MA3209 Metric and Topological Spaces
AY24/25 Semester 1
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Definitions

- **Topology**: collection $\mathcal{T}$ of subsets of X such that
  1. $\emptyset, X \in \mathcal{T}$
  2. (Closure under arbitrary union) $\{U_\alpha\}_{\alpha \in I} \in \mathcal{T} \implies \bigcup_{\alpha \in I} U_\alpha \in \mathcal{T}$
  3. (Closure under finite intersection) $\{U_1, \dots, U_n\} \in \mathcal{T} \implies \bigcap_{i=1}^n U_i \in \mathcal{T}$

  $(X, \mathcal{T})$ is a **topological space**, and $U \subset X$ is **open** if $U \in \mathcal{T}$
- **Basis**: collection $\mathcal{B}$ of X's subsets such that
  1. $\mathcal{B}$ covers X
  2. $\forall x \in X$ and $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$, $\exists B \in \mathcal{B}$ such that $x \in B \subset B_1 \cap B_2$

  The topology generated by $\mathcal{B}$ is

  $\mathcal{T} = \{U \subset X : \forall x \in U, \exists B \in \mathcal{B} \text{ such that } x \in B \subset U\}$
- $\mathcal{T}$ is **coarser** than $\mathcal{T}'$ if $\mathcal{T}' \subset \mathcal{T}$ (reverse is **finer**)
- **Subbasis** $\mathcal{S}$: collection of subsets of X whose union equals X. The **topology generated by** $\mathcal{S}$ is a collection $\mathcal{T}$ of all unions of finite intersection of sets in $\mathcal{S}$
- A **norm** on a $\mathbb{K}$-vector space V is a function $\|\cdot\| : V \to \mathbb{R}$ that satisfies
  1. (Nonnegativity) $\|x\| \geq 0 \; \forall x \in V$
  2. (Positive definiteness) $\|x\| = 0 \iff x = 0$
  3. (Absolute homogeneity) $\|\lambda x\| = |\lambda| \, \|x\| \; \forall \lambda \in \mathbb{K}$ and $\forall x \in V$
  4. (Triangle inequality) $\|x + y\| \leq \|x\| + \|y\|$
- Let A, B be nonempty subsets of a metric space (X, d).
  - **Distance**: $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$
  - **Diameter** of $A \subset X$ is $\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}$. $A \subset X$ is bounded if $\operatorname{diam}(A) < +\infty$
- The topology on X **induced** by a metric d is the topology generated by $\mathcal{B}_d$.
- A topology $\mathcal{T}$ on X is **metrizable** if there is a metric on X that induces $\mathcal{T}$
- Let $(Y, \mathcal{T}_Y)$ be a topological space and $X \subset Y$. Then $\mathcal{T}_X = \{U \cap X : U \in \mathcal{T}_Y\}$ is the **subspace topology** on X.
- Given a subset A of a metric space (X, d), the **restriction** of d to A is the metric $d_A(x, y) = d(x, y) \; \forall x, y \in A$. The topology induced by this metric is the subspace topology.
- $A \subset X$ is closed if $X \setminus A \in \mathcal{T}$
- Let $(X, \mathcal{T})$ be a topological space and $A \subset X$.
  1. **Interior** of A: $\mathring{A} = \bigcup_{U \in \mathcal{T}, U \subset A} U$.
  2. **Closure** of A: $\overline{A} = \bigcap_{X \setminus G \in \mathcal{T}, G \supset A} G$.
  3. **Boundary** of A: $\partial A = \overline{A} - \mathring{A}$.
- Let X be a topological space and $A \subset X$. A point $x \in X$ is a **limit point** of A if every open $U \subset X$ containing x intersects $A \setminus \{x\}$.
- A sequence $\{x_i\}_{i=1}^\infty$ in a topological space X converges to $x \in X$ (i.e. $x_i \to x$) if for any neighbourhood $U \ni x$, $\exists N > 0$ such that $x_k \in U$ for all $k > N$
- Let X and Y be topological spaces. $f : X \to Y$ is **continuous** if for any open $U \subset Y$, $f^{-1}(U) \subset X$ is open.
- Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \to Y$ is **uniformly continuous** on X if for any $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in X$ satisfy $d_X(x, y) < \delta$, then $d_Y(f(x), f(y)) < \epsilon$.
- Let $f_i : X \to Y$ be a sequence of maps from a set X to a metric space (Y, d):
  - $\{f_i\}_{i=1}^\infty$ **converges pointwise** to $f : X \to Y$ if $f_i(x) \to f(x)$ for any $x \in X$.
  - $\{f_i\}_{i=1}^\infty$ **converges uniformly** to $f : X \to Y$ if for any $\epsilon > 0$, there exists $N > 0$ such that for all $i \geq N$ and any $x \in X$, $d(f_i(x), f(x)) < \epsilon$.
- For all $\alpha \in \Lambda$, the map $\pi_{X_\alpha} : \prod_{\alpha \in \Lambda} X_\alpha \to X_\alpha$ defined by $(x_\alpha)_{\alpha \in \Lambda} \mapsto x_\alpha$ is the **projection** to the α-th factor.
- If $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in \Lambda}$ are topological spaces, the **product topology** on $\prod_{\alpha \in \Lambda} X_\alpha$ is the topology generated by the subbasis $\mathcal{S} = \{\pi_{X_\alpha}^{-1}(U_\alpha) : \alpha \in \Lambda, U_\alpha \in \mathcal{T}_\alpha\}$.
- If $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in \Lambda}$ are topological spaces, the **box topology** on $(X_\alpha, \mathcal{T}_\alpha)_{\alpha \in \Lambda}$ is the topology generated by the basis $\mathcal{B} = \{\prod_{\alpha \in \Lambda} U_\alpha : U_\alpha \subset X_\alpha \text{ is open}\}$.
- The product and box topologies are the **same for finite** product but **different for infinite** product.
- If $(X_1, d_{X_1}), \dots, (X_n, d_{X_n})$ are metric spaces, there are two common metrics on $X_1 \times \cdots \times X_n$:

$$d_1((x_1, \dots, x_n), (y_1, \dots, y_n)) = \sum_{i=1}^n d_{X_i}(x_i, y_i)$$

$$d_\infty((x_1, \dots, x_n), (y_1, \dots, y_n)) = \max_{i=1,\dots,n}(d_{X_i}(x_i, y_i))$$

- Let X and Y be topological spaces.
  - A surjective map $p : X \to Y$ is a **quotient map** if $V \subset Y$ is open $\iff p^{-1}(V) \subset X$ is open.
  - A continuous map $f : X \to Y$ is **open (closed)** if $f(U)$ is open (closed) for any open (closed) $U \subset X$.

