

# **Geometric Constraints in Algorithmic Design**

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# **Acknowledgments**

To anyone and everyone who supported and bore with me: thank you.



# **Abstract**

Modern Computer-Aided Design (CAD) applications need to employ, to a lesser or greater extent, geometric constraints that condition the geometric models being produced. As an example, consider that of a line that goes through a point and is parallel to another line, or constructing a circle from three points. The specification (and solving) of geometric constraints allows the creation of geometric shapes that would otherwise require substantially more complex descriptions. The inclusion, in a programming language, of primitive operations for creating geometric constraints augments the expressiveness of the language and frees the programmer from the specification of unnecessary details. Said primitive operations could include the definition of incidence relations, parallelism or perpendicularity, distances, and angles; to which the various parts in a geometric model must obey. The focus of this work is the creation and implementation, of geometric constraint primitives that facilitate the specification of geometric forms.

# **Keywords**

Algorithmic Design; Geometric Constraints; Geometric Constraint Solving; Parametric CAD; Programming Language.



# **Resumo**

Aplicações modernas de projecto assistido por computador precisam de empregar, em menor ou maior grau, restrições geométricas que condicionam os modelos produzidos. Como exemplos, considerem-se o de uma linha que atravessa um ponto e é paralela a outra linha, ou o da construção de um círculo recorrendo a três pontos. A especificação (e resolução) destas restrições permite a criação de formas geométricas que, de outro modo, exigiriam descrições substancialmente mais complexas. A inclusão, numa linguagem de programação, de operações primitivas capazes de criar restrições geométricas aumenta a expressividade da linguagem e liberta o programador de especificar detalhes desnecessários. Essas operações podem incluir a definição de relações de incidência, paralelismo ou perpendicularidade, distâncias, e ângulos; às quais o modelo geométrico deve obedecer. O foco deste trabalho consiste na criação e implementação de primitivas de restrições geométricas que facilitam a especificação de formas geométricas.

# **Palavras Chave**

Design Algorítmico; Restrições Geométricas; Resolução de Restrições Geométricas; CAD Paramétrico; Linguagem de Programação.



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# Acronyms and Abbreviations

|              |   |
|--------------|---|
| <b>AD</b>    | Algorithmic Design                                  |
| <b>AMD</b>   | Advanced Micro Devices                              |
| <b>API</b>   | Application Programming Interface                   |
| <b>B-Rep</b> | Boundary Representation                             |
| <b>BIM</b>   | Building Information Modeling                       |
| <b>CAD</b>   | Computer-Aided Design                               |
| <b>CAS</b>   | Computer Algebra System                             |
| <b>CGAL</b>  | the Computational Geometry Algorithms Library       |
| <b>CPU</b>   | Central Processing Unit                             |
| <b>CS</b>    | Computer Science                                    |
| <b>CSG</b>   | Constructive Solid Geometry                         |
| <b>CSP</b>   | Constraint Satisfaction Problem                     |
| <b>DDR</b>   | Double Data Rate                                    |
| <b>EPS</b>   | Encapsulated PostScript                             |
| <b>FFI</b>   | Foreign Function Interface                          |
| <b>GC</b>    | Geometric Constraint                                |
| <b>GCS</b>   | Geometric Constraint Solving                        |
| <b>GFL</b>   | Geometry Functions Library                          |
| <b>GUI</b>   | Graphical User Interface                            |
| <b>IDE</b>   | Integrated Development Environment                  |
| <b>LEDA</b>  | the Library for Efficient Data types and Algorithms |
| <b>PGF</b>   | Portable Graphics Format                            |

|                |  |
|----------------|--|
| <b>RAM</b>     | Random Access Memory                     |
| <b>RGC</b>     | Rhythmic Gymnastics Center               |
| <b>SDK</b>     | Software Development Kit                 |
| <b>SO-DIMM</b> | Small Outline Dual In-line Memory Module |
| <b>TikZ</b>    | TikZ ist <i>kein</i> Zeichenprogramm     |
| <b>TPL</b>     | Textual Programming Language             |
| <b>VBA</b>     | Visual Basic for Applications            |
| <b>VPL</b>     | Visual Programming Language              |

# 1

## Introduction

### Contents

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Modern Computer-Aided Design (CAD) tools include substantial support for parametric operations and Geometric Constraint Solving (GCS). These mechanisms have been developed over the past few decades [1] and are heavily and ubiquitously used across the Architecture, Engineering, and Construction industry.

Parametric modeling is used to design with constraints, whereby users express a set of parameters and interdependent operations, establishing restrictions between geometric entities. The resulting geometry can be controlled from input parameters using two computational mechanisms: (1) parametric operations, which build geometry that implicitly abides by constraints imposed when the user selects the operation and its inputs, and (2) GCS, which finds the positions of geometric entities that satisfy a set of constraints explicitly imposed by the user.

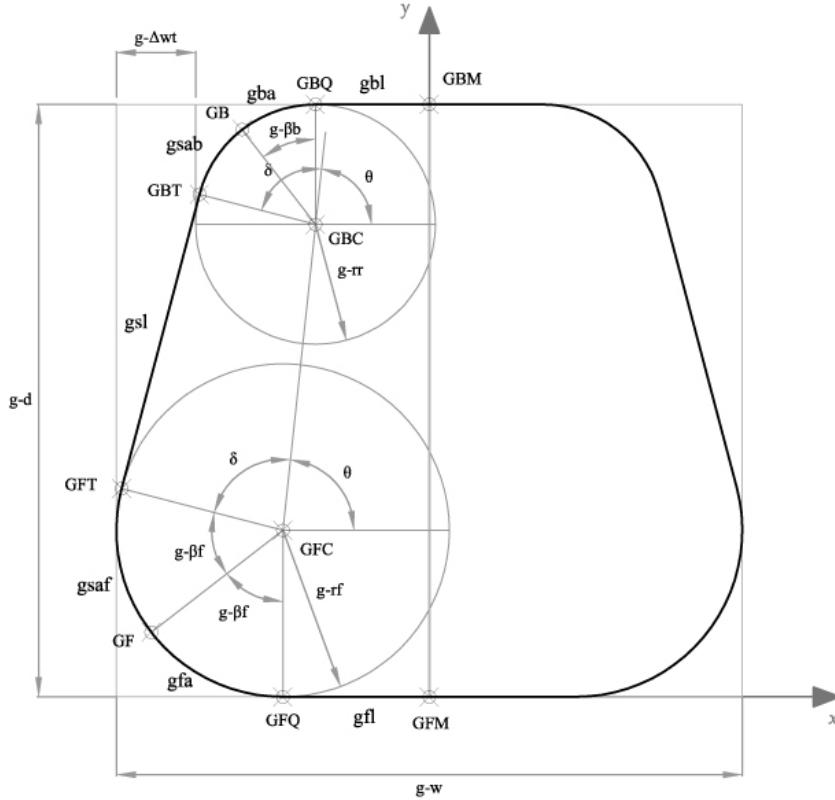
Nowadays, parametric operations in CAD software are mostly accessible through intuitive, robust, and easy-to-use direct-manipulation interfaces, offering a wide variety of different operations. These operations are created when a designer uses solid modeling operations, such as face extrusions or shape unions; and recorded into a user-controlled sequential history of construction steps that can be replayed in the event of changes, updating the modeled geometry. Alas, dependency propagation direction is fixed, forcing users to plan their model's features beforehand. In constraint solving, by contrast, dependency propagation direction isn't fixed. Instead, users introduce a set of parameters and geometric entities followed by specifying the constraints that relate these objects. Naturally, GCS fits in CAD software, having been the target of considerable research and development to implement efficient approaches and methodologies capable of solving Geometric Constraint (GC) problems. So much so that it has become standard in major CAD software, such as AutoCAD<sup>1</sup> from Autodesk®, which supports the ability to constrain objects in a variety of ways, e.g., point coincidence, line perpendicularity, and tangencies, among other kinds of constraints.

However, traditional interactive methods for parametric modeling suffer from the disadvantage that they do not scale properly when designing more complex ideas. In recent years, a novel approach to design named Algorithmic Design (AD) has emerged, allowing the specification of sketches and models through algorithms [2], leading to the creation and integration of AD tools into CAD software as well. Some use Visual Programming Languages (VPLs), others Textual Programming Languages (TPLs), or even a mixture of both. The latter overcomes a fundamental issue with VPLs which is the frequently disproportionate complexity between the program and the respective resulting model.

Dealing with GCs, regardless of the approach, can still prove to be an arduous task. Take as an example the sketch of a chair seat's outer frame, as seen in fig. 1.1, from a multi-purpose chair generation tool [3] where the chair's overall shape is controllable by specifying the values for a set of input parameters.

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<sup>1</sup><https://autodesk.com/autocad>



Source: Project source code, publicly unavailable (Jan 2019)

**Figure 1.1:** Sketch of a chair seat's outer frame, defined by 5 input parameters: (1) Width  $g-w$ , (2) depth  $g-d$ , (3) taper width  $g-\Delta wt$ , (4) front radius  $g-rf$ , and (5) rear radius  $g-rr$ .

The seat's corners are defined by circles whose respective front and rear radius' length,  $g-rf$ ,  $g-rr$ , is obtained by computing distances, from which the circles' centers, GFC and GBC, can be obtained. The circles are then connected through outer tangent lines,  $gsl$ , forming the outer frame of the chair's seat. Some of these operations, such as the radius computation, *tangency*, and *circumcenter*, depend on operations that query if a point is at a certain distance from an object, or if two points are coincident. Such operations must be handled carefully due to numerical robustness issues that may arise when performing fixed-precision arithmetic. As such, on top of the design process itself, the user must identify the GCs, resorting to trigonometry analysis, perform tolerance-based comparisons to determine point distance or if two points are coincident, among other techniques the user most likely is not aware he must rely upon to circumvent these issues, particularly, when we take into consideration that most AD practitioners are architects and designers without an extensive background in Computer Science (CS).

To overcome the limitations exposed above, this report proposes the implementation of GC primitives with specialized efficient solutions for different combinations of input objects. We additionally focus our work around TPLs, further making them more attractive, and easier to both adopt and use.

## 1.1 Document Structure

The present document is structured in 5 different chapters, namely:

**Introduction** Broken into several sections, including this one, presents: (1) a brief historical overview of the development of parametric operations in CAD software in section 1.2, (2) the main approaches to GCS in CAD, in section 1.3, (3) two simple algebraically formulated examples of GC problems and respective solutions along with code examples, in section 1.4, and (4) a section dedicated to further elaborating on AD and the benefits and drawbacks it introduces to the design process, in section 1.5.

**Related Work** An exposition of the related work in the form of (1) a comprehensive discussion about numerical robustness in computational processes, showcasing a set of software tools capable of handling these issues in the context of geometric computation, in section 2.1, (2) an overview of GC tools, presenting some of their benefits and drawbacks, in section 2.2, and (3) an overview of algorithmic design tools, similarly comparing them and addressing positive and negative points, in section 2.3.

**Solution** A solution proposal, followed by how it was implemented and how it is capable of efficiently handling the specification of GC problems, in chapter 3, going over its core components: (1) a *Exact Computation Geometry Library*, discussed in section 3.1.1, which provides the basic geometric entities and functions to work it, (2) an interoperability *Wrapper Code* layer that allows the target platform to repurpose the previous component's foreign constructs, detailed in section 3.1.2, and (3) *Geometric Constraint Primitives*, showcased in section 3.1.3 capable of solving a class of GC problems employing a Euclidean constructive approach to generating geometry.

**Evaluation** The methodology used to evaluate the proposed solution in chapter 4, involving (1) a performance comparison in section 4.1 with a similar project with a shared goal, (2) a performance, correctness, and effort estimation analysis, in section 4.2, comparing an implementation of a complex geometric algorithm in the target platform vs. a repurposed mature implementation, and (3) an extensive evaluation comparing two different approaches at solving GC problems present in four case studies inspired by existing designs.

**Conclusion** Concluding remarks that summarize our work, in chapter 5.

## 1.2 Parametric Operations in CAD

Ivan Sutherland introduced the world to Sketchpad [4] in 1963, an interactive 2D CAD program. Despite never using the word *parametric* in writing, Sutherland's Sketchpad was capable of establishing atomic constraints between objects which had all the essential properties of parametric equations, being the first of its kind and the prime ancestor of modern CAD programs. The earliest 3D system [5] dates from the

1970s. It used a Constructive Solid Geometry (CSG) [6, 7] binary tree, and Boundary Representation (B-Rep) [8] for representing solid objects. This system's parametric nature rested in the CSG tree, which acted as a rudimentary construction step history. The user could make modifications to the controlling parameters' values of a certain operation in the tree, reapply the modified history, and generate the newly updated model. Surfacing nearly a decade later, Pro/ENGINEER<sup>2</sup> [9] was the first system to be acknowledged as a parametric system. It enabled the establishment of relations between the objects' sizes and positions such that a change in a dimension between objects would automatically change affected objects accordingly. Unlike Sketchpad, it supported 3D geometry and changes would propagate over different drawings made by different users. This amid a sudden flux of activity and interest, GCS soon became standard in drawings by the early 1990s [10–12]. Efforts to expand the benefits of constraint solving beyond simple sketches were made, some systems having implemented constraint solving in 3D. Improvements from then on focused mostly on robustness and operation variety.

In recent decades, emphasis shifted to making parametric CAD software more interactive and user-friendly. The intent was to make it as simple as dragging a face of an object to where it should be instead of scrolling through a construction history in attempts to locate a specific operation, and hopefully changing the correct controlling parameter's value within that operation. This in itself is a tedious and error-prone process that can lead to undesired side effects instead of producing the intended changes. A variety of systems have been developed to mitigate this rigidity [13–15], but not without drawbacks, since direct-manipulation operations were just added to the construction history as transformation operations, oblivious to parent operations the new ones might depend on. Further limitations are discussed in [16], along with a proposal for future design software exempt of parametric operations. Nonetheless, parametric operations will still see continued usage for the foreseeable future.

### 1.3 Constraints in CAD

We have briefly seen how parametric operations in CAD software have evolved. These operations allow the user to create geometric objects that satisfy certain constraints *implicitly* imposed on the objects when the user selects the operations they want to use along with the respective operation's inputs. Naturally, GCS fits well in CAD applications. GCs allow the repositioning and scaling of geometric objects so that they satisfy constraints *explicitly* imposed on them by the user.

Constraint Satisfaction Problems (CSPs) are a well-known subject of research both in mathematics and in the CS field. GCS is a subclass of CSPs. More specifically, it is a CSP in a computational geometry setting. The abstract problem of GCS is often described as follows [1, pp. 6]:

*Given a set of geometric objects, such as points, lines, and circles; a set of geometric and*

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<sup>2</sup><https://www.ptc.com/en/products/creo/pro-engineer>

*dimensional constraints, such as distance, tangency, and perpendicularity; and an ambient space, usually the Euclidean plane; assign coordinates to the geometric objects such that the constraints are satisfied, or report that no such assignment has been found.*

One of the important features of a solver is its *competence*, which is related to the capability of reporting unsolvability: if in fact no solution for the problem at hand exists and the solver is capable of reporting unsolvability in that case, the solver is deemed fully competent. Since constraint solving is mostly an exponentially complex problem [17], partial competence suffices as long as decent solutions can be found in affordable time and space.

There are multiple approaches to constraint solving, but the most relevant ones are graph-based, logic-based, algebraic, and theorem prover-based, of which the first is the predominant one. It is important for these approaches that the GC system does not have too few or too many constraints. Summarily, a system can either be (1) under-constrained if the number of solutions is unbound due to lack of constraint coverage over the entities involved, (2) over-constrained if there are no solutions because of constraint contradictions, or (3) well-constrained if the number of solutions is bound to a finite positive number.

Some of the subjects approached here are briefed in [18]. The following sections present and briefly discuss the aforementioned approaches to constraint solving.

### 1.3.1 Graph-Based Approaches

In graph-based approaches, the problem is translated into a labeled *constraint graph*, where vertices are constrained geometric objects, and edges the constraints themselves. This approach is split into three main branches:

**Constructive Approaches** The graph is decomposed and recombined to extract basic construction steps that must be solved. In a subsequent phase, this is elaborated upon by employing algebraic and/or numerical methods. This has become the dominant approach to GCS, also becoming the target of considerable research and development.

**Degrees of Freedom Analysis** The graph's vertices are labeled with the degrees of freedom of the represented object. Each edge is labeled by the degrees of freedom the constraint cancels out. This graph is then analyzed for a solution strategy.

A symbolic solution method is derived using rules with geometric meaning, a method proved to be correct in [19]. It is further extended by using it along with numerical methods as a fallback if geometric reasoning fails [20].

Latham and Middleditch [21] decompose the graph into minimal connected components they call *balanced sets* that are solved by a geometric construction, falling back to a numerical solution

attempt. This method can deal with symbolic constraints and identifies under- and overconstrained problems, where the latter kind is approached by prioritizing the given constraints.

**Propagation Approaches** The graph's vertices represent variables and equations, and the edges are labeled with occurrences of the variables in equations. The goal is to orient the graph such that all incident edges to an equation vertex but one are incoming edges. If so, the equation system has been triangularized. Orientation algorithms include degree-of-freedom propagation and propagation of known values [22, 23] which can fail in the presence of orientation loops, but such situations are addressed [23] and they may resort to numerical solvers.

### 1.3.2 Logic-Based Approaches

Using logic-based approaches, the constraint problem is translated into a set of geometric assertions and axioms which is then transformed in such a way that specific solution steps are made explicit by applying geometric reasoning. The solver then takes a set of construction steps and assigns coordinate values to the geometric entities.

A geometric locus<sup>3</sup> at which constrained elements must be is obtained using first order logic to derive geometric information, applying a set of axioms from Hilbert's geometry [24–26]. Two different types of constraints are further considered [27, 28]: (1) sets of points placed with respect to a local coordinate frame, and (2) sets of straight line segments whose directions are fixed. The reasoning is performed by applying a rewriting system on the sets of constraints. Once every geometric element is in a unique set, the problem is solved.

### 1.3.3 Algebraic Approaches

In the case of an algebraic approach, the problem is translated into a system of equations where the variables are coordinates of geometric elements and the equations, which are generally nonlinear, express the constraints upon them. This approach's main advantage is its completeness and dimension independence. However, it is difficult to decompose the equation system into subproblems, and a general, complete solution of algebraic equations is inefficient. Nonetheless, small algebraic systems tend to appear in the other approaches and are routinely solved.

### 1.3.4 Symbolic Methods

Symbolic methods rely on general equation solvers which employ techniques to triangularize equation systems [29, 30] that emerge from employing an algebraic approach. A solver built on top of the Buchberger's algorithm is described in [31]; Kondo [32] further reports on a symbolic algebraic method.

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<sup>3</sup>In mathematics, a locus is a set of points that satisfy some condition. In layman's terms, a location or place.

These methods can produce generic solutions which can be evaluated for a different set of constraint assignments, then producing parameterized solutions. However, solvers are very slow and computation demands a lot of space, usually requiring exponential running time [33].

### 1.3.5 Numerical Methods

Among the oldest approaches to constraint solving, numerical methods solve large systems of equations iteratively. Methods like Newton iteration work properly if a good approximation of the intended solution can be supplied and the system is not ill-conditioned. Take, for example, a sketch of a model the user drew. If the starting point comes from said sketch, then it should follow that the result be close to what is intended. Alas, such methods may find only one solution, even in cases where there are many, and may not allow the user to select the one they are interested in. Such methods are called local methods, as opposed to global methods, exploring the problem space for every possible solution.

Relaxation methods [4, 34, 35] can be employed in attempts to partly minimize global error by perturbing the values assigned to the variables. However, in general, convergence to a solution is slow.

The Newton-Raphson iteration method, the most widely used one, is a local method and converges much faster than relaxation, but does not apply to over-constrained systems of equations unless expanded upon [36].

Global and guaranteed convergence can be had resorting to the *Homotopy continuation* family of methods [37]. Despite usage in GCS [33, 38], these are far less efficient than the Newton-Raphson method due to the latter's exhaustive nature.

### 1.3.6 Theorem Proving

GCS can be seen as a subproblem of geometric theorem proving, but the latter requires general techniques, therefore requiring much more complex methods than those required by the former.

Wen-Tsün Wu's method [39, 40] is an algebraic-based method that can be used to automatically find necessary conditions to obtain non-degenerated solutions. It can be used to prove novel geometric theorems [29]. Chou et al. [41, 42] develop on automatic geometric theorem proving, allowing the interpretation of the computed proof.

### 1.3.7 Other Areas

The following are briefly described key advances made during the past two decades that interface with other areas or that cannot be readily integrated into graph-constructive solvers. These techniques also constitute examples of further attempts to broaden the scope of GCS, proving that it is a strong field of research with many applications beyond CAD.

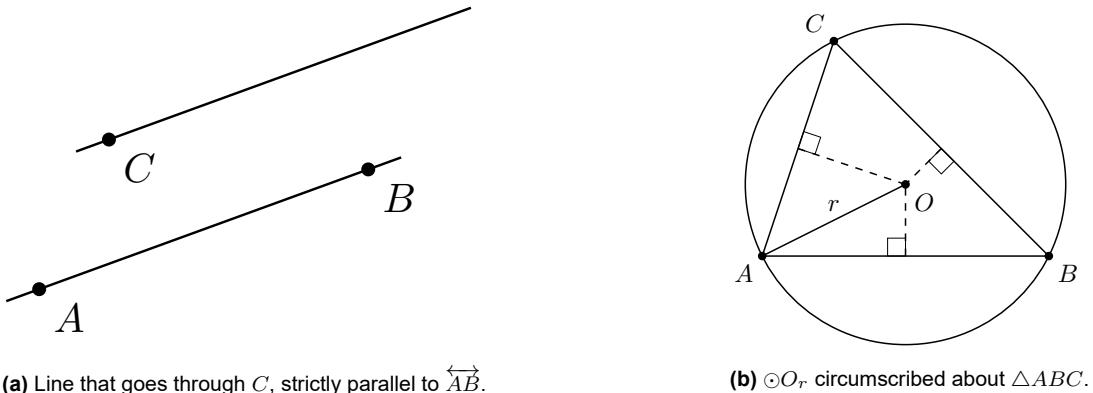
**Deformations** When restrictions are placed on the type of deformation, these problems can be seen as constraint solving. For example, Ahn et al. [43], Bao et al. [44], Moll and Kavraki [45] consider deformations that minimize strain energy; Xu et al. [46] entail surface deformation under area constraints. However, such techniques are rarely integrated with other GCs such as point distance or perpendicularity.

**Dynamic Geometry** The addition of constraints to a given under-constrained system can make it well-constrained, and such constraints can be seen as parameters when they are dimensional. Varying their values, different solutions arise, which can be wholly understood as a dynamic geometric configuration. Systems akin to Cinderella [47] can deal with these problems. Further literature exists on these problems from a constraint solving perspective [48].

