The True Squaring of the Circle and the Hyperbola

In particular the discovery of its remarkable proportions and a proof.

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Welcome,

Professors of Geometry

I was wondering to myself at length, dear Professor, whether analysis with its many operations might be sufficient, along with the general method of investigating all the proportions of variables, as Descartes is first seen to affirm in his Geometry; and indeed if so it may be that by its power it may be possible to demonstrate the so oft-repeated feat of squaring of the circle. Whenever I turn this over in my mind I easily perceive from the discoveries made so far on the properties of the circle that analysis may not be able to serve to establish such results. From there by searching others the following occurs to me: the first indeed is known in the general circle, whence I see that the sequence of polygons converges, of which by termination is the circular sector. . .

Definitiones.

1. Let two lines be drawn from the center of a circle, ellipse, or hyperbola to its perimeter. We call the piece bound by those two lines and the segment of the perimeter a sector.

2. Let the segment of the perimeter between the two lines be subtended any number of times, forming rectilinear triangles (where the common vertex is the center of the conic section and the bases are the subtending lines). Then if the conic section is a circle or ellipse we call the figure created by combining these triangles a regular inscribed triangle, if it is a hyperbola then we call the figure a regular circumscribed polygon.

3. Let the segment of the perimeter between the lines be made tangent to lines any number of times and let lines be drawn from the tangents to the center of the conic section, and further require that each of the quadrilaterals, understood as being composed from successive tangent lines and the lines to the center, be equal. If the conic section is a circle or an ellipse, then I call the figure created by the combination of these a regular circumscribed polygon. If it is a hyperbola then I call it a regular inscribed polygon.

4. Let all of the vertices of the subtending angles (except those to the center of the conic section) of the regular polygon tough all of the points of contact of the regular polygon with the tangents. I call this a complex polygon.

5. We say a magnitude is composed of magnitudes, when one magnitude makes other magnitudes by addition, subtraction, multiplication, division, root extraction, or any other operation imaginable.

6. When a magnitude is composed of magnitudes by addition, subtraction, multiplication, division, or root extraction, we say it is composed analytically.

7. When magnitudes can be composed analytically of magnitudes which are mutually commensurable, we say they are mutually analytic.

8. Let the magnitude X be composed of some magnitudes A, B, C, D, and E, and the magnitude Z be composed of magnitudes F, G, C, D, and E by the same method and operations as X only with the magnitudes F and G in place of A and B. If this is so, then we say that the magnitude X is composed of A and B by the same method as Z is composed of F and G.

9. Let there be two magnitudes A and B of which are composed two other magnitudes C and D, whose the difference is less than that of A and B. Also let E be composed from C and D by the same method as C is composed of A and B, and F be composed from C and D by the same method as D is composed of A and B. And further still let G be composed from E and F by the same method as E is composed of C and D and C is composed by A and B, and let H be composed from E and F by the same method as F is composed of C and D and C is composed by A and B. Continue thus. I call this a convergent series.

10. Terms being placed next to each other as A and B, or C and D, or E and F, or G and H, are called convergent terms.

Petitions.

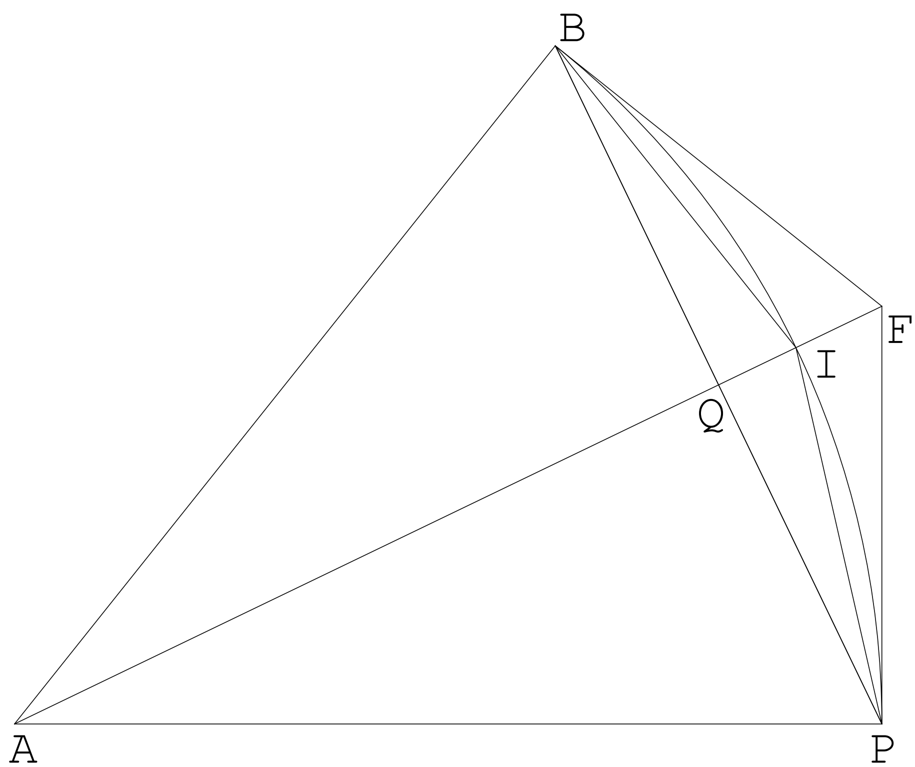
1. We desire that magnitudes composed of given mutually analytic magnitudes be mutually analytic themselves as well as analytic with the given magnitudes.

