MA3209 Metric and Topological Spaces
AY24/25 Semester 1
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Chapter 1

Topological Spaces

Definition. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X such that

1. $\emptyset, X \in \mathcal{T}$
2. (Closure under arbitrary union) $\{U_\alpha\}_{\alpha \in I} \in \mathcal{T} \implies \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$
3. (Closure under finite intersection) $\{U_1, \ldots, U_n\} \in \mathcal{T} \implies \bigcap_{i=1}^n U_i \in \mathcal{T}$

$(X, \mathcal{T})$ is a **topological space**, and $U \subset X$ is **open** if $U \in \mathcal{T}$

Example. Let X be any set.

- $\mathcal{T} = \{\emptyset, X\}$ is the **trivial topology**
- $\mathcal{T} = \{\text{subsets of } X\}$ is the **discrete topology**
- $\mathcal{T} = \{X - U : U \subset X \text{ is finite}\} \cup \{\emptyset\}$ is the **cofinite topology**

Definition. A **basis** for a topology of a set X is a collection $\mathcal{B}$ of X's subsets such that

1. $\mathcal{B}$ covers X
2. $\forall x \in X$ and $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$, $\exists B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$

The topology generated by $\mathcal{B}$ is

$$\mathcal{T} = \{U \subset X : \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subset U\}$$

Remark.
- If $\mathcal{B}$ is a basis of topology $\mathcal{T}$, then $\mathcal{T}$ is the collection of all unions of elements in $\mathcal{B}$
- The topology on $\mathbb{R}^n$ generated by the open balls is the **standard topology**

Definition. Let X be a set and $\mathcal{T}, \mathcal{T}'$ be topologies on X. $\mathcal{T}$ is **finer** than $\mathcal{T}'$ (equivalently, $\mathcal{T}'$ is **coarser** than $\mathcal{T}$) if $\mathcal{T}' \subset \mathcal{T}$. The topology generated by a basis $\mathcal{B}$ is the coarsest topology containing $\mathcal{B}$.

Proposition. Let $\mathcal{B}, \mathcal{B}'$ be bases of topologies $\mathcal{T}, \mathcal{T}'$ respectively on X. TFAE:

1. $\mathcal{T}'$ is finer than $\mathcal{T}$
2. $\forall B \in \mathcal{B}, \forall x \in B, \exists B' \in \mathcal{B}'$ such that $x \in B' \subset B$

Definition. A **subbasis** $\mathcal{S}$ of a set X is a collection of subsets of X whose union equals X. The **topology generated by** $\mathcal{S}$ is a collection $\mathcal{T}$ of all unions of finite intersection of sets in $\mathcal{S}$

Metric spaces

Definition. A **metric** on a set X is a function $d : X \times X \to \mathbb{R}$ such that

1. (Nonnegativity) $d(x, y) \geq 0 \; \forall x, y \in X$
2. (Positive definiteness) $d(x, y) = 0 \iff x = y$
3. (Symmetry) $d(x, y) = d(y, x)$
4. (Triangle inequality) $d(x, y) + d(y, z) \geq d(x, z) \; \forall x, y, z \in X$

(X, d) is a **metric space**. If only 1, 3, and 4 hold, and $d(x, x) = 0$ for all $x \in X$, then d is a **pseudo-metric**. If only 1, 2, and 4 hold, d is a **quasi-metric**.

Definition. A **norm** on a $\mathbb{K}$-vector space V is a function $\|\cdot\| : V \to \mathbb{R}$ that satisfies

1. (Nonnegativity) $\|x\| \geq 0 \; \forall x \in V$
2. (Positive definiteness) $\|x\| = 0 \iff x = 0$
3. (Absolute homogeneity) $\|\lambda x\| = |\lambda| \, \|x\| \; \forall \lambda \in \mathbb{K}$ and $\forall x \in V$
4. (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$

Example.
- The **discrete metric** is
$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$
- The l_p**-norm** is $V = \mathbb{K}^n, p \geq 1, \|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}, x \in \mathbb{K}^n$
- The l_∞**-norm** is $V = \mathbb{K}^n, \|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}, x \in \mathbb{K}^n$

Definition. Let A, B be nonempty subsets of a metric space (X, d).

