The True Squaring of the Circle and the Hyperbola

In its own species of proportion,
Discovered and proved.

James Gregory

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Aberdeen, Scottland

Welcome, Reader of Geometry

I was contemplating at length, dear Reader, whether analysis with its many operations, and the general method of investigating all proportions of a quantity, are sufficient, as Descartes claims at the beginning in his Geometry. If it were indeed so, it could be that it is possible to use it to demonstrate the oft-sung feat of squaring of the circle. Whenever I turn this over in my mind I readily see from the discoveries made so far on the properties of the circle that no one can establish such a result by use of analysis. Thus it occurred to me to search out the method that follows, first by considering the common circle. In doing so I hit upon the convergent sequence of polygons, the limit of which is the circular sector. I saw here at once a trace of analysis. It followed that the convergent series applies naturally not only in the case of the circle, but likewise in more general consideration. Thus from the properties of the circle followed the cases of ellipse and the hyperbola with no effort, and thus the infallible quadrature of every conic section was revealed to me. However the convergent series of polygons soon turned the tables on me, and I was forced to ply all of my arts to the insuperable difficulty of discovering its limit. By pondering its very nature, I saw that the purpose of analysis as well as of common algebra is not only to resolve problems, but likewise to demonstrate their impossibility (when its useful). After this first difficulty was put to rest, I also conquered the more general case, as I claim above. Indeed I uncovered the true and legitimate quadrature, in its own species of proportion, not only of the circle (which I had set out to do at the beginning) but of all of the conic sections, and I revealed an entire species of proportion previously unknown to the field of Geometry. I can calculate this proportion as closely as needed in relation to the dimension of the conic section, and can easily demonstrate the very rapid extraction of square roots of solid surds (unless I am very much mistaken). In fact, such approximations apply mathematically to every incommensurable proportion, such that we might better understand the nature of the proportion. I might speak of a proportion insofar as it is derived by some analytic operations (i.e. of the common arithmetic), and the proportion is then known to us in numbers or in the continuation of some discrete quantities. I am not afraid to render these operations geometrically in the style of Decartes.

First note that we always know how to relate commensurable quantities, or those having a common number between them. We cannot perceive incommensurable proportions unless they themselves are relatively commensurable, for the infinitude renders us ignorant, blunting our minds and impeding our simple notions. Two of the five arithmetic operations, addition and subtraction, are the most elementary. Multiplication is composed of addition, division from subtraction. Square root extraction, which in general is nothing more than finding commensurable proportions, and which closely approaches an analytically incommensurable proportion, is composed from the aforementioned four. Our sixth operation, which in general is nothing more than finding commensurable proportions closely approximating a non-analytic proportion, is composed of the previous five.

It will be noted that just as no fractional numbers are produced from integers by addition, subtraction or multiplication, but only by division, and just as no incommensurable numbers are produced from commensurables by addition subtraction, multiplication or division, but only by square root extraction, so non-analytic quantities are not born from analytic numbers by addition, subtraction, multiplication, division, or square root extraction, but rather from this sixth operation. As such this new operation of ours may be added to arithmetic and a new species of ratio may be added to geometry, which (as I demonstrate in this treatise) allows us to understand the analyticity of the ratio of the circle to the square of its diameter as a kind of ratio we currently know how to produce, just as square root extraction allows us to understand the ratio between the side of the square and its diameter in terms of commensurability.

It seems clear to me that the whole of the proof cannot be reduced to gemometric language, for in order to accomplish this we would use no small number of analytic magnitudes that are mutually related and incommensurable in kind. I am amazed that this idea has never been written about, as it opens wide the plains of discovery. For from this approach comes proof that the mesolabe* cannot be achieved by the power of the straightedge and compass, likewise that not always can affected equations be reduced to pure ones†, or such similar things which even by the most outstanding mathematicians are impossible to show by means of analysis, and are sought in vain daily by the uninititated. Euclid dedicated the entirety of his tenth book (except for a few minor general propositions) on incommensurables produced from the extraction of square roots. I don't know to what extent this material is covered in any other treatise, although

*An apparatus of Eratosthenes used to construct *n*th roots.

†Pure equations contain only one term raised to a power, as with $x^7 = c$. Affected equations contain multiple such terms, as with $x^9 - ax^b + x = c$.

it be not only very useful to speculative geometry but is also an extremely remarkable theory in its own right. For example, in a geometric progression in which the first term is given by a commensurable length or a power of such, and the second is any other whichever, be it binomial, trinomial, etc., it is impossible that the sequences progressing infinitely from these two terms will be mutually commensurable by a length or a power of such. I have more to offer, but perhaps I shall reserve this for a more favorable time. It is enough for the timebeing that this matter has been proven analytically. And if indeed analysis agreeing to such a glaring degree isn't compelling enough, there is the geometry, neither refuting nor being refuted by analysis, which in the past was proved by geometry. From this invention I deduce some new angular sections and a doctrine of logarithms, some easy, most expeditious in practice and fortified by geometric demonstration. As far as the legnthy construction of logarithms is concerned, a more preferable conjecture is seen than what is known, as well as the division of an angle into more than five equal parts. Counting by prime numbers can be done only with great difficulty. I demonstrate all of these sums (or those that I can) quickly and clearly. And I am not careful in citations, in so far as foreign books of such work are lacking, and indeed I suppose you are not poorly versed in geometry, otherwise you will acquire no fruit by your labor. It remains for me to warn you that I always aspire to the demonstration of utmost generality, not only in this but also in my other works.

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Definitions.

1. Let two lines be drawn from the center of a circle, ellipse, or hyperbola to its perimeter. We call the piece bound by those two lines and the segment of the perimeter a sector.

2. Let the segment of the perimeter between the two lines be subtended any number of times, forming rectilinear triangles (where the common vertex is the center of the conic section and the bases are the subtending lines). Then if the conic section is a circle or ellipse we call the figure created by combining these triangles a regular inscribed triangle, if it is a hyperbola then we call the figure a regular circumscribed polygon.

3. Let the segment of the perimeter between the lines be made tangent to lines any number of times and let lines be drawn from the tangents to the center of the conic section, and further require that each of the quadrilaterals, understood as being composed from successive tangent lines and the lines to the center, be equal. If the conic section is a circle or an ellipse, then I call the figure created by the combination of these a regular circumscribed polygon. If it is a hyperbola then I call it a regular inscribed polygon.

4. Let all of the vertices of the subtending angles (except those to the center of the conic section) of the regular polygon touch all of the points of contact of the regular polygon with the tangents. I call this a complex polygon.

5. We say a magnitude is composed of magnitudes, when one magnitude makes other magnitudes by addition, subtraction, multiplication, division, root extraction, or any other operation imaginable.

6. When a magnitude is composed of magnitudes by addition, subtraction, multiplication, division, or root extraction, we say it is composed analytically.
7. When magnitudes can be composed analytically of magnitudes which are mutually commensurable, we say they are mutually analytic.
8. Let the magnitude X be composed of some magnitudes A, B, C, D, and E, and the magnitude Z be composed of magnitudes F, G, C, D, and E by the same method and operations as X only with the magnitudes F and G in place of A and B. If this is so, then we say that the magnitude X is composed of A and B by the same method as Z is composed of F and G.
9. Let there be two magnitudes A and B of which are composed two other magnitudes C and D, whose the difference is less than that of A and B. Also let E be composed from C and D by the same method as C is composed of A and B, and F be composed from C and D by the same method as D is composed of A and B. And further still let G be composed from E and F by the same method as E is composed of C and D and C is composed by A and B, and let H be composed from E and F by the same method as F is composed of C and D and C is composed by A and B. Continue thus. I call this a convergent series.
10. Terms being placed next to each other as A and B, or C and D, or E and F, or G and H, are called convergent terms.

Assumptions.

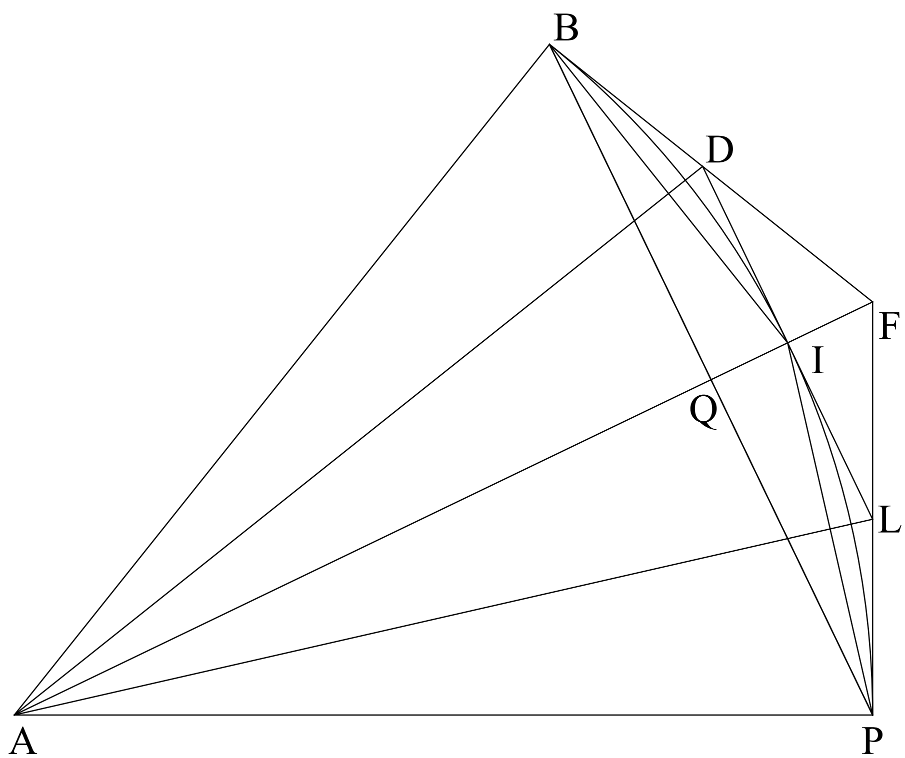
1. We assume that magnitudes composed of given mutually analytic magnitudes be mutually analytic themselves as well as analytic with the given magnitudes.

2. Likewise we assume that magnitudes that cannot be analytically composed from given mutually analytic magnitudes not be analytic with the given magnitudes.

The preceding assumptions may perhaps be seen by some as obscure, but they will clarified by an analysis of the elements.

The True Squaring of the Circle and the Hyperbola.

Let BIP be a segment of a circle, ellipse, or hyperbola with center A. The triangle ABP may be completed, and from the points B and P on the segment tangents BF and PF may be drawn, which will meet each other at the point F. The line AF is thus produced, which intersects the segment at point I and the line BP at the point Q. From this the lines BI and PI are joined.



Proposition I. Theorem.

The quadrilateral $BAPI$ is half of the quadrilateral $BAPF$ plus the triangle BAP.

The line AQ is drawn through F meeting with the the two lines FB and FP, which are tangent to the segment at the points B and P. Therefore the line AQ contacts the line BP at its bisector Q. We see from this that the triangle AQB is equal to the triangle AQP, the triangle FBQ equals triangle FQP, and the triangle ABF equals triangle APF. Therefore the triangle ABF is half of the quadrilateral $ABFP$. Similarly the triangle ABI is half of the quadrilateral $ABIP$ and the triangle ABQ is half of triangle ABP. ABF, ABI, and ABQ each have the same altitude and share one base, but the other bases AF, AI, and AQ progress arithmetically. Therefore the two quadrilaterals $ABFP$ and $ABIP$ and the triangle ABP are clearly in arithmetic progression with a ratio of AF to AI.

Let the line DL be drawn tangent to the segment at the point I so that it will meet the lines BF and PF at the points D and L, completing the polygon $ABDLP$.

Proposition II. Theorem.

The quadrilateral $ABFP$ plus quadrilateral $ABIP$ is to the double of quadrilateral $ABIP$ as quadrilateral $ABFP$ is to polygon $ABDLP$.

The line AF is drawn through the point of tangency of the line DL with the segment, and is likewise drawn through the meeting point of the two lines FB and FP, which terminate the line DL and touch the segment in two points. Therefore the line DL is bisected at the point I. Because of this the triangle FDI equals the triangle FIL and the triangle ABF equals the triangle APF. Thus the quadrilateral $ABDI$ equals quadrilateral $APLI$, and further quadrilateral $APLI$ is half of the polygon $ABDLP$. It is obvious from the above demonstration that the triangle AIL equals triangle ALP, but triangle ALF is to triangle ALI as FA is to AI and FA is to AI as quadrilateral $ABPF$ is to quadrilateral $ABIP$. Thus quadrilateral $ABFP$ is to quadrilateral $ABIP$ as triangle ALF is to triangle AIL. So putting it together, quadrilateral $ABFP$ plus $ABIP$ is to quadrilateral $ABIP$ as triangle AFL plus triangle AIL, i.e. triangle AFP, is to triangle AIL. Doubling this result, $ABFP$ plus $ABIP$ is to the double of $ABIP$ as triangle AFP is to quadrilateral $AILP$. Note that triangle AFP is half of quadrilateral $ABFP$ and quadrilateral $AILP$ is half of polygon $ABDLP$. Therefore quadrilateral $ABFP$ plus quadrilateral $ABIP$ is to the double of $ABIP$ as quadrilateral $ABFP$ is to polygon $ABDLP$.

