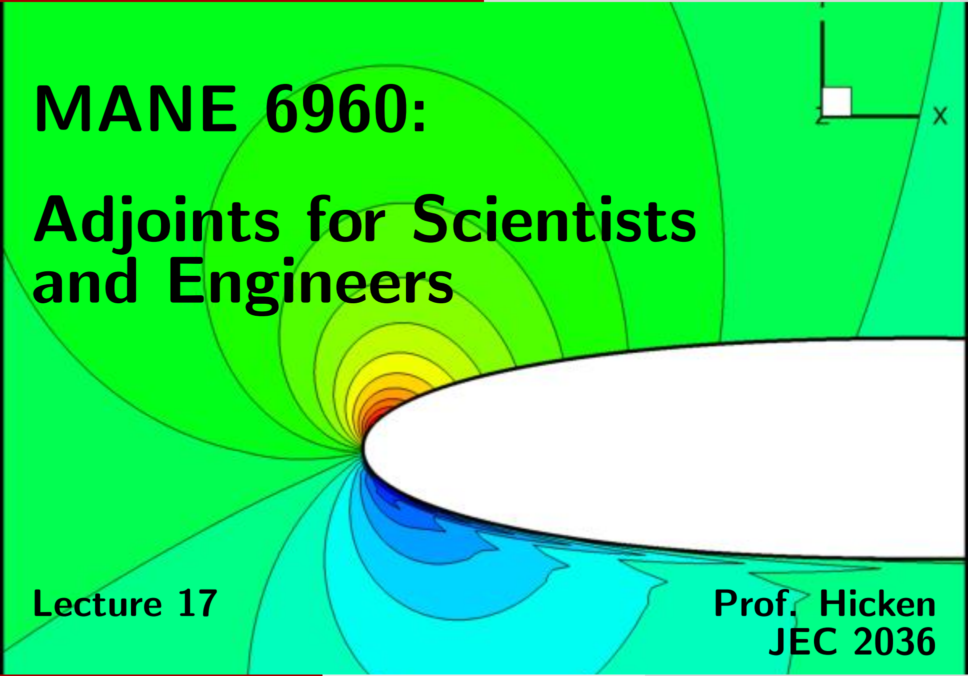

MANE 6960:

Adjoints for Scientists and Engineers

Lecture 17

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Overview

The focus of this lecture is on a more detailed adjoint analysis of the quasi-one-dimensional Euler equations.

- I hope this will complement your work with the equations in the assignments.
- Furthermore, this analysis highlights some surprising (i.e. counter-intuitive) behavior about adjoints.
- The analysis is closely based on that of [GP01].

Subsequently, we will examine some exact and numerical solutions that verify the analysis.

Adjoint Analysis of the Quasi-1D Euler Equations

Quasi-1D Euler Equations

Recall the quasi-1D Euler equations described in Assignment 2:

$$N(q) \equiv \frac{\partial}{\partial x}[F(q)] - G(q) = 0, \qquad \forall\, x \in [0,1]$$

where the flux and source are

$$F(q) = \left(\rho u A, \quad (\rho u^2 + p)A, \quad u(e+p)A\right)^T,$$
$$G(q) = \left(0, \quad p\frac{\partial A}{\partial x}, \quad 0\right)^T.$$

- state is $q = (\rho, \rho u, e)^T$
- $A(x)$ denotes the spatially varying nozzle area
- $p(q) = (\gamma - 1)(e - \frac{1}{2}\rho u^2)$ is the pressure

A(x) is fixed, so we ignore dependence in N(q), F(q), etc.

Quasi-1D Euler Equations (cont.)

The boundary conditions are imposed by setting the fluxes at the left and right boundaries to appropriate numerical flux functions:

$$F(q(0)) = \hat{F}(q(0), q_L), \qquad \text{and} \qquad F(q(1)) = \hat{F}(q(1), q_R).$$

- q_L and q_R are the boundary states at $x = 0$ and $x = 1$.
- $\hat{F}$ is a upwinding numerical flux function.
- These conditions have the effect of setting the incoming characteristic variables to the appropriate value.

Quasi-1D Euler Equations (cont.)

For generality, we will assume there is a shock at $x_s \in (0, 1)$.

