The True Squaring of the Circle and the Hyperbola

On its own kind of proportion,
Discovered and proved.

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Welcome, Reader of Geometry

I was wondering to myself at length, dear Reader, whether analysis with its many operations might be sufficient, and the general method of investigating all proportions of a magnitude, as Descartes is first seen to affirm in his Geometry; and indeed if so it may be that by its power it may be possible to demonstrate the so oft-sung feat of squaring of the circle. Whenever I turn this over in my mind I easily perceive from the discoveries made so far on the properties of the circle that analysis may not be able to serve to establish such results. Then by searching others this second thing occurs to me, the first indeed having been known in the common circle. From this I perceived the convergent sequence of polygons, the limit of which is the circular sector. I saw here at once a trace of analysis. Next, the convergent series having been started, not only by the particular easy case, but likewise to engender by consideration, and by the aforementioned quality of the circle was reduced by no effort to the ellipse and the hyperbola, the infallible quadrature of every conic section was revealed to me. On the other hand while I am thither converting as the convergent series of polygons the limit, insuperable difficulty in its limitation discovering after all arts and trying other attempts. However recalling to mind the office of analysis is just as the common algebra, not only to resolve sets of problems, but likewise by the same to demonstrate the impossibility (if it be useful). And when at first the decided difficulty was put to the test, to second driven back by me, which surely above was promised. Indeed not only of the circle (which I had proposed at the beginning) but all of the true and legitimate conic sections the quadrature of the proportions in its own type, and the entire type of proportion I reveal before the unknown of the orb to the Geometer, which proportion likewise I recude in relation to the dimension of the conic section to the comensurability of the true as near, easily done, demonstrable, and the very brief extraction of roots of the solid (if I'm not mistaken). Indeed in every proportion incommensurable to such approximations Mathematicians recur, as however we may better see by the proportion born, I may speak on the proportion so far as it has been born by some analytic operations, that is of the common arithmetic, the proportion indeed well known to us is in numbers or in a discrete quantity, which in continua, and not do I respect after Decartes these operations to lead in geometry.

. . .

Definitiones.

1. Let two lines be drawn from the center of a circle, ellipse, or hyperbola to its perimeter. We call the piece bound by those two lines and the segment of the perimeter a sector.

2. Let the segment of the perimeter between the two lines be subtended any number of times, forming rectilinear triangles (where the common vertex is the center of the conic section and the bases are the subtending lines). Then if the conic section is a circle or ellipse we call the figure created by combining these triangles a regular inscribed triangle, if it is a hyperbola then we call the figure a regular circumscribed polygon.

3. Let the segment of the perimeter between the lines be made tangent to lines any number of times and let lines be drawn from the tangents to the center of the conic section, and further require that each of the quadrilaterals, understood as being composed from successive tangent lines and the lines to the center, be equal. If the conic section is a circle or an ellipse, then I call the figure created by the combination of these a regular circumscribed polygon. If it is a hyperbola then I call it a regular inscribed polygon.

4. Let all of the vertices of the subtending angles (except those to the center of the conic section) of the regular polygon tough all of the points of contact of the regular polygon with the tangents. I call this a complex polygon.

5. We say a magnitude is composed of magnitudes, when one magnitude makes other magnitudes by addition, subtraction, multiplication, division, root extraction, or any other operation imaginable.

6. When a magnitude is composed of magnitudes by addition, subtraction, multiplication, division, or root extraction, we say it is composed analytically.

7. When magnitudes can be composed analytically of magnitudes which are mutually commensurable, we say they are mutually analytic.

8. Let the magnitude X be composed of some magnitudes A, B, C, D, and E, and the magnitude Z be composed of magnitudes F, G, C, D, and E by the same method and operations as X only with the magnitudes F and G in place of A and B. If this is so, then we say that the magnitude X is composed of A and B by the same method as Z is composed of F and G.

9. Let there be two magnitudes A and B of which are composed two other magnitudes C and D, whose the difference is less than that of A and B. Also let E be composed from C and D by the same method as C is composed of A and B, and F be composed from C and D by the same method as D is composed of A and B. And further still let G be composed from E and F by the same method as E is composed of C and D and C is composed by A and B, and let H be composed from E and F by the same method as F is composed of C and D and C is composed by A and B. Continue thus. I call this a convergent series.

10. Terms being placed next to each other as A and B, or C and D, or E and F, or G and H, are called convergent terms.

Petitions.

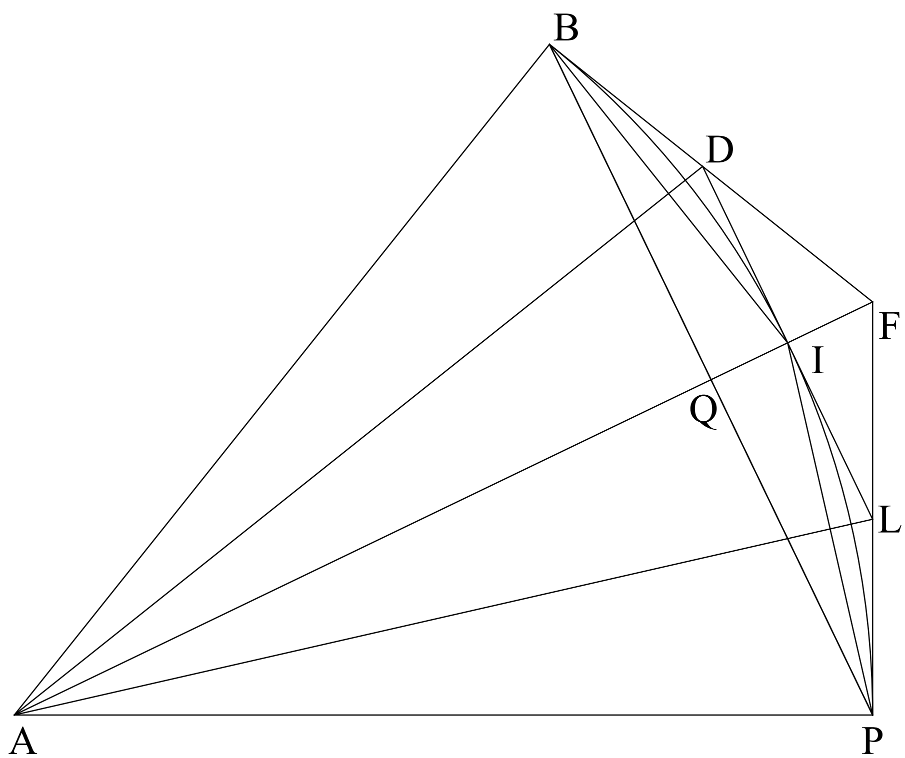
1. We desire that magnitudes composed of given mutually analytic magnitudes be mutually analytic themselves as well as analytic with the given magnitudes.

2. Likewise we desire that magnitudes that cannot be analytically composed from given mutually analytic magnitudes not be analytic with the given magnitudes.

The preceding desires may perhaps be seen by some as obscure, but will be made clear by an analysis of the elements.

The True Squaring of the Circle and the Hyperbola.

Let BIP be a segment of a circle, ellipse, or hyperbola with center A. The triangle ABP may be completed, and from the points B and P on the segment tangents BF and PF may be drawn, which will meet each other at the point F. The line AF is thus produced, which intersects the segment at point I and the line BP at the point Q. From this the lines BI and PI are joined.



Proposition I. Theorem.

The quadrilateral $BAPI$ is half of the quadrilateral $BAPF$ plus the triangle BAP.

The line AQ is drawn through F meeting with the the two lines FB and FP, which are tangent to the segment at the points B and P. Therefore the line AQ contacts the line BP at its bisector Q. We see from this that the triangle AQB is equal to the triangle AQP, the triangle FBQ equals triangle FQP, and the triangle ABF equals triangle APF. Therefore the triangle ABF is half of the quadrilateral $ABFP$. Similarly the triangle ABI is half of the quadrilateral $ABIP$ and the triangle ABQ is half of triangle ABP. ABF, ABI, and ABQ each have the same altitude and share one base, but the other bases AF, AI, and AQ progress arithmetically. Therefore the two quadrilaterals $ABFP$ and $ABIP$ and the triangle ABP are clearly in arithmetic progression with a ratio of AF to AI, Q.E.D.

Let the line DL be drawn tangent to the segment at the point I so that it will meet the lines BF and PF at the points D and L, completing the polygon $ABDLP$.

Proposition II. Theorem.

The quadrilateral $ABFP$ plus quadrilateral $ABIP$ is to the double of quadrilateral $ABIP$ as quadrilateral $ABFP$ is to polygon $ABDLP$.

The line AF is drawn through the point of tangency of the line DL with the segment, and is likewise drawn through the meeting point of the two lines FB and FP, which terminate the line DL and touch the segment in two points. Therefore the line DL is bisected at the point I. Because of this the triangle FDI equals the triangle FIL and the triangle ABF equals the triangle APF. Thus the quadrilateral $ABDI$ equals quadrilateral $APLI$, and further quadrilateral $APLI$ is half of the polygon $ABDLP$. It is obvious from the above demonstration that the triangle AIL equals triangle ALP, but triangle ALF is to triangle ALI as FA is to AI and FA is to AI as quadrilateral $ABPF$ is to quadrilateral $ABIP$. Thus quadrilateral $ABFP$ is to quadrilateral $ABIP$ as triangle ALF is to triangle AIL. So putting it together, quadrilateral $ABFP$ plus $ABIP$ is to quadrilateral $ABIP$ as triangle AFL plus triangle AIL, i.e. triangle AFP, is to triangle AIL. Doubling this result, $ABFP$ plus $ABIP$ is to the double of $ABIP$ as triangle AFP is to quadrilateral $AILP$. Note that triangle AFP is half of quadrilateral $ABFP$ and quadrilateral $AILP$ is half of polygon $ABDLP$. Therefore quadrilateral $ABFP$ plus quadrilateral $ABIP$ is to the double of $ABIP$ as quadrilateral $ABFP$ is to polygon $ABDLP$, Q.E.D.

Proposition III. Theorem.

The triangle BAP plus the quadrilateral ABIP is to the quadrilateral ABIP as the double of the quadrilateral ABIP is to the polygon ABDLP.

In the preceding proposition it is shown that the sum of $ABFP$ and $ABIP$ is to the double of quadrilateral $ABIP$ as quadrilateral $ABFP$ is to polygon $ABDLP$. By permuting we see that quadrilateral $ABFP$ plus $ABIP$ is to quadrilateral $ABFP$ as the double of quadrilateral $ABIP$ is to polygon $ABDLP$. Since quadrilaterals $ABFP$ and $ABIP$ and triangle ABP are in arithmetic progression, we have that quadrilateral $ABIP$ is to quadrilateral $ABFP$ as triangle ABP is to quadrilateral $ABIP$. Putting these together, quadrilateral $ABIP$ plus $ABFP$ is to quadrilateral $ABFP$ as triangle ABP plus quadrilateral $ABIP$ is to quadrilateral $ABIP$. On the other hand $ABIP$ plus $ABFP$ is to quadrilateral $ABFP$ as the double of quadrilateral $ABIP$ is to polygon $ABDLP$. And therefore triangle ABP plus quadrilateral $ABIP$

is to quadrilateral $ABIP$ as the double of quadrilateral $ABIP$ is to polygon $ABDLP$, Q.E.D.

Let the lines AD and AL be drawn to meet the segment at the points E and O and intersecting the lines BI and IP at H and M. From this are joined the lines BE, EI, IO, and OP to complete the polygon $ABEIOP$.

Proposition IV. Theorem.

The polygon $ABEIOP$ is half of $ABDLP$ plus the quadrilateral $ABIP$.

From the previous theorem it is obvious that the quadrilateral $AILP$, quadrilateral $AIOP$, and triangle AIP are in arithmetic progression, and from the previous proposition one can gather easily enough that the quadrilateral $AILP$ is half of polygon $ABDLP$, quadrilateral $AIOP$ is half of polygon $ABEIOP$, and triangle AIP is half of quadrilateral $ABIP$. Thus by doubling the terms, polygon $ABDLP$, polygon $ABEIOP$, and quadrilateral $ABIP$ are in arithmetic progression, Q.E.D.

