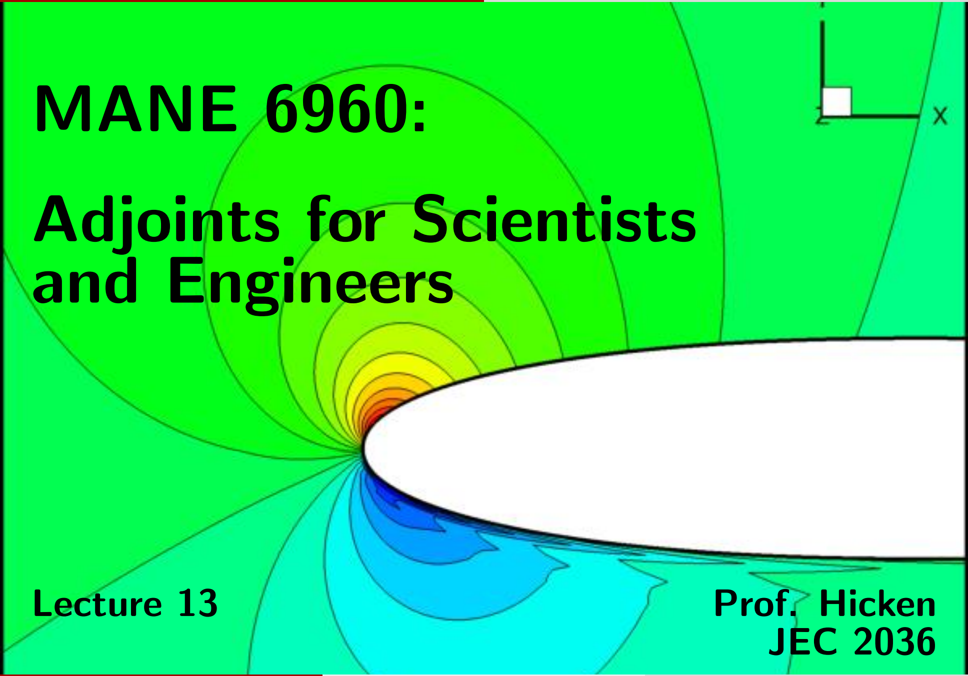


MANE 6960:

Adjoints for Scientists and Engineers

Lecture 13

Prof. Hicken
JEC 2036

Motivation

Thus far, I have focused on using the adjoint for sensitivity analysis, that is, computing total derivatives like $DJ_h/D\alpha$

Today, we begin to investigate a different application of the adjoint: output-based error estimation.

Motivation (cont.)

We often solve (numerically) PDEs in order to predict quantities of interest that depend on the state, that is, functionals.

- output, quantity of interest, and functional are synonymous in this context

Examples:

- lift, drag, and moment on an aerodynamic body
- maximum stress in a structure
- average power produced by an I.C. engine

Motivation (cont.)

Given the central role of outputs/functionals, it is important to be able to estimate the numerical error

$$\delta J_h \equiv J_h(u_h) - J(u),$$

since such an error estimate can tell us if the prediction can be trusted, or if the mesh needs to be refined.

How can we estimate δJ_h without knowing the exact solution?

Adjoint-Weighted-Residual Method for Linear Problems

Coarse and Fine Spaces

We cannot, in general, use the exact solution in order to estimate the functional error. However, we can approximate the exact solution using a computational mesh that is more resolved than the baseline mesh.

Therefore, suppose we have two meshes: a coarse mesh and a fine mesh.

- More generally, we just need a coarse- and fine-solution spaces, which might exist on the same mesh.
- We will use H to denote the coarse mesh/space and h to denote the fine mesh/space.

Coarse and Fine Spaces (cont.)

