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Mathematics in Computer Science

On the Location of Pseudozeros of a Complex Interval Polynomial

Hiroshi Sekigawa and Kiyoshi Shirayanagi

Abstract. Given a univariate complex interval polynomial F, we provide a rigorous method for deciding whether there exists a pseudozero of F in a prescribed closed complex domain D. Here a pseudozero of F is defined to be a zero of some polynomial in F. We use circular intervals and assume that the boundary C of D is a simple curve and that C is the union of a finite number of arcs, each of which is represented by a rational function. When D is not bounded, we assume further that all the polynomials in F are of the same degree. Examples of such domains are the outside of an open disk and a half-plane with boundary. Our decision method uses the representation of C and the property that a polynomial in F is of degree 1 with respect to each coefficient regarded as a variable.

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1. Introduction

There are two premises for using numeric or approximate computation in symbolic computation. One is that the exact values are known but approximate computation is used for efficiency. An example is the theory of stabilizing algebraic algorithms [17–19]. The other premise is that inexact values are given. In this paper, we treat problems on the latter premise. Specifically, we treat problems regarding zeros of complex polynomials with perturbations. Let F be a set of polynomials that represents a polynomial and its perturbations. We consider the following types of problems.

- Does there exist a polynomial $f \in F$ that has a zero in the prescribed complex domain?
- What is the union of the sets of zeros of polynomials $f \in F$?

We have already studied these problems when F is a set of real polynomials [13,16]. In the present paper we treat a set of complex polynomials whose coefficients belong to prescribed circular intervals. That is, we treat a set of complex polynomials described in a weighted l^{∞}-norm. This work is an extension of that reported earlier [14, 15].

The above problems are of practical importance because polynomial coefficients obtained through measurements or observations may contain errors. As described in [3], these types of problems have already been treated in control theory and some results have been obtained (see, for example, Kharitonov's theorem [6] or the edge theorem [1]). In research on symbolic-numeric algorithms, too, there are a lot of topics in this category, such as approximate GCDs [11] and approximate factorizations [12]. Here we only cite what seems to be the earliest work on each topic.

Two important differences between related works in control theory and the method we propose for the first of the aforementioned problems are that Kharitonov's theorem [6] and its extension to complex polynomials treat only the left half-plane as a domain and that the edge theorem [1] considers only real polynomials and in its original form is thus not directly applicable to complex polynomials described in a weighted l^{∞}-norm. Problems similar to the aforementioned two are considered in [4] using the technique of parametric minimization, but there perturbations are described in a weighted l^{2}-norm and, as Hitz and Kaltofen write in Section 7 of [4], "For the general case, finding the *parametric* minimum, in the l^{∞}-norm in particular, seems to be an open problem at current time." Numerical computation with guaranteed accuracy, as in the method proposed in [9], which uses linear programming and Krawczyk's method [7], can be used for the first problem, but this approach may require division of the original domain. It is guaranteed that the dividing procedure stops in a finite number of steps, but the procedure may cause very fine divisions for ill-conditioned polynomials.

With regard to the second problem, properties of the union of the sets, such as the boundedness and the number of roots in each connected component, are investigated in [8]. A pseudovariety, a generalization of the concept of a pseudozero (see Definition 2.2 below) is proposed in [2], which describes a method to visualize it that uses some numerical parameterization (i.e., solving a differential equation numerically). On the other hand, we can obtain a rough shape of the union of the sets by using the algorithm we propose for the first problem.

The rest of this article is organized as follows. Section 2 introduces interval polynomials and pseudozeros and explains the problems. Section 3 describes the theorems that support the proposed algorithms, whose details are explained in Section 4. Section 5 shows examples, and Section 6 concludes the paper by mentioning directions in which future work might proceed.

2. Definitions, notations and problems

In this section, after introducing an interval polynomial to describe a set of polynomials with perturbations and a pseudozero to describe a zero of an interval polynomial, we explain the problems treated in this article.