Proposition III. Theorem.

The triangle BAP plus the quadrilateral $ABIP$ is to the quadrilateral $ABIP$ as the double of the quadrilateral $ABIP$ is to the polygon $ABDLP$.

In the preceding proposition it is shown that the sum of $ABFP$ and $ABIP$ is to the double of quadrilateral $ABIP$ as quadrilateral $ABFP$ is to polygon $ABDLP$. By permuting we see that quadrilateral $ABFP$ plus $ABIP$ is to quadrilateral $ABFP$ as the double of quadrilateral $ABIP$ is to polygon $ABDLP$. Since quadrilaterals $ABFP$ and $ABIP$ and triangle ABP are in arithmetic progression, we have that quadrilateral $ABIP$ is to quadrilateral $ABFP$ as triangle ABP is to quadrilateral $ABIP$. Putting these together, quadrilateral $ABIP$ plus $ABFP$ is to quadrilateral $ABFP$ as triangle ABP plus quadrilateral $ABIP$ is to quadrilateral $ABIP$. On the other hand $ABIP$ plus $ABFP$ is to quadrilateral $ABFP$ as the double of quadrilateral $ABIP$ is to polygon $ABDLP$. And therefore triangle ABP plus quadrilateral $ABIP$

is to quadrilateral $ABIP$ as the double of quadrilateral $ABIP$ is to polygon $ABDLP$.

Let the lines AD and AL be drawn to meet the segment at the points E and O and intersecting the lines BI and IP at H and M. From this are joined the lines BE, EI, IO, and OP to complete the polygon $ABEIOP$.

Proposition IV. Theorem.

The polygon $ABEIOP$ is half of $ABDLP$ plus the quadrilateral $ABIP$.

From the previous theorem it is obvious that the quadrilateral $AILP$, quadrilateral $AIOP$, and triangle AIP are in arithmetic progression, and from the previous proposition one can gather easily enough that the quadrilateral $AILP$ is half of polygon $ABDLP$, quadrilateral $AIOP$ is half of polygon $ABEIOP$, and triangle AIP is half of quadrilateral $ABIP$. Thus by doubling the terms, polygon $ABDLP$, polygon $ABEIOP$, and quadrilateral $ABIP$ are in arithmetic progression.

Let lines CG and KN be drawn tangent to the segment at the points E and O and let the lines DL, DB, and LP intersect at the points C, G, K, and N to complete the polygon $ABCGKNP$.

Proposition V. Theorem.

The quadrilateral $ABIP$ plus the polygon $ABEIOP$ are to the polygon $ABEIOP$ as the double of the polygon $ABEIOP$ is to the polygon $ABCGKNP$.

From the third theorem it is obvious that the triangle ABI plus the quadrilateral $ABEI$ is to the quadrilateral $ABEI$ as the double of quadrilateral $ABEI$ is to polygon $ABCGI$. From the previous proposition it is easily concluded that triangle ABI is half of quadrilateral $ABIP$, quadrilateral $ABEI$ is half of polygon $ABEIOP$, and polygon $ABCGI$ is half of polygon $ABCGKNP$. Therefore by doubling the terms, quadrilateral $ABIP$ plus $ABEIOP$ is to polygon $ABEIOP$ as the double of polygon $ABEIOP$ is to polygon $ABCGKNP$.

From this one can easily see that the polygon $ABCGKNP$ is the harmonic mean between polygons $ABEIOP$ and $ABDLP$, which is sufficient to suggest that this may be demonstrated in perpetuity.

Scholium.

The two preceding propositions can be proved by the same method for any such complex polygons in place of the complex polygons $ABIP$ and $ABDLP$. Indeed the tangent polygon contains as many equal quadrilaterals as the subtending polygon contains triangles. And so it is evident that these ratios of the polygons continue themselves to infinity, drawing lines AN, AK, AG, and AC through points R, T, S, and V and composing other lines and polygons inside and outside of these. Note that we may say of these inscribed and circumscribed polygons that they double by inscription and circumscription.

From the previous propositions it is obvious (if we let triangle $ABP = a$ and quadrilateral $ABFP = b$) that quadrilateral $ABIP = \sqrt{ab}$ and polygon $ABDLP = \frac{2ab}{a+\sqrt{ab}}$. By the same method, let quadrilateral $ABIP = c$ and polygon $ABDLP = d$ and we have that polygon $ABEIOP = \sqrt{cd}$ and polygon $ABCGKNP = \frac{2cd}{c+\sqrt{cd}}$. It is evindent from this observation that the series of polygons converges.

And so, continuing this to infinity gives a magnitude equal to that of the sector of the circle, the ellipse, or the hyperbola given by $ABEIOP$. Indeed the difference of the complex polygons in the series always diminishes, so that all of the magnitudes may be made smaller, as we shall demonstrate in the theorems following the Scholium.

Therefore if the aforementioned series of polygons terminates, that is, if one may find a final inscribed polygon (if we may call it that) equal to the final circumscribed polygon, one would infallibly have the quadrature of the circle and the hyperbola. But since it has proved difficult, and in geometry it is perhaps altogether impossible for such a series to terminate, certain propositions are given from which to find this kind of limit of the series. After these (if it is possible) the general method for finding all of the limits of convergent series will be given.

Proposition VI. Theorem.

The difference between the triangle ABP and the quadrilateral $ABFP$ is greater than twice the difference between the quadrilateral $ABIP$ and the polygon $ABDLP$.

Denote the triangle ABP by A, quadrilateral $ABFP$ by B, quadrilateral $ABIP$ by C, and polygon $ABDLP$ by D. Since A is to C as C is to B, the difference between A and C is to A as the difference between C and B is to C. Permuting, the difference between A and C is to the difference between C and B as A is to C. Now putting these together, the difference between A and C plus the difference between C and B, that is, the difference between A and B is to the difference between C and B as $A+C$ is to C. But it has been shown that $A+C$ is to C as $2C$ is to D and the difference between A and B is to the difference between C and B as $2C$ is to D. Since $A+C$ is to C as $2C$ is to D, permuting gives $A+C$ is to $2C$ as C is to D. Dividing, the difference between A and C is to $2C$ as the difference between C and D is to D. Again permuting, the difference between A and C is to the difference between C and D as $2C$ is to D. But now the difference between A and B has been shown to be to the difference between C and B as $2C$ is to D. So from this the difference between A and B is to the difference between C and B as the difference between A and C is to the difference between C and D. However the difference between A and B is greater than the difference between C and B so that the difference between A and C is greater than the difference between C and D. Permuting the previous ratio, the difference between A and B is to the difference between A and C as the difference between C and B is to the difference between C and D. The difference between A and B is greater than the difference between A and C and so the difference between C and B is greater than the difference between C and D. But the difference between A and B is equal to the difference between A and C plus the difference between C and B. Therefore either of them is greater than the difference between C and D and so it is obvious that the difference between A and B is greater than the double of the difference between C and D, that is, the difference between the triangle ABP and the quadrilateral $ABFP$ is greater than twice the difference between the quadrilateral $ABIP$ and the polygon $ABDLP$.

Scholium.

The same method may be used to demonstrate that the difference between quadrilateral $ABIP$ and polygon $ABDLP$ is greater than twice the difference between polygons $ABEIOP$ and $ABCGKNP$. By the same method one can show that this difference is always exceeded in our halving of the complex polygon to infinity. In fact the difference between the prior inscribed and circumscribed polygons is always greater than twice the difference between the subsequent inscribed and circumscribed polygons. Thus more than half of the difference of the prior polygons is carried over to that of the difference of the subsequent ones. Therefore continuing the halving of the polygon, two complex polygons are found such that their difference is made less than any given magnitude, as we obtained in the preceding Scholium.

Let there be two unknown magnitudes, a and b, with a less than b, and let there be two given inequalities with c greater than d and c greater than e. From here we have that c is to d as $b-a$ is to $\frac{bd-ad}{c}$. Then when a is then added to the latter we get $\frac{ca+bd-ad}{c}$, where the magnitude is immediately denoted a. And also c is to e as $b-a$ is to $\frac{be-ae}{c}$. Then when this last magnitude is subtracted from b it yields $\frac{bc-be+ae}{c}$, which is then denoted b.

One may continue the convergent series from here in which the first terms are a, b and the second are $\frac{ca+bd-ad}{c}, \frac{bc-be+ae}{c}$. It is obvious that the term $\frac{ca+bd-ad}{c}$ is greater than the term a since the term a is added to $\frac{bd-ad}{c}$ giving $\frac{ca+bd-ad}{c}$. It is also clear that the term $\frac{ca+bd-ad}{c}$ is less than the term b since the difference between a and b is greater than the difference between a and $\frac{ca+bd-ad}{c}$. It is evident that the term $\frac{bc-be+ae}{c}$ is less that the term b since $\frac{be-ae}{c}$ is subtracted from b giving $\frac{bc-be+ae}{c}$. And it is further obvious that the term $\frac{bc-be+ae}{c}$ is greater than a since the difference between a and b is greater than the difference between $\frac{bc-be+ae}{c}$ and b. Therefore it is evident that the difference between the convergent terms a and b is greater than the difference between the convergent terms $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$.

However since the convergent terms a and b were given as unknown, a and b can be used in place of any of the convergent terms of the whole of the series. So by putting a and b for any of the terms, it necessarily follows from the composition of the series that $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$ are the subsequent convergent terms. And again since the difference between the terms a and b is greater than the difference between the terms $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$, it is clear that the difference between the prior convergent term is always greater than the difference between the subsequent convergent term. Because this difference always diminishes proportionally in the ratio

$b-a$ to $\frac{bc-be+ae-ca-bd+ad}{c}$, one can see that the terms of this convergent series are always decreasing. Therefore imagining this series continuing to infinity, we are able to imagine the final convergent terms being equal, where we call these equal terms the limit of the series.

Proposition VII. Problem.

To find the limit of the aforementioned series.

In order to satisfy these problems, we first want to find a magnitude that is composed by the same method from the convergent terms a and b as from the convergent terms $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$*, which follows easily from the following method. The magnitude is found by a multiplication with a and addition of b times a magnitude m, and the same may be obtained by multiplication with $\frac{ca+bd-ad}{c}$ and addition of $\frac{bc-be+ae}{c}$ times a magnitude m. Let the magnitude be z, then $za+bm$ is equal to $\frac{zca+zbd-zad+mbc-mbe+mae}{c}$ and the equation reduces to $z=\frac{mae-mbe}{ad-bd}$. This magnitude whether multiplied by a and added to mb, or multiplied by $\frac{ca+bd-ad}{c}$ and added to $\frac{mbc-mbe+mae}{c}$ produces the same magnitude in either case, namely $\frac{maae-mbae+mbad-mbbd}{ad-bd}$†. Thus the aforementioned magnitude is composed by the same method from the convergent terms a and b as from the convergent terms $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$. Because a and b are unknown magnitudes they can be any of the convergent terms of the series, where the subsequent convergent terms are $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$. Thus the magnitude $\frac{maae-mbae+mbad-mbbd}{ad-bd}$ is composed by the same method from any of the convergent terms of the series given as a and b. Therefore the this magnitude is composed in precisely the same way from its final convergent terms, which are equal. Let this final term be x, which multiplied by $\frac{mae-mbe}{ad-bd}$ and by m produces xm and $\frac{xmae-xmbe}{ad-bd}$. Sum-

*That is, an invariant expression.

†Here there is a typo in both editions of the book, giving the denominator as $cd-bd$.

ming the factors yields that $\frac{xmae-xmbe+xmad-xmbd}{ad-bd} = \frac{maae-mbae+mbad-mbbd}{ad-bd}$, and the equation reduces to $x = \frac{aae-bae+bad-bbd}{ae-be+ad-bd}$, which we wanted to find.

In order to make this problem less obscure we will illustrate with a numerical exercise: Let $c = 7$, $d = 2$, $e = 3$, $a = 28$, and $b = 42$. Then the second convergent terms are 32 and 36, the third are $33\frac{1}{7}$ and $34\frac{2}{7}$, and the limit is $33\frac{3}{5}$.

Changing nothing, if a is less than b then $\frac{ca+bd-ad}{c}$ may be greater than $\frac{bc-be+ae}{c}$, indeed in analysis greater can be subtracted by lesser, which is shown by this example: Let $c = 7$, $d = 5$, $e = 4$, $a = 28$, and $b = 42$. The second convergent terms will be 38 and 34, the third $35\frac{1}{7}$ and $36\frac{2}{7}$, and the limit $35\frac{7}{9}$.

