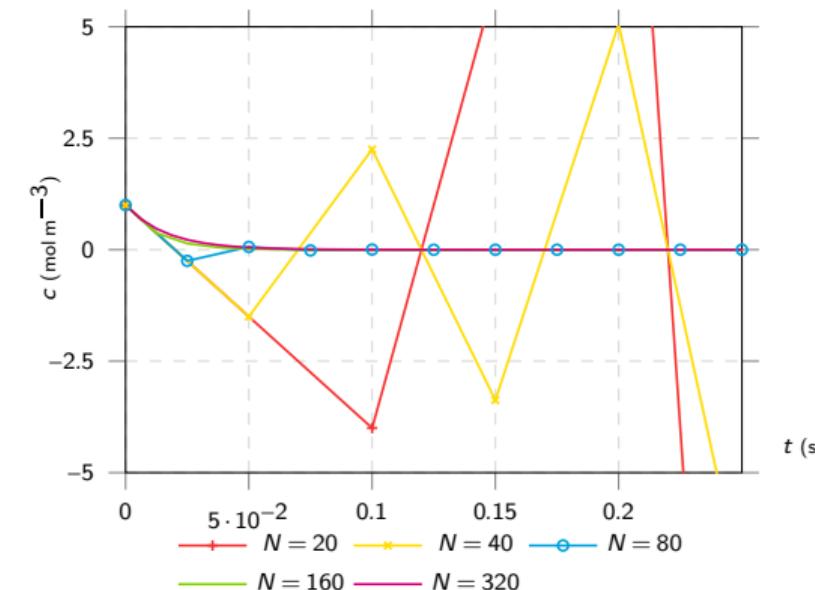
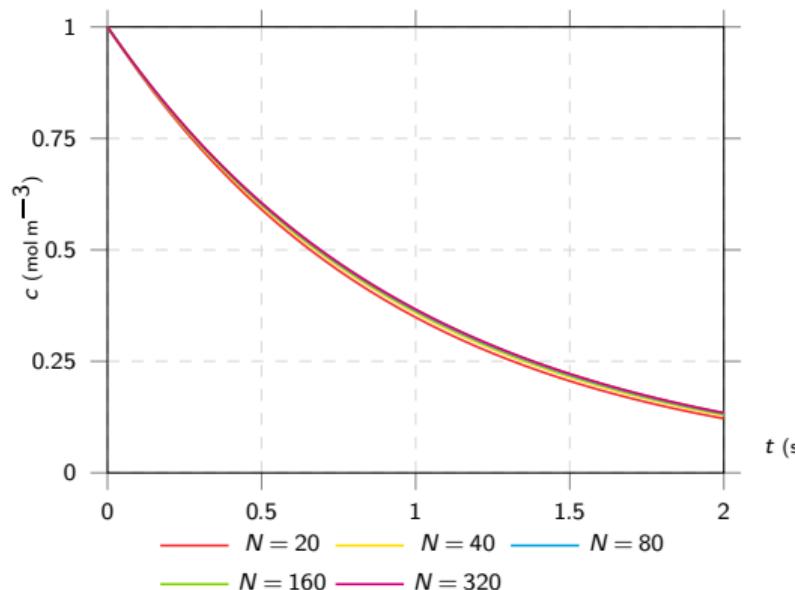
Euler's method - example



Problems with Euler's method

The question is: What step size, or how many steps to use?

- ① Accuracy \Rightarrow need information on numerical error!
- ② Stability \Rightarrow need information on stability limits!

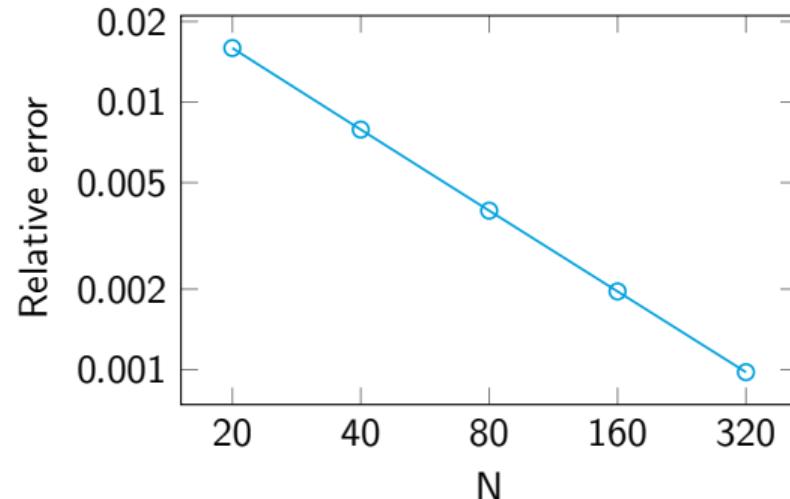


Accuracy

Comparison with analytical solution for $k = 1 \text{ s}^{-1}$:

$$c(t) = c_0 \exp(-kt) \Rightarrow \zeta = 1 - \exp(-kt) \Rightarrow \zeta_{\text{analytical}} = 0.864665$$

N	ζ	$\frac{\zeta_{\text{numerical}} - \zeta_{\text{analytical}}}{\zeta_{\text{analytical}}}$
20	0.878423	0.015912
40	0.871488	0.007891
80	0.868062	0.003929
160	0.866360	0.001961
320	0.865511	0.000979



Accuracy

For Euler's method: Error halves when the number of grid points is doubled, i.e. error is proportional to Δt : first order method.

Error estimate:

$$\left. \frac{dx}{dt} \right|_{t_0} = \frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t} + \frac{1}{2} \left. \frac{d^2 x}{dt^2} \right|_{t_0} (\Delta t) + \mathcal{O}(\Delta t)^2$$

$$\frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t} = f(x_0, t_0) - \frac{1}{2} \left. \frac{d^2 x}{dt^2} \right|_{t_0} (\Delta t) + \mathcal{O}(\Delta t)^2$$

Errors and convergence rate

Convergence rate (or: order of convergence) r

$$\epsilon = \lim_{\Delta x \rightarrow 0} c(\Delta x)^r$$

- A first order method reduces the error by a factor 2 when increasing the number of steps by a factor 2
- A second order method reduces the error by a factor 4 when increasing the number of steps by a factor 2

What to do when there is no analytical solution available? Compare to calculations with different number of steps: $\epsilon_1 = c(\Delta x_1)^r$ and $\epsilon_2 = c(\Delta x_2)^r$ and solve for r :

$$\frac{\epsilon_2}{\epsilon_1} = \frac{c(\Delta x_2)^r}{c(\Delta x_1)^r} = \left(\frac{\Delta x_2}{\Delta x_1} \right)^r \Rightarrow \log \left(\frac{\epsilon_2}{\epsilon_1} \right) = \log \left(\frac{\Delta x_2}{\Delta x_1} \right)^r$$

$$\Rightarrow r = \frac{\log \left(\frac{\epsilon_2}{\epsilon_1} \right)}{\log \left(\frac{\Delta x_2}{\Delta x_1} \right)} = \frac{\log \left(\frac{\epsilon_2}{\epsilon_1} \right)}{\log \left(\frac{N_1}{N_2} \right)} \quad \text{in the limit of } \Delta x \rightarrow 0 \quad \text{or} \quad N \rightarrow \infty$$

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Errors and convergence rate

L_2 norm (Euclidean norm)

$$\|\mathbf{v}\|_2 = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2} = \sqrt{\sum_{i=1}^n v_i^2}$$

L_∞ norm (maximum norm)

$$\|\mathbf{v}\|_\infty = \max(|v_1|, \dots, |v_n|)$$

Absolute difference

$$\epsilon_{\text{abs}} = \|\mathbf{y}_{\text{numerical}} - \mathbf{y}_{\text{analytical}}\|_{2,\infty}$$

Relative difference

$$\epsilon_{\text{rel}} = \left\| \frac{\mathbf{y}_{\text{numerical}} - \mathbf{y}_{\text{analytical}}}{\mathbf{y}_{\text{analytical}}} \right\|_{2,\infty}$$

Errors and convergence rate

Convergence rate (or: order of convergence) r

$$\epsilon = \lim_{\Delta x \rightarrow 0} c(\Delta x)^r$$

- A first order method reduces the error by a factor 2 when increasing the number of steps by a factor 2
- A second order method reduces the error by a factor 4 when increasing the number of steps by a factor 2

Computing the rate of convergence

When the analytical solution is available, choose ① or ② for a particular number of grid points N :

- ① Compute the relative or absolute error vector $\bar{\varepsilon}$. Take the norm to compute a single error value ϵ following:

- Based on L_1 -norm: $\epsilon = \frac{\|\bar{\varepsilon}\|_1}{N}$
- Based on L_2 -norm: $\epsilon = \frac{\|\bar{\varepsilon}\|_2}{\sqrt{N}}$
- Based on L_∞ -norm: $\epsilon = \|\bar{\varepsilon}\|_\infty$

- ② Compute the relative or absolute error at a single indicative points (e.g. middle of domain, outlet).

Compare to calculations with different number of steps: $\epsilon_1 = c(\Delta x_1)^r$ and $\epsilon_2 = c(\Delta x_2)^r$ and solve for r :

$$\frac{\epsilon_2}{\epsilon_1} = \frac{c(\Delta x_2)^r}{c(\Delta x_1)^r} = \left(\frac{\Delta x_2}{\Delta x_1} \right)^r \Rightarrow \log \left(\frac{\epsilon_2}{\epsilon_1} \right) = \log \left(\frac{\Delta x_2}{\Delta x_1} \right)^r$$

$$\Rightarrow r = \frac{\log \left(\frac{\epsilon_2}{\epsilon_1} \right)}{\log \left(\frac{\Delta x_2}{\Delta x_1} \right)} = \frac{\log \left(\frac{\epsilon_2}{\epsilon_1} \right)}{\log \left(\frac{N_1}{N_2} \right)} \quad \text{in the limit of } \Delta x \rightarrow 0 \text{ or } N \rightarrow \infty$$

Computing the rate of convergence

When the analytical solution is **not** available:

- ① Compute the solution with $N+1$, N , $N-1$ and $N-2$ grid points
- ② Select a single indicative grid point (e.g. middle of domain, outlet) that lies at exactly the same position in each computation
- ③ Use the solution c at this grid point for various grid sizes to compute:

$$r = \frac{\log \left| \frac{c_{N+1} - c_N}{c_N - c_{N-1}} \right|}{\log \left| \frac{c_N - c_{N-1}}{c_{N-1} - c_{N-2}} \right|}$$

- ④ Alternative for simulations with $2N$, N and $\frac{N}{2}$ grid points:

$$r = \frac{\log \left| \frac{c_{2N} - c_N}{c_N - c_{\frac{N}{2}}} \right|}{\log \left| \frac{N}{2N} \right|}$$

Example: Euler's method — order of convergence

N	ζ	$\frac{\zeta_{\text{numerical}} - \zeta_{\text{analytical}}}{\zeta_{\text{analytical}}}$	$r = \frac{\log\left(\frac{\epsilon_i}{\epsilon_{i-1}}\right)}{\log\left(\frac{N_{i-1}}{N_i}\right)}$
20	0.878423	0.015912	—
40	0.871488	0.007891	1.011832
80	0.868062	0.003929	1.005969
160	0.866360	0.001961	1.002996
320	0.865511	0.000979	1.001500

⇒ Euler's method is a first order method (as we already knew from the truncation error analysis)

Wouldn't it be great to have a method that can give the answer using much less steps? ⇒ Higher order methods

Today's outline

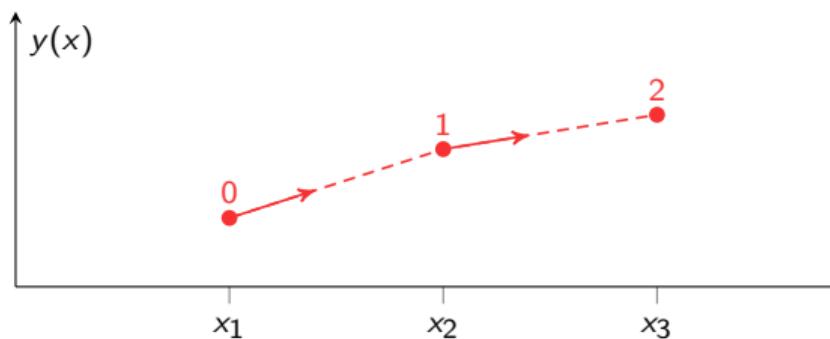
- Introduction
- Euler's method
 - Forward Euler
- Rates of convergence
- Runge-Kutta methods
 - RK2 methods
 - RK4 method
- Step size control
- Solving ODEs in Matlab

Runge-Kutta methods

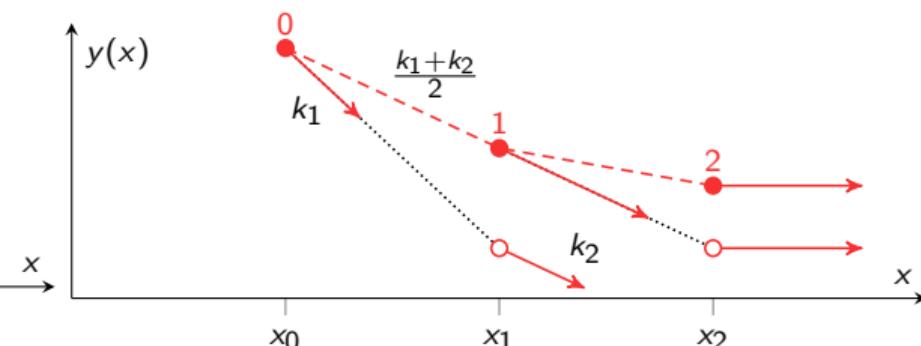
Propagate a solution by combining the information of several Euler-style steps (each involving one function evaluation) to match a Taylor series expansion up to some higher order.

Euler: $y_{i+1} = y_i + hf(x_i, y_i)$ with $h = \Delta x$, i.e. slope = $k_1 = f(x_i, y_i)$.