Examples of coarse and fine spaces:

h enrichment

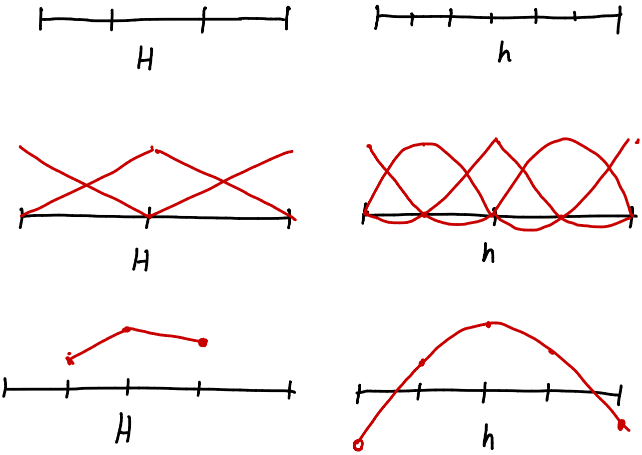
FV/FD or FE

FE

p enrichment

FD or spline FE

High order interpolation



Adjoint-Weighted-Residual Method

Now, suppose we have solved the discretized BVP of interest on mesh/space H:

$$L_H u_H = f_H.$$

We want to estimate the error in the (linear) functional

$$J_H(u_H) = (g_H, u_H)_H.$$

That is, we want to estimate the quantity

$$\delta J_H = J_H(u_H) - J(u).$$

Adjoint-Weighted-Residual Method (cont.)

We begin by replacing the exact functional with an approximation based on the fine mesh/space:

$$\delta J_H \approx J_H(u_H) - J_h(u_h)$$

$$= (g_H, u_H)_H - (g_h, u_h)_h$$

To proceed, I would like to group these products as

$$(g_?, u_H - u_h)_?$$

but g and $(\,,\,)$ are different, in general

Adjoint-Weighted-Residual Method (cont.)

In order to relate the two (discrete) functional values, we need to work in the same space.

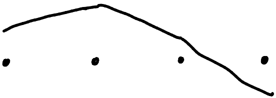
- Moving entirely to the coarse mesh/space will, in general, result in a loss of information.
- We need to represent the coarse solution, u_H, on the fine mesh/space.

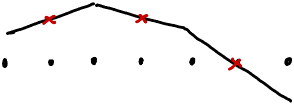
Therefore, we introduce a prolongation operator I_h^H from the coarse to the fine mesh/space, and we define

$$u_h^H \equiv I_h^H u_H.$$

linear interpolate

Example





Adjoint-Weighted-Residual Method (cont.)

Then we have

$$(g_H, u_H)_H - (g_h, u_h)_h = (g_h, u_h^H - u_h)_h$$

$$= (L_h^T \psi_h, u_h^H - u_h)_h \quad , \quad \because L_h^T \psi_h = g_h$$

$$= (\psi_h, L_h(u_h^H - u_h))_h - \underbrace{(\psi_h, f_h - f_h)_h}_{\text{zero}}$$

$$= (\psi_h, L_h u_h^H - f_h)_h - (\psi_h, \cancel{L_h u_h - f_h})_h \xrightarrow{} 0$$

u_h solves $L_h u_h = f_h$

Adjoint-Weighted-Residual Method (cont.)

Theorem: Adjoint-Weighted Residual (AWR)

Let u_H denote the solution to the discretized BVP $L_H u_H = f_H$ on a coarse space. Analogously, let u_h denote the solution of $L_h u_h = f_h$, the fine-space discretization of the same BVP. Then the difference

$$J_H(u_H) - J_h(u_h) = (g_H, u_H)_H - (g_h, u_h)_h = (\psi_h, L_h u_h^H - f_h)_h,$$

where ψ_h is the solution to the fine-space adjoint equation

$$L_h^T \psi_h = g_h.$$

Remarks on the AWR

hence the name AWR

This result says that the output/functional error can be estimated by weighting the residual $L_h u_h^H - f_h$ by the fine-space adjoint ψ_h.

While interesting, this result is not yet practical:

- solving for ψ_h is as expensive as solving for u_h, at least in the linear BVP case; and
- if we had u_h, we could just as easily compute $J_H(u_H) - J_h(u_h)$ directly.