2. Likewise we desire that magnitudes that cannot be analytically composed from given mutually analytic magnitudes not be analytic with the given magnitudes.

The preceding desires may perhaps be seen by some as obscure, but will be made clear by an analysis of the elements.

The True Squaring of the Circle and the Hyperbola.

Let BIP be a segment of a circle, ellipse, or hyperbola with center A. The triangle ABP may be completed, and from the points B and P on the segment tangents BF and PF may be drawn, which will meet each other at the point F. The line AF is thus produced, which intersects the segment at point I and the line BP at the point Q. From this the lines BI and PI are joined.



Proposition I. Theorem.

The quadrilateral $BAPI$ is half of the quadrilateral $BAPF$ plus the triangle BAP.

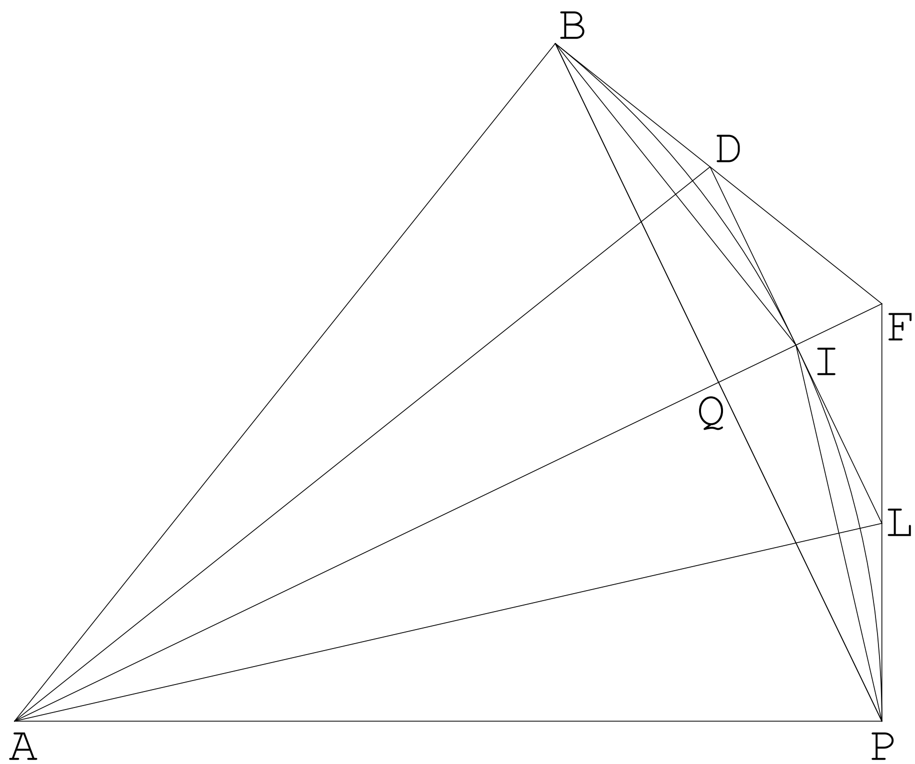
The line AQ is drawn through F meeting with the the two lines FB and FP, which are tangent to the segment at the points B and P. Therefore the line AQ contacts the line BP at its bisector Q. We see from this that the triangle AQB is equal to the triangle AQP, the triangle FBQ equals triangle FQP, and the triangle ABF equals triangle APF. Therefore the triangle ABF is half of the quadrilateral $ABFP$. Similarly the triangle ABI is half of the quadrilateral $ABIP$ and the triangle ABQ is half of triangle ABP. ABF, ABI, and ABQ each have the same altitude and share one base, but the other bases AF, AI, and AQ progress arithmetically. Therefore the two quadrilaterals $ABFP$ and $ABIP$ and the triangle ABP are clearly in arithmetic progression with a ratio of AF to AI, Q.E.D.

Let the line DL be drawn tangent to the segment at the point I so that it will meet the lines BF and PF at the points D and L, completing the polygon $ABDLP$.

Proposition II. Theorem.

The quadrilateral $ABFP$ plus quadrilateral $ABIP$ is to the double of quadrilateral $ABIP$ as quadrilateral $ABFP$ is to polygon $ABDLP$.