Euler's method



RK2 method



Classical second order Runge-Kutta (RK2) method

This method is also called Heun's method, or improved Euler method:

- ① Approximate the slope at x_i : $k_1 = f(x_i, y_i)$
- ② Approximate the slope at x_{i+1} : $k_2 = f(x_{i+1}, y_{i+1})$ where we use Euler's method to approximate $y_{i+1} = y_i + hf(x_i, y_i) = y_i + hk_1$
- ③ Perform an Euler step with the average of the slopes: $y_{i+1} = y_i + h\frac{1}{2}(k_1 + k_2)$

In pseudocode:

```
x = x0, y = y0
while x < xend do
    xi+1 = xi + h
    k1 = f(xi, yi)
    k2 = f(xi + h, yi + hk1)
    yi+1 = yi + h1/2(k1 + k2)
end while
```

Runge-Kutta methods — derivation

$$\frac{dy}{dx} = f(x, y(x))$$

Using Taylor series expansion: $y_{i+1} = y_i + h \left. \frac{dy}{dx} \right|_i + \frac{h^2}{2} \left. \frac{d^2y}{dx^2} \right|_i + \mathcal{O}(h^3)$

$$\left. \frac{dy}{dx} \right|_i = f(x_i, y_i) \equiv f_i$$

$$\left. \frac{d^2y}{dx^2} \right|_i = \left. \frac{d}{dx} f(x, y(x)) \right|_i = \left. \frac{\partial f}{\partial x} \right|_i + \left. \frac{\partial f}{\partial y} \right|_i \left. \frac{\partial y}{\partial x} \right|_i = \left. \frac{\partial f}{\partial x} \right|_i + \left. \frac{\partial f}{\partial y} \right|_i f_i \quad (\text{chain rule})$$

Substitution gives:

$$y_{i+1} = y_i + h f_i + \frac{h^2}{2} \left(\left. \frac{\partial f}{\partial x} \right|_i + \left. \frac{\partial f}{\partial y} \right|_i f_i \right) + \mathcal{O}(h^3)$$

$$y_{i+1} = y_i + \frac{h}{2} f_i + \frac{h}{2} \left(f_i + h \left. \frac{\partial f}{\partial x} \right|_i + h f_i \left. \frac{\partial f}{\partial y} \right|_i \right) + \mathcal{O}(h^3)$$

Runge-Kutta methods — derivation

Note multivariate Taylor expansion:

$$\begin{aligned}f(x_i + h, y_i + k) &= f_i + h \left. \frac{\partial f}{\partial x} \right|_i + k \left. \frac{\partial f}{\partial y} \right|_i + \mathcal{O}(h^2) \\&\Rightarrow \frac{h}{2} \left(f_i + h \left. \frac{\partial f}{\partial x} \right|_i + hf_i \left. \frac{\partial f}{\partial y} \right|_i \right) = \frac{h}{2} f(x_i + h, y_i + hf_i) + \mathcal{O}(h^3)\end{aligned}$$

Concluding:

$$y_{i+1} = y_i + \frac{h}{2} f_i + \frac{h}{2} f(x_i + h, y_i + hf_i) + \mathcal{O}(h^3)$$

Rewriting:

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + hk_1)$$

$$\Rightarrow y_{i+1} = y_i + \frac{h}{2}(k_1 + k_2)$$

Runge-Kutta methods — derivation

Generalization: $y_{i+1} = y_i + h(b_1 k_1 + b_2 k_2) + \mathcal{O}(h^3)$

with $k_1 = f_i$, $k_2 = f(x_i + c_2 h, y_i + a_{2,1} h k_1)$

(Note that classical RK2: $b_1 = b_2 = \frac{1}{2}$ and $c_2 = a_{2,1} = 1.$)

Bivariate Taylor expansion:

$$f(x_i + c_2 h, y_i + a_{2,1} h k_1) = f_i + c_2 h \left. \frac{\partial f}{\partial x} \right|_i + a_{2,1} h k_1 \left. \frac{\partial f}{\partial y} \right|_i + \mathcal{O}(h^2)$$

$$\begin{aligned} y_{i+1} &= y_i + h(b_1 k_1 + b_2 k_2) + \mathcal{O}(h^3) \\ &= y_i + h \left[b_1 f_i + b_2 f(x_i + c_2 h, y_i + a_{2,1} h k_1) \right] + \mathcal{O}(h^3) \\ &= y_i + h \left[b_1 f_i + b_2 \left\{ f_i + c_2 h \left. \frac{\partial f}{\partial x} \right|_i + a_{2,1} h k_1 \left. \frac{\partial f}{\partial y} \right|_i + \mathcal{O}(h^2) \right\} \right] + \mathcal{O}(h^3) \\ &= y_i + h(b_1 + b_2)f_i + h^2 b_2 \left(c_2 \left. \frac{\partial f}{\partial x} \right|_i + a_{2,1} f_i \left. \frac{\partial f}{\partial y} \right|_i \right) + \mathcal{O}(h^3) \end{aligned}$$

Comparison with Taylor:

$$y_{i+1} = y_i + h f_i + \frac{h^2}{2} \left(\left. \frac{\partial f}{\partial x} \right|_i + \left. \frac{\partial f}{\partial y} \right|_i f_i \right) + \mathcal{O}(h^3)$$

Using $b_1 + b_2 = 1$, $c_2 b_2 = \frac{1}{2}$, $a_{2,1} b_2 = \frac{1}{2} \Rightarrow 3$ eqns and 4 unknowns \Rightarrow multiple possibilities!

Runge-Kutta methods — derivation

$$y_{i+1} = y_i + h(b_1 + b_2)f_i + h^2 b_2 \left(c_2 \left. \frac{\partial f}{\partial x} \right|_i + a_{2,1} f_i \left. \frac{\partial f}{\partial y} \right|_i \right) + \mathcal{O}(h^3)$$

$$y_{i+1} = y_i + hf_i + \frac{h^2}{2} \left(\left. \frac{\partial f}{\partial x} \right|_i + \left. \frac{\partial f}{\partial y} \right|_i f_i \right) + \mathcal{O}(h^3)$$

⇒ 3 eqns and 4 unknowns ⇒ multiple possibilities!

- ① Classical RK2:

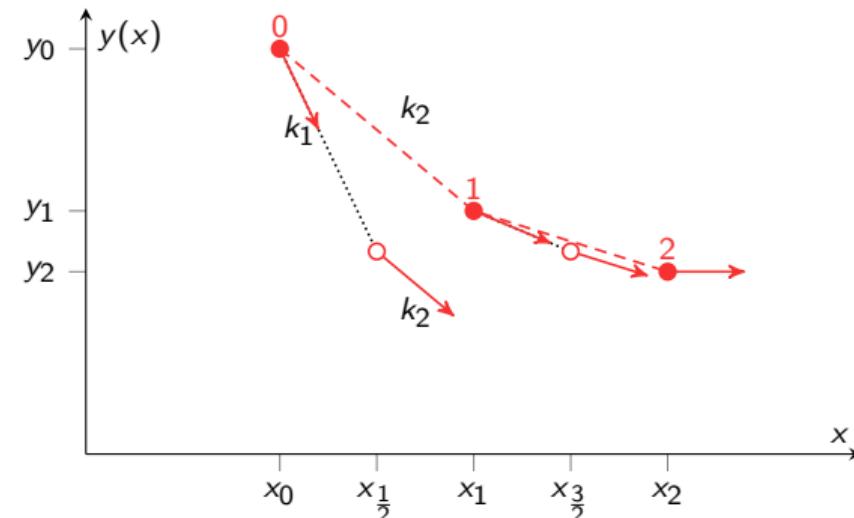
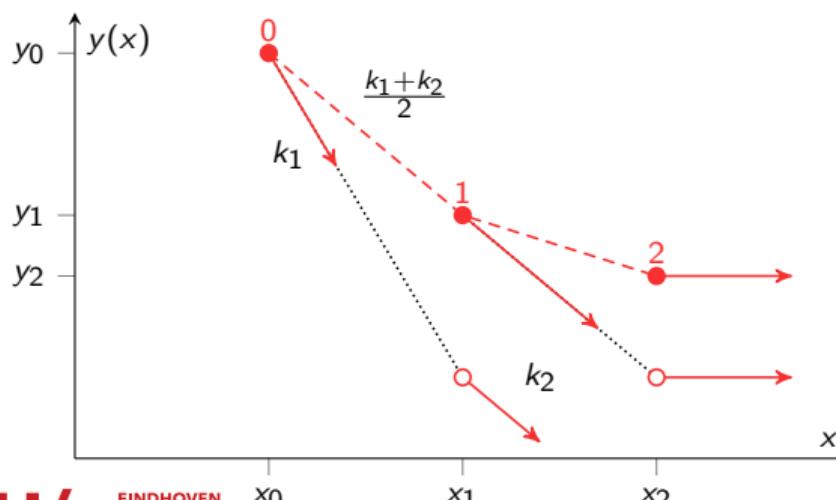
$$b_1 = b_2 = \frac{1}{2} \text{ and } c_2 = a_{2,1} = 1$$

- ② Midpoint rule (modified Euler):

$$b_1 = 0, b_2 = 1, c_2 = a_{2,1} = \frac{1}{2}$$

Second order Runge-Kutta methods

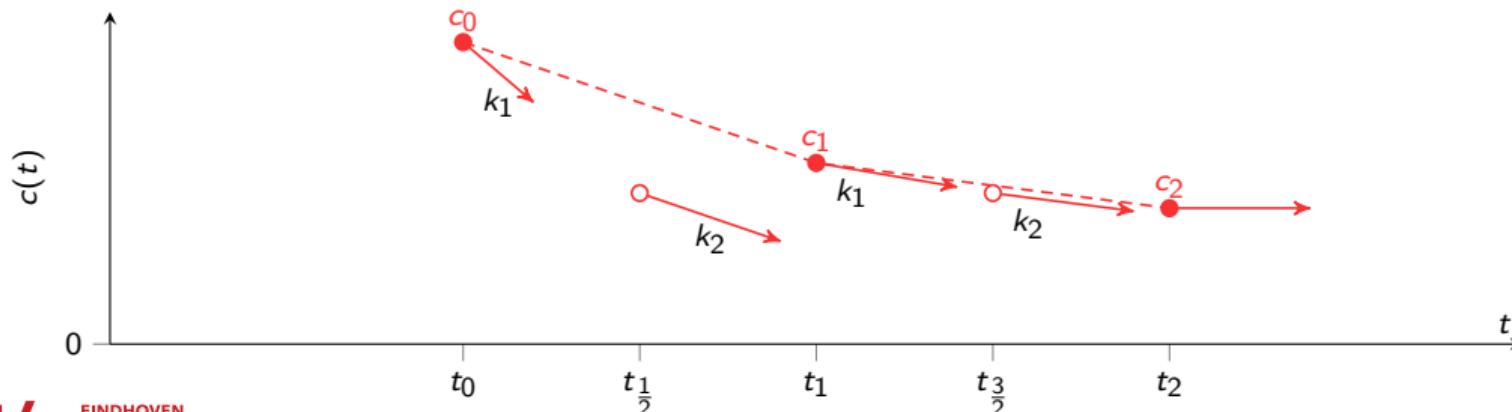
Classical RK2 method (= Heun's method, improved Euler method)	Explicit midpoint rule (modified Euler method)
$k_1 = f_i$	$k_1 = f_i$
$k_2 = f(x_i + h, y_i + hk_1)$	$k_2 = f(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_1)$
$y_{i+1} = y_i + \frac{1}{2}h(k_1 + k_2)$	$y_{i+1} = y_i + hk_2$



Second order Runge-Kutta method — Example

First order reaction in a batch reactor: $\frac{dc}{dt} = -kc$ with $c(t=0) = 1 \text{ mol m}^{-3}$, $k = 1 \text{ s}^{-1}$, $t_{\text{end}} = 2 \text{ s}$.

Time [s]	$C \text{ [mol m}^{-3}]$	$k_1 = hf(x_i, y_i)$	$k_2 = hf(x_i + \frac{1}{2}h, y_n + \frac{1}{2}k_1)$
0	1.00	$0.1 \cdot (-1 \cdot 1) = -0.1$	$0.1 \cdot (-1 \cdot (1 - 0.5 \cdot 0.1)) = -0.095$
0.1	$1 - 0.095 = 0.905$	$0.1 \cdot (-1 \cdot 0.905) = -0.0905$	$0.1 \cdot (-1 \cdot (0.905 - 0.5 \cdot 0.0905)) = -0.085975$
...
2	0.1358225	-0.0135822	-0.0129031



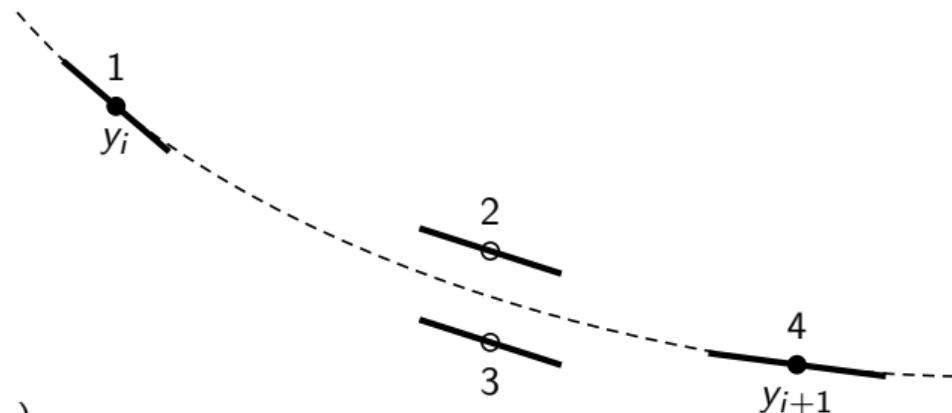
RK2 method — order of convergence

N	ζ	$\frac{\zeta_{\text{numerical}} - \zeta_{\text{analytical}}}{\zeta_{\text{analytical}}}$	$r = \frac{\log\left(\frac{\epsilon_i}{\epsilon_{i-1}}\right)}{\log\left(\frac{N_{i-1}}{N_i}\right)}$
20	0.864178	5.634×10^{-4}	—
40	0.864548	1.355×10^{-4}	2.056
80	0.864636	3.323×10^{-5}	2.028
160	0.864658	8.229×10^{-6}	2.014
320	0.864663	2.048×10^{-6}	2.007

⇒ RK2 is a second order method. Doubling the number of cells reduces the error by a factor 4!

Can we do even better?

RK4 method (classical fourth order Runge-Kutta method)



$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_1\right)$$

$$k_3 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_2\right)$$

$$k_4 = f(x_i + h, y_i + hk_3)$$

$$y_{i+1} = y_i + h \left(\frac{1}{6}k_1 + \frac{1}{3}(k_2 + k_3) + \frac{1}{6}k_4 \right)$$

RK4 method — order of convergence

N	ζ	$\frac{\zeta_{\text{numerical}} - \zeta_{\text{analytical}}}{\zeta_{\text{analytical}}}$	$r = \frac{\log\left(\frac{\epsilon_i}{\epsilon_{i-1}}\right)}{\log\left(\frac{N_{i-1}}{N_i}\right)}$
20	0.864664472	2.836×10^{-7}	—
40	0.864664702	1.700×10^{-8}	4.060
80	0.864664716	1.040×10^{-9}	4.030
160	0.864664717	6.435×10^{-11}	4.015
320	0.864664717	4.001×10^{-12}	4.007

⇒ RK4 is a fourth order method: Doubling the number of cells reduces the error by a factor 16!

Can we do even better?

Today's outline

- Introduction
- Euler's method
 - Forward Euler
- Rates of convergence
- Runge-Kutta methods
 - RK2 methods
 - RK4 method
- Step size control
- Solving ODEs in Matlab

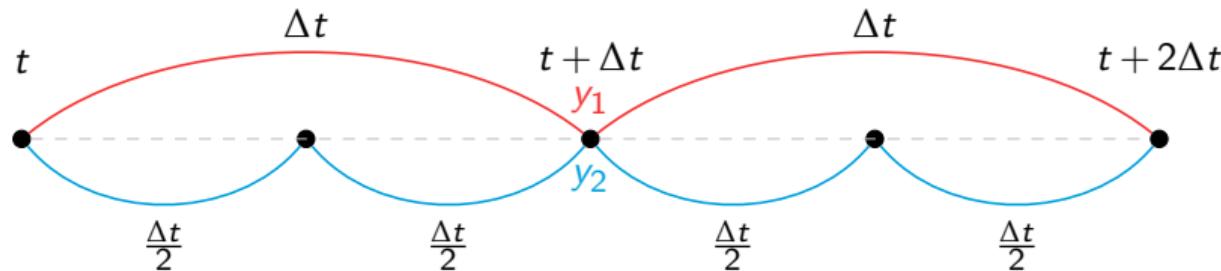
Adaptive step size control

The step size (be it either position, time or both (PDEs)) cannot be decreased indefinitely to favour a higher accuracy, since each additional grid point causes additional computation time. It may be wise to adapt the step size according to the computation requirements.