Definition 2.1 (Univariate complex interval polynomials). Let $\langle c, r\rangle$ denote the circular interval $\{\, z \in \mathbb{C} \mid |z - c| \leq r \,\}$, where c is a complex number and r is a nonnegative real number.

For $j = 1, \ldots, n$, let $e_j(x)$ be a nonzero polynomial in $\mathbb{C}[x]$ and $A_j = \langle c_j, r_j\rangle$ be a circular interval. A univariate complex interval polynomial is defined to be the set of polynomials

$$\left\{ \sum_{j=1}^{n} a_j e_j(x) \;\middle|\; a_j \in A_j \right\} . \tag{2.1}$$

A_j is said to be an interval coefficient.

For simplicity, the set described by (2.1) may be denoted as follows.

$$A_1 e_1(x) + A_2 e_2(x) + \cdots + A_n e_n(x) .$$

In this paper we refer to a univariate complex interval polynomial as a complex interval polynomial or simply an interval polynomial. Note that from the definition an interval polynomial F is a convex set in $\mathbb{C}[x]$.

Definition 2.2 (Pseudozeros). Let F be an interval polynomial. We define a point $\alpha \in \mathbb{C}$ as a pseudozero of F if and only if there exists $f \in F$ such that $f(\alpha) = 0$. We write all pseudozeros of F as $Z(F)$.

Let F be an interval polynomial as described by (2.1) and D be a domain in $\mathbb{C}$. We consider the following problems.

Problem 1. Does there exist a pseudozero of F in D?

Problem 2. Compute $Z(F)$.

It is difficult to determine $Z(F)$ exactly, but we can obtain a rough shape by using an algorithm for Problem 1 as follows. If $Z(F)$ is bounded and a rectangle containing $Z(F)$ is given, we divide it into four congruent rectangles by dividing each side into two equal parts and examine whether each of them intersects $Z(F)$. Similar computations are performed recursively for the rectangles that intersect $Z(F)$. If all polynomials in F have the same degree, we can compute an initial rectangle using, for example, the Cauchy bound for an algebraic equation.

In this paper we assume that D is a closed domain in $\mathbb{C}$ whose boundary $C \subset D$ is a simple curve. When D is not bounded, we further assume that the degree of $f \in F$ is constant. When C is not a closed curve, the domain D is not bounded. Therefore, from the above assumption on the degree, we can construct a new closed domain $D' \subset D$ such that the following conditions are satisfied.

- The boundary of D' is a simple and closed curve.
- $Z(F) \cap D = Z(F) \cap D'$.

We can therefore assume that C is a simple and closed curve. We also assume the following condition.

Condition 1. C is of finite length and $C = \cup_{m=1}^{M} C_m$ ($M < \infty$), where each C_m is expressed by an injective function as

$$\varphi_m(s)\,, \qquad s \in S_m \subset \mathbb{R}\,.$$

Here $\varphi_m(s) \in \mathbb{C}(s)$ and S_m is either of type $[a, b]$, $[a, \infty)$, $(-\infty, b]$ or $\mathbb{R}$.

For simplicity when computing, we restrict the real and the imaginary parts of complex numbers to rational numbers and use exact computation unless mentioned otherwise. For example, in Definition 2.1 the coefficients of polynomials $e_j(x)$ and the centers c_j belong to $\mathbb{Q}(\sqrt{-1})$ and the radii r_j belong to $\mathbb{Q}$. Both theoretically and practically, we can extend rational numbers to real algebraic numbers.

3. Theorems

The purpose of this section is to prove the following theorem. If we obtain computation methods for the preconditions of the theorem and for the second condition of the statement, we can establish an algorithm for Problem 1. We describe the computation methods in the next section.

