GRAPH COLOURING

chp 6

6.1

Four color Theorem

↳ all maps can be colored with ≤ 4 colors

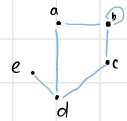
↳ Planar graph

Plannar graph

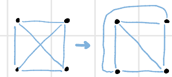
if edge crossing exists → check if it can be drawn another way

↳ dosent cross edges

↳ $n \geq 4$ then planar possible



Plannar

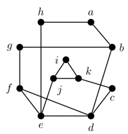


K-coloring

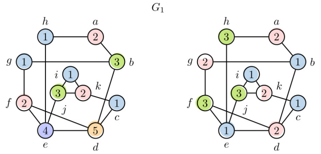
↳ no adjacent vertices have same color

↳ exactly k colors are used

colors are refered by numbers



G_1



proper 5-coloring

proper 3-coloring

color classes

↳ a set S: { all vertices of color i }

↳ graph should be a proper K-coloring

e.g G_1 color classes

$S_1: \{c, g, h, i\}$

$S_2: \{a, f, k\}$

$S_3: \{b, j\}$

$S_4: \{e\}$

$S_5: \{d\}$

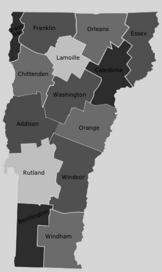
Independence number $\alpha(G)$

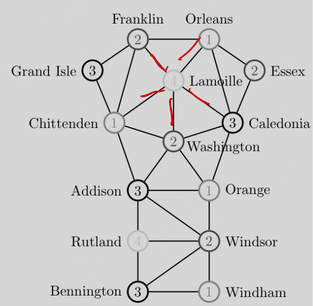
↳ $\alpha(G) = n$

↳ if n vertices have no edges in b/w but every n+1 vertice has atleast 1 edge

Example 6.1 Find a coloring of the map of the counties of Vermont and explain why three colors will not suffice.

Solution: First note that each county is given a vertex and two vertices are adjacent in the graph when their respective counties share a border. One possible coloring is shown below.





Note that Lamoille County is surrounded by five other counties. If we try to alternate colors amongst these five counties, for example Orleans – 1, Franklin – 2, Chittenden – 1, Washington – 2, we still need a third color for the fifth county (Caledonia – 3). Since Lamoille touches each of these counties, we know it needs a fourth color.

6.2

vertex coloring

↳ no loops in graphs

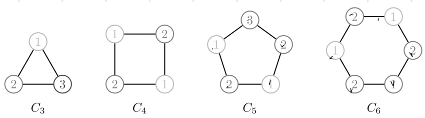
↳ adjacent vertices have different colors

cycles C_n

→ n = no of vertices

↳ n is even ⟶ 2 colors

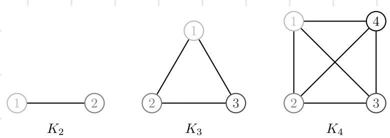
↳ n is odd ⟶ 3 colors



complete graph K_n

→ n = no of vertices

↳ every vertice connected to every vertice



↳ n ⟶ n colors

clique size $w(G)$

↳ a complete subgraph

↳ largest n of complete subgraph

↳ $w(G) = 2$ ⟶ triangle free

$X(G)$ of graphs

↳ $X(C_n) = 2$, n even

↳ $X(C_n) = 3$, n odd

↳ $X(K_n) = n$

↳ $X(W_n) = 3$, n even

↳ $X(W_n) = 4$, n odd

chromatic number $X(G)$

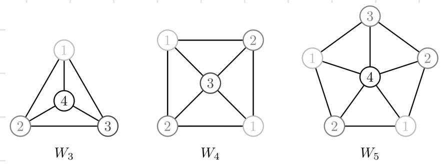
↳ K with smallest proper K - coloring

wheel W_n

→ n = no of vertices around the center vertice → n = n-1

↳ vertices form a cycle

↳ a center vertice connected to all vertices



↳ n is even ⟶ 3 colors

↳ n is odd ⟶ 4 colors

→ n here is diff than the n in cycle

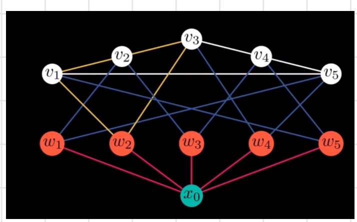
mycielski's construction

↳ finds triangle free graphs G_K → $w(G) = 2$

such that $X(G_K) \geq K$ → can have a infinitely high chromatic no.