- The **distance** between A and B is $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$
- The **diameter** of a set $A \subset X$ is $\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}$
- A set $A \subset X$ is bounded if $\operatorname{diam}(A) < +\infty$

Definition.
- The topology on X **induced** by a metric d is the topology generated by $\mathcal{B}_d$.
- A topology $\mathcal{T}$ on X is metrizable if there is a metric on X that induces $\mathcal{T}$

Remark.
- The discrete metric generates the discrete topology
- Every l_p-metric on $\mathbb{R}^n$ generates the standard topology

Subspaces of topological spaces

Definition. Let $(Y, \mathcal{T}_Y)$ be a topological space and $X \subset Y$. Then $\mathcal{T}_X = \{U \cap X : U \in \mathcal{T}_Y\}$ is the **subspace topology** on X.

Definition. Let $(Y, \mathcal{T}_Y)$ be a topological space, $X \subset Y$ be a subset, and $\mathcal{T}_X$ be the subspace topology. Then X is the **subspace** of Y with respect to $\mathcal{T}_X$.

Definition. Given a subset A of a metric space (X, d), the **restriction** of d to A is the metric $d_A(x, y) = d(x, y) \; \forall x, y \in A$. The topology induced by this metric is the subspace topology.

Remark.
- If $\mathcal{T}_X$ is a topology and $\mathcal{B}$ is a basis for $\mathcal{T}_Y$, then $\{B \cap X : B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_X$
- If $X \subset Y$ is open and $U \subset X$ is open, then $U \subset Y$ is open

Closed sets, closure, and limit points

Definition. Let $(X, \mathcal{T})$ be a topological space. A subset $A \subset X$ is closed if $X \setminus A \in \mathcal{T}$.

Proposition. Let X be a topological space.

- If $\{G_\alpha\}_{\alpha \in I}$ is an arbitrary collection of closed sets in X, then $\bigcap_{\alpha \in I} G_\alpha \subset X$ is closed.
- If $G_1, \ldots, G_n$ are closed sets in X, then $\bigcup_{i=1}^n G_i \subset X$ is closed.
- If $Y \subset X$, then $A \subset Y$ is closed is equivalent to $A = G \cap Y$ for some closed $G \subset X$.
- If $Y \subset X$ is closed and $A \subset Y$ is closed, then $A \subset X$ is closed.

Definition. Let $(X, \mathcal{T})$ be a topological space and $A \subset X$.

1. The **interior** of A is $\mathring{A} = \bigcup_{U \in \mathcal{T}, U \subset A} U$.
2. The **closure** of A is $\overline{A} = \bigcap_{X \setminus G \in \mathcal{T}, G \supset A} G$.
3. The **boundary** of A is $\partial A = \overline{A} - \mathring{A}$.

Remark.
- $\mathring{A} \subset A \subset \overline{A}$
- $\mathring{A} = A \iff A$ is open.
- $\overline{A} = A \iff A$ is closed.

Definition. Let X be a topological space and $A \subset X$. A point $x \in X$ is a **limit point** of A if every open $U \subset X$ containing x intersects $A \setminus \{x\}$.

Proposition. Let X be a topological space, $A \subset X$.

1. $x \in \overline{A} \iff \forall$ open U containing x, $U \cap A \neq \emptyset$.
2. If A' is the set of limit points of A, then $\overline{A} = A \cup A'$.

Definition. A sequence $(x_1, x_2, x_3, \ldots) = \{x_i\}_{i=1}^\infty$ of points in a topological space X converges to $x \in X$ if for any neighbourhood U containing x, $\exists N > 0$ such that $x_k \in U$ for all $k > N$. This is written as $x_i \to x$. x being a limit point of the sequence does not imply that $x_i \to x$, and $x_i \to x$ does not imply that x is a limit point of the sequence either.

Continuity

Definition. Let X and Y be topological spaces. A map $f : X \to Y$ is **continuous** if for any open set $U \subset Y$, $f^{-1}(U) \subset X$ is open.