**Evolutionary Methods** Consist of re-interpreting the problem as an optimization problem, attacking it using genetic, particle-swarm or other evolutionary methods [49, 50].

## 1.4 Geometric Constraint Problem Examples

This section presents two simple examples of geometric models that are defined through the specification of GCs, and the respective solutions using intuitive algebraic formulation, accompanied by programmatic solutions using `TikZ` [51] and `Eukleides` [52]. Depictions of the aforementioned models can be seen in fig. 1.2. The examples are limited to the two-dimensional Euclidean plane over real numbers,  $\mathbb{R}^2$ . Solutions for analogous problems in three-dimensional Euclidean space,  $\mathbb{R}^3$ , exist as well.



**Figure 1.2:** Geometric models defined using GC relations: (a) showcases line parallelism, and (b) showcases a circle circumscription about a triangle.

The first problem is that of a parallelism constraint: specifying a line that goes through a given point while also being strictly parallel to another already defined line. The second problem is a circumscription constraint: defining a circle that tightly wraps around a triangle, i.e., the circle's circumference goes

through three given non-collinear points.

### 1.4.1 Parallel lines

Let  $A, B, C \in \mathbb{R}^2$  such that  $C$  is a point in the line which is strictly parallel to the line  $\overleftrightarrow{AB}$  (see fig. 1.2a).

A line in  $\mathbb{R}^2$  can be described by the parametric equation

$$P_Q = Q + \lambda \vec{u} \Rightarrow \begin{cases} x = x_Q + \lambda u_x \\ y = y_Q + \lambda u_y \end{cases}, \lambda \in \mathbb{R} \quad (1.1)$$

where  $Q = (x_Q, y_Q)$  is a point on the line that goes through  $P_Q = (x, y)$ , and  $\vec{u} = (u_x, u_y)$  is the vector that drives the line. To then describe the line that goes through  $C$  and is parallel to  $\overleftrightarrow{AB}$ , one must compute the base point  $Q$ , trivially  $C$ , and the directional vector  $\vec{u}$ , which can be obtained from  $\overleftrightarrow{AB}$ . Let  $Q = C$ , and  $\vec{u} = B - A$ , such that

$$P_C = C + \lambda \vec{u}, \lambda \in \mathbb{R}.$$

Listing 1.1 shows the code used to produce the example shown in fig. 1.2a using TikZ with the tkz-euclide<sup>4</sup> L<sup>A</sup>T<sub>E</sub>X package, using `tkzDefLine`, which takes two points  $A, B$ , with the parallel transformation option. This option takes the point  $C$  the resulting line goes through. The result is a point  $D = C + \vec{u}$ , which can be obtained using `tkzGetPoint` to later draw the line.

---

```

1 \begin{tikzpicture}[rotate=20]
2   \tkzDefPoints{0/0/A,3/0/B,1/1/C}
3   \tkzDefLine[parallel=through C](A,B) \tkzGetPoint{D}
4   \tkzDrawLines[add=.1 and .1](A,B C,D)
5   \tkzDrawPoints(A,B,C)
6   \tkzLabelPoints(A,B,C)
7 \end{tikzpicture}

```

---

**Listing 1.1:** Parallel lines example from fig. 1.2a using tkz-euclide. The highlighted line shows how to define the line  $L_C$  parallel to  $\overleftrightarrow{AB}$ .

Listing 1.2 shows the code used to produce an identical figure using Eukleides. In Eukleides, the parallel line  $L_C$  can be obtained through the `parallel` function, which takes the line  $\overleftrightarrow{AB}$  it is parallel to and the point  $C$  it goes through.

### 1.4.2 Circumcenter

Let  $A, B, C, O \in \mathbb{R}^2$  be points such that  $O$  is the center point of a circle of radius  $r$ ,  $\odot O_r$ , that is circumscribed about the triangle  $\triangle ABC$  (see fig. 1.2b).

---

<sup>4</sup><https://ctan.org/pkg/tkz-euclide>

---

```

1 A B C triangle 3, pi/4 rad, pi/6 rad, 20 deg
2 AB = line(A, B)
3 lc = parallel(AB, C)
4 draw
5   AB; lc
6   A; B; C
7 end
8 label
9   A -pi/4 rad
10  B -pi/4 rad
11  C -pi/4 rad
12 end

```

---

**Listing 1.2:** Parallel lines example from fig. 1.2a using Eukleides. The highlighted line shows how to define the line  $l_C$  parallel to  $\overleftrightarrow{AB}$ .

A precondition for this computation is that  $\triangle ABC$  is not degenerate, i.e., its vertices are non-collinear. That can be verified by computing the cross product of any two distinct vectors that drive  $\triangle ABC$ 's edges and verifying it does not equate to zero.

To draw  $\odot O_r$ , we must compute both its center and radius. Its radius  $r$  can be trivially defined as the distance of the center  $O$  to any of the  $\triangle ABC$ 's vertices, i.e.,  $r = \overline{OA} = \overline{OB} = \overline{OC}$ . To determine  $O$ , one must compute the intersection of the perpendicular bisectors of the triangle's edges. Said bisectors are the mediators between an edge's vertices, which can be described by eq. (1.1), where  $P$  is the midpoint between the vertices, and  $\vec{u}$  is a vector normal to the edge. The midpoint  $M_{P_1 P_2}$  of two points  $P_1, P_2 \in \mathbb{R}$  is given by

$$M_{P_1 P_2} = \frac{P_1 + P_2}{2} = \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right). \quad (1.2)$$

Further, the scalar product of two vectors  $\vec{u}, \vec{v} \in \mathbb{R}^2$  is given by

$$\vec{u} \cdot \vec{v} = (u_x, u_y) \cdot (v_x, v_y) = u_x v_x + u_y v_y. \quad (1.3)$$

The normal vector  $\vec{n}$  is such that, for some vector  $\vec{u}$ ,

$$\vec{u} \cdot \vec{n} = 0.$$

A vector  $\vec{n} \in \mathbb{R}^2$  normal to another vector  $\vec{u}$  can be easily obtained by swapping the components of  $\vec{u}$  while negating one of them, a property easily verified by applying eq. (1.3).

Computing the edges' midpoints and respective normal vectors, we can then describe the mediators. Let  $M_{AB}, M_{AC}, M_{BC} \in \mathbb{R}^2$  be the midpoints, by eq. (1.2), of the respective edges, and  $\vec{u}_1, \vec{u}_2, \vec{u}_3$  the

edges' normal vectors, such that

$$\begin{aligned} P_{M_{AB}} &= M_{AB} + \lambda_1 \vec{u}_1 \\ P_{M_{AC}} &= M_{AC} + \lambda_2 \vec{u}_2, \quad \lambda_i \in \mathbb{R}. \\ P_{M_{BC}} &= M_{BC} + \lambda_3 \vec{u}_3 \end{aligned}$$

This problem can be further simplified by eliminating one of the redundant bisectors. Since the intersection of two lines already yields a single point, we can eliminate one of the equations. Say we discard the mediator of line  $\overleftrightarrow{BC}$ . We then require that

$$P_{M_{AB}} = P_{M_{AC}} \xrightarrow{(1.1)} \begin{cases} x_{M_{AB}} + \lambda_1 u_{1x} = x_{M_{AC}} + \lambda_2 u_{2x} \\ y_{M_{AB}} + \lambda_1 u_{1y} = y_{M_{AC}} + \lambda_2 u_{2y} \end{cases}.$$

Every variable is known except for  $\lambda_1$  and  $\lambda_2$ , but the equation system can be solved in order to assign values to both of them since we have exactly two equations that relate them. Finally, we can define  $O$  using one of the equations with the respectively found  $\lambda$ , i.e., using  $L_{M_{AB}}$ , for instance, we have

$$O = M_{AB} + \lambda_1 \vec{u}.$$

Listing 1.3 shows the code used to produce the example in Figure 1.2b using TikZ with the tkz-euclide L<sup>A</sup>T<sub>E</sub>X package. To compute the center point of  $\odot O_r$ , one can use `tkzCircumCenter`, which takes three points  $A, B, C$ , and generates the result  $O$ , obtainable using `tkzGetPoint`.

---

```

1 \begin{tikzpicture}
2   \tkzDefPoints{0/0/A,4/0/B,1/3/C}
3   \tkzCircumCenter(A,B,C) \tkzGetPoint{O}
4   \tkzDefMidPoint(A,B) \tkzGetPoint{AB}
5   \tkzDefMidPoint(A,C) \tkzGetPoint{AC}
6   \tkzDefMidPoint(B,C) \tkzGetPoint{BC}
7   \tkzDrawSegments[style=dashed](AB,O AC,O BC,O)
8   \tkzMarkRightAngles(A,AB,O B,BC,O C,AC,O)
9   \tkzDrawPolygon(A,B,C)
10  \tkzDrawCircle(O,A)
11  \tkzDrawSegment(O,A)
12  \tkzDrawPoints(A,B,C,O)
13  \tkzLabelLine[above](O,A){$r$}
14  \tkzLabelPoints[below left](A)
15  \tkzLabelPoints[below right](B)
16  \tkzLabelPoints[above left](C)
17  \tkzLabelPoints(O)
18 \end{tikzpicture}

```

---

**Listing 1.3:** Circumcenter example from fig. 1.2b using TikZ alongside tkz-euclide. The highlighted line shows how to obtain the center of  $\odot O_r$  via the non-degenerate triangle  $\triangle ABC$ .

Listing 1.4 shows the code that produces an identical figure using Eukleides. In Eukleides, one can use the `circle` function, which similarly takes three points  $A, B, C$ , and generates the circle  $\odot O_r$ , circumscribed about  $\triangle ABC$ , while  $O$  can be obtained using the `center` function.

---

```

1 A.B.C = point(0, 0).point(1, 3).point(4, 0)
2 Or = circle(A, B, C)
3 O = center(Or)
4 AB.AC.BC = midpoint(A.B).midpoint(A.C).midpoint(B.C)
5 draw
6   AB.O dashed
7   AC.O dashed
8   BC.O dashed
9   (A.B.C); Or; O.A
10  A; B; C; O
11 end
12 label
13  A -3*pi/4 rad
14  B 3*pi/4 rad
15  C -pi/4 rad
16  O -pi/4 rad
17  A, AB, O right
18  B, BC, O right
19  C, AC, O right
20 end

```

---

**Listing 1.4:** Circumcenter example from Figure 1.2b using Eukleides. The highlighted line shows how to obtain the center of  $\odot O_r$  via the non-degenerate triangle  $\triangle ABC$ .

Both languages used to produce the examples' solutions provide a sensible set of constraint primitives. However, in the particular case of `tkz-euclide`, the syntax required for describing the models is outdated, rigid, and may cause confusion. For example, in listings 1.1 and 1.3, command results can not be used directly as inputs to other commands and must instead be obtained using another command to create a permanent symbol associated with the resulting value. By contrast, functions and expressions' results in modern languages can be used directly as well as stored by using a far friendlier assignment syntax. Nonetheless, the underlying ideas can be repurposed and adapted, implementing them in a modern and more expressive language.

## 1.5 Algorithmic Design

In spite of the improved usability and pervasiveness of parametric features in modern CAD applications, along with the immense strides made in the area of GCS, approaches reliant on these tools tend to not scale well with design complexity. Correctly applying modifications to existing models becomes cumbersome when experimenting with generating different variants of a model or adapting it to new requirements. Users have to spend most of their time and effort unnecessarily tweaking and changing

their design's parameters' values, which can be, as mentioned, an error-prone process, hindering their capability to efficiently produce novel designs.

AD consists in the generation of CAD and Building Information Modeling (BIM) models through the specification of algorithmic descriptions [2], opposed to more classical approaches in which users directly interact with the geometric model being produced. Furthermore, the algorithms used to describe the idealized models are naturally parametric, which allows for the generation of multiple variants of said model by adjusting the algorithm parameters' values, enabling users to make changes to their models in a much more effortless and efficient manner when compared to direct-manipulation methods [53]. The parametric nature of the algorithmic specifications implicitly imposes constraints on the model since dependencies within the description are changed if an ancestor parameter's value changes upon re-execution, propagating the updates in a downwards fashion. This is advantageous since users can easily create more complex designs, hence also deeming AD a more scalable alternative to traditional approaches.

Such an approach also lead to the creation and integration of programming tools into existing CAD and BIM software such as Grasshopper<sup>5</sup> for Rhinoceros<sup>6</sup> or Dynamo<sup>7</sup> for Revit<sup>8</sup>. Some tools, like Rosetta [54], offer a distinctly portable solution in contrast to the likes of the aforementioned ones, enabling the generation of several identical models for a variety of different CAD and BIM applications through a single specification [55] while also giving users room to experiment with a series of different available programming languages.

Despite the benefits that come with the integration of AD tools in CAD and BIM software, it is key that these tools also provide a highly expressive platform to further boost user productivity. This means these tools should provide a variety of primitive constructs, abstraction mechanisms, high-level concepts, among other capabilities, making it easier for users to create sophisticated models and designs [56]. Generally, the more expressive the platform is, the better it is with respect to usage, also making it easier to learn, a crucial point when migrating from traditional direct-manipulation user interfaces. This quality becomes all the more important when generating a geometric model riddled with constraints users have to manually specify and figure out, potentially introducing calculation or logical errors during the process. Thus, the inclusion of GC concepts in such tools would make working with constraints easier, in turn mitigating (ideally nullifying) error propagation throughout the algorithm, and increase the tool's expressive power.

---

<sup>5</sup><https://www.grasshopper3d.com>

<sup>6</sup><https://www.rhino3d.com>

<sup>7</sup><https://dynamobim.org>

<sup>8</sup><https://autodesk.com/revit>



# 2

## Related Work

### Contents

---

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---



In this chapter, we expose and discuss numerical accuracy issues that arise when performing computations with fixed-precision arithmetic. We then proceed to naming some precautions and steps in order to obtain practical solutions, followed by a brief mention of some software libraries dedicated to overcoming these issues through a series of exact algorithms and data structures.

We follow that by a comparative analysis of a set of GCS-capable programming tools along different dimensions, such as supported language paradigm, native GCS capabilities, 2D and 3D support. Of those tools, Eukleides [52], GeoSolver [57], and TikZ & PGF [51] are extensively discussed.

Similarly, we analyze AD tools. Some of them are integrated within CAD applications while others are standalone applications. These tools and their capabilities are summarized in table 2.2. Furthermore, Dynamo and Grasshopper are expanded upon.

The chapter closes with small remarks on VPLs' poorer scalability with increasing project complexity when compared to TPLs, showcasing the Rhythmic Gymnastics Center (RGC) as an example.

## 2.1 Robustness

The correctness proofs of nearly all geometric algorithms presented in theoretical papers assumes exact computation with real numbers [58]. However, floating-point numbers are represented with fixed precision in computers, making them inexact, which leads to inaccurate representations of the conceptual real number counterparts. For example, the rational number one-tenth ( $\frac{1}{10}$ ) cannot be accurately represented as a floating-point number, nor is it guaranteed to be truly equal to another seemingly identical number. Such comparisons must be performed relying on tolerances, i.e., if  $a$  and  $b$  are two floating-point numbers, they are considered *the same* if  $|a - b| \leq \epsilon$  for a given tolerance  $\epsilon$ .

As an example, consider the problem of finding the closest of two points to the origin. The distance between two points  $P, Q \in \mathbb{R}^2$  can be expressed by

$$d(P, Q) = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2}. \quad (2.1)$$

Let  $A, B \in \mathbb{R}^2$  be two arbitrary points, and  $O \in \mathbb{R}^2$  the origin. To determine which point,  $A$  or  $B$ , is closest to the origin  $O$ , we compare the former's distances to the latter's. That is, if

$$d(A, O) < d(B, O)$$

holds,  $A$  is the closest to the origin. Otherwise, they are either equidistant or  $B$  is closer. However, applying the square root operation in the distance computation is a step that will introduce errors. Given that we are only interested in comparing distances, and not use their actual value, we can, instead, compare the squared distances. As such, we avoid the square root, thus improving robustness, and

speeding up the process because the square root is a computationally heavy operation. Mei et al. [59] further discuss the issues with numerical robustness in geometric computation, namely how they arise, and propose practical solutions.

When used without care, fixed-precision arithmetic almost always leads to unwanted results due to marginal error accumulation caused by rounding (*roundoff*), propagated throughout a series of calculations. As seen above, careful observations must be made before proceeding with computations as simple as distance calculation. To help solve this problem, more robust numerical constructs and concepts can be used. In particular, exact numbers, such as rational numbers or arbitrary precision numbers. The latter, also known as *bignums*, allow arbitrary-precision arithmetic, capable of representing numbers with virtually infinite precision with the drawback that arithmetic operations are slower, however mitigating precision issues, providing more accurate constructs and improving code robustness.

Several libraries already strive to implement robust geometric computation. One such example is the Computational Geometry Algorithms Library (CGAL) [60]. CGAL is a comprehensive library that employs an exact computation paradigm [61], producing correct results despite roundoff errors and properly handling *degenerate* situations (e.g., 3D points on a 2D plane), relying on numbers with arbitrary precision to do so. Moreover, other libraries, such as the Library for Efficient Data types and Algorithms (LEDA) [62], and CORE [63] and its successor [64], also deal with robustness problems in geometric computation, offering simpler interfaces when compared to CGAL. However, CGAL arguably remains the *de facto* standard library for robust exact geometric computation.

## 2.2 Geometric Constraint Tools

Constraint-based programming comes in a wide variety of ways, following a diverse set of programming paradigms, using different approaches to problem solving briefly detailed in section 1.3. Some of them also support an associative programming model, such as DesignScript [65], further discussed in section 2.3.1, allowing for the propagation of changes made to a variable to others that depended on the former.

Table 2.1 succinctly analyzes tools capable of solving geometric constraints. From this table, Eukleides, GeoSolver, and the *TikZ & PGF* system are further discussed: Eukleides for its elegant declarative language, similar to some of the languages outlined in table 2.2; GeoSolver for its helpful analysis Graphical User Interface (GUI), along with the fact it is implemented in Python, a well established and easy to use language, already used in some competence in CAD software (see table 2.2); and *TikZ* for its wide support, development, usage, and collection of packages that extend it, enabling the specification of graphics and geometry in a variety of simple distinct ways.

**Table 2.1:** Table of tools and languages with GCS capabilities.

| Tool                           | TPL | VPL | Assoc <sup>†</sup> | Decl <sup>‡</sup> | Imp* | 2D | 3D |
|--------------------------------|-----|-----|--------------------|-------------------|------|----|----|
| DesignScript [65]              | ✓   | ✗   | ✓                  | ✗                 | ✓    | ✓  | ✓  |
| Eukleides [52]                 | ✓   | ✗   | ✗                  | ✓                 | ✓    | ✓  | ✗  |
| GeoGebra [66]                  | ✓   | ✓   | ✗                  | ✗                 | ✓    | ✓  | ✓  |
| GeoSolver [57]                 | ✓   | ✓   | ✗                  | ✗                 | ✓    | ✓  | ✓  |
| Kaleidoscope <sup>¶</sup> [67] | ✓   | ✗   | ✓                  | ✗                 | ✓    | ≈  | ≈  |
| ThingLab [35]                  | ✗   | ✓   | ✓                  | ✓                 | ✗    | ✓  | ✓  |
| TikZ & PGF [51]                | ✓   | ✗   | ✗                  | ✗                 | ✓    | ✓  | ✗  |

<sup>¶</sup> — Doesn't natively support GCS, but can be extended to solve this class of constraint problems. <sup>†</sup> — Associative model / *change-propagation* mechanism; <sup>‡</sup> — Declarative paradigm; \* — Imperative paradigm

## 2.2.1 Eukleides

Devoted to elementary plane geometry, Eukleides is a simple, full-featured, and mainly declarative programming language, capable of handling basic data types, such as numbers and strings, and most importantly, geometric data types, such as points, vectors, lines, and circles. Like most languages, it provides control flow structures, allows user functions and module definitions, making it easily extendable.

Eukleides provides a wide variety of functions and constructions that easily allow the user to specify geometric constraints between objects, as demonstrated by listings 1.2 and 1.4. Among the listed ones, it includes functions to build parallel and perpendicular lines with respect to another line or segment, determine a line's bisector, tangent lines to a circle, shape intersection, and so on. It can generate Encapsulated PostScript (EPS) files or produce macros, enabling the embedding of Eukleides figures in  $\text{\LaTeX}$  documents.

Alas, the lack of 3D geometry support arguably constitutes Eukleides' primary disadvantage. It is also a TPL, which means that, while being a very simple language, it is less intuitive than a VPL. The first version of Eukleides included a GUI, xeukleides, but one is not yet available for the current version.

Additionally, it has not seen any development for the last decade, while TikZ is still being actively maintained. The latter still dominates diagram and graphic production in  $\text{\LaTeX}$  documents, some people<sup>1</sup> going as far as suggesting opting for it instead of using the former.