Steps

1. make G_K in a line → V
2. make only vertices of G_K → W
   ↳ connect them to x_0
3. check what V_n is connected with $V_1 - V_2$
   then connect W_n with the same vertice $W_1 - V_2$
4. Repeat for all vertices



Brooks Theorem

↳ connected graph

↳ Δ = max degree G

1. $\chi(G) \leq \Delta$
   - OR
   - ↳ not complete graph
   - ↳ not an odd cycle

2. $\chi(G) = \Delta + 1$
   - OR
   - ↳ complete graph
   - ↳ an odd cycle

$$\omega(G) \leq \chi(G) \leq \Delta(G) + 1$$

Equitable colouring

↳ minimal proper coloring

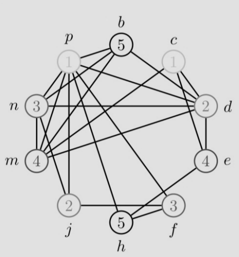
↳ no of vertices of each color defer by 1

Example 6.4 Find an equitable coloring for the graph from Example 6.3.

Solution: We begin with the 5-coloring obtained in Example 6.3. Note that colors 2 and 3 are each used three times, color 4 twice, and colors 1 and 5 each once. This implies we should try to move one vertex each from color 2 and color 3 and assign either color 1 or color 5. One possible solution is shown below.

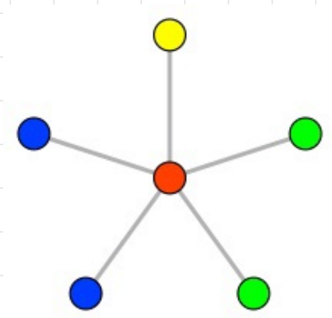
BEFORE

1 = P
2 = D, H, J
3 = C, F, N
4 = E, M
5 = B

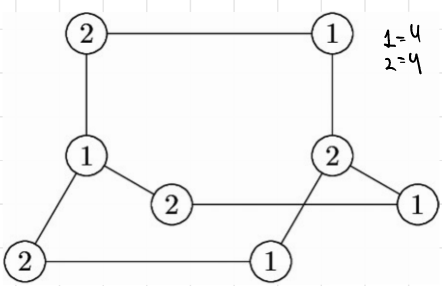


AFTER

1 = c, p
2 = d, j
3 = f, n
4 = e, m
5 = b, h



B=2
G=2
Y=1
R=1



1=4
2=4

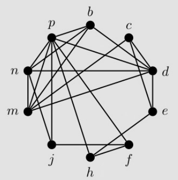
coloring strategies

↳ start with highest degree vertice

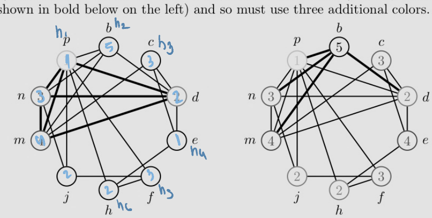
↳ look for force color choosing (cliques, wheels, odd cycles) → look for highest degree

↳ Repeat

↳ after these, avoid using additional colors

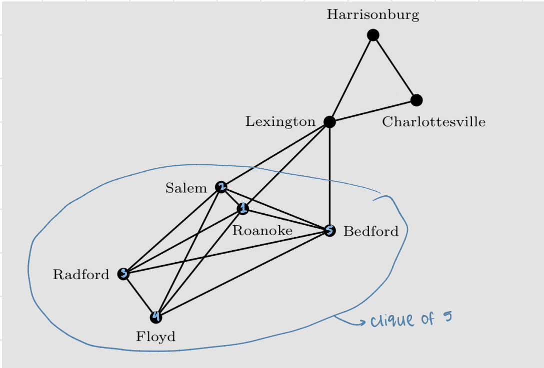


At our initial step, we want to find a vertex of highest degree (p) and give it color 1. Once p has been assigned a color, we look at its neighbors with high degree as well, namely d (degree 6), m and n (both of degree 5). These four vertices are also all adjacent to each other (forming a K_4 shown in bold below on the left) and so must use three additional colors.



Finally, b has the next highest degree (4) and is also adjacent to all the previously colored vertices (forming a K_5) and so a fifth color is needed. The remaining vertices all have degree 3 and can be colored without introducing any new colors. One possible solution is shown above on the right. This solution translates into the following teams:

Team	Members		
1	Pete		
2	Dan	Henry	Judy
3	Carl	Frank	Nell
4	Edith	Marie	
5	Betty		