Globally two different approaches can be used:

- ① Step doubling: compare solutions when taking one full step or two consecutive halve steps
- ② Embedded methods: Compare solutions when using two approximations of different order

Adaptive step size control: step doubling



- RK4 with one large step of h : $y_{i+1} = y_1 + ch^5 + \mathcal{O}(h^6)$
- RK4 with two steps of $\frac{1}{2}h$: $y_{i+1} = y_2 + 2c(\frac{1}{2}h)^5 + \mathcal{O}(h^6)$

Adaptive step size control: step doubling

- Estimation of truncation error by comparing y_1 and y_2 :

$$\Delta = y_2 - y_1$$

- If Δ too large, reduce step size for accuracy
- If Δ too small, increase step size for efficiency.
- Ignoring higher order terms and solving for c :

$$\Delta = \frac{15}{16}ch^5 \Rightarrow ch^5 = \frac{16}{15}\Delta \Rightarrow y_{i+1} = y_2 + \frac{\Delta}{15} + \mathcal{O}(h^6)$$

(local Richardson extrapolation)

Note that when we specify a tolerance tol , we can estimate the maximum allowable step size

as: $h_{\text{new}} = \alpha h_{\text{old}} \left| \frac{\text{tol}}{\Delta} \right|^{\frac{1}{5}}$ with α a safety factor (typically $\alpha = 0.9$).

Adaptive step size control: embedded methods

Use a special fourth and a fifth order Runge Kutta method to approximate y_{i+1}

- The fourth order method is special because we want to use the same positions for the evaluation for computational efficiency.
- RK45 is the preferred method (minimum number of function evaluations) (this is built in Matlab as `ode45`).

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Solving ODEs in Matlab

Matlab provides convenient procedures to solve (systems of) ODEs automatically.

The procedure is as follows:

- ① Create a function that specifies the ODE(s). Specifically, this function returns the $\frac{dy}{dx}$ value (vector).
- ② Initialise solver variables and settings (e.g. step size, initial conditions, tolerance), in a separate script
- ③ Call the ODE solver function, using a *function handle* to the ODE function described in point 1.
 - The ODE solver will return the vector for the independent variable, and a solution vector (matrix for systems of ODEs).

Solving ODEs in Matlab: example 1

We solve the system: $\frac{dx}{dt} = -k_1 x + k_2, k_1 = 0.2, k_2 = 2.5$

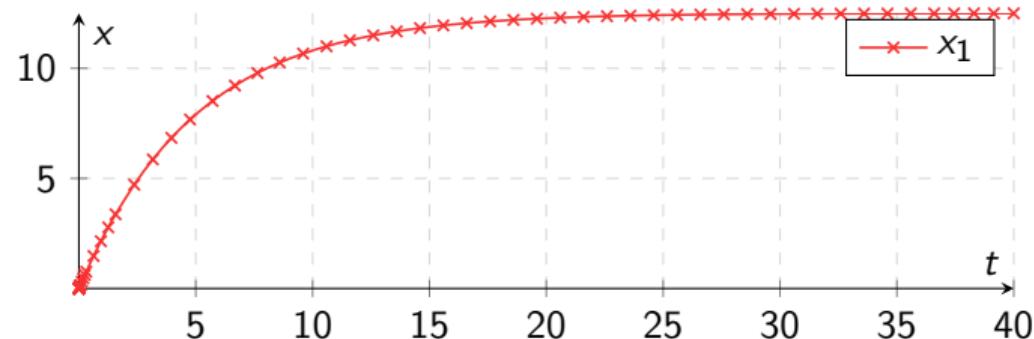
- Create an anonymous function handle:

```
>> myEqn = @(t,x) (-0.2*x + 2.5)
```

- Solve with a call to `ode45(function_handle, timespan, initial_condition)`:

```
>> ode45(myEqn, [0 40], 0);
```

- By omitting the output of this function, the graph is automatically drawn.



Solving ODEs in Matlab: example 2

We solve the system: $\frac{dx}{dt} = \begin{cases} -\frac{k_1}{x^2} & t \leq 10 \\ \frac{k_2}{x} - \frac{k_1}{x^2} & t > 10 \end{cases}$ with $k_1 = 0.5$, $k_2 = 1$, $x(0) = 2$

Create an ODE function

```
function [dxdt] = myEqnFunction(t,x)
k1 = 0.5;
k2 = 1;
dxdt = (t>10)*k2/x - k1/x^2;
```

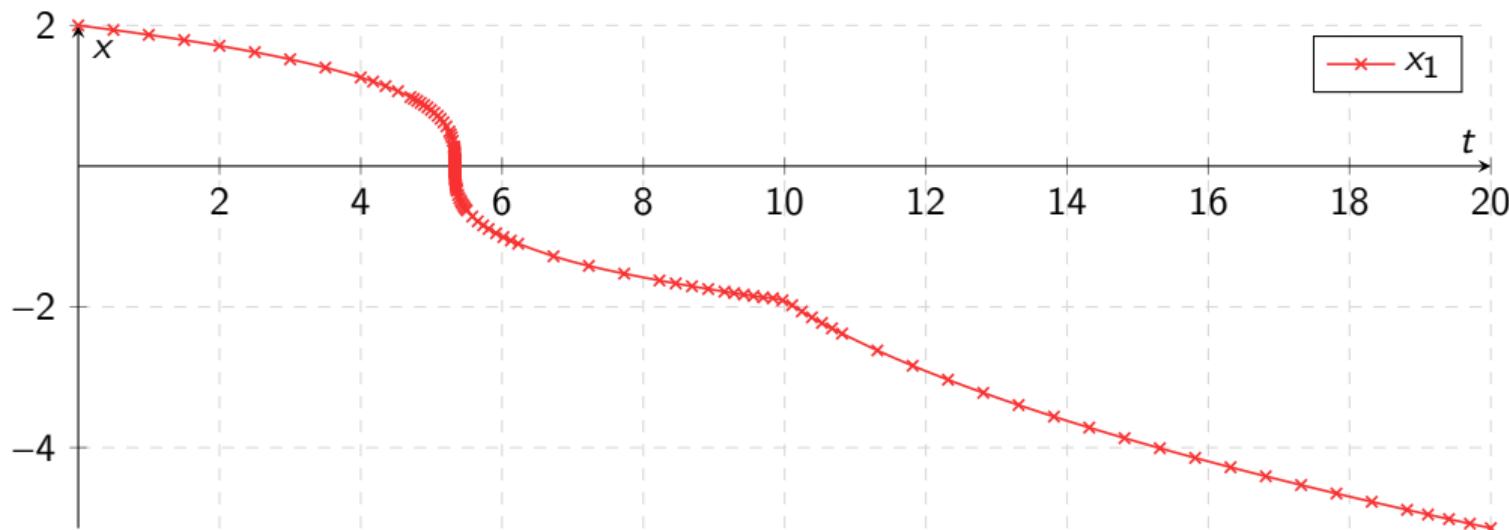
Create a solution script

```
tspan = [0 20]; % Time span
xinit = 2; % Initial condition
options = odeset('RelTol',1e-8,'AbsTol',1e-8);
[t,x] = ode45(@myEqnFunction,tspan,x_init,options);
```

Solving ODEs in Matlab: example 2

Plot the solution:

```
plot(t,x,'r-x')
```



Note the refinement in regions where large changes occur.

Solving ODEs in Matlab: example

A few notes on working with `ode45` and other solvers. If we want to give additional arguments (e.g. `k1` and `k2`) to our ODE function, we can list them in the function line:

```
function [dxdt] = myEqn(t,x,k1,k2)
```

The additional arguments can now be set in the solver script by *adding them after the options*:

```
[t,x] = ode45(@myEqn,tspan,x_0,options,k1,k2);
```

- Of course, in the solver script, the variables do not have to be called `k1` and `k2`:

```
[t,x] = ode45(@myODE,tspan,x_0,options,q,u);
```

- These variables may be of any type (vectors, matrix, struct). Especially a struct is useful to carry many values in 1 variable.

Solving systems of ODEs in Matlab: example

You have noticed that the step size in t varied. This is because we have given just the begin and end times of our time span:

```
tspan = [0 10];
```

You can also solve at specific steps, by supplying all steps explicitly, e.g.:

```
tspan = linspace(0,10,101);
```

This example provides 101 explicit time steps between 0 and 10 seconds.

Note that the results are interpolated to these data points afterwards; you do not influence the efficiency and accuracy of the solver algorithm this way!

Ordinary differential equations 2

Implicit methods, systems of ODEs and boundary value problems

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Numerical Methods (6E5X0), 2020-2021

Today's outline

● Introduction

- Backward Euler
- Implicit midpoint method

● Systems of ODEs

- Solution methods for systems of ODEs
- Solving systems of ODEs in Matlab
- Stiff systems of ODEs

● Boundary value problems

- Shooting method

● Conclusion

Problems with Euler's method: instability

Consider the ODE:

$$\frac{dy}{dx} = f(x, y(x)) \quad \text{with} \quad y(x=0) = y_0$$

First order approximation of derivative: $\frac{dy}{dx} = \frac{y_{i+1}-y_i}{\Delta x}$.

Where to evaluate the function f ?

- ① Evaluation at x_i : Explicit Euler method (forward Euler)
- ② Evaluation at x_{i+1} : Implicit Euler method (backward Euler)

Problems with Euler's method: instability – forward Euler

Explicit Euler method (forward Euler)

- Use values at x_i :

$$\frac{y_{i+1} - y_i}{\Delta x} = f(x_i, y_i) \Rightarrow y_{i+1} = y_i + h f(x_i, y_i).$$
 - This is an explicit equation for y_{i+1} in terms of y_i .
 - It can give instabilities with large function values.

Consider the first order batch reactor

$$\frac{dc}{dt} = -kc \Rightarrow c_{i+1} = c_i - k \textcolor{red}{c}_i \Delta t \Rightarrow \frac{c_{i+1}}{c_i} = 1 - k \Delta t$$

It follows that unphysical results are obtained for $k\Delta t \geq 1$!

Stability requirement

$$k\Delta t <$$

(but probably accuracy requirements are more stringent here!)

Problems with Euler's method: instability – backward Euler

Implicit Euler method (backward Euler)

- Use values at x_{i+1} : $\frac{y_{i+1} - y_i}{\Delta x} = f(x_{i+1}, y_{i+1}) \Rightarrow y_{i+1} = y_i + h f(x_{i+1}, y_{i+1})$.
 - This is an implicit equation for y_{i+1} , because it also depends on terms of y_{i+1} .

Consider the first order batch reactor

$$\frac{dc}{dt} = -kc \Rightarrow c_{i+1} = c_i - k \textcolor{red}{c_{i+1}} \Delta t \Rightarrow \frac{c_{i+1}}{c_i} = \frac{1}{1 + k \Delta t}$$

This equation does never give unphysical results

The implicit Euler method is *unconditionally stable* (but maybe not very accurate or efficient).

Semi-implicit Euler method

Usually f is a non-linear function of y , so that linearization is required (recall Newton's method).

$$\frac{dy}{dx} = f(y) \Rightarrow y_{i+1} = y_i + hf(y_{i+1}) \quad \text{using} \quad f(y_{i+1}) = f(y_i) + \left. \frac{df}{dy} \right|_i (y_{i+1} - y_i) + \dots$$

$$\Rightarrow y_{i+1} = y_i + h \left[f(y_i) + \left. \frac{df}{dy} \right|_i (y_{i+1} - y_i) \right]$$

$$\Rightarrow \left(1 - h \left. \frac{df}{dy} \right|_i \right) y_{i+1} = \left(1 - h \left. \frac{df}{dy} \right|_i \right) y_i + hf(y_i)$$

$$\Rightarrow y_{i+1} = y_i + h \left(1 - h \left. \frac{df}{dy} \right|_i \right)^{-1} f(y_i)$$

For the case that $f(x, y(x))$ we could add the variable x as an additional variable $y_{n+1} = x$. Or add one fully implicit Euler step (which avoids the computation of $\frac{\partial f}{\partial x}$):

$$y_{i+1} = y_i + hf(x_{i+1}, y_{i+1}) \Rightarrow y_{i+1} = y_i + h \left(1 - h \left. \frac{df}{dy} \right|_i \right)^{-1} f(x_{i+1}, y_i)$$

Semi-implicit Euler method - example

Second order reaction in a batch reactor

$$\frac{dc}{dt} = -kc^2 \text{ with } c_0 = 1 \text{ mol m}^{-3}, k = 1 \text{ m}^3 \text{ mol}^{-1} \text{ s}^{-1}, t_{\text{end}} = 2 \text{ s}$$

Analytical solution: $c(t) = \frac{c_0}{1 + kc_0 t}$

Define $f = -kc^2$, then $\frac{df}{dc} = -2kc \Rightarrow c_{i+1} = c_i - \frac{hkc_i^2}{1+2hkc_i}$

N	ζ	$\frac{\zeta_{\text{numerical}} - \zeta_{\text{analytical}}}{\zeta_{\text{analytical}}}$	$r = \frac{\log\left(\frac{\epsilon_j}{\epsilon_{j-1}}\right)}{\log\left(\frac{N_{j-1}}{N_j}\right)}$
20	0.654066262	1.89×10^{-2}	—
40	0.660462687	9.31×10^{-3}	1.02220
80	0.663589561	4.62×10^{-3}	1.01162
160	0.665134433	2.30×10^{-3}	1.00594
320	0.665902142	1.15×10^{-3}	1.00300

Second order implicit method: Implicit midpoint method

Implicit midpoint rule (second order)	Explicit midpoint rule (modified Euler method)
$y_{i+1} = y_i + hf\left(x_i + \frac{1}{2}h, \frac{1}{2}(y_i + y_{i+1})\right)$	$y_{i+1} = y_i + hf\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}hk_1\right)$

in case $f(y)$ then:

$$f\left(\frac{1}{2}(y_i + y_{i+1})\right) = f_i + \frac{df}{dy}\Bigg|_i \left(\frac{1}{2}(y_i + y_{i+1}) - y_i\right) = f_i + \frac{1}{2} \frac{df}{dy}\Bigg|_i (y_{i+1} - y_i)$$

Implicit midpoint rule reduces to

$$y_{i+1} = y_i + h f_i + \frac{h}{2} \left. \frac{df}{dy} \right|_i (y_{i+1} - y_i)$$

$$\Rightarrow \left(1 - \frac{h}{2} \left. \frac{df}{dy} \right|_i \right) y_{i+1} = \left(1 - \frac{h}{2} \left. \frac{df}{dy} \right|_i \right) y_i + h t$$

$$\Rightarrow y_{i+1} = y_i + h \left(1 - \frac{h}{2} \frac{df}{dy} \Big|_i \right)^{-1} f_i$$

Implicit midpoint method — example

Second order reaction in a batch reactor:

$\frac{dc}{dt} = -kc^2$ with $c_0 = 1 \text{ mol m}^{-3}$, $k = 1 \text{ m}^3 \text{ mol}^{-1} \text{ s}^{-1}$, $t_{\text{end}} = 2 \text{ s}$ (Analytical solution:
 $c(t) = \frac{c_0}{1+kc_0t}$).