The line AF is drawn through the point of tangency of the line DL with the segment, and is likewise drawn through the meeting point of the two lines FB and FP, which terminate the line DL and touch the segment in two points. Therefore the line DL is bisected at the point I. Because of this the triangle FDI equals the triangle FIL and the triangle ABF equals the triangle APF. Thus the quadrilateral $ABDI$ equals quadrilateral $APLI$,



and further quadrilateral *APLI* is half of the polygon *ABDLP*. It is obvious from the above demonstration that the triangle *AIL* equals triangle *ALP*, but triangle *ALF* is to triangle *ALI* as *FA* is to *AI* and *FA* is to *AI* as quadrilateral *ABPF* is to quadrilateral *ABIP*. Thus quadrilateral *ABFP* is to quadrilateral *ABIP* as triangle *ALF* is to triangle *AIL*. So putting it together, quadrilateral *ABFP* plus *ABIP* is to quadrilateral *ABIP* as triangle *AFL* plus triangle *AIL*, i.e. triangle *AFP*, is to triangle *AIL*. Doubling this result, *ABFP* plus *ABIP* is to the double of *ABIP* as triangle *AFP* is to quadrilateral *AILP*. Note that triangle *AFP* is half of quadrilateral *ABFP* and quadrilateral *AILP* is half of polygon *ABDLP*. Therefore quadrilateral *ABFP* plus quadrilateral *ABIP* is to the double of *ABIP* as quadrilateral

ABFP is to polygon *ABDLP*, Q.E.D.

Proposition III. Theorem.

The triangle BAP plus the quadrilateral ABIP is to the quadrilateral ABIP as the double of the quadrilateral ABIP is to the polygon ABDLP.

In the preceding proposition it is shown that the sum of *ABFP* and *ABIP* is to the double of quadrilateral *ABIP* as quadrilateral *ABFP* is to polygon *ABDLP*. By permuting we see that quadrilateral *ABFP* plus *ABIP* is to quadrilateral *ABFP* as the double of quadrilateral *ABIP* is to polygon *ABDLP*. Since quadrilaterals *ABFP* and *ABIP* and triangle *ABP* are in arithmetic progression, we have that quadrilateral *ABIP* is to quadrilateral *ABFP* as triangle *ABP* is to quadrilateral *ABIP*. Putting these together, quadrilateral *ABIP* plus *ABFP* is to quadrilateral *ABFP* as triangle *ABP* plus quadrilateral *ABIP* is to quadrilateral *ABIP*. On the other hand *ABIP* plus *ABFP* is to quadrilateral *ABFP* as the double of quadrilateral *ABIP* is to polygon *ABDLP*. And therefore triangle *ABP* plus quadrilateral *ABIP* is to quadrilateral *ABIP* as the double of quadrilateral *ABIP* is to polygon *ABDLP*, Q.E.D.

Let the lines *AD* and *AL* be drawn to meet the segment at the points *E* and *O* and intersecting the lines *BI* and *IP* at *H* and *M*. From this are joined the lines *BE*, *EI*, *IO*, and *OP* to complete the polygon *ABEIOP*.

Proposition IV. Theorem.

The polygon ABEIOP is half of ABDLP plus the quadrilateral ABIP.

From the previous theorem it is obvious that the quadrilateral *AILP*, quadrilateral *AIOP*, and triangle *AIP* are in arithmetic progression, and from the previous proposition one can gather easily enough that the quadrilateral $AILP$ is half of polygon $ABDLP$, quadrilateral $AIOP$ is half of polygon $ABEIOP$, and triangle AIP is half of quadrilateral $ABIP$. Thus by doubling the terms, polygon $ABDLP$, polygon $ABEIOP$, and quadrilateral $ABIP$ are in arithmetic progression, Q.E.D.

Let lines CG and KN be drawn tangent to the segment at the points E and O and let the lines DL, DB, and LP intersect at the points C, G, K, and N to complete the polygon $ABCGKNP$.

Proposition V. Theorem.

The quadrilateral $ABIP$ plus the polygon $ABEIOP$ are to the polygon $ABEIOP$ as the double of the polygon $ABEIOP$ is to the polygon $ABCGKNP$.

From the third theorem it is obvious that the triangle ABI plus the quadrilateral $ABEI$ is to the quadrilateral $ABEI$ as the double of quadrilateral $ABEI$ is to polygon $ABCGI$. From the previous proposition it is easily concluded that triangle ABI is half of quadrilateral $ABIP$, quadrilateral $ABEI$ is half of polygon $ABEIOP$, and polygon $ABCGI$ is half of polygon $ABCGKNP$. Therefore by doubling the terms, quadrilateral $ABIP$ plus $ABEIOP$ is to polygon $ABEIOP$ as the double of polygon $ABEIOP$ is to polygon $ABCGKNP$, Q.E.D.

From this one can easily see that the polygon $ABCGKNP$ is the harmonic mean between polygons $ABEIOP$ and $ABDLP$, which is sufficient to suggest that this may be demonstrated perpetually.

Scholium.

The two preceding propositions can be proved by the same method for whichever complex polygons in place of the complex polygons $ABIP$ and $ABDLP$. Indeed the tangent polygon contains as many equal quadrilaterals as the subtending polygon contains triangles. And so it is evident that these ratios of the polygons continue themselves to infinity, drawing lines AN, AK, AG, and AC through points R, T, S, and V and composing other lines and polygons inside and outside of these. Note that we may say of these inscribed and circumscribed polygons that they double by inscription and circumscription.