Notice that the solution to this problem may even be obtained by the same method if a is zero. For example, let $c = 8$, $d = 3$, $e = 4$, $a = 0$, and $b = 24$. Then the second convergent terms will be 9 and 12, the third $10\frac{1}{8}$ and $10\frac{1}{2}$, and the limit of the series $10\frac{2}{7}$.

Indeed the limits of these series can be found in Gregorie of St. Vincent's book on geometric progression, although his way of proceeding differs greatly from the one presented here.

Proposition VIII. Problem.

Let the two quantities A and B be given and the ratio of C to D be given. We want to find another magnitude so that the ratio of it to A is the multiplicate of the ratio of B to A in the ratio of C to D.[*]

First, let the ratio of C to D be commensurable, and let E be a common measure of C and D. For as often as E is contained in D let the ratio of F to A be the submultiplicate of the ratio of B to A in such ratio . Also as often as E is contained in C let the ratio of G to A be the multiplicate of the ratio F to A in such ratios . I claim that G is the desired magnitude. The ratio of G to A is the multiplicate of the ratio of F to A in the ratio of C to E, and the ratio of F to A is the multiplicate of the ratio of B to A in the ratio of E to D. Therefore by equality, the ratio of G to A is the multiplicate of the ratio B to A in the ratio C to D, which is what we wanted to show.

If the ratio of C to D is incommensurable, then I am convinced that in practice this problem is geometrically impossible. However it can be accomplished by approximation, assuming a commensurable ratio that approaches it.

Let there be a convergent series such that the first terms are A and B, the second C and D, and the third E and F. Let the second terms be made by the first, where B is greater than A, and the ratio of B to A is the multiplicate of the ratio of C to A in the ratio of M to N, with M greater than N, and likewise the ratio of B to A is the multiplicate of the ratio of D to A in the ratio of M to O, with M greater than O. Let it follow that the third terms are made from the second as the second are made from the first, and so the

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[*]If a ratio x is the "multiplicate" of the ratio y in the ratio z, then in modern notation $x = y^z$. Likewise, if a number x is the "submultiplicate" of the ratio y in the ratio z, then in modern notation $x = y^{\frac{1}{z}}$. See [1, p.286].

series continues.

Proposition IX. Problem.

To find the limit of the aforementioned series.

Let G be zero, that is the exponent of the ratio of equality, or of the ratio of A to A. Also let H satisfy the exponent of the ratio of B to A. Let the ratio of M to N be the ratio of the difference between G and H, which is H itself, or the exponent of the ratio $B : A$, to the difference between G and I, which is I itself. But the ratio of M to N is the ratio by which the ratio of B to A is the multiplicate of the ratio of C to A. Therefore the excess of I over G, which is I itself, is the exponent of the ratio of C to A. Let the ratio of M to O be as the ratio of the excess of H over G, which is H, to the excess of K over G, which is K. But then the ratio of M to O is the ratio by which the ratio of B to A is the multiplicate of the ratio of D to A. Whenever H is the exponent of the ratio of B to A, K will be the exponent of the ratio of D to A. Therefore if I is the exponent of the ratio of C to A and K is the exponent of the ratio of D to A, then the excess of K over I will be the exponent of the ratio of D to C. From here let the ratio of M to N be as the excess of K over I, or the exponent of the ratio of D to C, is to the excess of R over I. Then the ratio of M to N is the ratio, from the composition of the series, by which the ratio of D to C is the multiplicate of the ratio of E to C, and so the excess of K over I is the exponent of the ratio of D to C. Thus the excess of R over I is the exponent of the ratio of E to C and I is the exponent of the ratio of C to A. Therefore R is the exponent of the ratio of E of A. From here let the ratio of M to O be as the excess of K over I is to the excess of S over I. Then the ratio of M to O is the ratio, from the composition of the series, by which the ratio of D to C is the multiplicate of the ratio of F to C, where the excess of K over I is the exponent of the ratio of D to C. The excess of S over I will be the exponent of the ratio of F to C and I is the exponent of the ratio of C to A. Thus S is the exponent of the ratio of F to A. Therefore when R is the exponent of the ratio of E to A and S is the exponent of the ratio of F to A, the excess of S over R will be the exponent of the ratio of F to E. Continuing the series, it may be shown as before that T is the exponent of the ratio of X to A and V the exponent of the ratio of Y to A. Finally it will always be the case that the convergent terms of the series of exponents are exponents of the ratios of any convergent terms of the series may be found in the same way by the initial values, and specifically of the convergent terms of the proposed series by the first magnitude A of the series. Thus by the limit of the series of exponents is found by 7. For example, let L be the limit of the proposed series with the first term A, then it will be the exponent of the ratio. Therefore the ratio of Z to A may be found, and is the multiplicate of the given ratio of B to A in the ratio of L to H, and Z will be the desired limit, which we wanted to find.

To illustrate this problem in numbers, let $M = 4$, $N = 2$, $O = 1$, $A = 6$, and $B = 10$. Then the second convergent terms shall be $\sqrt{60}$ and $(2160)^{\frac{1}{4}}$, the third convergent terms $(7776000)^{\frac{1}{8}}$ and $(100776960000000)^{\frac{1}{16}}$, and the limit of the series $(360)^{\frac{1}{3}}$.

As another example, let $M = 6$, $N = 2$, $O = 3$, $A = 5$, and $B = 10$. Then the second convergent terms of the series shall be $(250)^{\frac{1}{3}}$ and $\sqrt{50}$, the third $(488281250000000)^{\frac{1}{18}}$ and $(7812500000)^{\frac{1}{12}}$, and the limit of the series $(12500)^{\frac{1}{5}}$. So far all of the limits of the convergent series can be made either by a single arithmetic proportion or a single geometric proportion. Now I shall add to the method, and by the power of this addition the limits of all convergent series may be found.

Proposition X. Problem.

To find the limit of a given convergent series from a given magnitude composed by the same method from two convergent terms of the series as from the subsequent convergent terms of the same series.

Let the convergent series be of any two convergent terms a and b and the subsequent convergent terms $\sqrt{ab}$ and $\frac{aa}{\sqrt{ab}}$. The sum of the convergent terms $a+b$ multiplied by the first convergent term a gives $aa+ab$. The sum of the subsequent convergent terms $\sqrt{ab}+\frac{a^2}{\sqrt{ab}}$ multiplied by the first convergent term $\sqrt{ab}$ likewise gives $aa+ab$. From this is discovered the limit of the convergent series. It is clear that the magnitude $aa+ab$ is made by the same method from the convergent terms a and b as from the subsequent convergent terms $\sqrt{ab}$ and $\frac{aa}{\sqrt{ab}}$, and because the magnitudes a and b were arbitrary terms of the convergent series, it is evident that the sum of any proposed convergent terms of the series multiplied by the first convergent term will give that same magnitude, which is likewise the sum of the subsequent convergent terms multiplied by the first convergent term. Since two convergent terms are always followed by two convergent terms, it is clear that the sum of any two of the convergent terms multiplied by the first convergent term will be $aa+ab$. And so the final convergent terms are equal. Therefore let the final term of this series be the limit z, which is added to itself and the sum multiplied by itself to give $2zz$, which equals the magnitude $aa+ab$, and solving this equation for z yields the limit of the series $\sqrt{\frac{aa+ab}{2}}$, which we wanted to find.

And therefore in order to find the limit of any convergent series it is necessary only to discover a magnitude composed by the same method from the first convergent terms as from the second convergent terms.

Corollary.

Since it is not important to the problem whether the convergent terms a and b are the first, second, third, etc., it is clear that all of the convergent terms of the series are composed by the same method from the first convergent terms as by the second, third, fourth, etc. convergent terms.

Proposition XI. Theorem.

The sector of the circle, ellipse, or hyperbola $ABIP$ is not composed analytically by the triangle ABP and the quadrilateral $ABFP$.

Let the triangle $ABP = a$ and the quadrilateral $ABFP = b$. It is clear from the preceding propositions that the quadrilateral $ABIP = \sqrt{ab}$ and the polygon $ABDLP = \frac{2ab}{a+\sqrt{ab}}$. The sector $ABIP$ is the limit of this convergent series. So that the radical signs and fractions may be removed from the terms of the series, for the first convergent terms of the series a and b, that is, for the triangle ABP and the quadrilateral $ABFP$, put $a^3 + a^2b$ and $a^2b + b^3$. Then the second convergent terms of the series, which are the quadrilateral $ABIP$ and the polygon $ABDLP$, will be $ba^2 + b^2a$ and $2b^2a$. I claim that the limit of the convergent series (where the first convergent terms of the series are $a^3 + a^2b$ and $a^2b + b^3$ and the second are $ba^2 + b^2a$ and $2b^2a$) is not composed analytically of the terms $a^3 + a^2b$ and $a^2b + b^3$*. Indeed, if the aforementioned limit is composed analytically of the terms $a^3 + a^2b$ and $a^2b + b^3$, then the limit would itself be analytic and would be composed in

*Here Gregory mens that he cannot, as in the previous example, devise an invariant expression in which he can solve for the limit, which is to say he cannot devise an analytic expression for the limit where he would obtain the same expression whether substituting the first pair of terms or the second pair of terms of the series.

the same way from the convergent terms $ba^2 + b^2a$ and $2b^2a$. Therefore the limit would be composed analytically in the same way from $a^3 + a^2b$ and $a^2b + b^3$ as it is composed from $ba^2 + b^2a$ and $2b^2a$, but no magnitude may be composed analytically in the same way from $a^3 + a^2b$ and $a^2b + b^3$ as it is composed from $ba^2 + b^2a$ and $2b^2a$, which I now demonstrate. If a magnitude may be composed analytically in the same way from $a^3 + a^2b$ and $a^2b + b^3$ as it is composed from $ba^2 + b^2a$ and $2b^2a$, then it would be produced by the adding, subtracting, multiplying, dividing, and the extraction of square roots from the terms $a^3 + a^2b$ and $a^2b + b^3$ as if in the same way the terms $ba^2 + b^2a$ and $2b^2a$ were added, subtracted, multiplied, divided, and the square roots extracted. However the latter is not possible to do, so neither can be the former. Thus I prove the latter: if the same magnitude would be made by addition, subtraction, multiplication, division, and the extraction of square roots of the terms $a^3 + a^2b$ and $a^2b + b^3$, which themselves are made by addition, subtraction, multiplication, division, and the extraction of square roots of the terms $ba^2 + b^2a$ and $2b^2a$, then by adding, or subtracting, or multiplying, or dividing equal magnitudes by or to the terms $a^3 + a^2b$ and $a^2b + b^3$, or by the extraction of square roots, these analytic operations being substituted in some way, by their reiteration or by both of these or by doing none of them, the two terms can be made into the final result, one from the term $a^2b + b^3$ and the other from the term $2b^2a$. Thus the final result from the term $a^3 + a^2b$ with the final result from $a^2b + b^3$ is the same as the final result by the term $ba^2 + b^2a$ with the final result from the term $2b^2a$ in the same way added, subtracted, multiplied, divided, and square roots extracted. This, however, is absurd, and therefore so is the earlier claim. That which follows is mostly made clear by the eighth definition, by which prove the rest. In the term $a^3 + a^2b$ is found a power of a, namely a^3, which is higher than any power of a found in the term ba^2+b^2a, and therefore also when equal magnitudes by addition, subtraction, multiplication, division, etc. to the terms a^3+a^2b and ba^2+b^2a. Just as it has been declared above, it will always be that the powers of a in the final result from the term a^3+a^2b will be higher than the powers of a in the final product from the term ba^2+b^2a because by adding, subtracting, multiplying, dividing, etc. to those with equal magnitudes always results in the making the same powers, and in the same way multiplying to those always elevates the higher powers in the term a^3+a^2b higher still than the lower powers in the term ba^2+b^2a, and also in extracting the same square roots from these, when a is higher in power, it will be higher also in the square root. And so because the highest power of a is found in the term ab^2+b^3 **than which is found in the term $2b^2a$, it is shown as before the highest power of a in the final result from the term ab^2+b^3 is the same as the highest power of a in the final result from the term $2b^2a$. Therefore in the final result of the term a^3+a^2b is found higher powers of a than in the final result of the term ba^2+b^2a, and in the final result of the term ab^2+b^3 the highest powers of a is the same with the highest power of a in the final result of the term $2b^2a$. And therefore the final result from the terms a^3+a^2b and ab^2+b^3, added, subtracted, multiplied, divided, etc. between them in the same way, will always produce a magnitude in which is found higher powers of a than any of those which can be found in the given magnitude by straight forward addition, subtraction, multiplication, division, etc. of the results from the terms ba^2+b^2a and $2b^2a$ because the higer powers taken with another power always makes a higher power than the lower powers with the same other power. And therefore these two magnitudes cannot be identically equal, when higher powers of a may be found the in one term than in the other. And so it is clear that the sector of the circle, ellipse, or hyperbola $ABIP$ cannot be composed analytically from the triangle ABP and the quadrilateral $ABFP$, as was to be shown.

