**Due December 14, 2010**

# MTH 2140 Final

Instructions: This is a self-scheduled final. You are not allowed to work in groups or discuss problems with other people and you are not allowed to use notes, books, web browsers, calculators, etc.
If you get stuck and cannot proceed on a problem, you may do the following for partial credit: (1) describe why you are stuck; (2) describe what principle, idea, or computational method you think the problem is testing; (3) write down another problem which tests the same thing; (4) solve the problem that you have written down.
You may work for up to 2.5 hours in a closed-book environment. When you have completed the test (either because time is up or because you've gotten as far as you're going to get in a closed book environment), feel free to change your pen color and continue to work for up to an hour with open notes, books, computers, etc. (but still no human interaction). You may take a break between the closed book and open book portion of the test. Please place the test in my mailbox (in MH250) or in the box outside of my office (MH 257) sometime before I get into work on Wednesday morning.

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1. (**10 points**) Find the general solution to $\dot{y} = -4ty + 2t$ and describe how solutions behave as $t \to \infty$.

2. (**10 points**) Find the general solution to

$$\dot{Y} = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix} Y$$

3. (**10 points**) Consider the fourth order differential equation

$$y'''' + 5y'' + 4 = e^{i\omega t} \tag{1}$$

We have not studied this equation in class, but it is similar in some respects to second order equations that we have studied. The frequency $\omega_{res}$ is said to be a *resonant* frequency for (1) if the *gain* $C$ for the particular solution $y_p(t) = Ce^{i\omega t}$ goes to $\infty$ as $\omega$ approaches $\omega_{res}$. What are the resonant frequencies for (1)?

4. (**20 points**) The following equation models the population of savory cod which is harvested at a rate $G$.

$$\dot{y} = y(5 - y) - G$$

Here $y$ is measured in millions of individual savory cod. Since $y$ represents a population, if $y$ ever falls below zero we regard the population as extinct.

   (a) (**10 points**) What is the largest rate of fishing that a sufficiently large initial population can tolerate without going extinct? (*Its not an integer. Don't freak out.*)

   (b) (**10 points**) What is the largest rate of fishing that an initial population of $y(0) = 1$ can tolerate without going extinct?

5. (**20 points**) Consider now the situation where the invasive bitter cod, which is not harvested, competes with the savory cod for the same ecological niche:

$$\begin{cases} \dot{x} = x(16 - 4x - 4y) \\ \dot{y} = y(5 - x - y) - 2 \end{cases} \tag{2}$$

Here $x$ represents the population of bitter cod while $y$ represents the population of savory cod. Since $x$ and $y$ represent populations, if either $x$ or $y$ ever falls below zero we regard that population as being extinct. For the this problem we have fixed the harvesting rate to be $G = 2$.

(a) (**5 points**) Find an equilibrium where the bitter and savory cod coexist.

(b) (**10 points**) Determine the stability type of this equilibrium value.

(c) (**5 points**) Assume that there is a solution $(x_*(t), y_*(t))$ such that $\lim_{t\to\infty}(x_*(t), y_*(t)) = (2, 2)$ with $x_*(0) > 2$ and $y_*(0) > 2$. What can you say about the long-term behavior of the initial value problem (2) with $x(0) = x_*(0)$ and $y(0) < y_*(0)$? *Its valuable to talk both about reasonable educated guesses and what you can determine with mathematical certainty. Please distinguish these two types of conclusions from each other.*

6. (**5 points**) (Grandiose version) How has this course changed the way you see the world? (Modest version) What, if anything, that you have learned in this course do you expect to remember five years from now?