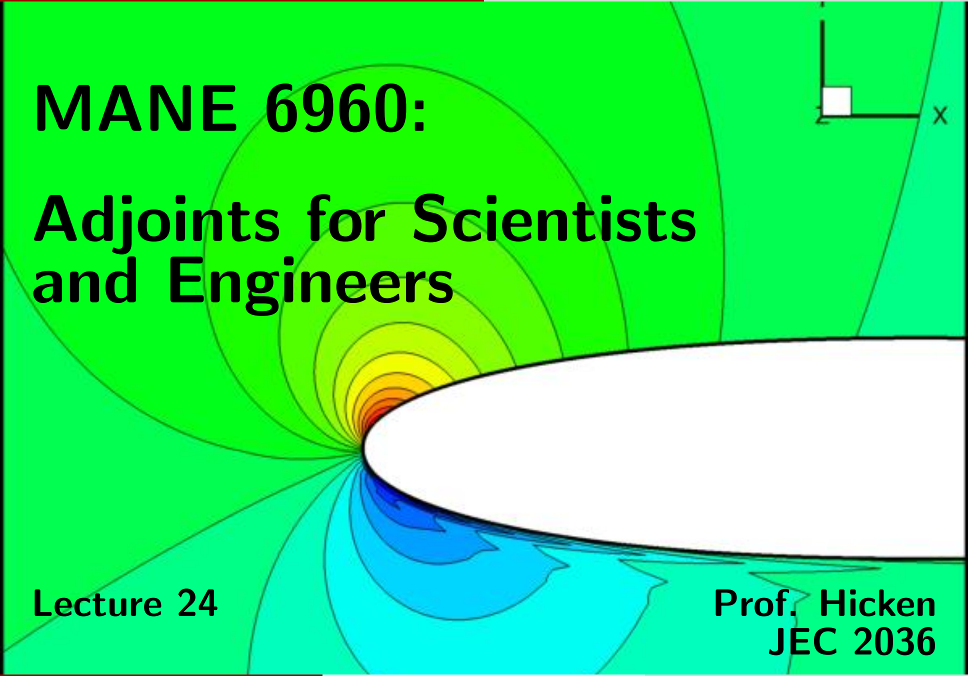


MANE 6960:

Adjoints for Scientists and Engineers

Lecture 24

Prof. Hicken
JEC 2036

Chaotic Dynamics and its Implications for the Adjoint

The Lorenz System: a model problem

Last lecture we saw that the incompressible Navier-Stokes equations produce an adjoint that is linearly unstable, in general.

This is a general result for the adjoint of chaotic systems.

To better understand the source of this instability and its implications for sensitivity analysis, we will investigate the Lorenz dynamical system.

The Lorenz System: a model problem (cont.)

The Lorenz dynamical system is defined by the following ODE:

$$\frac{dx}{dt} = \sigma(y - x), \quad , x(0) = x_0$$
$$\frac{dy}{dt} = x(\rho - z) - y, \quad , y(0) = y_0$$
$$\frac{dz}{dt} = xy - \beta z, \quad , z(0) = z_0$$

or, more concisely, $\frac{du}{dt} = f(u)$, where $u = (x, y, z)^T$.

- σ, ρ, and β are parameters
- simple model for convection in atmosphere
- its behavior was studied by E. Lorenz in 1963

What is Chaos?

The Lorenz system, like the incompressible NS equations, can display chaotic behavior for certain parameter choices.

What exactly is chaotic behavior?

While there are different ways of characterizing chaotic systems, the most straightforward to understand is the following:

> *The solutions of a chaotic dynamical system are highly sensitive to perturbations in their initial conditions.*

What is Chaos? (cont.)

The notion of being "sensitive to the initial conditions" can be made more precise as follows:

Consider two trajectories, $u(t)$ and $u(t) + \Delta u(t)$, whose initial conditions differ by Δu_0. Then, for sufficiently small t,

$$\|\Delta u(t)\| \approx e^{\lambda t} \|\Delta u_0\|,$$

where λ is the maximal Lyapunov exponent.