Define $f = -kc^2$, then $\frac{df}{dc} = -2kc$.

Substitution:

$$\begin{aligned} c_{i+1} &= c_i + h \left(1 - \frac{h}{2} \cdot (-2kc_i) \right)^{-1} \cdot (-kc_i^2) \\ &= c_i - \frac{hkc_i^2}{1 + hkc_i} = \frac{c_i + hkc_i^2 - hkc_i^2}{1 + hkc_i} \Rightarrow c_{i+1} = \frac{c_i}{1 + hkc_i} \end{aligned}$$

You will find that this method is exact for all step sizes h because of the quadratic source term!

Implicit midpoint method — example

Second order reaction in a batch reactor:

$$\frac{dc}{dt} = -kc^2 \text{ with } c_0 = 1 \text{ mol m}^{-3}, k = 1 \text{ m}^3 \text{ mol}^{-1} \text{ s}^{-1}, t_{\text{end}} = 2 \text{ s}$$

Analytical solution: $c(t) = \frac{c_0}{1 + kc_0 t}$

$$c_{i+1} = \frac{c_i}{1 + hkc_i}$$

N	ζ	$\frac{\zeta_{\text{numerical}} - \zeta_{\text{analytical}}}{\zeta_{\text{analytical}}}$	$r = \frac{\log\left(\frac{\epsilon_i}{\epsilon_{i-1}}\right)}{\log\left(\frac{N_{i-1}}{N_i}\right)}$
20	0.6666666667	1.665×10^{-16}	—
40	0.6666666667	0	—
80	0.6666666667	0	—
160	0.6666666667	0	—
320	0.6666666667	0	—

Implicit midpoint method — example

Third order reaction in a batch reactor: $\frac{dc}{dt} = -kc^3$

Analytical solution: $c(t) = \frac{c_0}{\sqrt{1+2kc_0^2 t}}$

$$c_{i+1} = c_i - \frac{h k c_i^3}{1 + \frac{3}{2} h k c_i^2}$$

N	ζ	$\frac{\zeta_{\text{numerical}} - \zeta_{\text{analytical}}}{\zeta_{\text{analytical}}}$	$r = \frac{\log\left(\frac{\epsilon_i}{\epsilon_{i-1}}\right)}{\log\left(\frac{N_{i-1}}{N_i}\right)}$
20	0.5526916174	1.71×10^{-4}	—
40	0.5527633731	4.17×10^{-5}	2.041
80	0.5527807304	1.03×10^{-5}	2.021
160	0.5527849965	2.55×10^{-6}	2.011
320	0.5527860538	6.34×10^{-7}	2.005

Today's outline

● Introduction

- Backward Euler
- Implicit midpoint method

● Systems of ODEs

- Solution methods for systems of ODEs
- Solving systems of ODEs in Matlab
- Stiff systems of ODEs

● Boundary value problems

- Shooting method

● Conclusion

Systems of ODEs

A system of ODEs is specified using vector notation:

$$\frac{dy}{dx} = \mathbf{f}(x, \mathbf{y}(x))$$

for

$$\frac{dy_1}{dx} = f_1(x, y_1(x), y_2(x)) \quad \text{or} \quad f_1(x, y_1, y_2)$$

$$\frac{dy_2}{dx} = f_2(x, y_1(x), y_2(x)) \quad \text{or} \quad f_2(x, y_1, y_2)$$

The solution techniques discussed before can also be used to solve systems of equations.

Systems of ODEs: Explicit methods

Forward Euler method

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\mathbf{f}(x_i, \mathbf{y}_i)$$

Improved Euler method (classical RK2)

$$\mathbf{y}_{i+1} = \mathbf{y}_i + \frac{h}{2}(\mathbf{k}_1 + \mathbf{k}_2) \quad \text{using} \quad \begin{aligned} \mathbf{k}_1 &= \mathbf{f}(x_i, \mathbf{y}_i) \\ \mathbf{k}_2 &= \mathbf{f}(x_i + h, \mathbf{y}_i + h\mathbf{k}_1) \end{aligned}$$

Modified Euler method (midpoint rule)

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\mathbf{k}_2 \quad \text{using} \quad \begin{aligned} \mathbf{k}_1 &= \mathbf{f}(x_i, \mathbf{y}_i) \\ \mathbf{k}_2 &= \mathbf{f}\left(x_i + \frac{h}{2}, \mathbf{y}_i + \frac{h}{2}\mathbf{k}_1\right) \end{aligned}$$

Systems of ODEs: Explicit methods

Classical fourth order Runge-Kutta method (RK4)

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h \left(\frac{\mathbf{k}_1}{6} + \frac{1}{3} (\mathbf{k}_2 + \mathbf{k}_3) + \frac{\mathbf{k}_4}{6} \right)$$

$$\mathbf{k}_1 = \mathbf{f}(x_i, \mathbf{y}_i)$$

$$\mathbf{k}_2 = \mathbf{f}\left(x_i + \frac{h}{2}, \mathbf{y}_i + \frac{h}{2} \mathbf{k}_1\right)$$

using

$$\mathbf{k}_3 = \mathbf{f}\left(x_i + \frac{h}{2}, \mathbf{y}_i + \frac{h}{2} \mathbf{k}_2\right)$$

$$\mathbf{k}_4 = \mathbf{f}(x_i + h, \mathbf{y}_i + h \mathbf{k}_3)$$

Solving systems of ODEs in Matlab

Solving systems of ODEs in Matlab is completely analogous to solving a single ODE:

- ① Create a function that specifies the ODEs. This function returns the $\frac{dy}{dx}$ vector.
- ② Initialise solver variables and settings (e.g. step size, initial conditions, tolerance), in a separate script. Initial conditions and tolerances should be given per-equation, i.e. as a vector.
- ③ Call the ODE solver function, using a *function handle* to the ODE function described in point 1.
 - The ODE solver will return the vector for the independent variable, and a solution matrix, with a column as the solution for each equation in the system.

Solving systems of ODEs in Matlab: example

We solve the system: $\frac{dx_1}{dt} = -x_1 - x_2, \quad \frac{dx_2}{dt} = x_1 - 2x_2$

Create an ODE function

```
function [dxdt] = myODEFunction(t,x)
dxdt(1) = -x(1) - x(2);
dxdt(2) = x(1) - 2*x(2);
dxdt=dxdt'; % Transpose to column vector
return
```

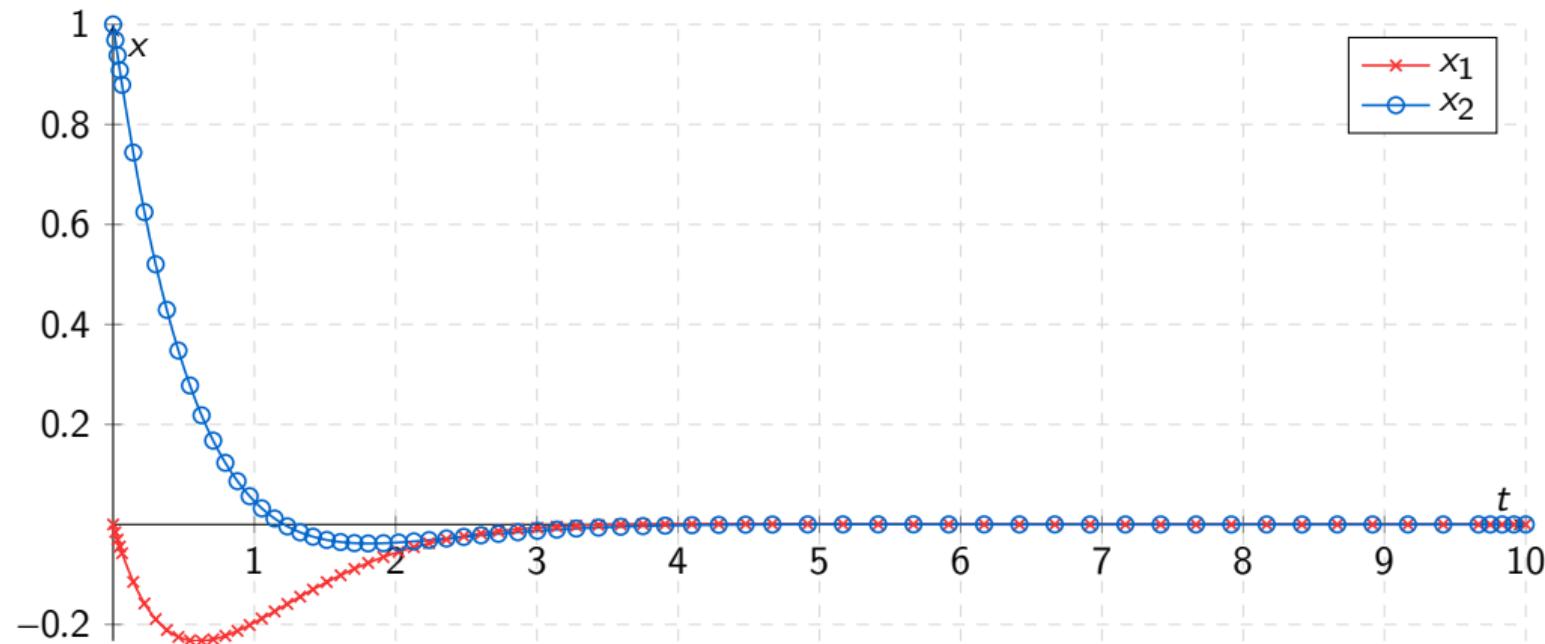
Create a solution script

```
x_init = [0 1]; % Initial conditions
tspan = [0 10]; % Time span
options = odeset('RelTol',1e-4,'AbsTol',[1e-4 1e-4]);
[t,x] = ode45(@myODEfunction,tspan,x_init,options);
```

Solving systems of ODEs in Matlab: example

Plot the solution:

```
plot(t,x(:,1),'r-x',t,x(:,2),'b-o')
```



Solving systems of ODEs in Matlab: repeated notes

A few notes on working with `ode45` and other solvers. If we want to give additional arguments (e.g. `a`, `b` and `c`) to our ODE function, we can list them in the function line:

```
function [dxdt] = myODE(t,x,a,b,c)
```

The additional arguments can now be set in the solver script by *adding them after the options*:

```
[t,x] = ode45(@myODE,tspan,x_0,options,a,b,c);
```

- Of course, in the solver script, the variables do not need to be called `a`, `b` and `c`:

```
[t,x] = ode45(@myODE,tspan,x_0,options,k1,phi,V);
```

- These variables may be of any type (vectors, matrix, struct). Especially a struct is useful to carry many values in 1 variable.
- Alternative way to define a function handle with more arguments than the default (`t,x`) required by `ode45`:

```
[t,x] = ode45(@(t,x) myode(t,x,k1,phi,V),tspan,x_0,options);
```

Solving systems of ODEs in Matlab: example

You may have noticed that the step size in t varied. This is because we have given the begin and end times of our time span, and `ode45` uses adaptive step size for efficiency:

```
tspan = [0 10];
```

You can also retrieve the solution at specific steps, by supplying all steps explicitly as a vector, e.g.:

```
tspan = linspace(0,10,101);
```

This example provides 101 explicit time steps between 0 and 10 seconds.

Note that this is an interpolated result. The solver uses, in the background, still the adaptive step size functionality!

Systems of ODEs: Implicit methods

Backward Euler method

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h \left(\mathbf{I} - h \frac{d\mathbf{f}}{d\mathbf{y}} \Big|_i \right)^{-1} \mathbf{f}(\mathbf{y}_i)$$

Implicit midpoint method

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h \left(\mathbf{I} - \frac{h}{2} \frac{d\mathbf{f}}{d\mathbf{y}} \Big|_i \right)^{-1} \mathbf{f}(\mathbf{y}_i)$$

Stiff systems of ODEs

A system of ODEs can be stiff and require a different solution method. For example:

$$\frac{dc_1}{dt} = 998c_1 + 1998c_2 \quad \frac{dc_2}{dt} = -999c_1 - 1999c_2$$

with boundary conditions $c_1(t=0) = 1$ and $c_2(t=0) = 0$.

The analytical solution is:

$$c_1 = 2e^{-t} - e^{-1000t} \quad c_2 = -e^{-t} + e^{-1000t}$$

For the explicit method we require $\Delta t < 10^{-3}$ despite the fact that the term is completely negligible, but essential to keep stability.

The “disease” of stiff equations: we need to follow the solution on the shortest length scale to maintain stability of the integration, although accuracy requirements would allow a much larger time step.

Demonstration with example

Forward Euler (explicit)

$$\frac{c_{1,i+1} - c_{1,i}}{dt} = 998c_{1,i} + 1998c_{2,i}$$

$$\frac{c_{2,i+1} - c_{2,i}}{dt} = -999c_{1,i} - 1999c_{2,i}$$

$$\Rightarrow \begin{aligned} c_{1,i+1} &= (1 + 998\Delta t) c_{1,i} + 1998\Delta t c_{2,i} \\ c_{2,i+1} &= -999\Delta t c_{1,i} + (1 - 1999\Delta t) c_{2,i} \end{aligned}$$

Demonstration with example

Backward Euler (implicit)

$$\frac{c_{1,i+1} - c_{1,i}}{\Delta t} = 998c_{1,i+1} + 1998c_{2,i+1}$$

$$\frac{c_{2,i+1} - c_{2,i}}{\Delta t} = -999c_{1,i+1} - 1999c_{2,i+1}$$

$$\Rightarrow \begin{aligned} (1 - 998\Delta t)c_{1,i+1} - 1998\Delta t c_{2,i} &= c_{1,i} \\ 999\Delta t c_{1,i+1} + (1 + 999\Delta t)c_{2,i+1} &= c_{2,i} \end{aligned}$$

$$A\mathbf{c}_{i+1} = \mathbf{c}_i \text{ with } A = \begin{pmatrix} 1 - 998\Delta t & -1998\Delta t \\ 999\Delta t & 1 + 1999\Delta t \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} c_{1,i} \\ c_{2,i} \end{pmatrix}$$

Demonstration with example

Backward Euler (implicit) $A\mathbf{c}_{i+1} = \mathbf{c}_i$ with $A = \begin{pmatrix} 1 - 998\Delta t & -1998\Delta t \\ 999\Delta t & 1 + 1999\Delta t \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} c_{1,i} \\ c_{2,i} \end{pmatrix}$

Cramers rule:

$$c_{1,i+1} = \frac{\begin{vmatrix} c_{1,i} & -1998\Delta t \\ c_{2,i} & 1 + 1999\Delta t \end{vmatrix}}{\det |A|} = \frac{(1+1999\Delta t)c_{1,i} + 1998\Delta t c_{2,i}}{(1-998\Delta t)(1+1999\Delta t) + 1998 \cdot 999\Delta t^2}$$

$$c_{2,i+1} = \frac{\begin{vmatrix} 1 - 998\Delta t & c_{1,i} \\ 999\Delta t & c_{2,i} \end{vmatrix}}{\det |A|} = \frac{-999\Delta t c_{1,i} + (1-998\Delta t) c_{2,i}}{(1-998\Delta t)(1+1999\Delta t) + 1998 \cdot 999\Delta t^2}$$

Forward Euler: $\Delta t \leq 0.001$ for stability

Backward Euler: always stable, even for $\Delta t > 100$ (but then not very accurate!)