Let lines CG and KN be drawn tangent to the segment at the points E and O and let the lines DL, DB, and LP intersect at the points C, G, K, and N to complete the polygon $ABCGKNP$.

Proposition V. Theorem.

The quadrilateral $ABIP$ plus the polygon $ABEIOP$ are to the polygon $ABEIOP$ as the double of the polygon $ABEIOP$ is to the polygon $ABCGKNP$.

From the third theorem it is obvious that the triangle ABI plus the quadrilateral $ABEI$ is to the quadrilateral $ABEI$ as the double of quadrilateral $ABEI$ is to polygon $ABCGI$. From the previous proposition it is easily concluded that triangle ABI is half of quadrilateral $ABIP$, quadrilateral $ABEI$ is half of polygon $ABEIOP$, and polygon $ABCGI$ is half of polygon $ABCGKNP$. Therefore by doubling the terms, quadrilateral $ABIP$ plus $ABEIOP$ is to polygon $ABEIOP$ as the double of polygon $ABEIOP$ is to polygon $ABCGKNP$, Q.E.D.

From this one can easily see that the polygon $ABCGKNP$ is the harmonic mean between polygons $ABEIOP$ and $ABDLP$, which is sufficient to suggest that this may be demonstrated perpetually.

Scholium.

The two preceding propositions can be proved by the same method for whichever complex polygons in place of the complex polygons $ABIP$ and $ABDLP$. Indeed the tangent polygon contains as many equal quadrilaterals as the subtending polygon contains triangles. And so it is evident that these ratios of the polygons continue themselves to infinity, drawing lines AN, AK, AG, and AC through points R, T, S, and V and composing other lines and polygons inside and outside of these. Note that we may say of these inscribed and circumscribed polygons that they double by inscription and circumscription.

From the previous propositions it is obvious (if we let triangle $ABP = a$ and quadrilateral $ABFP = b$) that quadrilateral $ABIP = \sqrt{ab}$ and polygon $ABDLP = \frac{2ab}{a+\sqrt{ab}}$. By the same method, let quadrilateral $ABIP = c$ and polygon $ABDLP = d$ and we have that polygon $ABEIOP = \sqrt{cd}$ and polygon $ABCGKNP = \frac{2cd}{c+\sqrt{cd}}$, and it is evindent from this that the series of polygons converges.

And so, continuing this to infinity, it is obvious that in the end we have shown that the magnitude of the sector of the circle, the ellipse, or the hyperbola equals $ABEIOP$. Indeed the difference of the complex polygons in the series always diminishes, so that all of the magnitudes may be made smaller, and so in the following theorems we shall demonstrate the Scholium.

Therefore if the aforementioned series of polygons terminates, that is, if one may find a final inscribed polygon (if we may call it that) equal to the final circumscribed polygon, one would infallibly have the quadrature of the circle and the hyperbola. But since it has proved difficult, and in geometry it is perhaps altogether impossible for such a series to terminate, certain propositions are permitted from which to find this kind of limit of the series. And eventually (if it is possible) the general method for finding all of the limits of convergent series.

Proposition VI. Theorem.

The difference between the triangle ABP and the quadrilateral $ABPF$ is greater than twice the difference between the quadrilateral $ABIP$ and the polygon $ABDLP$.

Denote the triangle ABP by A, quadrilateral $ABFP$ by B, quadrilateral $ABIP$ by C, and polygon $ABDLP$ by D. Since A is to C as C is to B, the difference between A and C is to A as the difference between C and B is to B. Permuting, the difference between A and C is to the difference between C and B as A is to C. Now putting these together, the difference between A and C plus the difference between C and B, that is, the difference between A and B is to the difference between C and B as $A+C$ is to C. But $A+C$ is to C as $2C$ is to D and the difference between A and B is to the difference between C and B as $2C$ is to D. Since $A+C$ is to C as $2C$ is to D, permuting gives $A+C$ is to $2C$ as C is to D. Dividing, the difference between A and C is to $2C$ as the difference between C and D is to D. Again permuting, the difference between A and C is to the difference between C and D as $2C$

is to D. But now the difference between A and B has been demonstrated to be to the difference between C and B as $2C$ is to D, and from this the difference between A and B is to the difference between C and B as the difference between A and C is to the difference between C and D. However the difference between A and B is greater than the difference between C and B and the difference between A and C is greater than the difference between C and D. Permuting the previous ratio, the difference between A and B is to the difference between A and C as the difference between C and B is to the difference between C and D. The difference between A and B is greater than the difference between A and C and the difference between C and B is greater than the difference between C and D, and the difference between A and B is equal to the difference between A and C plus the difference between C and B. Therefore either of them is greater than the difference between C and D and it is obvious that the difference between A and B is greater than the double of the difference between C and D, that is, the difference between the triangle ABP and the quadrilateral $ABFP$ is greater than twice the difference between the quadrilateral $ABIP$ and the polygon $ABDLP$, Q.E.D.

Scholium.

The same method may by all means be used to demonstrate that the difference between quadrilateral $ABIP$ and polygon $ABDLP$ is greater that twice the difference between polygons $ABEIOP$ and $ABCGKNP$. From here by the same method one is able to demonstrate this difference is always exceeded in our doubling to infinity of the complex polygon. In fact the difference between the prior inscribed and circumscribed polygons is always greater than twice the difference between the subsequent inscribed and circumscribed polygons. Thus it bears more than half of the difference of the prior polygons to that of the difference of the subsequent. Therefore continuing the subdoubling of the polygon, we discover the two complex polygons, where the difference is made less than whatever magnitude, as we assumed in the preceding Scholium.

Let there be two magnitudes, a and b, with a less than b, and let there be two inequalities c is greater than d and c is greater than e. From here he have that c is to d as $b-a$ is to $\frac{bd-ad}{c}$ to which a is then added to the magnitude so that $\frac{ca+bd-ad}{c}$, where one immediately assigns the magnitude as a. And also c is to e as $b-a$ is to $\frac{be-ae}{c}$, where the magnitude is subtracted from b yielding $\frac{bc-be+ae}{c}$ which is then assigned to b.

One may continue the convergent series from here in which the first terms are a,b, the second $\frac{ca+bd-ad}{c}$, $\frac{bc-be+ae}{c}$, and it is obvious that the term $\frac{ca+bd-ad}{c}$ is greater than the term a since the term a is added to $\frac{bd-ad}{c}$ giving $\frac{ca+bd-ad}{c}$. It is also clear that the term $\frac{ca+bd-ad}{c}$ is less than the term b since the difference between a and b is greater than the difference between a and $\frac{ca+bd-ad}{c}$. It is evident that the term $\frac{bc-be+ae}{c}$ is less that the term b since $\frac{be-ae}{c}$ is subtracted from b giving $\frac{bc-be+ae}{c}$. And it is further obvious that the term $\frac{bc-be+ae}{c}$ is greater than a since the difference between a and b is greater than the difference between $\frac{bc-be+ae}{c}$ and b. Therefore it is evident that the difference between the convergent terms a and b is greater than the difference between the convergent terms $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$.

However since the convergent terms a and b where given as indefinite, a and b can be selected to be in the location of whichever of the convergent terms of the whole of the series. So by putting a and b for whichever terms of the convergent series, it necessarily follows from the composition of the series that $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$ are the immediately following convergent terms. And again since the difference between the terms a and b is greater than the difference between the terms $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$, it is clear that the difference between the prior convergent term is always greater than the difference between the subsequent convergent term. Because this difference always diminishes proportionally in the ratio $b - a$ is to $\frac{bc-be+ae-ca-bd+ad}{c}$, one can see that the terms of this convergent series are progressively less. Therefore imagining this series continuing to infinity, we are able to imagine the final convergent terms being equal, where we call these equal terms the limit of the series.

Proposition VII. Problem.

To find the limit of the aforementioned series.

So that this set of problems may be satisfied, we want to first find the magnitude that is composed by the same method from the convergent terms a and b as from the convergent terms $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$, which follows easily from the following method. The magnitude may be obtained by multiplication by a and addition by b times a magnitude m, and the same may be obtained by multiplication by $\frac{ca+bd-ad}{c}$ and addition by $\frac{bc-be+ae}{c}$ times a magnitude m. Let the magnitude be z, then $za + bm$ is equal to $\frac{zca+zbd-zad+mbc-mbe+mae}{c}$ and the equation reduces to $z = \frac{mae-mbe}{ad-bd}$. This magnitude whether multiplied by a and added to mb, or multiplied by $\frac{ca+bd-ad}{c}$ and added to $\frac{mbc-mbe+mae}{c}$ produces the same magnitude in either case, namely $\frac{maae-mbae+mbad-mbbd}{ad-bd}$. And so the aforementioned magnitude is composed by the same method from the convergent terms a and b as from the convergent terms $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$, and because a and b are indefinite magnitudes they can be any convergent terms whatsoever of the series, where the convergent terms immediately following are $\frac{ca+bd-ad}{c}$ and $\frac{bc-be+ae}{c}$. Thus the magnitude $\frac{maae-mbae+mbad-mbbd}{cd-bd}$ is composed by the same method from any of the convergent terms of the series what are composed of the convergent terms a and b. Therefore the aforementioned magnitude is composed by the same method from its final convergent terms, which are equal. Let this final term be x, which multiplied by $\frac{mae-mbe}{ad-bd}$ and by m produces xm and $\frac{xmae-xmbe}{ad-bd}$. Summing the factors yields $\frac{xmae-xmbe+xmad-xmbd}{ad-bd}$ is equal to $\frac{maae-mbae+mbad-mbbd}{ad-bd}$, and the equation reduces to x is equal to the term $\frac{aae-bae+bad-bbd}{ae-be+ad-bd}$, which we wanted to find.

In order to make this problem less obscure by an exercise, we illustrate in numbers: Let $c=7$, $d=2$, $e=3$, $a=28$, and $b=42$. Then the second convergent terms are 32 and 36, the third are $33\frac{1}{7}$ and $34\frac{2}{7}$, and the limit is $33\frac{3}{5}$.

Changing nothing, if a is less than b then $\frac{ca+bd-ad}{c}$ may be greater than $\frac{bc-be+ae}{c}$, indeed greater can be subtracted analytically by lesser, which will not bear showing in the example. Let $c=7$, $d=5$, $e=4$, $a=28$, and $b=42$. The second convergent terms will be 38 and 34, the third $35\frac{1}{7}$ and $36\frac{2}{7}$, and the limit $35\frac{7}{9}$.

The solution to this problem may even be obtained by this same method is a is zero, or exactly nothing. For example, let $c=8$, $d=3$, $e=4$, $a=0$, and $b=24$. Then the second convergent terms will be 9 and 12, the third $10\frac{1}{8}$ and $10\frac{1}{2}$, and the limit of the series $10\frac{2}{7}$.

Indeed the limits of these series can be found in Gregorie of St. Vincent's book on geometric progression, although his way of proceeding differs greatly from the one presented here.

Proposition VIII. Problem.

Let the two quantities A and B be given and $C : D$ be any given ratio. We want to find another magnitude so that the ratio of it to A is the multiplicate of $B : A$ in the ratio $C : D$.[1]

First, let the ratio $C : D$ be commensurable, and let E be a common measure of C and D. For as often as E is contained in D let the ratio $F : A$ be the submultiplicate of $B : A$ in such ratio[2]. Also as often as E is contained in C let the ratio $G : A$ be the multiplicate of the ratio $F : A$ in such ratio[3]. I claim that G is the desired magnitude. The ratio $G : A$ is the multiplicate of the ratio $F : A$ in the ratio $C : E$, and the ratio $F : A$ is the multiplicate of the ratio $B : A$ in the ratio $E : D$. Therefore by equality, the ratio $G : A$ is the multiplicate of the ratio $B : A$ in the ratio $C : D$, which is what we wanted to show.

If the ratio $C : D$ is incommensurable, then I am convinced that in practice this problem is geometrically impossible. However it can be accomplished by approximation, assuming a commensurable ratio that approaches it.