  If a surjective continuous map is open or closed $\implies$ it is a quotient map. Composition of quotient maps is also a quotient map.
- Let $f : X \to Y$ be a surjective continuous map and $A \subset X$. Then A is a **saturated set** wrt f if $A = f^{-1}(S)$ for some $S \subset Y$. Equivalently $A = f^{-1}(f(A))$.
- Let X be a topological space and let X^* be the cells of a partition of X. Let $p : X \to X^*$ be the surjective map that sends each point in X to the subset that contains it. X^* equipped with the quotient topology induced by p is a **quotient space** of X.
- A topological space X is T_1 if for any distinct $x, y \in X$, there exists an open set $U \subset X$ such that $x \in U$ but $y \notin U$.
- X is T_2 or **Hausdorff** if for any distinct $x, y \in X$, there exist open neighbourhoods U, V of x, y respectively such that they are disjoint.
- Let X be a topological space. $\forall x \in X$, a **countable basis of** X **at** x is a countable collection $\mathcal{B}$ of open sets in X that contain x such that every open set in X that contains x also contains some $B \in \mathcal{B}$.
- X is **first countable** if there is a countable basis of X at x for every $x \in X$.
- X is **compact** if every open cover ($\{U_\alpha\}_{\alpha \in \Lambda}$ s.t. their union is X) of X admits a finite subcover.
- $Y \subset X$ is a compact subspace $\iff$ every collection $\mathcal{U}$ of open sets in Y such that $Y \subset \bigcup_{U \in \mathcal{U}} U$ admits a finite sub-collection $\mathcal{U}' \subset \mathcal{U}$ such that $Y \subset \bigcup_{U \in \mathcal{U}'} U$.
- A collection $\mathcal{G}$ of subsets of X has the **finite intersection property** if every finite sub-collection $\{G_1, \dots, G_n\} \subset \mathcal{G}$ satisfies $\bigcap_{i=1}^n G_i \neq \emptyset$.
- A point x in a topological space is **isolated** if $\{x\}$ is open in X.
- A topological space X is **limit point compact** if every infinite subset of X has a limit point in X. Converse: No limit points $\implies X$ finite
- Let X be a topological space. X is **sequentially compact** if every sequence in X has a convergent subsequence.
- Let X be a metric space, and let $\mathcal{U}$ be an open cover of X. A number $\delta > 0$ is a **Lebesgue number** for $\mathcal{U}$ if for all subsets $S \subset X$ such that $\operatorname{diam}(S) < \delta$, there exists $U \in \mathcal{U}$ such that $S \subset U$.
- A metric space X is **totally bounded** if for all $\epsilon > 0$, there exists a finite cover of X by balls of radius ϵ.
- A sequence of points $(x_i)_{i=1}^\infty$ in a metric space is a **Cauchy sequence** if $\forall \epsilon > 0, \exists N > 0$ such that $d(x_n, x_m) < \epsilon, \forall m, n > N$.
- A metric space is **complete** if every Cauchy sequence converges
- A topological space is **locally compact at** $x \in X$ if $\exists$ compact $C \subset X$ and open $U \subset X$ s.t. $x \in\subset U \subset C$. If X locally compact at all $x \in X$ then it is locally compact
- $f : X \to Y$ is a **homeomorphism** if f is a bijective continuous map with continuous inverse f^{-1}
- If Y is compact Hausdorff and $\exists$ map $h : X \to Y$ s.t. $\overline{h(X)} = Y$ and h is a homeomorphism onto its image, then Y is a **compactification** of X. If $Y \setminus h_Y(X)$ is a point, then Y is a **one-point compactification** of X
- Let (Y, d) be a metric space and ρ the metric on Y given by

$$\rho(x, y) = \frac{d(x, y)}{1 + d(x, y)}.$$

  The *uniform metric* on $Y^\Lambda = \prod_{\alpha \in \Lambda} Y$ is the metric given by

$$\bar{\rho}(x, y) = \sup\{\rho(\pi_\alpha(x), \pi_\alpha(y)) : \alpha \in \Lambda\}.$$

  The *uniform topology* on Y^Λ is the topology generated by the uniform metric. It is finer than the product topology and coarser than the box topology; these three are all different if Λ is infinite
- Let X be a topological space and (Y, d) a metric space. $\mathcal{C}(X, Y) = \{f \in Y^X : f \text{ is continuous}\}$, $\mathcal{B}(X, Y) = \{f \in Y^X : f(X) \subset Y \text{ has bounded diameter}\}$
- Let (X, d_X) and (Y, d_Y) be two metric spaces. We say $f : (X, d_X) \to (Y, d_Y)$ is an isometric embedding if for all $a, b \in X$, $d_X(a, b) = d_Y(f(a), f(b))$. We say f is an isometry if it is a surjective isometric embedding.
- If (X, d_X) is a metric space, then a *metric completion* of X is a complete metric space (Y, d_Y) and an isometric embedding $\phi : X \to Y$ such that $\overline{\phi(X)} = Y$.
- A **separation** of a topological space X is a pair U, V of disjoint, nonempty open subsets whose union is X. X is **connected** if there does not exist a separation of X
- Given $x, y \in X$, a **path** from x to y is a continuous map $f : [a, b] \to X$ s.t. $f(a) = x$ and $f(b) = y$. X is **path connected** if $\forall x, y \in X$, there is a path from x to y
- Let X be a topological space. $\forall x, y \in X$, define $x \sim y$ if $\exists$ connected subset $C \subset X$ s.t. $x, y \in C$. The equivalence classes of $\sim$ are the **connected components** of X

