Linear systems - iterative solvers

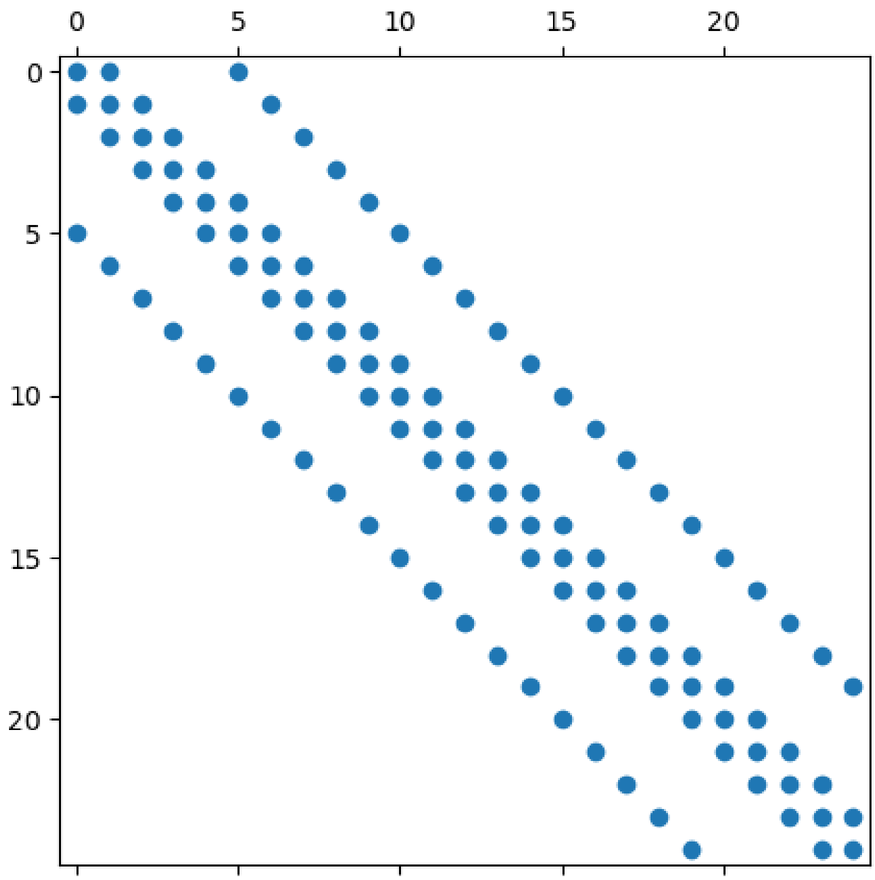
Set up matrix

```python
import numpy as np
from scipy.sparse.linalg import spsolve
from scipy.sparse import diags, lil_matrix,csr_matrix,csc_matrix
from matplotlib import cm
import matplotlib.pyplot as plt

Nx,Ny = 5,5 # Number of grid points along x,y direction
Nc = Nx*Ny    # Total number of points

e = np.ones(Nc)
A = diags([e, e, -4*e, e, e], [-Nx, -1, 0, 1, Nx], shape=(Nc,Nc))
b = np.zeros(Nc)
print(A)
plt.spy(A, marker='o',markersize=6)
plt.tight_layout()

print(np.arange(Nc).reshape((Nx,Ny))[::-1])
print(type(A))
```

```
(5, 0)    1.0
(6, 1)    1.0
(7, 2)    1.0
(8, 3)    1.0
(9, 4)    1.0
(10, 5)   1.0
(11, 6)   1.0
(12, 7)   1.0
(13, 8)   1.0
(14, 9)   1.0
```

```
  (15, 10)	1.0
  (16, 11)	1.0
  (17, 12)	1.0
  (18, 13)	1.0
  (19, 14)	1.0
  (20, 15)	1.0
  (21, 16)	1.0
  (22, 17)	1.0
  (23, 18)	1.0
  (24, 19)	1.0
  (1, 0)	1.0
  (2, 1)	1.0
  (3, 2)	1.0
  (4, 3)	1.0
  (5, 4)	1.0
  : :
  (19, 20)	1.0
  (20, 21)	1.0
  (21, 22)	1.0
  (22, 23)	1.0
  (23, 24)	1.0
  (0, 5)	1.0
  (1, 6)	1.0
  (2, 7)	1.0
  (3, 8)	1.0
  (4, 9)	1.0
  (5, 10)	1.0
  (6, 11)	1.0
  (7, 12)	1.0
  (8, 13)	1.0
  (9, 14)	1.0
  (10, 15)	1.0