## 2.2.2 GeoSolver

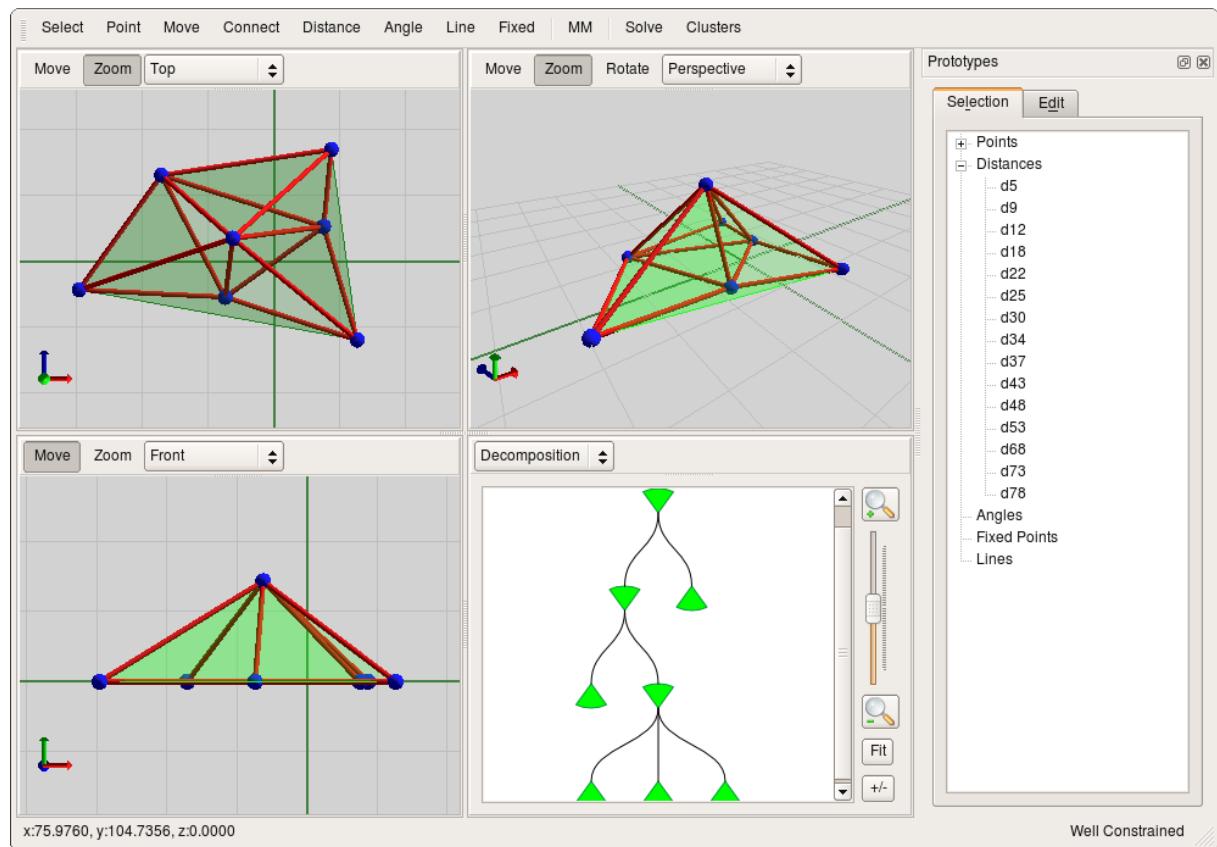
GeoSolver is an open-source Python package that provides classes and functions for specifying, analyzing, and solving geometric constraint problems. It features a set of 3D geometric constraint problems

<sup>1</sup><https://tex.stackexchange.com/a/208412/178614>

consisting of point variables, two-point distance and three-point angle constraints. Problems with other geometric variables can be mapped to these basic constraints on point variables.

The solutions found by GeoSolver are generic and parametric, and can be used to derive specific solutions. Since generic solutions are exponentially hard to find, GeoSolver also allows different ways of reducing the number of solutions that would be generated, consequently reducing computation time. In order to efficiently find a solution, GeoSolver employs a cluster rewriting-based approach described in [57], capable of handling non-rigid clusters contrasting with typical graph constructive-based approaches.

A GUI interactive tool called GCS Workbench [68] (see fig. 2.1) is distributed along with the GeoSolver package. With it, the user can easily edit, analyze and solve geometric constraint problems. The latter features are obviously supported by GeoSolver, and 3D interactivity is supported via additional libraries, such as pyQt and pyOpenGL. Although an excellent tool for understanding how a geometric constraint problem is decomposed in GeoSolver, GCS Workbench is not efficient for complex design tasks when compared with its programmatic supporting package.



Source: <http://geosolver.sourceforge.net> (Jan 2019)

**Figure 2.1:** Depiction of the GCS Workbench's GUI with two separate panes: (1) showcasing different perspectives of the model and the constraint problem's decomposition, and (2) a prototyping pane, destined for constraint analysis and edition.

### 2.2.3 TikZ & PGF

Originally a small  $\text{\LaTeX}$  style created by Till Tantau for his Ph.D. thesis, TikZ [51], along with its underlying lower-level Portable Graphics Format (PGF) system, is a fully featured graphics language, basically consisting of a series of  $\text{\TeX}$  commands that draw graphics. TikZ stands for “TikZ ist *kein* Zeichenprogramm”, a recursive acronym, which translates to “TikZ is no drawing program”. As mentioned, the user instead programmatically describes their drawings.

On its own, TikZ already includes a series of commands capable of handling geometric constraints, such as tangency, perpendicularity, intersection; but may appear daunting to the user in its raw form. Several packages have been built on top of it to facilitate the generation of drawings using a simpler syntax, such as tkz-2d and its successor tkz-euclide<sup>2</sup>. The package tkz-euclide was designed for easy access to the programming of Euclidean geometry using a Cartesian coordinate system with TikZ. It was used to produce fig. 1.2 with the respective code listed in listings 1.1 and 1.3.

Like Eukleides, an obvious limitation they share is the lack for 3D modeling support. Unlike it, a plethora of resources and usage examples exist, along with an immense amount of packages that layer on top of it for a panoply of diverse use cases. It still undoubtedly remains the go-to graphics system within the  $\text{\TeX}$  typesetting community. However, again comparing it to Eukleides, even using something as tkz-euclide, it can look syntactically appalling, even for the adept  $\text{\TeX}$  user, instead of following a simpler and established familiar syntax akin to other declarative or imperative programming languages.

## 2.3 Algorithmic Design Tools

As discussed in section 1.5, AD tools have been integrated into several modern CAD and BIM applications, using TPLs, VPLs, or even a mixture of both approaches.

Other tools, like JSCAD<sup>3</sup> and ImplicitCAD<sup>4</sup>, are standalone CAD software hosted on the web. Being cloud-based is advantageous in many fronts: it is inherently portable and removes the additional typical installation steps required for desktop applications. Alas, being relatively new, they lack features in comparison to the immense feature-set of applications such as AutoCAD.

Table 2.2 succinctly summarizes a list of CAD software that supports the usage of a programming language, as well as other AD tools that live detached from existing CAD software. From there, Dynamo and Grasshopper are further comparatively discussed, being relatively similar tools integrated within CAD/BIM software. Moreover, both include TPL and VPL support in different forms.

---

<sup>2</sup><https://ctan.org/pkg/tkz-euclide>

<sup>3</sup><https://openjscad.xyz>

<sup>4</sup><https://implicitcad.org>

**Table 2.2:** CAD/BIM software with programmatic capabilities and AD software/tools. Added notes per tool shortly outline deemed significant characteristics.

| Application             | Tool               | TPL | VPL | Note   |
|-------------------------|--------------------|-----|-----|--|
| AutoCAD                 | .NET API           | ✓   | ✗   | Powerful, but very verbose; C# & VB.NET                                  |
|                         | ActiveX Automation | ✓   | ✗   | Deprecated, bundled separately; VBA                                      |
|                         | Visual LISP        | ✓   | ✗   | IDE; AutoLISP extension  |
| Dynamo Studio           | Dynamo             | ✓   | ✓   | Data flow paradigm; Associative programming support through DesignScript |
| Revit                   |                    |     |     |  |
| Archicad                | Grasshopper        | ✓   | ✓   | Data flow paradigm; Rhino SDK access, C# & VB.NET                        |
| Rhinoceros3D            | Python Scripting   | ✓   | ✗   | Simple language; Create custom Grasshopper components                    |
|                         | RhinoScript        | ✓   | ✗   | VBScript based   |
| Standalone <sup>†</sup> | ImplicitCAD        | ✓   | ✗   | Web hosted; OpenSCAD inspired  |
|                         | JSCAD              | ✓   | ✗   | Web hosted; JavaScript   |
|                         | OpenSCAD           | ✓   | ✗   | Solid 3D models; Simple domain language                                  |
|                         | Rosetta [54]       | ✓   | ✗   | Portable tool; Multiple front- and back-end support                      |

<sup>†</sup>These tools are standalone software, i.e., not directly integrated into any specific CAD application.

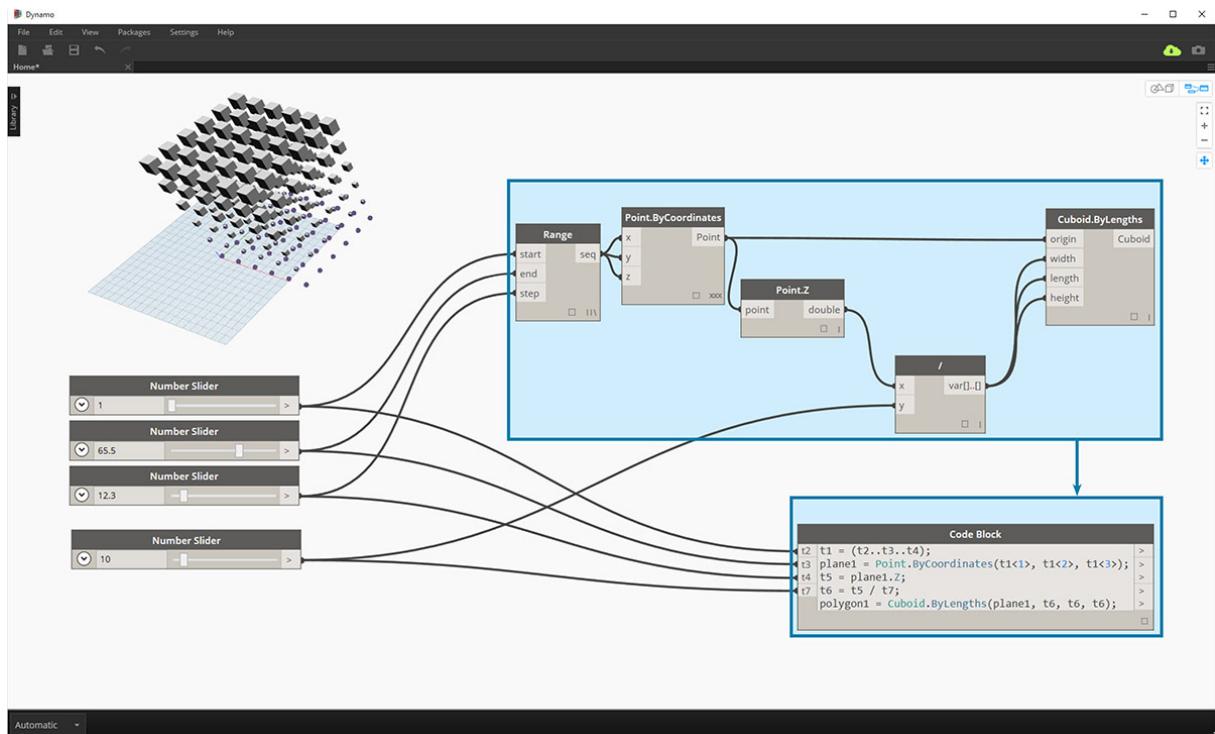
### 2.3.1 Dynamo

An open source AD tool available as a plug-in for Revit or by itself within Dynamo Studio, Dynamo extends BIM with the data and logic environment of a graphical algorithm editor. Dynamo can be used through both a VPL and a TPL, showcased in fig. 2.2.

In its visual form, Dynamo offers a wide variety of functions, called nodes, most of them capable of generating an even wider variety of geometry through node combination, wiring one's outputs to another's inputs, and resorting to predefined mutable parameters which can serve as some of the nodes' initial inputs. The workflow itself is the final product: a visual program, usually designed to execute a specific task. Dynamo further allows extension through the creation of custom nodes which can be shared as packages.

One of the nodes in Dynamo, aptly named code block, allows the usage of a TPL; a language called DesignScript.<sup>5</sup> Originally developed by Robert Aish [65], DesignScript is a multi-paradigm domain-specific language and is the programming language at the core of Dynamo itself. So much so that entire workflows can be reduced to one code block (see fig. 2.2).

<sup>5</sup>[https://primer.dynamobim.org/10\\_Custom-Nodes/10-4\\_Python.html](https://primer.dynamobim.org/10_Custom-Nodes/10-4_Python.html) — Dynamo is also capable of interpreting Python code.



Source: [http://primer.dynamobim.org/en/07\\_Code-Block/7-2\\_Design-Script-syntax.html](http://primer.dynamobim.org/en/07_Code-Block/7-2_Design-Script-syntax.html) (Jan 2019)

**Figure 2.2:** Showcase of Dynamo's visual interface containing a workflow that produces the model on the top left. The figure also shows Dynamo's capability of converting a workflow into a single DesignScript code block.

DesignScript is an associative language, which maintains a graph of dependencies with variables. Executing a script will effectively propagate the variables' values accordingly. By default, code blocks in Dynamo follow an associative paradigm. The user can, however, switch to an imperative paradigm approach instead effortlessly if needed.

This *change-propagation* mechanism in DesignScript, consequently present in Dynamo, makes Dynamo a great tool for dealing with constraints. However, most users might not fully exercise DesignScript's associative capabilities and instead approach the problem with the mindset of an imperative programming paradigm given its overwhelming presence in and adoption by major well-known TPLs.

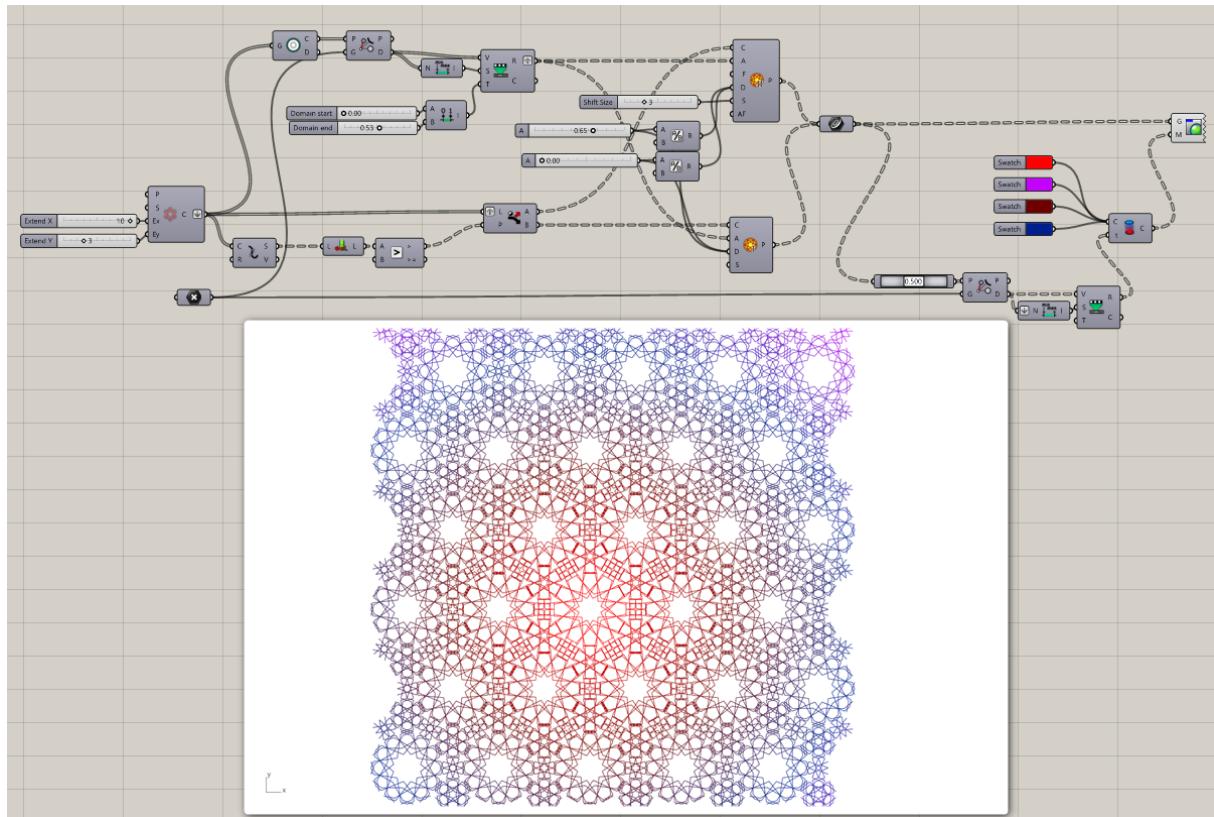
### 2.3.2 Grasshopper

Grasshopper is a graphical algorithm editor tightly integrated with Rhinoceros3D, destined for designers who are exploring generative algorithms. In spite of tight integration with Rhino, a CAD application, it is possible to use Grasshopper along with Archicad<sup>6</sup>, a BIM tool, as well as Revit, through Hummingbird<sup>7</sup>

<sup>6</sup><https://graphisoft.com/solutions/archicad>

<sup>7</sup><https://www.food4rhino.com/en/app/hummingbird>

or Lyrebird<sup>8</sup>. Figure 2.3 shows a simple example of a Grasshopper workflow.



Source: <https://www.grasshopper3d.com/photo/islamic-pattern-parakeet> (Jan 2019)

**Figure 2.3:** Islamic Pattern, by Esmaeil Mottaghi. On top is the Grasshopper workflow to produce the pattern below it, aided by Parakeet<sup>9</sup>.

It is a closed-source product, designed by David Rutten and developed by McNeel and Associates, Rhino's developers. Its VPL is simple to use, which is crucial for users who are not familiar with programming using TPLs. Nonetheless, it offers a TPL alternative by way of custom programmatic components. Using C# or VB.NET, the user can create custom code components with access to Rhino's Software Development Kit (SDK) and openNURBS<sup>10</sup> within Rhino. Alternatively, through GhPython<sup>11</sup>, the user can also write Python code. Unlike Dynamo's DesignScript, Python and the .NET languages don't support an associative programming model.

Functions in Grasshopper are called components and work just like Dynamo's nodes; a wide variety of them exist, most of them capable of producing geometry, and are composable.

<sup>8</sup><https://www.food4rhino.com/en/app/lyrebird>

<sup>9</sup><https://www.food4rhino.com/app/parakeet>

<sup>10</sup><https://developer.rhino3d.com/guides/opennurbs/what-is-opennurbs>

<sup>11</sup><https://www.food4rhino.com/app/ghpython>

### 2.3.3 Visual Programming Scalability

Both Dynamo and Grasshopper's visual approach suffer from the disproportionate scalability between the code and the respective model's complexity [53]. Sophisticated modeling workflows tend to become difficult to create, and harder for a human to understand when compared to a textual approach.

As an example, consider the Irina Viner-Usmanova RGC, a project developed by TPO Pride<sup>12</sup>. The RGC, built in Moscow's Luzhniki Stadium, is an arena that houses training sessions and competitions while also comprising several other premises. Figure 2.4 depicts an outside view of the RGC and its prominent overarching roof covering. The roof covering was designed using a combination of Rhino3D and Grasshopper. Grasshopper was used from a conceptual stage all the way through to production of construction drawings.



Source: <https://www.grasshopper3d.com/photo/rhythmic-gymnastics-center-moscow-russia-5> (Jul 2021)

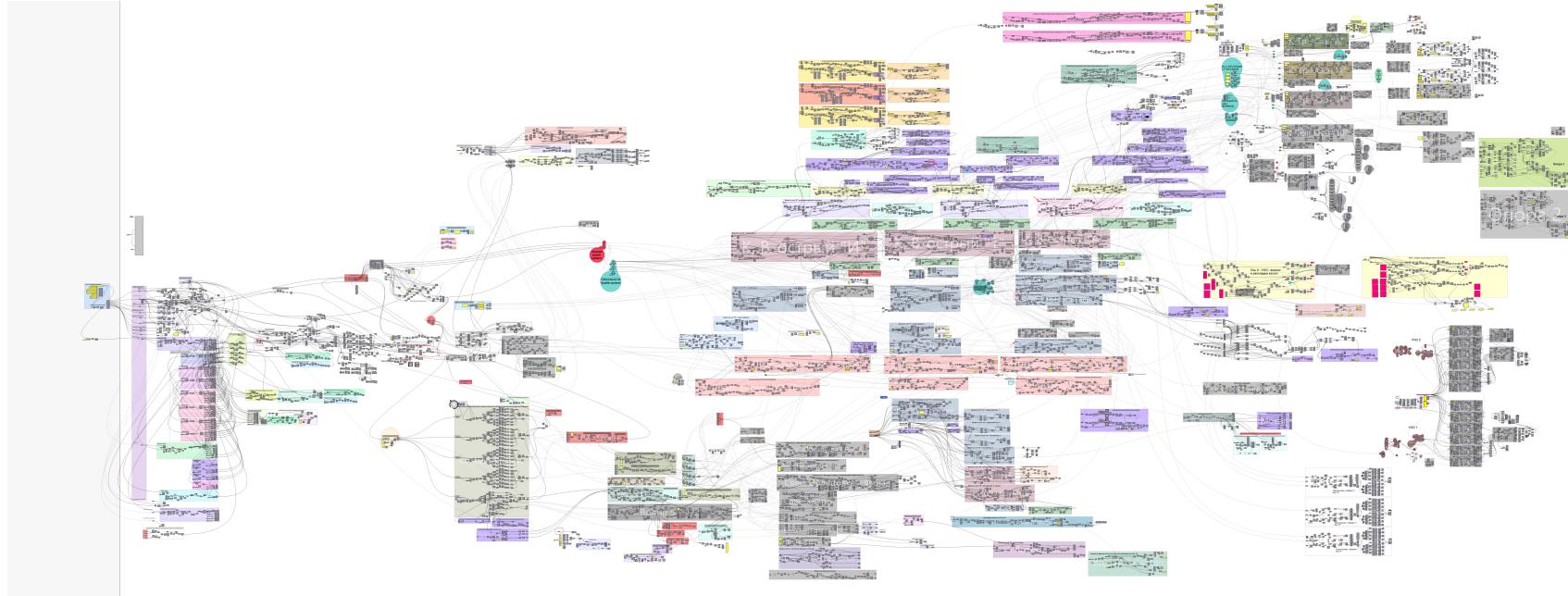
**Figure 2.4:** Irina Viner-Usmanova RGC in the Luzhniki Complex, Moscow, Russia. The roof covering was designed using Rhino/Grasshopper.

Developing a roof covering with such a contour lends itself well to AD since it resembles a sine wave whose amplitude is progressively damped along the length of the building. Such a shape can be easily described through a relatively simple mathematical function to obtain the general wave's shape.

<sup>12</sup><http://prideproject.pro> (July 2, 2021)

With parametrization in mind, one could then easily fluctuate input variables in order to achieve different variations of the roof covering's shape, e.g., varying the wave's frequency, amplitude, or dampening rate.

Such complex AD projects tend to require complex AD programs that become overly difficult to develop and understand with VPLs. The final Grasshopper definition of the RGC roof's covering can be seen in fig. 2.5. This further reinforces the statement that complex workflows become exponentially difficult to comprehend due to the added dimensionality of the constructs used in VPLs. This disadvantage, however, is mitigated with their respective TPL alternatives which, despite project complexity, scale relatively better than VPLs on account of abstraction mechanisms.



Source: <https://www.grasshopper3d.com/photo/final-definition> (Jul 2021)

**Figure 2.5:** Final Grasshopper definition of the Irina Viner-Usmanova RGC roof covering.



# 3

## Solution

### Contents

---

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---



Despite strides in enhancing performance and efficiency of geometric constraint solving approaches, briefly discussed in section 1.3, the core issue lies in the generality of geometric constraint solvers. Although several approaches employ efficient methods to find a solution, they resort to solving potentially well-known problems generically when computationally lighter solutions exist. Instead of delegating the problem to a solver, a more efficient approach would be to pinpoint the type of geometric constraint itself, specializing a solution for several applicable entities. Take the tangency constraint as an example: positioning two circles tangent to each other or a line tangent to an ellipse. Depending on the inputs, these constraints might have particularly efficient solutions for each case, in kind making the computation more efficient.