- Let X be a topological space. $\forall x, y \in X$, define $x \overset{p}{\sim} y$ if $\exists$ a path in X from x to y. $\overset{p}{\sim}$ are called **path components**
- X is locally (path) connected at x if $\forall$ open $U \subset X$ containing x, $\exists$ (path) connected open set $V \subset X$ s.t. $x \in V \subset U$. X is locally (path) connected if it is locally (path connected) at every $x \in X$
- A topological space X is **second countable** if it has a countable basis, i.e. there exists some countable collection $\mathcal{U}$ of open sets in X s.t. every open subset of X can be written as a union of elements in $\mathcal{U}$. Second countable $\implies$ first countable
- X is a **Lindelof** space if every open cover has a countable subcover
- A T_1 topological space X is **regular** or T_3 if $\forall x \in X$ and every closed $B \subset X$ s.t. $x \notin B$, $\exists$ disjoint open sets $U, V \subset X$ s.t. $x \in U$ and $B \subset V$.
- A T_1 topological space X is **normal** or T_4 if $\forall$ closed and disjoint $A, B \subset X$ $\exists$ disjoint open sets $U, V \subset X$ s.t. $A \subset U$ and $B \subset V$
- Let $A, B \subset X$. A and B are separated by a continuous function if $\exists$ continuous $f : X \to [0, 1]$ s.t. $f(A) = 0$ and $f(B) = 1$.
- X is **completely regular** if it is T_1 and $\forall x \in X$ and closed $A \subset X$ s.t. $x \notin A$, $\{x\}$ and A are separated by a continuous function
- X is **completely normal** if it is T_1 and $\forall A, B \subset X$ closed disjoint, A and B are separated by a continuous function
- Let X, Y be two topological spaces and $f : X \to Y$ be an injective continuous map. f is a topological embedding if f is a homeomorphism between X and $f(X)$
- Let X be a topological space and (Y, d) a metric space. Given $f \in Y^X$, compact $C \subset X$, and $\epsilon > 0$, $B(C, f, \epsilon) = \{g \in Y^X : \sup_{x \in C} d(f(x), g(x)) < \epsilon\}$ is the topology of compact convergence.
- Let X, Y be topological spaces. $\forall$ compact $C \subset X$ and open $U \subset Y$, $S(C, U) = \{g \in \mathcal{C}(X, Y) : g(C) \subset U\}$ forms a basis of the compact-open topology.
- A topological space X is compactly generated if it satisfies either
  - $A \subset X$ open $\iff$ $A \cap C$ open in C for every compact $C \subset X$
  - $B \subset X$ closed $\iff$ $B \cap C$ closed in C for every compact $C \subset X$
- Let (Y, d) be a metric space and $\mathcal{Y} \subset \mathcal{C}(X, Y)$. If $x_0 \in X$, then $\mathcal{Y}$ is equicontinuous at x_0 if $\forall \epsilon > 0$, $\exists$ open $U \subset X$ containing x_0 s.t. $\forall x \in U, \forall f \in \mathcal{Y}$, $d(f(x), f(x_0)) < \epsilon$. $\mathcal{Y}$ is equicontinuous if it is equicontinuous at every $x_0 \in X$