Let there be a convergent series such that the first terms are A and B, the second C and D, and the third E and F. Let the second terms be made by the first, where B is greater than A, as the multiplicate of the ratio $C : A$ in the ratio of $M : N$, with $M \geq N$, and the ratio of $B : A$ is the multiplicate of the ratio $D : A$ in the ratio $M : O$, with $M \geq O$. Further, the third terms are made from the second as the second are made from the first, and so the

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[1] If a ratio x is the "multiplicate" of the ratio y in the ratio z, then in modern notation $x = y^z$. Likewise, if a number x is the "submultiplicate" of the ratio y in the ratio z, then in modern notation $x = y^{\frac{1}{z}}$. See [1, p.286].

[2] That is, $\frac{F}{A} = \left(\frac{B}{A}\right)^{\frac{E}{D}}$

[3] That is, $\frac{G}{A} = \left(\frac{B}{A}\right)^{\frac{C}{E}}$

series continues.

Proposition IX. Problem.

To find the limit of the aforementioned series.

Set $G = 0$, that is the exponent of the ratio of equality, or of the ratio $A : A$. Also let H satisfy the exponent of the ratio $B : A$. Let $M : N$ be as the difference between G and H, that is H itself, or the exponent of the ratio $B : A$, is to the excess of I over G, that is I itself, but $M : N$ is the ratio by which $B : A$ is the multiplicate of the ratio $C : A$. Therefore the excess of I over G, that is I itself, is the exponent of the ratio $C : A$. Let $M : O$ be as the excess of H over G, that is H, is to the excess of K over G, that is K, but $M : O$ is the ratio by which $B : A$ is the multiplicate of the ratio $D : A$. Whenever H is the exponent of the ratio $B : A$, K will be the exponent of the ratio $D : A$. Therefore if I is the exponent of the ratio $C : A$ and K is the exponent of the ratio $D : A$, then the excess of K over I will be the exponent of the ratio $D : C$. From here let $M : N$ be as the excess of K over I, or the exponent of the ratio $D : C$, is to the excess of R over I, but $M : N$ is the ratio, from the composition of the series, by which $D : C$ is the multiplicate of $E : C$, and so the excess of K over I is the exponent of the ratio $D : C$. Thus the excess of R over I is the exponent of the ratio $E : C$ and I is the exponent of the ratio $C : A$. Therefore R is the exponent of the ratio $E : A$. From here let $M : O$ as the excess of K over I is to the excess of S over I, but $M : O$ is the ratio, from the composition of the series, by which $D : C$ is the multiplicate of $F : C$, where the excess of K over I is the exponent of the ratio $D : C$. The excess of S over I will be the exponent of the ratio $F : C$ and I is the exponent of the ratio $C : A$. Thus S is the exponent of the ratio $F : A$. Therefore when R is the exponent of $E : A$ and S is the exponent of the ratio $F : A$, the excess of S over R will be the exponent of the ratio $F : E$. Continuing whichever series, it may be demonstrated as before that T be the exponent of $X : A$ and V the exponent of the ratio $Y : A$. Finally it will always be shown that the convergent terms of the series of exponents are exponents of the ratios, and specifically of the convergent terms of the proposed series by the first magnitude, A of the series, of whichever convergent terms of the series may be found in the same way by the initial values. Thus by the term of the series of exponents through this 7 found. For example, let L, it will be the exponent of the ratio, be the limit of the proposed series with the first term A. Therefore $Z : A$ may be found that is the multiplicate of the given $B : A$ in the ratio $L : H$, and Z will be the desired limit, which we wanted to find.

To illustrate this problem in numbers, let $M = 4$, $N = 2$, $O = 1$, $A = 6$, and $B = 10$. Then the second convergent terms shall be $\sqrt{60}$ and $(2160)^{\frac{1}{4}}$, the third convergent terms $(7776000)^{\frac{1}{8}}$ and $(100776960000000)^{\frac{1}{16}}$, and the limit of the series $(360)^{\frac{1}{3}}$.

As another example, let $M = 6$, $N = 2$, $O = 3$, $A = 5$, and $B = 10$. Then the second convergent terms of the series shall be $(250)^{\frac{1}{3}}$ and $\sqrt{50}$, the third $(488281250000000)^{\frac{1}{18}}$ and $(7812500000)^{\frac{1}{12}}$, and the limit of the series $(12500)^{\frac{1}{5}}$. Thus far all of the limits of the convergent series can be made either by a single arithmetic proportion or a single geometric proportion. Now I shall add to the method, and by the power of this the limits of all convergent series may be found.

Proposition X. Problem.

To find the limit of a given series from a given magnitude composed by the same method from two convergent terms of any convergent series in the same way as from the subsequent convergent terms of the same series.

Let the convergent series be of any two convergent terms a and b and the the subsequent convergent terms $\sqrt{ab}$ and $\frac{aa}{\sqrt{ab}}$. The sum of the convergent terms $a+b$ multiplied by the first convergent term a gives $aa+ab$. The sum of the subsequent convergent terms $\sqrt{ab}+\frac{a^2}{\sqrt{ab}}$ multiplied by the first convergent term $\sqrt{ab}$ likewise gives $aa+ab$. From this is discovered the limit of the convergent series. It is clear that the magnitude $aa+ab$ is made by the same method from the convergent terms a and b as from the subsequent convergent terms $\sqrt{ab}$ and $\frac{aa}{\sqrt{ab}}$, and because the magnitudes a and b were arbitrarily chosen terms of the convergent series, it is evident that the sum of any proposed convergent terms of the series multiplied by the first convergent term will give that same magnitude, which is likewise the sum of the subsequent convergent terms multiplied by the first convergent term. Since two convergent terms are always followed by two convergent terms, it is clear that the sum of any two of the convergent terms multiplied by the first convergent term will be $aa+ab$. And so the final convergent terms are equal. Therefore let the final term of this series be the limit z, which is added to itself and the sum multiplied by itself to give $2zz$, which equals the magnitude $aa+ab$, and solving this equation for z yields the limit of the series $\sqrt{\frac{aa+b^2}{2}}$, which we wanted to find.

And therefore in order to find the limit of any convergent series it is necessary only to discover a magnitude composed by the same method from the first convergent terms as is likewise composed from the second convergent terms.

Conclusions.

Since it is not important to the problem whether the convergent terms a and b are the first, second, third, etc., it is clear that all of the convergent terms of the series are composed by the same method from the first convergent terms as by the second, third, fourth, etc. convergent terms.

Proposition XI. Theorem.

The sector of the circle, ellipse, or hyperbola $ABIP$ is not composed analytically by the triangle ABP and the quadrilateral $ABFP$.

Let the triangle $ABP = a$ and the quadrilateral $ABFP = b$. It is clear from the preceding propositions that the quadrilateral $ABIP = \sqrt{ab}$ and the polygon $ABDLP = \frac{2ab}{a+\sqrt{ab}}$. The sector $ABIP$ is the limit of this convergent series. So that the signs of radicals and fractions may be removed from the terms of the series, for the first convergent terms of the series a and b, that is, for the triangle ABP and the quadrilateral $ABFP$, put a^3+a^2b and a^2b+b^3. Then the second convergent terms of the series, that is, the quadrilateral $ABIP$ and the polygon $ABDLP$, will be ba^2+b^2a and $2b^2a$. I claim that the limit of the convergent series (where the first convergent terms of the series are a^3+a^2b and a^2b+b^3 and the second are ba^2+b^2a and $2b^2a$) is not composed analytically of the terms a^3+a^2b and a^2b+b^3. Indeed, if the aforementioned limit is composed analytically of the terms a^3+a^2b and a^2b+b^3, then the limit would itself be analytic and would be composed by the same method from the convergent terms ba^2+b^2a and $2b^2a$. Therefore the limit would be composed analytically by the same method from a^3+a^2b and a^2b+b^3 as it is composed from ba^2+b^2a and $2b^2a$, but no magnitude may be composed analytically by the same method from a^3+a^2b and a^2b+b^3 as it is composed from $ba^2 + b^2a$ and $2b^2a$, which I thus demonstrate. If a magnitude may be composed analytically by the same method from $a^3 + a^2b$ and a^2b+b^3 as it is composed from ba^2+b^2a and $2b^2a$, then the same magnitude would be made by adding, subtracting, multiplying, dividing, and extracting roots from the terms $a^3 + a^2b$ and $a^2b + b^3$ as if by the same method the terms $ba^2 + b^2a$ and $2b^2a$ were added, subtracted, multiplied, divided, and roots extracted. However the latter is not possible to do, so neither can be the former. Thus I prove less***, if the same magnitude is made by addition, subtraction, multiplication, division, and root extraction of the terms $a^3 + a^2b$ and $a^2b + b^3$, which themselves are made by addition, subtraction, multiplication, division, and root extraction of the terms $ba^2 + b^2a$ and $2b^2a$, then by adding, or subtracting, or multiplying, or dividing equal magnitudes by or to the terms $a^3 + a^2b$ and $a^2b + b^3$, or by root extraction, these analytic operations turning into others, by reiterating any of them or doing none of them, the two terms can be made into the final product, one from the term $a^2b + b^3$ and the other from the term $2b^2a$, so that the final product from the term $a^3 + a^2b$ with the final product from $a^2b + b^3$ is the same as the final product by the term $ba^2 + b^2a$ with the final product from the term $2b^2a$ in the same way added, subtracted, multiplied, divided, and roots extracted.

... Yikes! ...

And therefore these two magnitudes cannot be equal, where the other may be obtained by a itself in one as in the other. And so it is evident that a sector of a circle, ellipse, or hyperbola $ABIP$ cannot be composed analytically from the triangle ABP and the quadrilateral $ABFP$, QED.

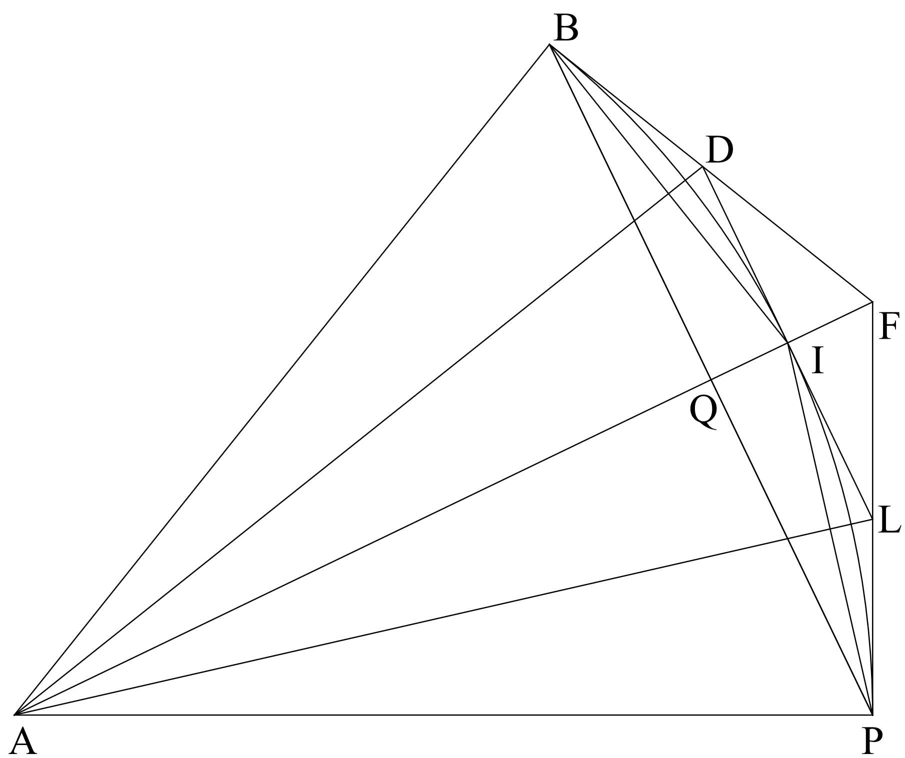
However, so that the purpose may be made evident, I subject it to another more brief and easier proof stemming from another means of attack. A magnitude cannot be composed analytically from the terms $a^3 + a^2b$ and

$a^2b + b^3$ by the same method as the same magnitude is composed from the terms $ba^2 + b^2a$ and $2b^2a$ because by adding, subtracting, multiplying, and dividing $a^3 + a^2b$ and $a^2b + b^3$, and by extracting roots,

Scholium.