From the previous propositions it is obvious (if we let triangle $ABP = a$ and quadrilateral $ABFP = b$) that quadrilateral $ABIP = \sqrt{ab}$ and polygon $ABDLP = \frac{2ab}{a+\sqrt{ab}}$. By the same method, let quadrilateral $ABIP = c$ and polygon $ABDLP = d$ and we have that polygon $ABEIOP = \sqrt{cd}$ and polygon $ABCGKNP = \frac{2cd}{c+\sqrt{cd}}$, and it is evindent from this that the series of polygons converges.

And so, continuing this to infinity, it is obvious that in the end we have shown that the magnitude of the sector of the circle, the ellipse, or the hyperbola equals $ABEIOP$. Indeed the difference of the complex polygons in the series always diminishes, so that all of the magnitudes may be made smaller, and so in the following theorems we shall demonstrate the Scholium.

Therefore if the aforementioned series of polygons terminates, that is, if one may find a final inscribed polygon (if we may call it that) equal to the final circumscribed polygon, one would infallibly have the quadrature of the circle and the hyperbola. But since it has proved difficult, and in geometry it is perhaps altogether impossible for such a series to terminate, certain propositions are permitted from which to find this kind of limit of the series. And eventually (if it is possible) the general method for finding all of the limits of convergent series.

Proposition VI. Theorem.

The difference between the triangle ABP and the quadrilateral $ABPF$ is greater than twice the difference between the quadrilateral $ABIP$ and the polygon $ABDLP$.

Denote the triangle ABP by A, quadrilateral $ABFP$ by B, quadrilateral $ABIP$ by C, and polygon $ABDLP$ by D. Since A is to C as C is to B, the difference between A and C is to A as the difference between C and B is to B. Permuting, the difference between A and C is to the difference between C and B as A is to C. Now putting these together, the difference between A and C plus the difference between C and B, that is, the difference between A and B is to the difference between C and B as $A+C$ is to C. But $A+C$ is to C as $2C$ is to D and the difference between A and B is to the difference between C and B as $2C$ is to D. Since $A+C$ is to C as $2C$ is to D, permuting gives $A+C$ is to $2C$ as C is to D. Dividing, the difference between A and C is to $2C$ as the difference between C and D is to D. Again permuting, the difference between A and C is to the difference between C and D as $2C$ is to D. But now the difference between A and B has been demonstrated to be to the difference between C and B as $2C$ is to D, and from this the difference between A and B is to the difference between C and B as the difference between A and C is to the difference between C and D. However the difference between A and B is greater than the difference between C and B and the difference between A and C is greater than the difference between C and D. Permuting the previous ratio, the difference between

A and B is to the difference between A and C as the difference between C and B is to the difference between C and D. The difference between A and B is greater than the difference between A and C and the difference between C and B is greater than the difference between C and D, and the difference between A and B is equal to the difference between A and C plus the difference between C and B. Therefore either of them is greater than the difference between C and D and it is obvious that the difference between A and B is greater than the double of the difference between C and D, that is, the difference between the triangle ABP and the quadrilateral $ABFP$ is greater than twice the difference between the quadrilateral $ABIP$ and the polygon $ABDLP$, Q.E.D.

Scholium.

The same method may by all means be used to demonstrate that the difference between quadrilateral $ABIP$ and polygon $ABDLP$ is greater that twice the difference between polygons $ABEIOP$ and $ABCGKNP$. From here by the same method one is able to demonstrate this difference is always exceeded in our doubling to infinity of the complex polygon. In fact the difference between the prior inscribed and circumscribed polygons is always greater than twice the difference between the subsequent inscribed and circumscribed polygons. Thus it bears more than half of the difference of the prior polygons to that of the difference of the subsequent. Therefore continuing the subdoubling of the polygon, we discover the two complex polygons, where the difference is made less than whatever magnitude, as we assumed in the preceding Scholium.

Let there be two magnitudes, a and b, with a less than b, and let there be two inequalities c is greater than d and c is greater than e. From here he have that c is to d as $b-a$ is to $\frac{bd-ad}{c}$ to which a is then added to the magnitude so that $\frac{ca+bd-ad}{c}$, where one immediately assigns the magnitude as a. And also c is to e as $b-a$ is to $\frac{be-ae}{c}$, where the magnitude is subtracted from b yielding $\frac{bc-be+ae}{c}$ which is then assigned to b.

One may continue the convergent series from here in which the first terms are a, b, the second $\frac{ca+bd-ad}{c}, \frac{bc-be+ae}{c}$, and it is obvious that the term $\frac{ca+bd-ad}{c}$ is greater than the term a since the term a is added to $\frac{bd-ad}{c}$ giving $\frac{ca+bd-ad}{c}$. It is also clear that the term $\frac{ca+bd-ad}{c}$ is less than the term b since the difference between a and b is greater than the difference between a and $\frac{ca+bd-ad}{c}$. It is evident that the term $\frac{bc-be+ae}{c}$ is less that the term b since $\frac{be-ae}{c}$ is subtracted from b giving $\frac{bc-be+ae}{c}$. And it is further obvious that the term $\frac{bc-be+ae}{c}$ is greater than a since the difference between a and b is greater than the difference between $\frac{bc-be+ae}{c}$ and b. Therefore it is evident that the difference between the convergent terms a and b is greater than the difference between the convergent terms $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$.