Proposition. Let X and Y be topological spaces and $f : X \to Y$. TFAE:

1. f is continuous
2. $\forall A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$
3. For any closed set $B \subset Y$, $f^{-1}(B) \subset X$ is closed
4. For any $x \in X$ and any open set $V \subset Y$ containing $f(x)$, there exists open $U \subset X$ containing x such that $f(U) \subset V$

Proposition (Pasting Lemma). Let $X = A \cup B$ where $A, B \subset X$ are both closed (or both open). Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x), & \text{if } x \in A \\ g(x), & \text{if } x \in B \end{cases}$$

is continuous.

Remark. Given a topology $\mathcal{T}_Y$ on Y and a map $f : X \to Y$, the **pull back** topology on X is defined as $\mathcal{T}_X = \{f^{-1}(U) : U \in \mathcal{T}_Y\}$. This is the coarsest topology on X such that f is continuous.

Definition. Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \to Y$ is **uniformly continuous** on X if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in X$ satisfy $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \epsilon$.

Proposition. Let (X, d_X) and (Y, d_Y) be metric spaces. A map $f : X \to Y$ is uniformly continuous iff for any two sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ in X such that $d_X(x_i, y_i) \to 0$, we have $d_Y(f(x_i), f(y_i)) \to 0$.

Definition. Let $f_i : X \to Y$ be a sequence of maps from a set X to a metric space (Y, d):

- $\{f_i\}_{i=1}^\infty$ **converges pointwise** to $f : X \to Y$ if $f_i(x) \to f(x)$ for any $x \in X$.
- $\{f_i\}_{i=1}^\infty$ **converges uniformly** to $f : X \to Y$ if for any $\epsilon > 0$, there exists $N > 0$ such that for all $i \geq N$ and any $x \in X$, $d(f_i(x), f(x)) < \epsilon$.

Standard constructions

Product of topological spaces

Definition. Let $\{X_\alpha\}_{\alpha \in \Lambda}$ be nonempty sets.

- The **product** is defined as $\prod_{\alpha \in \Lambda} X_\alpha = \{(x_\alpha)_{\alpha \in \Lambda} : x_\alpha \in X_\alpha, \forall \alpha \in \Lambda\}$
- For all $\alpha \in \Lambda$, the map $\pi_{X_\alpha} : \prod_{\alpha \in \Lambda} X_\alpha \to X_\alpha$ defined by $(x_\alpha)_{\alpha \in \Lambda} \mapsto x_\alpha$ is the **projection** to the α-th factor.

Definition.

- If $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in\Lambda}$ are topological spaces, the **product topology** on $\prod_{\alpha\in\Lambda} X_\alpha$ is the topology generated by the subbasis $\mathcal{S} = \{\pi_{X_\alpha}^{-1}(U_\alpha) : \alpha \in \Lambda, U_\alpha \in \mathcal{T}_\alpha\}$.
- If $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in\Lambda}$ are topological spaces, the **box topology** on $(X_\alpha, \mathcal{T}_\alpha)_{\alpha\in\Lambda}$ is the topology generated by the basis $\mathcal{B} = \{\prod_{\alpha\in\Lambda} U_\alpha : U_\alpha \subset X_\alpha \text{ is open}\}$.

The product and box topologies are the **same for finite** product but **different for infinite** product.

Proposition. Let $\{X_\alpha\}_{\alpha\in\Lambda}$ be topological spaces. For any $\alpha \in \Lambda$, let $\pi_{X_\alpha} : \prod_{\alpha\in\Lambda} X_\alpha \to X_\alpha$ be the projection to the α-th factor:

1. The product topology on $\prod_{\alpha\in\Lambda} X_\alpha$ is the coarsest topology such that π_{X_α} is continuous for any $\alpha \in \Lambda$.
2. Let Y be a topological space, and for any $\alpha \in \Lambda$, let $f_\alpha : Y \to X_\alpha$. The map $f = \prod_{\alpha\in\Lambda} f_\alpha : Y \to \prod_{\alpha\in\Lambda} X_\alpha$ defined by $y \mapsto (f_\alpha(y))_{\alpha\in\Lambda}$ is continuous iff f_α is continuous for every $\alpha \in \Lambda$.

Proposition. Let X be a topological space and $f, g : X \to \mathbb{R}$ be continuous. Then $f+g$, $f-g$, and $f \cdot g$ are continuous. Also, if $0 \notin g(X)$, then $\frac{f}{g}$ is continuous.