It will be seen as most obscure...

Proposition XII. Theorem.



Let the quadrilateral $ABIP$ be A, the polygon $ABEIOP$ be C, the polygon $ABCGKNP$ be D, and the polygon $ABDLP$ be B. I claim that D is the harmonic mean of C and B. Combining proposition 4 and $A : C :: C : B$

gives $A+C : C :: C+B : B$. But then from proposition 5 $A+C : C :: 2C : D$ and also $C+B : B :: 2C : D$. Permuting gives $B+C : 2C :: B : D$ and by dividing, the difference between B and C is to $2C$ as the difference between B and D is to D. Again by permuting, the difference between B and C is to the difference between B and D as $2C$ is to D, that is, as $C+B$ is to B. And by dividing, the difference between D and C is to the difference between B and D as C is to B. Therefore D is the harmonic mean between C and B, Q.E.D.

This proposition is valid by the same method for every complex polygon, so that it follows from the Scholium of Proposition 5.

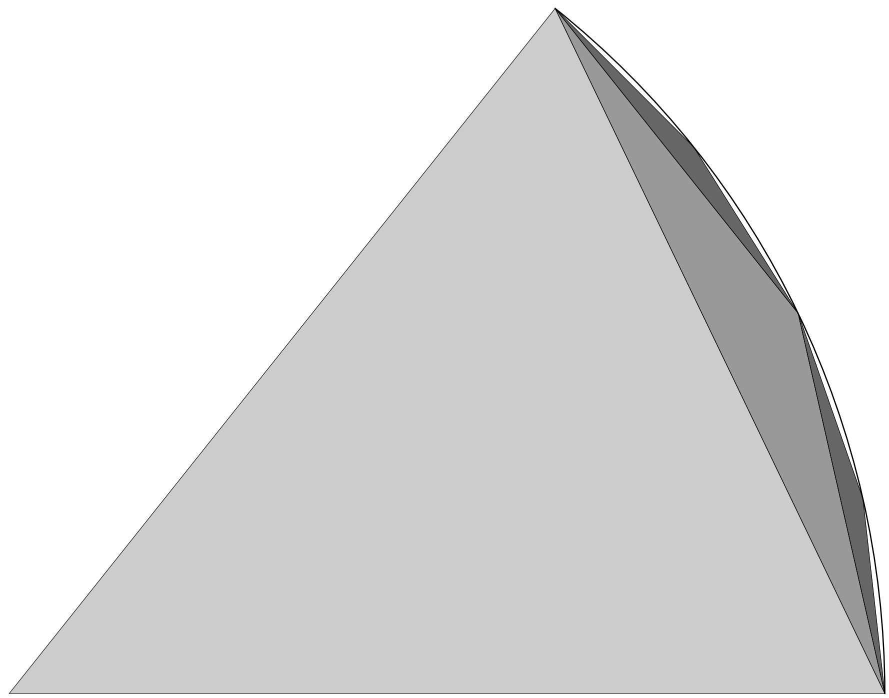
Proposition XIII. Theorem.

Let C be the arithmetic mean, D the geometric mean, and E the harmonic mean of the two magnitudes A and B. I claim that C, D, and E are continuously proportional[4]. Because A, E, and B are in harmonic ratio, the difference between A and E shall be to the difference between E and B as A is to B. By combining, the difference between A and B shall be to the difference between E and B as $A+B$ is to B. From here, by permuting and combining, $2A : A+B :: E : B$, but $2A$ is twice A and $A+B$ is twice C, so that $A : C :: E : B$. Thus $CE = AB$ and $AB = DD$, and so $CE = DD$. Therefore $C : D :: D : E$, Q.E.D.

Proposition XIV. Theorem.

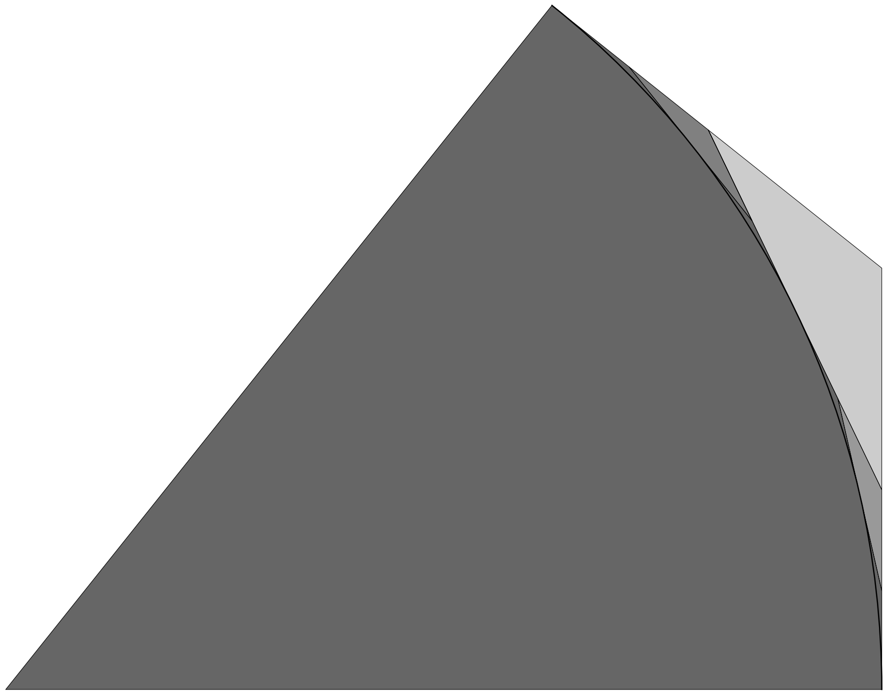
Let A and B be two complex polygons with A inscribed in the sector of a circle or ellipse and B circumscribed. A convergent series of these complex polygons may be continued according to our method of drawing

[4]That is, $C : D :: D : E$.



Series of inscribed polygons.

the subdouble, so that the polygons inscribed in the circle are A, C, E, etc. and those circumscribed are B, D, F, etc. I claim that $A+E$ is less than $2C$. This is clear from the previous propositions by the following analogies. First, A, C, and B are continuously proportional, and second C, D, and B are in harmonic proportion. Therefore the excess of C over A, that is $C-A$, is to the excess of D over C, or $D-C$, in a ratio composed from the proportion $A:C$ and from the proportion $A+C:A$, that is in the ratio $A+C:C$. And $A+C$ is greater than C, so that the excess of C over A is greater than the excess of D over C. However D is greater than E and so the excess of C over A is much greater than the excess of E over C. Therefore $A+E$ is less than $2C$, Q.E.D.



Series of circumscribed polygons.

Proposition XV. Theorem.

By the same assumptions, I claim that the excess of C over A is less than the quadruple of the excess of E over C. This is clear from the previous propositions by analogy with the following three analogies. First, A, C, and B are continuously proportional, second C, D, and B are in harmonic proportion, and third C, E, and D are continuously proportional. And so the excess of C over A, that is $C-A$, is to the excess of E over C, or $E-C$, as $AC+EC+AE+CC$ is to CC, and B is greater than E. So AB, or CC, is greater than AE, and thus $AE+CC$ is less than $2CC$. And so $AC+EC$ is to $2CC$ as $A+E$ is to $2C$, but $A+E$ is less than $2C$ so that $AC+EC$ is less than $2CC$. Thus $AC+EC+AE+CC$ is less than $4CC$. Therefore $C-A$ is less than the quadruple of $E - C$, Q.E.D.

Proposition XVI. Theorem.

Let A and B be two complex polygons with A circumscribed about the sector of a hyperbola and B inscribed. A convergent series of these complex polygons may be continued according to our method of drawing the subdouble, so that the polygons circumscribed about the hyperbola are A, C, E, etc. and those inscribed are B, D, F, etc. I claim that $A+E$ is greater than $2C$. This is clear from the previous propositions by the following analogies. First, A, C, and B are continuously proportional, and second C, D, and B are in harmonic proportion. Therefore the excess of A over C, that is $A - C$, is to the excess of C over D, or $C - D$, in a ratio composed from the proportion $A : C$ and from the proportion $A + C : A$, that is in the ratio $A + C : C$. And $A + C$ is greater than C, so that the excess of A over C is greater than the excess of C over D. However E is greater than D and so the excess of A over C is much greater than the excess of C over E. Therefore $A + E$ is greater than $2C$, Q.E.D.

Proposition XVII. Theorem.

By the same assumptions, I claim that the excess of A over C is greater than the quadruple of the excess of C over E. This is clear from the previous propositions by analogy with the following three analogies. First, A, C, and B are continuously proportional, second C, D, and B are in harmonic proportion, and third C, E, and D are continuously proportional. And so the excess of A over C, that is $A - C$, is to the excess of C over E, or $C - E$, in a ratio composed of the proportions $A : C$, $A + C : A$, and $E + C : C$. And so $A - C$ is to $C - E$ as $AC + EC + AE + CC$ is to CC, and B is less than E. So

AB, or CC, is less than AE, and thus $AE+CC$ is greater than $2CC$. And so $AC+EC$ is to $2CC$ as $A+E$ is to $2C$, but $A+E$ is greater than $2C$ so that $AC+EC$ is greater than $2CC$. Thus $AC+EC+AE+CC$ is greater than $4CC$. Therefore $C-A$ is greater than the quadruple of $E-C$, Q.E.D.

Proposition XVIII. Theorem.

Let A and B be two magnitudes such that A is less than B. Let C be their geometric mean and D their arithmetic mean. I claim that D is greater than C. Since B, C, and A are continuously proportional, by dividing, permuting, and combining, it shall be that the excess of B over A is to the excess of C over A as $A+C$ is to A. And so $A+C$ is greater than twice A. Thus the excess of B over A is greater than twice the excess of D over A, so that the excess of D over A is greater than the excess of C over A. Therefore D is greater than C, Q.E.D.

Proposition XIX. Theorem.

By the same assumptions, let E be the harmonic mean of A and B. I claim that C is greater than E. From proposition 13, D is to C as C is to E, but D is greater than C. Therefore C is greater than E, Q.E.D.

Conclusion.

From the two preceding propositions it is obvious that D is greater than E, that is, that the arithmetic mean of two magnitudes is greater than the harmonic mean of the same.

Proposition XX. Theorem.

Let A and B be two complex polygons with A inscribed in the sector of a circle or ellipse and B circumscribed. A convergent series of these complex polygons may be continued according to our method of drawing the subdouble, so that the polygons inscribed in the circle are A, C, E, K, etc. and those circumscribed are B, D, F, L, etc. Also let Z be the limit of the convergent series, that is, the sector of the circle or ellipse. I claim that Z is greater than C plus one third of the excess of C over A. Let the excess of G over C be a fourth part of the excess of C over A and the excess of H over G be a fourth part of the excess of G over C. This series may be continued infinitely, so let X be the limit of this process. The excess of C over A is less than the quadruple of the excess of E over C, and so the excess of E over C is greater than the excess of G over C, and therefore E is greater than G. Now the excess of E over C is less than the quadruple of the excess of K over E, and so the excess of G over C is much less than the excess of K over E, and therefore the excess of K over E is greater than the excess of H over G. Since E is greater than G, it is obvious K is greater than H. It is demonstrated by the same method in every series A, C, E and A, C, G, by continuation to however many terms. Each term of the series A, C, E is greater than the corresponding term of the series A, C, G. And so the limit of the series A, C, E, that is Z, will be greater than the limit of the series A, C, G, that is X. And from Archimedes the quadrature of the parabola fixed as X is equal to C plus one third of the excess of C over A, and therefore Z is greater than it, Q.E.D.

Proposition XXI. Theorem.