* $\chi(G) \le \frac{1}{2} + \sqrt{2m + \frac{1}{4}}$

edges

* $\chi(G) \le 1 + \ell(G)$ length of longest path

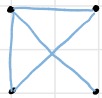
* $\chi(G) \le 1 + \max_H \delta(H)$

min degree of H

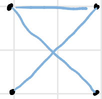
induced subgraph

$V(H) \subseteq V(G)$

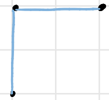
$E(H) \subseteq E(G)$



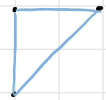
G



H_1



H_2



H_3

Subgraph

↳ subset vertices

↳ subset edges

Spanning subgraph

↳ same vertices

↳ subset edges

↳ H_1

Induced subgraph

↳ subset vertices

↳ same edges

↳ H_3

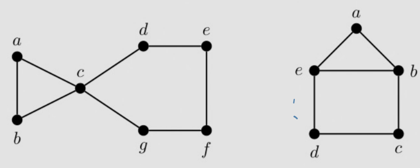
Perfect graph

↳ $\chi(H) = \omega(H)$

for all induced subgraphs H

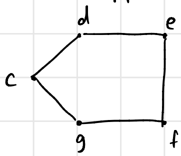
* G is Perfect iff
  - ↳ $\overline{G}$ is perfect
  - OR ↳ no induced subgraph of G
  - OR ↳ $\overline{G}$ is an odd cycle ≥ 5

Example 6.6 Determine if either of the two graphs below are perfect.



G_2 G_3

H

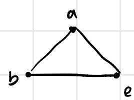


$C_5(H) \longrightarrow \chi(H): 3$

$\omega(H): 2$

Hence NOT Perfect

H



$K_3(H): 3$

$\omega(H) = 3$

HENCE PERFECT

PERFECT GRAPH EXAMPLES

↳ Trees

↳ Bipartite graphs

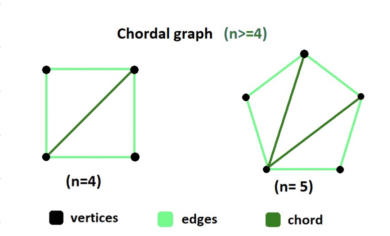
↳ chordal graphs

↳ interval graphs

Chordal graph

↳ cycle ≥ 4

↳ an edge (chord) b/w 2 non adjacent vertices



Interval graph

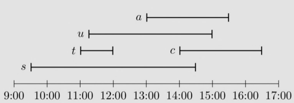
↳ vertex = intervals

↳ edges: overlapping intervals

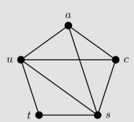
Example 6.7 Five student groups are meeting on Saturday, with varying time requirements. The staff at the Campus Center need to determine how to place the groups into rooms while using the fewest rooms possible. The times required for these groups is shown in the table below. Model this as a graph and determine the minimum number of rooms needed.

Student Group	Meeting Time
Agora	13:00–15:30
Counterpoint	14:00–16:30
Spectrum	9:30–14:30
Tupelos	11:00–12:00
Upstage	11:15–15:00

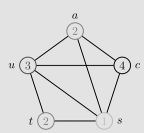
Solution: First we display the information in terms of the intervals. Although this step is not necessary, sometimes the visual aids in determining which vertices are adjacent.



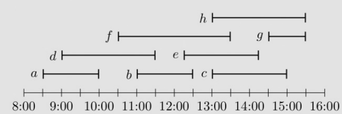
Below is the graph where each vertex represents a student group and two vertices are adjacent if their corresponding intervals overlap.



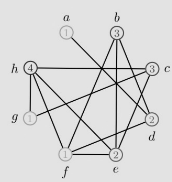
A proper coloring of this graph is shown below. Note that four colors are required since there is a K_4 subgraph with a, c, s, and u.



Example 6.8 Eight meetings must occur during a conference this upcoming weekend, as noted below. Determine the minimum number of rooms that must be reserved.



Solution: Each meeting is represented by a vertex, with an edge between meetings that overlap and colors indicating the room in which a meeting will occur. If we color the vertices according to their start time (so in the order a, d, f, b, e, c, h, g), we get the coloring below.



Note that four meeting rooms are needed since there is a point at which four meetings are all in session, which is demonstrated by the K_4 among the vertices c, e, f, and h.

6.3

Edge coloring

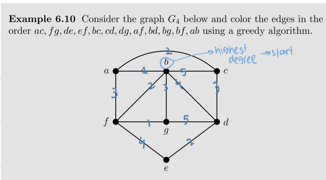
↳ no loop graphs

↳ adjacent edges have different colors

↳ start with highest degree vertices' edges

Vizings Theorem

↳ $\Delta(G) \le \chi'(G) \le \underbrace{\Delta(G)}_{\text{max degree of } G} + 1$



Example 6.10 Consider the graph G_4 below and color the edges in the order $ac, fg, de, ef, bc, cd, dg, af, bd, bg, bf, ab$ using a greedy algorithm.