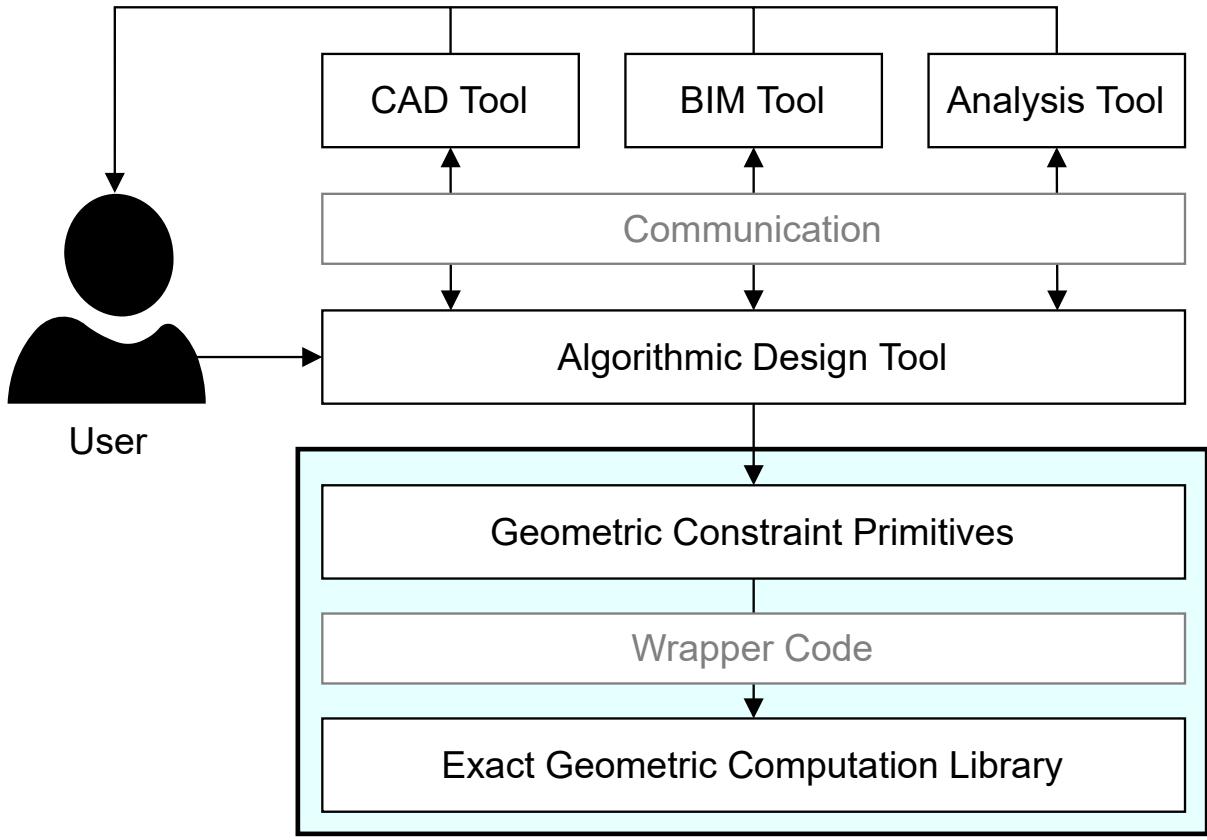
Classical numerical methods constitute alluring alternatives to the predominant graph-based approaches. Having been studied for several decades, even if the provided solution does not encompass all the possible values within the problem’s domain, they can be used to target specific problems efficiently. Nonetheless, these suffer from robustness issues discussed in section 2.1, effectively yielding inaccurate solutions if precautions aren’t taken. A similar argument can be made about algebraic methods.

This work aims to implement a series of geometric constraint primitives in an already expressive TPL to overcome the need for the specification of unnecessary details when modeling geometrically constrained entities, promoting an easier and more efficient usage. Choosing to implement these in a TPL further avoids the poor scalability with increasing code complexity that arises from what could be analogous specifications in a VPL, a subject previously discussed in section 1.5.

Moreover, by relying on an exact geometry computation library, one of the core features of this solution lies in the capability of transparently dealing with plenty of the previously addressed robustness issues. The user can then resort to these primitives, and, by composing them, elegantly specify the set of geometric constraints necessary in order to produce the idealized model. Since the available primitives will implement specialized solutions for a finite set of shapes the user can utilize in whichever combination possible during the design process, the solution will be exempt of a generic solver component, potentially boosting performance of design generation.

Figure 3.1 shows the typical AD workflow and how the proposed solution could be integrated with the AD tool. The encapsulated modules in the figure represent the underlying computation library as an external component, featuring the geometric constraint primitives library and the code wrapping the computation library.

The following sections go over the components in fig. 3.1, namely the *Exact Computation Geometric Library*, the *Wrapper Code*, and the *Geometric Constraint Primitives*. Additionally, we discuss a few trade-offs from tackling the problem in this fashion as opposed to potential alternative routes, describing advantages and disadvantages of our approach.



**Figure 3.1:** General overview of the solution’s architecture encapsulated within the blue colored box beneath a depiction of the typical AD workflow.

### 3.1 Implementation

This section details implementation choices with regard to the chosen platforms for realizing the initially proposed general solution architecture, previously illustrated in fig. 3.1. Following a brief analysis, we expand specifically on the concrete components corresponding to the ones within fig. 3.1’s light blue rectangle.

Examining the AD workflow portion of fig. 3.1, there are depictions of CAD, BIM, and analysis tools, of which examples could be Rhinoceros3D, Autodesk’s Revit, and Radiance, respectively, with no particular focus on any of them. Digging a layer deeper, we find the AD tool, which, by means made available by the tools above it, produces models specific to those tools from a description provided by the user. The AD tool we’ve chosen was Khepri<sup>1</sup> [69], a text-based tool written in the Julia programming language [70]. Khepri is the successor of another text-based AD tool named Rosetta [54], a tool written in the Racket programming language [71]. It follows that the *Geometric Constraint Primitives* were implemented in the Julia language as well, supported by an *Exact Computation Geometry Library*. The library chosen for the

<sup>1</sup><https://github.com/aptmcl/Khepri.jl>

effect was CGAL [60], a highly performant and robust geometric library written in the C++ programming language [72].

This language disparity between the *Geometric Constraint Primitives* module and the *Exact Computation Geometry Library* requires a solution for language interoperation. In other words, we need to make CGAL available to the Julia language. Fortunately, the Julia language already possesses facilities that allow it to invoke functionality within libraries written in the C [73] or the Fortran [74] programming languages. This interfacing mechanism is commonly known as Foreign Function Interface (FFI). It allows for the repurposing of mature software libraries in foreign languages without the need for a complete rewrite or adaptation.<sup>2</sup> This mechanism can also in turn be leveraged and built upon to interface with other programming languages, e.g., Java, Python, MATLAB, and, C++.<sup>3</sup>

Overcoming the language interoperability hurdle, we can now start focusing on the implementation of the *Geometric Constraint Primitives*. These primitives build on top of the functionality available in CGAL, some of which is directly inherited from it, substantially helping us in the process, e.g., intersections. We further enriched the pool with a few more functions, illustrating a constructive approach to GCS, similar and inspired by the approach of tkz-euclide mentioned in section 2.2.3. By creating an abstraction over more primitive functionality, we aimed to provide an easy to understand and utilize set of tools so that users can avoid reimplementing it themselves, which is an error-prone process. Furthermore, we level the playing field by working at a conceptual level which is more familiar to and understood by traditional CAD software users rather than falling back to the more analytic approach programming languages naturally offer.

The following sections will elaborate further on the components emphasized in the previous paragraphs, adopting a bottom-up-like approach. We will discuss CGAL and what constructs and functionality it can provide to aid our goal, as well as some added benefits of building on top of a very mature and comprehensive library. That will be followed by a section detailing how it was possible to map said functionality to the Julia language, of which the result was a Julia package aptly named CGAL.jl<sup>4</sup> [75]. Finally, we showcase how we leveraged CGAL.jl to build the aforementioned *Geometric Constraint Primitives*, a set of functionalities that implements specialized yet comprehensible constructive approach solutions to GC problems.

---

<sup>2</sup>The decision to include such a mechanism at the language's core by the language designers makes it so that the language can rapidly evolve. As such, one can avoid reimplementing several facilities and software libraries in, but not limited to, scientific and numerical computing areas. Arguably, it may be one of the fundamental features that made the language as popular as it is and kept it afloat, unlike other similar historical examples that might have lacked such a mechanism.

<sup>3</sup>There is an entire GitHub organization with projects dedicated to foreign language interoperation at <https://github.com/JuliaInterop> (July 8, 2021)

<sup>4</sup>Packages in the Julia ecosystem are conventionally terminated with a .jl suffix, the extension used for Julia files. This is reminiscent of a familiar convention followed in the Java ecosystem where libraries and tools are usually prefixed with the letter J, e.g., JUnit, JMeter, JDeveloper, among others.

### 3.1.1 Computational Geometry Algorithms Library

CGAL is a software project that provides easy access to efficient and reliable geometric algorithms in the form of a C++ library. It offers a multitude of data structures and algorithms, such as triangulations, Voronoi diagrams, and convex hull algorithms, to name a few. The library is broken up into three parts [76]:

1. The kernel, which consists of geometric primitive objects and operations on these objects. The objects are represented both as (a) stand-alone classes parameterized by a representation class that specifies the underlying number types used for computation, and as (b) members of the kernel classes, which allows for more flexibility and adaptability of the kernel;
2. Basic geometric data structures and algorithms, parameterized by traits classes that define the interface between the data structure or algorithm and the primitives they use;
3. Non-geometric support facilities, such as circulators, random sources, and I/O support for debugging and for interfacing CGAL to various visualization tools.

Listing 3.1 showcases an example of a very simple CGAL program, demonstrating the construction of some points and a segment, and performing some basic operations on them.

As mentioned, geometric primitive types are defined in the kernel. The chosen kernel in the example uses double precision floating point numbers for the Cartesian coordinates of the point.

We can also see some predicates, such as testing the orientation of three points, and constructions, like the distance<sup>5</sup> and midpoint computation. Predicates typically produce a boolean logical value or one of a discrete set of possible results, whereas constructions produce either a number or another geometric entity.

It is worth noting that floating point-based geometric computation can lead to surprising results since we are relying on inexact constructions. If one must ensure that the numbers get interpreted at their full precision, CGAL offers kernels that perform exact predicates and exact constructions. Revisiting listing 3.1, it is as simple as switching the Simple\_cartesian kernel with one that provides exact constructions, e.g., Exact\_predicates\_exact\_constructions\_kernel or Epeck for short.

CGAL is arguably considered as the industry's *de facto* geometric library, used in well-known projects such as OpenSCAD. It is a very mature software library with decades of Ph.D.-grade research results, still being actively maintained to this day. Being an Open Source project, one can easily contribute to it by reporting issues or bugs in the software as well as directly submitting patches.<sup>6</sup>

---

<sup>5</sup>It is worth noting CGAL does not compute the absolute distance, offering instead to compute the squared distance, avoiding the additional square root computation. This preserves exactness and avoids a potentially unnecessary heavy computation.

<sup>6</sup>The library's source is hosted on GitHub at <https://github.com/CGAL/cgal>. To illustrate the ease with which one can contribute, here is a pull request the author submitted: <https://github.com/CGAL/cgal/pull/4705>.

**Listing 3.1:** An example CGAL program illustrating how to construct some points and a line segment, and perform some basic operations on them. It uses `double` precision floating point numbers for Cartesian coordinates.

---

```
1 #include <iostream>
2 #include <CGAL/Simple_cartesian.h>
3
4 typedef CGAL::Simple_cartesian<double> Kernel;
5 typedef Kernel::Point_2 Point_2;
6 typedef Kernel::Segment_2 Segment_2;
7
8 int main()
9 {
10     Point_2 p(1,1), q(10,10);
11
12     std::cout << "p = " << p << std::endl;
13     std::cout << "q = " << q.x() << " " << q.y() << std::endl;
14
15     std::cout << "sqdist(p,q) = "
16             << CGAL::squared_distance(p,q) << std::endl;
17
18     Segment_2 s(p,q);
19     Point_2 m(5, 9);
20
21     std::cout << "m = " << m << std::endl;
22     std::cout << "sqdist(Segment_2(p,q), m) = "
23             << CGAL::squared_distance(s,m) << std::endl;
24
25     std::cout << "p, q, and m ";
26     switch (CGAL::orientation(p,q,m)) {
27     case CGAL::COLLINEAR:
28         std::cout << "are collinear\n";
29         break;
30     case CGAL::LEFT_TURN:
31         std::cout << "make a left turn\n";
32         break;
33     case CGAL::RIGHT_TURN:
34         std::cout << "make a right turn\n";
35         break;
36     }
37
38     std::cout << " midpoint(p,q) = " << CGAL::midpoint(p,q) << std::endl;
39     return 0;
40 }
```

---

These factors, among others, justify our choice for our solution's *Exact Computation Geometric Library* component. We chose CGAL because of its comprehensiveness and decades of work instead of relying on less mature software, as well as the critical mass of maintainers behind it. That is not to say less mature software cannot be used in its stead, though it is unlikely they can match CGAL, be it in terms of performance, quality, or breadth.

However, CGAL is a terribly complex library under the hood, presenting many challenges when it comes to mapping it to the Julia language. Firstly, it is a C++ library. Despite Julia's native capabilities for interoperating with C, there's additional work to be done to reach C++ code. Secondly, a partial problem also shared by C, is memory management, which differs between C/C++ and Julia, potentially leading to memory leaks and other related issues if not properly tended to. Finally, CGAL makes extensive use of C++ templates, proving sometimes difficult to transparently map some of its constructs.

Fortunately, there are both methods and additional libraries that aid us by transparently overcoming some of those issues. In the next section, we go over how we overcame these issues, demonstrating it by reproducing the example in listing 3.1 in Julia.

### 3.1.2 From C++ to Julia

Having established CGAL as our *Exact Computation Geometry Library* of choice, we must now overcome the quite literal language barrier between Julia and C++. Fortunately, the former possesses FFI mechanisms that can aid us in resolving this issue. Here is an excerpt from the language's manual about Julia's C and Fortran FFI capabilities:<sup>7</sup>

*To allow easy use of (...) existing code, Julia makes it simple and efficient to call C and Fortran functions. (...) This is accomplished just by making an appropriate call with `ccall` syntax, which looks like an ordinary function call.*

To illustrate how this `ccall` construct can be used in the context of our problem, we created a small wrapper around CGAL exposing the squared\_distance function. The original function we are interested in takes two instances of 3D points. To facilitate the wrapping, we create the points on the C++ side of things, instead passing primitive `double` values representing the points' coordinates. The C++ wrapper library is listed in listing 3.2.

To invoke our wrapper function, we must precede it with `extern "C"` as is illustrated. This is important because C++ compilers mangle function names. This technique is employed to solve a series of problems in order to support features like function overloading. By preventing this from happening, we can then refer to our wrapper function by its declared name, just like a C function.

---

<sup>7</sup>From <https://docs.julialang.org/en/v1/manual/calling-c-and-fortran-code/>

---

```

1 #include <CGAL/Simple_cartesian.h>
2 #include <CGAL/squared_distance_3.h> // squared_distance
3
4 extern "C" // C function to be invoked in Julia using `ccall`
5 double squared_distance(double x1, double y1, double z1,
6                         double x2, double y2, double z2) {
7     typedef CGAL::Simple_cartesian<double>::Point_3 Point_3;
8     Point_3 p(x1, y1, z1), q(x2, y2, z2);
9     return CGAL::squared_distance(p, q);
10 }
```

---

**Listing 3.2:** Example C library code that wraps CGAL's squared\_distance global function. The original function takes in instances of Point\_3 classes, so we instantiate them from our `double` coordinate inputs.

After compiling the library, we can invoke the newly wrapped function from Julia using `ccall`, as showcased in listing 3.3.

---

```

1 const lib = joinpath(@__DIR__, "libsqdist") # path to the compiled library
2
3 # julia wrapper function around C function
4 squared_distance(x1::Real, y1::Real, z1::Real
5                   , x2::Real, y2::Real, z2::Real) =
6     ccall((:squared_distance, lib)      # qualified function name
7           , Float64                  # return type
8           , (Float64, Float64, Float64
9           , Float64, Float64, Float64) # parameter types
10          , x1, y1, z1, x2, y2, z2) # arguments
11
12 let p = (0, 0, 0), q = (3, 4, 0)
13     @info "Squared distance" p q squared_distance(p..., q...) # = 25.0
14 end
```

---

**Listing 3.3:** Example Julia program that invokes the functionality from the library whose source is listed in listing 3.2. Julia's `ccall` construct converts the input arguments' types to the types specified in the native C function's parameter types.

Though this looks like a good start, the number passing strategy soon shows its limitations. For example, think of the combinatorial explosion problem that may arise when a function requires a number  $M$  of  $N$ -dimensional points. We would then have to create a wrapper that takes  $N \cdot M$  coordinates. Such an approach does not scale. It is possible to overcome this limitation by mirroring C `structs` in Julia.

As an example, we consider the `circumcenter` function from CGAL, a function that takes three points as its parameters, returning the circumcenter about the given points, under the assumption the points are not collinear. We could try and directly mirror CGAL's `Point_3` type, but that would require that we know its layout. Even then, we would be breaking an abstraction barrier that could prove detrimental if CGAL's developers decide to change the type's internal representation. To circumvent this completely, we can just create a `struct` that opaquely wraps around CGAL's type. The C++ wrapper code for this

example is listed in listing 3.4.

---

```
1 #include <CGAL/Simple_cartesian.h>
2 #include <CGAL/Kernel/global_functions.h> // circumcenter
3
4 struct Point { double x, y, z; }; // C struct wrapping Point_3
5
6 extern "C" // C function to be invoked in Julia using `ccall`
7 Point circumcenter(Point p, Point q, Point r) {
8     typedef CGAL::Simple_cartesian<double>::Point_3 Point_3;
9     Point_3 _p(p.x, p.y, p.z), _q(q.x, q.y, q.z), _r(r.x, r.y, r.z)
10    , _s = CGAL::circumcenter(_p, _q, _r);
11    return Point{_s.x(), _s.y(), _s.z()};
12 }
```

---

**Listing 3.4:** Example C shared library source code that wraps CGAL's circumcenter global function. In this instance, we use an additional struct to wrap around CGAL's Point\_3 class to facilitate data transfer.

The wrapper code looks very similar to the previous example in listing 3.2. This time, however, we introduced a new **struct Point** that we will mirror in Julia so that we can seamlessly pass instances of it to our externalized C++ function.<sup>8</sup> All the wrapper function does is take in our Points and creates new Point\_3 objects, using them to invoke CGAL's circumcenter function. The returned Point\_3 is then used to create a Point, which is then sent back upstream to Julia.

On the Julia side of things, the process is much the same as before with the addition of a new Point type that contains three **Float64** fields. The previous example showed the latter Julia type's correspondence to the C/C++ **double** type. The Julia code for this example is listed in listing 3.5.

We are once again met with a very familiar snippet of code. Much like with its respective wrapper library, there is a new **struct Point** that mirrors the one we created in C++. From then on, the process is exactly the same.

So far, we were able to extract relatively useful functions from CGAL. In fact, the latter already solves the problem exemplified in a previous chapter in section 1.4.2. Although we could incrementally build upon this approach, not only does it become cumbersome, but it proves impractical, given CGAL's complexity.

Fortunately, the Julia community has explored methods of interoperating with many other languages, one of them being C++. That exploration resulted in packages like `CxxWrap.jl`<sup>9</sup>. `CxxWrap.jl` adopts an approach to language interoperation similar to that of `BOOST.Python` [77] or `pybind11`<sup>10</sup>. The user writes the code for the Julia wrapper in C++, and then simply issue an instruction on the Julia side to initialize the library, making it available there. The Julia package is supported by a C++ component known

<sup>8</sup>Note that, unlike the function, the **struct** was not externalized. This is not necessary because we do not need to refer it by name. We need only to match its field layout.

<sup>9</sup><https://github.com/JuliaInterop/CxxWrap.jl>

<sup>10</sup><https://github.com/pybind/pybind11>

---

```

1 const lib = joinpath(@__DIR__, "libcirc") # path to the compiled library
2
3 struct Point # julia equivalent struct
4     x::Float64
5     y::Float64
6     z::Float64
7 end
8
9 # julia wrapper function around C function
10 circumcenter(p1::Point, p2::Point, p3::Point) =
11     ccall((:circumcenter, lib) # qualified function name
12         , Point # return type
13         , (Point, Point, Point) # parameter types
14         , p1, p2, p3) # argument types
15
16 let p = Point(1,2,3), q = Point(1,1,1), r = Point(0,1,2)
17     @info "Circumcenter" p q r circumcenter(p,q,r) # = Point(1.0, 1.5, 2.0)
18 end

```

---

**Listing 3.5:** Example Julia program that invokes the functionality from the library listed in listing 3.4. We use an additional Julia struct that's equivalent to the one specified in C to facilitate data transfer.

as libcxxwrap-julia, or the friendlier name, JICxx. This component is what the C++ wrapper code depends on. Listing 3.6 constitutes the code to wrap necessary functionality to reproduce the example CGAL program in listing 3.1.

We direct our focus to the lines that are highlighted in the figure. We define a function that is later invoked by `CxxWrap.jl`. In it, we start registering the required types and functions that we need in a declarative fashion<sup>11</sup>, reminiscent of the builder design pattern. Notice the `JLCXX_MODULE` symbol preceding the function definition. That symbol takes care of properly externalizing the function regardless of the system the library is built for.<sup>12</sup>

After compiling the wrapper code, we can load it on the Julia side resorting to `CxxWrap.jl`. This process is reminiscent of and analogous to the ones illustrated earlier with simpler examples using `ccall`. Listing 3.7 shows a bare-bones CGAL Julia module.

All that is necessary to make the functionality we wrapped on the C++ side available is (1) tell `CxxWrap.jl` where the library is, (2) tell `CxxWrap.jl` to initialize itself when the module is loaded, and (3) export the mapped functionality. The rest of the non-highlighted code, both in this example and the previous only serves the purpose of obtaining a human-readable representation of the C++ objects in Julia.

Finally, we reach a point where we are able to reproduce the example listed in listing 3.1 in the Julia language, having mapped all the necessary functionality to do so. Listing 3.8 shows the example,

---

<sup>11</sup>Order with which types and functions are registered matters. This means we cannot add a function that depends on a type we did not yet register.