  (11, 16)	1.0
  (12, 17)	1.0
  (13, 18)	1.0
  (14, 19)	1.0
  (15, 20)	1.0
  (16, 21)	1.0
  (17, 22)	1.0
  (18, 23)	1.0
  (19, 24)	1.0
[[20 21 22 23 24]
 [15 16 17 18 19]
```

```
 [10 11 12 13 14]
 [ 5  6  7  8  9]
 [ 0  1  2  3  4]]
<class 'scipy.sparse._dia.dia_matrix'>
```



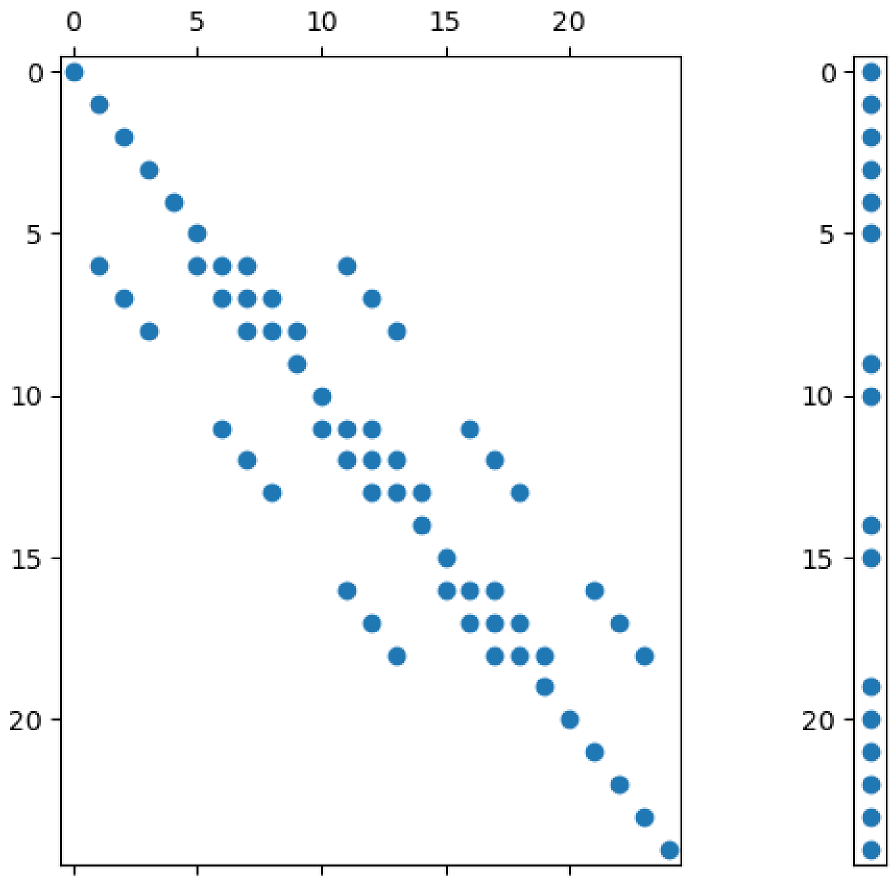
Boundary conditions

```
bnd_bottom = np.arange(Nx)
bnd_left = np.arange(Ny) * Nx
bnd_right = bnd_left + Nx - 1
bnd_top = bnd_bottom + Nx*(Ny-1)
print(bnd_bottom, bnd_left, bnd_right, bnd_top)

bnd_all = np.unique(np.concatenate((bnd_bottom,bnd_left,bnd_right,bnd_top)))
```

```
print(bnd_all)

Tb = {'bottom': 300, 'left': 1000, 'right': 1000, 'top': 500}


A = lil_matrix(A)
# For all equations that represent a boundary, reset the coefficient row to zero
# consequently add a 1 only on the main diagonal
A[bnd_all,:] = 0
A[bnd_all,bnd_all] = 1

b[bnd_bottom] = Tb['bottom']
b[bnd_left] = Tb['left']
b[bnd_right] = Tb['right']
b[bnd_top] = Tb['top']
```

```
[0 1 2 3 4] [ 0  5 10 15 20] [ 4  9 14 19 24] [20 21 22 23 24]
[ 0  1  2  3  4  5  9 10 14 15 19 20 21 22 23 24]
```

```
ax1 = plt.subplot(121); plt.spy(A, marker='o',markersize=6, aspect="auto")
ax2 = plt.subplot(122); plt.spy(b[:,None], marker='o',markersize=6);