$5 \le \chi'(G) \le 6$

5

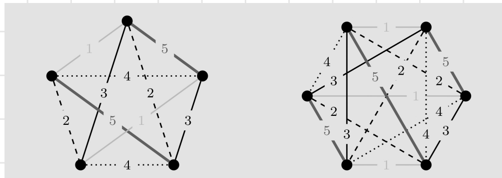
chromatic index $\chi'(G)$

↳ smallest possible edge coloring

$\chi'(K_n) = n-1$, n even

$\chi'(K_n) = n$, n odd

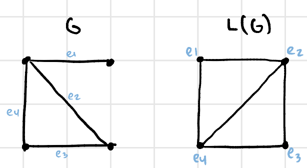
→ for complete graph



Line graph L(G)

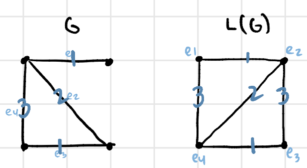
↳ vertice: edges of G

↳ edges: edges in G having same vertice



$\chi'(G) = \chi(L(G))$

3 = 3



1 Factorization

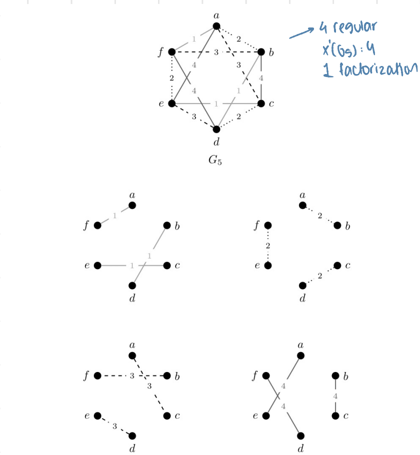
↳ edges partitioned into 1 disjoint factor

1 Factorization Represented as edge coloring

↳ give each edge of 1 factor, the same color

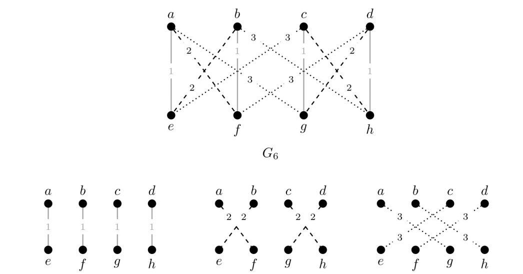
↳ k-regular graph

↳ $\chi'(G) = k$ — 4



→ 4 regular
$\chi'(G_5) = 4$
1 factorization

Proposition 5.22 states that every k-regular bipartite graph has a 1-factorization. The proof is not constructive, but does illustrate that we could find a 1-factorization of a bipartite by simply recursively removing perfect matchings from the graph, as shown in graph G_6.



* grap G has 2n vertices

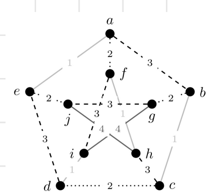
↳ degree n ⟶ n odd

↳ degree n-1 ⟶ n even

Then G has 1 factorization

* complete graphs
on odd number of vertices
dont have 1 factorization

* Petersen graph → 3 cubic regular graph
dosent have 1 factorization



edge-coloring of Petersen graph

Ramsey Numbers R(m,n)

↳ min no of vertices r

↳ so all simple graph on r contain

- ↳ clique of size m
- ↳ independent set of size n → group of n vertices with no edge in between

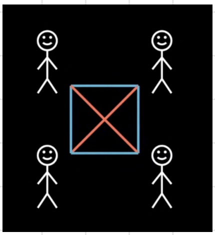
What is the smallest number of people needed in a room to guarantee that either three people know each other or three do not know each other?