Demonstration with example

Cure for stiff problems: use implicit methods! To find out whether your system is stiff: check whether one of the eigenvalues have an imaginary part

Implicit methods in Matlab

Matlab offers a stabilized solver, `ode15s`, for stiff problems.

$$\frac{dc_1}{dt} = 998c_1 + 1998c_2 \quad \frac{dc_2}{dt} = -999c_1 - 1999c_2, c_1(0) = 1, c_2(0) = 0$$

- Create the ode function

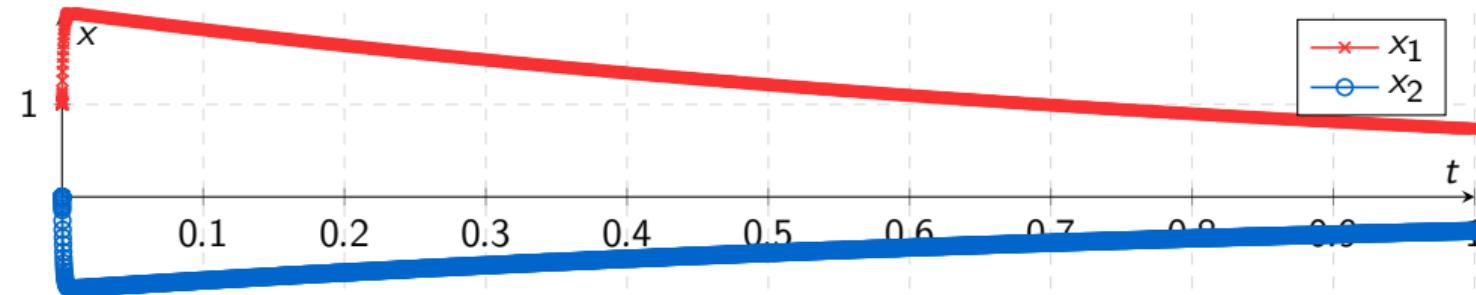
```
function [dcdt] = stiff_ode(t,c)
dcdt = zeros(2,1); % Pre-allocation
dcdt(1) = 998 * c(1) + 1998*c(2);
dcdt(2) = -999 * c(1) - 1999*c(2);
return
```

- Compare the resolution of the solutions

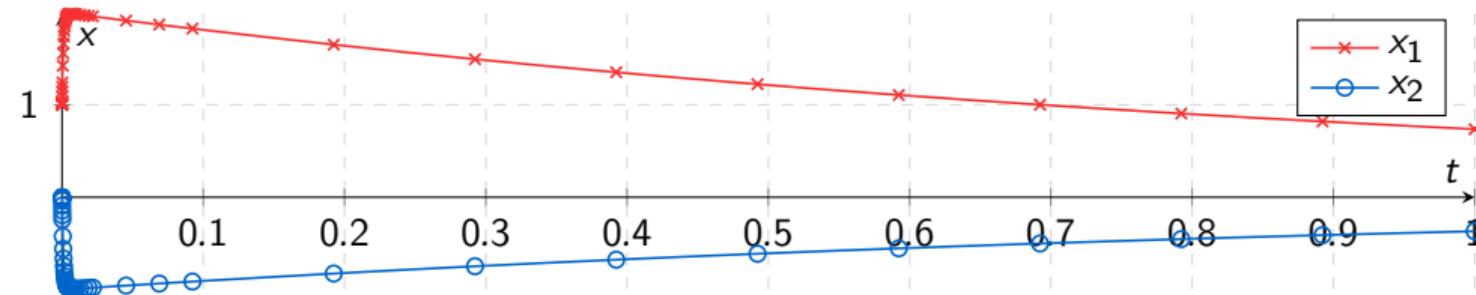
```
subplot(2,1,1);
ode45(@stiff_ode, [0 1], [1 0]);
subplot(2,1,2);
ode15s(@stiff_ode, [0 1], [1 0]);
```

Implicit methods in Matlab

ode45



ode15s



The explicit solver requires 1245 data points (default settings), the implicit solver just 48!

Implicit methods in Matlab: Generic backward Euler

How to make a generic Backward Euler implementation? Recall the update formula:

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h \left(\mathbf{I} - h \frac{d\mathbf{f}}{d\mathbf{y}} \Big|_i \right)^{-1} \mathbf{f}(\mathbf{y}_i)$$

- Set up input: Number of steps, end time, initial conditions
- Preallocate and calculate: create a full time vector, calculate the step size h , preallocate \mathbf{y} with zeros and store the initial condition as the first \mathbf{y} .
- Loop over the number of iterations:
 - Compute the Jacobian: calculate the function both for \mathbf{y}_i as well as for $\mathbf{y}_i + s$, where s is a very small number. Recall:

$$\frac{df}{dy} = \frac{f(y+s) - f(y)}{s}$$

- Compute the update formula for \mathbf{y}_{i+1} . Use `eye`, `inv`.

Today's outline

● Introduction

- Backward Euler
- Implicit midpoint method

● Systems of ODEs

- Solution methods for systems of ODEs
- Solving systems of ODEs in Matlab
- Stiff systems of ODEs

● Boundary value problems

- Shooting method

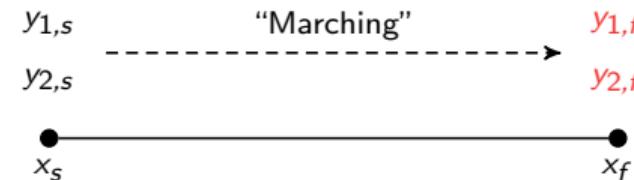
● Conclusion

Importance of boundary conditions

The nature of boundary conditions determines the appropriate numerical method. Classification into 2 main categories:

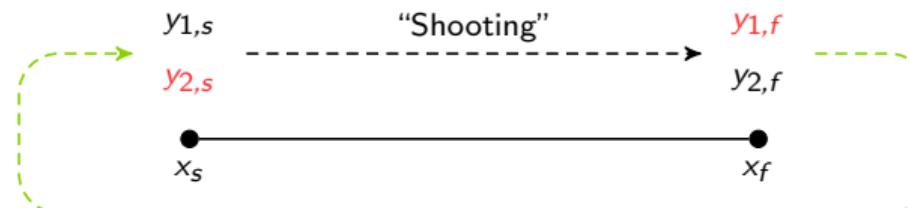
- *Initial value problems (IVP)*

We know the values of all y_i at some starting position x_s , and it is desired to find the values of y_i at some final point x_f .



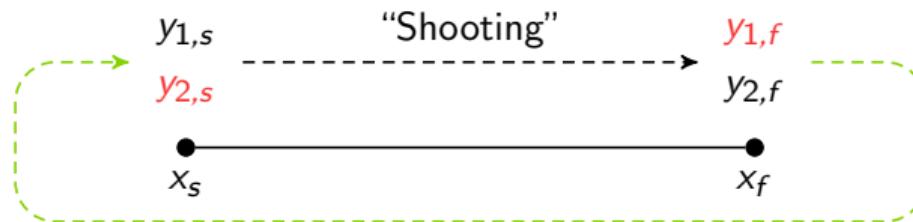
- *Boundary value problems (BVP)*

Boundary conditions are specified at more than one x . Typically, some of the BC are specified at x_s and the remainder at x_f .



Shooting method

How to solve a BVP using the shooting method:



- Define the system of ODEs
- Provide an initial guess for the unknown boundary condition
- Solve the system and compare the resulting boundary condition to the expected value
- Adjust the guessed boundary value, and solve again. Repeat until convergence.
 - Of course, you can subtract the expected value from the computed value at the boundary, and use a non-linear root finding method

BVP: example in Excel

Consider a chemical reaction in a liquid film layer of thickness δ :

$$\mathcal{D} \frac{d^2 c}{dx^2} = k_R c \text{ with } \begin{aligned} c(x=0) &= C_{A,i,L} = 1 && \text{(interface concentration)} \\ c(x=\delta) &= 0 && \text{(bulk concentration)} \end{aligned}$$

Question: compute the concentration profile in the film layer.

Step 1: Define the system of ODEs

This second-order ODE can be rewritten as a system of first-order ODEs, if we define the flux q as:

$$q = -\mathcal{D} \frac{dc}{dx}$$

Now, we find:

$$\frac{dc}{dx} = -\frac{1}{\mathcal{D}} q$$

$$\frac{dq}{dx} = -k_R c$$

BVP: example in Excel

Consider a chemical reaction in a liquid film layer of thickness δ :

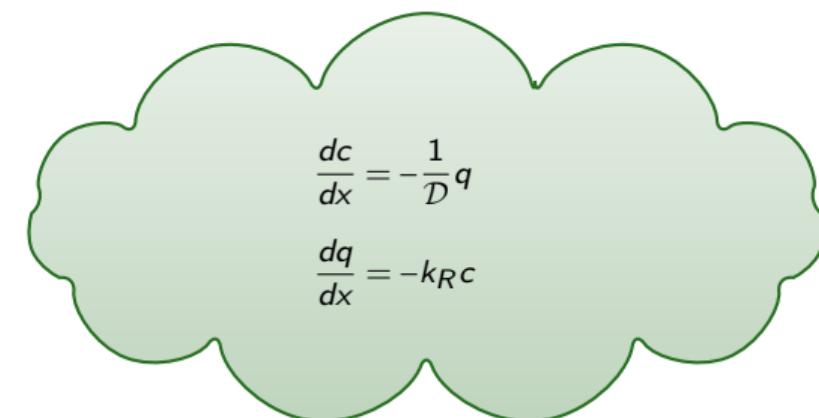
$$\mathcal{D} \frac{d^2 c}{dx^2} = k_R c \text{ with } \begin{aligned} c(x=0) &= C_{A,i,L} = 1 && \text{(interface concentration)} \\ c(x=\delta) &= 0 && \text{(bulk concentration)} \end{aligned}$$

Question: compute the concentration profile in the film layer.

Step 2: Set the boundary conditions

The boundary conditions for the concentrations at $x = 0$ and $x = \delta$ are known.

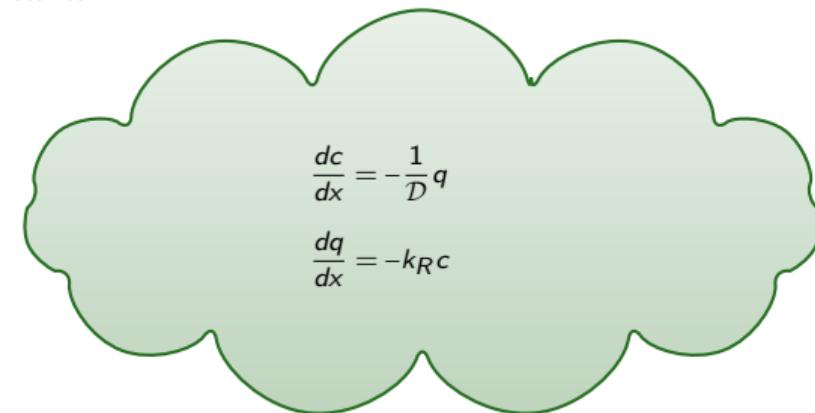
The flux at the interface, however, is not known, and should be solved for.


$$\begin{aligned} \frac{dc}{dx} &= -\frac{1}{\mathcal{D}} q \\ \frac{dq}{dx} &= -k_R c \end{aligned}$$

BVP: example in Excel

Solving the two first-order ODEs in Excel. First, the cells with constants:

	A	B	C
1	CAiL	1	mol/m ³
2	D	1e-8	m ² /s
3	kR	10	1/s
4	delta	1e-4	m
5	N	100	
6	dx	=B4/B5	



Now, we program the forward Euler (explicit) schemes for c and q below:

	A	B	C
10	x	c	q
11	0	=B1	10
12	=A11+\$B\$6	=B11+\$B\$6*(-1/\$B\$2*C11)	=C11+\$B\$6*(-\$B\$3*B11)
13	=A12+\$B\$6	=B12+\$B\$6*(-1/\$B\$2*C12)	=C12+\$B\$6*(-\$B\$3*B12)
...
111	=A110+\$B\$6	=B110+\$B\$6*(-1/\$B\$2*C110)	=C110+\$B\$6*(-\$B\$3*B110)

BVP: example in Excel

- We now have profiles for c and q as a function of position x .
- The concentration $c(x = \delta)$ depends (eventually) on the boundary condition at the interface $q(x = 0)$
- We can use the solver to change $q(x = 0)$ such that the concentration at the bulk meets our requirement: $c(x = \delta) = 0$

BVP: example in Matlab

We first program the system of ODEs in a separate function:

$$\frac{dc}{dx} = -\frac{1}{D}q$$
$$\frac{dq}{dx} = -k_R c$$

```
function [dydx] = diffReactSystem(t,y,ps)
c = y(1);
q = y(2);

dcdx = -q/ps.D;
dqdx = -ps.kR*c;

dydx = [dcdx; dqdx];
end
```

Note that we pass a variable (type: struct) that contains required parameters: ps.

BVP: example in Matlab

Let's first try to solve the ODE system using [ode45](#):

```
% Set up parameters
ps.D = 1e-8;      % Diffusion constant [m2/s]
ps.kR = 10;       % Reaction rate [1/s]
ps.delta = 1e-4;  % Film thickness [m]
ps.C_a_L = 1;     % Liquid side interface concentration [mol/m3]
ps.q0 = 1e-3;     % Initial guess for flux at x=0 [mol/m2/s]

%% Solve ODE system
[t,y] = ode45(@diffReactSystem, [0 ps.delta], [ps.C_a_L ps.q0],[],ps);

%% Postprocessing
c = y(:,1);
q = y(:,2);

plot(t,c);
xlabel('Position in film layer [m]');
ylabel('Concentration value');
yyaxis right % We can use 2 y axes to plot data on different scales
plot(t,q);
ylabel('Flux')
```

BVP: example in Matlab

That seems to work! Now we want to fit the value for q at $x = 0$ (defined below as `bcq`), such that the concentration at $x = \delta$ equals zero. We create a function with the output defined as the deviation from the target value:

```
function f = diffReactCrit(bcq, ps)
% Solve ODE using fitting parameter bcq (boundary condition for q),
% other parameters are defined in ps
[t,y] = ode45(@diffReactSystem, [0 ps.delta], [ps.C_a_L bcq],[],ps);
% Extract concentration and flux for clarity
c = y(:,1);
q = y(:,2);
% We subtract the desired value from the concentration at x=delta (0 in this case)
f = c(end) - 0;
end
```

Note the following:

- We use the interval $0 \leq x \leq \delta$
- Boundary conditions are given as: $c(x = 0) = 1$ and $q(x = 0) = \text{bcq}$, which is given as an argument to the function (i.e. changeable from 'outside'!)
- The function returns `f`, the difference between the computed and desired concentration at $x = \delta$.

