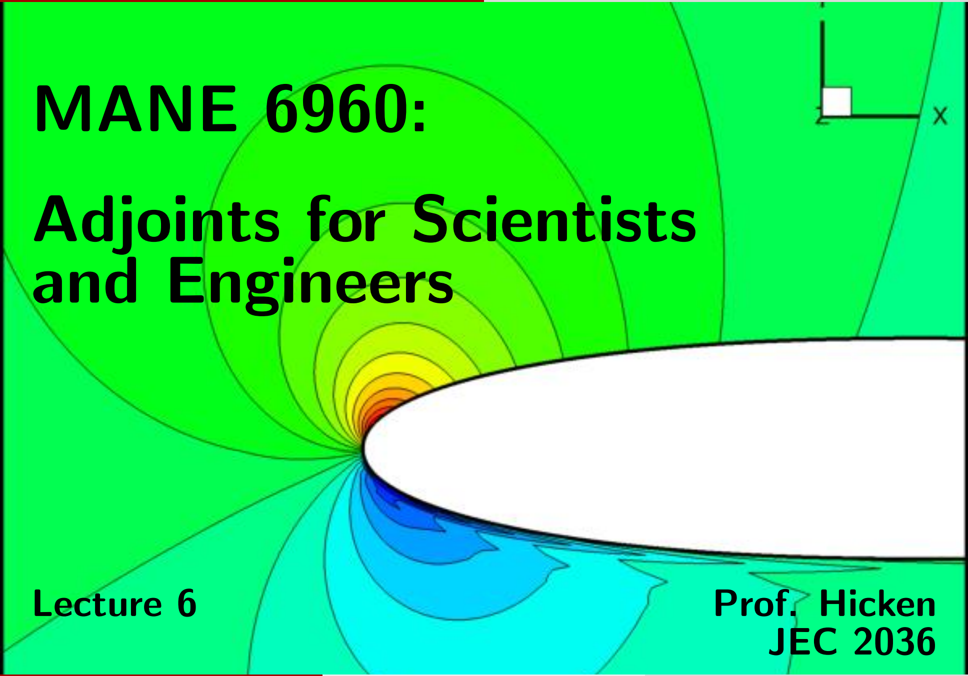


# MANE 6960:

# Adjoints for Scientists and Engineers

**Lecture 6**

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**JEC 2036**

# Adjoints and Over-determined Boundary Value Problems

# Lecture Objective

Today's lecture is a bit of a digression whose purpose is to illustrate an application of the adjoint.

Specifically, we will learn how solutions to the homogeneous adjoint BVP can be used to determine compatibility conditions on the source $f$ and boundary data $b$.

# Motivation

To motivate today's topic, consider the linear (matrix) problem

$$Ax = b, \qquad \begin{bmatrix} A \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

where $A$ is an $m \times n$ matrix, with $m \geq n$. In other words, we consider potentially over-determined linear systems.

Now, suppose that $A^T y = 0$ for some $y \neq 0$. Then

$$y^T A x = \cancel{(A^T y)}^{0} x = y^T b$$

$$\Longrightarrow \quad \boxed{y^T b = 0}$$

# Motivation (cont.)

Thus, in order for $Ax = b$ to have a solution, the right-hand-side vector $b$ must be orthogonal to every nontrivial solution of the homogeneous equation $A^T y = 0$.

The solution to the homogeneous adjoint BVP,

$$\begin{aligned} L^* \psi &= 0, \qquad \forall x \in \Omega, \\ B^* \psi &= 0, \qquad \forall x \in \Gamma, \end{aligned} \tag{Adj0}$$

plays a similar role to $y$ above, as we will now show.

# Compatibility in Over-determined Systems

**Theorem: Compatibility**

The primal boundary-value problem

$$\begin{aligned} Lu &= f, \qquad \forall x \in \Omega, \\ Bu &= b, \qquad \forall x \in \Gamma, \end{aligned}$$

has a solution only if

$$(\psi, f)_\Omega + (C^* \psi, b)_\Gamma = 0,$$

for all nontrivial solutions to the homogeneous adjoint BVP (Adj0), where $C^*$ is the boundary operator from Green's identity.

# Compatibility in Over-determined Systems (cont.)

Proof: From Green's extended identity we have

$$(\psi, Lu)_\Omega - (u, L^*\psi)_\Omega = (B^*\psi, Cu)_\Gamma - (C^*\psi, Bu)_\Gamma$$

If $\psi$ satisfies the homogeneous adjoint BVP, (Adj0), the above simplifies to

$$(\psi, Lu)_\Omega = -(C^*\psi, Bu)_\Gamma \quad \leftarrow \text{“} y^T b = 0 \text{”}$$

Substituting $Lu = f$ and $Bu = b$ gives the desired result.

