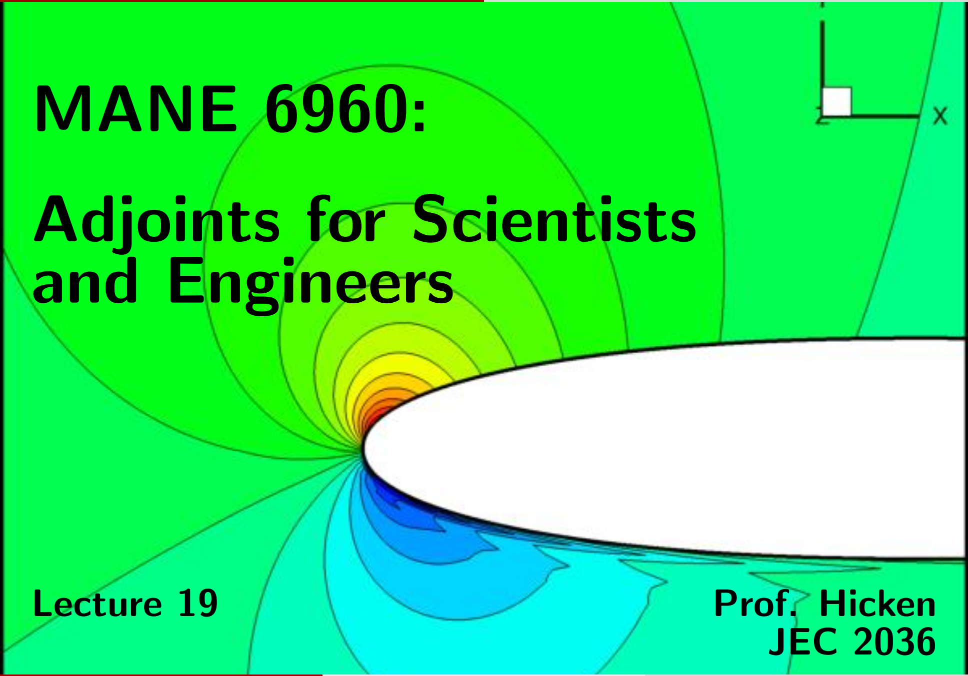

MANE 6960:

Adjoints for Scientists and Engineers

Lecture 19

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Overview

While many engineering systems can be modeled using steady boundary-value problems, there are certainly many more that require unsteady models.

Therefore, today we begin extending the adjoint methodology to unsteady problems.

- In many respects, the mathematics and methods extend easily
- In other respects, we will find that the methods require more careful bookkeeping, and
- In some cases, we will even find that the method breaks down completely

The Adjoint for Linear Initial Boundary-Value Problems

Generic Linear IBVP

To introduce the adjoint for unsteady problems, we will focus our attention on the following generic initial boundary-value problem (IBVP):

$$\begin{aligned} \frac{\partial u}{\partial t} + Lu &= f, && \forall x \in \Omega, t \in [0, T], \\ Bu &= b, && \forall x \in \Gamma, t \in [0, T], \\ u &= u_0, && \forall x \in \Omega, t = 0. \end{aligned} \tag{$\star$}$$

- The operators L and B, as well as the functions f and b, are functions of time t, in general.

Generic Linear Functional

The functional we consider is a time-integrated version of the one we considered for BVPs, plus a term that depends on the state at the terminal time T:

$$J(u) = \int_0^T (g, u)_\Omega \, dt + \int_0^T (c, Cu)_\Gamma \, dt + \left.(g_T, u)_\Omega\right|_{t=T}. \qquad (\dagger)$$

- As with the IBVP, g, c, and C may be functions of t.

Approach I

We will consider two approaches, or perspectives, in deriving the adjoint IBVP.

1. The first works well when only $\partial u/\partial t$ appears in the IBVP;
2. The second perspective is better for more general PDEs, e.g. the wave equation.

In either case, it is useful to recall the compatibility conditions:

$$(\psi, Lu)_\Omega - (u, L^*\psi)_\Omega = (Cu, B^*\psi)_\Gamma - (Bu, C^*\psi)_\Gamma.$$

Approach I (cont.)