<sup>12</sup>Think similar to `extern "C"`, but slightly more robust.

**Listing 3.6:** C++ wrapper code powered by Jlcxx that maps the types and functions needed from CGAL to reproduce the example shown in listing 3.1 in Julia.

---

```

1 #include <CGAL/Exact_predicates_inexact_constructions_kernel.h> // Epick
2 #include <CGAL/enum.h> // Orientation, alias of Sign
3 #include <CGAL/IO/io.h> // set_pretty_mode
4
5 #include <jlcxx/jlcxx.hpp>
6
7 // helper for generating CGAL global function wrappers
8 #define WRAPPER(F) \
9     template<typename ...TS> \
10     inline auto F(const TS&... ts) { return CGAL::F(ts...); }
11
12 WRAPPER(midpoint)
13 WRAPPER(orientation)
14 WRAPPER(squared_distance)
15
16 template<typename T> // used in julia to pretty print types
17 std::string to_string(const T& t) {
18     std::ostringstream oss("");
19     CGAL::set_pretty_mode(oss);
20     oss << t;
21     return oss.str();
22 }
23
24 JLCXX_MODULE define_julia_module(jlcxx::Module& m) {
25     typedef CGAL::Epick Kernel;
26     typedef Kernel::Point_2 Point_2;
27     typedef Kernel::Segment_2 Segment_2;
28
29     // types
30     m.add_type<Point_2>("Point2")
31         .constructor<double, double>()
32         .method("x", &Point_2::x)
33         .method("y", &Point_2::y)
34         .method("_tostring", &to_string<Point_2>());
35
36     m.add_type<Segment_2>("Segment2")
37         .constructor<const Point_2&, const Point_2&>()
38         .method("_tostring", &to_string<Segment_2>());
39
40     m.add_bits<CGAL::Orientation>("Orientation", jlcxx::julia_type("CppEnum"));
41     m.set_const("COLLINEAR", CGAL::COLLINEAR);
42     m.set_const("LEFT_TURN", CGAL::LEFT_TURN);
43     m.set_const("RIGHT_TURN", CGAL::RIGHT_TURN);
44
45     // functions
46     m.method("midpoint", &midpoint<Point_2,Point_2>());
47     m.method("orientation", &orientation<Point_2,Point_2,Point_2>());
48     m.method("squared_distance", &squared_distance<Point_2,Point_2>());
49     m.method("squared_distance", &squared_distance<Segment_2,Point_2>());
50 }
```

---

---

```

1 module CGAL
2
3 using CxxWrap
4 export Point2, Segment2,
5     COLLINEAR, LEFT_TURN, RIGHT_TURN,
6     x, y, midpoint, orientation, squared_distance
7
8 @wrapmodule "$( __DIR__ )/libcgal_julia" # path to shared library
9
10 __init__() = @initcxx # initialize CxxWrap
11
12 # `x` and `y` return references, this pretty prints them
13 Base.show(io::IO, x::CxxWrap.CxxBaseRef{<:Real}) = print(io, x[])
14
15 for m in methods(CGAL._tostring) # for pretty printing
16     T = m.sig.parameters[2]
17     @eval Base.show(io::IO, x::$T) = print(io, _tostring(x))
18 end
19
20 end # CGAL

```

---

**Listing 3.7:** An example Julia module that mimics CGAL.jl, wrapping the library produced from listing 3.6. It initializes the library and exports the mapped functionality.

translated from C++.

We now have available some core functionality on which we can continue to incrementally build upon following a similar approach to that in listing 3.6. Doing so, with relatively low effort, will let us obtain the primitive objects on which our *Geometric Constraint Primitives* will be supported on. The final implementation section goes over how we effectively used the functionality from CGAL.jl to implement constructive solutions for GC problems. We showcase this approach with implementations of the example problems introduced in section 1.4.

### 3.1.3 Geometric Constraint Primitives

## 3.2 Trade-offs

Since virtually anything comes without trade-offs and compromises, it is paramount we address our implementation's qualities, negative and positive.

Relying on a library such as CGAL proves to be as great as it can be daunting. As mentioned in section 3.1.1, CGAL is a very comprehensive and mature software library, arguably even far exceeding our solution's needs, yet fitting it perfectly.

It is, however, an external component, and with every such component, we do not hold as much

---

```
1  using CGAL
2
3  p, q = Point2(1,1), Point2(10,10)
4
5  println("p = $p")
6  println("q = $(x(q)) $(y(q))")
7
8  println("sqdist(p,q) = $(squared_distance(p,q))")
9
10 s = Segment2(p,q)
11 m = Point2(5, 9)
12
13 println("m = $m")
14 println("sqdist(Segment2(p,q), m) = $(squared_distance(s, m))")
15
16 print("p, q, and m ")
17 let o = orientation(p,q,m)
18     if      o == COLLINEAR println("are collinear")
19     elseif o == LEFT_TURN println("make a left turn")
20     elseif o == RIGHT_TURN println("make a right turn")
21     end
22 end
23
24 println(" midpoint(p,q) = $(midpoint(p,q))")
```

---

**Listing 3.8:** The example program as seen in listing 3.1 written in the Julia programming language using CGAL.jl. The kernel instantiation is hidden away in the C++ layer of the wrapper code.

---

```

1  using Khepri
2  import CGAL: Point2, Segment2, to_vector
3
4  # implementation
5  parallel(l::Segment2, p::Point2) = Segment2(p, p + to_vector(l))
6
7  # conversion
8  parallel(l, p) = convert(Line, parallel(convert(Segment2, l)
9                           , convert(Point2, p)))
10
11 begin
12     backend(autocad)
13     delete_all_shapes()
14
15     with(current_cs, cs_from_o_phi(u0(), deg2rad(20))) do
16         A, B, C = u0(), x(3), xy(1,1)
17         v = .1(B - A) # small offset
18         AB = line(A - v, B + v)
19         parallel(AB, C - v)
20         surface_circle.((A, B, C), 3e-2)
21         text("A", add_pol(in_world(A), .3, -π/3), .2)
22         text("B", add_pol(in_world(B), .3, -π/3), .2)
23         text("C", add_pol(in_world(C), .3, -π/3), .2)
24     end
25 end

```

---

**Listing 3.9:** Implementation of the parallel lines example illustrated in fig. 1.2a using Khepri alongside our solution backed by CGAL.jl.

control over it if it was internal instead. For example, in the advent a bug is found within CGAL, one cannot *immediately* fix it by altering its source code and use this fixed version. Important emphasis on bugs are not *immediately* fixable lest we forget CGAL is still an Open Source project arguably anyone can contribute to. Alas, said contribution deployments are still out of our control. In hindsight, however, it can be considered just an inconvenience since it is a project that is actively maintained by certainly more knowledgeable people in the computer graphics and mathematics fields.

CGAL is also a highly generic library, making use and further abusing C++ templates. Although its design makes usage an elegant experience (as elegant as C++ can be), the same cannot be said with as much *gusto* when trying to wrap its constructs to another language, especially a language with different memory management paradigm which could lead to some nasty low-level ordeals.

Regarding our wrapper code, there is one small problem we still technically did not solve. We are still mapping CGAL types in an opaque fashion, pre-setting the kernel on the C++ side. Alternative ways of supplying different kernels were explored, including deploying a different library compiled with a different kernel. Despite not entirely solving the problem, it offered the user the choice of using exact constructions. However, this alternative became unmaintainable and virtually impossible to successfully

employ. In part due to Julia's package pre-compilation, it is no longer viable to attempt and use the other library.

Ideally, the geometric objects would be parametric, and the kernel (or kernels) mapped to Julia as well, but we were met with complications that would make it unfeasible for us to implement our solution. As a compromise, `CGAL.jl` fixed a kernel that provides exact predicates with inexact constructions, favoring performance over some robustness loss. Nonetheless, in practical terms, it suffices for our use case. In fact, we later evaluate how having followed such an approach can benefit other scenarios where more complex algorithms with GC properties can be obtained with little effort.

As an alternative to wrapping CGAL, we could have explored alternatives in the still growing Julia ecosystem. Some of the work seems promising,<sup>13</sup> but not only are some libraries still catching up, it is highly unlikely they would ever meet the quality of CGAL.

---

<sup>13</sup><https://github.com/JuliaGeometry>

---

```

1  using Khepri
2  import CGAL: Point2, circumcenter
3
4  # conversion
5  circumcenter(p, q, r) =
6      convert(Loc
7          , circumcenter(convert.(Point2, (p, q, r))...))
8
9  right_angle(p, q, r; scale=.2) =
10     with(current_cs, cs_from_o_vx_vy(q, p - q, r - q)) do
11         line(y(scale), xy(scale, scale), x(scale))
12     end
13
14 begin
15     backend(autocad)
16     delete_all_shapes()
17
18     A, B, C = u0(), xy(1, 3), x(4)
19     O = circumcenter(A, B, C)
20     AB = intermediate_loc(A, B)
21     AC = intermediate_loc(A, C)
22     BC = intermediate_loc(B, C)
23     line.((AB, AC, BC), O)
24     foreach(t -> right_angle(t...))
25         , ((A,AB,O), (B,BC,O), (C,AC,O)))
26     polygon(A, B, C)
27     circle(O, distance(O, A))
28     line(O, A)
29     surface_circle.((A, B, C, O), 3e-2)
30     text("r", intermediate_loc(O, A) + vy(.1), .2)
31     text("A", A + .2vxxy(-1, -1), .2)
32     text("B", B + .1vxxy(-2, 1), .2)
33     text("C", C + .1vxxy( 1, -2), .2)
34     text("O", O + .1vxxy( 1, -2), .2)
35 end

```

---

**Listing 3.10:** Implementation of the circumcenter example illustrated in fig. 1.2b using Khepri alongside our solution backed by CGAL.jl. In this particular case, we can leverage CGAL's facilities directly.



# 4

## Evaluation

### Contents

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---



In this chapter, we evaluate our solution by measuring the qualities of the approach we took to tackle GCS.

Firstly, we aim to benchmark our solution's performance by comparing it to a similar project called ConstraintGM [78]. We were able to reproduce the project's original benchmark tests and create an analogous test suite for our solution so we can compare both projects.

Secondly, we explore our approach's potential regarding repurposing more complex geometric algorithms with intrinsic GC relations in contrast with re-implementing a version of said algorithms from scratch. Specifically, we set out to repurpose CGAL's 2D Delaunay Triangulation and Voronoi Diagram algorithms and further compare the resulting mapping's performance and correctness when compared to a native Julia implementation of the algorithms as provided by the package VoronoiDelaunay.jl<sup>1</sup>, that, in the past, was once benchmarked against CGAL<sup>2</sup>. Additionally, we set out to estimate the effort it took to develop VoronoiDelaunay.jl and compare that to the effort it took to extract an analogous algorithm from CGAL.

Finally, we end our evaluation by showcasing four different case studies inspired by existing designs. This section of the evaluation focuses on comparing different approaches to solving GC problems present in each case study by adopting two different approaches: (1) an analytic approach, one programming naturally begs for, and (2) a constructive approach, essentially adding a conceptual abstraction layer over the former. We aim to show that, by following the latter approach, resulting programs become both easier to understand and reproducible as the set of instructions can be used to recreate the resulting geometry by hand, using a ruler and a compass.

Benchmarks were performed on a Lenovo® ThinkPad® E595 laptop computer with the following system specifications:

- AMD Ryzen™ 5 3500U CPU @ 2.1GHz<sup>3</sup>;
- 1×16GB SO-DIMM of DDR4 RAM @ 2400MT/s.
- Arch<sup>4</sup> Linux® x86\_64 Kernel 5.12.15-zen1<sup>5</sup>

---

<sup>1</sup><https://github.com/JuliaGeometry/VoronoiDelaunay.jl>

<sup>2</sup><https://gist.github.com/skariel/3d2018f9341a058e00fc>

<sup>3</sup>Base clock frequency. Can boost up to 3.7GHz.

<sup>4</sup><https://archlinux.org>

<sup>5</sup><https://github.com/zen-kernel/zen-kernel/tree/v5.12.15-zen1>

## 4.1 ConstraintGM

ConstraintGM is a domain-specific language developed with the shared goal of tackling GC problem specification using a TPL (Racket, to be precise). This solution heavily and blindly relied on Maxima [79], a generic Computer Algebra System (CAS) to solve GC problems. Geometric entities were serialized into their algebraic equation representation coupled with the problem's constraints and sent to Maxima. Maxima would then attempt to solve the system of equations and return a result that was parsed and sent back to Racket.

This approach came at a severe performance cost, the reason being two-fold: (1) the communication between ConstraintGM and Maxima was slow, and (2) Maxima is a *generic* solver and could not take advantage of the geometric characteristics of the problem at hand. The considerable performance penalty of this approach are hard to justify in the case of simple geometric problems for which there are well-known efficient solutions. This lead to an *impromptu* implementation of some GC problem solutions, creating the Geometry Functions Library (GFL). As expected, the latter approach revealed that contextually specialized solutions had much better performance than relying on a generic solver.

It is worth noting that relying on Maxima meant ConstraintGM was intrinsically exact and robust since symbolic GCS methods, used by Maxima, automatically provide those features. The GFL, when compared to the Maxima-based approach, is more fragile in that regard because it will depend on the underlying constructs' number type. If an exact arbitrary precision number type is used, exactness and robustness will be preserved, but at a performance cost since arbitrary precision arithmetic is computationally heavy. Relying on inexact number types can eventually lead to erroneous results if one is not careful to avoid error-inducing computations, but, by contrast, offers better performance. That said, some computations are unavoidable, such as the computation of the squared root, for example, when the effective distance is necessary to operate with. In the end, it is a matter of making a compromise and evaluating which fits the case at end the best.

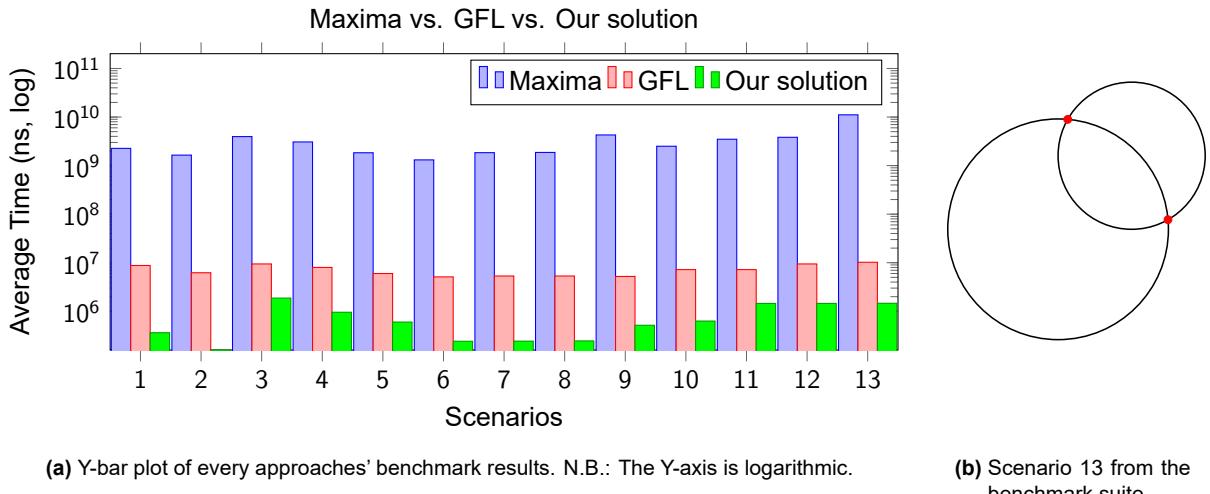
The project's benchmark involve three different GC problems focused around object intersection, namely (1) line-line intersection, (2) circle-line intersection, and (3) circle-circle intersection. These problems expanded into thirteen different scenarios that consisted of rearranging the geometric entities' disposition in order to evaluate the intersection operation's results and measure each individual scenario's performance.

We measured real execution time instead of CPU time both for ConstraintGM and for our solution. We ran ConstraintGM's benchmarks a total of nine times to obtain a relatively decent sample of results, while our solution's benchmarks were aided by BenchmarkTools.jl [80], a package that considerably facilitates benchmarking Julia code tremendously. The source code used to benchmark both ConstraintGM and our solution is listed in listings A.1 and A.2 respectively. The benchmark's results are gathered in table 4.1. To facilitate comprehension, these are also plotted in fig. 4.1.

**Table 4.1:** Performance comparison between ConstraintGM’s solutions, both using Maxima and GFL, and our solution.

| Scenario | Execution time (mean $\pm \sigma$ ) |                                |  | Our solution                         | GC problem           |
|----------|-------------------------------------|--------------------------------|--|--------------------------------------|----------------------|
|          | Maxima                              | GFL                            |  |                                      |                      |
| 1        | 2.265 s $\pm$ 142.344 ms            | 8.778 ms $\pm$ 1.641 ms        |  | 362.191 $\mu$ s $\pm$ 4.841 $\mu$ s  | Line $\cap$ Line     |
| 2        | 1.645 s $\pm$ 197.824 ms            | 6.222 ms $\pm$ 440.959 $\mu$ s |  | 159.567 $\mu$ s $\pm$ 4.634 $\mu$ s  |                      |
| 3        | 3.958 s $\pm$ 295.673 ms            | 9.444 ms $\pm$ 1.333 ms        |  | 1.871 ms $\pm$ 4.892 ms              |                      |
| 4        | 3.070 s $\pm$ 243.768 ms            | 8.000 ms $\pm$ 0.000 ns        |  | 955.815 $\mu$ s $\pm$ 3.710 ms       | Circle $\cap$ Line   |
| 5        | 1.839 s $\pm$ 84.460 ms             | 6.000 ms $\pm$ 0.000 ns        |  | 600.386 $\mu$ s $\pm$ 19.807 $\mu$ s |                      |
| 6        | 1.311 s $\pm$ 59.139 ms             | 5.111 ms $\pm$ 333.333 $\mu$ s |  | 241.531 $\mu$ s $\pm$ 12.504 $\mu$ s |                      |
| 7        | 1.847 s $\pm$ 50.466 ms             | 5.333 ms $\pm$ 500.000 $\mu$ s |  | 242.780 $\mu$ s $\pm$ 5.692 $\mu$ s  |                      |
| 8        | 1.868 s $\pm$ 37.650 ms             | 5.333 ms $\pm$ 500.000 $\mu$ s |  | 245.425 $\mu$ s $\pm$ 11.840 $\mu$ s |                      |
| 9        | 4.277 s $\pm$ 37.053 ms             | 5.222 ms $\pm$ 440.959 $\mu$ s |  | 515.829 $\mu$ s $\pm$ 4.881 ms       | Circle $\cap$ Circle |
| 10       | 2.506 s $\pm$ 238.696 ms            | 7.222 ms $\pm$ 440.959 $\mu$ s |  | 629.341 $\mu$ s $\pm$ 5.360 ms       |                      |
| 11       | 3.493 s $\pm$ 258.715 ms            | 7.222 ms $\pm$ 440.959 $\mu$ s |  | 1.453 ms $\pm$ 5.070 ms              |                      |
| 12       | 3.830 s $\pm$ 150.402 ms            | 9.444 ms $\pm$ 527.046 $\mu$ s |  | 1.455 ms $\pm$ 5.088 ms              |                      |
| 13       | 11.111 s $\pm$ 81.302 ms            | 10.222 ms $\pm$ 1.093 ms       |  | 1.463 ms $\pm$ 5.055 ms              |                      |

Observing the results, we can see the disparity between the approach reliant on Maxima when compared to both the GFL and our solution, which was to be expected. Even so, amidst relatively consistent results, scenario 13 made Maxima slug more than usual. That scenario consists of two circles intersecting at two different points, illustrated in fig. 4.1b. It is not the only scenario of the set that involves two circles that intersect. However, this is the one that produces relatively more complex results, which could justify why Maxima took relatively longer to compute the solution than it did for the scenarios that preceded this one.



**Figure 4.1:** ConstraintGM benchmark results collected from table 4.1 in a multi-series plot (a) beside the depiction of scenario 13 (b) from the benchmark suite.

Analyzing the results further, we can see our solution also outdoes ConstraintGM’s GFL by a significant margin. This is most likely due to the fact that we are relying on CGAL, a library implemented in C++. The latter is notoriously known for being a high-performance language, considerably outperforming

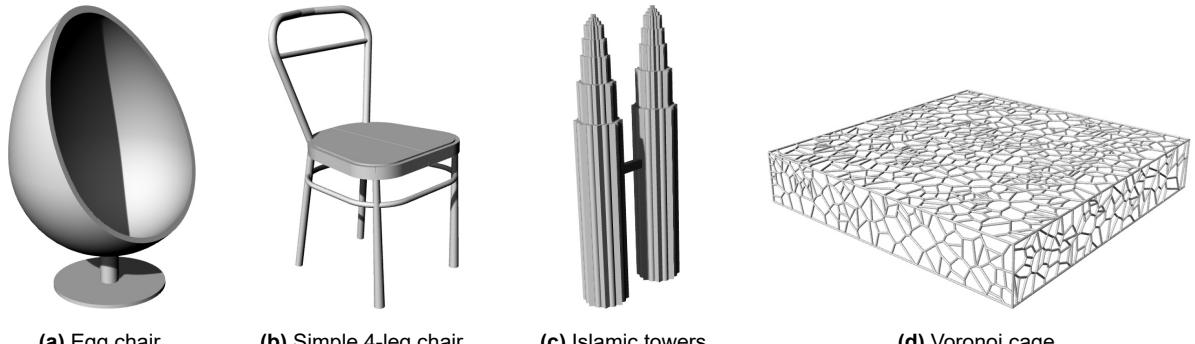
Racket in a series of benchmarks. Nevertheless, despite some overhead in the case of Julia, the results are still positive, on average beating the GFL by an order of magnitude.

In conclusion, our solution proves capable and performant, having surpassed ConstraintGM's GFL by an entire order of magnitude on average.

## 4.2 VoronoiDelaunay.jl

## 4.3 Case Studies

In this section, we aim to demonstrate our solution when applied to four different case studies, each presenting a parametric geometric shape: an egg, a rounded trapezoid, a star with semicircles, and a Voronoi diagram. Each case, illustrated in fig. 4.2, was inspired by an existing design: (1) Eero Aarnio's Egg chair, (2) Thonet 214 chair seat, (3) César Pelli's Petronas tower section, and (4) PTW Architects' Beijing National Aquatics Center. These cases present a set of GC problems involving circles and lines, such as *tangent line to two circles* and *circle-line intersection*. These problems were solved employing both an *analytic* approach, an approach TPLs naturally demand, and a *constructive* approach, the one made possible by relying on our solution.