# Compatibility in Over-determined Systems (cont.)

Some remarks are in order regarding this result:

- Although our focus in previous lectures has been adjoints in the context of functionals, this result does not depend on an explicit functional, just Green's identity.
  (The functional "hiding" here is the trivial one, $J(u) = 0$)
- If the homogeneous adjoint BVP has only the trivial solution, $\psi = 0$, then the condition

  $$(\psi, f)_\Omega + (C^* \psi, b)_\Gamma = 0,$$

  is satisfied for all $f$ and $b$.

# Example 1: over-constrained point mass

Consider a point mass governed by Newton's second law,

$$m\ddot{x}(t) = f(t), \qquad \forall t \in [0, T]$$

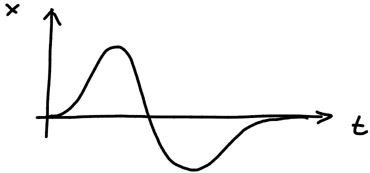
and subject to the initial and terminal conditions

$$\begin{aligned} x(0) &= 0, \qquad & \dot{x}(0) &= 0 \\ x(T) &= 0, \qquad & \dot{x}(T) &= 0. \end{aligned}$$

What are the compatibility conditions on the external force $f(t)$?

# Example 1: over-constrained point mass (cont.)

Aside: the particle must behave as, e.g.


Let's find Green's identity here:

$$\int_{t=0}^{T} y m \frac{d^2 x}{dt^2} dt = \int_{t=0}^{T} \frac{d}{dt}\left(y m \frac{dx}{dt}\right) dt - \int_{t=0}^{T} \frac{dy}{dt} m \frac{dx}{dt} dt$$

$$= \left[ y m \frac{dx}{dt} \right]_{t=0}^{T} - \int_{t=0}^{T} \frac{d}{dt}\left(\frac{dy}{dt} m x\right) dt + \int_{t=0}^{T} \frac{d^2 y}{dt^2} m x \, dt$$

# Example 1: over-constrained point mass (cont.)

So,
$$\int_{t=0}^{T} y\, m\frac{d^2x}{dt^2}\,dt = \int_{t=0}^{T}\left(m\frac{d^2y}{dt^2}\right)x\,dt + \cancel{\left[y\, m\frac{dx}{dt}\right]_{t=0}^{T}} \to 0$$
$$- \cancel{\left[\frac{dy}{dt}\, m\, x\right]_{t=0}^{T}} \to 0$$

∴ initial and terminal condition

∴ homogeneous adjoint BVP is

$$m\frac{d^2y}{dt^2} = 0, \quad \forall\, t \in [0,T]$$

There are two nontrivial solutions:

$$y = 1, \qquad y = t$$

(or $y = c_0 + c_1 t \quad \forall\, c_0, c_1 \in \mathbb{R}$)

# Example 1: over-constrained point mass (cont.)

$$\text{"}(\psi, f)_{\Omega} + (C^{*}\psi, b)_{\Gamma}\text{"} = (y, f)_{[0,T]} + 0 = 0$$

Therefore, the applied force must satisfy

$$\int_{t=0}^{T} f(t)\, dt = 0$$

and $$\int_{t=0}^{T} f(t)\, t\, dt = 0$$

# Example 2: loaded elastic free bar

Consider the following ODE for an elastic bar:

$$
\begin{aligned}
I(x)\frac{d^2u}{dx^2} - \sigma &= 0, \qquad && \forall x \in [0,l] \\
\frac{d^2\sigma}{dx^2} &= q(x), \qquad && \forall x \in [0,l],
\end{aligned}
$$

Are there any conditions on the load $q(x)$ if the BCs are

$$
\begin{aligned}
\sigma(0) &= 0, \qquad & \frac{d\sigma}{dx}(0) &= \rho_0, \\
\sigma(l) &= 0, \qquad & \frac{d\sigma}{dx}(l) &= \rho_l?
\end{aligned}
$$

# Example 2: loaded elastic free bar (cont.)

As before, we start by deriving Green's identity:

$$\int_{x=0}^{\ell} [\psi \quad \phi] \begin{bmatrix} I(x)\frac{d^2u}{dx^2} - \sigma \\ \frac{d^2\sigma}{dx^2} \end{bmatrix} dx$$

$$= \int_{x=0}^{\ell} \psi\left(I\frac{d^2u}{dx^2} - \sigma\right) dx + \int_{x=0}^{\ell} \phi \frac{d^2\sigma}{dx^2} dx$$