By the same assumptions as above, I claim that Z, which is a sector of a circle or ellipse, is less than the greater of the two continuously proportional arithmetic means of A and B. Let G be the arithmetic mean of A and

B and H be the arithmetic mean between G and B. Likewise let M be the arithmetic mean of G and H and N be the arithmetic mean of M and H. This convergent series, with terms AB, GH, MN, OP, may be continued infinitely, so that its limit is X. It is clear from the preceding propositions that G is greater than C, and H, the arithmetic mean of G and B, is greater than the harmonic mean of G and B. However the harmonic mean of G and B is greater than D, the harmonic mean of C and B, since G is greater than C. And so the arithmetic mean of G and B, that is H, is greater than D, the harmonic mean of C and B. By the same method M, the arithmetic mean of G and H is greater than the geometric mean between G and H. And since G is greater than C, and H is greater than D, the geometric mean of G and H is greater than E, the geometric mean of C and D. Thus M is greater than E. Now N, the arithmetic mean of M and H, is greater than the harmonic mean of the same, and since H is greater than D and M is greater than E, the harmonic mean of M and H is greater than F, the harmonic mean of E and D. And so N is greater than F. Continuing the series by the same method to infinity, one may always show that the terms of the series AB, CD are less than the corresponding terms of the series AB, GH. Therefore the limit, Z, of the series AB, CD will be less than the limit, X, of the series AB, GH. Also, from Proposition 7, the limit, X, of the series AB, GH is equal to the greater of the two continuously proportional arithmetic means of A and B, and so Z is the less than the same, Q.E.D.

Proposition XXII. Theorem.

By the same assumptions as above, I claim that I claim that Z, which is a sector of a circle or ellipse, is less than the greater of the two continuously proportional geometric means of A and B. Let G be the geometric mean of A and B and H be the geometric mean between G and B. Likewise let M

be the geometric mean of G and H and N be the geometric mean of M and H. This convergent series, with terms AB, GH, MN, OP, may be continued infinitely, so that its limit is X. It is clear from the preceding propositions that C and G are equals, and H is greater than D. By this reasoning, M, the geometric mean of G and H, is greater than E, the geometric mean of C and D. Now N, the geometric mean of M and H, is greater than the harmonic mean of the same, and since M is greater than E and H is greater than D, the harmonic mean of M and H is greater than F, the harmonic mean of E and D. And so N is greater than F. Continuing the series by the same method to infinity, one may always show that the terms of the series AB, CD are less than the corresponding terms of the series AB, GH. Therefore the limit, Z, of the series AB, CD will be less than the limit, X, of the series AB, GH. Also, from Proposition 9, the limit, X, of the series AB, GH is equal to the greater of the two continuously proportional geometric means of A and B, and so Z is the less than the same, Q.E.D.

Scholium.

It is not much work for me to show that the greater of the two continuously proportional arithmetic means of two unequal magnitudes is greater than the greater of the two continuously proportional geometric means of the same magnitudes. Therefore it is a more exact approximation of the previous proposition, if it be carried out. However I use the preceding proposition for its ease.

Proposition XXIII. Theorem.

Let A and B be two complex polygons with A circumscribed in the sector of a hyperbola and B inscribed. A convergent series of these complex polygons may be continued according to our method of drawing the subdouble, so that the polygons circumscribed in the circle are A, C, E, K, etc. and those inscribed are B, D, F, L, etc. Also let Z be the limit of the convergent series, that is, the sector of the hyperbola. I claim that Z is greater than C subtracted from one third of the excess of A over C. Let the excess of C over G be a fourth part of the excess of A over C and the excess of G over H be a fourth part of the excess of C over G. This series may be continued infinitely, so let X be the limit of this process. The excess of A over C is greater than the quadruple of the excess of C over E, and so the excess of C over E is less than the excess of C over G, and therefore E is greater than G. Now the excess of C over E is greater than the quadruple of the excess of E over K, and so the excess of C over G is much greater than the excess of E over K, and therefore the excess of G over H is greater than the excess of E over K. Since E is greater than G, it is obvious K is greater than H. It is demonstrated by the same method in every series A, C, E, K and A, C, G, H by continuation to however many terms. Each term of the series A, C, E is greater than the corresponding term of the series A, C, G. And so the limit of the series A, C, E, that is Z, will be greater than the limit of the series A, C, G, that is X. And from Archimedes the quadrature of the parabola fixed as X is equal to C plus one third of the excess of C over A, and therefore Z is greater than it, Q.E.D.

Proposition XXIV. Theorem.

By the same assumptions as above, I claim that Z, which is a sector of a hyperbola, is less than the lesser of the two continuously proportional arithmetic means of A and B. Let G be the arithmetic mean of A and B and H be the arithmetic mean between G and B. Likewise let M be the arithmetic mean of G and H and N be the arithmetic mean of M and H.

This convergent series, with terms AB, GH, MN, OP, may be continued infinitely, so that its limit is X. It is clear from the preceding propositions that G is greater than C, and H, the arithmetic mean of G and B, is greater than the harmonic mean of G and B. However the harmonic mean of G and B is greater than D, the harmonic mean of C and B, since G is greater than C. And so the arithmetic mean of G and B, that is H, is greater than D, the harmonic mean of C and B. By the same method M, the arithmetic mean of G and H is greater than the geometric mean between G and H. And since G is greater than C, and H is greater than D, the geometric mean of G and H is greater than E, the geometric mean of C and D. Thus M is greater than E. Now N, the arithmetic mean of M and H, is greater than the harmonic mean of the same, and since H is greater than D and M is greater than E, the harmonic mean of M and H is greater than F, the harmonic mean of E and D. And so N is greater than F. Continuing the series by the same method to infinity, one may always show that the terms of the series AB, CD are less than the corresponding terms of the series AB, GH. Therefore the limit, Z, of the series AB, CD will be less than the limit, X, of the series AB, GH. Also, from Proposition 7, the limit, X, of the series AB, GH is equal to the lesser of the two continuously proportional arithmetic means of A and B, and so Z is the less than the same, Q.E.D.

Proposition XXV. Theorem.

By the same assumptions as above, I claim that I claim that Z, which is a sector of a hyperbola, is less than the lesser of the two continuously proportional geometric means of A and B. Let G be the geometric mean of A and B and H be the geometric mean between G and B. Likewise let M be the geometric mean of G and H and N be the geometric mean of M and H. This convergent series, with terms AB, GH, MN, OP, may be continued infinitely, so that its limit is X. It is clear from the preceding propositions that C and G are equals, and H is greater than D. By this reasoning, M, the geometric mean of G and H, is greater than E, the geometric mean of C and D. Now N, the geometric mean of M and H, is greater than the harmonic mean of the same, and since M is greater than E and H is greater than D, the harmonic mean of M and H is greater than F, the harmonic mean of E and D. And so N is greater than F. Continuing the series by the same method to infinity, one may always show that the terms of the series AB, CD are less than the corresponding terms of the series AB, GH. Therefore the limit, Z, of the series AB, CD will be less than the limit, X, of the series AB, GH. Also, from Proposition 9, the limit, X, of the series AB, GH is equal to the lesser of the two continuously proportional geometric means of A and B, and so Z is the less than the same, Q.E.D.

It is clear from this claim that this approximation is that one demonstratred in the preceding proposition, if this one might be a little more laborious. One will not ignore, however, that the two series can have equal limits, and such that however many terms of one series would be always be greater than the corresponding terms of the other series. But in such series that are carried out a long way, the difference is less than it of the same number by means of the number of terms. But the counter of our series that are carried out a long way differs more greatly by the number of terms, as one can very easily show.

I observe by experiment that the difference between the second of two proportional arithmetic means and the second of two proportional geometric means is always much greater than the difference between the second of two proportional geometric means and the sector of the circle, ellipse, or hyperbola. That noted I consider appropriate, indeed this sector is obtained differing scarcely beyond one from the second of the proportional arithmetic means, when the arithmetic mean does not exceed the the geometric mean beyond one, which exceedingly noted, for from this it is clear that the approximation is boldly employed, when the series is continued to that the midpoint of first of the noted is the same in either convergent term, which experience likewise have evinced. In fact the sector never in this case differs by unity from the second of the two continuously proportional arithmetic means.

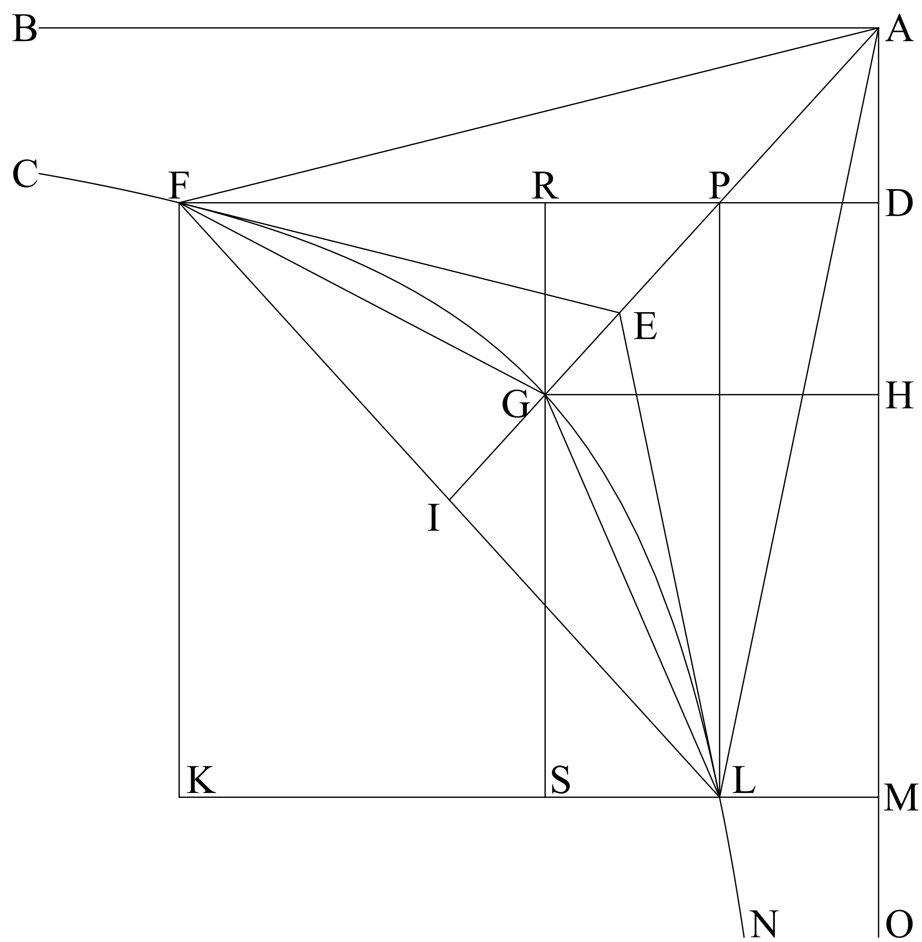
Similarly another approximation is altogether most brief and most astonishing, although it may not turn out to strengthen that geometric demonstration to me. Namely if the first third of the noted in whichever term be converging to the same, the sector of the circle, ellipse, or hyperbola always differs within unity from the greatest quarter by the arithmetic continous proportion the other of the terms of our approximation.

Proposition XXVI. Theorem.

Let CFN be any hyperbola with center A and asymptotes AB and AO. Also, let $AFGL$ be its sector, with circumscribed triangle AFL. Let lines FD and LM be drawn parallel to the asymptote AB and complete parallelograms $FDMK$ and $PLMD$. I claim that the triangle AFL is the arithmetic mean of parallelograms $FDMK$ and $PLMD$. Gregorie of St. Vincent shows in his Libra de Hyperbola that the triangle AFL is equal to the quadrilateral $DFLM$, but it is obvious that quadrilateral $DFLM$ is the arithmetic mean of parallelograms $FDMK$ and $PLMD$, Q.E.D.

Proposition XXVII. Theorem.