Products of metric spaces

Definition. If $(X_1, d_{X_1}), \ldots, (X_n, d_{X_n})$ are metric spaces, there are two common metrics on $X_1 \times \cdots \times X_n$:

$$d_1((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \sum_{i=1}^{n} d_{X_i}(x_i, y_i)$$

$$d_\infty((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \max_{i=1,\ldots,n}(d_{X_i}(x_i,y_i))$$

Remark.

- If $\mathcal{B}_1, \ldots, \mathcal{B}_n$ are bases for the topological spaces $(X_1, \mathcal{T}_1), \ldots, (X_n, \mathcal{T}_n)$ respectively, then $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$ is a basis for $X_1 \times \cdots \times X_n$ that generates the product topology.
- If $(X_1, d_{X_1}), \ldots, (X_n, d_{X_n})$ are metric spaces that induce topologies $\mathcal{T}_1, \ldots, \mathcal{T}_n$ on $X_1, \ldots, X_n$ respectively, then the metrics $d_1$and d_∞ on $X_1 \times \cdots \times X_n$ both induce the product topology.
- In the infinite product case, let $(X_i, d_{X_i})_{i=1}^\infty$ be metric spaces. Given the metric d_∞ above, we define $d_\infty : \prod_{i=1}^\infty X_i \times \prod_{i=1}^\infty X_i \to \mathbb{R}$ by $d_\infty(x,y) = \sup\{d_{X_i}(x_i,y_i) : i \in \mathbb{Z}^+\}$. But this is not well-defined as $d_{X_i}(x_i, y_i)$ might be unbounded as $i \to \infty$.

Proposition. Let (X, d) be a metric space. Then $\rho : X \times X \to \mathbb{R}$ given by $\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$ is a metric and its diameter is less than 1. Furthermore ρ and d induce the same topology on X.

Quotient of topological spaces

Definition. Let X and Y be topological spaces.

- A surjective map $p : X \to Y$ is a **quotient map** if $V \subset Y$ is open $\iff$ $p^{-1}(V) \subset X$ is open.
- A continuous map $f : X \to Y$ is **open (closed)** if $f(U)$ is open (closed) for any open (closed) $U \subset X$.

If a surjective continuous map is open or closed, then it is a quotient map. The composition of quotient maps is also a quotient map.

Definition. Let $f : X \to Y$ be a surjective continuous map and $A \subset X$. Then A is a **saturated set** wrt f if $A = f^{-1}(S)$ for some $S \subset Y$. Equivalently $A = f^{-1}(f(A))$.

Definition. Let $f : X \to Y$ be a surjective continuous map.

1. f is a quotient map $\iff$ f sends every saturated (wrt f) open (closed) set to an open (closed) set.
2. If f is a quotient map and $A \subset X$ is saturated and open (closed), then $f|_A : A \to f(A)$ is also a quotient map.

Proposition. If X is a topological space, $A \subset X$ and $p : X \to A$ is surjective, then $\exists!$ topology on A (called the **quotient topology**) such that p is a quotient map.

Definition. Let X be a topological space and let X^* be the cells of a partition of X. Let $p : X \to X^*$ be the surjective map that sends each point in X to the subset that contains it. X^* equipped with the quotient topology induced by p is a **quotient space** of X.

Chapter 2

T_1 and T_2 spaces

Definition. Let X be a topological space.

- X is T_1 if for any distinct $x, y \in X$, there exists an open set $U \subset X$ such that $x \in U$ but $y \notin U$.
- X is T_2 or **Hausdorff** if for any distinct $x, y \in X$, there exist open neighbourhoods U, V of x, y respectively such that they are disjoint.

Examples

- Any Hausdorff space is T_1
- Any metric space is Hausdorff
- If $|X| \geq 2$, then the trivial topology is not T_1
- The discrete topology is Hausdorff
- The cofinite topology is T_1. The cofinite topology is Hausdorff iff X is finite
- If X is infinite, then the cofinite topology on X is not metrizable

Proposition. X is T_1 $\iff$ $\forall x \in X$, $\{x\}$ is closed. It follows that finite sets in metric spaces are closed.