Remarks on the AWR (cont.)

In the case of finite-difference and finite-volume discretizations, the AWR can be made practical by approximating the fine-space adjoint using the prolongation operator:

$$\psi_h \approx I_h^H \psi_H = \psi_h^H$$

Thus we have the estimate

$$J_H(u_H) - J_h(u_h) \approx (\psi_h^H, L_h u_h^H - f_h)_h$$

- This approach requires us to solve for the adjoint on the coarse mesh/space only.

Remarks on the AWR (cont.)

Unfortunately, this technique does not work for Galerkin finite-element methods since

$$(\psi_h^H, L_h u_h^H - f_h)_h = (\psi_H, L_H u_H - f_H)_H$$

$$= 0$$

(assumes no "variational crimes")

Follows from Galerkin orthogonality, since residual is perpendicular to all functions in coarse space, including ψ_H.

Example 1: Linear Advection [Hic12]

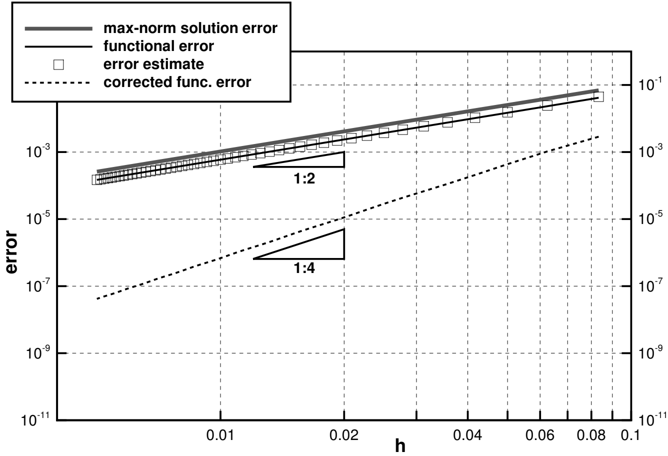
As a simple example, consider 2D linear advection as the BVP:

$$\begin{aligned} \nabla \cdot (\lambda u) = f, &\qquad \forall x \in [0,1]^2 \\ u(x) = b(x) &\qquad \forall x \in \Gamma_-. \end{aligned}$$

- functional is an integral over the outlet
- SBP-SAT finite-difference discretization
- fine "space" is based on using a higher-order FD operator on the same mesh
- thus, I_h^H is the identity matrix here

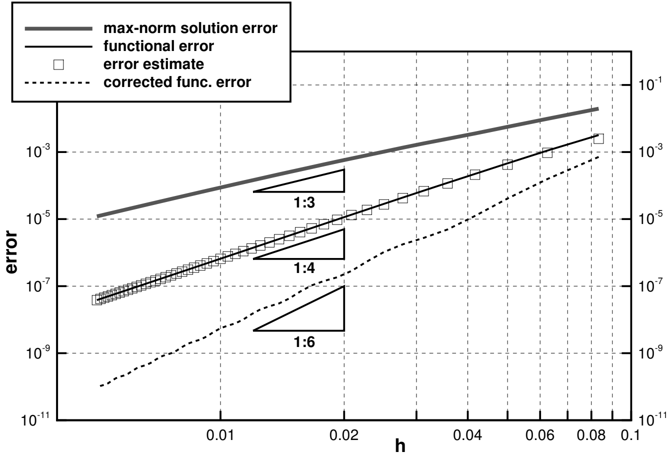
Example 1: Linear Advection [Hic12] (cont.)

2nd-order coarse space; 3rd-order fine space



Example 1: Linear Advection [Hic12] (cont.)