↳ $R(m,n) = R(n,m)$

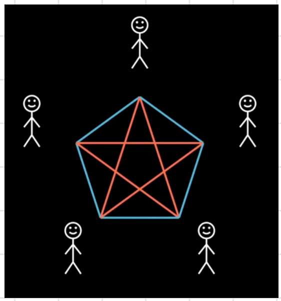
↳ $R(2,n) = n$

triangle with 1 color

no monochronomatic cycles



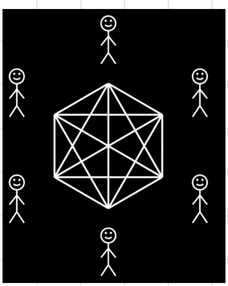
no monochronomatic cycles

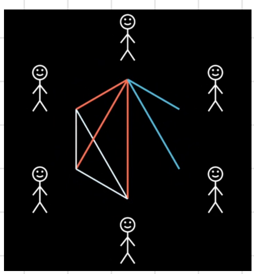


monochronomatic triangle

blue K_3 red K_3

$R(3,3) = 6$ → no of vertices req to form a complete graph





→ there will always exist a monochromatic triangle

6.4 coloring variations

First Fit Coloring Algo

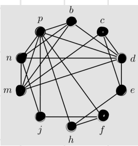
uses first avl color

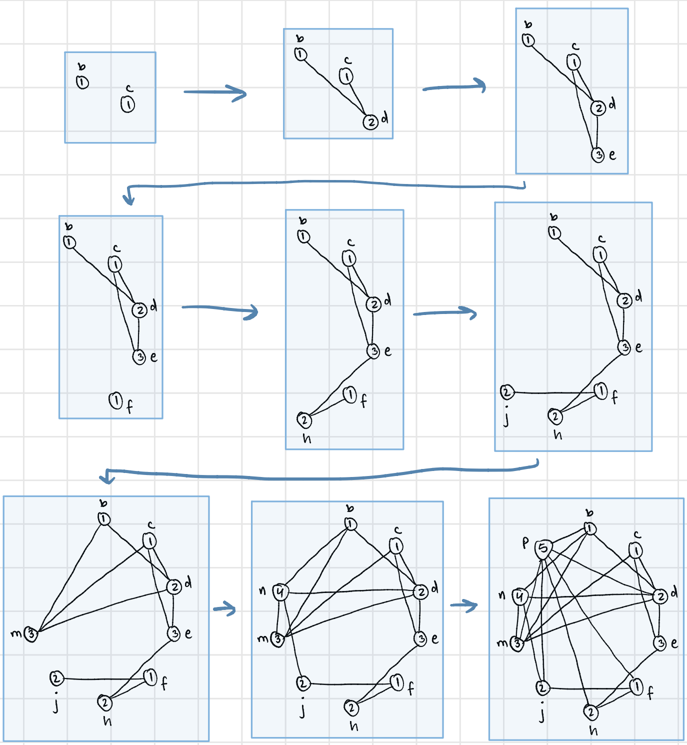
- ↳ x_1 → color 1
- ↳ x_2 → color 1 if x_1, x_2 not adjacent; color 2 otherwise
- ↳ x_3 ....
- ↳ repeat one by one
- ↳ worst case for on-line algos — little knowledge about previous vertices treated
- ↳ good for interval graphs

on-line coloring

Definition 6.24 Consider a graph G with the vertices ordered as $x_1 \prec x_2 \prec \cdots \prec x_n$. An ***on-line algorithm*** colors the vertices one at a time where the color for x_i depends on the induced subgraph $G[x_1, \ldots, x_i]$ which consists of the vertices up to and including x_i. The maximum number of colors a specific algorithm $\mathcal{A}$ uses on any possible ordering of the vertices is denoted $\chi_{\mathcal{A}}(G)$.

Q)

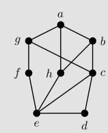




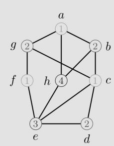
Example 6.16 Color the graph below using First-Fit using the two different vertex orders listed and determine if either finds the optimal coloring for the graph.

(a) $a \prec b \prec c \prec d \prec e \prec f \prec g \prec h$

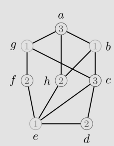
(b) $e \prec h \prec b \prec d \prec c \prec g \prec f \prec a$



Solution: Below are the two First-Fit colorings of the graph based on the order given. Only the vertex order from (b) gives the optimal coloring since $\omega(G) = 3$.



vertex order (a)



vertex order (b)

↓ gives optimal solution

weighted coloring

can be solved using First Fit

↳ $G = (V, E, w)$

↳ weight: no of colors assigned to vertex

↳ if 2 vertices adjacent, their set of colors is disjoint

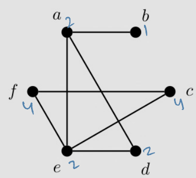
steps

↳ solve cliques first, biggest first

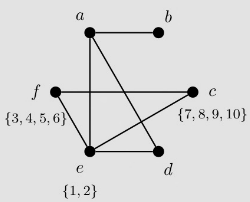
↳ fill the rest

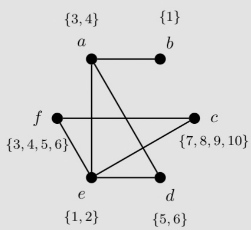
Example 6.18 Find an optimal weighted coloring for the graph below where the vertices have weights as shown below.

$w(a) = 2$
$w(b) = 1$
$w(c) = 4$
$w(d) = 2$
$w(e) = 2$
$w(f) = 4$



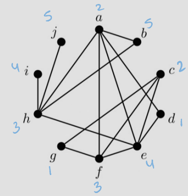
Solution: Note that $a, d,$ and e form a K_3 with total weight 8, and $c, e,$ and f form a K_3 with total weight 10. We begin by assigning weights to the vertices from the second K_3, and since e appears twice we assign it the first set of colors $\{1, 2\}$. From there we are forced to use another 4 colors on f ($\{3, 4, 5, 6\}$) and another 4 colors on c ($\{7, 8, 9, 10\}$), as shown on the left below. We can color the remaining three vertices using colors 1 through 10 as shown on the right.



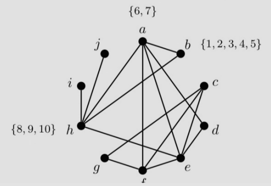


Example 6.19 Suppose the ten families needing train tickets have the same underlying graph as that from Example 6.15 and the size of each family is noted below. Determine the minimum number of seats needed to accommodate everyone's travels.