(a) Egg chair.

(b) Simple 4-leg chair.

(c) Islamic towers.

(d) Voronoi cage.

**Figure 4.2:** Case study designs inspired by the Eero Aarnio's Egg chair (a), Thonet 214 chair (b), César Pelli's Petronas Twin Towers (c), and PTW Architects' Beijing National Aquatics Center (d).

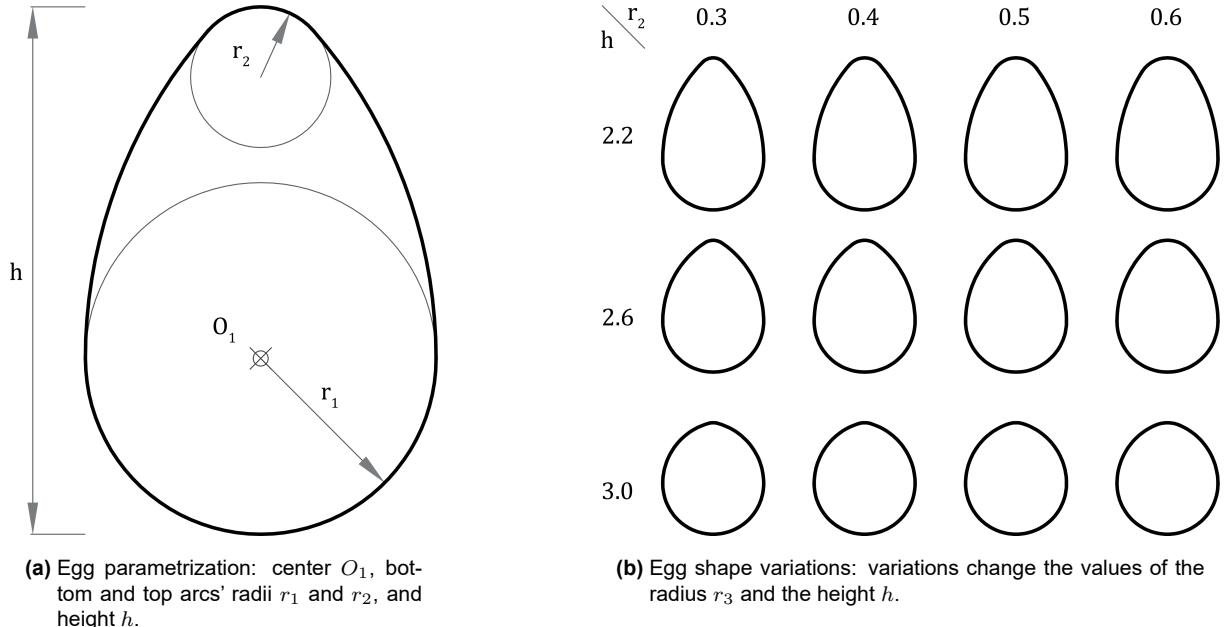
### 4.3.1 Egg

The first case study is an egg shape (which can be considered a particular case of an oval shape). Examples of applications of this shape in design include the Eero Aarnio's Egg chair, James Law's Cybertecture Egg building, and RAU Architects and Ro&Ad Architects' Tij observatory.

The egg shape is a broad concept, as it can refer to a wide range of curves resembling an egg [81]. In this case, our shape is bilaterally symmetric and is composed of four arcs, the side arcs being tangent

to the bottom and top arcs.

Our parametrization of the egg defines four parameters: the bottom arc's center  $O_1$ , the bottom and top arcs' radii  $r_1$  and  $r_2$  respectively, and the egg's height  $h$  (fig. 4.3a). Different egg shapes, more or less elongated, can be obtained by varying the parameters' values (fig. 4.3b). One of those variations, where  $r_2 = (2 - \sqrt{2})r_1$  and  $h = 2r_1 + r_2$ , corresponds to the Moss' Egg (fig. 4.3, the egg with  $h = 2.6$  and  $r_2 = 0.6$ ).



**Figure 4.3:** Egg problem: (a) shows our parametrization of the egg which can be used to generate shape variations, some of them shown in (b).

In this case, the major problem was to define the side arc  $A_3$ , which is given by the center  $O_3$ , radius  $r_3$ , and amplitude  $\alpha$  (fig. 4.4a). The *analytic solution* is described below. The angle  $\alpha$  and the radius  $r_3$  are devised from the triangle  $\triangle O_1O_2O_3$  (fig. 4.4b), and the center  $O_3$  is given by a translation of  $O_1$  through the unit vector  $\vec{e}_x$ , parallel to the X-axis, scaled by the factor  $r_1 - r_3$ . It is important to note that, despite their simplicity, the derivations needed to arrive at the following formulas are not obvious.

$$1. \alpha = 2 \arctan \frac{r_1 - r_2}{h - r_1 - r_2}$$

$$2. r_3 = \frac{r_1 - r_2 \cos \alpha}{1 - \cos \alpha}$$

$$3. O_3 = O_1 + (r_1 - r_2) \vec{e}_x$$

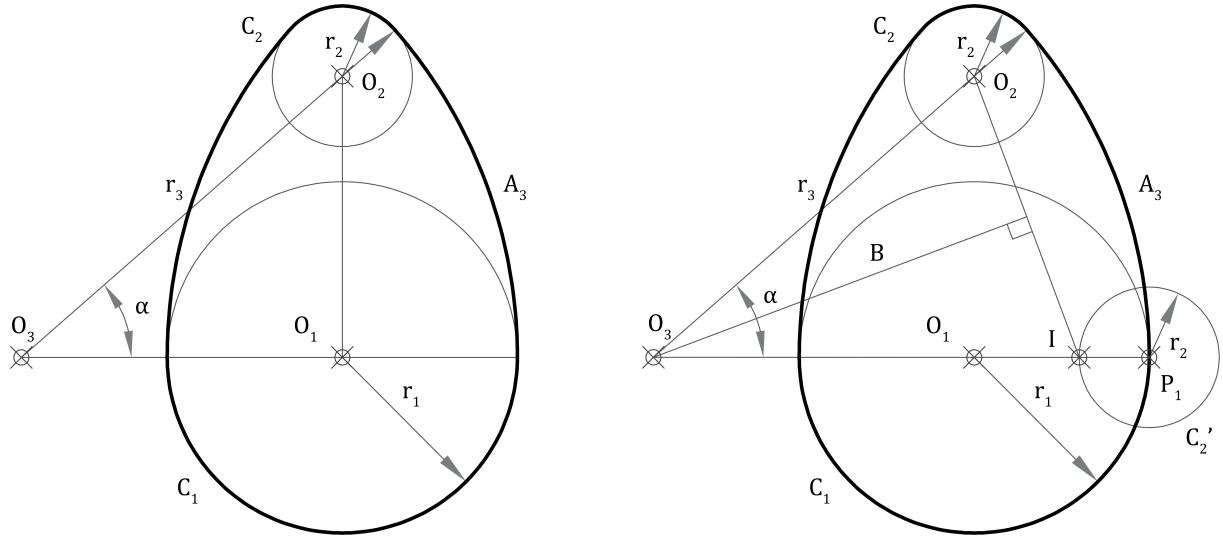
The *constructive solution* is based on the geometric problem of determining the *tangent line to two circles*. In fact, the side arc  $A_3$  is tangent to two circles,  $C_1$  and  $C_2$ , and passes through point  $P_1$ . Two GC primitives from our solution were employed in this solution, namely *intersection* and *bisector*. The solution can be computed according to the following procedure (fig. 4.4b):

1.  $C'_2 = \text{circle}(P_1, r_2)$
2.  $I = \text{intersection}(C'_2, \overline{O_1 P_2})$
3.  $B = \text{bisector}(\overline{IO_2})$
4.  $O_3 = \text{intersection}(B, O_1 P_1)$
5.  $r_3 = \text{length}(\overline{O_3 P_1})$
6.  $\alpha = \text{angle}(O_2, O_3, O_1)$

We can further increase the expressive level of the solution by defining the `tangent_circle` functionality. Given two circles,  $C_1$  and  $C_2$ , and a point  $P_1$  on  $C_1$ , `tangent_circle` produces the circle tangent to both  $C_1$  and  $C_2$  that goes through  $P_1$ . This way, the solution can be simplified by replacing steps 1 through 5 by using the `tangent_circle` functionality as follows:

1.  $O_3, r_3 = \text{tangent_circles}(C_1, C_2, P_1)$
2.  $\alpha = \text{angle}(O_2, O_3, O_1)$

In the egg's case,  $C_1$  must be larger than  $C_2$ , and  $P_1$  is one of the intersection points between the horizontal line that passes through  $O_1$  and the circle  $C_1$ .

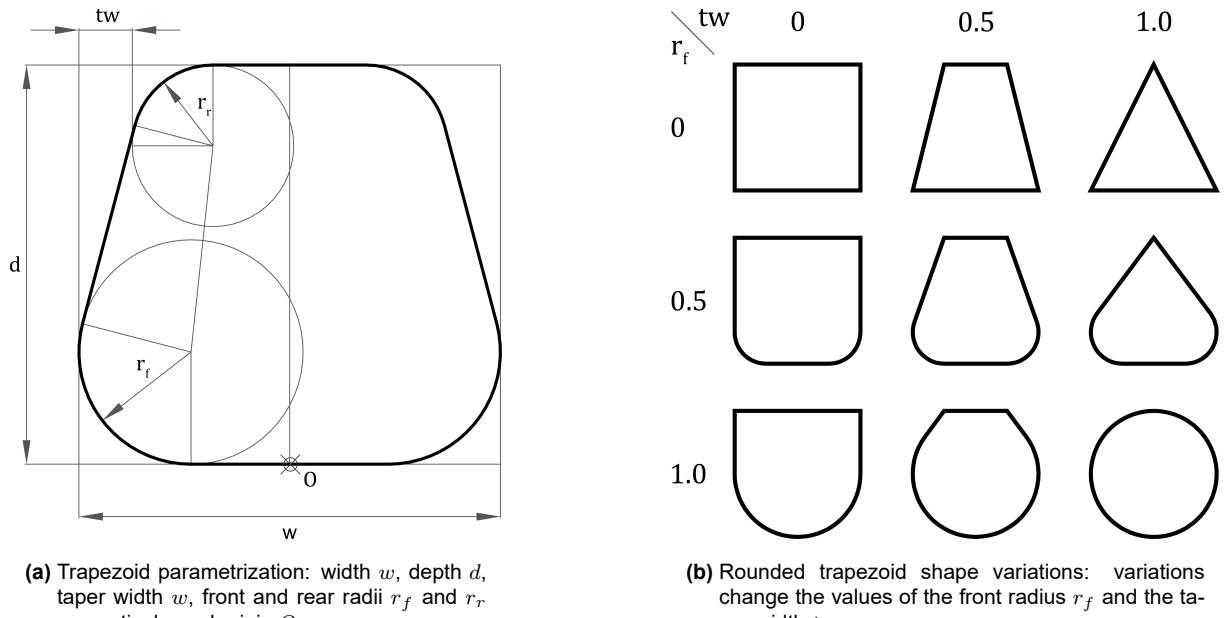


**Figure 4.4:** Solutions to the Egg problem using two different approaches.

Achieving the equations in the *analytic solution* is not a straightforward task for designers, since it involves considerable logical reasoning and the use of non-trivial formulas (such as trigonometric half-angle identities). It is also unclear how those equations were derived. By contrast, in the *constructive solution*, all the steps are clearly externalized, which makes it much more comprehensible.

### 4.3.2 Rounded Trapezoid

The second case study is a rounded trapezoid shape. This parametric shapes is used in the contour of a variety of different chair seats, such as the Thonet 214 and the Zig Zag chairs [82]. This shape is bilaterally symmetric and is defined by six parameters: width  $w$ , depth  $d$ , taper width  $tw$ , front radius  $r_f$ , rear radius  $r_r$ , and origin  $O$  (fig. 4.5a). The taper width and front and rear radii are ratios of the width and the depth. By varying these parameters' values, one can obtain different shapes, such as a square, a trapezoid, a triangle, a rounded rectangle, a drop-like shape, and a circle, among others (fig. 4.5b).



(a) Trapezoid parametrization: width  $w$ , depth  $d$ , taper width  $tw$ , front and rear radii  $r_f$  and  $r_r$  respectively, and origin  $O$ .

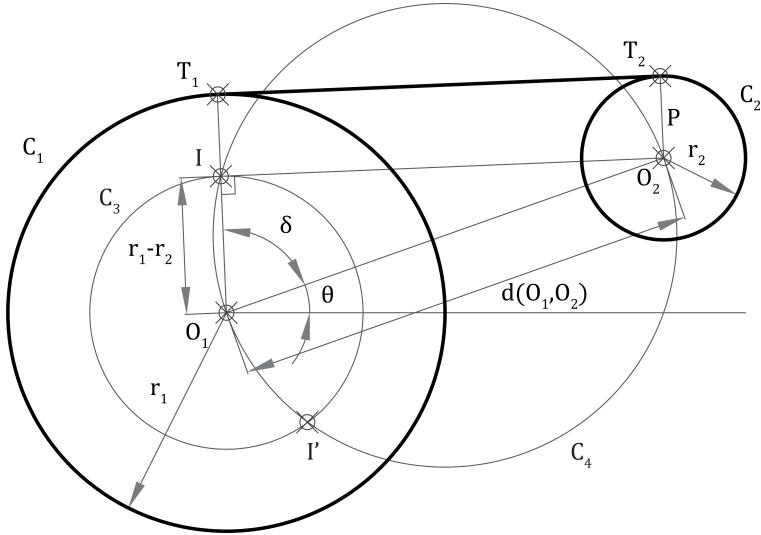
(b) Rounded trapezoid shape variations: variations change the values of the front radius  $r_f$  and the taper width  $tw$ .

**Figure 4.5:** Rounded trapezoid problem: (a) shows our parametrization of the trapezoid which can be used to generate shape variations, some of them shown in (b).

One side of the chair is obtained by two circles and the *tangent line to two circles*, while the other side is a reflection of the former side. Given two circles  $C_1$  and  $C_2$ , centered on  $O_1$  and  $O_2$  with radii  $r_1$  and  $r_2$  respectively (fig. 4.6), two solutions can be delineated that can accurately reproduce the tangent line  $\overline{T_1 T_2}$ .

The *analytic solution* is achieved by calculating the angle  $\delta$  between the line segment  $\overline{O_1 O_2}$  and the line perpendicular to the tangent line between the two circles passing through  $O_1$ . The angle  $\theta$  is given by the slope of  $\overline{O_1 O_2}$ . The point  $T_1$  is given by a translation of  $O_1$  through a vector  $(r_1, \angle\delta + \theta)$  where  $r_1$  is its length and  $\angle\delta + \theta$  is its polar angle. A similar approach is used to obtain  $T_2$ .

1.  $\delta = \arccos \frac{r_1 - r_2}{\|O_2 - O_1\|}$
2.  $\theta = \angle \overrightarrow{O_2 - O_1}, \vec{e}_x$
3.  $T_2 = O_1 + (r_1, \angle\delta + \theta)$



**Figure 4.6:** Both analytic and constructive solutions to the rounded trapezoid problem.

4.  $T_2 = O_2 + (r_1, \angle\delta + \theta)$

The *constructive solution* follows a sequence of steps that reflects the geometric characteristics of the problem (fig. 4.6). Four GC primitives were employed in this solution, namely *midpoint*, *intersection*, and *parallel lines*.

1.  $C_3 = \text{circle}(O_1, r_1 - r_2)$

2.  $M = \text{midpoint}(O_1, O_2)$

3.  $d = \text{length}(\overline{O_1O_2})$

4.  $C_4 = \text{circle}(M, \frac{d}{2})$

5.  $I, I' = \text{intersection}(C_3, C_4)$

6.  $P = \text{parallel}(\overline{IO_1}, O_2)$

7.  $T_1 = \text{intersection}(IO_1, C_1)$

8.  $T_2 = \text{intersection}(P, C_2)$

Despite the adequacy of the *analytic solution* for the design problem at hand, it only produces the external tangent needed to solve this specific problem, which works well if the circle's center is on the 1<sup>st</sup> and 2<sup>nd</sup> quadrants: however, in the remaining quadrants, it produces an unintended tangent. In contrast, our *constructive approach* is capable of producing a solution independently of how the circles are arranged.

The advantage of using well-established functions from CGAL is that they deal with degenerate cases better than re-implementations of the same functionality from scratch. For instance, in the case of concentric circles, the intersection between those circles does not produce an error; instead, it produces an empty result. In the case of the proposed *analytic solution*, the distance between the circles' centers is zero, which leads to a division-by-zero error.

Architects and designers are used to manipulating GCs, such as *tangency*, *parallelism*, and *intersection*, and less inclined to deal with the mathematical intricacies of analytic geometry, which makes the second approach more appealing to them. To that end, we can introduce the `tangentlines` functionality, which, given two circles  $C_1$  and  $C_2$ , produces a sequence of tangent lines between the circles, allowing the user to select the lines that best suit the problem. This allows us to reduce the solution to the single step

$$\overline{T_1T_2}, \dots = \text{tangent}_{\text{lines}}(C_1, C_2)$$

where  $\overline{T_1T_2}$  is one of the lines of the sequence.

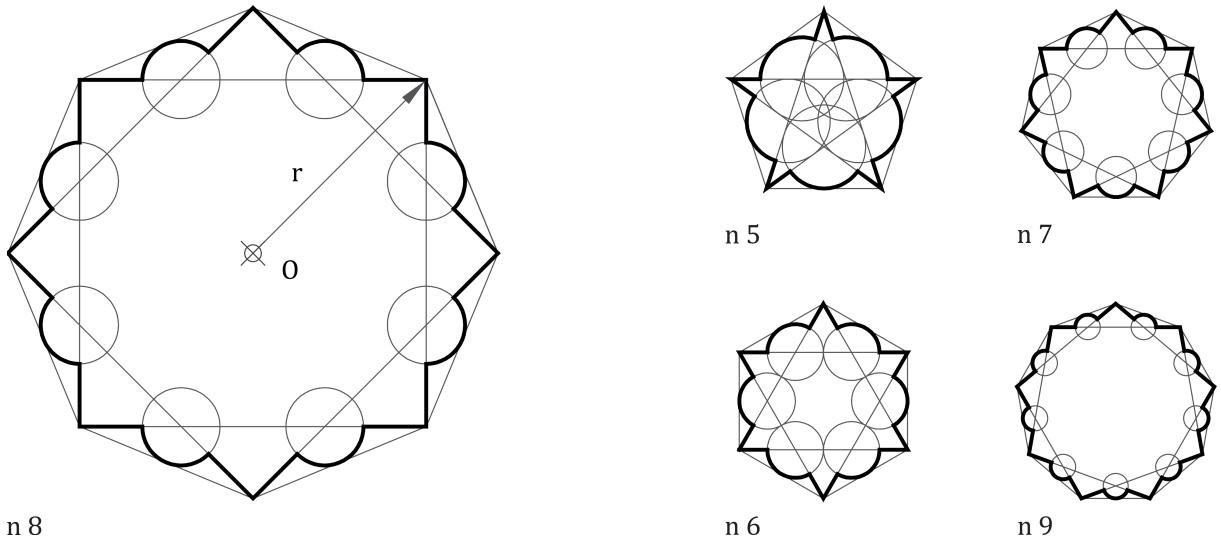
Depending on how the circles are arranged, there can be multiple solutions: (1) no segments, if one circle contains the other, (2) two segments, if the circles intersect, or (3) four segments, if the circles are disjoint. The advantage of having the `tangentlines` functionality is that it is capable of finding every solution to the generic *tangent line to two circles* problem. Hence, the user can reutilize this functionality each time a different problem of a similar nature arises.

### 4.3.3 Star with Semicircles

The third case study is a star shape with semicircles. This shape was inspired by César Pelli's Petronas tower floor plan, which in turn mimics Islamic patterns. The contour of the Petronas tower floor plan is formed by two overlapping congruent squares, forming an octagram known as Star of Lakshmi, and by eight circles centered on each of the eight intersection points and tangent to the bounding octagon. This shape can be generalized to a parametric shape defined by three parameters: origin  $O$ , radius  $r$ , and number of vertices  $n$  (fig. 4.7a). Note that the number of vertices cannot be less than five. Five variations, comprising stars with 5 to 9 vertices, are illustrated in fig. 4.7.

Both analytic and constructive solutions are based on computing one side of the star, composed of the line segment  $\overline{V_1I_1}$ , the arc centered on  $O_1$  from  $I_1$  to  $I_2$ , with radius  $r_1$ , and the line segment  $\overline{I_2V_2}$  (fig. 4.8). The remaining sides result from the recursive application of the preceding side with a rotation transformation around the center.

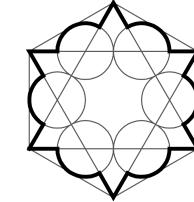
The *analytic solution* is described below. The radius  $r_1$  of the circle  $C_1$  is calculated by step 1, the radius  $r_2$  is defined by step 2, and its center  $O_1$  is given by step 3. The intersection points  $I_1$  and  $I_2$  are given by a rotation around  $O_1$  (fig. 4.8).