Results

- Let $\mathcal{B}, \mathcal{B}'$ be bases of topologies $\mathcal{T}, \mathcal{T}'$ respectively on X. TFAE:
  1. $\mathcal{T}'$ is finer than $\mathcal{T}$
  2. $\forall B \in \mathcal{B}$, $\forall x \in B$, $\exists B' \in \mathcal{B}'$ such that $x \in B' \subset B$
- $A \subset X$ open $\iff$ $\forall a \in A$ $\exists$ open $U_a \subset A$ such that $a \in U_a$ (all $a \in A$ are interior points)
- $A \subset X$ is closed $\iff$ to $\forall$ open U containing $a \in A$, $U \cap A \neq \emptyset$ (all limit points are contained within A)
- $\mathring{A} \subset A \subset \overline{A}$, $\mathring{A} = A$ $\iff$ A is open, $\overline{A} = A$ $\iff$ A is closed.
- Let X be a topological space, $A \subset X$.
  1. $x \in \overline{A}$ $\iff$ $\forall$ open U containing x, $U \cap A \neq \emptyset$.
  2. If A' is the set of limit points of A, then $\overline{A} = A \cup A'$.
- f is continuous $\iff$ $\forall A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$ $\iff$ For any closed set $B \subset Y$, $f^{-1}(B) \subset X$ is closed $\iff$ For any $x \in X$ and any open set $V \subset Y$ containing $f(x)$, there exists open $U \subset X$ containing x such that $f(U) \subset V$
- (**Pasting Lemma**) Let $X = A \cup B$ where $A, B \subset X$ are both closed (or both open). Let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then $h : X \to Y$ defined by
$$h(x) = \begin{cases} f(x), & \text{if } x \in A \\ g(x), & \text{if } x \in B \end{cases}$$
is continuous.
- Let (X, d_X) and (Y, d_Y) be metric spaces. $f : X \to Y$ is uniformly continuous iff for any two sequences $\{x_i\}_{i=1}^\infty$ and $\{y_i\}_{i=1}^\infty$ in X such that $d_X(x_i, y_i) \to 0$, $d_Y(f(x_i), f(y_i)) \to 0$.
- Let $\{X_\alpha\}_{\alpha \in \Lambda}$ be topological spaces. For any $\alpha \in \Lambda$, let $\pi_{X_\alpha} : \prod_{\alpha \in \Lambda} X_\alpha \to X_\alpha$ be the projection to the α-th factor:
  1. The product topology on $\prod_{\alpha \in \Lambda} X_\alpha$ is the coarsest topology such that π_{X_α} is continuous for any $\alpha \in \Lambda$.
  2. Let Y be a topological space, and for any $\alpha \in \Lambda$, let $f_\alpha : Y \to X_\alpha$. The map $f = \prod_{\alpha \in \Lambda} f_\alpha : Y \to \prod_{\alpha \in \Lambda} X_\alpha$ defined by $y \mapsto (f_\alpha(y))_{\alpha \in \Lambda}$ is continuous iff f_α is continuous for every $\alpha \in \Lambda$.
- Let X be a topological space and $f, g : X \to \mathbb{R}$ be continuous $\implies$ $f + g$, $f - g$, and $f \cdot g$ are continuous. Also, if $0 \notin g(X)$, then $\frac{f}{g}$ is continuous.
- Let (X, d) be a metric space. Then $\rho : X \times X \to \mathbb{R}$ given by $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$ is a metric and its diameter is less than 1. Furthermore ρ and d induce the same topology on X.
- Let $(X_i, d_{X_i})_{i=1}^\infty$ be metric spaces for all i and let $\rho_{X_i} = \frac{d_{X_i}(x,y)}{1+d_{X_i}(x,y)}$ for all $x, y \in X_i$. Then $d : \prod_{i=1}^\infty X_i \times \prod_{i=1}^\infty X_i \to \mathbb{R}$ given by
$$d(x, y) = \sup\{\frac{1}{i}\rho_{X_i}(x_i, y_i) : i \in \mathbb{Z}\}$$
induces the product topology on $\prod_{i=1}^\infty X_i$
- f is a quotient map $\iff$ f sends every saturated (wrt f) open (closed) set to an open (closed) set.
- If f is a quotient map and $A \subset X$ is saturated and open (closed), then $f|_A : A \to f(A)$ is also a quotient map.
- X is T_1 $\iff$ $\forall x \in X$, $\{x\}$ is closed. Hence finite sets in metric spaces are closed.
- Let X be a topological space, and $A \subset X$. If there exists a sequence $(x_i)_{i=1}^\infty \subset A$ such that $x_i \to x$ as $i \to \infty$, then $x \in \overline{A}$. The converse is true if X is first countable.
- Let $f : X \to Y$. If f is continuous, then for any sequence $(x_i)_{i=1}^\infty \subset X$ such that $x_i \to x$ as $n \to \infty$, we have $f(x_i) \to f(x)$ as $i \to \infty$. The converse holds if X is first countable.
- Every closed subspace of a compact space is compact.
- (**Tube Lemma**) Let X be a topological space and Y be a compact topological space. If $N \subset X \times Y$ is an open set that contains $\{(x_0, y) : y \in Y\}$, then N contains $W \times Y$ for some $W \subset X$ that contains x_0.
- If X and Y are compact topological spaces, then $X \times Y$ is compact.
- X compact $\iff$ X has the following property: Let $\mathcal{G}$ be a collection of closed sets in X. If $\mathcal{G}$ has the finite intersection property, then $\bigcap_{G \in \mathcal{G}} G \neq \emptyset$. Idea: consider $\mathcal{U} = \{X \setminus G : G \in \mathcal{G}\}$, use FIP, note $X \setminus (\bigcap_{G \in \mathcal{G}}) = \emptyset$
- If X is compact and $\{G_i\}_{i=1}^\infty$ is a nested (i.e. $G_{i+1} \subset G_i$ for all $i \in \mathbb{Z}^+$) sequence of closed subsets in X, then $\bigcap_{i=1}^\infty G_i \neq \emptyset$.
- Let X be a non-empty, compact, Hausdorff space. If X has no isolated points, then X is uncountable.
- Compact $\implies$ limit point compact
- If X is (any type of) compact metric space, then every open cover of X has a Lebesgue number.
- (Any type of) compactness and metrizable $\implies$ totally bounded. Idea: contradiction, construct sequence $x_n \in X \setminus (\bigcup_{i=1}^{n-1} B_{\epsilon_0}(x_i))$ which has no convergent subsequence
- If X is metrizable, compact $\iff$ limit point compact $\iff$ sequentially compact.
- Let $f : (X, d_X) \to (Y, d_Y)$ be continuous. If X is compact, then f is uniformly continuous.
- Totally bounded $\implies$ finite diameter
- X locally compact and Hausdorff iff $\exists$ compact Hausdorff space Y and map $h_Y : X \to Y$ s.t. (1) h_Y is a homeomorphism onto its range and (2) $Y \setminus h_Y(X)$ is a single point. Furthermore, if (Y, h_Y) and $(Y', h_{Y'})$ are two such spaces and maps, $\exists$ homeomorphism $f : Y \to Y'$ s.t. $f|_{h_Y(X)} = h_{Y'} \circ h_Y^{-1}|_{h_Y(X)} : h_Y(X) \to h_{Y'}(X)$
- X is Hausdorff, non-compact, locally compact $\implies$ it admits a unique one-point compactification
- Let X be a Hausdorff topological space. Then X is locally compact is equivalent to for any $x \in X$, for any open $U \subset X$ such that $x \in U$, there exists open $V \subset X$ such that $x \in V$, $\bar{V} \subset U$ and $\bar{V}$ is compact.
- Let X be a locally compact space. $A \subset X$ is closed or X is Hausdorff and A is open $\implies$ A locally compact
- X is a homeomorphism to an open subset of a compact Hausdorff space $\iff$ X is locally compact and Hausdorff
- (Y, d) complete $\implies$ $(Y^\Lambda, \overline{\rho})$ complete
- $\mathcal{C}(X, Y), \mathcal{B}(X, Y) \subset Y^X$ are closed in the uniform topology. In particular, (Y, d) complete $\implies$ $(\mathcal{C}(X, Y), \overline{\rho})$ and $(\mathcal{B}(X, Y), \overline{\rho})$ complete
- Let (X, d) be a metric space. Then there is an isometric embedding ϕ of X into a complete metric space Y such that $\phi(X) \subset Y$ is dense. Furthermore, if $(Y', d_{Y'})$ is a complete metric space and $\phi' : X \to Y'$ is an isometric embedding such that $\overline{\phi'(X)} = Y'$, then there exists an isometry $f : Y \to Y'$ such that
$$f|_{\phi(X)} = \phi' \circ \phi^{-1} : \phi(X) \to \phi'(X).$$
- X is connected $\iff$ the only sets in X that are open and closed are $\emptyset$ and X
- $\{A_\alpha\}_{\alpha \in \Lambda}$ is a collection of connected subsets of X s.t. $\bigcap_{\alpha \in \Lambda} A_\alpha \neq \emptyset$ $\implies$ $\bigcup_{\alpha \in \Lambda} A_\alpha \subset X$ is connected
- $A \subset X$ connected and $A \subset B \subset \overline{A}$ $\implies$ B connected
- $f : X \to Y$ continuous and $A \subset X$ connected $\implies$ $f(A) \subset Y$ connected
- X, Y connected $\implies$ $X \times Y$ connected
- Every connected component of X is connected
- X is locally (path) connected $\iff$ $\forall$ open $U \subset X$, each (path) connected component of U is open in X
- If X is locally path connected, then connected components and path components are the same

- Suppose X is second countable. Then (1) X is Lindelof and (2) there exists a countable subset $A \subset X$ that is dense, i.e. $\overline{A} = X$. The converse of both of them are true if X is metrizable
- X is regular $\iff \forall x \in X, \forall U \subset X$ containing x, $\exists$ open $V \subset X$ containing x s.t. $\overline{V} \subset U$
- Every metrizable space is normal
- X is a regular topological space with a countable basis $\implies$ X is normal
- (Urysohn's metrization theorem) X is regular with countable basis $\implies$ X is metrizable
- (Urysohn's lemma) X is normal $\implies$ X is completely normal
- (Tychonoff's theorem) The product of compact spaces is compact, i.e. if $\{X_\alpha\}_{\alpha\in\Lambda}$ is a family of compact spaces, then $X = \prod_{\alpha\in\Lambda} X_\alpha$ is compact wrt product topology
- (Arzela-Ascoli theorem) Let X be a topological space and (Y, d) a metric space. Equip $\mathcal{C}(X,Y)$ with the compact open topology and let $\mathcal{Y} \subset \mathcal{C}(X,Y)$.
  1. If $\mathcal{Y}$ is equicontinuous under d and $\mathcal{Y}_a = \{f(a) : f \in \mathcal{Y}\}$ has compact closure for each $a \in X$, then $\overline{\mathcal{Y}} \subset \mathcal{C}(X,Y)$ is compact
  2. Converse holds if X is locally compact and Hausdorff