However since the convergent terms a and b where given as indefinite, a and b can be selected to be in the location of whichever of the convergent terms of the whole of the series. So by putting a and b for whichever terms of the convergent series, it necessarily follows from the composition of the series that $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$ are the immediately following convergent terms. And again since the difference between the terms a and b is greater than the difference between the terms $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$, it is clear that the difference between the prior convergent term is always greater than the difference between the subsequent convergent term. Because this difference always diminishes proportionally in the ratio $b-a$ is to $\frac{bc-be+ae-ca-bd+ad}{c}$, one can see that the terms of this convergent series are progressively less. Therefore imagining this series continuing to infinity, we are able to imagine the final convergent terms being equal, where we call these equal terms the limit of the series.

Proposition VII. Problem.

To find the limit of the aforementioned series.

So that this set of problems may be satisfied, we want to first find the magnitude that is composed by the same method from the convergent terms a and b as from the convergent terms $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$, which follows easily from the following method. The magnitude may be obtained by multiplication by a and addition by b times a magnitude m, and the same may be obtained by multiplication by $\frac{ca+bd-ad}{c}$ and addition by $\frac{bc-be+ae}{c}$ times a magnitude m. Let the magnitude be z, then $za+bm$ is equal to $\frac{zca+zbd-zad+mbc-mbe+mae}{c}$ and the equation reduces to $z = \frac{mae-mbe}{ad-bd}$. This magnitude whether multiplied by a and added to mb, or multiplied by $\frac{ca+bd-ad}{c}$ and added to $\frac{mbc-mbe+mae}{c}$ produces the same magnitude in either case, namely $\frac{maae-mbae+mbad-mbbd}{ad-bd}$. And so the aforementioned magnitude is composed by the same method from the convergent terms a and b as from the convergent terms $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$, and because a and b are indefinite magnitudes they can be any convergent terms whatsoever of the series, where the convergent terms immediately following are $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$. Thus the magnitude $\frac{maae-mbae+mbad-mbbd}{cd-bd}$ is composed by the same method from any of the convergent terms of the series what are composed of the convergent terms a and b. Therefore the aforementioned magnitude is composed by the same method from its final convergent terms, which are equal. Let this final term be x, which multiplied by $\frac{mae-mbe}{ad-bd}$ and by m produces xm and $\frac{xmae-xmbe}{ad-bd}$. Summing the factors yields $\frac{xmae-xmbe+xmad-xmbd}{ad-bd}$ is equal to $\frac{maae-mbae+mbad-mbbd}{ad-bd}$, and the equation reduces to x is equal to the term $\frac{aae-bae+bad-bbd}{ae-be+ad-bd}$, which we wanted to find.

In order to make this problem less obscure by an exercise, we illustrate in numbers: Let $c = 7$, $d = 2$, $e = 3$, $a = 28$, and $b = 42$. Then the second convergent terms are 32 and 36, the third are $33\frac{1}{7}$ and $34\frac{2}{7}$, and the limit is $33\frac{3}{5}$.

Changing nothing, if a is less than b then $\frac{ca+bd-ad}{c}$ may be greater than $\frac{bc-be+ae}{c}$, indeed greater can be subtracted analytically by lesser, which will not bear showing in the example. Let $c = 7$, $d = 5$, $e = 4$, $a = 28$, and $b = 42$. The second convergent terms will be 38 and 34, the third $35\frac{1}{7}$ and $36\frac{2}{7}$, and the limit $35\frac{7}{9}$.

The solution to this problem may even be obtained by this same method is a is zero, or exactly nothing. For example, let $c = 8$, $d = 3$, $e = 4$, $a = 0$, and $b = 24$. Then the second convergent terms will be 9 and 12, the third $10\frac{1}{8}$ and $10\frac{1}{2}$, and the limit of the series $10\frac{2}{7}$.

Indeed the limits of these series can be found in Gregorie of St. Vincent's book on geometric progression, although his way of proceeding differs greatly from the one presented here.

Proposition VIII. Problem.

Let the two quantities A and B be given and $C : D$ be any given ratio. We want to find another magnitude so that the ratio of it to A is the multiplicate of $B : A$ in the ratio $C : D$.[1]

First, let the ratio $C : D$ be commensurable, and let E be a common measure of C and D. For as often as E is contained in D let the ratio $F : A$ be the submultiplicate of $B : A$ in such ratio[2]. Also as often as E is contained

[1] If a ratio x is the "multiplicate" of the ratio y in the ratio z, then in modern notation $x = y^z$. Likewise, if a number x is the "submultiplicate" of the ratio y in the ratio z, then in modern notation $x = y^{\frac{1}{z}}$. See [1, p.286].