BVP: example in Matlab

Finally, we should solve the system so that we obtain the right boundary condition $q = b_{cq}$ such that $c(x = \delta) = 0$. We can use the built-in function `fzero` to do this:

```
% Set up parameters
ps.D = 1e-8; % Diffusion constant
ps.kR = 10; % Reaction rate
ps.delta = 1e-4; % Film thickness
ps.C_a_L = 1; % Liquid side interface concentration
ps.q0 = 2e-4; % Initial guess for q(x=0)

% Fit boundary condition for q on x=0 such that c(end)=0
fitted_q = fzero(@diffReactCrit,ps.q0,[],ps)

% Solve ODE once more such that we can plot the final data
[t,y] = ode45(@diffReactSystem, [0 ps.delta], [ps.C_a_L fitted_q],[],ps);
```

Postprocessing of the data can be done similar to the example in slide 473.

BVP example: analytical solution

Compare with the analytical solution:

$$q = k_L E_A C_{A,i,L} \quad \text{with}$$

$$E_A = \frac{\text{Ha}}{\tanh \text{Ha}} \quad (\text{Enhancement factor})$$

$$\text{Ha} = \frac{\sqrt{k_R D}}{k_L} \quad (\text{Hatta number})$$

$$k_L = \frac{D}{\delta} \quad (\text{mass transfer coefficient})$$

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Other methods

Other explicit methods:

- Bulirsch-Stoer method (Richardson extrapolation + modified midpoint method)

Other implicit methods:

- Rosenbrock methods (higher order implicit Runge-Kutta methods)
- Predictor-corrector methods

Summary

- Several solution methods and their derivation were discussed:
 - Explicit solution methods: Euler, Improved Euler, Midpoint method, RK45
 - Implicit methods: Implicit Euler and Implicit midpoint method
 - A few examples of their spreadsheet implementation were shown
- We have paid attention to accuracy and instability, rate of convergence and step size
- Systems of ODEs can be solved by the same algorithms. Stiff problems should be treated with care.
- An example of solving ODEs with Matlab was demonstrated.

Partial differential equations

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Numerical Methods (6E5X0), 2020-2021

Today's outline

● Introduction

● Instationary diffusion equation

- Discretization
- Solving the diffusion equation
- Non-linear source terms

● Convection

- Discretization
- Central difference scheme
- Upwind scheme

● Conclusions

- Other methods
- Summary

Overview

Main question

How to solve parabolic PDEs like:

$$\frac{\partial c}{\partial t} = \mathcal{D} \frac{\partial^2 c}{\partial x^2} - u \frac{\partial c}{\partial x} + R$$

$$t = 0; 0 \leq x \leq \ell \Rightarrow c = c_0$$

with

$$t > 0; x = 0 \Rightarrow -\mathcal{D} \frac{\partial c}{\partial x} + uc = u_{\text{in}} c_{\text{in}}$$
$$t > 0; x = \ell \Rightarrow \frac{\partial c}{\partial x} = 0$$

accurately and efficiently?

What is a PDE?

Partial differential equation

An equation containing a function and their derivatives to multiple independent variables.

Order of PDE

The highest derivative appearing in the PDE

General second order PDE:

$$A \frac{\partial^2 f}{\partial x^2} + B \frac{\partial^2 f}{\partial x \partial y} + C \frac{\partial^2 f}{\partial y^2} + D \frac{\partial f}{\partial x} + E \frac{\partial f}{\partial y} + Ff = G$$

- Linear equation: Coefficients A, B, \dots, G do not depend on x and y .
 - Non-linear equation: Coefficients A, B, \dots, G are a function of x and y .

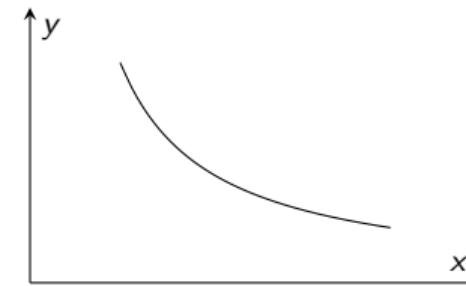
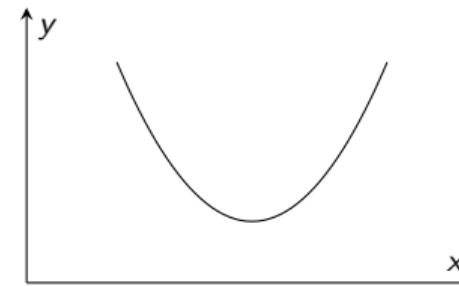
Classification of PDE's

$$A \frac{\partial^2 f}{\partial x^2} + B \frac{\partial^2 f}{\partial x \partial y} + C \frac{\partial^2 f}{\partial y^2} + D \frac{\partial f}{\partial x} + E \frac{\partial f}{\partial y} + Ff = G$$

The discriminant Δ of a quadratic polynomial is computed as (note: only the higher order coefficients are important):

$$\Delta \equiv B^2 - 4AC$$

- $\Delta < 0 \Rightarrow$ Elliptic equation
(e.g. Laplace equation for stationary diffusion in 2D)
 - $\Delta = 0 \Rightarrow$ Parabolic equation
(e.g. instationary heat penetration in 1D)
 - $\Delta > 0 \Rightarrow$ Hyperbolic equation
(e.g. wave equation)

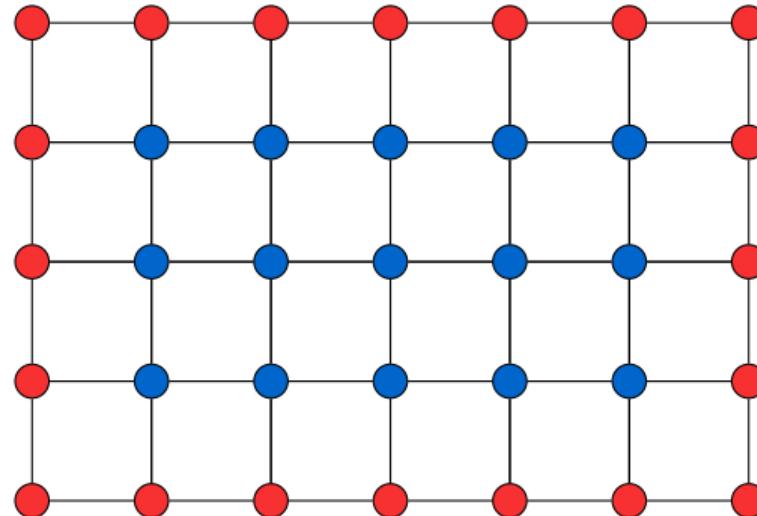


Importance of classification

Different PDE types require different solution techniques because of the difference in range of influence:

- *Characteristics*
Curves in xy -domain along with signal propagation takes place
- *Domain of dependence of point P*
points in xy -domain which influence the value of f in point P
- *Range of influence of point P*
points in xy -domain which are influenced by the value of f in point P

Example elliptic PDE (boundary value problems: BVP)

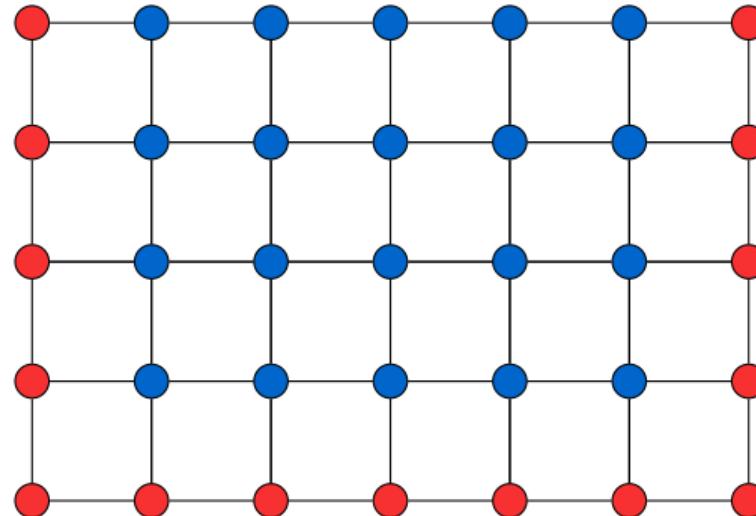


- Grid point at which dependent variable has to be computed
- Grid point at which boundary condition is specified

Typical example: Poisson equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = f(x, y)$$

Example parabolic PDE (initial value problem: IVP)



- Grid point at which dependent variable has to be computed
- Red Grid point at which boundary condition is specified

Typical example: Poisson equation

$$\frac{\partial c}{\partial t} + u \frac{\partial c}{\partial x} = \mathcal{D} \frac{\partial^2 c}{\partial x^2} + R$$

TU/e Eindhoven University of Technology Stability (in numerical sense) of the numerical method is of crucial importance.

Boundary conditions

- Dirichlet or fixed condition: prescribed value of f at boundary

$$f = f_0 \quad f_0 \text{ is a known function}$$

- Neumann condition: prescribed value of derivative of f at boundary

$$\frac{\partial f}{\partial n} = q \quad q \text{ is a known function}$$

- Mixed or Robin condition: relation between f and $\frac{\partial f}{\partial n}$ at boundary

$$a \frac{\partial f}{\partial n} + bf = c \quad a, b \text{ and } c \text{ are known functions}$$

Numerical solution method

Finite differences (method of lines, MOL):

- ① Discretize spatial domain in discrete grid points
- ② Find suitable approximation for the spatial derivatives
- ③ Substitute approximations in PDE, which gives a system of ODE's, one for every grid points
- ④ Advance in time with a suitable ODE solver

Alternative methods: collocation, Galerkin or Finite Element methods

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Instationary diffusion equation (Fick's second law)

$$\frac{\partial c}{\partial t} = D \frac{\partial^2 c}{\partial x^2}, \quad \text{with} \quad \begin{aligned} t = 0; 0 \leq x \leq \ell &\Rightarrow c = c_0 \\ t > 0; x = 0 &\Rightarrow c = c_L \\ t > 0; x = \ell &\Rightarrow c = c_R \end{aligned}$$

Second derivative $\frac{\partial^2 c}{\partial x^2}$



$$c_{i+1} = c_i + \left. \frac{\partial c}{\partial x} \right|_i \Delta x + \frac{1}{2} \left. \frac{\partial^2 c}{\partial x^2} \right|_i \Delta x^2 + \frac{1}{6} \left. \frac{\partial^3 c}{\partial x^3} \right|_i \Delta x^3 + \dots$$

$$c_{i-1} = c_i - \left. \frac{\partial c}{\partial x} \right|_i \Delta x + \frac{1}{2} \left. \frac{\partial^2 c}{\partial x^2} \right|_i \Delta x^2 - \frac{1}{6} \left. \frac{\partial^3 c}{\partial x^3} \right|_i \Delta x^3 + \dots$$

$$c_{i+1} + c_{i-1} = 2c_i + \left. \frac{\partial^2 c}{\partial x^2} \right|_i \Delta x^2 + \mathcal{O}(\Delta x^4)$$

$$\Rightarrow \left. \frac{\partial^2 c}{\partial x^2} \right|_i = \frac{c_{i+1} - 2c_i + c_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

Due to symmetric discretization: second order (central discretization).

Instationary diffusion equation (Fick's second law)

An alternative discretization

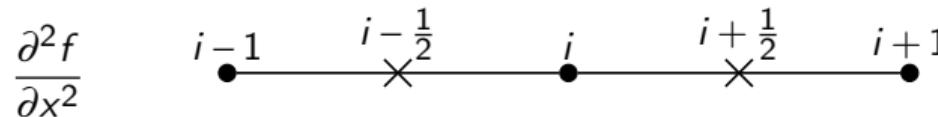
$$\frac{\partial^2 c}{\partial x^2} \Big|_i = \frac{\frac{\partial c}{\partial x} \Big|_{i+\frac{1}{2}} - \frac{\partial c}{\partial x} \Big|_{i-\frac{1}{2}}}{\Delta x} + \mathcal{O}(\Delta x^2)$$

$$= \frac{\frac{c_{i+1} - c_i}{\Delta x} - \frac{c_i - c_{i-1}}{\Delta x}}{\Delta x} = \frac{c_{i+1} - 2c_i + c_{i-1}}{\Delta x^2}$$

This is convenient for the derivation of $\frac{\partial}{\partial x} \left(D \frac{\partial c}{\partial x} \right)$

$$\begin{aligned}\frac{\partial}{\partial x} \left(D \frac{\partial c}{\partial x} \right) &= \frac{D_{i+\frac{1}{2}} \frac{\partial c}{\partial x} \Big|_{i+\frac{1}{2}} - D_{i-\frac{1}{2}} \frac{\partial c}{\partial x} \Big|_{i-\frac{1}{2}}}{\Delta x} = \frac{D_{i+\frac{1}{2}} \frac{c_{i+1} - c_i}{\Delta x} - D_{i-\frac{1}{2}} \frac{c_i - c_{i-1}}{\Delta x}}{\Delta x} \\ &= \frac{D_{i+\frac{1}{2}} c_{i+1} - \left(D_{i+\frac{1}{2}} + D_{i-\frac{1}{2}} \right) c_i + D_{i-\frac{1}{2}} c_{i-1}}{(\Delta x)^2}\end{aligned}$$

Instationary diffusion equation (Fick's second law)



$$f_{i+\frac{1}{2}} = f_i + \frac{1}{2} \Delta x \left. \frac{\partial f}{\partial x} \right|_i \Delta x + \frac{1}{2} \left(\frac{1}{2} \Delta x \right)^2 \left. \frac{\partial^2 f}{\partial x^2} \right|_i + \mathcal{O}(\Delta x^3)$$

$$f_{i-\frac{1}{2}} = f_i - \frac{1}{2} \Delta x \left. \frac{\partial f}{\partial x} \right|_i \Delta x + \frac{1}{2} \left(\frac{1}{2} \Delta x \right)^2 \left. \frac{\partial^2 f}{\partial x^2} \right|_i + \mathcal{O}(\Delta x^3)$$

$$f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}} = \Delta x \frac{\partial f}{\partial x} + \mathcal{O}(\Delta x^3)$$

$$\Rightarrow \left. \frac{\partial f}{\partial x} \right|_i = \frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{\Delta x} + \mathcal{O}(\Delta x^2)$$

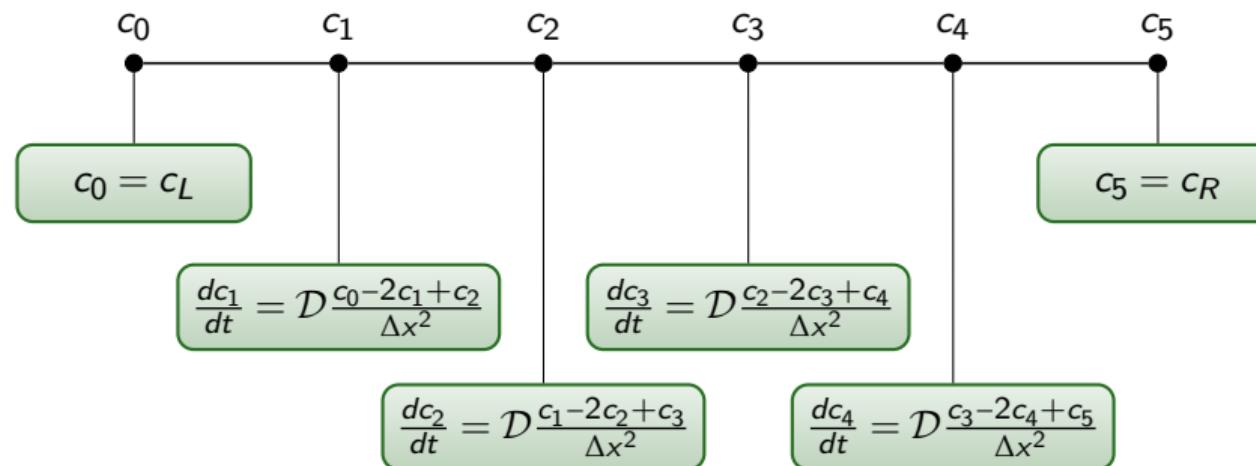
Symmetric discretization yields second order!