ax1.sharey
plt.xticks([])
plt.tight_layout()
plt.savefig('sparse_python_bnds.pdf')
```




```
A = A.tocsc()
T = spsolve(A,b)
```

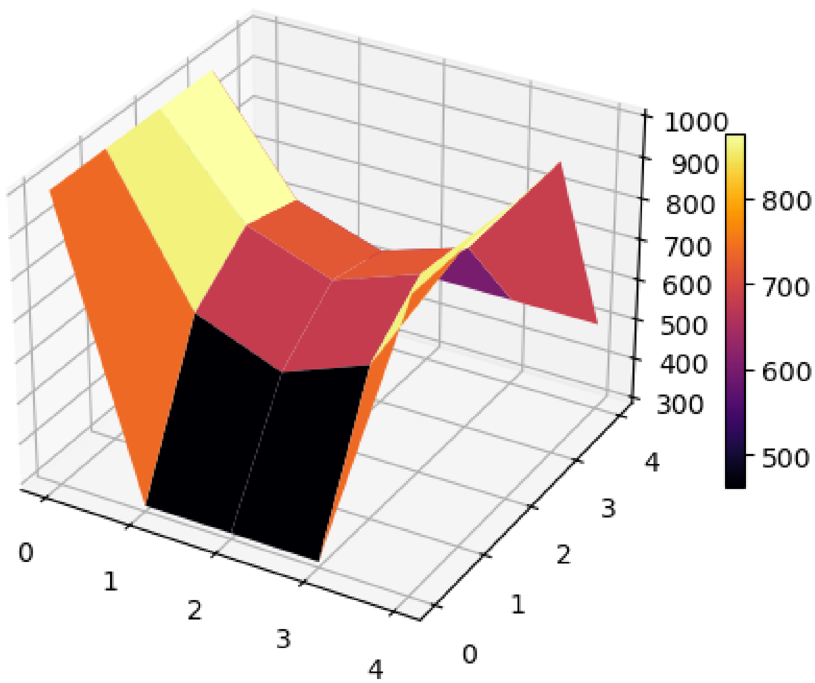
```
print(T.reshape(Nx,Ny)[::-1])
```

```
[[ 500.          500.          500.          500.           500.        ]
 [1000.         735.71428571  667.85714286  735.71428571 1000.        ]
 [1000.         775.          700.          775.         1000.        ]
 [1000.         664.28571429  582.14285714  664.28571429 1000.        ]
 [1000.         300.          300.          300.         1000.        ]]
```

```
x,y = np.meshgrid(np.arange(Nx),np.arange(Ny))
```

```
Tnum = T.reshape(Nx,Ny)
fig, ax = plt.subplots(subplot_kw={"projection": "3d"})
surf = ax.plot_surface(x,y,Tnum,cmap=cm.inferno)
fig.colorbar(surf, shrink=0.5)
```

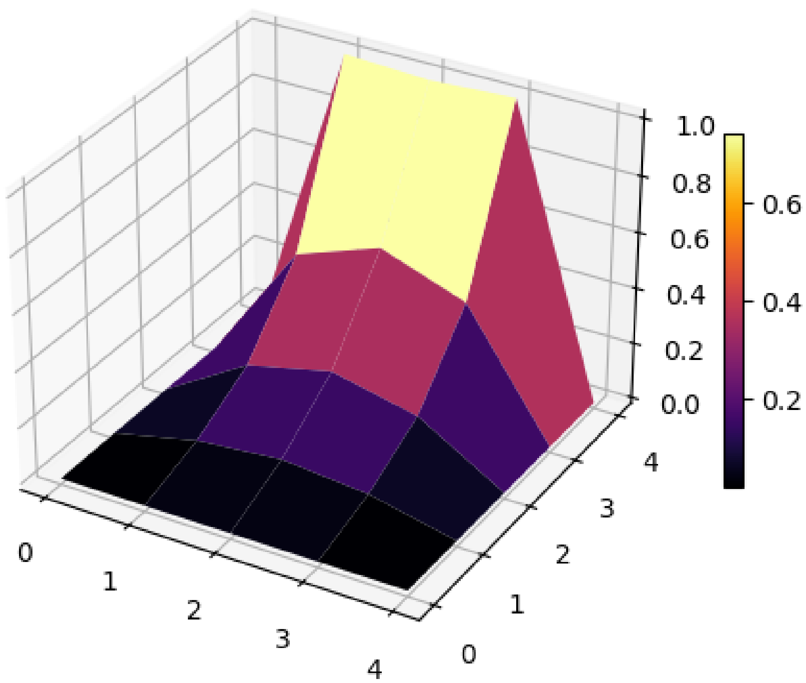
```
<matplotlib.colorbar.Colorbar at 0x7f7f62641dd0>
```



Verification

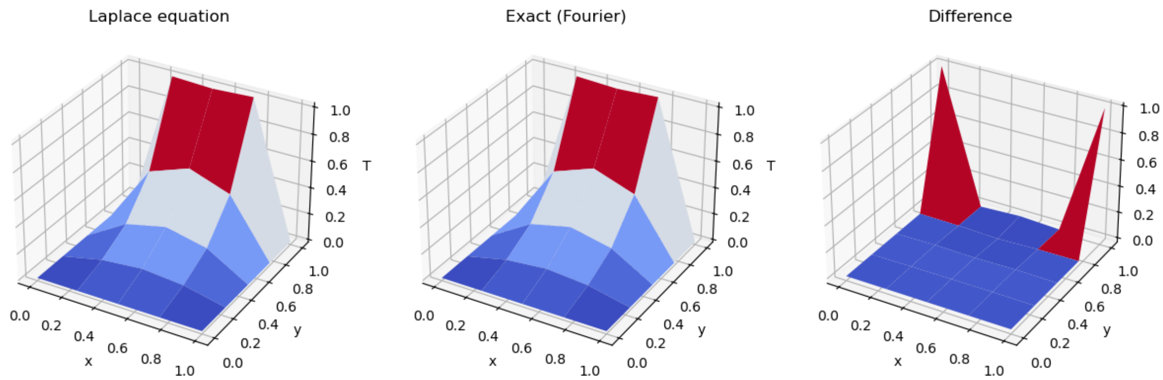
```
import numpy as np
import matplotlib.pyplot as plt
import matplotlib.cm as cm

Nx = Ny = 5

xf,yf = np.meshgrid(np.linspace(0,1,Nx),np.linspace(0,1,Ny))
term = np.zeros_like(xf)
N = 100
```

```
for m in range(1,N,2):