[2] That is, $\frac{F}{A} = \left(\frac{B}{A}\right)^{\frac{E}{D}}$

in C let the ratio $G : A$ be the multiplicate of the ratio $F : A$ in such ratio[3]. I claim that G is the desired magnitude. The ratio $G : A$ is the multiplicate of the ratio $F : A$ in the ratio $C : E$, and the ratio $F : A$ is the multiplicate of the ratio $B : A$ in the ratio $E : D$. Therefore by equality, the ratio $G : A$ is the multiplicate of the ratio $B : A$ in the ratio $C : D$, which is what we wanted to show.

If the ratio $C : D$ is incommensurable, then I am convinced that in practice this problem is geometrically impossible. However it can be accomplished by approximation, assuming a commensurable ratio that approaches it.

Let there be a convergent series such that the first terms are A and B, the second C and D, and the third E and F. Let the second terms be made by the first, where B is greater than A, as the multiplicate of the ratio $C : A$ in the ratio of $M : N$, with $M \geq N$, and the ratio of $B : A$ is the multiplicate of the ratio $D : A$ in the ratio $M : O$, with $M \geq O$. Further, the third terms are made from the second as the second are made from the first, and so the series continues.

Proposition IX. Problem.

To find the limit of the aforementioned series.

Set $G = 0$, that is the exponent of the ratio of equality, or of the ratio $A : A$. Also let H satisfy the exponent of the ratio $B : A$. Let $M : N$ be as the difference between G and H, that is H itself, or the exponent of the ratio $B : A$, is to the excess by which I surpasses G, that is I itself, but $M : N$ is the ratio by which $B : A$ is the multiplicate of the ratio $C : A$. Therefore the excess by which I surpasses G, that is I itself, is the exponent of the ratio

[3]That is, $\frac{G}{A} = \left(\frac{B}{A}\right)^{\frac{C}{E}}$

$C : A$. Let $M : O$ be as the excess by which H surpasses G, that is H, is to the excess by which K surpasses G, that is K, but $M : O$ is the ratio by which $B : A$ is the multiplicate of the ratio $D : A$. Whenever H is the exponent of the ratio $B : A$, K will be the exponent of the ratio $D : A$. Therefore if I is the exponent of the ratio $C : A$ and K is the exponent of the ratio $D : A$, then the excess by which K surpasses I will be the exponent of the ratio $D : C$. From here let $M : N$ be as the excess by which K surpasses I, or the exponent of the ratio $D : C$, is to the excess by which R surpasses I, but $M : N$ is the ratio, from the composition of the series, by which $D : C$ is the multiplicate of $E : C$, and so the excess by which K surpasses I is the exponent of the ratio $D : C$. Thus the excess by which R surpasses I is the exponent of the ratio $E : C$ and I is the exponent of the ratio $C : A$. Therefore R is the exponent of the ratio $E : A$. From here let $M : O$ as the excess by which K surpasses I is to the excess by which S surpasses I, but $M : O$ is the ratio, from the composition of the series, by which $D : C$ is the multiplicate of $F : C$, where the excess by which K surpasses I is the exponent of the ratio $D : C$. The excess by which S surpasses I will be the exponent of the ratio $F : C$ and I is the exponent of the ratio $C : A$. Thus S is the exponent of the ratio $F : A$. Therefore when R is the exponent of $E : A$ and S is the exponent of the ratio $F : A$, the excess by which S surpasses R will be the exponent of the ratio $F : E$. Continuing whichever series, it may be demonstrated as before that T be the exponent of $X : A$ and V the exponent of the ratio $Y : A$. Finally it will always be shown that the convergent terms of the series of exponents are exponents of the ratios, and specifically of the convergent terms of the proposed series by the first magnitude, A of the series, of whichever convergent terms of the series may be found in the same way by the initial values. Thus by the term of the series of exponents through this 7 found. For example, let L, it will be the exponent of the ratio, be the limit of the proposed series with the first term A. Therefore $Z : A$ may be found that is the multiplicate of the given $B : A$ in the ratio $L : H$, and Z will be the desired limit, which we wanted to find.

To illustrate this problem in numbers, let $M = 4$, $N = 2$, $O = 1$, $A = 6$, and $B = 10$. Then the second convergent terms shall be $\sqrt{60}$ and $(2160)^{\frac{1}{4}}$, the third convergent terms $(7776000)^{\frac{1}{8}}$ and $(100776960000000)^{\frac{1}{16}}$, and the limit of the series $(360)^{\frac{1}{3}}$.

As another example, let $M = 6$, $N = 2$, $O = 3$, $A = 5$, and $B = 10$. Then the second convergent terms shall be $(250)^{\frac{1}{3}}$ and $\sqrt{50}$, the third $(488281250000000)^{\frac{1}{18}}$ and $(7812500000)^{\frac{1}{12}}$, and the limit of the series $(12500)^{\frac{1}{5}}$. Thus far all of the limits of the convergent series can be made either by a single arithmetic proportion or a single geometric proportion. Now I shall add to the method, and by the power of this the limits of all convergent series may be found.