By the same assumptions, let line AI be drawn bisecting FL at I and intersecting the hyperbola at the point G. Also let $AFGL$ be a circumscribed



quadrilateral of the sector. I claim that this is the geometric mean of parallelograms $FDMK$ and $PLMD$. From the proof of Gregorie of St. Vincent it is evident that quadrilateral $AFGL$ is equal to $DFGLM$. Because AGI bisects the line FL at I, from the Libra de Hyperbola of Gregorie of St. Vincent it is clear that the lines LM, GH, and FD are continually proportional in the same ratio with the three continuous proportionals AD, AH, and AM. Let the line RGS be drawn through the point G parallel to the asymptote AO, meeting the lines FD and MK at the points R and S. Because the lines FD, GH, and LM are continuously propotional, by dividing and permuting we obtain FR is to SL as GH is to LM. Likewise, since the lines MA, HA, and DA are continuously proportional, by dividing and permuting we obtain MH is to HD, that is SG is to GR, as HA is to DA, or GH is to LM. Thus FR is to SL as SG it to GR, and when the angles FRG and GLS are equal, on account of parallels FR and SL being equal, the triangles FRG and GLS shall be equal. Therefore parallelogram $RDMS$ is equal to polygon $DFGLM$, or quadrilateral $AFGL$. However, parallelogram $RDMS$ is the geometric mean of parallelograms $PDML$ and $FDMK$ since in having the same height and the bases LM, SM and KM are continuously proportional. And so the quadrilateral $AFGL$ is the geometric mean of parallelograms $PDML$ and $FDMK$, Q.E.D.

Proposition XXVIII. Theorem.

By the same assumptions, let the lines FE and LE be drawn tangent to the hyperbola at points F and L in order to complete the quadrilateral $AFEL$. I claim this is the harmonic mean of parallelograms $PDML$ and $FDMK$. Triangle AFL, quadrilateral $AFGL$, and the harmonic mean of parallelograms $PDML$ and $FDMK$ are continuously proportional since triangle AFL is the arithmetic mean and quadrilateral $AFGL$ the geometric mean of these parallelograms, as is clear from Proposition 13. However triangle AFL, quadrilateral $AFGL$ and quadrilateral $AFEL$ are continuously proportional by Proposition 11. Therefore quadrilateral $AFEL$ is the harmonic mean of parallelograms $PDML$ and $FDMK$, Q.E.D.

Proposition XXIX. Problem.

To find a square equal to a given circle.

Let the square circumscribed by the circle be $4 \cdot 10^{15}$, then the inscribed square is $2 \cdot 10^{15}$, between which 2828427124746190 is the octagonal geometric mean. Now let the harmonic mean be between the octagon inside the circle and the square about it, which by trivial labor is found by dividing the double of the area of the octagonal inside the circle, or the double of the rectangle of the areas inside and about the circle, by the sum of the square and the octagon within. Then I find 3313708498984760 to be the harmonic mean, the circumscribed octagon. Continuing this converging series of the complex polygons where the midpoint of the first term is the same in whichever convergent term, it is easy to do so up to the polygon of 16384 sides. Indeed the inscribed is 3141592576586860 and the circumscribed 3141592692091258. This is not considered the final term, since in division and root extraction we always stray in some small part from the true value, which the last imperfect term renders closely. From this is employed the approximation from the proofs of Propositions 20 and 21, and the terms found within will determine the true measure of the circle, positing the square of the diamter as $4 \cdot 10^{15}$, the smaller circle 31415926535897 89, and the larger 3141592653589792. Therefore the measure of the circle may no longer hide, I deliver this series of polygons, which we wanted to find.

	Inside the circle	About the circle
4	2000000000000000	4000000000000000
8	2828427124746190	3313708498984760
16	3061467458920718	3182597878074527
32	3121445152258051	3151724907429255
64	3136548490545938	3144118385245904
128	3140331156954752	3142223629942456
256	3141277250932772	3141750369168965
512	3141513801144299	3141632080703181
1024	3141572940367090	3141602510256808
2048	3141587725277158	3141595117749588
4096	3141591421543029	3141593269613390
8192	3141592345578073	3141592807595664
16384	3141592576586860	3141592692091258

The circle consists of the following terms

3141592653589789 3141592653589792

and by the same method altogether is obtained the equivalent polygon to whichever circular or elliptic sector inscribed by a known triangle and circumscribed by a quadrilateral.

Proposition XXX. Problem.

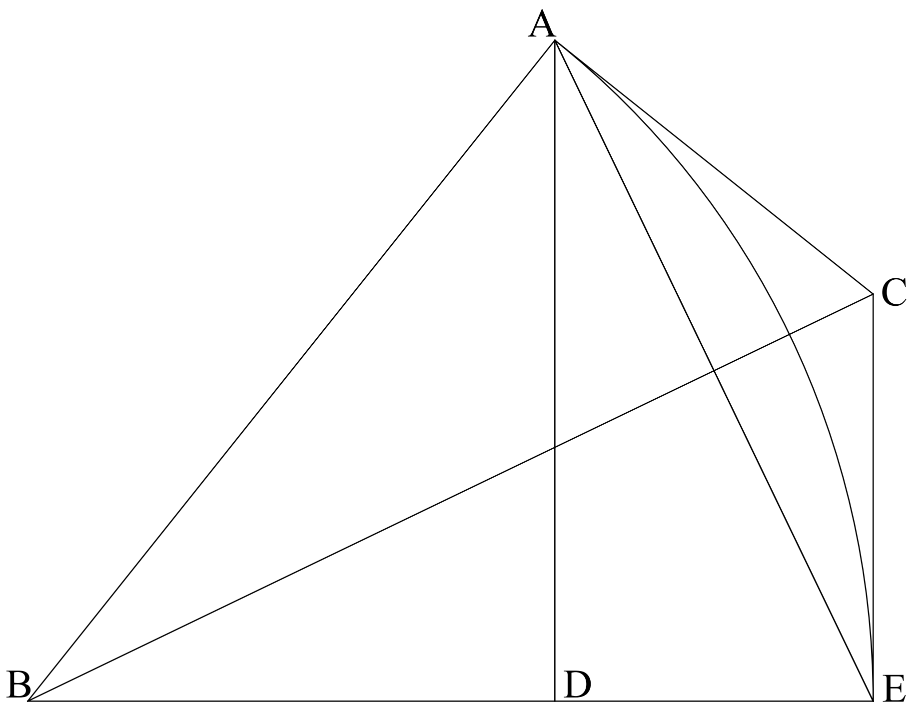
To find an arc from a given sine.

Let AE be the arc of the circle described about the center B. The radius of this arc is of course AB and the sine is AD. We want to find which proportion of the arc itself has to the entire circumference of the circle. Let AE be the chord of the arc and its half-tangents be AC and CE. From the square of the radius *AB* is produced the square of the sine *AD*, and square of the cosine *BD* remains, and thus *BD* is given. Therefore the area of the triangle *ABD* is given by the rectangle and likewise the area of the triangle *ABE* is given, namely the rectangle of the given sine *AD* and half of the radius *BE*. From this it is seen that as the sum of the triangles *ABD* and *ABE* is to the triangle *ABE* as the double of the triangle *ABE* is to the quadrilateral *ABEC*, which is given by Proposition 5. From the given inscribed triangle *ABE* and the circumscribed quadrilateral *ABEC* the sector *ABE* itself may be found by the preceding Proposition, which to the given entire circle has the desired proportion of the arc *AE* to the total circumference, which we wanted to find.

Proposition XXXI. Problem.

To find a sine from a given arc.

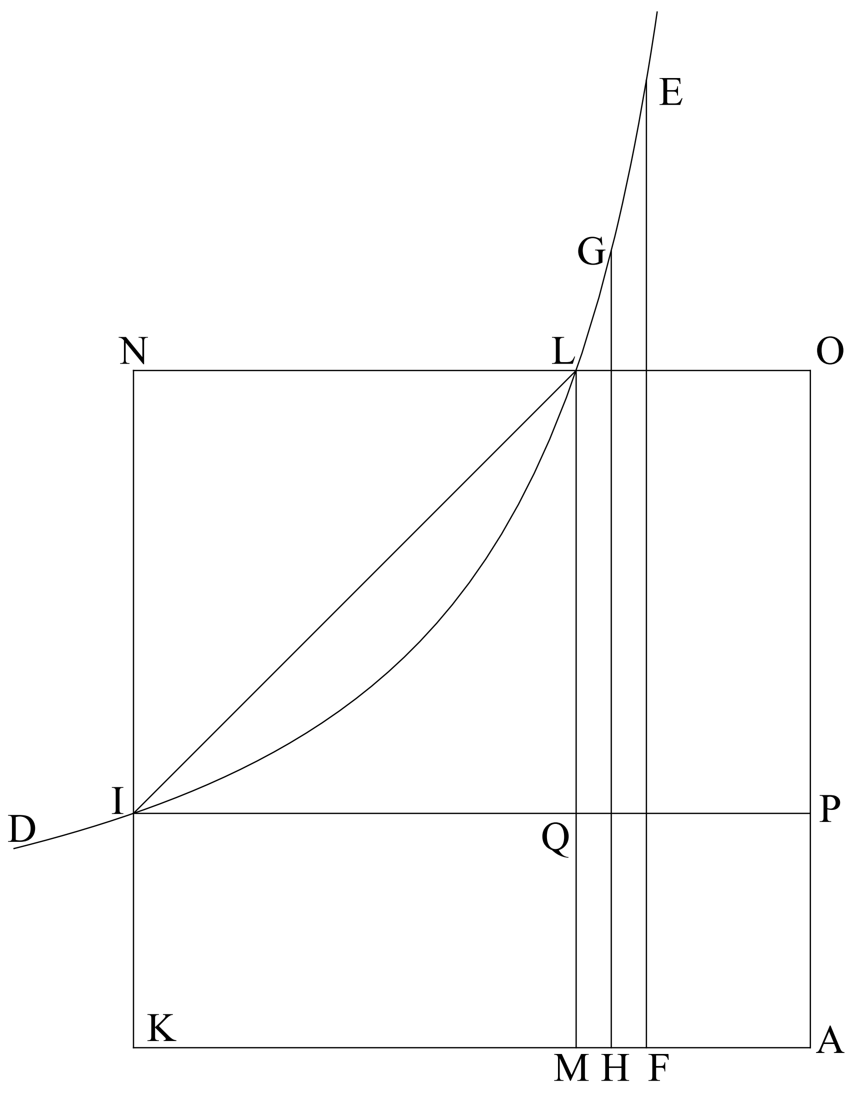
From the given arc it is clear how to give the area of the sector. Therefore given the sector one may consider from how many arithmetic marks the entire sine is made. Now part of such a given sector having been supposed, that is the sector *ABE*, when inscribed triangle *ABE* and circumscribed quadrilateral *ABEC* repeat however many times, as often as the given sector is a multiple of the sector *ABE* agreeing in every arithmetic mark as the square root of the whole sine contains. Indeed this can easily be shown from consideration of the table of Proposition 29. Precision is not required in this process, for if large radii are used then it makes no difference if the difference be off by a few parts. The radius *BA* is given from the known sector *ABE*, by which the arc *AE* is likewise known. Let its sine *AD* be given by z. Thus from the given sine and radius, the inscribed triangle *ABE* is given as in the previous Proposition, as well as the circumscribed quadrilateral *ABEC*. And so the sector itself is given, it is the second of the



two continuously proportional arithmetic means of the inscribed triangle and the circumscribed quadrilateral. Thus is given the equation between the double of the quadrilateral *ABEC* plus triangle *ABE* and the triple of the known sector *ABE*, from whose resolution the value of the unknown magnitude z is obvious, which is sine *AD*. And by the given arc *AE* and its sine likewise the sine of all of the repetitions of the arc *AE* is given from the common doctrine of the angle of a sector. Therefore the sine of the arc initially proposed cannot hide, when it be in a given multiple of the arc *AE*, which we wanted to find.

Proposition XXXII. Problem.

To find a square equal to the hyperbolic area bound by a hyperbolic curve, one asymptote, and two a lines parallel to the other asymptote, which space is equal to the hyperbolic sector having for its base the same curve.