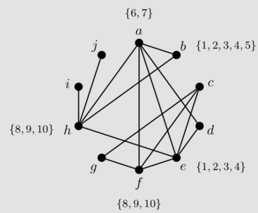
$w(a) = 2$
$w(b) = 5$
$w(c) = 2$
$w(d) = 1$
$w(e) = 4$
$w(f) = 3$
$w(g) = 1$
$w(h) = 3$
$w(i) = 4$
$w(j) = 5$



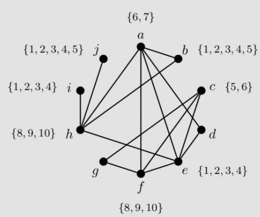
Solution: As with the example above, we begin by looking for the largest total weight for a clique. Since the clique size is three, we want to find K_3 subgraphs with high total weight. The largest of these is with $a, b,$ and h with total 10. Since b has the largest weight among these, we give it colors 1 through 5, assign a colors 6 and 7, and colors 8, 9, and 10 to h.



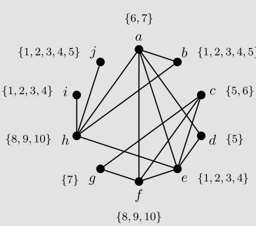
The next two cliques with a high total weight (of 9) are formed by a, e, f and c, e, f. Since a has already been assigned colors, we begin with the first clique. We can fill in the colors for e and f without introducing new colors, as shown below. Note that since e is adjacent to both a and h it could only use four consecutive colors chosen from those used on b.



At this point we can fill in the colors for c by choosing two consecutive colors from those available (5, 6, and 7). We also fill in the colors for i and j by choosing the correct number of consecutive colors from any of 1 to 7.

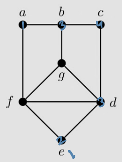


Finally, we need one color each for d and g. These can be any color not already used by one of their neighbors.



List coloring

Example 6.20 The table on the right below shows two different lists for the vertices in the graph on the left. Find a proper coloring for each version of the lists or explain why none exists.



Vertex	List 1 (no proper coloring)	List 2 (possible)
a	$\{1, 2\}$	$\{1\}$
b	$\{1, 3, 5\}$	$\{1, 4, 5\}$
c	$\{3, 4\}$	$\{3, 4\}$
d	$\{3, 4\}$	$\{3, 4\}$
e	$\{2, 3\}$	$\{2, 3\}$
f	$\{1, 2, 3\}$	$\{1, 2, 3\}$
g	$\{4, 5\}$	$\{4, 5\}$

no value of f

Solution: A possible coloring from the first set of lists can be given as

$$a-1, b-4, c-3, d-4, e-3, f-2, g-5.$$

However, there is no proper coloring possible from the second set of lists. First note a must be given color 1 and so f can only be colored 2 or 3. But since those are the same colors available for e, we know one will be given color 2 and the other color 3. In either case, since d is adjacent to both f and e we know it cannot be given color 3, and so must be given color 4, forcing g to be color 5 and c to be color 3. But now b is adjacent to vertices of color 1, 3, and 5, and so cannot be given a color from its list.

?

K-choosable

- all lists of size K
- list coloring exists

choosability ch(G)

- min value of K, of K-choosable

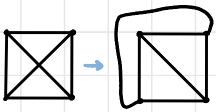
* for any simple graph

- $\chi(G) \leq ch(G) \leq \Delta(G) + 1$

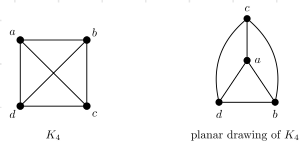
PLANARITY

Planar

↳ edeges dont cross



even if edge crossing exists it can still be considered planar graph as it can be drawn another way with no edge crossings



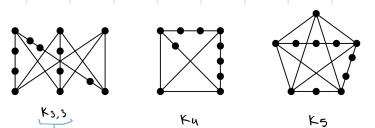
7.1 Kuratowski's Theorem

↳ G is planar iff

↳ no subdivision of $K_{3,3}$ or K_5

Subdivision

↳ adding vertices b/w u and y

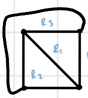


bipartite graph of $K_{3,3}$

$\Delta(G) = 4$ of all

Region

↳ portion bounded by edges



R_4 → outer region

PROOF

↳ By induction on e

if $e = 1$, then since graph is connected

↳ tree, $v = 2$, $r = 1$

or

↳ loop, $v = 1$, $r = 2$

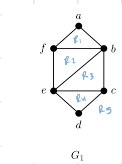
either case euler formula holds

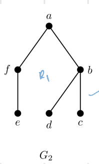
Eulers Formula

↳ connected planar graph

↳ $r = e - v + 2$ or $n - m + r = 2$

(r: region, e: edges, v: vertices; n: vertices, m: edges, r: regions)





