$$\frac{\Delta(\alpha)}{(1+|\alpha|^2)^2} = \Re\left\{\langle\phi\,|\,V_1\,|\,\psi\rangle\,\langle\psi\,|\,\sigma_1\,|\,\phi\rangle + \langle\phi\,|\,V_3\,|\,\psi\rangle\,\langle\psi\,|\,\sigma_3\,|\,\phi\rangle\right\} \tag{1}$$

$$|\psi\rangle = \frac{|0\rangle + \alpha\,|1\rangle}{\sqrt{1+|\alpha|^2}}\,, \qquad |\phi\rangle = \frac{\alpha^*\,|0\rangle - |1\rangle}{\sqrt{1+|\alpha|^2}} \tag{2}$$

$$\sigma_3\,|0\rangle = |0\rangle\ , \qquad \sigma_3\,|1\rangle = -\,|1\rangle\,. \tag{3}$$

$$\begin{aligned}\Delta(\alpha) = \Re\Big\{&(\alpha^{*2}-1)\,\big(\alpha(\langle 0\,|\,V_1\,|\,0\rangle - \langle 1\,|\,V_1\,|\,1\rangle) + \alpha^2\,\langle 0\,|\,V_1\,|\,1\rangle - \langle 1\,|\,V_1\,|\,0\rangle\big) + \\ &2\alpha^*\,\big(\alpha(\langle 0\,|\,V_3\,|\,0\rangle - \langle 1\,|\,V_3\,|\,1\rangle) + \alpha^2\,\langle 0\,|\,V_3\,|\,1\rangle - \langle 1\,|\,V_3\,|\,0\rangle\big)\Big\}\end{aligned} \tag{4}$$

$$V_1 = \int_0^{+\infty} \mathrm{d}\tau\, G(\tau)\, U_{\mathrm{eff}}(\tau)\, \widetilde{\sigma}_1(-\tau)\, U^\dagger_{\mathrm{eff}}(\tau)\ , \quad V_2 = \sigma_1 \tag{5}$$

$$V_3 = \int_0^{+\infty} \mathrm{d}\tau\, G(\tau)\, U_{\mathrm{eff}}(\tau)\, \widetilde{\sigma}_3(-\tau)\, U^\dagger_{\mathrm{eff}}(\tau)\ , \quad V_4 = \sigma_3\ . \tag{6}$$

$$V_1 = \int_0^{+\infty} \mathrm{d}\tau\, G(\tau)\,\{\cos\Omega\tau\cos\omega_{\mathrm{eff}}\tau\,\sigma_1 - \cos\Omega\tau\sin\omega_{\mathrm{eff}}\tau\,\sigma_2 - \sin\Omega\tau\,\sigma_3\} \tag{7}$$

$$V_3 = \int_0^{+\infty} \mathrm{d}\tau\, G(\tau)\,\{\sin\Omega\tau\cos\omega_{\mathrm{eff}}\tau\,\sigma_1 - \sin\Omega\tau\sin\omega_{\mathrm{eff}}\tau\,\sigma_2 - \cos\Omega\tau\,\sigma_3\} \tag{8}$$

$$G(\tau) = \int_0^{+\infty} \mathrm{d}\omega\, J(\omega)\left(\cos\omega\tau\coth\frac{\beta\omega}{2} - i\sin\omega\tau\right) \tag{9}$$

Taking $\alpha \in \mathbb{R}$,

$$\begin{aligned}\Delta(\alpha) = (\alpha^2-1)\,&\big(2\alpha\Re g_{s0} + \alpha^2(\Re g_{cc} + \Im g_{cs}) - \Re g_{cc} + \Im g_{cs}\big) + \\ &2\alpha\,\big(-2\alpha\Re g_{c0} - \alpha^2(\Re g_{sc} + \Im g_{ss}) + \Re g_{sc} - \Im g_{ss}\big)\end{aligned} \tag{10}$$