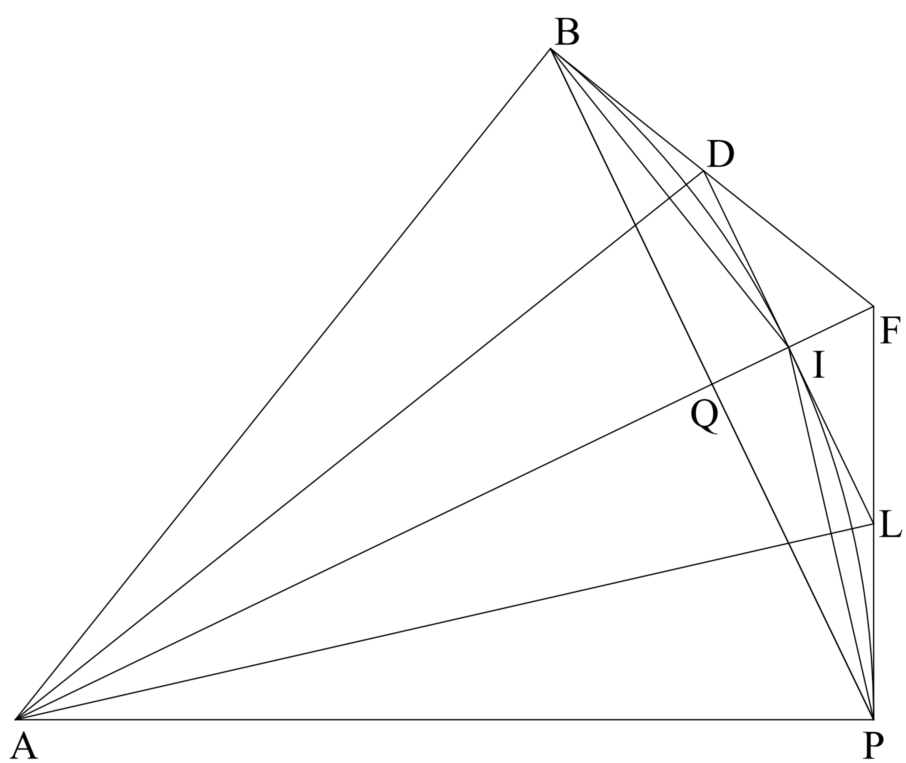
However, so that the proposition may be made perfectly clear, I subject it to a more brief and easier proof stemming from another means of attack. A magnitude cannot be composed analytically from the terms $a^3 + a^2b$ and a^2b+b^3 in the same way as it is composed from the terms ba^2+b^2a and $2b^2a$ because by the adding, subtracting, multiplying, and dividing of the two numbers $a^3 + a^2b$ and $a^2b + b^3$, and by extracting square roots, more terms are made in the end result than if the binomial $ba^2 + b^2a$ and the simple magnitude $2b^2a$ are added, subtracted, multiplied, divided, or square roots extracted. So if more terms are in one result than the other, it is impossible that they are identically equal, which was proposed. The rest of the claim can be obtained from the prior proof.

Scholium.

This theorem will perhaps be seen as most obscure because of the many unusual expressions that need to be employed, and because of the many lemmas assumed, which were indeed irksome to prove, since to any analyst they are obvious upon a first reading, as they follow entirely from the nature of analytic operations.

This position requires that I speak to some degree on the proportion between the triangle ABP and the quadrilateral $ABIP$, because as it so happens, drawing attention to the truest axiom of the philosophers, all of our inquiry originates in the perceiving. Indeed between the kinds of proportions, the perceiving of the the commensurable only needing to be touched upon in order to be understood completely by the human mind, but the incommensurable thus far being only barely contemplated by mathematicians, so far as a commensurable ratio of a certain thing is subdublicated, subtriplicated, etc. or born from addition, subtraction, etc. That is, a magnitude which is incommensurable to a given magnitude from it is only barely contemplated by human minds, since the commensurable can be composed from other of the known and given magnitudes by addition, subtraction, multiplication, division, and square root extraction. And from the proofs thus far it is clear that the sector $ABIP$ cannot be composed from addition, subtraction, multiplication, division, and square root extraction from the triangle ABP and the quadrilateral $ABFP$. However the triangle ABP and the quadrilateral $ABFP$ we suppose to be mutually analytic magnitudes, and thus the sector $ABIP$ cannot be analytic to these, which is to say it cannot be composed analytically from the magnitudes ABP and $ABFP$ by addition, subtraction, multiplication, division, and the extraction of square roots. Therefore because of this, noone can exhibit the ratio between the triangle ABP and the sector $ABIP$, since it is clear that this ratio is not analytic. But it will be claimed strongly by some that the ratio between the triangle ABP and the sector $ABIP$ can be changed in any way, and thus the ratio between them could be analytic or even commensurable depending on how they are given. I respond that if this were the case the ratio between the triangle ABP and the quadrilateral $ABFP$ will not be analytic. Therefore the triangle ABP will never be given in analytic terms from a given circle, ellipse, or hyperbola, which is most clear from the preceding. However, even if from the preceding we cannot fully comprehend the ratio between the triangle ABP and the sector $ABIP$, we can come to know it to a great extent because the sector $ABIP$ is the limit of the given convergent series. And from this consideration it is possible to find a commensurable magnitude whose difference from the sector $ABIP$ will be less than any given magnitude. It always returns to this when the practicioners work with incommensurable magnitudes, and in this our practice of approximation will not be more laborious than in the many other approximations of analytic magnitudes, and indeed it will be even easier, shorter, and better equipped than that of the angular section of Viète, which the greatest mathematicians are beginning to use again in practice. Thus I do not see how the quadrature of the circle is to be considered unknown any longer. Since indeed it has been shown that the ratio of the circle to the square of the diameter is not analytic, it will certainly be vain and inept to try to find, as it were, such an impostor. And in face of the rejection of such an analytic magnitude, I hardly believe that any can be better known than by this convergent series of ours, insofar as it is most plainly given by such a sequence.

Proposition XII. Theorem.



Let the quadrilateral *ABIP* be *A*, the polygon *ABEIOP* be *C*, the polygon $ABCGKNP$ be D, and the polygon $ABDLP$ be B. I claim that D is the harmonic mean of C and B. Combining proposition 4 and $A : C :: C : B$ gives $A + C : C :: C + B : B$. But then from proposition 5, $A + C : C :: 2C : D$ and also $C + B : B :: 2C : D$. Permuting gives $B + C : 2C :: B : D$ and by dividing, the difference between B and C is to $2C$ as the difference between B and D is to D. Again by permuting, the difference between B and C is to the difference between B and D as $2C$ is to D, that is, as $C + B$ is to B. And by dividing, the difference between D and C is to the difference between B and D as C is to B. Therefore D is the harmonic mean between C and B, as was to be shown.

This proposition holds in the same way for each complex polygon, as the Scholium of Proposition 5 explains.

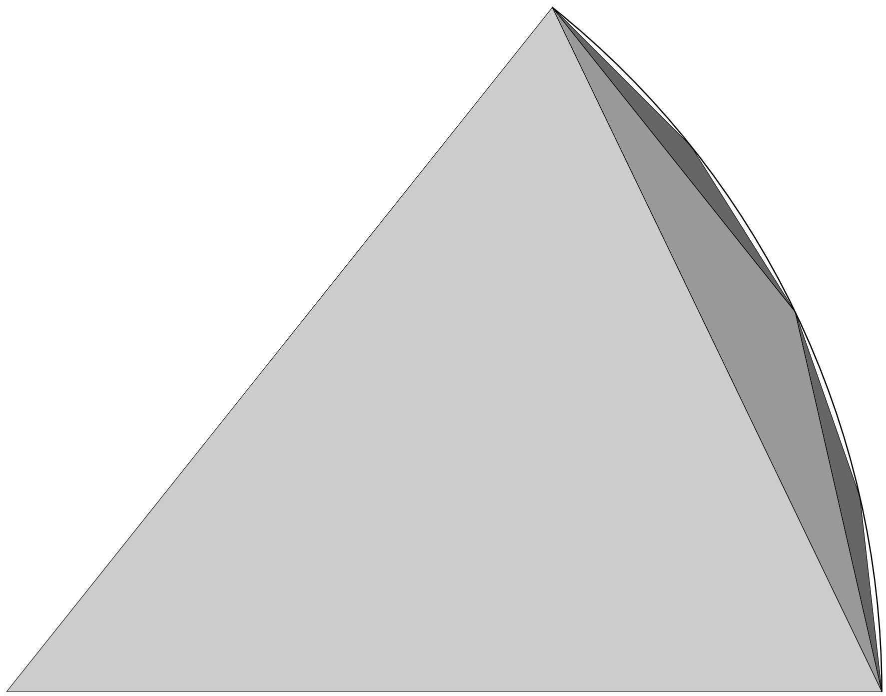
Proposition XIII. Theorem.

Let C be the arithmetic mean, D the geometric mean, and E the harmonic mean of the two magnitudes A and B. I claim that C, D, and E are continuously proportional *. Because A, E, and B are in harmonic ratio, the difference between A and E shall be to the difference between E and B as A is to B. Combining these, the difference between A and B shall be to the difference between E and B as $A + B$ is to B. Now by permuting and combining, $2A : A + B :: E : B$, but $2A$ is twice A and $A + B$ is twice C, so that $A : C :: E : B$. Thus $CE = AB$ and $AB = DD$, and so $CE = DD$. Therefore $C : D :: D : E$, as was to be shown.

Proposition XIV. Theorem.

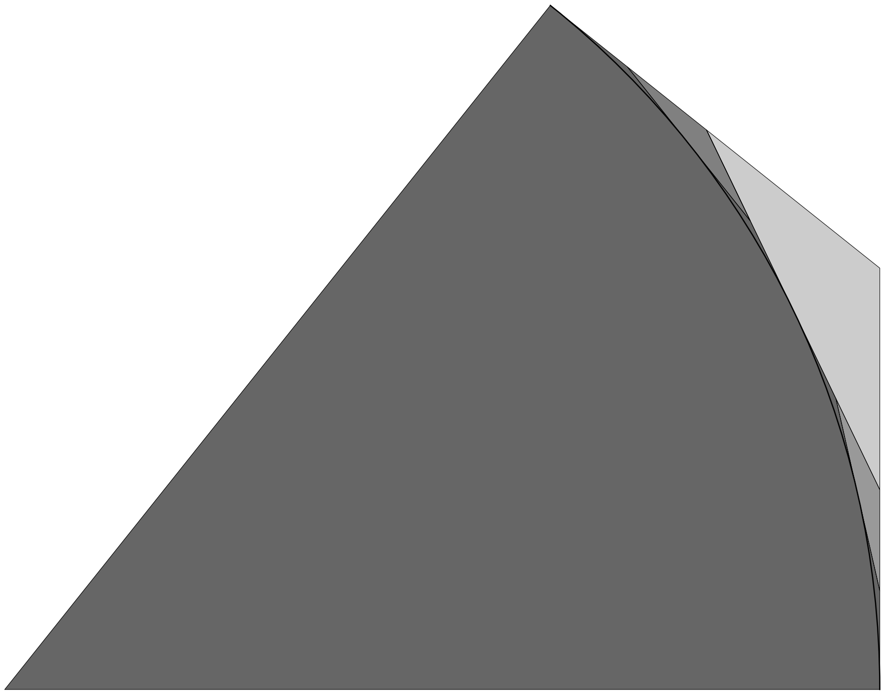
Let A and B be two complex polygons with A inscribed in the sector of a circle or ellipse and B circumscribed. A convergent series of these

*That is, $C : D :: D : E$.



Series of inscribed polygons.

complex polygons may be continued according to our method of drawing the subdouble, so that the polygons inscribed in the circle are A, C, E, etc. and those circumscribed are B, D, F, etc. I claim that $A+E$ is less than $2C$. This is clear from the previous propositions by the following analogies. First, A, C, and B are continuously proportional, and second C, D, and B are in harmonic proportion. Therefore the excess of C over A, that is $C-A$, is to the excess of D over C, or $D-C$, in a ratio composed from the proportion $A:C$ and from the proportion $A+C:A$, that is in the ratio $A+C:C$. And $A+C$ is greater than C, so that the excess of C over A is greater than the excess of D over C. However D is greater than E and so the excess of C over A is much greater than the excess of E over C. Therefore $A+E$ is less than $2C$, as was to be shown.



Series of circumscribed polygons.

Proposition XV. Theorem.