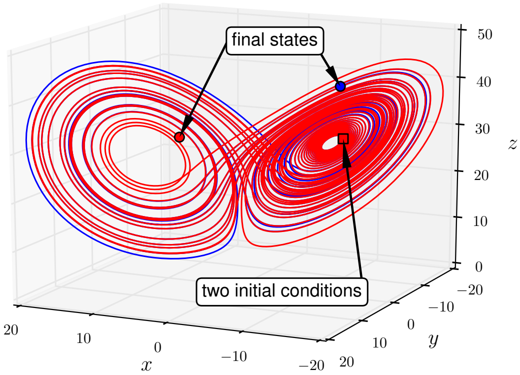
$$\lambda \equiv \lim_{T\to\infty} \lim_{\|\Delta u_0\|\to 0} \frac{1}{T} \ln \frac{\|\Delta u(T)\|}{\|\Delta u_0\|}.$$

What is Chaos? (cont.)

Thus, if the maximal Lyapunov exponent is positive, $\lambda > 0$, two trajectories that are initially "close" will diverge from each other exponentially.

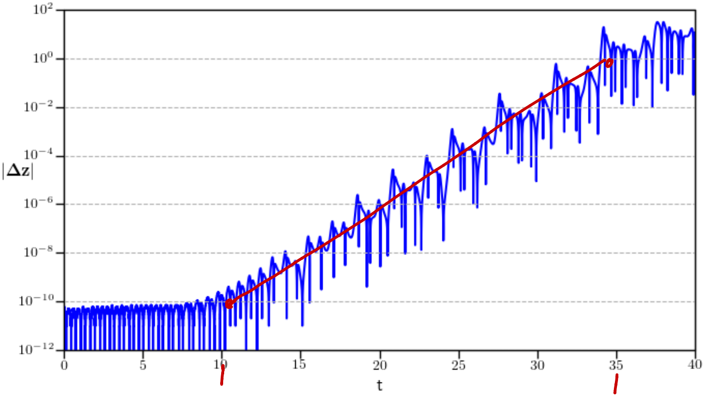
Thus, a positive λ is a necessary condition for a dynamical system to be considered chaotic.

What is Chaos? (cont.)



Two trajectories of the Lorenz system that start within 10^{-10} of each other; here we use $(\sigma, \beta, \rho) = (10, 8/3, 28)$.

What is Chaos? (cont.)



Evolution of $|\Delta z(t)|$ in the Lorenz system for the two trajectories on the previous slide.

$$\lambda \simeq \frac{1}{25} \ln \frac{1}{10^{-10}} = 0.92$$

Implications for Derivatives

Because the Lorenz system is chaotic (for particular parameters), any associated adjoint ODE will be unstable, in general.

Obviously, this means the adjoint can grow unbounded as it evolves backward in time.

What are the implications of this for applications that need the adjoint, like sensitivity analysis for optimization?

Implications for Derivatives (cont.)

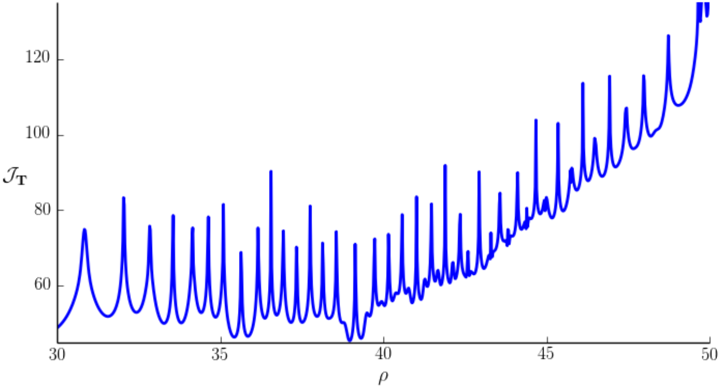
Consider the objective function

$$\mathcal{J}_T(\rho) \equiv \frac{1}{2T} \int_0^T (z(t,\rho) - z_{\mathsf{targ}})^2 \, dt,$$

where $z_{\mathsf{targ}} = 35$, and $z(t,\rho)$ is the z-component from the Lorenz system.