Theorem 3.1. *Let $F = \langle c_1, r_1\rangle e_1(x) + \cdots + \langle c_n, r_n\rangle e_n(x)$ be a complex interval polynomial and D be a complex domain with boundary C that satisfies Condition 1. When D is unbounded we assume that the degrees of all polynomials in F are equal. Suppose that there exists a polynomial $f_0 \in F$ such that f_0 has no zero in D and that there exists a point $\alpha_0 \in C$ such that α_0 is not a pseudozero of F. Then the following two conditions are equivalent.*

- *F has a pseudozero in D.*
- *There exists a point $\alpha \in C$ such that the following equality holds.*

$$\left|\sum_{j=1}^{n} c_j e_j(\alpha)\right| = \sum_{j=1}^{n} r_j |e_j(\alpha)|\,.$$

To prove Theorem 3.1, we need some lemmas, propositions and theorems.

If a complex interval polynomial contains a polynomial that has no zero in D, then Problem 1 is equivalent to asking whether there is a pseudozero of F on C.

Proposition 3.2. *Suppose that a polynomial $f_0 \in F$ has no zero in D. When D is unbounded we assume that the degrees of all polynomials in F are equal. Then the following two statements are equivalent.*

1. *F has a pseudozero in D.*
2. *F has a pseudozero on C.*

Proof. It is sufficient to prove that the first statement implies the second statement. Assume that $f \in F$ has a zero in D but no zero on C. Let g_t be $(1-t)f_0 + tf$. Then $g_0 = f_0$, $g_1 = f$ and $g_t \in F$ for any t $(0 \le t \le 1)$. We prove the statement by contradiction. Suppose that any g_t $(0 \le t \le 1)$ has no zero on C.

When D is bounded, Rouché's theorem (see below) implies that the number of zeros of g_0 in D is equal to that of g_1. This contradicts the assumption.

When D is unbounded, the assumption that C is of finite length implies that the compliment D^c of D is bounded. Therefore, the number of zeros of g_0 in D^c is equal to that of g_1. Since $\deg g_0 = \deg g_1$, the number of zeros of g_0 in D is equal to that of g_1. This contradicts the assumption. □

The following is a version of Rouché's theorem.

Theorem 3.3 (Rouché's theorem). *Let C be a simple closed curve of finite length in a domain $\Omega \subset \mathbb{C}$ and let the inside of C be in Ω. Suppose that $f(z)$ and $g(z)$ are holomorphic on Ω and that $f(z) + tg(z)$ has no zero on C for any t $(0 \le t \le 1)$. Then the number of zeros of $f(z)$ inside C is equal to that of $f(z) + g(z)$, where each zero is counted as many times as its multiplicity.*

Because this version is not described in standard textbooks, we provide a proof in the appendix.

To examine whether a given point is a pseudozero, we use the following theorem as described in [5, 8, 20] and [10].

Theorem 3.4. *Let F be an interval polynomial $\langle c_1, r_1\rangle e_1(x) + \cdots + \langle c_n, r_n\rangle e_n(x)$. Then α is a pseudozero of F if and only if the following inequality holds.*

$$\left|\sum_{j=1}^{n} c_j e_j(\alpha)\right| \le \sum_{j=1}^{n} r_j |e_j(\alpha)|\,.$$

Using Theorem 3.4, we can prove Theorem 3.1 as follows.

Proof. Suppose that there exists a point $\alpha \in C$ such that

$$\left|\sum_{j=1}^{n} c_j e_j(\alpha)\right| = \sum_{j=1}^{n} r_j |e_j(\alpha)|$$

holds. Then from Theorem 3.4, $\alpha \in C \subset D$ is a pseudozero of F. That is, the second statement of Theorem 3.1 implies the first one.

Now we prove the opposite direction. Since there exists a point $\alpha_0 \in C$ such that α_0 is not a pseudozero of F, the following inequality holds from Theorem 3.4.

$$\left|\sum_{j=1}^{n} c_j e_j(\alpha_0)\right| > \sum_{j=1}^{n} r_j |e_j(\alpha_0)|\,.$$

Because $f_0 \in F$ has no zero in D, we can use Proposition 3.2. Since F has a pseudozero in D, there exists $\alpha \in C$ such that

$$\left| \sum_{j=1}^{n} c_j e_j(\alpha) \right| \leq \sum_{j=1}^{n} r_j |e_j(\alpha)| . \tag{3.1}$$