→ a tree has only 1 region as no cycles

$6 - 9 + 5 = 2$

$2 = 2$

$6 - 5 + 1 = 2$

$2 = 2$

↳ Suppose Eulers Formula holds for all graph with $m > 1$

Consider graph with $e = m + 1$

if $G' = G - e$ is not connected for any edge in e

↳ e must be a bridge of G

hence G must be a tree

so $n = m + 1$, $r = 1$

$n - (m - 1) + (r - 1) = 2$

$n - m + r = 2$

if $G' = G - e$ is connected

↳ e must be part of a cycle in G

hence G' must have

$r = r - 1$, $e = m - 1$ by removal of e

$n - (m - 1) + (r - 1) = 2$

$n - m + r = 2$

So Euler formula holds for all planar graphs

Maximally Planar

↳ adding an edge = non planar for 2 non adjacent vertices

1. Euler formula: $V - e + r = 2$
2. K_5: 5v of atleast degree 4
3. $K_{3,3}$ 3v of degree 3

if

↳ simple plannar graph

↳ $v \geq 3$ vertices

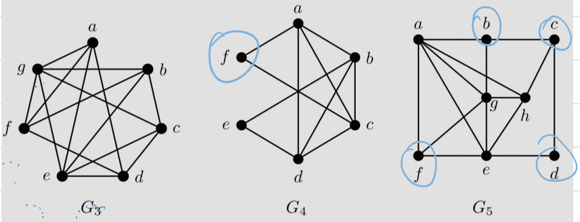
THEN

↳ $e \leq 3v - 6$

↳ maximally planar, $e = 3v - 6$

↳ no cycles of length 3, $e \leq 2v - 4$

Example 7.3 Determine which of the following graphs are planar. If planar, give a drawing with no edge crossings. If nonplanar, find a $K_{3,3}$ or K_5 subdivision.



G_3 G_4 G_5

1. ↳ Euler formula: $v - e + r = 2$

$7 - 15 + 27 = 2$

$19 = 2$

×

$6 - 10 + 11 = 2$

$7 = 2$

×

$8 - 15 + 11 = 2$

$4 = 2$

×

2. ↳ look for K_5 subdivision

every v has atleast degree 4

so K_5 possible

4v of degree 4

needs to be 5

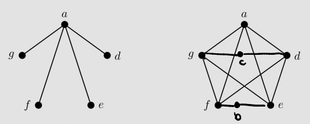
so K_5 not possible

4v of atleast degree 4

needs 5

so K_5 not possible

To do this, we start by picking a vertex and looking at its neighbors, hoping to find as many as possible that form a complete subgraph. Beginning with a as a main vertex in K_5, we see the other main vertices would have to be either its neighbors or vertices reachable by a short path. We will start by choosing the other main vertices of the K_5 to be the neighbors of a, namely d, e, f, and g. A starting graph is shown below on the left. Next we fill in the edges between these four vertices, as shown on the right below.



At this point, we are only missing two edges in forming K_5, namely dg and ef. We have two vertices available to use for paths between these nonadjacent vertices, and using them we find a K_5 subdivision. Thus G_3 is nonplanar.

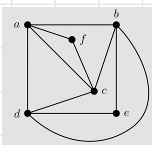
3. ↳ look for $K_{3,3}$ subdivision

not enough atleast degree 3 vertices

f = degree 2

so $K_{3,3}$ not possible

PLANAR

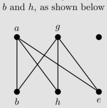


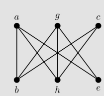
3v of degree 3 → f, b, c

$K_{3,3}$ possible

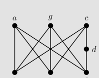
Again, we start by selecting vertex a to be one of the main vertices of a possible $K_{3,3}$ subdivision. At the same time, we will look for another vertex that is adjacent to three of the neighbors of a. We see that a and g are both adjacent to b, h, e, and f. Let us begin with b, h, and e for the vertices on the other side of the $K_{3,3}$, as shown below on the left.

We now search among the remaining vertices (f, d, c) for one that is adjacent to as many of b, h, and e as possible and find c is adjacent to both b and h, as shown below on the right.





At this point we are only missing one edge to form a $K_{3,3}$ subdivision, namely ce. Luckily we can form a path from c to e using the available vertex d. Thus we have found a $K_{3,3}$ subdivision and proven that G_5 is nonplanar.



cycle chord method

↳ to find planar graph

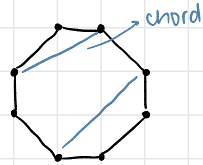
STEPS

↳ find a cycle

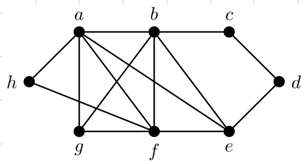
↳ make max chords in the cycle → make sure they don't cross

↳ draw the left over edges

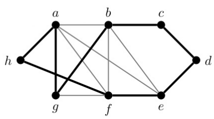
chord



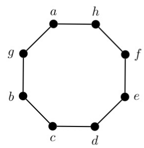
Q)



G_6

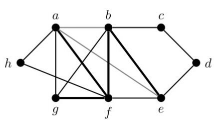


highlighted portion of G_6

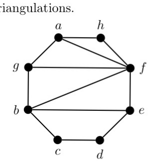


planar drawing

Once we have created the spanning cycle, we attempt to place as many edges in the interior of the cycle as possible so that they do not cross. This should resemble chords of a circle. In the example G_6 below, we are making most of the interior regions of the cycle into triangulations.