    term = term + (np.sin(m*np.pi*xf)*np.sinh(m*np.pi*yf)) / (m*np.sinh(m*np.pi))

# Exact solution
Tex = term * 4 / np.pi
fig, ax = plt.subplots(subplot_kw={"projection": "3d"})
surf = ax.plot_surface(x,y,Tex,cmap=cm.inferno)
fig.colorbar(surf, shrink=0.5)
plt.show()
```




```
from laplace_demo import create_laplace_coefficient_matrix, set_boundary_conditions

Tb = {'bottom': 0, 'left': 0, 'right': 0, 'top': 1}
A,b = create_laplace_coefficient_matrix(Nx,Ny)
A,b = set_boundary_conditions(A, b, Tb, Nx, Ny)

Tnum = spsolve(A,b).reshape((Nx,Ny))
```

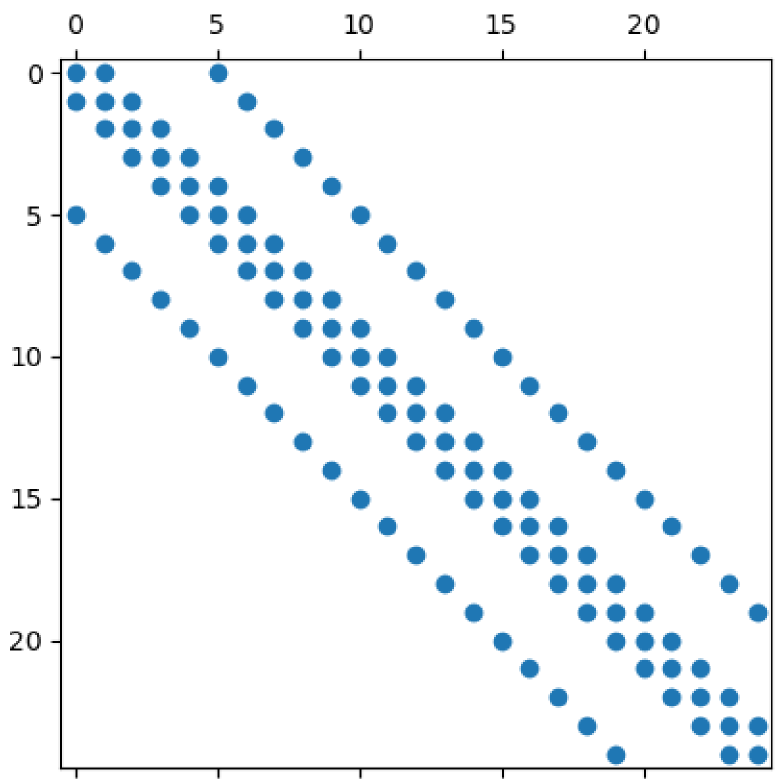
```
fig, axs = plt.subplots(1, 3, figsize=(15, 5), subplot_kw=dict(projection='3d'))
# Plot the numerical
axs[0].plot_surface(xf, yf, Tex, cmap = "coolwarm")
axs[0].set_xlabel('x'); axs[0].set_ylabel('y'); axs[0].set_zlabel('T')
axs[0].set_title("Laplace equation ")
# Plot exact (Fourier)
axs[1].plot_surface(xf, yf, Tex, cmap='coolwarm')
axs[1].set_xlabel('x'); axs[1].set_ylabel('y'); axs[1].set_zlabel('T')
axs[1].set_title("Exact (Fourier)")
# Plot difference
axs[2].plot_surface(xf, yf, Tnum - Tex, cmap='coolwarm')
axs[2].set_xlabel('x'); axs[2].set_ylabel('y'); axs[2].set_zlabel('T')