Examples

- $\mathcal{T} = \{\emptyset, X\}$ is the **trivial topology**
- $\mathcal{T} = \{\text{subsets of } X\}$ is the **discrete topology**. Non-compact, has Lebesgue number (midterm question)
- $\mathcal{T} = \{X - U : U \subset X \text{ is finite}\} \cup \{\emptyset\}$ is the **cofinite topology**
- $X = \{a,b,c\}$, possible topologies include $\{\{a\},\{a,b\},\emptyset,X\}$, $\{\{b,c\},\emptyset,X\}$ (and more)
- $X = \mathbb{R}$, $\mathcal{T} = \{(-\alpha,\alpha) : \alpha \in \mathbb{R}^+\} \cup \{\emptyset, \mathbb{R}\}$
- (**HW**) Collection of unions of arithmetic sequences
- (**HW**) **Co-countable topology**: U is open if $U = \emptyset$ or $X \setminus U$ is countable. It is not comparable with the standard topology, but finer than the co-finite topology
- The **discrete metric** is
$$d(x,y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$
- (**HW**) Let X the space of all closed subsets of $\mathbb{R}^n$. Let $B_\epsilon(A) = \bigcup_{a\in A} B_\epsilon(a)$ be an ϵ-neighbourhood of A. Then the **Hausdorff metric** $d_H(A,B) = \inf\{\epsilon > 0 : A \subset B_\epsilon(B) \text{ and } B \subset B_\epsilon(A)\}$ is a metric on X. This is not a metric on the space of all subsets of $\mathbb{R}^n$, e.g. $A = [0,1]^n$, $B = (0,1)^n$
- The l_p**-norm** is $V = \mathbb{K}^n, p \geq 1, \|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{1/p}, x \in \mathbb{K}^n$.
- The l_∞**-norm** is $V = \mathbb{K}^n, \|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}, x \in \mathbb{K}^n$
- $[a,b] \subset \mathbb{R}$ is closed wrt standard topology on $\mathbb{R}$
- Let $X = [0,1] \cup (2,3) \subset \mathbb{R}$. $[0,1]$ is both open and closed in X wrt subspace topology on X
- $\{0\} \cup (1,2) \subset \mathbb{R}$ has $[1,2]$ as its set of limit points wrt standard topology on $\mathbb{R}$
  - 0 is not a limit point of A, e.g. $(-1/2, 1/2)$ open but doesn't intersect $A \setminus \{0\}$
  - $x \in \mathbb{R} \setminus (\{0\} \cup [1,2])$ is not a limit pt of A
  - Every $x \in [1,2]$ is a limit pt of A
- x being a lim pt of $\{x_i\} \not\Rightarrow x_i \to x$. E.g. $\{(-1)^n + \frac{1}{n}\}$ doesn't converge but has lim pts $\{-1, 1\}$
- $x_i \to x \not\Rightarrow x$ is a lim pt of $\{x_i\}$. E.g. $(1,1,\ldots)$ converges to 1 but $\{1\}$ has no lim pt
- Given a topology $\mathcal{T}_Y$ on Y and a map $f : X \to Y$, the **pull back** topology on X is $\mathcal{T}_X = \{f^{-1}(U) : U \in \mathcal{T}_Y\}$. This is the coarsest topology on X such that f is continuous.
- If (X,d) is a metric space with $A \subset X$ nonempty, then $f : X \to \mathbb{R}$ defined by $x \mapsto d(x,A)$ is uniformly continuous. Idea: $d(x,A) \leq d(x,y) + d(y,A)$, let $\epsilon = \delta$
- (**Midterm**) Let $\mathbb{R}^{\mathbb{N}}$ be equipped with the box topology, and consider $f : \mathbb{R} \to \mathbb{R}^{\mathbb{N}}$ as $x \mapsto (x,x,\ldots)$. All component functions are the identity and hence continuous, but f is not continuous. Idea: Let $U = \prod_{n=1}^\infty (-\frac{1}{n}, \frac{1}{n})$. If f continuous, there should exist $\epsilon > 0$ s.t. $(-\epsilon,\epsilon) \subset f^{-1}(U)$, but $\frac{\epsilon}{2} > \frac{1}{n}$
- Let $p : [0,1] \cup [2,3] \to [0,2]$ be a map defined by
$$x \mapsto \begin{cases} x & \text{if } x \in [0,1], \\ x-1 & \text{if } x \in [2,3]. \end{cases}$$
  p is closed but not open. By the pasting lemma, we know that p is continuous. Let $A \subset [0,1] \cup [2,3]$ be closed, then
$$A_1 = A \cap [0,1] \subset [0,1], \quad A_2 = A \cap [2,3] \subset [2,3]$$
  are closed sets. This together with the definition of p, $[0,1]$ and $[1,2]$ are closed in $[0,2]$ shows that $p(A_1)$ and $p(A_2)$ are closed subsets of $[0,2]$ with respect to its topology. As a result we show that $p(A)$ is closed and thus p is a closed map. But p is not open. Let $B = (0,1)$, then B is open wrt. the subspace topology in $[0,1] \cup [2,3]$. However $p((0,1)) = (0,1) \subset [0,2]$ is not open.
- (**Midterm**) $p : (0,1) \cup (2,3) \to (0,2)$ is open but not closed by a similar argument
- Let $X = \mathbb{R}^2 \setminus \{(x,y) : 0 \leq x < 1, 0 < y < 1\}$ and $f : X \to \mathbb{R}$ be defined as $f(x,y) = x$. Then f is surjective, continuous, not open, not closed, but a quotient map.
- If X is a topological space, $A \subset X$ and $p : X \to A$ is surjective, then $\exists!$ topology on A (called the **quotient topology**) such that p is a quotient map where $\mathcal{T} = \{U \subset A : p^{-1}(U) \subset X \text{ is open}\}$
- Let $p : \mathbb{R} \to \{a,b,c\}$ be a map defined as
$$x \mapsto \begin{cases} a, & \text{if } x > 0, \\ b, & \text{if } x = 0, \\ c, & \text{if } x < 0. \end{cases}$$