By the same assumptions, I claim that the excess of C over A is less than the quadruple of the excess of E over C. This is clear from the previous propositions by analogy with the following three analogies. First, A, C, and B are continuously proportional, second C, D, and B are in harmonic proportion, and third C, E, and D are continuously proportional. And so the excess of C over A, that is $C-A$, is to the excess of E over C, or $E-C$, as $AC+EC+AE+CC$ is to CC, and B is greater than E. So AB, or CC, is greater than AE, and thus $AE+CC$ is less than $2CC$. And so $AC+EC$ is to $2CC$ as $A+E$ is to $2C$, but $A+E$ is less than $2C$ so that $AC+EC$ is less than $2CC$. Thus $AC+EC+AE+CC$ is less than $4CC$. Therefore $C-A$ is less than the quadruple of $E - C$, as was to be shown.

Proposition XVI. Theorem.

Let A and B be two complex polygons with A circumscribed about the sector of a hyperbola and B inscribed. A convergent series of these complex polygons may be continued according to our method of drawing the subdouble, so that the polygons circumscribed about the hyperbola are A, C, E, etc. and those inscribed are B, D, F, etc. I claim that $A+E$ is greater than $2C$. This is clear from the previous propositions by the following analogies. First, A, C, and B are continuously proportional, and second C, D, and B are in harmonic proportion. Therefore the excess of A over C, that is $A - C$, is to the excess of C over D, or $C - D$, in a ratio composed from the proportion $A : C$ and from the proportion $A + C : A$, that is in the ratio $A + C : C$. And $A + C$ is greater than C, so that the excess of A over C is greater than the excess of C over D. However E is greater than D and so the excess of A over C is much greater than the excess of C over E. Therefore $A + E$ is greater than $2C$, as was to be shown.

Proposition XVII. Theorem.

By the same assumptions, I claim that the excess of A over C is greater than the quadruple of the excess of C over E. This is clear from the previous propositions by analogy with the following three analogies. First, A, C, and B are continuously proportional, second C, D, and B are in harmonic proportion, and third C, E, and D are continuously proportional. And so the excess of A over C, that is $A - C$, is to the excess of C over E, or $C - E$, in a ratio composed of the proportions $A : C$, $A + C : A$, and $E + C : C$. And so $A - C$ is to $C - E$ as $AC + EC + AE + CC$ is to CC, and B is less than E. So

AB, or CC, is less than AE, and thus $AE + CC$ is greater than $2CC$. And so $AC + EC$ is to $2CC$ as $A + E$ is to $2C$, but $A + E$ is greater than $2C$ so that $AC + EC$ is greater than $2CC$. Thus $AC + EC + AE + CC$ is greater than $4CC$. Therefore $C - A$ is greater than the quadruple of $E - C$, as was to be shown.

Proposition XVIII. Theorem.

Let A and B be two magnitudes such that A is less than B. Let C be their geometric mean and D their arithmetic mean. I claim that D is greater than C. Since B, C, and A are continuously proportional, by dividing, permuting, and combining, it shall be that the excess of B over A is to the excess of C over A as $A + C$ is to A. And so $A + C$ is greater than twice A. Thus the excess of B over A is greater than twice the excess of D over A, so that the excess of D over A is greater than the excess of C over A. Therefore D is greater than C, as was to be shown.

Proposition XIX. Theorem.

By the same assumptions, let E be the harmonic mean of A and B. I claim that C is greater than E. From proposition 13, D is to C as C is to E, but D is greater than C. Therefore C is greater than E, as was to be shown.

Corollary.

From the two preceding propositions it is obvious that D is greater than E, that is, that the arithmetic mean of two magnitudes is greater than the harmonic mean of the same.

Proposition XX. Theorem.

Let A and B be two complex polygons with A inscribed in the sector of a circle or ellipse and B circumscribed. A convergent series of these complex polygons may be continued according to our method of drawing the subdouble, so that the polygons inscribed in the circle are A, C, E, K, etc. and those circumscribed are B, D, F, L, etc. Also let Z be the limit of the convergent series, that is, the sector of the circle or ellipse. I claim that Z is greater than C plus one third of the excess of C over A. Let the excess of G over C be a fourth part of the excess of C over A and the excess of H over G be a fourth part of the excess of G over C. This series may be continued to infinity, so let X be the limit of this process. The excess of C over A is less than the quadruple of the excess of E over C, and so the excess of E over C is greater than the excess of G over C, and therefore E is greater than G. Now the excess of E over C is less than the quadruple of the excess of K over E, and so the excess of G over C is much less than the excess of K over E, and therefore the excess of K over E is greater than the excess of H over G. Since E is greater than G, it is obvious K is greater than H. It is demonstrated in the same way for every series A, C, E and A, C, G, by continuation to however many terms. Each term of the series A, C, E is greater than the corresponding term of the series A, C, G. And so the limit of the series A, C, E, that is Z, will be greater than the limit of the series A, C, G, that is X. And from Archimedes, the quadrature of the parabola corresponding to X is equal to C plus one third of the excess of C over A, and therefore Z is greater than this, as was to be shown.

Proposition XXI. Theorem.

By the same assumptions as above, I claim that Z, which is a sector of a circle or ellipse, is less than the greater of the two continuously proportional arithmetic means of A and B. Let G be the arithmetic mean of A and

B and H be the arithmetic mean between G and B. Likewise let M be the arithmetic mean of G and H and N be the arithmetic mean of M and H. This convergent series, with terms AB, GH, MN, OP, may be continued infinitely, so that its limit is X. It is clear from the preceding propositions that G is greater than C, and H, the arithmetic mean of G and B, is greater than the harmonic mean of G and B. However the harmonic mean of G and B is greater than D, the harmonic mean of C and B, since G is greater than C. And so the arithmetic mean of G and B, that is H, is greater than D, the harmonic mean of C and B. By the same method M, the arithmetic mean of G and H is greater than the geometric mean between G and H. And since G is greater than C, and H is greater than D, the geometric mean of G and H is greater than E, the geometric mean of C and D. Thus M is greater than E. Now N, the arithmetic mean of M and H, is greater than the harmonic mean of the same, and since H is greater than D and M is greater than E, the harmonic mean of M and H is greater than F, the harmonic mean of E and D. And so N is greater than F. Continuing the series in the same way to infinity, one may always show that the terms of the series AB, CD are less than the corresponding terms of the series AB, GH. Therefore the limit, Z, of the series AB, CD will be less than the limit, X, of the series AB, GH. Also, from Proposition 7, the limit, X, of the series AB, GH is equal to the greater of the two continuously proportional arithmetic means of A and B, and so Z is the less than the same, as was to be shown.

Proposition XXII. Theorem.

By the same assumptions as above, I claim that I claim that Z, which is a sector of a circle or ellipse, is less than the greater of the two continuously proportional geometric means of A and B. Let G be the geometric mean of A and B and H be the geometric mean between G and B. Likewise let M

be the geometric mean of G and H and N be the geometric mean of M and H. This convergent series, with terms AB, GH, MN, OP, may be continued infinitely, so that its limit is X. It is clear from the preceding propositions that C and G are equals, and H is greater than D. By this reasoning, M, the geometric mean of G and H, is greater than E, the geometric mean of C and D. Now N, the geometric mean of M and H, is greater than the harmonic mean of the same, and since M is greater than E and H is greater than D, the harmonic mean of M and H is greater than F, the harmonic mean of E and D. And so N is greater than F. Continuing the series in the same way to infinity, one may always show that the terms of the series AB, CD are less than the corresponding terms of the series AB, GH. Therefore the limit, Z, of the series AB, CD will be less than the limit, X, of the series AB, GH. Also, from Proposition 9, the limit, X, of the series AB, GH is equal to the greater of the two continuously proportional geometric means of A and B, and so Z is the less than the same, as was to be shown.

Scholium.

It is not much work to show that the greater of the two continuously proportional arithmetic means of two unequal magnitudes is greater than the greater of the two continuously proportional geometric means of the same magnitudes. Therefore it is a more exact approximation of the previous proposition, if it should be carried out. However I use the preceding proposition for its ease.

Proposition XXIII. Theorem.

Let A and B be two complex polygons with A circumscribed in the sector of a hyperbola and B inscribed. A convergent series of these complex polygons may be continued according to our method of drawing the subdouble, so that the polygons circumscribed in the circle are A, C, E, K, etc. and those inscribed are B, D, F, L, etc. Also let Z be the limit of the convergent series, that is, the sector of the hyperbola. I claim that Z is greater than C subtracted from one third of the excess of A over C. Let the excess of C over G be a fourth part of the excess of A over C and the excess of G over H be a fourth part of the excess of C over G. This series may be continued infinitely, so let X be the limit of this process. The excess of A over C is greater than the quadruple of the excess of C over E, and so the excess of C over E is less than the excess of C over G, and therefore E is greater than G. Now the excess of C over E is greater than the quadruple of the excess of E over K, and so the excess of C over G is much greater than the excess of E over K, and therefore the excess of G over H is greater than the excess of E over K. Since E is greater than G, it is obvious K is greater than H. It is shown in the same way for every series A, C, E, K and A, C, G, H by continuation to however many terms. Each term of the series A, C, E is greater than the corresponding term of the series A, C, G. And so the limit of the series A, C, E, that is Z, will be greater than the limit of the series A, C, G, that is X. And from Archimedes the quadrature of the parabola corresponding to X is equal to C plus one third of the excess of C over A, and therefore Z is greater than it, as was to be shown.

Proposition XXIV. Theorem.

By the same assumptions as above, I claim that Z, which is a sector of a hyperbola, is less than the lesser of the two continuously proportional arithmetic means of A and B. Let G be the arithmetic mean of A and B and H be the arithmetic mean between G and B. Likewise let M be the arithmetic mean of G and H and N be the arithmetic mean of M and H.

This convergent series, with terms AB, GH, MN, OP, may be continued infinitely, so that its limit is X. It is clear from the preceding propositions that G is greater than C, and H, the arithmetic mean of G and B, is greater than the harmonic mean of G and B. However the harmonic mean of G and B is greater than D, the harmonic mean of C and B, since G is greater than C. And so the arithmetic mean of G and B, that is H, is greater than D, the harmonic mean of C and B. By the same method M, the arithmetic mean of G and H is greater than the geometric mean between G and H. And since G is greater than C, and H is greater than D, the geometric mean of G and H is greater than E, the geometric mean of C and D. Thus M is greater than E. Now N, the arithmetic mean of M and H, is greater than the harmonic mean of the same, and since H is greater than D and M is greater than E, the harmonic mean of M and H is greater than F, the harmonic mean of E and D. And so N is greater than F. Continuing the series by the same method to infinity, one may always show that the terms of the series AB, CD are less than the corresponding terms of the series AB, GH. Therefore the limit, Z, of the series AB, CD will be less than the limit, X, of the series AB, GH. Also, from Proposition 7, the limit, X, of the series AB, GH is equal to the lesser of the two continuously proportional arithmetic means of A and B, and so Z is the less than the same, as was to be shown.

Proposition XXV. Theorem.

By the same assumptions as above, I claim that I claim that Z, which is a sector of a hyperbola, is less than the lesser of the two continuously proportional geometric means of A and B. Let G be the geometric mean of A and B and H be the geometric mean between G and B. Likewise let M be the geometric mean of G and H and N be the geometric mean of M and H. This convergent series, with terms AB, GH, MN, OP, may be continued infinitely, so that its limit is X. It is clear from the preceding propositions that C and G are equals, and H is greater than D. By this reasoning, M, the geometric mean of G and H, is greater than E, the geometric mean of C and D. Now N, the geometric mean of M and H, is greater than the harmonic mean of the same, and since M is greater than E and H is greater than D, the harmonic mean of M and H is greater than F, the harmonic mean of E and D. And so N is greater than F. Continuing the series in the same way to infinity, one may always show that the terms of the series AB, CD are less than the corresponding terms of the series AB, GH. Therefore the limit, Z, of the series AB, CD will be less than the limit, X, of the series AB, GH. Also, from Proposition 9, the limit, X, of the series AB, GH is equal to the lesser of the two continuously proportional geometric means of A and B, and so Z is the less than the same, as was to be shown.

It is clear from this claim that the approximation given here is the one shown in the preceding proposition, although this one might be a little more laborious. One will not ignore, however, that the two series can have equal limits, and be such that all of the terms of one series would be always be greater than the corresponding terms of the other series. But in extending out such series further, the difference becomes less by the number of terms. But to the contrary, our series being extended out further differ to a greater extent depending on the number of terms, as can be very easily shown.

I observe by experience that the difference between the second of two proportional arithmetic means and the second of two proportional geometric means is always much greater than the difference between the second of two proportional geometric means and the sector of the circle, ellipse, or hyperbola. By this observation I think it fitting that this sector is obtained differing by scarcely more than one from the second of the two proportionally continued arithmetic means when the arithmetic mean does not exceed the the geometric mean by more than one, which is remarkable, for from this it is clear that the approximation is confidently employed when such a series is continued just as the midpoint of first of the terms is the same in either convergent term, which experience has likewise shown. In fact in this case the sector never differs by unity from the second of the two continuously proportional arithmetic means.