If the equality in (3.1) holds, then the proof is finished. Otherwise, let $g(x)$ be

$$\left| \sum_{j=1}^{n} c_j e_j(x) \right| - \sum_{j=1}^{n} r_j |e_j(x)| .$$

Then $g(x)$ is a continuous function from $\mathbb{C}$ to $\mathbb{R}$. Therefore $g(C)$, the image of the boundary C, is a connected subset of $\mathbb{R}$ and contains a positive value $g(\alpha_0)$ and a negative value $g(\alpha)$. Therefore there exists a point $\beta \in C$ such that $g(\beta) = 0$. That is,

$$\left| \sum_{j=1}^{n} c_j e_j(\beta) \right| = \sum_{j=1}^{n} r_j |e_j(\beta)|$$

holds. □

4. Algorithms

In this section we first present an outline of the algorithm for Problem 1 and then describe it in detail.

Algorithm 1.
Input: an interval polynomial F and a domain D with boundary C.
Output: the answer to the question "Does F have a pseudozero in D?"

1. Take a polynomial $f \in F$. If f has a zero in D, then return "yes."
2. Take a point $\alpha \in C$. If α is a pseudozero of F, then return "yes."
3. Examine whether the equality of the second condition of Theorem 3.1 holds. If it does, then return "yes."
   Otherwise, return "no."

The computational methods for Step 1 are as follows. We can determine whether $f \in F$ has a zero on C by using a real root counting algorithm such as the Sturm algorithm, the sign variation method, or some other improved algorithm. The detailed procedure is as follows. For a polynomial $g(x) \in \mathbb{C}[x]$, we denote by $\overline{g}(x)$ the polynomial whose coefficients are the complex conjugates of the coefficients of $g(x)$ and for a rational function $h(x) = h_1(x)/h_2(x)$ ($h_1(x)$, $h_2(x) \in \mathbb{C}[x]$), we denote by $\overline{h}(x)$ the rational function $\overline{h_1}(x)/\overline{h_2}(x)$. We apply a real root counting algorithm to the numerator of $f(\varphi_m(s))\overline{f}(\overline{\varphi_m}(s)) \in \mathbb{Q}(s)$, which is a real polynomial. The equation $f(\varphi_m(\sigma)) = 0$ holds if and only if the equation $f(\varphi_m(\sigma))\overline{f}(\overline{\varphi_m}(\overline{\sigma})) = 0$ holds. Note that the multiplicity of each real zero is doubled by this transformation. If f does not have a zero on C, then we can examine whether f has a zero in the interior of D by using the argument principle. Let d be the number of zeros of f in D with multiplicity counted and w be the number of times $f(C)$ winds around the origin. Then d is equal to w or $\deg f - w$ depending on whether D is bounded or unbounded. The computation method for w is as follows.

Subalgorithm 1.
Input: a complex polynomial f and a simple closed curve $C \subset \mathbb{C}$.
Output: the number of times $f(C)$ winds around the origin.

1. Let L be an empty list.
2. Take a point γ_0 on C. Move a point γ counterclockwise on C from γ_0 until it reaches γ_0 again.
   When $f(\gamma)$ is on the real axis,
   - if the last element of L is "r", then delete it;
   - otherwise add "r" to L.

   When $f(\gamma)$ is on the imaginary axis,
   - if the last element of L is "i", then delete it;
   - otherwise add "i" to L.
3. Return the number (the length of the list L)/4.

In Step 2 of Subalgorithm 1, the symbols "r" and "i" stand for "crossing the real axis" and "crossing the imaginary axis", respectively.

For a simple closed curve C satisfying Condition 1, we can implement Step 2 of Subalgorithm 1 as follows. Let $g(s)$ be the numerator of $f(\varphi_m(s)) + \overline{f}(\overline{\varphi_m}(s))$ and $h(s)$ be the numerator of $(f(\varphi_m(s)) - \overline{f}(\overline{\varphi_m}(s)))/i$. The point $f(\varphi_m(\sigma))$ is on the real axis if and only if $h(\sigma) = 0$ and the point $f(\varphi_m(\tau))$ is on the imaginary axis if and only if $g(\tau) = 0$. Note that g and h are polynomials with rational coefficients.