3rd-order coarse space; 4th-order fine space



Example 2: Poisson BVP [Hic12]

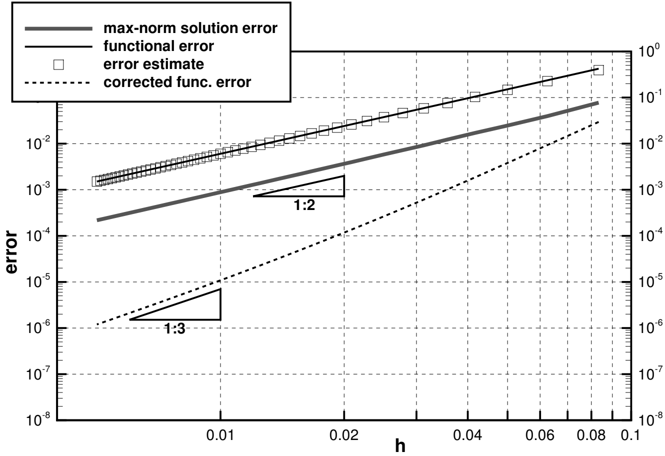
Next, consider the 2D Poisson BVP

$$\begin{aligned} -\nabla \cdot (\gamma \nabla u) &= f, \qquad \forall x \in [0,1]^2 \\ u(x) &= b(x) \qquad \forall x \in \Gamma. \end{aligned}$$

- functional is an integral over the entire boundary
- SBP-SAT finite-difference discretization
- fine "space" is based on using a higher-order FD operator on the same mesh
- thus, I_h^H is the identity matrix here

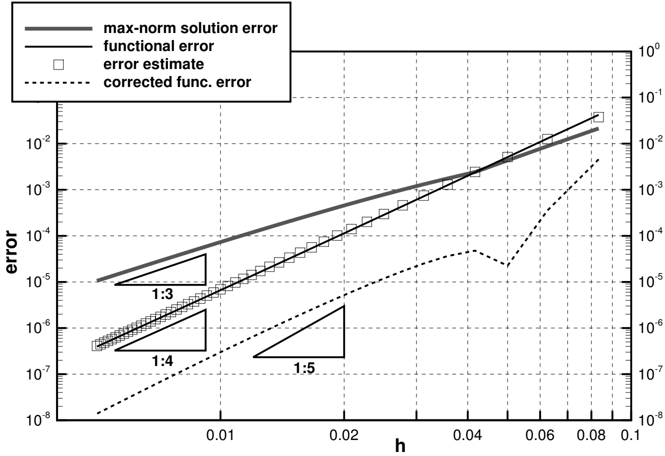
Example 2: Poisson BVP [Hic12] (cont.)

2nd-order coarse space; 3rd-order fine space



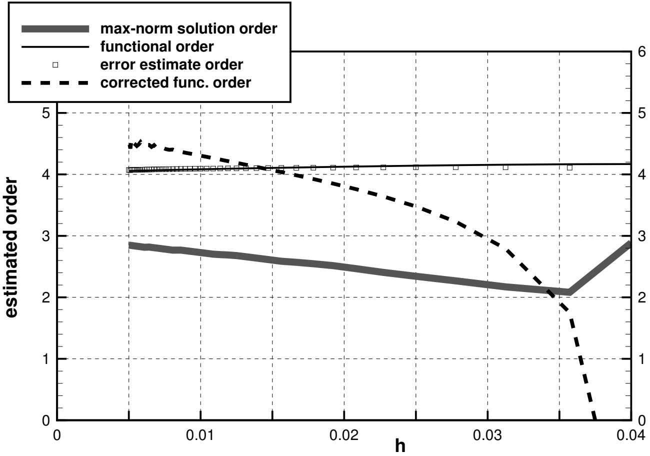
Example 2: Poisson BVP [Hic12] (cont.)

3rd-order coarse space; 4th-order fine space



Example 2: Poisson BVP [Hic12] (cont.)

Asymptotically, the error estimate appears to approach 5th order



References

[Hic12] Jason E. Hicken, *Output error estimation for summation-by-parts finite-difference schemes*, Journal of Computational Physics **231** (2012), no. 9, 3828–3848.