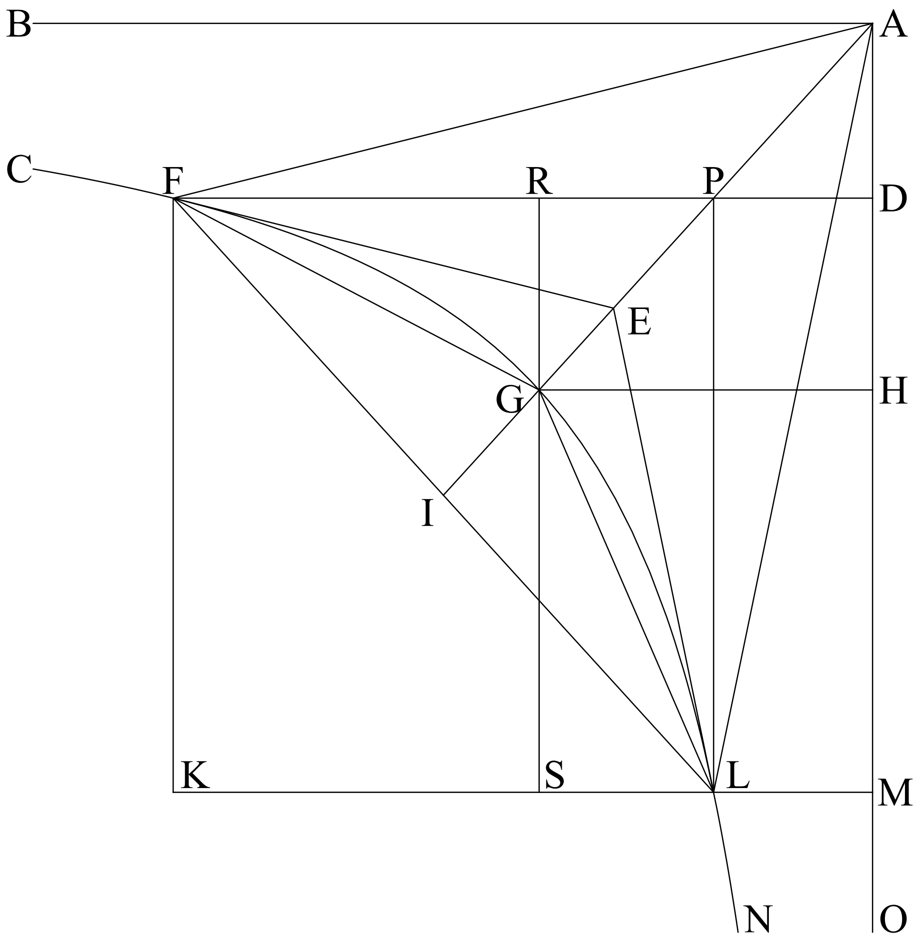
Likewise another approximation is altogether most brief and most astonishing, although it not happen to strengthen the geometric demonstration. Namely if the first third part of the terms in either convergent be the same, then the sector of the circle, ellipse, or hyperbola always differs within unity from the greatest fourth part by arithmetically continous proportion between the terms of our approximation.

Proposition XXVI. Theorem.

Let CFN be any hyperbola with center A and asymptotes AB and AO. Also, let $AFGL$ be its sector, with circumscribed triangle AFL. Let lines FD and LM be drawn parallel to the asymptote AB and complete parallelograms $FDMK$ and $PLMD$. I claim that the triangle AFL is the arithmetic mean of parallelograms $FDMK$ and $PLMD$. Gregorie of St. Vincent shows in his Book of the Hyperbola that the triangle AFL is equal to the quadrilateral $DFLM$, but it is obvious that quadrilateral $DFLM$ is the arithmetic mean of parallelograms $FDMK$ and $PLMD$, as was to be shown.

Proposition XXVII. Theorem.

By the same assumptions, let line AI be drawn bisecting FL at I and intersecting the hyperbola at the point G. Also let $AFGL$ be a circumscribed



quadrilateral of the sector. I claim that this is the geometric mean of parallelograms $FDMK$ and $PLMD$. From the proof of Gregorie of St. Vincent it is evident that quadrilateral $AFGL$ is equal to $DFGLM$. Because AGI bisects the line FL at I, from Gregorie of St. Vincent's Book of the Hyperbola it is clear that the lines LM, GH, and FD are continually proportional in the same ratio with the three continuous proportionals AD, AH, and AM. Let the line RGS be drawn through the point G parallel to the asymptote AO, meeting the lines FD and MK at the points R and S. Because the lines FD, GH, and LM are continuously propotional, by dividing and permuting we obtain FR is to SL as GH is to LM. Likewise, since the lines MA, HA, and DA are continuously proportional, by dividing and permuting we obtain MH is to HD, that is SG is to GR, as HA is to DA, or GH is to LM. Thus FR is to SL as SG it to GR, and when the angles FRG and GLS are equal, on account of parallels FR and SL being equal, the triangles FRG and GLS shall be equal. Therefore parallelogram $RDMS$ is equal to polygon $DFGLM$, or quadrilateral $AFGL$. However, parallelogram $RDMS$ is the geometric mean of parallelograms $PDML$ and $FDMK$ since by having the same height, the bases LM, SM and KM are continuously proportional. Therefore the quadrilateral $AFGL$ is the geometric mean of parallelograms $PDML$ and $FDMK$, as was to be shown.

Proposition XXVIII. Theorem.

By the same assumptions, let the lines FE and LE be drawn tangent to the hyperbola at points F and L in order to complete the quadrilateral $AFEL$. I claim this is the harmonic mean of parallelograms $PDML$ and $FDMK$. Triangle AFL, quadrilateral $AFGL$, and the harmonic mean of parallelograms $PDML$ and $FDMK$ are continuously proportional since triangle AFL is the arithmetic mean and quadrilateral $AFGL$ the geometric mean of these parallelograms, as is clear from Proposition 13. However triangle AFL, quadrilateral $AFGL$ and quadrilateral $AFEL$ are continuously proportional by Proposition 11. Therefore quadrilateral $AFEL$ is the harmonic mean of parallelograms $PDML$ and $FDMK$, as was to be shown.

Proposition XXIX. Problem.

To find a square equal to a given circle.

Let the square circumscribed by the circle be $4 \cdot 10^{15}$, then the inscribed square is $2 \cdot 10^{15}$, between which 2828427124746190 is the octagonal geometric mean. Now let the harmonic mean be between the octagon inside the circle and the square about it, which by trivial labor is found by dividing the double of the area of the octagonal inside the circle, or the double of the rectangle of the areas inside and about the circle, by the sum of the square and the octagon within. Then I find 3313708498984760 to be the harmonic mean, the circumscribed octagon. Continuing this convergent series of complex polygons where the midpoint of the first term is the same in each convergent term is easy to do up to the polygon of 16384 sides. Indeed the inscribed is 3141592576586860 and the circumscribed 31415926 92091258. This is not considered the final term, since in division and root extraction we always stray in some small part from the true value, which is closely approximated by this last imperfect term. Now the approximation from the proofs of Propositions 20 and 21 is used and the terms thus found will determine the true measure of the circle, letting the square of the diamter be $4 \cdot 10^{15}$, the smaller circle 3141592653589789, and the larger 31 41592653589792. Thus the true measure of the circle may no longer hide, as was to be shown. I set out the following series of polygons.

	Inside the circle	About the circle
4	2000000000000000	4000000000000000
8	2828427124746190	3313708498984760
16	3061467458920718	3182597878074527
32	3121445152258051	3151724907429255
64	3136548490545938	3144118385245904
128	3140331156954752	3142223629942456
256	3141277250932772	3141750369168965
512	3141513801144299	3141632080703181
1024	3141572940367090	3141602510256808
2048	3141587725277158	3141595117749588
4096	3141591421543029	3141593269613390
8192	3141592345578073	3141592807595664
16384	3141592576586860	3141592692091258

The circle lies between the following terms

3141592653589789 3141592653589792

and in precisely the same way an equivalent polygon to any circular or elliptic sector inscribed by a known triangle and circumscribed by a quadrilateral is obtained.

Proposition XXX. Problem.

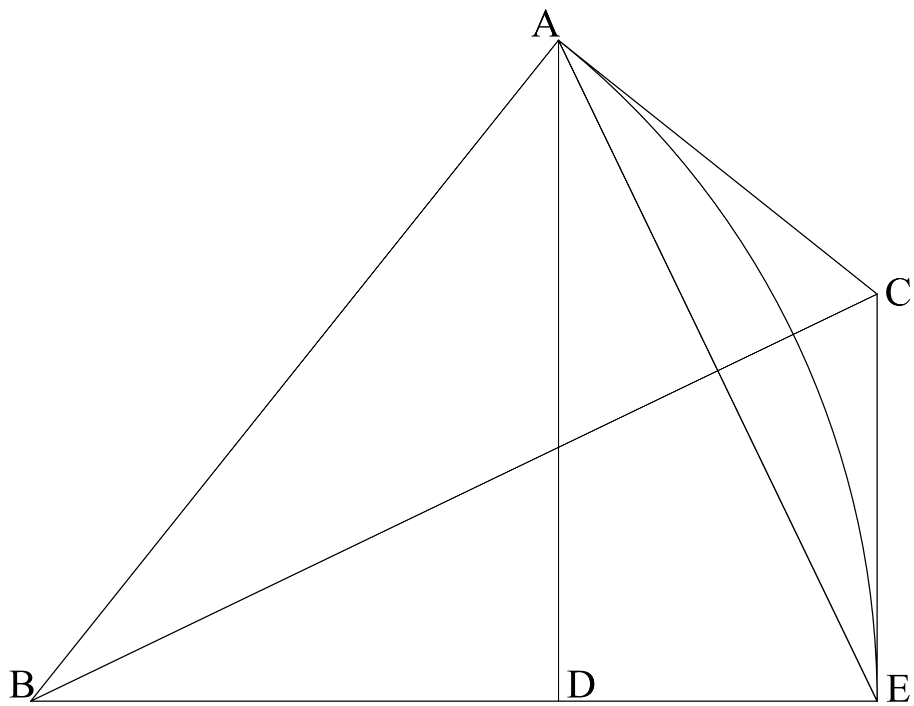
To find an arc from a given sine.

Let AE be the arc of the circle described about the center B. The radius of this arc is of course AB and the sine is AD. We want to find which proportion the arc itself has to the entire circumference of the circle. Let AE be the chord of the arc and its half-tangents be AC and CE. From the square of the radius *AB* is produced the square of the sine *AD*, and square of the cosine *BD* remains, and thus *BD* is given. Therefore the area of the triangle *ABD* is given by the rectangle and likewise the area of the triangle *ABE* is given, namely the rectangle of the given sine *AD* and half of the radius *BE*. From this it is seen that the sum of the triangles *ABD* and *ABE* is to the triangle *ABE* as the double of the triangle *ABE* is to the quadrilateral *ABEC*, which is given by Proposition 5. From the given inscribed triangle *ABE* and the circumscribed quadrilateral *ABEC*, the sector *ABE* itself may be found by the preceding proposition, which to the given entire circle has the desired proportion of the arc *AE* to the total circumference, which was to be shown.

Proposition XXXI. Problem.

To find a sine from a given arc.