Let DIL be a hyperbola with asymptotes AO and AK that meet at the right angle OAK. Consider the hyperbolic area $ILMK$ bounded by the hyperbolic curve IL, the asymptotic segment KM, and the two lines AK and LM, which are parallel to the other asymptote AO. Choose the line IK to be 10^{12}, LM to be 10^{13}, and AM to be 10^{12} so that the line KM is $9 \cdot 10^{12}$. We want to find the measure of the area $ILMK$. Let the lines IK and OL be extended and draw the line IP in order to complete the rectangles $LNKM$ and $QIKM$. It is clear that rectangle $LNKM$ has area $9 \cdot 10^{25}$ and $QIKM$ has area $9 \cdot 10^{24}$, and that the area of quadrilateral $LIKM$ is the arithmetic mean of these rectangles, being $4.95 \cdot 10^{25}$. The geometric mean between $LNKM$ and $QIKM$ is found to be 28460498941515413987990042 which is the regular circumscribed pentagon of the hyperbolic area $LIKM$. Now as the quadrilateral $LIKM$ is to this circumscribed pentagon, so the double of the pentagon is to the regular inscribed hexagon of the hyperbolic area $LIKM$, namely 20779754131836628160009835. This gives the complex of the regular hexagon with the aforementioned pentagon, of which the two areas bring about the first terms of the convergent series. Between this is the geometric mean by which the double of the square is divided by the same geometric mean plus the greater term, or the circumscribed pentagon. And they give the geometric mean and bearing whatever proportion of the second convergent terms. Thus this convergent series of complex polygons may be continued, while the first midpoint of the terms is the same in both convergent terms, namely up to the twentieth term, where the circumscribed polygon is 230258509299312032995896153417 3864 and the inscribed is 230258509299312035931811124. From here the approximation used is that proved in Propositions 23 and 24, and the inscribed terms are discovered, describing the true hyperbolic area of $LIKM$, being bound below by 230258509299404562401786 81 and the same area to be bound above by 230258509299404562401787 04. And so the area can no longer hide, which was to be shown. I thus deliver the entire series of polygons plus the number of lines subtending the hyperbolic curve in whichever circumscribed polygon.

	Circumscribed Area	Inscribed Area
2	28460498941515413987990042	20779754131836628160009835
4	24318761696971474416609403	22410399968461612921314879
8	23345088913234727934949897	22868197570682058351436953
16	23105412906351426185065096	22986193244865462241217428
32	23045725982658962868047234	23015921117139340153267671
64	23030818728479610745741910	23023367512879647736902891
128	23027092819292183214705676	23025230015404383009313933
256	23026161398510805910921810	23025695697539046352276636
512	23025928546847571901068394	23025812121604634087915779
1024	23025870334152518169052273	23025841227841783762272302
2048	23025855780992551911165543	23025848504414868310197241
4096	23025852142703422669729927	23025850323559001769499206
8192	23025851233131194254554390	23025850778345089029496888
16384	23025851005738140519209367	23025850892041614212944994
32768	23025850948889877295901163	23025850920465745719335070
65536	23025850934677811503232115	23025850927571778609090592
131072	23025850931124795055887228	23025850929348286832351848
262144	23025850930236540944102405	23025850929792413888218560
524288	23025850930014477416159412	23025850929903445652188450
1048576	23025850929958961534173864	23025850929931203593181124

The hyperbolic sector consists of the following terms

23025850929940456240178681 23025850929940456240178704

It is therefore possible without danger of error to assume the following number for the sector of the hyperbola, of which the multiples of the number up to ten, facilitating division thanks to the composition of the logarithm, I reveal this. For in fact in long division it is better to use repeated subtraction for repetition of division than ordinary division, as agrees the expert of arithmetic.

It is clear that this problem can be resolved by the same method even if the asymptotes AO and AK are not at a right angle. However we assumed so that the problem would be made easier and more readily used in the doctrine of logarithms, which was first discovered by our most noble Napier, and which we have now elevated (unless I am mistaken) to the highest peak of perfection.

1	2302585092994045624017870 0
2	4605170185988091248035740 0
3	6907755278982136872053610 0
4	9210340371976182496071480 0
5	11512925464970228120089350 0
6	13815510557964273744107220 0
7	16118095650958319368125090 0
8	18420680743952364992142960 0
9	20723265836945410616160830 0
10	23025850929940456240178700 0

Proposition XXXIII. Problem.

To find the logarithm of any given number.

By the same assumptions as in the preceding proposition, it is clear that, taking IK to be unity, ML is ten. Therefore, IK being unity, let GH be any parallel to the asymptote AO, the logarithm is desiered of this proposed number. It is clear from the given line GH to get KF, and from the preceding to get likewise the hyperbolic area $GIKH$, which hyperbolic area I claim is the logarithm of the proposed number GH. I take by the area $LIKM$ the logarithm of the number ten. Indeed (from Gregorie of St. Vincent) the area $GHKI$ is in the same ratio to the area $LMKI$, in which ratio GH to IK is a multiple of the ratio LM to IK. However the ratio GH to IK is a multiple of the ratio LM to IK in the same ratio as the number GH is a multiple of the number LM, since it is contained itself in both ratios. Thus the area $GIKH$ is in the same ratio to the area $LIKM$, in which the number GH is a multiple of the number LM, and so (since by hypothesis the area $LIKM$ is the logarithm of the number LM, or ten) the area $GIKH$ shall be the logarithm of the proposed number GH, since this is the essential property of logarithms, as they be among themselves in the same direct ratio, in which they are one another multiplied by the same number. And the logarithm of ten is generally given as a one with some arbitrary number of zeros. If this be done, the area $LIKM$ is to the area $GIKH$ as the arbitrary logarith of ten is to the other number. That number shall be found to be the logarithm of the proposed number GH, which we wanted to find.

Scholium.

The exercise of the preceding problem set is long and laborious. Thus in order to abbreviate our labor when composing tables of logarithms, it has been understood that we merely need to work on the discovery of logarithms of prime numbers. Indeed the logarithms of composite numbers are found without effort from the primes by addition and subtraction. However as the logarithms of prime numbers may be easily found, the order progressing from the priors to the latters, so that from the arbitrary logarithm of 10 to each prime number 2, and from 10 and 2 to 3, likewise from 10, 2, and 3 to 7, likewise from 10, 2, 3, and 7 to 11, and thus hereafter. Next two composite numbers differing by very little are found, of which one is composed from a number having a known logarithm, and so having the given logarithm, the other number is composed from only a prime number (of which the logarithm is found) or from that plus another number having known logarithm. Now these composite numbers are drawn (which may be, e.g., GH and EF) to the hyperbola as parallels to the asymptote OA, and the hyperbolic area $EGHF$ is found according to Proposition 32, which is done quickly from GH and EF, which differ by very little. By assumption, the logarithm of one of the numbers, e.g. GH, is given, and so the ratio of its logarithm to the arbitrary logarithm of ten is given, which is the same (from the proofs to this point) as the ratio of the hyperbolic area $GIKH$ to the hyperbolic area $LIKM$. However the area LIKM is given by Proposition 32, and so the area $IKHG$ is known, and with the given area $EGHF$ $EIKF$ is given. Thus the logarithm of the composite number EF is given. And when by assumption the logarithms of every number composing the number EF may be given, except that prime number of which the logarithm is desired, that logarithm of the prime number will be given, which we wanted to find. For example, Let it be proposed to find the logarithm of the number two, supposing arbitrarily the logarithm of the number ten, but given as one with 25 zeros, the two composite numbers, differing by very little, are 1000 and 1024. The logarithm of the number 1000, or the triple of the area found above as 23025850929940 456240178700, namely that area given by the arbitrary logarithm of the number ten.

	Circumscribed Area	Inscribed Area
2	23717082451262844989917	237162487062045867846886
4	237166655750699903737556	237164571388o54419219371
8	237165613567087322970403	237165092476425954356426
16	237165353021613523599438	237165222748948181485250
32	237165287885271907848389	237165255317105572320456
64	237165271601188181041012	237165263459146597159038

The hyperbolic sector consists of the following terms

237165266173160272103220 237165266173160458453029

Let be between these terms the four greatest of the continuously arithmetic proportionals 237165266173160 421183067, which hence shall be the true sector of the hyperbola in the proposed number of the noted, since the first third of the noted is the same in both of the convergent terms.

The logarithm of the number 1024 is unknown, indeed is composed from only the prime number 2, namely it is multiplied by ten. These composite numbers are drawn to the hyperbola, as has been said, letting GH be 1000 and EF be 1024. But since IK is 1000000000000, GH shall be 1000000 000000000 and EF 1024000000000000, and by Proposition 32 the area $EGHF$ is found to be 237165266173160421183067 (I give this convergent series for the profit of the reader), or the logarithm of the number $1\frac{24}{1000}$ by the proposed arbitrary logarithm of ten 230258509299404562401787 00. Next, by the same assumed arbitrary logarithm of ten, the logarithm of the number 1000 is added, or the triple of the logarithm of ten, to the logarithm of the number $1\frac{24}{1000}$, and will give the logarithm of the sum of the number 1024, of which a tenth part will be the logarithm of the number two through the same arbitrary logarithm of ten, or 693147180559945291 4171917. So it will be that the logarithm of the number ten 23025850929 940456240178700 is to the logarithm of the number two corresponding to 6931471805599452914171917 as the proposed arbitrary logarithm of the number ten, or 1000000000000000000000000 is to the logarithm of the sought number two 3010299956639811952405804, which we wanted to find[5]. By the same method the logarithm of three is found to be 47712125 47196624373502993, etc.

In order to show these composite numbers, differing very little among themselves, through one of the prime numbers, I present this table for each prime number up to 100, as well as one rule for prime numbers between 100 and 1000 and another for prime numbers above 1000, which have all been contrived so that the true logarithm of any prime number can be found by the corresponding arbitrary logarithm of ten 10000000000000 000000000000 by only one multiplication, two divisions, and one square

[5]That is, $\log_1 0(2) = \frac{\log(2)}{\log(10)}$

root extraction, as well as some little effort.

2	$1000 = 10^3$ $1024 = 2^{10}$
3	$32805 = 5 \cdot 6561 = 5 \cdot 3^8$ $32768 = 2^{15}$
7	$2400 = 3 \cdot 32 = 3 \cdot 2^5$ $2401 = 7^4$
11	$9800 = 2 \cdot 49 \cdot 100 = 2 \cdot 7^2 \cdot 10^2$ $9801 = 121 \cdot 81 = 11^2 \cdot 3^4$
13	$123200 = 7 \cdot 11 \cdot 25 \cdot 64 = 7 \cdot 11 \cdot 5^2 \cdot 2^6$ $123201 = 169 \cdot 729 = 13^2 \cdot 3^6$
17	$2600 = 13 \cdot 8 \cdot 25 = 13 \cdot 2^3 \cdot 5^2$ $2601 = 9 \cdot 289 = 3^2 \cdot 17^2$
19	$28899 = 169 \cdot 9 \cdot 19 = 13^2 \cdot 3^2 \cdot 19$ $28900 = 100 \cdot 289 = 10^2 \cdot 17^2$
23	$25920 = 10 \cdot 32 \cdot 81 = 10 \cdot 2^5 \cdot 3^2$ $25921 = 49 \cdot 529 = 7^2 \cdot 23^2$
29	$613088 = 17 \cdot 23 \cdot 32 \cdot 49 = 17 \cdot 23 \cdot 2^5 \cdot 7^2$ $613089 = 729 \cdot 841 = 3^6 \cdot 29^2$
31	$116280 = 10 \cdot 17 \cdot 19 \cdot 4 \cdot 9 = 10 \cdot 17 \cdot 19 \cdot 2^2 \cdot 3^2$ $116281 = 121 \cdot 961 = 11^2 \cdot 31^2$
37	$165648 = 3 \cdot 7 \cdot 17 \cdot 29 \cdot 16 = 3 \cdot 7 \cdot 17 \cdot 29 \cdot 2^4$

$165649 = 121 \cdot 1369 = 11^2 \cdot 37^2$

41
$1413720 = 7 \cdot 10 \cdot 11 \cdot 17 \cdot 4 \cdot 27 = 7 \cdot 10 \cdot 11 \cdot 17 \cdot 2^2 \cdot 3^3$
$1413721 = 1681 \cdot 841 = 41^2 \cdot 29^2$