  Then the quotient topology on $\{a,b,c\}$ is
$$\mathcal{T} = \{\{a\},\{c\},\{a,c\},\{a,b,c\},\emptyset\}.$$
- Partition $X = \mathbb{R} = \mathbb{R}^- \cup \{0\} \cup \mathbb{R}^+$. Then the quotient space $X^* = \{\mathbb{R}^-, \{0\}, \mathbb{R}^+\}$.
- Let $X = \{(x,y) : x^2 + y^2 \leq 1\}$ and decompose it as the union of $\bigcup_{(x,y):x^2+y^2<1} \{(x,y)\}$ and $\{(x,y) : x^2+y^2 = 1\}$. Then X^* is
$$\{\{(x,y)\} : x^2+y^2 < 1\} \cup \{\{(x,y) : x^2+y^2 = 1\}\}.$$
- Let $X = \mathbb{R}$ and p defined as that p sends x to $x+n$ for some $n \in \mathbb{Z}$ such that $x + n \in [0,1)$. It is clear that such n is unique for a fixed $x \in \mathbb{R}$.
  In this setting, $X^* = [0,1)$. We may also identify X^* as S^1 (unit circle) or $\mathbb{R}/\mathbb{Z}$.
- Any Hausdorff space is T_1
- Any metric space is Hausdorff
- If $|X| \geq 2$, then the trivial topology is not T_1
- The discrete topology is Hausdorff
- The cofinite topology is T_1. The cofinite topology is Hausdorff iff X is finite. Idea: for T_2, break into finite/infinite cases, for infinite case let U, V be X minus finite number of elements, these cannot be disjoint
- If X is infinite, then the cofinite topology on X is not metrizable
- Metric spaces are first countable: $\forall x \in X$, $\{B_{1/i}(x) : i \in \mathbb{Z}^+\}$ is a countable basis of X at x
- The cofinite topology on an uncountable set X, e.g. $\mathbb{R}$ is not first countable. Idea: suppose countable basis exists, $B_i = X \setminus F_i$ for some finite $F_i \subset X$, consider $y \in X \setminus (\{x\} \cup \bigcup_{i=1}^\infty F_i)$, show that B_i not subset of U
- $X = \{\frac{1}{n} : n \in \mathbb{Z}^+\} \subset \mathbb{R}$ is not compact since $\{\{\frac{1}{n}\} : n \in \mathbb{Z}^+\} \subset \mathbb{R}$ is an open cover of X wrt subspace topology but does not have finite subcover.
- $X = \{\frac{1}{n} : n \in \mathbb{Z}^+\} \cup \{0\} \subset \mathbb{R}$ is compact. Let $\mathcal{U}$ be any open cover, then $\exists U \in \mathcal{U}$ that contains 0 and $\exists N > 0$ such that $\frac{1}{n} \in U$ for $n \geq N$. For each $n < N$, let $U_n \in \mathcal{U}$ such that $\frac{1}{n} \in U_n$. All together we obtain a finite subcover of $\mathcal{U}$ as: $\{U_1, \ldots, U_{N-1}, U\} \subset \mathcal{U}$
- Any metric space X of infinite diameter is not compact. Let $x \in X$. Then $\{B_n(x) : n \in \mathbb{Z}^+\}$ is an open cover of X which does not have a finite subcover.‘
- $S = \{(x,y) \in \mathbb{R}^2 : |x| \leq \frac{1}{y^2+1}\} \subset \mathbb{R}^2$ contains $\{0\} \times \mathbb{R}$ but not a tube
- Any unbounded metric space is not limit point compact. Idea: Pick $x_1 \in X$ and $x_i \in B_i(x_1) \setminus B_{i-1}(x_1)$, $\{x_1, x_2, \ldots\}$ has no limit points
- Let Y (where $|Y| \geq 2$) be equipped with the trivial topology and $\mathbb{Z}^+$ be equipped with discrete topology. Let $X = \mathbb{Z}^+ \times Y$, then the product topology on X is $\{A \times Y : A \subset \mathbb{Z}^+\}$. Every non-empty subset of $\mathbb{Z}^+ \times Y$ has a limit point, so X is limit point compact. However, $\{\{a\} \times Y : a \in A\}$ is a cover of X with no finite subcover, so X is not compact
- $\mathbb{R}^n$ with ℓ_p metric has infinite diamter, so it is not totally bounded
- $\mathbb{R}^n$ with respect to l_p metric for $p \in [1,\infty]$ is complete.
- If $(x_i)_{i=1}^\infty$ is a Cauchy sequence in $\mathbb{R}^n$, then $\exists M > 0$ such that $x_i \in B'_M(0) = \{x \in \mathbb{R}^n : d(x,0) \leq M\}$ for every i. Since $B'_M(0)$ is compact and thus sequentially compact, $(x_i)_{i=1}^\infty$ has a convergent subsequence, so $(x_i)_{i=1}^\infty$ converges.
- Equipped with the standard metric on $\mathbb{R}$ restricted to $\mathbb{Q}$, $\mathbb{Q}$ is not complete. Since $\mathbb{Q} \subset \mathbb{R}$ is dense, there are sequences in $\mathbb{Q}$ that converge in $\mathbb{R}$ to an irrational number. Such sequences are Cauchy but do not have a convergent subsequence in $\mathbb{Q}$.
- Let d be the standard metric on $\mathbb{R}$, ρ the metric on $\mathbb{R}$ given by
$$\rho(x,y) = \frac{d(x,y)}{1 + d(x,y)},$$
  D the metric on $\prod_{\mathbb{Z}} \mathbb{R} = \mathbb{R}^\omega$ given by
$$D(x,y) = \sup\left\{\frac{\rho(\pi_k(x), \pi_k(y))}{k} : k \in \mathbb{Z}^+\right\},$$
  where $\pi_k : \mathbb{R}^\omega \to \mathbb{R}$ is the projection to the k-th factor.
- (**HW**) $(\mathbb{R}^w, D)$ is complete
- $\mathbb{R}^n$ is locally compact. Let $U = B_\epsilon(x)$ and $C = \overline{U}$
- $\mathbb{Q} \subset \mathbb{R}$ is not locally compact
- $\mathbb{R}^w$ with product topology is not locally compact