axs[2].set_title("Difference")
plt.show()
plt.savefig('laplace_exact_comparison.pdf')
```

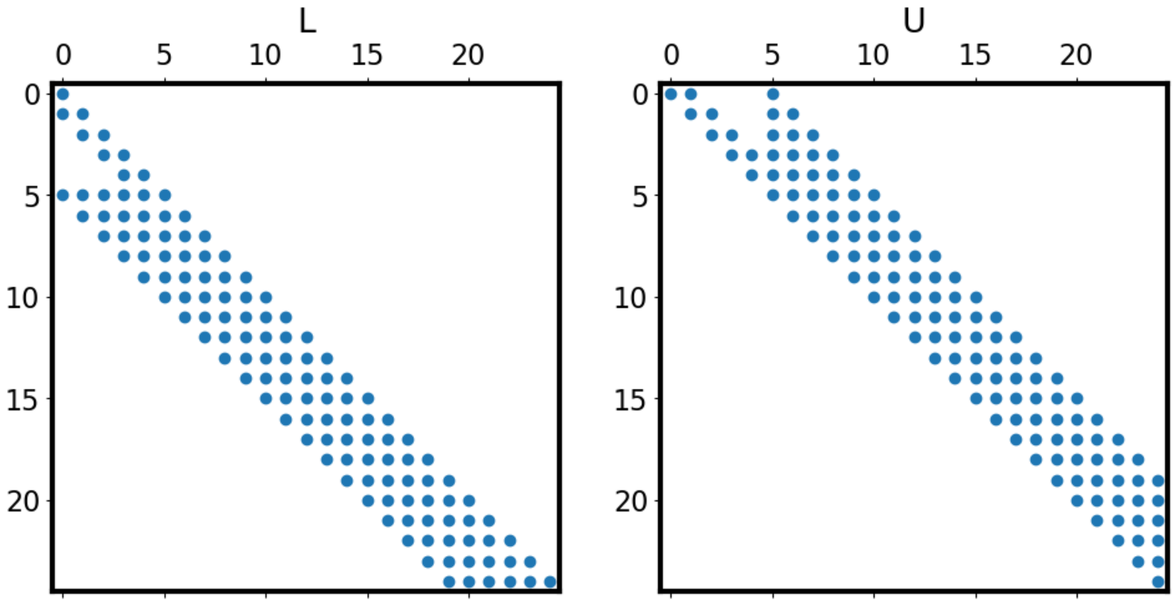



```
<Figure size 640x480 with 0 Axes>
```

```
import numpy as np
from scipy.linalg import lu
import matplotlib.pyplot as plt
from laplace_demo import create_laplace_coefficient_matrix

A,b = create_laplace_coefficient_matrix(5,5)
plt.spy(A,marker='o',markersize=6)

# Perform LU decomposition
P,L,U = lu(A.toarray())
```

```
plt.rcParams.update({'font.size': 16})
plt.rcParams['axes.linewidth'] = 3
# Plot the sparsity patterns of L and U
plt.figure(figsize=(10, 5))
plt.subplot(121)
plt.spy(L,marker='o',markersize=6)
plt.title('L')
plt.subplot(122)
plt.spy(U,marker='o',markersize=6)
plt.title('U')
plt.tight_layout()
plt.savefig('sparse_lu.pdf')

plt.rcdefaults()
```






```
%reset -f
```

Iterative solvers

```
from laplace_demo import create_laplace_coefficient_matrix, set_boundary_conditions
import matplotlib.pyplot as plt
from scipy.sparse.linalg import spsolve
import numpy as np
import matplotlib.cm as cm

Nx = Ny = 20

Tb = {'bottom': 20, 'left': 40, 'right': 80, 'top': 100}
A,b = create_laplace_coefficient_matrix(Nx,Ny)
A,b = set_boundary_conditions(A, b, Tb, Nx, Ny)