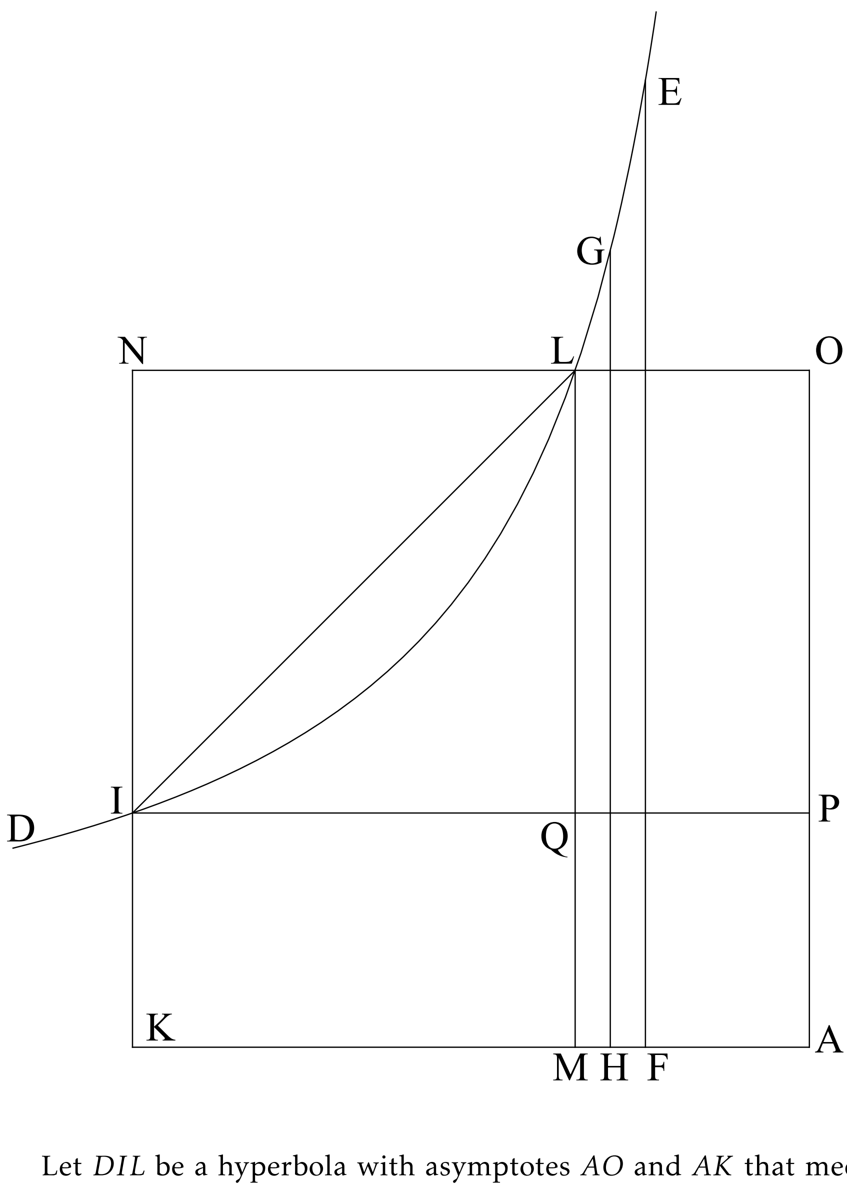
From the given arc it is clear how to give the area of the sector. Therefore given the sector one may consider from how many arithmetic marks the entire sine is made. Now part of such a given sector having been supposed, namely the sector *ABE*, when the inscribed triangle *ABE* and circumscribed quadrilateral *ABEC* are repeated as many times as desired, the given sector is a multiple of the sector *ABE* agreeing in every arithmetic term as the square root of the whole sine contains. Indeed this can easily be shown from consideration of the table of Proposition 29. Precision is not required in this process, for if large radii are used then it makes no difference if the difference is off by a few parts. The radius *BA* is given from the known sector *ABE*, by which the arc *AE* is likewise known. Let its sine *AD* be given by z. Thus from the given sine and radius, the inscribed triangle *ABE* is given as in the previous proposition, as well as the circumscribed quadrilateral *ABEC*. And so the sector itself is given, it is the second of the



two continuously proportional arithmetic means of the inscribed triangle and the circumscribed quadrilateral. Thus is given the equation between the double of the quadrilateral *ABEC* plus triangle *ABE* and the triple of the known sector *ABE*, from whose resolution the value of the unknown magnitude z is obvious, which is the sine of *AD*. And by the given arc *AE* and its sine likewise the sine of all of the repetitions of the arc *AE* is given from the common doctrine on the angle of a sector. Therefore the sine of the proposed arc cannot hide, when it be in a given multiple of the arc *AE*, which was to be shown.

Proposition XXXII. Problem.

To find a square equal to the hyperbolic area bound by a hyperbolic curve, one asymptote, and two a lines parallel to the other asymptote, which space is equal to the hyperbolic sector having for its base the same curve.



Let DIL be a hyperbola with asymptotes AO and AK that meet at the right angle OAK. Consider the hyperbolic area $ILMK$ bounded by the hyperbolic curve IL, the asymptotic segment KM, and the two lines AK and LM, which are parallel to the other asymptote AO. Choose the line IK to be 10^{12}, LM to be 10^{13}, and AM to be 10^{12} so that the line KM is $9 \cdot 10^{12}$. We want to find the measure of the area $ILMK$. Let the lines IK and OL be extended and draw the line IP in order to complete the rectangles $LNKM$ and $QIKM$. It is clear that rectangle $LNKM$ has area $9{\cdot}10^{25}$ and $QIKM$ has area $9{\cdot}10^{24}$, and that the area of quadrilateral $LIKM$ is the arithmetic mean of these rectangles, being $4.95 \cdot 10^{25}$. The geometric mean between $LNKM$ and $QIKM$ is found to be 28460498941515413987990042 which is the regular circumscribed pentagon of the hyperbolic area $LIKM$. Now as the quadrilateral $LIKM$ is to this circumscribed pentagon, so the double of the pentagon is to the regular inscribed hexagon of the hyperbolic area $LIKM$, namely 20779754131836628160009835. This gives the complex of the regular hexagon with the aforementioned pentagon, of which the two areas bring about the first terms of the convergent series. Between this is the geometric mean by which the double of the square is divided by the same geometric mean plus the greater term, or the circumscribed pentagon. And they give the geometric mean and bear whatever proportion of the second convergent terms. Thus this convergent series of complex polygons may be continued, while the first midpoint of the terms is the same in both convergent terms, namely up to the twentieth term, where the circumscribed polygon is 2302585092993120329958961534173864 and the inscribed is 2 3025850929931203593181124. From here the approximation used is that proved in Propositions 23 and 24, and the inscribed terms are discovered, describing the true hyperbolic area of $LIKM$, being bound below by 2302 5850929940456240178681 and the same area to be bound above by 23025 850929940456240178704. And so the area can no longer hide, which was to be shown. I thus deliver the entire series of polygons plus the number of lines subtending the hyperbolic curve in any circumscribed polygon.

	Circumscribed Area	Inscribed Area
2	28460498941515413987990042	20779754131836628160009835
4	24318761696971474416609403	22410399968461612921314879
8	23345088913234727934949897	22868197570682058351436953
16	23105412906351426185065096	22986193244865462241217428
32	23045725982658962868047234	23015921117139340153267671
64	23030818728479610745741910	23023367512879647736902891
128	23027092819292183214705676	23025230015404383009313933
256	23026161398510805910921810	23025695697539046352276636
512	23025928546847571901068394	23025812121604634087915779
1024	23025870334152518169052273	23025841227841783762272302
2048	23025855780992551911165543	23025848504414868310197241
4096	23025852142703422669729927	23025850323559001769499206
8192	23025851233131194254554390	23025850778345089029496888
16384	23025851005738140519209367	23025850892041614212944994
32768	23025850948889877295901163	23025850920465745719335070
65536	23025850934677811503232115	23025850927571778609090592
131072	23025850931124795055887228	23025850929348286832351848
262144	23025850930236540944102405	23025850929792413888218560
524288	23025850930014477416159412	23025850929903445652188450
1048576	23025850929958961534173864	23025850929931203593181124

The hyperbolic sector lies between the following terms

23025850929940456240178681 23025850929940456240178704

It is therefore possible without danger of error to assume the following number for the sector of the hyperbola, of which the multiples of the number up to ten, facilitating division thanks to the composition of the logarithm, I reveal by this. For indeed in long division it is better to use repeated subtraction for repetition of division than ordinary division, as any master of arithmetic will agree.

It is clear that this problem can be resolved in the same way even if the asymptotes AO and AK are not at a right angle. However we made assumptions so that the problem would be easier and more readily used in the doctrine of logarithms, which was first discovered by our most noble Napier, and which we have now elevated (unless I am mistaken) to the highest peak of perfection.

1	2302585092994045624017870O
2	46051701859880912480357400
3	69077552789821368720536100
4	92103403719761824960714800
5	115129254649702281200893500
6	138155105579642737441072200
7	161180956509583193681250900
8	184206807439523649921429600
9	207232658369454106161608300
10	230258509299404562401787000

Proposition XXXIII. Problem.

To find the logarithm of any given number.

By the same assumptions as in the preceding proposition, it is clear that, taking IK to be unity, ML is ten. Therefore, IK being unity, let GH be any parallel to the asymptote AO, then the logarithm of this given number is desired. It is clear from the given line GH to get KF, and from the preceding propositions how to get the hyperbolic area $GIKH$, and I claim that this hyperbolic area the logarithm of the proposed number GH. I take by the area $LIKM$ the logarithm of the number ten. Indeed (from Gregorie of St. Vincent) the area $GHKI$ is in the same ratio to the area $LMKI$, as the ratio GH to IK is a multiple of the ratio LM to IK. However the ratio GH to IK is a multiple of the ratio LM to IK in the same ratio as the number GH is a multiple of the number LM, since it is contained itself in both ratios. Thus the area $GIKH$ is in the same ratio to the area $LIKM$, in which the number GH is a multiple of the number LM, and so (since by hypothesis the area $LIKM$ is the logarithm of the number LM, or ten) the area $GIKH$ shall be the logarithm of the proposed number GH, since this is the essential property of logarithms, as they be among themselves in the same direct ratio, in which they are one another multiplied by the same number. And the logarithm of ten is generally given as a one with some arbitrary number of zeros. If this be done, the area $LIKM$ is to the area $GIKH$ as the arbitrary logarith of ten is to the other number. That number shall be found to be the logarithm of the proposed number GH, which we wanted to find.

Scholium.

The exercise of the preceding set of problems is long and laborious. Thus in order to abbreviate our labor when composing tables of logarithms, it has been understood that we merely need to work on the discovery of logarithms of prime numbers. Indeed the logarithms of composite numbers are found without effort from the primes by addition and subtraction. However as the logarithms of prime numbers may be easily found, the order progressing from the priors to the latters, so that from the arbitrary logarithm of 10 to each prime number 2, and from 10 and 2 to 3, likewise from 10, 2, and 3 to 7, likewise from 10, 2, 3, and 7 to 11, and thus hereafter. Next two composite numbers differing by very little are found, of which one is composed from a number having a known logarithm, and so having the given logarithm, the other number is composed from only a prime number (of which the logarithm is found) or from that plus another number having known logarithm. Now these composite numbers are drawn (which may be, e.g., GH and EF) to the hyperbola as parallels to the asymptote OA, and the hyperbolic area $EGHF$ is found according to Proposition 32, which is done quickly from GH and EF, which differ by very little. By assumption, the logarithm of one of the numbers, e.g. GH, is given, and so the ratio of its logarithm to the arbitrary logarithm of ten is given, which is the same (from the proofs to this point) as the ratio of the hyperbolic area $GIKH$ to the hyperbolic area $LIKM$. However the area LIKM is given by Proposition 32, and so the area $IKHG$ is known, and with the given area $EGHF$ $EIKF$ is given. Thus the logarithm of the composite number EF is given. And when by assumption the logarithms of every number composing the number EF may be given, except that prime number of which the logarithm is desired, that logarithm of the prime number will be given, which we wanted to find. For example, Let it be proposed to find the logarithm of the number two, supposing arbitrarily the logarithm of the number ten, but given as one with 25 zeros, the two composite numbers, differing by very little, are 1000 and 1024. The logarithm of the number 1000, or the triple of the area found above as 23025850929940 456240178700, namely that area given by the arbitrary logarithm of the number ten.

	Circumscribed Area	Inscribed Area
2	23717082451262844989917	23716248706204586784688 6
4	23716665575069990373755 6	23716457138805441921937 1
8	23716561356708732297040 3	23716509247642595435642 6
16	23716535302161352359943 8	23716522274894818148525 0
32	23716528788527190784838 9	23716525531710557232045 6
64	23716527160118818104101 2	23716526345914659715903 8

The hyperbolic sector lies between the following terms

23716526617316027210322 0   23716526617316045845302 9

Let the four greatest of the continuously arithmetic proportionals of 23716526617316042118306 7 be between these terms, which hence shall be the true sector of the hyperbola in the proposed number of the noted, since the first third of the noted is the same in both of the convergent terms.

Let the logarithm of the number 1024 be unknown, it is composed from only the prime number 2, namely it is multiplied ten times. These composite numbers are drawn to the hyperbola, as has been said, letting GH be 1000 and EF be 1024. But since IK is 1000000000000, GH shall be 10000 00000000000 and EF 1024000000000000, and by Proposition 32 the area $EGHF$ is found to be 237165266173160421183067 (I give this convergent series for the profit of the reader), or the logarithm of the number $1\frac{24}{1000}$ by the proposed arbitrary logarithm of ten 230258509299404562401787 0. Next, by the same assumed arbitrary logarithm of ten, the logarithm of the number 1000 is added, or the triple of the logarithm of ten, to the logarithm of the number $1\frac{24}{1000}$, and will give the logarithm of the sum of the number 1024, of which a tenth part will be the logarithm of the number two through the same arbitrary logarithm of ten, or 693147180559945291 4171917. So it will be that the logarithm of the number ten 23025850929 940456240178700 is to the logarithm of the number two corresponding to 6931471805599452914171917 as the proposed arbitrary logarithm of the number ten, or 10000000000000000000000000 is to the logarithm of the sought number two 3010299956639811952405804, which we wanted to find*. By the same method the logarithm of three is found to be 47712125 47196624373502993, etc.

In order to show these composite numbers, differing very little among themselves, through one of the prime numbers, I present this table for each prime number up to 100, as well as one rule for prime numbers between 100 and 1000 and another for prime numbers above 1000, which have all been contrived so that the true logarithm of any prime number can be found by the corresponding arbitrary logarithm of ten 10000000000000 000000000000 by only one multiplication, two divisions, and one square

*That is, $\log_1 0(2) = \frac{\log(2)}{\log(10)}$

root extraction, as well as some little effort.