- Consequently, we must also satisfy the Rankine-Hugoniot jump condition

$$[F(q)]_{x_s^-}^{x_s^+} = 0$$

- This condition states that the conservative flux is continuous across the shock.

$$[F(q)]_{x_s^-}^{x_s^+} \equiv \left.\begin{pmatrix} \rho u A \\ (\rho u^2 + p)A \\ (e + p)uA \end{pmatrix}\right|_{x=x_s^+} - \left.\begin{pmatrix} \rho u A \\ (\rho u^2 + p)A \\ (e + p)uA \end{pmatrix}\right|_{x=x_s^-}$$

$$f(x_s^-) = \lim_{x \to x_s^-} f(x) \quad , \quad f(x_s^+) = \lim_{x \to x_s^+} f(x)$$

Quasi-1D Euler Equations (cont.)

For the functional we use the integrated pressure:

$$\begin{aligned} J(q) &= (1, p(q))_\Omega \\ &= \int_0^1 p\, dx \\ &= \int_0^{x_s^-} p\, dx + \int_0^{x_s^+} p\, dx. \end{aligned}$$

- This is chosen because it is similar to the lift/drag functional

Deriving the Adjoint Equation

Rather than deriving the adjoint operator based on Green's extended identity, I will show you the (closely related) Lagrangian approach to obtaining the adjoint PDE and boundary conditions.

As we have done in the discrete case, we define a Lagrangian by weighting the relevant equations by multipliers (adjoints) and adding them to the functional:

$$L(q, x_s, \psi, \psi_s) = \int_0^1 p\, dx + \int_0^1 \psi^T N(q)\, dx + \psi_s^T [F(q)]_{x_s^-}^{x_s^+} \\ + [\psi^T (F(q) - \hat{F}(q, q_L))]_{x=0} - [\psi^T (F(q) - \hat{F}(q, q_R))]_{x=1}.$$

Deriving the Adjoint Equation (cont.)

- Note the inclusion of the Rankine-Hugoniot conditions and an associated adjoint $\psi_s \in \mathbb{R}^3$.
- We have also included the boundary conditions in terms of the flux functions.

To obtain the adjoint PDE, we Fréchet differentiate with respect to q in the direction v, and require the result to be zero for all v.

$$L'[q]v = \int_0^1 \frac{\partial p}{\partial q} v \, dx + \int_0^{x_s^-} \psi^T N'[q] v \, dx$$

$$+ \int_{x_s^-}^1 \psi^T N'[q] v \, dx + \psi_s^T \left[\frac{\partial F}{\partial q} v \right]_{x_s^-}^{x_s^+}$$

$$+ \left[\psi^T \left(\frac{\partial F}{\partial q} - \frac{\partial \hat{F}}{\partial q} \right) v \right]_{x=0} - \left[\psi^T \left(\frac{\partial F}{\partial q} - \frac{\partial \hat{F}}{\partial q} \right) v \right]_{x=1}$$

Deriving the Adjoint Equation (cont.)

Now, $N'[q]v = \left(\frac{\partial}{\partial x}\left[\frac{\partial F}{\partial q} v\right] - \frac{\partial G}{\partial q} v \right)$

So, using integration by parts gives us

$$\int_0^{x_s^-} \psi^T N'[q] v \, dx + \int_{x_s^+}^{1} \psi^T N'[q] v \, dx$$

$$\left.\begin{aligned} &= \int_0^{x_s^-} v^T \left[-\left(\frac{\partial F}{\partial q}\right)^T \frac{\partial \psi}{\partial x} - \left(\frac{\partial G}{\partial q}\right)^T \psi \right] dx \\ &\quad + \int_{x_s^+}^{1} v^T \left[-\left(\frac{\partial F}{\partial q}\right)^T \frac{\partial \psi}{\partial x} - \left(\frac{\partial G}{\partial q}\right)^T \psi \right] dx \end{aligned}\right\} \text{merge this}$$

$$+ \left[v^T \left(\frac{\partial F}{\partial q}\right)^T \psi \right]_{x=x_s^-} - \left[v^T \left(\frac{\partial F}{\partial q}\right)^T \psi \right]_{x=0}$$

$$+ \left[v^T \left(\frac{\partial F}{\partial q}\right)^T \psi \right]_{x=1} - \left[v^T \left(\frac{\partial F}{\partial q}\right)^T \psi \right]_{x=x_s^+}$$

Deriving the Adjoint Equation (cont.)