For the first approach to deriving the adjoint IBVP, we first need the adjoint operator for the time-derivative. For this, we follow the familiar process used for spatial operators:

$$\int_0^T \psi \frac{\partial u}{\partial t}\, dt = \int_0^T \frac{\partial}{\partial t}(\psi u)\, dt - \int_0^T \frac{\partial \psi}{\partial t} u \, dt$$

$$= (\psi u)\Big|_{t=T} - \underbrace{(\psi u)}_{(\psi u_0)|_{t=0}}\Big|_{t=0} - \int_0^T u \frac{\partial \psi}{\partial t}\, dt$$

Approach I (cont.)

Next, as we did for BVPs, we subtract the adjoint-weighted residual from the functional (note the inner product is over $\Omega \times [0, T]$):

$$J(u) = \int_0^T (g, u)_\Omega \, dt + \int_0^T (c, Cu)_\Gamma \, dt + (g_T, u)_\Omega \big|_{t=T} - \int_0^T \left(\psi, \frac{\partial u}{\partial t} + Lu - f \right)_\Omega dt$$

$$= \int_0^T (g, u)_\Omega \, dt + \int_0^T (c, Cu)_\Gamma \, dt + (g_T, u)_\Omega \big|_{t=T}$$

$$- \int_0^T \int_\Omega \psi \frac{\partial u}{\partial t} \, d\Omega \, dt + \int_0^T (\psi, f)_\Omega \, dt$$

$$- \int_0^T (u, L^*\psi)_\Omega \, dt - \int_0^T (Cu, B^*\psi)_\Gamma \, dt + \int_0^T (\overset{b}{\cancel{B}}u, C^*\psi)_\Gamma \, dt$$

Approach I (cont.)

$$J(u) = \int_0^T (f, \psi)_\Omega \, dt + \int_0^T (b, C^* \psi)_\Gamma \, dt + (u_0, \psi)_\Omega \Big|_{t=0}$$

$$\left.\begin{aligned} & - \int_0^T \left(u, -\frac{\partial \psi}{\partial t} + L^* \psi - g\right)_\Omega dt \\ & - \int_0^T (Cu, B^* \psi - c)_\Gamma \, dt \\ & - (u, \psi - g_T)_\Omega \Big|_{t=T} \end{aligned}\right\}$$

Want these to vanish for duality

Definition: Adjoint Problem (linear IBVP)

Let u be the solution to the linear IBVP ($\star$). For the functional defined by ($\dagger$), the associated adjoint IBVP is

$$\begin{aligned} -\frac{\partial \psi}{\partial t} + L^* \psi &= g, & \forall x \in \Omega, t \in [0,T], \\ B^* \psi &= c, & \forall x \in \Gamma, t \in [0,T], \\ \psi &= g_T, & \forall x \in \Omega, t = T, \end{aligned} \tag{Adj}$$

and the adjoint-based functional is

$$J(\psi) = J(u) = \int_0^T (f, \psi)_\Omega \, dt + \int_0^T (b, C^* \psi)_\Gamma \, dt + (u_0, \psi)_\Omega \big|_{t=0} \,.$$

As with the adjoint BVP, the roles of the various operators and functions are worth highlighting:

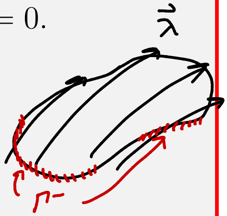
	primal	**adjoint**
spatial operator	L	L^*
temporal operator	$\frac{\partial}{\partial t}$	$-\frac{\partial}{\partial t}$
boundary op.	B	B^*
J boundary op.	C	C^*
source	f	g
J volume weight	g	f
initial/term. value	u_0	g_T
J term/initial weight	g_T	u_0
boundary value	b	c
J boundary weight	c	b

new

Exercise

Derive the adjoint IBVP and functional $J(\psi)$ for

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla \cdot (\lambda u) &= 0, && \forall x \in \Omega, t \in [0, T] \\ (\lambda \cdot n) u &= (\lambda \cdot n) u_{\mathsf{BC}}, && \forall x \in \Gamma^-, t \in [0, T], \\ u &= u_0, && \forall x \in \Omega, t = 0. \end{aligned}$$



and the primal functional

$$J(u) = \int_0^T \int_{\Gamma^+} u(\lambda \cdot n) \, d\Gamma \, dt,$$

where $\Gamma^- = \{\Gamma \,|\, \lambda \cdot \hat{n} < 0\}$ and $\Gamma^+ = \Gamma \setminus \Gamma^-$.