- Let $\mathbb{D} = \{(x,y) : x^2+y^2<1\}$. Then $\overline{\mathbb{D}}$ and $\mathbb{S}^2$ are compactifications of $\mathbb{D}$, and $\mathbb{S}^2$ is a one-point compactification
- $U_1 = \prod_{n\in\mathbb{N}}\{x=(x_n)_{n\in\mathbb{N}} : |x_n| < 2^{-n}\forall n \in \mathbb{N}\}$ is open in the box topology but not the uniform topology on $\mathbb{R}^{\mathbb{N}}$ (does not contain uniform ball)
- $U_2 = \{x \in \mathbb{R}^{\mathbb{N}} : \rho(x,0) < 0.01\}$ is open in the uniform topology but not in the product topology (does not contain any set of the form $\{x=(x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : x_{n_1} = \cdots = x_{n_k} = 0)$
- The supremum metric $d_{\sup}$ on $\mathcal{B}(X,Y)$ is $d_{\sup}(f,g) = \sup\{d(f(x),g(x)) : x\in X\}$. This is well defined since $\operatorname{diam}(f(X)\cup g(X))$ is bounded
- The trivial topology is connected.
- $[-1,0)\cup(0,1] \subset X$ is not connected wrt standard topology
- $\mathbb{Q}\subset\mathbb{R}$ is not connected wrt standard topology $(a,b)\subset\mathbb{R}$ is connected, as are $(a,b]$, $[a,b)$, $[a,b]$
- (Topologist's sine curve) Let S be defined as $S = \{(x,y)\in\mathbb{R}^2 : y = \sin(\frac{2\pi}{x}), 0<x\le 1\}$. Let $f:(0,1]\to S$ be defined as $t\mapsto(t,\sin\frac{2\pi}{t})$. This gives that $S = f((0,1])$ is path connected and thus connected. In particular, $\bar{S} = S\cup(\{0\}\times[-1,1])$ is connected. Idea: if a separation exists, then A contains all $(t,\sin 2\pi/t)$ so B contains only $(0,0)$. But any open set around $(0,0)$ intersects A. However, TSC is not path connected (it can never leave the y-axis)
- $(0,1)\subset\mathbb{R}$ is connected and locally connected
- $(0,1)\cup(1,2)$ is not connected but locally connected
- TSC is connected but not locally connected
- $\mathbb{Q}\subset\mathbb{R}$ is neither connected nor locally connected
- $\mathbb{R}^n$ is second countable since $\{B_r(x) : r\in\mathbb{Q}, x\in\mathbb{Q}^n\}$ is a countable basis
- $\mathbb{R}^w$ with product topology is second countable since $\{\prod_{n\in\Lambda}(a_n,b_n)\times\prod_{n\in\mathbb{Z}\setminus\Lambda}\mathbb{R} : \Lambda \text{ is finite}, a_n < b_n, a_n, b_n\in\mathbb{Q}\ \forall n\in\Lambda\}$