Instationary diffusion equation: spatial discretization

Substitution of spatial derivatives yields:

$$\frac{dc_i}{dt} = \mathcal{D} \frac{c_{i-1} - 2c_i + c_{i+1}}{\Delta x^2} \quad \text{for } i = 0, \dots, N$$

For example, using 6 (ridiculously low number!) grid points:



Instationary diffusion equation: boundary conditions

Two options:

- ① Keep boundary conditions as additional equations:

$$c_0 = c_L, \frac{dc_1}{dt} = \mathcal{D} \frac{c_0 - 2c_1 + c_2}{\Delta x^2}, \frac{dc_2}{dt} = \mathcal{D} \frac{c_1 - 2c_2 + c_3}{\Delta x^2},$$
$$\frac{dc_3}{dt} = \mathcal{D} \frac{c_2 - 2c_3 + c_4}{\Delta x^2}, \frac{dc_4}{dt} = \mathcal{D} \frac{c_3 - 2c_4 + c_5}{\Delta x^2}, c_5 = c_R$$

- ② Substitute boundary conditions to reduce number of equations:

$$\frac{dc_1}{dt} = \mathcal{D} \frac{c_L - 2c_1 + c_2}{\Delta x^2}, \frac{dc_2}{dt} = \mathcal{D} \frac{c_1 - 2c_2 + c_3}{\Delta x^2},$$
$$\frac{dc_3}{dt} = \mathcal{D} \frac{c_2 - 2c_3 + c_4}{\Delta x^2}, \frac{dc_4}{dt} = \mathcal{D} \frac{c_3 - 2c_4 + c_R}{\Delta x^2}$$

Instationary diffusion equation: temporal discretization

$$\frac{dc_i}{dt} = \mathcal{D} \frac{c_{i-1} - 2c_i + c_{i+1}}{\Delta x^2}$$

Time discretization: forward Euler (explicit)

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} = \mathcal{D} \frac{c_{i-1}^n - 2c_i^n + c_{i+1}^n}{\Delta x^2}$$

$$\Rightarrow c_i^{n+1} = Foc_{i-1}^n + (1 - 2Fo)c_i^n + Foc_{i+1}^n \quad \text{with } Fo = \frac{\mathcal{D}\Delta t}{\Delta x^2}$$

Straightforward updating (explicit equation), simple to implement in a program but stability constraint $Fo = \frac{\mathcal{D}\Delta t}{\Delta x^2} < \frac{1}{2}$!

Small $\Delta x \Rightarrow$ small $\Delta t \Rightarrow$ patience required ☺

Instationary diffusion equation: temporal discretization

$$\frac{dc_i}{dt} = \mathcal{D} \frac{c_{i-1} - 2c_i + c_{i+1}}{\Delta x^2}$$

Time discretization: backward Euler (implicit)

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} = \mathcal{D} \frac{c_{i-1}^{n+1} - 2c_i^{n+1} + c_{i+1}^{n+1}}{\Delta x^2}$$

$$\Rightarrow -\text{Fo}c_{i-1}^{n+1} + (1 + 2\text{Fo})c_i^{n+1} - \text{Fo}c_{i+1}^{n+1} = c_i^n \quad \text{with } \text{Fo} = \frac{\mathcal{D}\Delta t}{\Delta x^2}$$

Requires the solution of a system of linear equations, but no stability constraints!

Note: extension to higher order schemes (with time step adaptation) straightforward. Often second or third order optimal, because for each Euler-like step in the additional order an often large system needs to be solved (not treated in this course).

Solving the instationary diffusion equation: example

Solve the diffusion problem using explicit discretization:

$$\frac{\partial c_i}{\partial t} = \mathcal{D} \frac{\partial^2 c}{\partial x^2} \quad \text{with} \quad \begin{aligned} 0 &\leq x \leq \delta, \delta = 5 \cdot 10^{-3} \text{ m} \\ \delta/\Delta x &= 100 \text{ grid cells} \\ \mathcal{D} &= 1 \cdot 10^{-8} \text{ m}^2 \text{s}^{-1} \\ t_{\text{end}} &= 5000 \text{ s} \\ c_L &= 1 \text{ mol m}^{-3}, c_R = 0 \text{ mol m}^{-3} \end{aligned}$$

$$c_i^{n+1} = \text{Foc}_{i-1}^n + (1 - 2\text{Fo})c_i^n + \text{Foc}_{i+1}^n \quad \text{with } \text{Fo} = \frac{\mathcal{D}\Delta t}{\Delta x^2}$$

- ① Initialise variables
- ② Compute time step so that $\text{Fo} \leq \frac{1}{2} \Rightarrow \Delta t = 0.125 \text{ s}$!
- ③ Compute 40000 time steps times 100 grid nodes!
- ④ Store solution

Solving the instationary diffusion equation: example

Initialise the variables and matrices:

```
Nx = 100; % Nc grid points
Nt = 40000; % Nt time steps
D = 1e-8; % m/s
c_L = 1.0; c_R = 0; % mol/m3
t_end = 5000.0; % s
x_end = 5e-3; % m

% Time step and grid size
dt = t_end/Nt;
dx = x_end/Nx;

% Fourier number
Fo=D*dt/dx/dx

% Initial matrices for solutions (Nx times Nt)
c = zeros(Nt+1,Nx+1); % All concentrations are zero
c(:,1) = c_L; % Concentration at left side
c(:,Nx+1) = c_R; % Concentration at right side

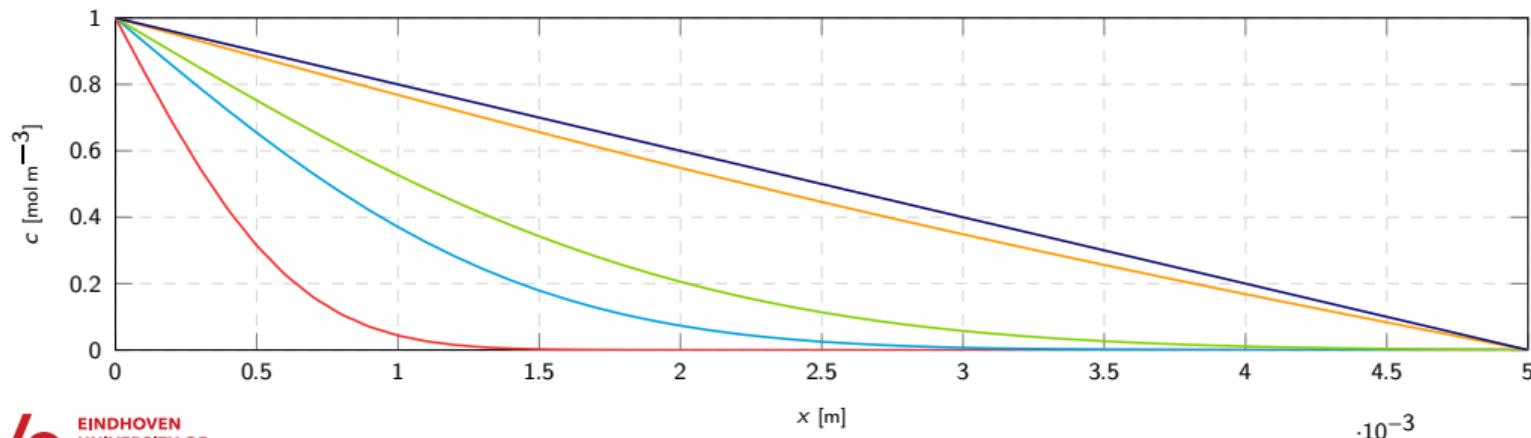
% Grid node and time step positions
x = linspace(0,x_end,Nx+1);
```

Solving the instationary diffusion equation: example

Compute the solution (nested time-and-grid loop):

- Create a time-loop
- Create a loop over *internal* grid points
- Update each node using $c_i^{n+1} = Foc_{i-1}^n + (1 - 2Fo)c_i^n + Foc_{i+1}^n$
- Plot the solution for selected time steps

Plotting the solution at $t = \{12.5, 62.5, 125, 625, 5000\}$ s.



Solving the instationary diffusion equation: example

A double-loop can impose serious computation times if the number of grid points increases:

```
for n = 1:Nt-1 % time loop
    for i = 2:Nx % Nested loop for grid nodes
        c(n+1,i) = Fo*c(n,i-1) + (1-2*Fo)*c(n,i) + Fo*c(n,i+1);
    end
end
```

Remedy: vectorization. Construct a 3-point stencil Laplacian matrix first, then use the matrix product to evolve the simulation:

```
% Construct sparse matrix
e = ones(Nx-1,1);
md = [1; (1-2*Fo)*e; 1];
ld = [Fo*e; 0; 0];
ud = [0; 0; Fo*e];
A = spdiags([ld md ud], [-1 0 1], Nx+1, Nx+1);

% Faster for row-wise operations, so transpose
c=c';
for n = 1:Nt-1 % time loop
    c(:,n+1) = A*c(:,n);
end
```

Solving the diffusion equation implicitly

Linear system $Ax = b$ from $-Foc_{i-1}^{n+1} + (1 + 2Fo)c_i^{n+1} - Foc_{i+1}^{n+1} = c_i^n$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots & 0 \\ -Fo & (1+2Fo) & -Fo & 0 & \dots & 0 \\ 0 & -Fo & (1+2Fo) & -Fo & \dots & 0 \\ 0 & 0 & -Fo & (1+2Fo) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \end{pmatrix} \begin{pmatrix} c_0^{n+1} \\ c_1^{n+1} \\ c_2^{n+1} \\ c_3^{n+1} \\ \vdots \\ c_m^{n+1} \end{pmatrix} = \begin{pmatrix} c_0^n \\ c_1^n \\ c_2^n \\ c_3^n \\ \vdots \\ c_m^n \end{pmatrix}$$

$$1 \times c_0^{n+1} = c_0^n \text{ (boundary condition)}$$

$$-Foc_0^{n+1} + (1 + 2Fo)c_1^{n+1} - Foc_2^{n+1} = c_1^n$$

$$-Foc_1^{n+1} + (1 + 2Fo)c_2^{n+1} - Foc_3^{n+1} = c_2^n$$

$$-Foc_2^{n+1} + (1 + 2Fo)c_3^{n+1} - Foc_4^{n+1} = c_3^n$$

$$1 \times c_m^{n+1} = c_m^n \text{ (boundary condition)}$$

Solving the diffusion equation implicitly in Matlab

To solve the linear system, we need to define matrix A . It is clear that storing many zeros is not efficient in terms of memory. We use a *sparse matrix* format. Two alternative ways to set up the matrix:

Set individual elements of the matrix:

```
% Bands in matrix (internal cells)
A = sparse(Nx+1,Nx+1);
for i=2:Nx
    A(i,i-1) = -Fo;
    A(i,i) = (1+2*Fo);
    A(i,i+1) = -Fo;
end

% Set boundary cells, only main diag:
A(1,1) = 1; % Left
A(Nx+1,Nx+1) = 1; % Right
```

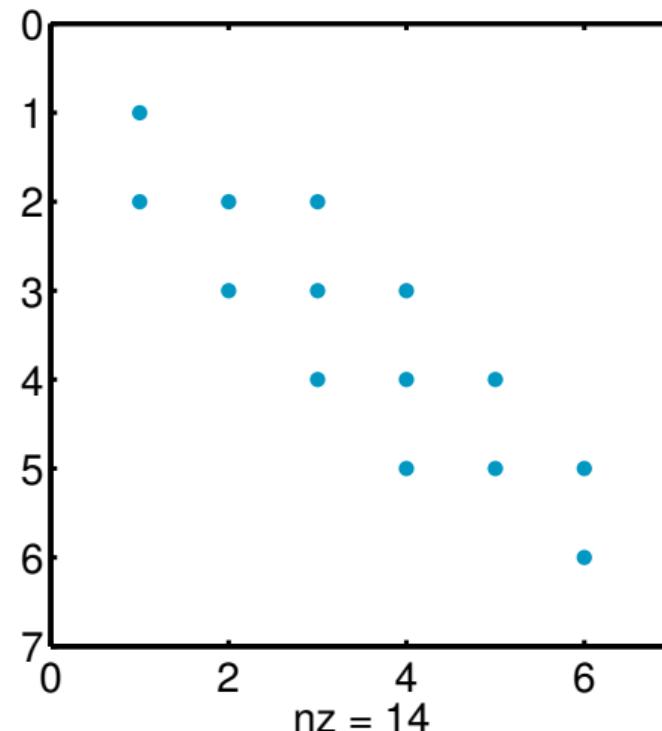
Set matrix using bands:

```
% Bands in matrix (internal cells)
e = ones(Nx-1,1); % Ones for internal cells
md = [1; e*(1+2*Fo); 1]; % Main diagonal
ld = [-e*Fo; 0; 0]; % Lower diagonal
hd = [0; 0; -e*Fo]; % Upper diagonal
A = spdiags([ld md hd], [-1 0 1], Nx+1, Nx+1)
```

Note: The first argument of `spdiags` defines each column as a diagonal, starting at row 1 (for lower-diagonal) or column 1 (for upper-diagonal).

Solving the diffusion equation implicitly in Matlab

The command `spy(A)` shows a figure with the non-zero positions.