Exercise (cont.)

Using Approach I, we just need the compatibility condition for the spatial operators:

$$\int_{\Omega} \psi \, \vec{\nabla} \cdot (\vec{\lambda} u) \, d\Omega = \int_{\Omega} \vec{\nabla} \cdot (\vec{\lambda} \psi u) \, d\Omega - \int_{\Omega} u \, \vec{\lambda} \cdot \vec{\nabla} \psi \, d\Omega$$

$$= - \int_{\Omega} u \, \vec{\lambda} \cdot \vec{\nabla} \psi \, d\Omega + \int_{\Gamma} \psi u (\vec{\lambda} \cdot \hat{n}) \, d\Gamma$$

Exercise (cont.)

Continuing:

$$\int_\Omega \psi\, \vec{\nabla}\cdot(\vec{\lambda} u)\, d\Omega = \underbrace{-\int_\Omega u\, \vec{\lambda}\cdot\vec{\nabla}\psi\, d\Omega}_{(u,\, L^*\psi)_\Omega} \rightarrow L^* = -\vec{\lambda}\cdot\vec{\nabla}$$

$$+ \underbrace{\int_{\Gamma^-} \psi\, (u(\vec{\lambda}\cdot\hat{n}))\, d\Gamma}_{-(Bu,\, C^*\psi)_{\Gamma^-}} \hookrightarrow C^*\psi = -\psi$$

$$+ \underbrace{\int_{\Gamma^+} \psi\, (u(\vec{\lambda}\cdot\hat{n}))\, d\Gamma}_{(B^*\psi,\, Cu)_{\Gamma^+}} \hookrightarrow B^*\psi = \psi$$

Exercise (cont.)

Identifying terms in (Adj) we have the following adjoint IBVP:

$$-\frac{\partial \psi}{\partial t} - \vec{\lambda} \cdot \vec{\nabla} \psi = 0 \quad , \quad \forall x \in \Omega, \; t \in [0,T]$$

$$\mathcal{B}^* \psi = \psi = 1 \quad , \quad \forall x \in \Gamma^+, \; t \in [0,T]$$

$$\psi(t=T) = 0$$

And

$$J(\psi) = \int_0^T \int_{\Gamma^-} -\psi \, u_{BC} \, d\Gamma \, dt + \int_{\Omega} u_0 \, \psi(t=0) \, d\Omega$$

Approach II

As long as the temporal operator is $\partial/\partial t$, we can extend the spatial adjoint BVP to the adjoint IBVP in a straightforward way, as illustrated above.

However, for more general temporal operators, it may be more useful to just treat t as another spatial variable.

Approach II (cont.)

Thus, in this approach to deriving the adjoint IBVP, we make the following associations:

- $\Omega = \Omega_x \times \Omega_t = \Omega_x \times [0, T]$ (space-time domain)
- $\Gamma = \Gamma_x \cup \Gamma_t$ (space-time boundary)
- L includes both the temporal and spatial operators
- B, C, B^*, and C^* are primal and adjoint space-time boundary operators
- b, c are primal and adjoint space-time boundary functions

Approach II (cont.)

From this perspective, the compatibility conditions are completely unchanged in form:

$$(\psi, Lu)_\Omega - (u, L^*\psi)_\Omega = (Cu, B^*\psi)_\Gamma - (Bu, C^*\psi)_\Gamma.$$

- What changes is the definition of the domains and the inner products.

To derive the IBVP using this approach, the first step is to derive the compatibility conditions associated with the space-time operators.

- This will be illustrated in the following example.