# Example 2: loaded elastic free bar (cont.)

$$= \int_{x=0}^{\ell} \frac{d}{dx}\left(\psi I \frac{du}{dx}\right) dx - \int_{x=0}^{\ell} \frac{d}{dx}(I\psi) \frac{du}{dx}\, dx - \int_{x=0}^{\ell} \psi \sigma\, dx$$

$$+ \int_{x=0}^{\ell} \frac{d}{dx}\left(\phi \frac{d\sigma}{dx}\right) dx - \int_{x=0}^{\ell} \frac{d\phi}{dx}\frac{d\sigma}{dx}\, dx$$

$$= \left[\psi I \frac{du}{dx}\right]_{x=0}^{\ell} + \left[\phi \frac{d\sigma}{dx}\right]_{x=0}^{\ell} - \int_{x=0}^{\ell} \psi \sigma\, dx$$

$$- \int_{x=0}^{\ell} \frac{d}{dx}\left(\frac{d}{dx}(I\psi) u\right) dx + \int_{x=0}^{\ell} \frac{d^2}{dx^2}(I\psi)\, u\, dx$$

$$- \int_{x=0}^{\ell} \frac{d}{dx}\left(\frac{d\phi}{dx}\sigma\right) dx + \int_{x=0}^{\ell} \frac{d^2\phi}{dx^2}\sigma\, dx$$

# Example 2: loaded elastic free bar (cont.)

$$= \left[\psi\, I\, \frac{du}{dx}\right]_{x=0}^{\ell} - \left[\frac{d}{dx}(I\psi)\, u\right]_{x=0}^{\ell} \Big\}\ \text{“}(B^*\psi,\ Cu)_\Gamma\text{”}$$

$$+ \left[\phi\, \frac{d\sigma}{dx}\right]_{x=0}^{\ell} - \left[\frac{d\phi}{dx}\,\sigma\right]_{x=0}^{\ell} \Big\}\ \text{“}(Bu,\ C^*\psi)_\Gamma\text{”}$$

$$\underbrace{+ \int_{x=0}^{\ell} \frac{d^2}{dx^2}(I\psi)\, u\, dx + \int_{x=0}^{\ell} \left(\frac{d^2\phi}{dx^2} - \psi\right)\sigma\, dx}_{\text{“}(u,\ L^*\psi)_\Omega\text{”}}$$

# Example 2: loaded elastic free bar (cont.)

First, identify the homogeneous adjoint BVP and its (nontrivial) solutions:

$$\frac{d^2\psi}{dx^2} - \psi = 0, \quad \forall x \in [0,\ell]$$

$$\frac{d^2}{dx^2}(I\psi) = 0, \quad \forall x \in [0,\ell]$$

$$\psi(0) = 0, \quad \psi(\ell) = 0$$

$$\frac{d}{dx}(I\psi)\Big|_{x=0} = 0, \quad \frac{d}{dx}(I\psi)\Big|_{x=\ell} = 0$$

$\therefore$ The nontrivial solutions are

$$\phi = 1, \quad \phi = x$$

$$\Longrightarrow \psi = 0$$

# Example 2: loaded elastic free bar (cont.)

Since we are considering solutions to the homogeneous adjoint BVP, the Green's identity becomes

$$\int_{x=0}^{\ell} [\psi, \phi] \underbrace{\begin{bmatrix} \overbrace{I \frac{d^2 u}{dx^2} - \sigma}^{0} \\ \frac{d^2 \sigma}{dx^2} \end{bmatrix}}_{q(x)} dx = \Big(\phi \overbrace{\frac{d\sigma}{dx}}^{\rho_\ell}\Big)_{x=\ell} - \Big(\phi \overbrace{\frac{d\sigma}{dx}}^{\rho_0}\Big)_{x=0}$$

$$\Longrightarrow \int_{x=0}^{\ell} \phi \, q \, dx = \phi(\ell)\rho_\ell - \phi(0)\rho_0$$

$$\Longrightarrow \boxed{\begin{aligned} \int_{x=0}^{\ell} q \, dx &= \rho_\ell - \rho_0 \\ \int_{x=0}^{\ell} q \, x \, dx &= \rho_\ell \, \ell \end{aligned}}$$

# References