So,

$$L'[q]v = \int_0^1 v^T \left[-\left(\frac{\partial F}{\partial q}\right)^T \frac{\partial \psi}{\partial x} - \left(\frac{\partial G}{\partial q}\right)^T \psi + \left(\frac{\partial p}{\partial q}\right)^T \right] dx$$

$$\left.\begin{aligned} &+ \left(\psi_s^T - \psi(x_s^+)^T\right) \left.\frac{\partial F}{\partial q}\right|_{x_s^+} v(x_s^+) \\ &- \left(\psi_s^T - \psi(x_s^-)^T\right) \left.\left(\frac{\partial F}{\partial q}\right)\right|_{x_s^-} v(x_s^-) \end{aligned}\right\} (*)$$

$$- \left[v^T \left(\frac{\partial \hat{F}}{\partial q}\right)^T \psi \right]_{x=0} + \left[v^T \left(\frac{\partial \hat{F}}{\partial q}\right)^T \psi \right]_{x=1} = 0$$

$$\forall \; v$$

Deriving the Adjoint Equation (cont.)

In summary, the adjoint PDE for the quasi-1D Euler equations is

$$\left(\frac{\partial G}{\partial q}\right)^T \psi$$

$$N'[q]^*\psi - g'[q] = -\left(\frac{\partial F}{\partial q}\right)^T \frac{\partial \psi}{\partial x} - \cancel{G^T}\psi + \left(\frac{\partial p}{\partial q}\right)^T = 0$$

and the boundary conditions are

$$\left[\left(\frac{\partial \hat{F}}{\partial q}\right)^T \psi\right]_{x=0} = 0, \qquad \left[\left(\frac{\partial \hat{F}}{\partial q}\right)^T \psi\right]_{x=1} = 0$$

$$\hat{F} = \frac{1}{2}\left(\frac{\partial F}{\partial q} q + \frac{\partial F}{\partial q} q_L\right) - \frac{1}{2}\left|\frac{\partial F}{\partial q}\right|(q - q_L)$$

$$\frac{\partial \hat{F}}{\partial q} = \frac{1}{2}\left(\frac{\partial F}{\partial q} - \left|\frac{\partial F}{\partial q}\right|\right) + O(\|q - q_L\|)$$

analogous to $\frac{1}{2}(\lambda - |\lambda|)$ in scalar case

Deriving the Adjoint Equation (cont.)

Furthermore, we have that the adjoint variables are continuous at the shock: (from (*) on slide 11)

$$\psi_s - \psi(x_s^+) = 0$$

$$\psi_s - \psi(x_s^-) = 0$$

$$\Rightarrow \quad \boxed{\psi(x_s^-) = \psi_s = \psi(x_s^+)}$$

Shock Conditions

We also need to consider the implications of the shock location on the adjoint variables. To do so, we take the Fréchet derivative of L with respect to x_s in the direction δ_s and set the result to zero.

$$L'[x_s]\delta_s = \frac{\partial}{\partial x_s}\left(\int_0^{x_s^-} p\,dx + \int_{x_s^+}^1 p\,dx\right)\delta_s \leftarrow \text{use Leibniz}$$

$$+ \frac{\partial}{\partial x_s}\left(\int_0^{x_s^-}\psi^T N(q)\,dx + \int_{x_s^+}^1 \psi^T N(q)\,dx\right)\delta_s$$

$$+ \frac{\partial}{\partial x_s}\psi_s^T\left[F(q)\big|_{x_s^+} - F(q)\big|_{x_s^-}\right]\delta_s = 0$$

(B.C.s do not depend on x_s)

Shock Conditions (cont.)

$$L'[x_s]\delta_s = p\big|_{x_s^-}\delta_s - p\big|_{x_s^+}\delta_s$$

$$+\left[\psi^T N(q)\right]\Big|_{x_s^-}\delta_s - \left[\psi^T N(q)\right]_{x_s^+}\delta_s$$

$$+\left[\psi_s^T \frac{\partial F}{\partial x}\right]_{x_s^+}\delta_s - \left[\psi_s^T \frac{\partial F}{\partial x}\right]_{x_s^-}\delta_s$$

$$= \Big\{ -\left[p\right]_{x_s^-}^{x_s^+} + \left[\psi^T\left(\cancel{\frac{\partial F}{\partial x}} - G\right)\right]_{x_s^-} - \left[\psi^T\left(\cancel{\frac{\partial F}{\partial x}} - G\right)\right]_{x_s^+}$$

$$- \cancel{\left[\psi_s^T \frac{\partial F}{\partial x}\right]_{x_s^-}} + \cancel{\left[\psi_s^T \frac{\partial F}{\partial x}\right]_{x_s^+}} \Big\}\delta_s$$

recall $\psi_s = \psi(x_s^-) = \psi(x_s^+)$ $\qquad \doteq 0$

Shock Conditions (cont.)