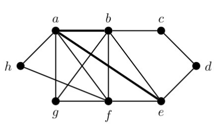


highlighted portion of G_6

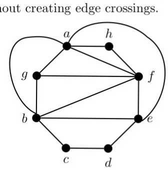


planar drawing

When placing these chords, we should take care to notice any vertices that have incident edges remaining, as those will need to be placed as curves along the outside of the cycle. For example, in G_6 only two edges remain to be placed outside the spanning cycle, namely ab and ae. Having multiple edges from the same vertex left to place is often a benefit since they can be drawn in the same or opposite directions of the cycle without creating edge crossings.



highlighted portion of G_6



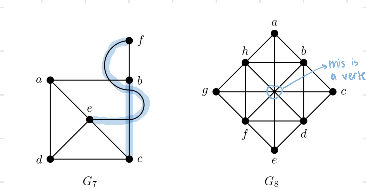
planar drawing

7.3 EDGE CROSSING

crossing number cr(G)

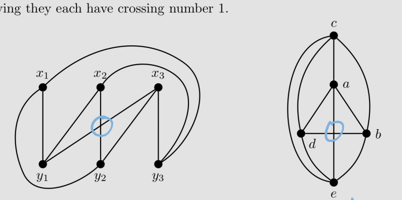
↳ min no of edge crossing satisfying the condition

1. cross the same edge **ONCE**
2. max 2 edges cross a given point



Example 7.4 Determine the crossing numbers for K_5 and $K_{3,3}$.

Solution: Since we know K_5 and $K_{3,3}$ are not planar, $cr(K_5), cr(K_{3,3}) \geq 1$. A drawing of each of these adhering to the criteria above is shown below, proving they each have crossing number 1.



cr = 1 cr = 1

G7 violates condition 1 as ef crosses bc more than once

cr = 0

G8 violates condition 2 as ae, bf, cg, ah all cross a point

cr = 0

if

* ↳ simple plannar graph

THEN

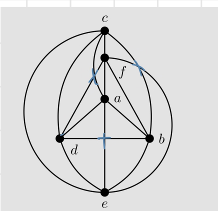
↳ $cr(G) \geq e - 3v + 6$

↳ bipartite, $cr(G) \geq e - 2v + 4$

Q) Find cr(G) of K_6

$cr \geq 15 - 3(6) + 6$

$cr \geq 3$

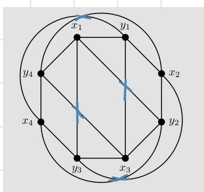


$cr(K_6) = 3$

Q) Find cr(G) of $K_{4,4}$

$cr \geq 16 - 2(8) + 4$

$cr \geq 4$



$cr(K_{4,4}) = 4$

$$cr(K_n) \leq \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor$$

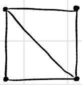
$$cr(K_{m,n}) \leq \left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor$$

Thickness $\theta(G)$

↳ min no of planar sub graphs whose edges union gives G

↳ $\theta(\text{planar graph}) = 1$

$\theta)$

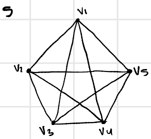


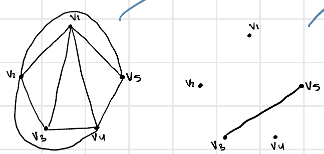
planar

$cr(G) = 0$

$\theta(G) = 1$

$\theta)\ K_5$



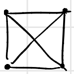


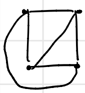
2 planar graph whose union: G

$cr(K_5) = 1$

$\theta(K_5) = 2$

$\theta)$





planar

$cr(G) = 0$

$\theta(G) = 1$

if G is simple graph

↳ $\theta(G) \geq \dfrac{e}{3v-6}$ ↳ $cr(G) \geq e - 3v + 6$

↳ bipartite, $\theta(G) \geq \dfrac{e}{2v-4}$ ↳ bipartite, $cr(G) \geq e - 2v + 4$

$$\theta(K_n) = \begin{cases} \left\lfloor \dfrac{n+7}{6} \right\rfloor & n \neq 9, 10 \\ 3 & n = 9, 10 \end{cases}$$

ishma hafeez notes

repshareit