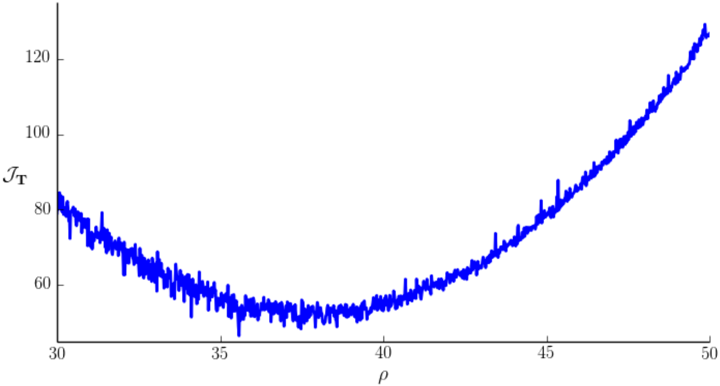
- Think of this as a model for some time-averaged objective in a flow (e.g. lift or drag).
- Ideally, we would like to take $T \to \infty$, but we are limited to finite solution times in practice.
- While the initial conditions influence J_T, they are not important to the general characteristics of the adjoint here.

Implications for Derivatives (cont.)



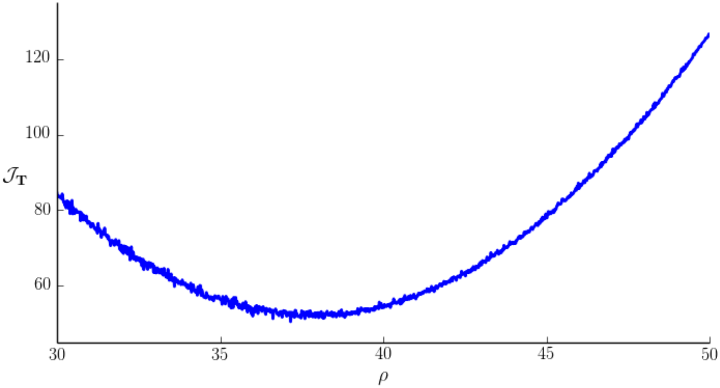
Objective function versus ρ for $T = 4$.

Implications for Derivatives (cont.)



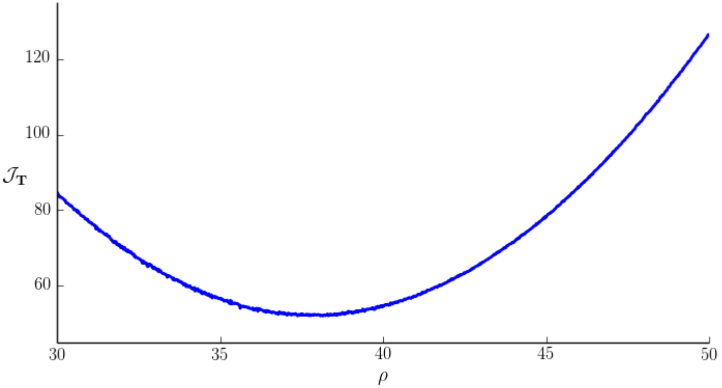
Objective function versus ρ for $T = 40$.

Implications for Derivatives (cont.)



Objective function versus ρ for $T = 400$.

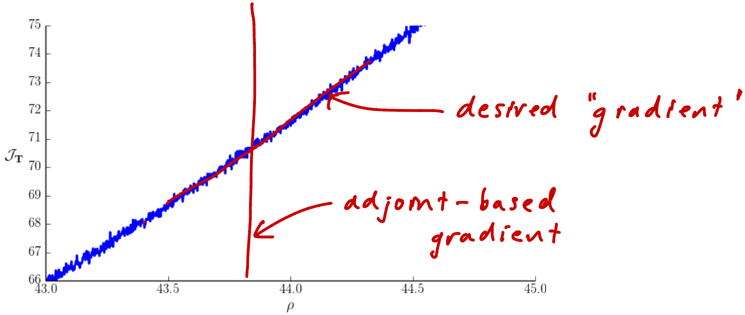
Implications for Derivatives (cont.)



Objective function versus ρ for $T = 4000$.

Implications for Derivatives (cont.)

With $T = 4000$, the objective looks relatively smooth with a clearly defined minimizer; however, if we look closer, we see that problems remain:



Close-up of objective function versus ρ for $T = 4000$.

Implications for Derivatives (cont.)

If the adjoint is used to compute the sensitivity of $\partial \mathcal{J}_T / \partial \rho$, it will correctly estimate the derivative.

- The adjoint "works" mathematically here.
- However, it fails from a practical perspective to detect the general trend.

What we need is to modify the adjoint, somehow, so that it can see the forest for the trees. . .

- This remains an active area of research.

References