Proposition X. Problem.

To find the limit of a given series, which is composed from a given magnitude by the same method by which two convergent terms of whatever convergent series compose the subsequent convergent terms of that series.

Let the convergent series ...

Proposition XXIX. Problem.

To find a square equal to a given circle.

Let $4 \cdot 10^{15}$ be the square circumscribed by the circle, then that of the inscribed square is $2 \cdot 10^{15}$, between which 2828427124746190 is the octagonal geometric mean. From this between the octagon within the circle and the square without is the harmonic mean, since by trivial labor is found by dividing the double of the octagonal area within the circle, or the double of the rectangle from the square within and without the circle, by the sum of the square and the octagon within. Then I find 3313708498984760 to be the harmonic mean to the circumscribed octagon. Continuing this converging series of the complex polygons where the midpoint of the first term is the same in whichever convergent term, it is easy to do so up to the polygon of 16384 sides. Indeed the inscribed is 3141592576586860 and the circumscribed 3141592692091258. This is not considered the final term, since in division and root extraction we always stray in some small part from the true value, which the last imperfect term renders closely. From this is employed the approximation from the proofs of 20 and 21, and the terms found within determine the true measure of the circle by having put the diameters by the square of $4 \cdot 10^{15}$, the smaller circle of 3141592653589789 and the larger 3141592653589792 and through this the measure of the circle may not hide. Given a circle to be found, I serve up this series of polygons.

	Inscribed Area	Circumscribed Area
4	2000000000000000	4000000000000000
8	2828427124746190	3313708498984760
16	3061467458920718	3182597878074527
32	3121445152258051	3151724907429255
64	3136548490545938	3144118385245904
128	3140331156954752	3142223629942456
256	3141277250932772	3141750369168965
512	3141513801144299	3141632080703181
1024	3141572940367090	3141602510256808
2048	3141587725277158	3141595117749588
4096	3141591421543029	3141593269613390
8192	3141592345578073	3141592807595664
16384	3141592576586860	3141592692091258

The circle consisting of the following terms

3141592653589789 3141592653589792

is discovered by the same method of equal lines applied to whichever circular or elliptic sector inscribed by a known triangle and circumscribed by a quadrilateral.

Proposition XXX. Problem.

To find an arc from a given sine.

Let the arc AE of the circle described about the center B. The radius of this arc is of course AB and the sine is AD. We want to find the proportion of the arc itself to the entire circumference of the circle. Let AE be the chord of the arc and AC or CE the tangent lines from A and E meeting at C. From the square of the radius *AB* is produced the square of the sine *AD*, and square of the cosine *BD* remains, and thus *BD* may be given. Therefore the area of the triangle *ABD* is given by the rectangle; the area of the triangle *ABE* is clearly given likewise from the rectangle of the given sine *AD* that splits the radius *BE* in two. From this it is seen that as the sum of the triangles *ABD* and *ABE* is to the triangle *ABE*, so the double of the triangle *ABE* is to the quadrilateral *ABEC*, and this gives the constant 5. From the given inscribed triangle *ABE* and the circumscribed quadrilateral *ABEC* it is found through the above sector *ABE* which to the sought for given entire circle has the proportion of the arc *AE* to the total circumference, which we wanted to find.

Proposition XXXI. Problem.

To find a sine from a given arc.

From the given arc it is clear how to give the area of the sector. Therefore by the given sector one may consider from how many arithemic terms the entire sine is made. It follows that some part of the given sector may be summed. For example the sector *ABE* itself, as the inscribed triangle *ABE* and the circumscribed quadrilateral *ABEC* may be repeated however many times, as often as the given sector is multiplied by the sector *ABE* agreeing in all arithmetic terms as contain the square root of the whole sine. Indeed this can easily be shown from consideration of table of 29. Precision is not required in this process, for if large radii are used then it makes no difference if the difference be off by a few parts. The radius *BA* is given from the known sector *ABE*, by which the arc *AE* is likewise known. Let its sine *AD* be given by z; and thus from the given sine and the radius is given as in the previous by the inscribed triangle of the sector *ABE* and that circumscribed quadrilateral *ABEC*. And so the sector itself is given, it is the second of the two arithmetic means of constant proportion between the inscribed triangle and the circumscribed quadrilateral. Thus is given the equation between the double of the quadrilateral $ABEC$ plus one of the triangle ABE and the triple of the known sector ABE, by which the resolution is clearly the value of the unknown magnitude z, which is the sine AD. And by the given arc AE and its sine is given from the common principle of the sine of angular sectors all repeating from its same arc AE. Thus the sine of the arc is not hidden in the beginning proposition where it be given in the rule of multiplication to the arc AE, which we wanted to find.

Proposition XXXII. Problem.

To find a square equal to the hyperbolic area contained under the hyperbolic curve, bound below by one asymptote and between two a lines parallel to the other asymptote, which is equal to the space of the hyperbolic sector having for its base the curve itself.