Exercises

- $\mathcal{T}$ is equal to the collection of all unions of elements in $\mathcal{B}$
- Open balls of radius r are a basis on $\mathbb{R}^n$, this is the **standard topology**. Proof idea: use definition of basis, covering is obvious, for intersection use two balls and triangle inequality
- Discrete metric generates the discrete topology. Proof idea: If $r<1$, $B_r(x) = \{x\}$, unions of the singletons produce all subsets of X.
- Every ℓ_p metric on $\mathbb{R}^n$ generates the standard topology. Forward: consider $d_p(x,y)<r$, square both sides, open brackets using inequality. Reverse: Let $\epsilon>0$, $\delta = \frac{\epsilon}{n^{1/p}}$, bound by $n\cdot\max\{|x_i-y_i|^2\}^{p/2}$, play games
- $\mathcal{B}$ is a basis for $\mathcal{T}_Y \implies \{B\cap X : B\in\mathcal{B}\}$ is a basis for $\mathcal{T}_X$
- If $X\subset Y$ is open and $U\subset X$ is open, then $U\subset Y$ is open.
- (**HW**) Topology induced by subspace metric is the subspace topology
- (**HW**) $\triangle = \{(x,x) : x\in\mathbb{R}\}\subset\mathbb{R}^2$ is closed wrt standard topology on $\mathbb{R}^2$. Idea: let r_p be the distance from $p\in\mathbb{R}^2\setminus\triangle$ to $\triangle$. Show that $\mathbb{R}^2\setminus = \bigcup_{p\in\mathbb{R}^2\setminus\triangle} B_{r_p}(p)$
- (**HW**) Let X be a topological space
  - If $\{G_\alpha\}_{\alpha\in I}$ is an arbitrary collection of closed sets in X, then $\bigcap_{\alpha\in I} G_\alpha\subset X$ is closed.
  - If $G_1,\ldots,G_n$ are closed sets in X, then $\bigcup_{i=1}^n G_i\subset X$ is closed.
  - If $Y\subset X$, then $A\subset Y$ is closed is equivalent to $A = G\cap Y$ for some closed $G\subset X$. Idea: consider subspace topology, then use $A = Y\setminus(Y\setminus A) = Y\setminus(H\cap Y) = Y\setminus H = (X\setminus H)\cap Y$
  - If $Y\subset X$ is closed and $A\subset Y$ and $A\subset Y$ is closed, then $A\subset X$ is closed.
- (**HW**) Find interior/closure/boundary of $\{(x,y)\in\mathbb{R}^2 : 0<x\le 1, 0<y\le 1\}$. Idea: Let $\epsilon = \min\{x,y,1-x,1-y\}$ for interior (and similar for closure). Play games
- If X equipped with discrete topology, all subsets of X have no limit pts. Idea: singletons $\{x\}$ are open but do not intersect any $A\setminus\{x\}$
- If (X,d) is a metric space, $\{x_i\}$ converging to x $\iff \forall\epsilon>0, \exists N>0$ s.t. $d(x_i,x)<\epsilon\forall i>N$. Idea: apply definition of convergence, note that balls form a basis
- If $\mathcal{S}$ is a subbasis for $\mathcal{T}_Y$, then $f:X\to Y$ continuous $\iff f^{-1}(S)\subset X$ is open for any $S\in\mathcal{S}$. Idea: just apply definition of subbasis
- Let X,Y,Z be topological spaces. Constant map, composition, inclusion map, restriction map are all continuous.
- Let (X,d_X) and (Y,d_Y) be two metric spaces, then $f:X\to Y$ is continuous wrt the topologies induced by these metrics $\iff \forall x\in X, \forall\epsilon>0, \exists\delta>0$ s.t. if $y\in X$ satisfies $d_X(x,y)<\delta$, then $d_Y(f(x),f(y))<\epsilon$. Idea: observe that any open set is just a union of open balls, consider δ ball around $x\in X$ and $x\in f^{-1}(V)$ and ϵ ball around $f(x)$
- (**Midterm**) Let $f_i : X\to Y$ be a sequence of continuous functions from topological space X to metric space (Y,d). $\{f_i\}$ converges uniformly to $f:X\to Y \implies f$ is continuous. Idea: $f^{-1}(U) = \bigcup_{i\ge N} f_i^{-1}(B_{\epsilon/2}(f(x)))$
- (**Tut**) $A\subset X, B\subset Y \implies$ product topology on $A\times B$ induced by subspace topologies on $A\subset X$, $B\subset Y$ same as subspace topology $A\times B\subset X\times Y$ induced by product topology on $X\times Y$
- Let $n\in\mathbb{Z}^+$ s.t. $n = m_1+\cdots+m_k$ where all $m_i\in\mathbb{Z}^+$. Then the product topology of standard topologies on $\mathbb{R}^{m_1}\times\cdots\times\mathbb{R}^{m_k} = \mathbb{R}^n$ is the standard topology on $\mathbb{R}^n$. Idea: forward direction, sum up all ϵ of each B_{m_i} to make a B_ϵ in $\mathbb{R}^n$
- If $\mathcal{B}_1,\ldots,\mathcal{B}_n$ are bases for the topological spaces $(X_1,\mathcal{T}_1),\ldots,(X_n,\mathcal{T}_n)$ respectively, then $\mathcal{B}_1\times\cdots\times\mathcal{B}_n$ is a basis for $X_1\times\cdots\times X_n$ that generates the product topology.
- If $(X_1,d_{X_1}),\ldots,(X_n,d_{X_n})$ are metric spaces that induce topologies $\mathcal{T}_1,\ldots,\mathcal{T}_n$ on $X_1,\ldots,X_n$ respectively, then the metrics $d_1$and d_∞ on $X_1\times\cdots\times X_n$ both induce the product topology.
- In the infinite product case, let $(X_i,d_{X_i})_{i=1}^\infty$ be metric spaces. Given the metric d_∞ above, we define $d_\infty : \prod_{i=1}^\infty X_i\times\prod_{i=1}^\infty X_i\to\mathbb{R}$ by $d_\infty(x,y) = \sup\{d_{X_i}(x_i,y_i) : i\in\mathbb{Z}^+\}$. But this is not well-defined as $d_{X_i}(x_i,y_i)$ might be unbounded as $i\to\infty$.
- (**Midterm**) Every compact subspace of a Hausdorff space is closed. Idea: show that no limit point can be found in the complement of the subspace
- (**HW**) If X is equipped with the cofinite topology, then every subset is compact but only the finite sets are closed. Idea: take one set in the open cover, only finite number of points not covered, so patch with finite number of open sets
- Sequentially compact $\implies$ limit point compact. Idea: $A\subset X$, convergent subsequence in A converges to $a\in X$, we have points in any neighbourhood of a, so this must be a limit point. (**HW**) Converse is not true, e.g. $\{(a,\infty) : a\in\mathbb{R}\}$
- (**HW**) Open cover of $\mathbb{R}$ with no Lebesgue number: $(n-\frac{1}{|n|+1}, n+\frac{1}{|n|+1})$
- If (X,d) is a metric space and $\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$, then (X,ρ) is totally bounded $\leftrightarrow (X,d)$ is totally bounded.
- If (X,d) and (X',d') are bi-Lipschitz (i.e. $\exists A>1$ such that $A^{-1}d(x,y)\le d'(x,y)\le Ad(x,y)$ for $\forall x,y\in X$), then (X,d) is totally bounded $\iff (X',d')$ is totally bounded
- Any subset in $(\mathbb{R}^n,l_p)$ is totally bounded $\leftrightarrow$ it is bounded.
- (**HW**) A Cauchy sequence converges iff it has a convergent subsequence.
- (**HW**) If d and d' are metrics on X that are bi-Lipschitz, then a sequence is Cauchy in (X,d) if and only if it is Cauchy in (X',d').
- For a metric space, compact $\iff$ complete and totally bounded. Idea: consider lim pt $x\in X$ and Cauchy sequence where $x_i\in B_{\frac{1}{i}}(x)$ Corollary: $G\subset\mathbb{R}^n$ compact $\iff G$ closed and bounded (Heine-Borel Thm)
- (**HW**) If $U,V\subset X$ is a separation of X and $Y\subset X$ is a connected subspace, then $Y\subset U$ or $Y\subset V$
- (**HW**) Path connected $\implies$ connected. Idea: continuity of path implies $[a,b]$ disconnected
- (**HW**) All path components are path connected and thus every path component lies in a connected component. Idea: path connected $\implies$ connected
- (**HW**) If X is locally path connected and $\tilde{X} = \{\text{connected components of } X\}$, then the quotient topology on $\tilde{X}$ is discrete. Idea: show U in quotient topology $\iff U$ is a CC of X
- (**HW**) First and second countability are preserved by taking products and subspaces
- (**HW**) $T_4\implies T_3\implies T_2\implies T_1$
- (**HW**) If X is copmact, $T_2\iff T_3\iff T_4$
- Let X be a topological space. X is normal $\iff\forall$ closed $A\subset X$, $\forall$ open $U\supset A$, $\exists$ open $V\supset A$ s.t. $\overline{V}\subset U$. Idea: forward: consider A and $X\setminus U$ which are closed and disjoint. Reverse: $X\setminus B$ open and $A\subset X\setminus B$, same for $X\setminus A$
- Completely regular $\implies$ regular. Completely normal $\implies$ normal

HW

- $S\subset X$ is dense in X means that $\forall x\in X$ and every neighbourhood U of x, $U\cap S\ne\emptyset$. This is equivalent to the interior of $X\setminus S$ being empty and $\overline{S} = X$
- The limit points of $\{\frac{1}{m}+\frac{1}{n} : m,n\in\mathbb{Z}^+\}\subset\mathbb{R}$ are $\{0\}\cup\{\frac{1}{k}\mid k\in\mathbb{N}\}$. The limit point of $\{\frac{\sin n}{n} : n\in\mathbb{Z}^+\}\subset\mathbb{R}$ is 0
- Topological space with compact subset but closure of subset is not compact: $\mathcal{T}$ is standard topology on $\mathbb{R}$,$X = \{U\cup\{0\}\mid U\in\mathcal{T}\}\cup\{\emptyset\}$, $\{0\}$ is compact but $\overline{\{0\}} = \mathbb{R}$ is not
- A sequence $\{x_n\}$ converges to a point x iff every subsequence has in turn a subsequence converging to x. Idea: forward: if no subsequence exists, then there are an infinite number of elements lying outside a neighbourhood of x.
- Product of two Hausdorff spaces is Hausdorff
- Subspace of Hausdorff space is Hausdorff
- $(-\sqrt{2},\sqrt{2})\cap\mathbb{Q}$ is a closed and bounded subset of $\mathbb{Q}$ that is not compact. Idea: to show not compact, consider the cover $A_n = (-\sqrt{2}+\frac{1}{n},\sqrt{2})\cap\mathbb{Q}$