Exercise

Consider the 1+1 dimensional wave equation

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= 0, && \forall x \in [0, L], t \in [0, T], \\ u(0, t) = u(L, t) &= 0, && \forall t \in [0, T], \\ u(x, 0) &= f(x), && \forall x \in [0, L], \\ \frac{\partial u}{\partial t}(x, 0) &= g(x), && \forall x \in [0, L]. \end{aligned}$$

1. Derive L^*, B^* and C^* using the second approach.

Exercise (cont.)

We begin by deriving the compatibility conditions for the space-time operators:

$$\int_0^T \int_0^L \psi \left(\frac{\partial^2 u}{\partial t} - c^2 \frac{\partial^2 u}{\partial x^2} \right) dx\, dt$$

$$= \int_0^L \int_0^T \frac{\partial}{\partial t}\left(\psi \frac{\partial u}{\partial t} \right) - \frac{\partial \psi}{\partial t} \frac{\partial u}{\partial t}\, dt\, dx$$

$$- c^2 \int_0^T \int_0^L \frac{\partial}{\partial x}\left(\psi \frac{\partial u}{\partial x} \right) - \frac{\partial \psi}{\partial x} \frac{\partial u}{\partial x}\, dx\, dt$$

Exercise (cont.)

Continuing

$$\int_0^T\int_0^L \psi\left(\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2}\right)dx\,dt$$

$$= \int_0^L \left(\psi\frac{\partial u}{\partial t}\right)\Big|_{t=0}^{T} dx - \int_0^L\int_0^T \frac{\partial}{\partial t}\left(\frac{\partial \psi}{\partial t}u\right)dt\,dx + \int_0^L\int_0^T u\frac{\partial^2 \psi}{\partial t^2}dt\,dx$$

$$-c^2\int_0^T\left(\psi\frac{\partial u}{\partial x}\right)\Big|_{x=0}^{L} dt + c^2\int_0^T\int_0^L \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}u\right)dx\,dt - c^2\int_0^T\int_0^L u\frac{\partial^2 \psi}{\partial x^2}dx\,dt$$

$$= \int_0^T\int_0^L u\underbrace{\left(\frac{\partial^2 \psi}{\partial t^2} - c^2\frac{\partial^2 \psi}{\partial x^2}\right)}_{L^*\psi}dx\,dt + \int_0^L\left(\psi\frac{\partial u}{\partial t} - \frac{\partial \psi}{\partial t}u\right)\Big|_{t=0}^{T} dx$$

$$+ \int_0^T\left(c^2\frac{\partial \psi}{\partial x}u - c^2\frac{\partial u}{\partial x}\psi\right)\Big|_{x=0}^{L} dt$$

Exercise (cont.)

$$-(Bu, C^* \psi)_\Gamma = \int_0^T \left(u \, c^2 \frac{\partial \psi}{\partial x} \right)\Big|_{x=0}^{L} dt + \int_0^L \left(\psi \frac{\partial u}{\partial t} - \frac{\partial \psi}{\partial t} u \right)\Big|_{t=0} dx$$

$$(B^* \psi, Cu)_\Gamma = \int_0^T -\left(c^2 \psi \frac{\partial u}{\partial x} \right)\Big|_{x=0}^{L} dt + \int_0^L \left(\psi \frac{\partial u}{\partial t} - \frac{\partial \psi}{\partial t} u \right)\Big|_{t=T} dx$$

Therefore

$$L^* \psi = \frac{\partial^2 \psi}{\partial t^2} - c^2 \frac{\partial^2 \psi}{\partial x^2}$$

$$B^* \psi = \underbrace{\begin{cases} \psi(x=0), & \forall t \\ \psi(x=L), & \forall t \\ \psi(t=T), & \forall x \\ \frac{\partial \psi}{\partial t}(t=T), & \forall x \end{cases}}_{\text{not unique here}} \qquad C^* \psi = \begin{cases} -c^2 \frac{\partial \psi}{\partial x}\Big|_{x=0}^{L} \\ -\psi(t=0), & x \in [0,L] \\ \frac{\partial \psi}{\partial t}(t=0), & x \in [0,L] \end{cases}$$

References