2	$1000 = 10^3$ $1024 = 2^{10}$
3	$32805 = 5 \cdot 6561 = 5 \cdot 3^8$ $32768 = 2^{15}$
7	$2400 = 3 \cdot 32 = 3 \cdot 2^5$ $2401 = 7^4$
11	$9800 = 2 \cdot 49 \cdot 100 = 2 \cdot 7^2 \cdot 10^2$ $9801 = 121 \cdot 81 = 11^2 \cdot 3^4$
13	$123200 = 7 \cdot 11 \cdot 25 \cdot 64 = 7 \cdot 11 \cdot 5^2 \cdot 2^6$ $123201 = 169 \cdot 729 = 13^2 \cdot 3^6$
17	$2600 = 13 \cdot 8 \cdot 25 = 13 \cdot 2^3 \cdot 5^2$ $2601 = 9 \cdot 289 = 3^2 \cdot 17^2$
19	$28899 = 169 \cdot 9 \cdot 19 = 13^2 \cdot 3^2 \cdot 19$ $28900 = 100 \cdot 289 = 10^2 \cdot 17^2$
23	$25920 = 10 \cdot 32 \cdot 81 = 10 \cdot 2^5 \cdot 3^2$ $25921 = 49 \cdot 529 = 7^2 \cdot 23^2$
29	$613088 = 17 \cdot 23 \cdot 32 \cdot 49 = 17 \cdot 23 \cdot 2^5 \cdot 7^2$ $613089 = 729 \cdot 841 = 3^6 \cdot 29^2$
31	$116280 = 10 \cdot 17 \cdot 19 \cdot 4 \cdot 9 = 10 \cdot 17 \cdot 19 \cdot 2^2 \cdot 3^2$ $116281 = 121 \cdot 961 = 11^2 \cdot 31^2$
37	$165648 = 3 \cdot 7 \cdot 17 \cdot 29 \cdot 16 = 3 \cdot 7 \cdot 17 \cdot 29 \cdot 2^4$

$165649 = 121 \cdot 1369 = 11^2 \cdot 37^2$

41
$1413720 = 7 \cdot 10 \cdot 11 \cdot 17 \cdot 4 \cdot 27 = 7 \cdot 10 \cdot 11 \cdot 17 \cdot 2^2 \cdot 3^3$
$1413721 = 1681 \cdot 841 = 41^2 \cdot 29^2$

43
$978120 = 10 \cdot 11 \cdot 13 \cdot 19 \cdot 4 \cdot 9 = 10 \cdot 11 \cdot 13 \cdot 19 \cdot 2^2 \cdot 3^2$
$978121 = 529 \cdot 1849 = 23^2 \cdot 43^2$

53
$664848 = 7 \cdot 19 \cdot 23 \cdot 8 \cdot 125 = 7 \cdot 19 \cdot 23 \cdot 2^3 \cdot 5^3$
$664849 = 9 \cdot 121 \cdot 2809 = 3^2 \cdot 11^2 \cdot 53^2$

57
$5851560 = 3 \cdot 5 \cdot 13 \cdot 31 \cdot 121 = 3 \cdot 5 \cdot 13 \cdot 31 \cdot 11^2$
$5851561 = 1681 \cdot 3481 = 41^2 \cdot 57^2$

61
$3575880 = 5 \cdot 7 \cdot 11 \cdot 43 \cdot 8 \cdot 27 \cdot = 5 \cdot 7 \cdot 11 \cdot 43 \cdot 2^3 \cdot 3^3$
$3575881 = 961 \cdot 3721 = 31^2 \cdot 61^2$

67
$1620528 = 3 \cdot 13 \cdot 16 \cdot 49 = 3 \cdot 13 \cdot 2^4 \cdot 7^2$
$1620529 = 361 \cdot 4489 = 19^2 \cdot 67^2$

71
$2016399 = 3 \cdot 11 \cdot 29 \cdot 43 \cdot 49 = 3 \cdot 11 \cdot 29 \cdot 43 \cdot 7^2$
$2016400 = 16 \cdot 25 \cdot 5041 = 2^4 \cdot 5^2 \cdot 71^2$

73
$5116644 = 4 \cdot 9 \cdot 169 \cdot 841 = 2^2 \cdot 3^2 \cdot 13^2 \cdot 29^2$
$5116645 = 7 \cdot 17 \cdot 19 \cdot 31 \cdot 73$

79
$5997600 = 17 \cdot 32 \cdot 9 \cdot 25 \cdot 49 = 17 \cdot 2^5 \cdot 3^2 \cdot 5^2 \cdot 7^2$
$5997601 = 961 \cdot 6241 = 31^2 \cdot 79^2$

83
$1164240 = 5 \cdot 11 \cdot 16 \cdot 27 \cdot 49 = 5 \cdot 11 \cdot 2^4 \cdot 3^3 \cdot 7^2$
$1164241 = 169 \cdot 6889 = 13^2 \cdot 83^2$

89
$2859480 = 5 \cdot 47 \cdot 8 \cdot 9 \cdot 169 \cdot = 5 \cdot 47 \cdot 2^3 \cdot 3^2 \cdot 13^2$

$$2859481 = 361 \cdot 7921 = 19^2 \cdot 89^2$$

97

$$1138488 = 3 \cdot 13 \cdot 41 \cdot 89 \cdot 8 = 3 \cdot 13 \cdot 41 \cdot 89 \cdot 2^3$$
$$1138489 = 121 \cdot 9409 = 11^2 \cdot 97^2$$

For prime numbers between 100 and 1000 let this be the rule: before the prime number of which the logarithm is desired, the two numbers immediately preceding are assumed, and the number following immediately after it, which three numbers with that prime are four numbers following one after another in natural order among themselves. Next the first number is multiplied by the cube of the third and the fourth by the cube of the second, and it will be that their difference equals the sum of the prime and the fourth or of the second and the third, as can easily be shown. These numbers have at least six prime factors between them, and thus they differ very little among themselves. Also the logarithms of the four of these numbers (except the third) are known from the preceding method, and thus are suitable to our abbreviation. So much apparatus is not useful in numbers beyond 1000, since the rectangle of the numbers, among which the prime number is understood immediately of which the logarithm is desired, which is only less the square of the prime number by one. And so these have at least six prime factors among them, and the logarithms of the first and third are obtained. Therefore the infinitude are available to us.

Proposition XXXIV. Problem.

From a given logarithm to find its number.

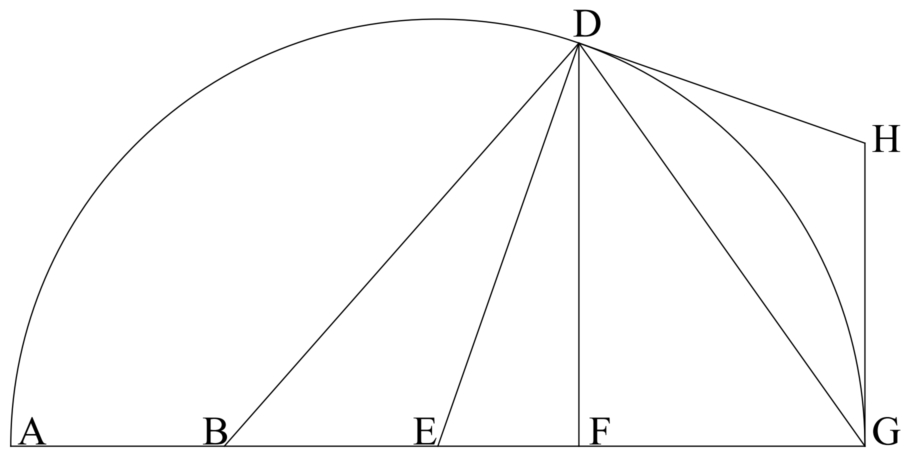
From the demonstration it is clear that this problem is the same as that which was proposed, namely from the given hyperbolic area and one line understood to be parallel to one of the asymptotes, to find another area and its parallel to the asymptote. It may be considered from however many arithmetic terms comprise the arbitrary logarithm of ten, and the logarithms or the given area, the area $LIKM$, being assumed so that the regular circumscribed polygons of the area $LIKM$ and the regular inscribed hexagons of the same be repeated however many times, the given area being repeated to the area $LIKM$, agreeing in all arithmetic terms, as many the square root of the arbitrary logarithm of ten contains. Indeed this can be done easily from the table of Proposition 32. Therefore the measure of the area $LIKM$ is obtained and the line IK is unity by assumption. Let LM be z. As in Proposition 32 the regular circumscribed pentagon and the regular inscribed hexagon give the area $LIKM$, between which the given area $LIKM$ is the second of the two continuously proportional arithmetic means. And so the double of the hexagon plus the pentagon is equal to the triple of the area, which equation the unknown z, or the number LM, clearly resolves, which repeated however many times, as many as the area $LIKM$ is submultiplied to the space, or the given logarithm, is the desired number, which was to be shown.

This is the same problem as Proposition 8, but more general and the method of this solution is far less work.

Proposition XXXV. Problem.

A line having been drawn through a given point on a diamter, to divide the semicircle into a given ratio.

Let ADG be a semicircle of given diameter AG, center E, and B be a given point on the diameter. Assume it is made as described, and let the line BD divide the semicircle into a given ratio. Since the measure of the



semicircle is given and the ratio into which it is divided, its portion, *DBG*, is given. Let z be the line *BD*. From the given lines *BD*, *BE*, and *ED*, the triangles *DEB*, *DEF*, and *DEG* are known. Next let it be that as *DEF* plus *DEG* is to *DEG* so the double of *DEG* is to the circumscribed quadrilateral *DEGH*. Setting *DEG* and *DEGH* as the first convergent term, the convergent series of complex ploygons may be continued, repeated as often as necessary according to the properties of the circle until the agreed upon approximation is met so that the sector *DEG* is reached, which itself plus the triangle *DBE* is euqal to the known value *DBG*, the equation of which is clearly resolved by the unknown magnitude z, or the line *BD*. The rest is obvious.

The same problem is resolved in precisely the same way in the ellipse, the hyperbola, or any sector given.

Scholium.

If a set of problems for the previous Propositions is desired for mechanical pratice, it will not be difficult to imitate the calculation, approximation, and resolution of equations to some extent according to the common rules of the practice of Geometry. Many such problem sets may be resolved by the power of analysis and by our rules of convergent series, which before may have been impossible to estimate. However, it will be strongly claimed by many that these solutions are not geometric. I respond that if the only practice understood by the geometer is the power of the straightedge and compass, not only will this be impossible but likewise will every problem set which cannot be reduced to a quadratic equation, as may easily be shown. And if the reduction of the problem to an analytic equation be understood by the geometer, all of this problem set are impossible to the geometer, where by this proof it is clear that such a reduction is cannot be done. If in truth this most simple method of every possibility be understood by the geometer, it will be found most strongly after timely consideration that the entirety of the above problem set may be resolved most geometrically. Carefully observing the whole doctrine of convergent series it is possible likewise by little effort to apply it to simple series. Indeed let A, B, C, D, E, etc. be a series of such nature that the third term C is composed by the same method from the first and second terms A and B as the fourth term D is composed from the second and third terms B and C, and the fifth E from the third and fourth C and D, and so on infinitely. Let also the difference of the aforementioned A and$B$ be always greater than the difference of the subsequent terms B and C. We may assume this series to continue infinitely until two of the adjacent terms are not different, and letting one of these terms be z, which we call the limit of the series. I claim that z is composed by the same method from A and B as from B and C or C and D. The proof scarcely differes from that of Proposition 10 and its Conclusions. If this ratio is put to a triangle, inscribed in a sector of a circle or ellipse or circumscribed to a sector of a hyperbola, a, and a quadrilateral, regularly inscribed in a circle or ellipse or circumscribed to a hyperbola, b, then the hexagon regularly inscribed in a sector of a circle or ellipse or circumscribed to a hyperbola will be $\sqrt{\frac{2b^3}{a+b}}$. Thus the sector of a circle, ellipse, or hyperbola is composed of the same method from a and b as from b and $\sqrt{\frac{2b^3}{a+b}}$. And so this likewise can be shown, that the ratio of the sector to its given triangle may not be analytic, according to Proposition 11. It would actually be possible to yet prove by another particular method that the circular arc does not have an analytic ratio to its given chord, but I do not add more, meanwhile advising geometers to growth by science. I myself might have discovered in certain figures (which Descartes called the second type) three foci, or three points, from which lines drawn to any point of the curve the sum or difference is always the same. Whence it appears to me to be true as all curves of the first type have two foci whether real or imaginary, as all of the second type have three, all third four, and so on infinitely. This speculation is certainly most worthy of scrutiny, and indeed it may be an extraordinary property of the geometric figures, and of the most useful mechanical practice of all equations.

FINIS.

Anno Dom: 1667.

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