First countable space

Definition. Let X be a topological space.

- $\forall x \in X$, a **countable basis of X at x** is a countable collection $\mathcal{B}$ of open sets in X that contain x such that every open set in X that contains x also contains some $B \in \mathcal{B}$.
- X is **first countable** if there is a countable basis of X at x for every $x \in X$.

Proposition. Let X be a topological space.

1. Let $A \subset X$. If there exists a sequence $(x_i)_{i=1}^\infty \subset A$ such that $x_i \to x$ as $i \to \infty$, then $x \in \overline{A}$. The converse is true if X is first countable.
2. Let $f : X \to Y$. If f is continuous, then for any sequence $(x_i)_{i=1}^\infty \subset X$ such that $x_i \to x$ as $n \to \infty$, we have $f(x_i) \to f(x)$ as $i \to \infty$. The converse holds if X is first countable.

Compactness

Definition. Let X be a topological space.

- An **open cover** of X is a collection of open sets $\{U_\alpha\}_{\alpha\in\Lambda}$ in X such that $\bigcup_{\alpha\in\Lambda} U_\alpha = X$.
- X is **compact** if every open cover of X admits a finite subcover.

Remark. $Y \subset X$ is a compact subspace $\iff$ every collection $\mathcal{U}$ of open sets in Y such that $Y \subset \bigcup_{U\in\mathcal{U}} U$ admits a finite sub-collection $\mathcal{U}' \subset \mathcal{U}$ such that $Y \subset \bigcup_{U\in\mathcal{U}'} U$.

Proposition. Every closed subspace of a compact space is compact.

Proposition. Every compact subspace of a Hausdorff space is closed.

Proposition (Tube lemma). Let X be a topological space and Y be a compact topological space. If $N \subset X \times Y$ is an open set that contains $\{(x_0, y) : y \in Y\}$, then N contains $W \times Y$ for some $W \subset X$ that contains x_0.

Corollary. If X and Y are compact topological spaces, then $X \times Y$ is compact.

Definition. A collection $\mathcal{G}$ of subsets of X has the **finite intersection property** if every finite sub-collection $\{G_1, \ldots, G_n\} \subset \mathcal{G}$ satisfies $\bigcap_{i=1}^n G_i \neq \emptyset$.

Proposition. A topological space X being compact is equivalent to X having the following property: Let $\mathcal{G}$ be a collection of closed sets in X. If $\mathcal{G}$ has the finite intersection property, then $\bigcap_{G\in\mathcal{G}} G \neq \emptyset$.

Corollary. If X is compact and $\{G_i\}_{i=1}^\infty$ is a nested (i.e. $G_{i+1} \subset G_i$ for all $i \in \mathbb{Z}^+$) sequence of closed subsets in X, then $\bigcap_{i=1}^\infty G_i \neq \emptyset$.

Definition. A point x in a topological space is **isolated** if $\{x\}$ is open in X.

Theorem. Let X be a non-empty, compact, Hausdorff space. If X has no isolated points, then X is uncountable.

Limit points, sequential compactness, and the Lebesgue number

Definition. A topological space X is **limit point compact** if every infinite subset of X has a limit point in X.

Proposition. If X is compact, then it is limit point compact.

Definition. Let X be a topological space. X is **sequentially compact** if every sequence in X has a convergent subsequence.

Definition. Let X be a metric space, and let $\mathcal{U}$ be an open cover of X. A number $\delta > 0$ is a **Lebesgue number** for $\mathcal{U}$ if for all subsets $S \subset X$ such that $\operatorname{diam}(S) < \delta$, there exists $U \in \mathcal{U}$ such that $S \subset U$.

Lemma. If X is a sequentially compact metric space, then every open cover of X has a Lebesgue number.

Definition. A metric space X is **totally bounded** if for all $\epsilon > 0$, there exists a finite cover of X by balls of radius ϵ.

Lemma. If X is sequentially compact and metrizable, then X is totally bounded.

Theorem. If X is metrizable, then TFAE:

1. X is compact.
2. X is limit point compact.
3. X is sequentially compact.

Corollary. Let $f : (X, d_X) \to (Y, d_Y)$ be continuous. If X is compact, then f is uniformly continuous.