Let DIL be an hyperbola with asymptotes AO and AK that meet at the right angle OAK. Consider the hyperbolic space $ILMK$ bounded by the hyperbolic curve IL, the asymptotic segment KM, and the two lines AK and LM, which are parallel to the other asymptote AO. Choose the line IK to be 10^{12}, LM to be 10^{13}, and AM to be 10^{12} so that the line KM is $9 \cdot 10^{12}$. We want to find the measure of the space $ILMK$. Let the lines IK and OL be extended and draw the line IP in order to complete the rectangles $LNKM$ and $QIKM$.

It is clear that rectangle $LNKM$ has area $9 \cdot 10^{25}$ and $QIKM$ has area $9 \cdot 10^{24}$, and that the area of trapezoid $LIKM$ is the arithmetic mean of these rectangles, having $4.95 \cdot 10^{25}$. The geometric mean between $LNKM$ and $QIKM$ is found to be 28460498941515413987990042 which is a regular pentagon circumscribed by the hyperbolic area $LIKM$. Now as the trapezoid *LIKM* is to this circumscribed pentagon, so the double of the pentagon is to the regular inscribed hexagon of the hyperbolic area *LIKM*, namely 20779754131836628160009835. This gives the complex of the regular hexagon with the aforementioned pentagon, in which the two line bring about the first terms of the convergent series. Between this is the geometric mean by which the double of the square is divided by the same geometric mean of one plus the major term or the circumscribed pentagon. And they give the geometric mean and bearing proportion to the second convergent terms. Thus this convergent series of complex polygons is continued, with the mean of the first term being in both places the converging term, up to the twentieth term, where the circumscribed polygon is 2302585092993120329958961534173864 and the inscribed is 23025850929931203593181124. From this the approximation is brought out to that proved in propositions 23 and 24, and the terms within are discovered to describe the true hyperbolic area of *LIKM* to be bound below by 23025850929940456240178681 and above by 23025850929940456240178704, and the area can no longer hide, which was to be shown. I thus deliver the entire series of polygons one with the number of lines subtending the hyperbolic curve and another in circumscriped polygons.

	Circumscribed Area	Inscribed Area
2	28460498941515413987990042	20779754131836628160009835
4	24318761696971474416609403	22410399968461612921314879
8	23345088913234727934949897	22868197570682058351436953
16	23105412906351426185065096	22986193244865462241217428
32	23045725982658962868047234	23015921117139340153267671
64	23030818728479610745741910	23023367512879647736902891
128	23027092819292183214705676	23025230015404383009313933
256	23026161398510805910921810	23025695697539046352276636
512	23025928546847571901068394	23025812121604634087915779
1024	23025870334152518169052273	23025841227841783762272302
2048	23025855780992551911165543	23025848504414868310197241
4096	23025852142703422669729927	23025850323559001769499206
8192	23025851233131194254554390	23025850778345089029496888
16384	23025851005738140519209367	23025850892041614212944994
32768	23025850948889877295901163	23025850920465745719335070
65536	23025850934677811503232115	23025850927571778609090592
131072	23025850931124795055887228	23025850929348286832351848
262144	23025850930236540944102405	23025850929792413888218560
524288	23025850930014477416159412	23025850929903445652188450
1048576	23025850929958961534173864	23025850929931203593181124

The hyperbolic sector is bounded by the following terms

23025850929940456240178681 23025850929940456240178704

It is therefore possible without danger of error to assume the following number through the sector of the hyperbola, of which the number you may multiply until the tenth, facilitating division by subtraction thanks to the composition of the logarithm. For in fact by long division

Appendix.

A modern rendering of the propositions.

Proposition I:

Let the area of $BAPF = a$, the area of $BAP = b$, and the area of $BAPI = c$. Then $c = \frac{a+b}{2} = \mathrm{AM}(a, b)$.

Proposition II:

Let the area of $AFBP = a$, the area of $ABIP = b$, and the area of $ABDLP = c$. Then $c = \frac{2ab}{a+b} = \frac{\mathrm{GM}(a,b)}{\mathrm{AM}(a,b)}$.

Proposition III:

Let the area of $ABIP = a$, the area of $BAP = b$, and the area of $ABDLP = c$. Then $c = \frac{2a^2}{a+b} = \frac{a^2}{\mathrm{AM}(a,b)}$.

Proposition IV:

Let the area of $ABIP = a$, the area of $ABDLP = b$, and the area of $ABEIOP = c$. Then $c = \frac{a+b}{2} = \mathrm{AM}(a, b)$.

Proposition V:

Let the area of $ABEIOP = a$, the area of $ABIP = b$, and the area of $ABCGKNP = c$. Then $c = \frac{2a^2}{a+b} = \frac{a^2}{\mathrm{AM}(a,b)}$.

Proposition VI:

Let the area $ABP = a$, the area of $ABFP = b$, the area of $ABIP = c$, and the area of $ABDLP = d$. Then $a - b \leq 2(c-d) = \frac{(a+b)^2}{a+3b} = \frac{2\text{AM}^2(a,b)}{\text{AM}(a,b)+b}$.

Proposition VII:

Let $a_0 = a$ and $b_0 = b$ where $a < b$ and

Bibliography

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