43
$978120 = 10 \cdot 11 \cdot 13 \cdot 19 \cdot 4 \cdot 9 = 10 \cdot 11 \cdot 13 \cdot 19 \cdot 2^2 \cdot 3^2$
$978121 = 529 \cdot 1849 = 23^2 \cdot 43^2$

53
$664848 = 7 \cdot 19 \cdot 23 \cdot 8 \cdot 125 = 7 \cdot 19 \cdot 23 \cdot 2^3 \cdot 5^3$
$664849 = 9 \cdot 121 \cdot 2809 = 3^2 \cdot 11^2 \cdot 53^2$

57
$5851560 = 3 \cdot 5 \cdot 13 \cdot 31 \cdot 121 = 3 \cdot 5 \cdot 13 \cdot 31 \cdot 11^2$
$5851561 = 1681 \cdot 3481 = 41^2 \cdot 57^2$

61
$3575880 = 5 \cdot 7 \cdot 11 \cdot 43 \cdot 8 \cdot 27 \cdot = 5 \cdot 7 \cdot 11 \cdot 43 \cdot 2^3 \cdot 3^3$
$3575881 = 961 \cdot 3721 = 31^2 \cdot 61^2$

67
$1620528 = 3 \cdot 13 \cdot 16 \cdot 49 = 3 \cdot 13 \cdot 2^4 \cdot 7^2$
$1620529 = 361 \cdot 4489 = 19^2 \cdot 67^2$

71
$2016399 = 3 \cdot 11 \cdot 29 \cdot 43 \cdot 49 = 3 \cdot 11 \cdot 29 \cdot 43 \cdot 7^2$
$2016400 = 16 \cdot 25 \cdot 5041 = 2^4 \cdot 5^2 \cdot 71^2$

73
$5116644 = 4 \cdot 9 \cdot 169 \cdot 841 = 2^2 \cdot 3^2 \cdot 13^2 \cdot 29^2$
$5116645 = 7 \cdot 17 \cdot 19 \cdot 31 \cdot 73$

79
$5997600 = 17 \cdot 32 \cdot 9 \cdot 25 \cdot 49 = 17 \cdot 2^5 \cdot 3^2 \cdot 5^2 \cdot 7^2$
$5997601 = 961 \cdot 6241 = 31^2 \cdot 79^2$

83
$1164240 = 5 \cdot 11 \cdot 16 \cdot 27 \cdot 49 = 5 \cdot 11 \cdot 2^4 \cdot 3^3 \cdot 7^2$
$1164241 = 169 \cdot 6889 = 13^2 \cdot 83^2$

89
$2859480 = 5 \cdot 47 \cdot 8 \cdot 9 \cdot 169 \cdot = 5 \cdot 47 \cdot 2^3 \cdot 3^2 \cdot 13^2$

$$2859481 = 361 \cdot 7921 = 19^2 \cdot 89^2$$

97
$$1138488 = 3 \cdot 13 \cdot 41 \cdot 89 \cdot 8 = 3 \cdot 13 \cdot 41 \cdot 89 \cdot 2^3$$
$$1138489 = 121 \cdot 9409 = 11^2 \cdot 97^2$$

For prime numbers between 100 and 1000 let this be the rule: before the prime number of which the logarithm is desired, the two numbers immediately preceding are assumed, and the number following immediately after it, which three numbers with that prime are four numbers following one after another in natural order among themselves. Next the first number is multiplied by the cube of the third and the fourth by the cube of the second, and it will be that their difference equals the sum of the prime and the fourth or of the second and the third, as can easily be shown. These numbers have at least six prime factors between them, and thus they differ very little among themselves. Also the logarithms of the four of these numbers (except the third) are known from the preceding method, and thus are suitable to our abbreviation. So much apparatus is not useful in numbers beyond 1000, since the rectangle of the numbers, among which the prime number is understood immediately of which the logarithm is desired, which is only less the square of the prime number by one. And so these have at least six prime factors among them, and the logarithms of the first and third are obtained. Therefore the infinitude are available to us.

Proposition XXXIV. Problem.

From a given logarithm to find its number.

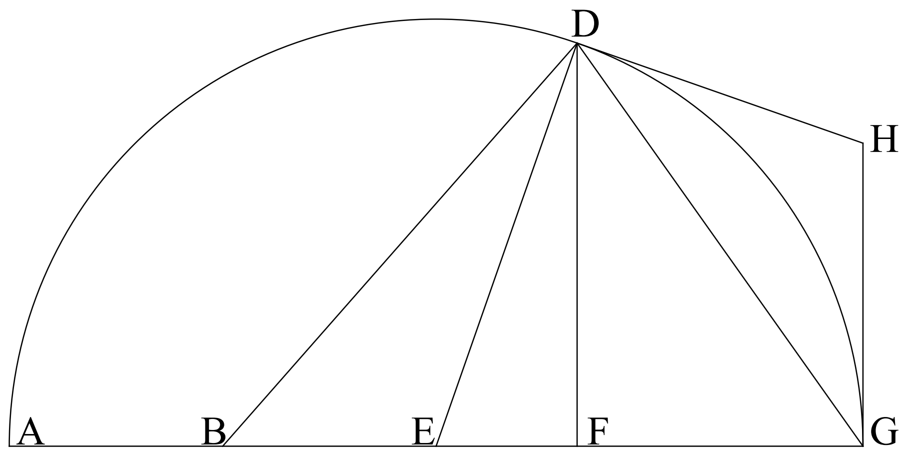
From the demonstration it is clear that this same problem be as if that was proposing. From the given hyperbolic area, and one line understood to be parallel to one of the asymptotes, to find another area and its parallel to the asymptote. It may be considered from however many arithmetic terms the arbitrary logarithm of ten is comprised, and some part of the logarithm, or the given area, is assumed, namely the area $LIKM$, so that the regular circumscribed polygons of the area $LIKM$ and the regular inscribed hexagons of the same be repeated however many times, as often as is repeated the given area to the area $LIKM$, agreeing in all arithmetic terms, as many the square root of the arbitrary logarithm of ten contains. Indeed this can be done easily from the table of Proposition 32. Therefore the measure of the area $LIKM$ is obtained and the line IK is unity by assumption. Let LM be z. As in Proposition 32 the regular circumscribed pentagon and the regular inscribed hexagon give the area $LIKM$, between which the given area $LIKM$ is the second of the two continuously proportional arithmetic means. And so the double of the hexagon plus the pentagon is equal to the triple of the area, of which equation the unknown z, or the number LM, clearly resolves, of which repeated however many times, as many as the area $LIKM$ is submultiplied to the space, or the given logarithm, is the desired number, which we wanted to find.

This is the same problem as Proposition 8, but more general and the method of this solution is far less work.

Proposition XXXV. Problem.

A line having been drawn through a given point on a diamter, to divide the semicircle into a given ratio.

Let ADG be a semicircle of given diameter AG, center E, and B be a given point on the diameter. Assume it is done as desired, and let the line BD divide the semicircle into a given ratio. Since the measure of the



semicircle is given and the ratio into which it is divided, so its portion, DBG, is given. Let z be the line BD. From the given lines BD, BE, and ED, the triangles DEB, DEF, and DEG are made known. Next let it be that as DEF plus DEG is to DEG so the double of DEG is to the circumscribed quadrilateral $DEGH$. Setting DEG and $DEGH$ as the first convergent term, the convergent series of complex ploygons may be continued, repeated as often as necessary according to the properties of the circle. Until the agreed upon approximation to employ such that the sector DEG is shown, which plus the triangle DBE is euqal to the known portion DBG, the equation of which clearly resolved the unknown magnitude z, or the line BD. The rest is obvious.

The same problem is resolved by altogether the same method in the ellipse, the hyperbola, or any sector given.

Scholium.

If a problem set of the previous Propositions is desired for mechanical pratice, it will not be difficult to imitate the calculation, approximation, and resolution of equations to some extent according to the common rules of the practice of Geometry. Many such problem sets may be resolved by the power of analysis and by our rules of convergent series, which before may have been impossible to estimate. However, it will be strongly said by anyone that these solutions are not geometric. I respond that if the only practice understood by the geometer is the power of the straightedge and compass, not only will this be impossible but likewise will every problem set which cannot be reduced to a quadratic equation, as may easily be shown. And if the reduction of the problem to an analytic equation be understood by the geometer, all of this problem set are impossible to the geometer, where by this proof it is clear that such a reduction is cannot be done. If in truth this most simple method of every possibility be understood by the geometer, it will be found most strongly after timely consideration that the entirety of the above problem set may be resolved most geometrically. Carefully observing the whole doctrine of convergent series it is possible likewise by little effort to apply it to simple series. Indeed let A, B, C, D, E, etc. be a series of such nature that the third term C is composed by the same method from the first and second terms A and B as the fourth term D is composed from the second and third terms B and C, and the fifth E from the third and fourth C and D, and so on infinitely. Let also the difference of the aforementioned A and$B$ be always greater than the difference of the subsequent terms B and C. We may assume this series to continue infinitely until two of the adjacent terms are not different, and letting one of these terms be z, which we call the limit of the series. I claim that z is composed by the same method from A and B as from B and C or C and D. The proof scarcely differes from that of Proposition 10 and its Conclusions. If this ratio is put to a triangle, inscribed in a sector of a circle or ellipse or circumscribed to a sector of a hyperbola, a, and a quadrilateral, regularly inscribed in a circle or ellipse or circumscribed to a hyperbola,

b, then the hexagon regularly inscribed in a sector of a circle or ellipse or circumscribed to a hyperbola will be $\sqrt{\frac{2b^3}{a+b}}$. Thus the sector of a circle, ellipse, or hyperbola is composed of the same method from a and b as from b and $\sqrt{\frac{2b^3}{a+b}}$. And so this likewise can be shown, that the ratio of the sector to its given triangle may not be analytic, according to Proposition 11. It would actually be possible to yet prove by another particular method that the circular arc does not have an analytic ratio to its given chord, but I do not add more, meanwhile advising geometers to growth by science. I myself have discovered in certain figures (which Descartes called the second type) three foci, or three points, from which lines drawn to any point of the curve the sum or difference is always the same. Whence it appears to me to be true as all curves of the first type have two foci whether real or imaginary, as all of the second type have three, all third four, and so on infinitely. This speculation is certainly most worthy of scrutiny, and indeed it may be an extraordinary property of the geometric figures, and of the most useful mechanical practice of all equations.

FINIS.

Anno Dom: 1667.

Appendix.

A modern rendering of the propositions.

Proposition I:

Let the area of $BAPF = a$, the area of $BAP = b$, and the area of $BAPI = c$. Then $c = \frac{a+b}{2} = \mathrm{AM}(a,b)$.

Proposition II:

Let the area of $AFBP = a$, the area of $ABIP = b$, and the area of $ABDLP = c$. Then $c = \frac{2ab}{a+b} = \frac{\mathrm{GM}(a,b)}{\mathrm{AM}(a,b)}$.

Proposition III:

Let the area of $ABIP = a$, the area of $BAP = b$, and the area of $ABDLP = c$. Then $c = \frac{2a^2}{a+b} = \frac{a^2}{\mathrm{AM}(a,b)}$.

Proposition IV:

Let the area of $ABIP = a$, the area of $ABDLP = b$, and the area of $ABEIOP = c$. Then $c = \frac{a+b}{2} = \mathrm{AM}(a,b)$.

Proposition V:

Let the area of $ABEIOP = a$, the area of $ABIP = b$, and the area of polygon $ABCGKNP = c$. Then $c = \frac{2a^2}{a+b} = \frac{a^2}{\mathrm{AM}(a,b)}$.

Proposition VI:

Let the area $ABP = a$, the area of $ABFP = b$, the area of $ABIP = c$, and the area of $ABDLP = d$. Then $a - b \leq 2(c - d) = \frac{(a+b)^2}{a+3b} = \frac{2\mathrm{AM}^2(a,b)}{\mathrm{AM}(a,b)+b}$.

Proposition VII:

Let $a_0 = a$ and $b_0 = b$ where $a < b$ and

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[1] Charles Hutton. *A Philosophical and Mathematical Dictionary*, volume II. Printed for the Author, 2nd edition, 1815.