Solving the diffusion equation implicitly in Matlab

The concentration matrix is initialised and the boundary conditions are set as follows:

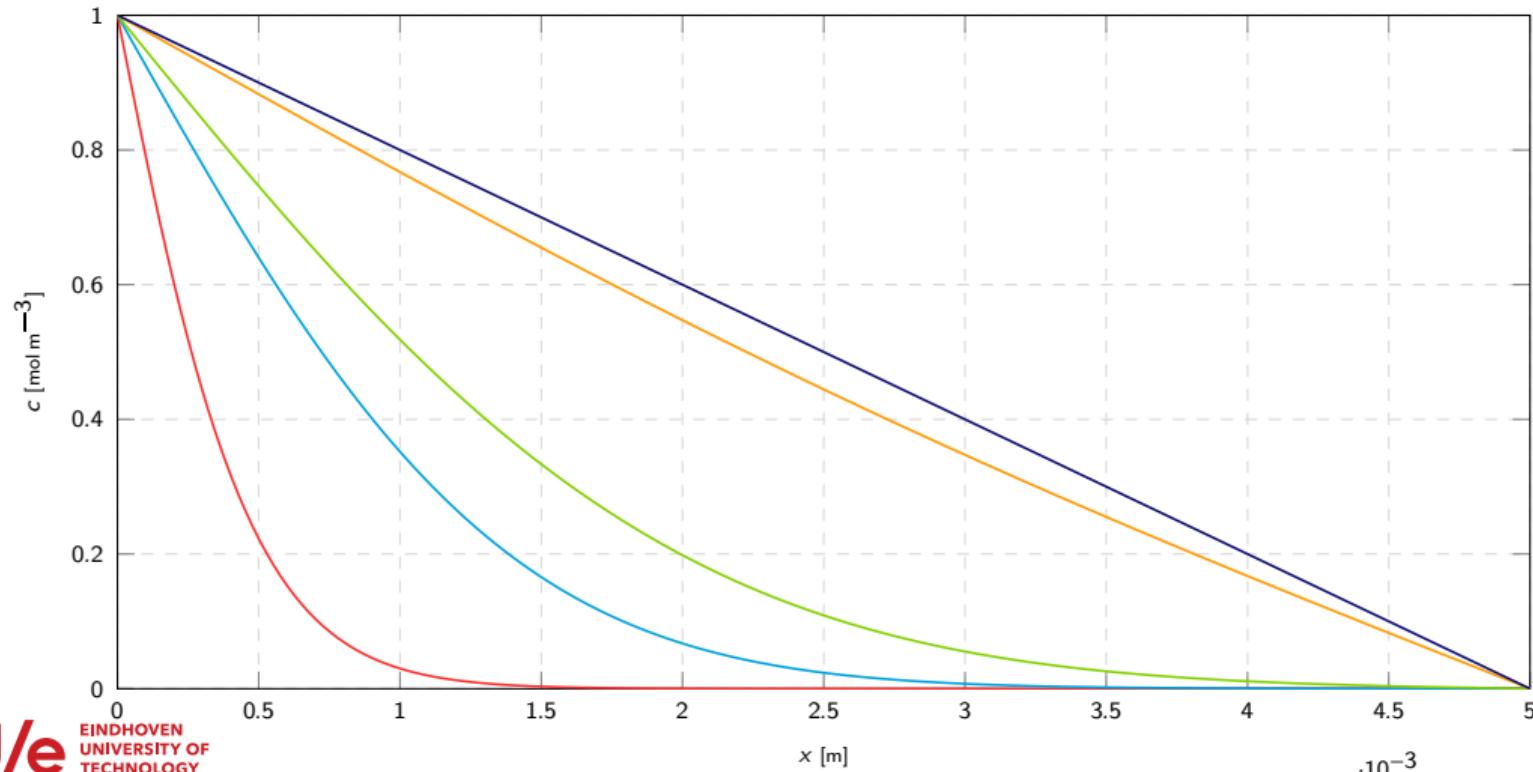
```
% Initial matrices for solutions (Nx times Nt)
c = zeros(Nt+1,Nx+1); % All concentrations are zero
c(:,1) = c_L;          % Concentration at left side
c(:,Nx+1) = c_R;       % Concentration at right side
```

The right hand side vector (b) can now be set during the time-loop:

```
for n = 1:Nt-1           % time loop
    b = c(n,:);           % Set right hand side
    solX = A\b;            % Solve linear system
    c(n+1,:) = solX;       % Store solution each time step
end
```

Solving the diffusion equation implicitly in Matlab

Plotting the solution at $t = \{12.5, 62.5, 125, 625, 5000\}$ s.



About explicit vs. implicit solutions

- Explicit solution:
 - Easy to implement
 - Very small time steps required.
 - This problem took about 0.5 s.
- Implicit solution:
 - Harder to implement, needs sparse matrix solver
 - No stability constraint
 - This problem took about 0.05 s
- The difference will become much larger for systems with e.g. more grid nodes!

Extension with non-linear source terms

$$\frac{\partial c}{\partial t} = \mathcal{D} \frac{\partial^2 c}{\partial x^2} + R(c) \quad \text{with} \quad \begin{aligned} t = 0; 0 \leq x \leq \ell &\Rightarrow c = c_0 \\ t > 0; x = 0 &\Rightarrow c = c_L \\ t > 0; x = \ell &\Rightarrow c = c_R \end{aligned}$$

- Forward Euler (explicit): simply add to right-hand side

$$\begin{aligned} \frac{c_i^{n+1} - c_i^n}{\Delta t} &= \mathcal{D} \frac{c_{i-1}^n - 2c_i^n + c_{i+1}^n}{\Delta x^2} + R(c_i^n) \\ \Rightarrow c_i^{n+1} &= \text{Fo} c_{i-1}^n + (1 - 2\text{Fo}) c_i^n + \text{Fo} c_{i+1}^n + R_i^n \Delta t \end{aligned}$$

- Backward Euler (implicit): linearization required

$$\begin{aligned} R(c_i^{n+1}) &= R(c_i^n) + \left. \frac{dR}{dc} \right|_i (c_i^{n+1} - c_i^n) \\ \frac{c_i^{n+1} - c_i^n}{\Delta t} &= \mathcal{D} \frac{c_{i-1}^{n+1} - 2c_i^{n+1} + c_{i+1}^{n+1}}{\Delta x^2} + R(c_i^{n+1}) \end{aligned}$$

$$\Rightarrow -\text{Fo} c_{i-1}^{n+1} + (1 + 2\text{Fo} - \left. \frac{dR}{dc} \right|_i \Delta t) c_i^{n+1} - \text{Fo} c_{i+1}^{n+1} = c_i^n + \left(R_i^n - \left. \frac{dR}{dc} \right|_i c_i^n \right) \Delta t$$

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Extension with convection terms

$$\frac{\partial c}{\partial t} = \mathcal{D} \frac{\partial^2 c}{\partial x^2} - u \frac{\partial c}{\partial x} + R$$

Discretization of first derivative $\frac{dc}{dx}$,
looks simple but is numerical headache!

Central discretization:

$$\frac{dc}{dx} = \frac{c_{i+1} - c_{i-1}}{2\Delta x}$$

⇒ simple and easy, too bad it doesn't work: yields unstable solutions if convection dominated.

Central difference scheme of 1st derivative

Unsteady convection:

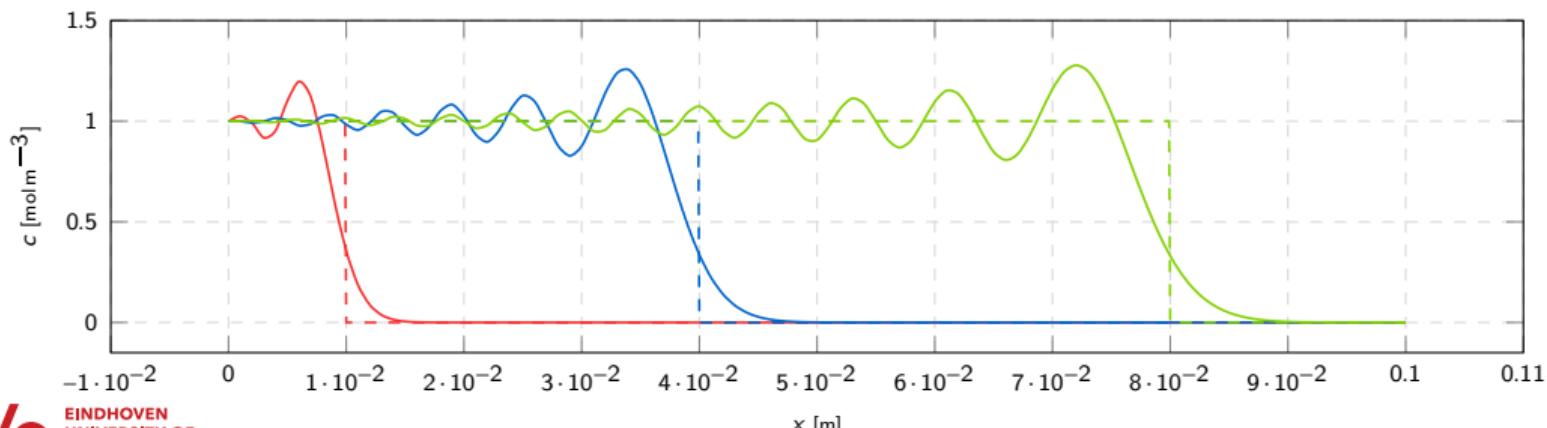
$$\frac{\partial c}{\partial t} = -u \frac{\partial c}{\partial x}$$

Central difference for first derivative:

$$\frac{dc}{dx} = \frac{c_{i+1} - c_{i-1}}{2\Delta x}$$

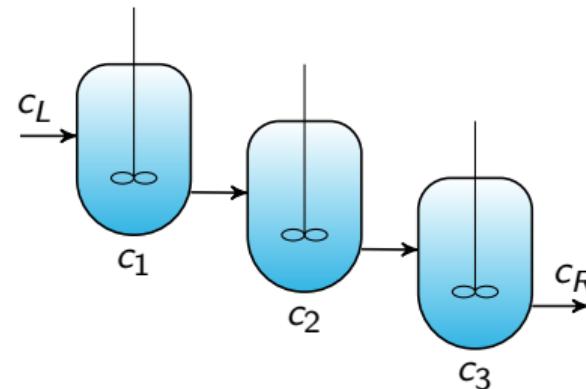
Forward Euler discretization of temporal and spatial domain:

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} = -u \frac{c_{i+1} - c_{i-1}}{2\Delta x} \Rightarrow c_i^{n+1} = c_i^n - u \frac{c_{i+1} - c_{i-1}}{2\Delta x} \Delta t$$



Extension with convection terms

Solution: upwind discretization, like CSTR's in series:



First order upwind: $-u \frac{dc}{dx} \Big|_i = \begin{cases} -u \frac{c_i - c_{i-1}}{\Delta x} & \text{if } u \geq 0 \\ -u \frac{c_{i+1} - c_i}{\Delta x} & \text{if } u < 0 \end{cases}$

Stable if $\text{Co} = \frac{u\Delta t}{\Delta x} < 1$ (with Co the Courant number).

Courant number). However, only 1st order accurate (large smearing of concentration fronts).

Higher order upwind requires TVD schemes (trick of the trade)...

First order upwind scheme of 1st derivative

Unsteady convection:

$$\frac{\partial c}{\partial t} = -u \frac{\partial c}{\partial x}$$

Upwind scheme for first derivative:

$$-u \frac{dc}{dx} \Big|_i = \begin{cases} -u \frac{c_i - c_{i-1}}{\Delta x} & \text{if } u \geq 0 \\ -u \frac{c_{i+1} - c_i}{\Delta x} & \text{if } u < 0 \end{cases}$$

Forward Euler discretization of temporal and spatial domain:

$$\frac{c_i^{n+1} - c_i^n}{\Delta t} = -u \frac{c_{i+1} - c_{i-1}}{2\Delta x}$$

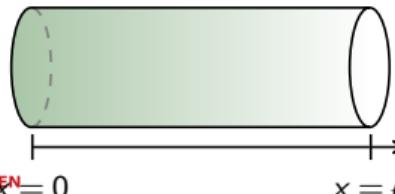
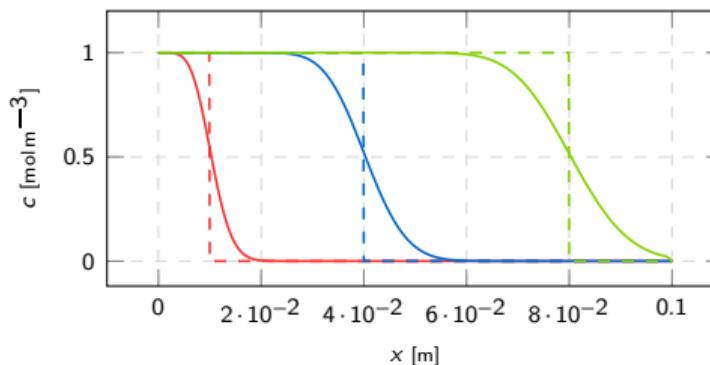
$$\Rightarrow c_i^{n+1} = \begin{cases} c_i^n - u\Delta t \frac{c_i - c_{i-1}}{\Delta x} & \text{if } u \geq 0 \\ c_i^n - u\Delta t \frac{c_{i+1} - c_i}{\Delta x} & \text{if } u < 0 \end{cases}$$

Upwind scheme: example

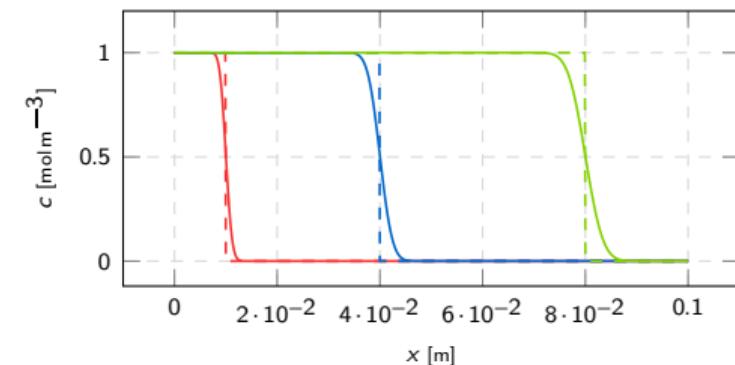
Unsteady convection through a pipe:

$$\frac{\partial c}{\partial t} = -u \frac{\partial c}{\partial x} \quad \text{with} \quad u = 0.1 \text{ m s}^{-1} \Rightarrow c_i^{n+1} = c_i^n - u \frac{c_i - c_{i-1}}{\Delta x} \Delta t$$

Using 100 grid cells



Using 1000 grid cells



Today's outline

- Introduction

- Instationary diffusion equation

- Discretization
- Solving the diffusion equation
- Non-linear source terms

- Convection

- Discretization
- Central difference scheme
- Upwind scheme

- Conclusions

- Other methods
- Summary

Extension to systems of PDE's

- Explicit methods: straightforward extension
- Implicit methods: yields block-tridiagonal matrix (note ordering of equations: all variables per grid cell)

Extension to 2D or 3D systems

Spatial discretization in 2 directions — different methods available:

- Explicit
- Fully implicit
 - 1D gives tri-diagonal matrix
 - 2D gives penta-diagonal matrix
 - 3D gives hepta-diagonal matrix

Use of dedicated matrix solvers (e.g. ICCG, multigrid, ...)

- Alternating direction implicit (ADI)
 - Per direction implicit, but still overall unconditionally stable

Further extensions for parabolic PDEs

- Higher order temporal discretization (multi-step) with time step adaptation
- Non-uniform grids with automatic grid adaptation
- Higher-order discretization methods, especially higher order TVD (flux delimited) schemes for convective fluxes (e.g. WENO schemes)
- Higher-order finite volume schemes (Riemann solvers)

Summary

- Several classes of PDEs were introduced
 - Elliptic, Parabolic, Hyperbolic PDEs
- Diffusion equation: discretization of temporal and spatial domain was discussed
 - Solutions of the diffusion equation using explicit and implicit methods
 - How to add non-linear source terms
- Convection: upwind vs. central difference schemes