$$-[p]_{x_s^-}^{x_s^+} + [\psi^T G]_{x_s^-}^{x_s^+} = 0$$

Recall $G = \begin{pmatrix} 0 \\ p\,dA/dx \\ 0 \end{pmatrix}$

$$\Rightarrow -[p]_{x_s^-}^{x_s^+} + \left[\psi_2\, p \frac{dA}{dx}\right]_{x_s^-}^{x_s^+} = 0,$$

but ψ_2 and $\frac{dA}{dx}$ are continuous

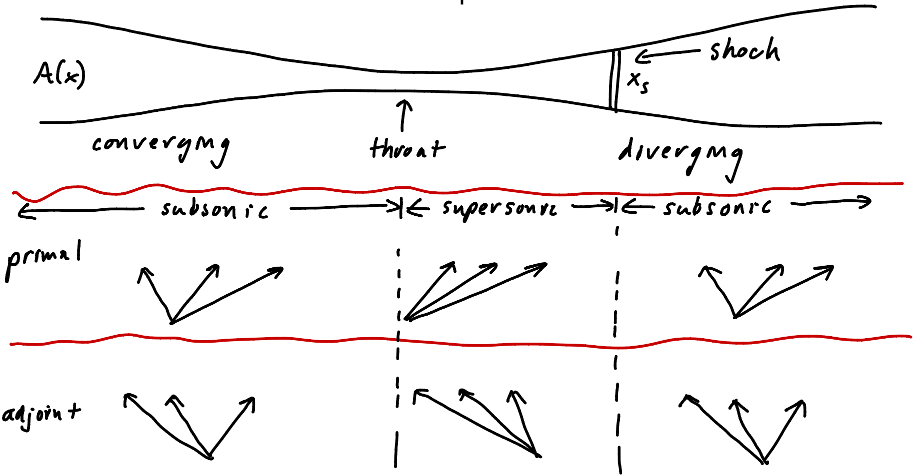
$$\Rightarrow \psi_2 \frac{dA}{dx} \underbrace{\cancel{[p]_{x_s^-}^{x_s^+}}}_{1} = \underbrace{\cancel{[p]_{x_s^-}^{x_s^+}}}_{1} \quad \Rightarrow \psi_2 \frac{dA}{dx} = 1$$

Thus, the adjoint variables have internal boundary conditions at the shock:

$$\boxed{\psi_2(x_s) = \left(\frac{\partial A}{\partial x}\right)^{-1}}$$

Shock Conditions (cont.)

The need for the internal BC on the adjoint is easily understood if we consider the characteristic wave speeds in the domain:



Singularity at a Sonic Point

A more detailed analysis of the adjoint shows that there is a logarithmic singularity at the sonic throat in the converging-diverging nozzle [GP01].

- The source of this issue is hinted at by the characteristic speeds on either side of the throat.
- No such singularity appears to exist in 2- and 3-D flows at sonic lines/surfaces; see [GP97] for a possible explanation.

Some Analytic and Numerical Solutions

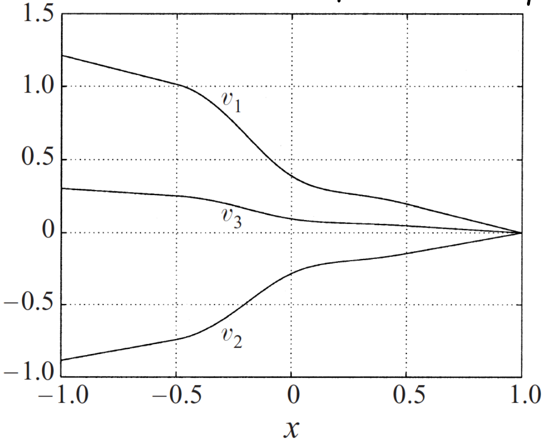
Quasi-1D Euler Adjoint Solutions

Giles and Pierce used a Green's function approach to derive Analytic solutions to the quasi-1D Euler equations [GP97].

- Correspond to particular choices of functional
- Verify the predictions of the theory

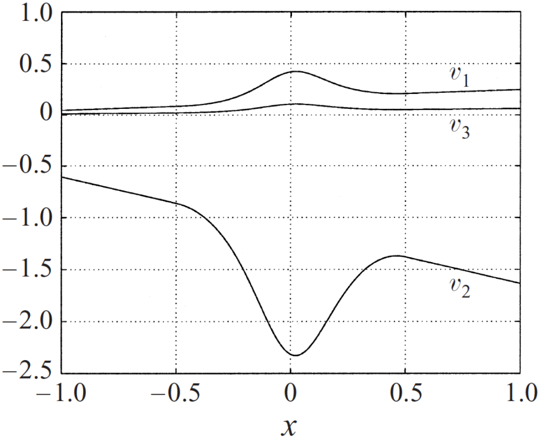
Quasi-1D Euler Adjoint Solutions (cont.)