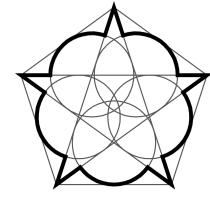
n 8

**(a)** Star parametrization: origin  $O$ , radius  $r$ , and number of vertices  $n$ .

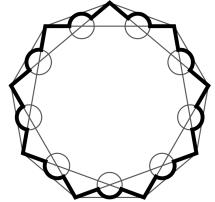
n 5



n 6



n 7



n 9

**(b)** Star shape variations: variations change the values of the number of vertices  $n$ .

**Figure 4.7:** Star with semicircles problem: (a) shows our parametrization of the star which can be used to generate shape variations, some of them shown in (b).

$$1. \quad r_1 = r \frac{\sin^2 \frac{\pi}{n}}{\cos \frac{\pi}{n}}$$

$$2. \quad r_2 = r \cos \frac{\pi}{n} - r_1$$

$$3. \quad O_1 = O + (r_2, \angle \frac{\pi}{n})$$

$$4. \quad I_1 = O_1 + (r_1, \angle \frac{2\pi}{n} - \frac{\pi}{2})$$

$$5. \quad I_2 = O_1 + (r_1, \angle \frac{\pi}{2})$$

The *construction solution* is based on finding the position and size of the circle  $C_1$  (see fig. 4.8). Two GC primitives were employed in this solution, namely *intersection*, and *tangent circle to one line*.

$$1. \quad O_1 = \text{intersection}(\overline{V_1 V_3}, \overline{V_2 V_n})$$

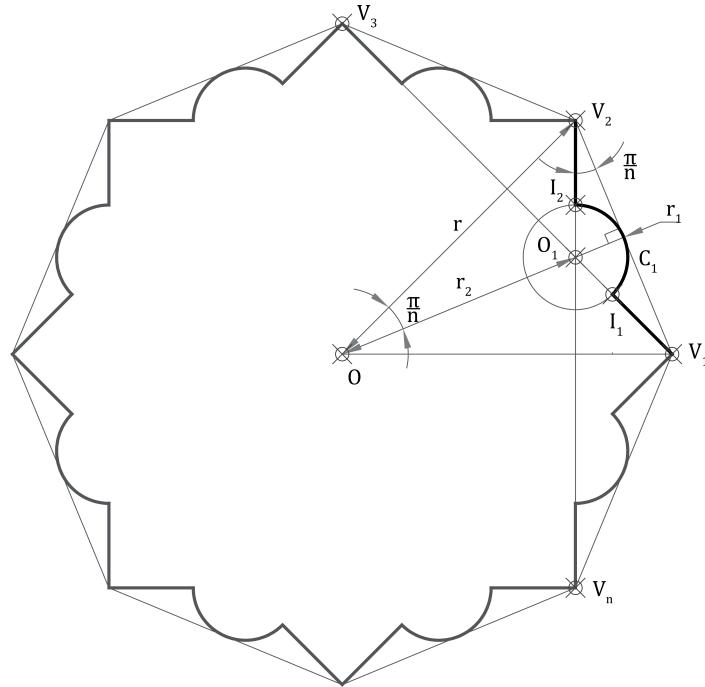
$$2. \quad C_1 = \text{tangent}_{\text{circle}}(O_1, \overline{V_1 V_2})$$

$$3. \quad P, r_1 = C_1$$

$$4. \quad I_1 = \text{intersection}(\overline{V_1 V_3}, C_1)$$

$$5. \quad I_2 = \text{intersection}(\overline{V_2 V_n}, C_1)$$

As seen in the other cases, the constructive solution is more understandable, as one can easily reproduce it step-by-step by hand, using a ruler and a compass.



**Figure 4.8:** Both analytic and construction solutions to the star with semicircles problem.

#### 4.3.4 Voronoi Diagram

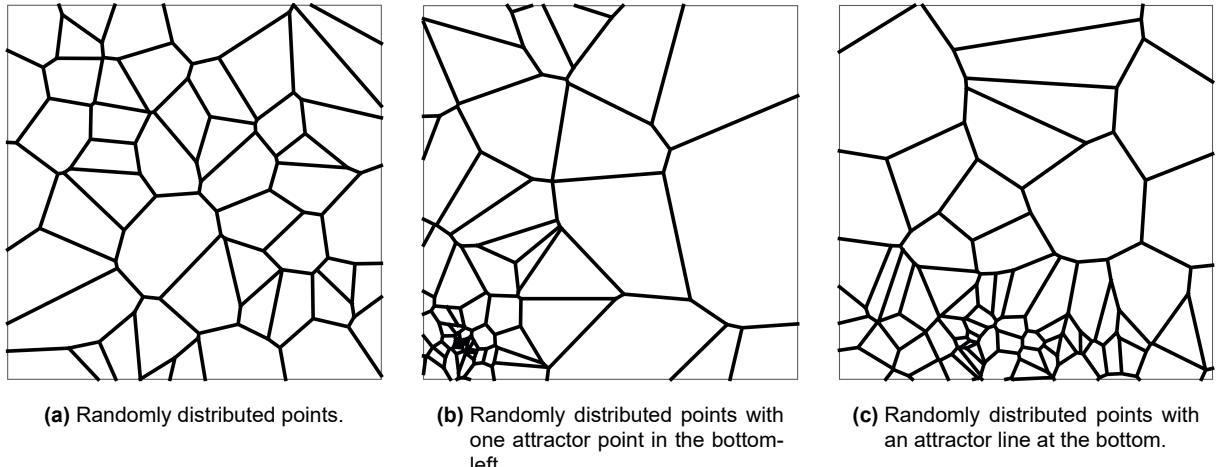
Our fourth case study is that of Voronoi diagrams, which are used in a variety of design fields. For instance, several facade designs exhibit a Voronoi appearance, such as PTW Architects' Beijing National Aquatics Center, ARM<sup>6</sup> Architecture's Melbourne Recital Centre, and Hassell's Alibaba Headquarters.

In its simplest version, a Voronoi diagram consists of partitioning a plane into regions from a set of points, called *sites*; for every site, each region contains every point in the plane closer to that site than to any other site. The sites are typically randomly distributed points, although they can follow other distributions. Figure 4.9 shows three Voronoi diagrams generated from entirely randomly distributed points (fig. 4.9a), from random points with one attractor point in the bottom-left corner (fig. 4.9b), and from random points with one attractor line at the bottom edge (fig. 4.9c). An attractor object is responsible for controlling the density of random points based on the distance to it.

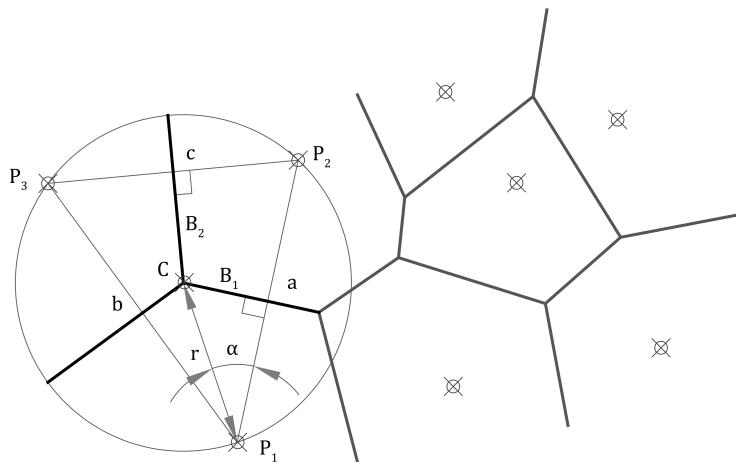
Both the analytic and constructive methods focus on computation of a vertex relies on the computation of the *circumcenter* of a triangle, for instance, triangle  $\triangle P_1P_2P_3$  (fig. 4.10).

There are several possible *analytic solution* to compute a triangle's circumcenter. One of them is based on the *circumradius* formula (step 4), where  $a$ ,  $b$ , and  $c$  are the lengths of the triangle's sides (step 1) and  $A$  is the triangle's area (step 3). The circumcenter  $C$  can then be easily computed by a translation (step 6) from  $P_1$  following the angle  $\alpha$  (step 5).

<sup>6</sup><https://armarchitecture.com.au/projects/melbourne-recital-centre/>. Not to be mistaken with the ARM family of computing architectures for computer processors.



**Figure 4.9:** Voronoi diagram problem: depicted are three Voronoi diagrams variations which are mostly generated at random. (a) is entirely random, (b) expands on the latter with an attractor point, and (c) goes even further by using an entire attractor line.



**Figure 4.10:** Analytic and constructive approaches to computing a Voronoi vertex via computing the circumcenter of every triangle in the Delaunay triangulation whose vertices are the Voronoi diagram's sites.

1.  $a, b, c = \|P_2 - P_1\|, \|P_3 - P_1\|, \|P_3 - P_2\|$
2.  $s = \frac{a+b+c}{2}$
3.  $A = \sqrt{s(s-a)(s-b)(s-c)}$
4.  $r = \frac{abc}{4A}$
5.  $\alpha = \arccos \frac{a}{2r}$
6.  $C = P_1 + (r, \angle \alpha)$

The *constructive solution* computes the circumcenter, given by the intersection of the edges' perpendicular bisectors. In fact, this has been exemplified in chapter 1. Two GC primitives were employed in

this solution, namely *bisector* and *intersection*.

$$1. B_1 = \text{bisector}(\overline{P_1 P_2})$$

$$2. B_2 = \text{bisector}(\overline{P_2 P_3})$$

$$3. C = \text{intersection}(B_1, B_2)$$

We can increase the abstraction level of the solution by using the *circumcenter* functionality implemented in our solution, which, in reality, is implemented in CGAL:

$$1. C = \text{circumcenter}(P_1, P_2, P_3)$$

The circumcenter problem is only a small part of the generation of a Voronoi diagram, since we first need to build a Delaunay triangulation. We can then apply the *circumcenter* functionality to find the Voronoi vertices and draw the diagram's edges. This set of steps can be abstracted away in a functionality called *voronoi* that, given a set of points  $P_S$ , produces a 2D Euclidean Voronoi diagram:

$$1. V = \text{voronoi}(P_S)$$

Implementing this functionality from scratch is a demanding and error-prone task. Fortunately, CGAL already has an algorithm that produces robust 2D Euclidean Voronoi diagrams that can handle degenerate cases, such as dealing with three or more collinear points. This algorithm, much like the *circumcenter* functionality, was entirely repurposed, and made available in CGAL.jl, and, thus, it is also available in our solution.



# 5

## Conclusion



The generation of highly constrained sophisticated designs is not viable through usage of interactive interfaces due to rigidity in the manipulation of existing models in order to generate multiple variants, or through VPLs because of the disproportionate relation between the resulting workflow and respective design complexity. However, working with geometric constraints in TPLs imposes a set of challenges, which can be overcome through the usage of GCS approaches to solve complex systems of constraints. To achieve that goal, several methods can be employed, but they mostly resort to generic GCS algorithms, but solvers, in general, have difficulty in identifying specific underlying subproblems for which efficiently computable and robust solutions might be available.

The prior analysis of the set of geometric constraints that must be dealt with, nonetheless, requires certain background knowledge on numerical robustness to mitigate fixed-precision arithmetic issues, such as *roundoff* error accumulation throughout calculation, as well as investigation on how to solve these specific constraint problems. The user will end up having to spend more time and effort in this process than in the design process itself.

Thus, in order to overcome these obstacles, an alternative approach is proposed in the form of the implementation of geometric constraint primitives in an expressive TPL supported by an exact geometric computation library. The latter provides a series of optimized geometrical algorithms and exact data structures that allow transparent handling of robustness issues, lifting this concern from the user's shoulders with the goal of improving constrained geometry specification efficiency as well as consequently facilitating the design process.



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# ConstraintGM

**Listing A.1:** Source code from ConstraintGM's benchmarks for thirteen different scenarios involving the intersection of differently arranged geometric entities.

Adapted from: <https://bitbucket.org/FabioPinheiro/geometric-constraints/src/master> (July 2021)

```
1 #lang racket
2 (require "../core/entities-declaration.rkt"
3          "../mathematical-module/geometric-restrictions.rkt"
4          "../core/dispatcher.rkt"
5          "../core/communication.rkt")
6
7 (define (benchmark-dispatcher e1 e2 [times 1000])
8   (displayln "# new benchmark"))
9   (define seq (stream->list (in-range 0 times 1)))
10  (collect-garbage)
11  (time (for ([i seq]) (intersection-dispatcher e1 e2)))
12  (collect-garbage)
13  (time (for ([i seq]) (intersection e1 e2))))
14
15 (define l-l
16   (list (cons (new-line (new-point 7 10) (new-point 7 -10))
17                (new-line (new-point 0 5) (new-point 15 5)))
18          (cons (new-line (new-point 1 0) (new-point 3 0))
19                (new-line (new-point 1 1) (new-point 3 1)))))
```

```

20 (define c-l
21   (list (cons (new-circle (new-point 0 0) 1)
22                (new-line (new-point -5 0) (new-point 5 0)))
23         (cons (new-circle (new-point 0 0) 1)
24                (new-line (new-point -5 1) (new-point 5 1))))
25         (cons (new-circle (new-point 0 0) 0.5)
26                (new-line (new-point -5 1) (new-point 5 1))))))
27 (define c-c
28   (list (cons (new-circle (new-point 0 0) 1)
29                (new-circle (new-point 0 0) 2))
30         (cons (new-circle (new-point 0 0) 10)
31                (new-circle (new-point 1 0) 1.5)))
32         (cons (new-circle (new-point 100 0) 1)
33                (new-circle (new-point 10 0) 2)))
34         (cons (new-circle (new-point 1 -2) 3)
35                (new-circle (new-point 1 -2) 3)))
36         (cons (new-circle (new-point 0 0) 1)
37                (new-circle (new-point 0 2) 1)))
38         (cons (new-circle (new-point 0 0) 1.5)
39                (new-circle (new-point 1 0) 1.5)))
40         (cons (new-circle (new-point 0 0) 1.5)
41                (new-circle (new-point 1 0) 1)))
42         (cons (new-circle (new-point 0 0) 1.5)
43                (new-circle (new-point 1 1) 1)))))

44
45 (displayln "### Time for start-maxima ###")
46 (time (start-maxima))
47 (displayln "### intersection-line-line ###")
48 (for ([i l-l]) (benchmark-dispatcher (car i) (cdr i)))
49 (displayln "### intersection-circle-line ###")
50 (for ([i c-l]) (benchmark-dispatcher (car i) (cdr i)))
51 (displayln "### intersection-circle-circle ###")
52 (for ([i c-c]) (benchmark-dispatcher (car i) (cdr i)))

```

---

**Listing A.2:** Benchmark code testing the same scenarios benchmarked by ConstraintGM using our solution.

---

```

1 import Base.Iterators: flatten
2
3 using BenchmarkTools
4 using CGAL
5
6 const ss = [
7     Segment2(Point2(7, 10), Point2(7, -10)) =>
8     Segment2(Point2(0, 5), Point2(15, 5)),
9     Segment2(Point2(1, 0), Point2(3, 0)) =>
10    Segment2(Point2(1, 1), Point2(3, 1)),
11 ]
12 const cs = [
13     Circle2(Point2(0, 0), 1)      => Segment2(Point2(-5, 0), Point2(5, 0)),
14     Circle2(Point2(0, 0), 1)      => Segment2(Point2(-5, 1), Point2(5, 1)),
15     Circle2(Point2(0, 0), 0.5^2) => Segment2(Point2(-5, 1), Point2(5, 1)),
16 ]
17 const cc = [
18     Circle2(Point2(0, 0), 1)      => Circle2(Point2(0, 0), 2^2),
19     Circle2(Point2(0, 0), 10^2)   => Circle2(Point2(1, 0), 1.5^2),
20     Circle2(Point2(100, 0), 1)   => Circle2(Point2(10, 0), 2^2),
21     Circle2(Point2(1, -2), 3^2)  => Circle2(Point2(1, -2), 3^2),
22     Circle2(Point2(0, 0), 1)      => Circle2(Point2(0, 2), 1),
23     Circle2(Point2(0, 0), 1.5^2) => Circle2(Point2(1, 0), 1.5^2),
24     Circle2(Point2(0, 0), 1.5^2) => Circle2(Point2(1, 0), 1),
25     Circle2(Point2(0, 0), 1.5^2) => Circle2(Point2(1, 1), 1),
26 ]
27
28 const scenarios = flatten([ss, cs, cc])
29
30 sample(a, b; times = 1000) = for i ∈ 1:times intersection(a, b) end
31
32 @info "===== STARTING BENCHMARKS ====="
33 for (i, (a, b)) ∈ enumerate(scenarios)
34     @info "Scenario $i" a b intersection(a, b)
35     display(@benchmark sample($a, $b))
36     println("\n")
37 end
38 @info "===== BENCHMARKS OVER ====="

```

---

**Listing A.3:** Data produced through the execution of the code in listing A.1 a total of nine times to obtain a slightly bigger sample size. Contains slight post-processing that involves grouping up the results and reducing them.

```

### intersection-line-line ###
# new benchmark
cpu time: 7 real time: 7 gc time: 0
cpu time: 7 real time: 7 gc time: 0
cpu time: 8 real time: 8 gc time: 0
cpu time: 8 real time: 8 gc time: 0
cpu time: 8 real time: 8 gc time: 0

```

```

cpu time: 9 real time: 9 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time: 12 real time: 12 gc time: 0
  cpu time:
    (min ... max): 7.000 ms ... 12.000 ms
    (median):     8.000 ms
    (mean ± σ):   8.778 ms ±  1.641 ms
  real time:
    (min ... max): 7.000 ms ... 12.000 ms
    (median):     8.000 ms
    (mean ± σ):   8.778 ms ±  1.641 ms
  gc time:
    (min ... max): 0.000 s ... 0.000 s
    (median):     0.000 s
    (mean ± σ):   0.000 s ± 0.000 s
cpu time: 1231 real time: 2140 gc time: 17
cpu time: 1232 real time: 2185 gc time: 15
cpu time: 1300 real time: 2226 gc time: 17
cpu time: 1361 real time: 2101 gc time: 17
cpu time: 1399 real time: 2125 gc time: 18
cpu time: 1405 real time: 2356 gc time: 18
cpu time: 1444 real time: 2318 gc time: 17
cpu time: 1453 real time: 2475 gc time: 18
cpu time: 1526 real time: 2457 gc time: 19
  cpu time:
    (min ... max): 1.231 s ...  1.526 s
    (median):      1.399 s
    (mean ± σ):   1.372 s ± 101.282 ms
  real time:
    (min ... max): 2.101 s ...  2.475 s
    (median):      2.226 s
    (mean ± σ):   2.265 s ± 142.334 ms
  gc time:
    (min ... max): 15.000 ms ... 19.000 ms
    (median):      17.000 ms
    (mean ± σ):   17.333 ms ±  1.118 ms

# new benchmark
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 7 real time: 7 gc time: 0
cpu time: 7 real time: 7 gc time: 0
  cpu time:
    (min ... max): 6.000 ms ...  7.000 ms
    (median):      6.000 ms
    (mean ± σ):   6.222 ms ± 440.959 µs
  real time:

```

```

(min ... max): 6.000 ms ... 7.000 ms
(median): 6.000 ms
(mean ± σ): 6.222 ms ± 440.959 μs
gc time:
(min ... max): 0.000 s ... 0.000 s
(median): 0.000 s
(mean ± σ): 0.000 s ± 0.000 s
cpu time: 569 real time: 1200 gc time: 9
cpu time: 629 real time: 1492 gc time: 10
cpu time: 689 real time: 1637 gc time: 10
cpu time: 702 real time: 1599 gc time: 10
cpu time: 739 real time: 1723 gc time: 11
cpu time: 783 real time: 1800 gc time: 12
cpu time: 792 real time: 1780 gc time: 12
cpu time: 795 real time: 1807 gc time: 12
cpu time: 808 real time: 1767 gc time: 15
cpu time:
(min ... max): 569.000 ms ... 808.000 ms
(median): 739.000 ms
(mean ± σ): 722.889 ms ± 83.047 ms
real time:
(min ... max): 1.200 s ... 1.807 s
(median): 1.723 s
(mean ± σ): 1.645 s ± 197.524 ms
gc time:
(min ... max): 9.000 ms ... 15.000 ms
(median): 11.000 ms
(mean ± σ): 11.222 ms ± 1.787 ms

### intersection-circle-line ###
# new benchmark
cpu time: 9 real time: 9 gc time: 0
cpu time: 9 real time: 9 gc time: 0
cpu time: 9 real time: 9 gc time: 0
cpu time: 9 real time: 9 gc time: 0
cpu time: 9 real time: 9 gc time: 0
cpu time: 9 real time: 9 gc time: 0
cpu time: 9 real time: 9 gc time: 0
cpu time: 9 real time: 9 gc time: 0
cpu time: 13 real time: 13 gc time: 0
cpu time:
(min ... max): 9.000 ms ... 13.000 ms
(median): 9.000 ms
(mean ± σ): 9.444 ms ± 1.333 ms
real time:
(min ... max): 9.000 ms ... 13.000 ms
(median): 9.000 ms
(mean ± σ): 9.444 ms ± 1.333 ms
gc time:
(min ... max): 0.000 s ... 0.000 s
(median): 0.000 s
(mean ± σ): 0.000 s ± 0.000 s
cpu time: 1597 real time: 3603 gc time: 14

```

```

cpu time: 1771 real time: 3832 gc time: 16
cpu time: 1777 real time: 3637 gc time: 15
cpu time: 1896 real time: 3821 gc time: 15
cpu time: 1898 real time: 4149 gc time: 15
cpu time: 2002 real time: 4271 gc time: 17
cpu time: 2011 real time: 4298 gc time: 16
cpu time: 2028 real time: 3704 gc time: 15
cpu time: 2048 real time: 4306 gc time: 16
  cpu time:
    (min ... max): 1.597 s ... 2.048 s
    (median): 1.898 s
    (mean ± σ): 1.892 s ± 151.672 ms
  real time:
    (min ... max): 3.603 s ... 4.306 s
    (median): 3.832 s
    (mean ± σ): 3.957 s ± 295.673 ms
  gc time:
    (min ... max): 14.000 ms ... 17.000 ms
    (median): 15.000 ms
    (mean ± σ): 15.444 ms ± 881.917 μs

# new benchmark
cpu time: 8 real time: 8 gc time: 0
cpu time: 8 real time: 8 gc time: 0
cpu time: 8 real time: 8 gc time: 0
cpu time: 8 real time: 8 gc time: 0
cpu time: 8 real time: 8 gc time: 0
cpu time: 8 real time: 8 gc time: 0
cpu time: 8 real time: 8 gc time: 0
cpu time: 8 real time: 8 gc time: 0
cpu time: 8 real time: 8 gc time: 0
  cpu time:
    (min ... max): 8.000 ms ... 8.000 ms
    (median): 8.000 ms
    (mean ± σ): 8.000 ms ± 0.000 s
  real time:
    (min ... max): 8.000 ms ... 8.000 ms
    (median): 8.000 ms
    (mean ± σ): 8.000 ms ± 0.000 s
  gc time:
    (min ... max): 0.000 s ... 0.000 s
    (median): 0.000 s
    (mean ± σ): 0.000 s ± 0.000 s
cpu time: 1217 real time: 2703 gc time: 13
cpu time: 1232 real time: 2710 gc time: 13
cpu time: 1296 real time: 2979 gc time: 14
cpu time: 1341 real time: 3050 gc time: 14
cpu time: 1343 real time: 3059 gc time: 15
cpu time: 1429 real time: 3287 gc time: 13
cpu time: 1458 real time: 3172 gc time: 14
cpu time: 1462 real time: 3386 gc time: 14
cpu time: 1482 real time: 3282 gc time: 15
  cpu time:

```

```

(min ... max): 1.217 s ... 1.482 s
(median): 1.343 s
(mean ± σ): 1.362 s ± 100.657 ms
real time:
(min ... max): 2.703 s ... 3.386 s
(median): 3.059 s
(mean ± σ): 3.069 s ± 243.768 ms
gc time:
(min ... max): 13.000 ms ... 15.000 ms
(median): 14.000 ms
(mean ± σ): 13.889 ms ± 781.736 μs

# new benchmark
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time:
(min ... max): 6.000 ms ... 6.000 ms
(median): 6.000 ms
(mean ± σ): 6.000 ms ± 0.000 s
real time:
(min ... max): 6.000 ms ... 6.000 ms
(median): 6.000 ms
(mean ± σ): 6.000 ms ± 0.000 s
gc time:
(min ... max): 0.000 s ... 0.000 s
(median): 0.000 s
(mean ± σ): 0.000 s ± 0.000 s
cpu time: 617 real time: 1775 gc time: 7
cpu time: 622 real time: 1672 gc time: 8
cpu time: 633 real time: 1786 gc time: 8
cpu time: 648 real time: 1931 gc time: 8
cpu time: 656 real time: 1835 gc time: 7
cpu time: 659 real time: 1866 gc time: 8
cpu time: 662 real time: 1880 gc time: 8
cpu time: 665 real time: 1858 gc time: 8
cpu time: 673 real time: 1944 gc time: 7
cpu time:
(min ... max): 617.000 ms ... 673.000 ms
(median): 656.000 ms
(mean ± σ): 648.333 ms ± 19.862 ms
real time:
(min ... max): 1.672 s ... 1.944 s
(median): 1.858 s
(mean ± σ): 1.839 s ± 84.460 ms
gc time:
(min ... max): 7.000 ms ... 8.000 ms

```

```

(median):    8.000 ms
(mean ± σ): 7.667 ms ± 500.000 μs

### intersection-circle-circle ###
# new benchmark
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time:
  (min ... max): 5.000 ms ... 6.000 ms
  (median):    5.000 ms
  (mean ± σ):  5.111 ms ± 333.333 μs
real time:
  (min ... max): 5.000 ms ... 6.000 ms
  (median):    5.000 ms
  (mean ± σ):  5.111 ms ± 333.333 μs
gc time:
  (min ... max): 0.000 s ... 0.000 s
  (median):    0.000 s
  (mean ± σ):  0.000 s ± 0.000 s
cpu time: 530 real time: 1290 gc time: 4
cpu time: 534 real time: 1213 gc time: 5
cpu time: 539 real time: 1304 gc time: 5
cpu time: 539 real time: 1249 gc time: 4
cpu time: 545 real time: 1353 gc time: 4
cpu time: 546 real time: 1278 gc time: 4
cpu time: 547 real time: 1382 gc time: 5
cpu time: 562 real time: 1359 gc time: 5
cpu time: 589 real time: 1373 gc time: 6
cpu time:
  (min ... max): 530.000 ms ... 589.000 ms
  (median):    545.000 ms
  (mean ± σ):  547.889 ms ± 17.934 ms
real time:
  (min ... max): 1.213 s ... 1.382 s
  (median):    1.304 s
  (mean ± σ):  1.311 s ± 59.139 ms
gc time:
  (min ... max): 4.000 ms ... 6.000 ms
  (median):    5.000 ms
  (mean ± σ):  4.667 ms ± 707.107 μs

# new benchmark
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0

```

```

cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time:
  (min ... max): 5.000 ms ... 6.000 ms
  (median): 5.000 ms
  (mean ± σ): 5.333 ms ± 500.000 µs
real time:
  (min ... max): 5.000 ms ... 6.000 ms
  (median): 5.000 ms
  (mean ± σ): 5.333 ms ± 500.000 µs
gc time:
  (min ... max): 0.000 s ... 0.000 s
  (median): 0.000 s
  (mean ± σ): 0.000 s ± 0.000 s
cpu time: 539 real time: 1761 gc time: 5
cpu time: 548 real time: 1843 gc time: 4
cpu time: 554 real time: 1852 gc time: 4
cpu time: 556 real time: 1932 gc time: 4
cpu time: 558 real time: 1802 gc time: 4
cpu time: 560 real time: 1885 gc time: 4
cpu time: 568 real time: 1855 gc time: 5
cpu time: 571 real time: 1813 gc time: 4
cpu time: 577 real time: 1879 gc time: 4
cpu time:
  (min ... max): 539.000 ms ... 577.000 ms
  (median): 558.000 ms
  (mean ± σ): 559.000 ms ± 11.758 ms
real time:
  (min ... max): 1.761 s ... 1.932 s
  (median): 1.852 s
  (mean ± σ): 1.847 s ± 50.466 ms
gc time:
  (min ... max): 4.000 ms ... 5.000 ms
  (median): 4.000 ms
  (mean ± σ): 4.200 ms ± 440.959 µs

# new benchmark
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time:
  (min ... max): 5.000 ms ... 6.000 ms
  (median): 5.000 ms
  (mean ± σ): 5.333 ms ± 500.000 µs

```

```

real time:
  (min ... max): 5.000 ms ... 6.000 ms
  (median): 5.000 ms
  (mean ± σ): 5.333 ms ± 500.000 μs
gc time:
  (min ... max): 0.000 s ... 0.000 s
  (median): 0.000 s
  (mean ± σ): 0.000 s ± 0.000 s
cpu time: 515 real time: 1794 gc time: 4
cpu time: 528 real time: 1829 gc time: 5
cpu time: 533 real time: 1869 gc time: 4
cpu time: 542 real time: 1865 gc time: 4
cpu time: 544 real time: 1893 gc time: 4
cpu time: 548 real time: 1867 gc time: 4
cpu time: 548 real time: 1892 gc time: 5
cpu time: 555 real time: 1886 gc time: 4
cpu time: 568 real time: 1921 gc time: 4
cpu time:
  (min ... max): 515.000 ms ... 568.000 ms
  (median): 544.000 ms
  (mean ± σ): 542.333 ms ± 15.516 ms
real time:
  (min ... max): 1.794 s ... 1.921 s
  (median): 1.869 s
  (mean ± σ): 1.868 s ± 37.650 ms
gc time:
  (min ... max): 4.000 ms ... 5.000 ms
  (median): 4.000 ms
  (mean ± σ): 4.200 ms ± 440.959 μs

# new benchmark
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 5 real time: 5 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time: 6 real time: 6 gc time: 0
cpu time:
  (min ... max): 5.000 ms ... 6.000 ms
  (median): 5.000 ms
  (mean ± σ): 5.222 ms ± 440.959 μs
real time:
  (min ... max): 5.000 ms ... 6.000 ms
  (median): 5.000 ms
  (mean ± σ): 5.222 ms ± 440.959 μs
gc time:
  (min ... max): 0.000 s ... 0.000 s
  (median): 0.000 s
  (mean ± σ): 0.000 s ± 0.000 s
cpu time: 644 real time: 4223 gc time: 5

```

```

cpu time: 662 real time: 4313 gc time: 4
cpu time: 665 real time: 4257 gc time: 5
cpu time: 666 real time: 4230 gc time: 5
cpu time: 669 real time: 4311 gc time: 5
cpu time: 677 real time: 4287 gc time: 6
cpu time: 686 real time: 4248 gc time: 4
cpu time: 700 real time: 4312 gc time: 5
cpu time: 702 real time: 4311 gc time: 6
  cpu time:
    (min ... max): 644.000 ms ... 702.000 ms
    (median): 669.000 ms
    (mean ± σ): 674.556 ms ± 18.789 ms
  real time:
    (min ... max): 4.223 s ... 4.313 s
    (median): 4.287 s
    (mean ± σ): 4.277 s ± 37.581 ms
  gc time:
    (min ... max): 4.000 ms ... 5.000 ms
    (median): 4.000 ms
    (mean ± σ): 4.200 ms ± 440.959 μs

# new benchmark
cpu time: 7 real time: 7 gc time: 0
cpu time: 7 real time: 7 gc time: 0
cpu time: 7 real time: 7 gc time: 0
cpu time: 7 real time: 7 gc time: 0
cpu time: 7 real time: 7 gc time: 0
cpu time: 7 real time: 7 gc time: 0
cpu time: 7 real time: 7 gc time: 0
cpu time: 8 real time: 8 gc time: 0
cpu time: 8 real time: 8 gc time: 0
  cpu time:
    (min ... max): 7.000 ms ... 8.000 ms
    (median): 7.000 ms
    (mean ± σ): 7.222 ms ± 440.959 μs
  real time:
    (min ... max): 7.000 ms ... 8.000 ms
    (median): 7.000 ms
    (mean ± σ): 7.222 ms ± 440.959 μs
  gc time:
    (min ... max): 0.000 s ... 0.000 s
    (median): 0.000 s
    (mean ± σ): 0.000 s ± 0.000 s
cpu time: 990 real time: 2249 gc time: 8
cpu time: 1011 real time: 2350 gc time: 9
cpu time: 1027 real time: 2493 gc time: 9
cpu time: 1044 real time: 2575 gc time: 8
cpu time: 1045 real time: 2482 gc time: 9
cpu time: 1060 real time: 2480 gc time: 9
cpu time: 1070 real time: 2256 gc time: 9
cpu time: 1076 real time: 2636 gc time: 8
cpu time: 1197 real time: 3035 gc time: 11
  cpu time:

```

```

(min ... max): 990.000 ms ... 1.197 s
(median):     1.045 s
(mean ± σ):   1.058 s ± 59.110 ms
real time:
(min ... max): 2.249 s ... 3.035 s
(median):     2.482 s
(mean ± σ):   2.506 s ± 238.696 ms
gc time:
(min ... max): 8.000 ms ... 11.000 ms
(median):     9.000 ms
(mean ± σ):   8.889 ms ± 927.961 μs

# new benchmark
cpu time: 9 real time: 9 gc time: 0
cpu time: 9 real time: 9 gc time: 0
cpu time: 9 real time: 9 gc time: 0
cpu time: 9 real time: 9 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time:
(min ... max): 7.000 ms ... 8.000 ms
(median):     7.000 ms
(mean ± σ):   7.222 ms ± 440.959 μs
real time:
(min ... max): 7.000 ms ... 8.000 ms
(median):     7.000 ms
(mean ± σ):   7.222 ms ± 440.959 μs
gc time:
(min ... max): 0.000 s ... 0.000 s
(median):     0.000 s
(mean ± σ):   0.000 s ± 0.000 s
cpu time: 1368 real time: 3237 gc time: 9
cpu time: 1383 real time: 3326 gc time: 10
cpu time: 1408 real time: 3233 gc time: 10
cpu time: 1416 real time: 3306 gc time: 9
cpu time: 1458 real time: 3484 gc time: 9
cpu time: 1501 real time: 3417 gc time: 10
cpu time: 1597 real time: 3882 gc time: 11
cpu time: 1602 real time: 3669 gc time: 11
cpu time: 1605 real time: 3882 gc time: 10
cpu time:
(min ... max): 1.368 s ... 1.605 s
(median):     1.458 s
(mean ± σ):   1.482 s ± 97.673 ms
real time:
(min ... max): 3.233 s ... 3.882 s
(median):     3.417 s
(mean ± σ):   3.493 s ± 258.715 ms
gc time:
(min ... max): 9.000 ms ... 11.000 ms

```

```

(median):    10.000 ms
(mean ± σ): 9.889 ms ± 781.736 µs

# new benchmark
cpu time: 9 real time: 9 gc time: 0
cpu time: 9 real time: 9 gc time: 0
cpu time: 9 real time: 9 gc time: 0
cpu time: 9 real time: 9 gc time: 0
cpu time: 9 real time: 9 gc time: 0
cpu time: 9 real time: 9 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time:
  (min ... max): 9.000 ms ... 10.000 ms
  (median):    9.000 ms
  (mean ± σ): 9.444 ms ± 527.046 µs
real time:
  (min ... max): 9.000 ms ... 10.000 ms
  (median):    9.000 ms
  (mean ± σ): 9.444 ms ± 527.046 µs
gc time:
  (min ... max): 0.000 s ... 0.000 s
  (median):    0.000 s
  (mean ± σ): 0.000 s ± 0.000 s
cpu time: 1520 real time: 3614 gc time: 10
cpu time: 1531 real time: 3652 gc time: 9
cpu time: 1551 real time: 3759 gc time: 11
cpu time: 1607 real time: 3791 gc time: 11
cpu time: 1621 real time: 3758 gc time: 10
cpu time: 1637 real time: 3961 gc time: 10
cpu time: 1668 real time: 4010 gc time: 10
cpu time: 1716 real time: 3912 gc time: 11
cpu time: 1753 real time: 4016 gc time: 11
cpu time:
  (min ... max): 1.520 s ... 1.753 s
  (median):    1.621 s
  (mean ± σ): 1.622 s ± 80.986 ms
real time:
  (min ... max): 3.614 s ... 3.016 s
  (median):    3.791 s
  (mean ± σ): 3.333 s ± 150.402 ms
gc time:
  (min ... max): 9.000 ms ... 11.000 ms
  (median):    10.000 ms
  (mean ± σ): 10.333 ms ± 707.107 µs

# new benchmark
cpu time: 9 real time: 9 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time: 10 real time: 10 gc time: 0

```

```

cpu time: 10 real time: 10 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time: 10 real time: 10 gc time: 0
cpu time: 13 real time: 13 gc time: 0
  cpu time:
    (min ... max): 9.000 ms ... 13.000 ms
    (median): 10.000 ms
    (mean ± σ): 10.222 ms ± 1.093 ms
  real time:
    (min ... max): 9.000 ms ... 13.000 ms
    (median): 10.000 ms
    (mean ± σ): 10.222 ms ± 1.093 ms
  gc time:
    (min ... max): 0.000 s ... 0.000 s
    (median): 0.000 s
    (mean ± σ): 0.000 s ± 0.000 s
cpu time: 2373 real time: 10990 gc time: 15
cpu time: 2397 real time: 10976 gc time: 15
cpu time: 2479 real time: 11129 gc time: 16
cpu time: 2496 real time: 11109 gc time: 15
cpu time: 2552 real time: 11087 gc time: 15
cpu time: 2573 real time: 11144 gc time: 16
cpu time: 2602 real time: 11191 gc time: 16
cpu time: 2639 real time: 11197 gc time: 15
cpu time: 2672 real time: 11176 gc time: 17
  cpu time:
    (min ... max): 2.373 s ... 2.672 s
    (median): 2.552 s
    (mean ± σ): 2.531 s ± 103.464 ms
  real time:
    (min ... max): 10.976 s ... 11.197 s
    (median): 11.129 s
    (mean ± σ): 11.111 s ± 81.302 ms
  gc time:
    (min ... max): 15.000 ms ... 17.000 ms
    (median): 15.000 ms
    (mean ± σ): 15.556 ms ± 726.486 µs

```

**Listing A.4:** More detailed data from benchmarking our solution by submitting it to a test suite analogous to ConstraintGM's. This is the output produced by listing A.2.

```

BenchmarkTools.Trial: 10000 samples with 1 evaluation.
Range (min ... max): 248.077 µs ... 311.634 ms | GC (min ... max): 0.00% ... 45.83%
Time (median): 266.237 µs | GC (median): 0.00%
Time (mean ± σ): 362.191 µs ± 4.841 ms | GC (mean ± σ): 10.69% ± 1.26%

```



Memory estimate: 15.62 KiB, allocs estimate: 1000.

```
BenchmarkTools.Trial: 10000 samples with 1 evaluation.
```

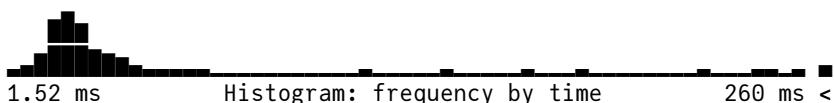
```
Range (min ... max): 149.461 µs ... 199.677 µs | GC (min ... max): 0.00% ... 0.00%
Time (median): 157.493 µs | GC (median): 0.00%
Time (mean ± σ): 159.567 µs ± 4.634 µs | GC (mean ± σ): 0.00% ± 0.00%
```



```
Memory estimate: 0 bytes, allocs estimate: 0.
```

```
BenchmarkTools.Trial: 2662 samples with 1 evaluation.
```

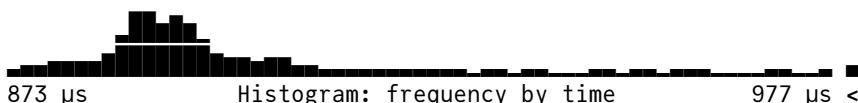
```
Range (min ... max): 1.520 ms ... 101.968 ms | GC (min ... max): 0.00% ... 28.94%
Time (median): 1.564 ms | GC (median): 0.00%
Time (mean ± σ): 1.871 ms ± 4.892 ms | GC (mean ± σ): 4.85% ± 1.83%
```



```
Memory estimate: 140.62 KiB, allocs estimate: 4000.
```

```
BenchmarkTools.Trial: 5196 samples with 1 evaluation.
```

```
Range (min ... max): 873.160 µs ... 265.349 ms | GC (min ... max): 0.00% ... 32.06%
Time (median): 893.135 µs | GC (median): 0.00%
Time (mean ± σ): 955.815 µs ± 3.710 ms | GC (mean ± σ): 2.35% ± 1.17%
```



```
Memory estimate: 15.62 KiB, allocs estimate: 1000.
```

```
BenchmarkTools.Trial: 8242 samples with 1 evaluation.
```

```
Range (min ... max): 576.334 µs ... 1.000 ms | GC (min ... max): 0.00% ... 0.00%
Time (median): 599.660 µs | GC (median): 0.00%
Time (mean ± σ): 600.386 µs ± 19.807 µs | GC (mean ± σ): 0.00% ± 0.00%
```



```
Memory estimate: 0 bytes, allocs estimate: 0.
```

```
BenchmarkTools.Trial: 10000 samples with 1 evaluation.
```

```
Range (min ... max): 228.941 µs ... 499.228 µs | GC (min ... max): 0.00% ... 0.00%
Time (median): 238.928 µs | GC (median): 0.00%
Time (mean ± σ): 241.531 µs ± 12.504 µs | GC (mean ± σ): 0.00% ± 0.00%
```



```
Memory estimate: 0 bytes, allocs estimate: 0.
```

```
BenchmarkTools.Trial: 10000 samples with 1 evaluation.
```

```
Range (min ... max): 231.595 µs ... 339.919 µs | GC (min ... max): 0.00% ... 0.00%
Time (median): 240.814 µs | GC (median): 0.00%
```

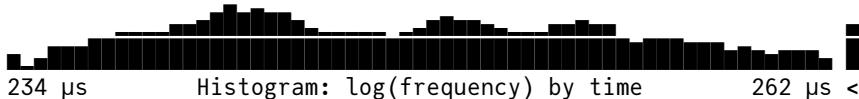
Time (mean  $\pm$   $\sigma$ ): 242.780  $\mu$ s  $\pm$  5.692  $\mu$ s | GC (mean  $\pm$   $\sigma$ ): 0.00%  $\pm$  0.00%



Memory estimate: 0 bytes, allocs estimate: 0.

BenchmarkTools.Trial: 10000 samples with 1 evaluation.

|   |  |
|---|--|
| Range (min ... max): 234.389 $\mu$ s ... 492.523 $\mu$ s          | GC (min ... max): 0.00% ... 0.00%            |
| Time (median): 242.840 $\mu$ s                                    | GC (median): 0.00%                           |
| Time (mean $\pm$ $\sigma$ ): 245.425 $\mu$ s $\pm$ 11.840 $\mu$ s | GC (mean $\pm$ $\sigma$ ): 0.00% $\pm$ 0.00% |



Memory estimate: 0 bytes, allocs estimate: 0.

BenchmarkTools.Trial: 9576 samples with 1 evaluation.

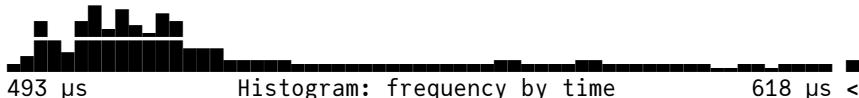
|   |  |
|---|--|
| Range (min ... max): 393.697 $\mu$ s ... 309.638 ms         | GC (min ... max): 0.00% ... 48.23%           |
| Time (median): 418.072 $\mu$ s                              | GC (median): 0.00%                           |
| Time (mean $\pm$ $\sigma$ ): 515.829 $\mu$ s $\pm$ 4.881 ms | GC (mean $\pm$ $\sigma$ ): 8.11% $\pm$ 1.29% |



Memory estimate: 15.62 KiB, allocs estimate: 1000.

BenchmarkTools.Trial: 8297 samples with 1 evaluation.

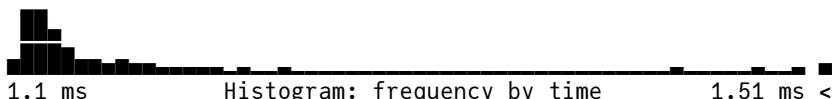
|   |  |
|---|--|
| Range (min ... max): 493.222 $\mu$ s ... 317.227 ms         | GC (min ... max): 0.00% ... 47.42%           |
| Time (median): 511.171 $\mu$ s                              | GC (median): 0.00%                           |
| Time (mean $\pm$ $\sigma$ ): 629.341 $\mu$ s $\pm$ 5.360 ms | GC (mean $\pm$ $\sigma$ ): 7.68% $\pm$ 1.42% |



Memory estimate: 15.62 KiB, allocs estimate: 1000.

BenchmarkTools.Trial: 3425 samples with 1 evaluation.

|  |  |
|--|--|
| Range (min ... max): 1.102 ms ... 103.956 ms         | GC (min ... max): 0.00% ... 28.37%           |
| Time (median): 1.121 ms                              | GC (median): 0.00%                           |
| Time (mean $\pm$ $\sigma$ ): 1.453 ms $\pm$ 5.070 ms | GC (mean $\pm$ $\sigma$ ): 6.83% $\pm$ 1.93% |



Memory estimate: 140.62 KiB, allocs estimate: 4000.

BenchmarkTools.Trial: 3422 samples with 1 evaluation.

|  |  |
|--|--|
| Range (min ... max): 1.097 ms ... 102.675 ms         | GC (min ... max): 0.00% ... 28.67%           |
| Time (median): 1.123 ms                              | GC (median): 0.00%                           |
| Time (mean $\pm$ $\sigma$ ): 1.455 ms $\pm$ 5.088 ms | GC (mean $\pm$ $\sigma$ ): 6.84% $\pm$ 1.93% |



Memory estimate: 140.62 KiB, allocs estimate: 4000.

BenchmarkTools.Trial: 3403 samples with 1 evaluation.

|                      |                         |                   |                  |
|----------------------|-------------------------|-------------------|------------------|
| Range (min ... max): | 1.105 ms ... 102.643 ms | GC (min ... max): | 0.00% ... 28.69% |
| Time (median):       | 1.132 ms                | GC (median):      | 0.00%            |
| Time (mean ± σ):     | 1.463 ms ± 5.055 ms     | GC (mean ± σ):    | 6.78% ± 1.94%    |



Memory estimate: 140.62 KiB, allocs estimate: 4000.