```

```
T_num = spsolve(A,b).reshape((Nx,Ny))
```

```
from it_methods import jacobi

if not isinstance(A,np.ndarray):
```

```
    A = A.toarray()

sol,n_it = jacobi(A,b,tol=1e-3)
print(f'Solved in {n_it} iterations!')

T_num = sol.reshape(Nx,Ny)
```

```
324
Solved in 324 iterations!
```

```
from it_methods import jacobi_vec

if not isinstance(A,np.ndarray):
    A = A.toarray()

sol,n_it = jacobi_vec(A,b,tol=1e-3,itmax=5)
print(f'Solved in {n_it} iterations!')

T_num = sol.reshape(Nx,Ny)
```

```
from it_methods import gaussseidel

if not isinstance(A,np.ndarray):
    A = A.toarray()

sol,n_it = gaussseidel(A,b,tol=1e-3)
print(f'Solved in {n_it} iterations!')

T_num = sol.reshape(Nx,Ny)
```

```
from it_methods import gaussseidel_vec

if not isinstance(A,np.ndarray):
    A = A.toarray()

sol,n_it = gaussseidel_vec(A,b,tol=1e-3)
print(f'Solved in {n_it} iterations!')

T_num = sol.reshape(Nx,Ny)
```

```
x,y = np.meshgrid(np.linspace(0,1,Nx),np.linspace(0,1,Ny))
fig, ax = plt.subplots(subplot_kw={"projection": "3d"})
surf = ax.plot_surface(x,y,T_num,cmap=cm.viridis,edgecolor='black',linewidth=0.5)
fig.colorbar(surf, shrink=0.5)
plt.tight_layout()
# plt.show()
plt.savefig('python_it5.pdf')
```