their $v = \psi$



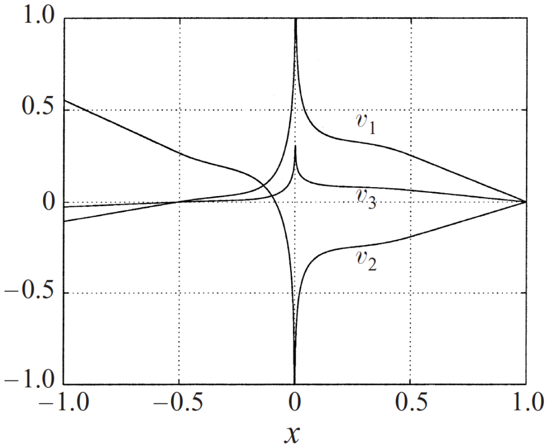
Adjoint solutions, supersonic flow [GP01]

Quasi-1D Euler Adjoint Solutions (cont.)



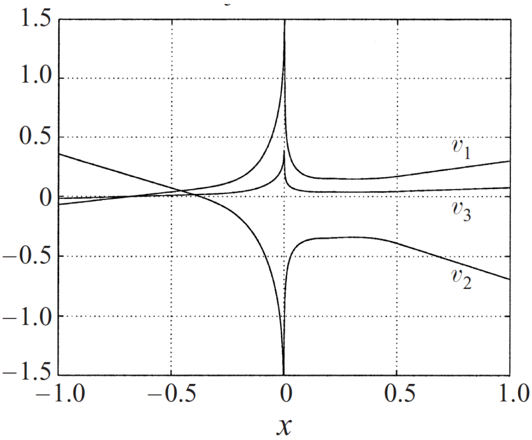
Adjoint solutions, subsonic flow [GP01]

Quasi-1D Euler Adjoint Solutions (cont.)



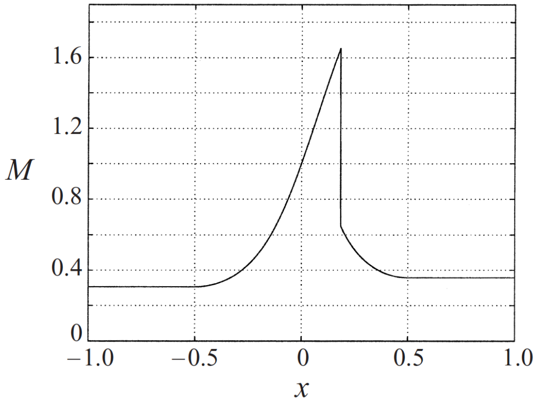
Adjoint solutions, transonic isentropic flow [GP01]

Quasi-1D Euler Adjoint Solutions (cont.)



Adjoint solutions, transonic shocked flow [GP01]

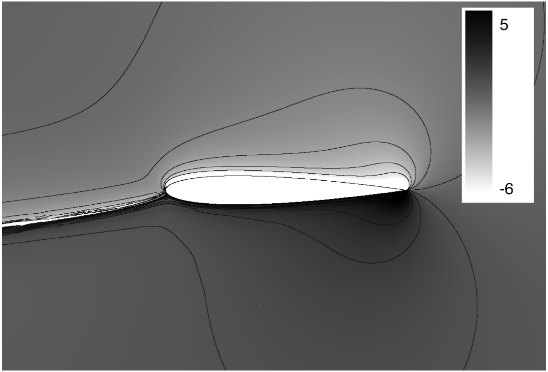
Quasi-1D Euler Adjoint Solutions (cont.)



Mach number for transonic shocked flow [GP01]

Singularity Along Stagnation Streamline

Another adjoint feature worth highlighting is the singularity along the stagnation streamline for some functionals, most notably lift.



ρu component of lift adjoint [FD11]

References

[FD11] Krzysztof J. Fidkowski and David L. Darmofal, *Review of output-based error estimation and mesh adaptation in computational fluid dynamics*, AIAA Journal **49** (2011), no. 4, 673–694.

[GP97] M. B. Giles and N. A. Pierce, *Adjoint equations in CFD: duality, boundary conditions, and solution behaviour*, 13th AIAA Computational Fluid Dynamics Conference (Snowmass Village, CO), no. AIAA–97–1850, June 1997.

[GP01] Michael B. Giles and Niles A. Pierce, *Analytic adjoint solutions for the quasi-one-dimensional Euler equations*, Journal of Fluid Mechanics **426** (2001), 327–345.