The following two subsections explain Steps 2 and 3 of Algorithm 1 in two cases: the general case where $e_j(x)$ is any nonzero polynomial in $\mathbb{C}[x]$, and the special case where it is x^{j-1}.

4.1. General case

A computation method for Step 2 is as follows. From the assumption $\varphi_m \in \mathbb{Q}(\sqrt{-1})(s)$, we can take a point $\alpha \in C$ whose real part and imaginary part are rational numbers. For α the inequality of Theorem 3.4 is of the form

$$\sqrt{q} \leq \sum_{j=1}^{n} \sqrt{q_j}\,, \tag{4.1}$$

where q and q_j are nonnegative rational numbers. Therefore to decide whether the inequality (4.1) holds, we first examine whether the equality holds by using exact computation. If it does not hold, we can decide whether the strict inequality holds by using approximate computation with error analysis – for example, interval computation – under the assumption that we can raise the precision as high as desired.

Computational methods for Step 3 are a little more complicated. Let a fixed but arbitrary $\alpha \in \mathbb{C}$ be given. Let K be $\mathbb{Q}(\sqrt{-1}, \alpha, \bar{\alpha}) \cap \mathbb{R}$, an extension field of $\mathbb{Q}$. To obtain a polynomial that has the square of the right-hand side of the equality in Theorem 3.1 as a zero, we construct polynomials $P_0, \ldots, P_{n+1}$ as follows.

$$\begin{aligned} P_0(x) &= x\,, \\ P_j(x) &= P_{j-1}\big(x - r_j|e_j(\alpha)|\big)P_{j-1}\big(x + r_j|e_j(\alpha)|\big) \qquad (j = 1, \ldots, n)\,, \\ P_{n+1}(x^2) &= P_n(x)\,. \end{aligned}$$

As described in the following proposition, P_j is a polynomial in x^2 for $j = 1, \ldots, n$.

Proposition 4.1. *Let K and P_j be as above.*

1. *$P_j \in K[x]$ and P_j is a polynomial in x^2 for $j = 1, \ldots, n$.*
2. *The degree of P_n is 2^n and the zeros of P_n are*
$$\pm r_1|e_1(\alpha)| \pm \cdots \pm r_n|e_n(\alpha)|\,,$$
where we take all combinations of $+$ and $-$.
3. *The degree of P_{n+1} is 2^{n-1} and the zeros of P_{n+1} are*
$$\big(r_1|e_1(\alpha)| \pm \cdots \pm r_n|e_n(\alpha)|\big)^2\,,$$
where we take all combinations of $+$ and $-$.

Proof. Let $f(x)$ belong to $K[x]$. Then for $a \in K$, the polynomial
$$f(x - \sqrt{a})f(x + \sqrt{a}) \tag{4.2}$$
belongs to $K[x]$. This is clear when $\sqrt{a} \in K$. If $\sqrt{a} \notin K$ the conjugates of $\sqrt{a}$ over K are $\pm\sqrt{a}$ and the polynomial (4.2) is unchanged under the substitution $-\sqrt{a}$ for $\sqrt{a}$. This means that the polynomial (4.2) belongs to $K[x]$. Thus P_j belongs to $K[x]$ because $(r_j|e_j(\alpha)|)^2 = r_j^2 e_j(\alpha)\overline{e_j(\alpha)}$ belongs to K.

A polynomial $f(x) \in \mathbb{C}[x]$ is a polynomial in x^2 if and only if $f(-x) = f(x)$ holds. Suppose that $f(x)$ is a polynomial in x^2. Let $g(x)$ be $f(x - b)f(x + b)$. Then $g(x)$ is also a polynomial in x^2 since the following equalities hold.
$$g(-x) = f(-x - b)f(-x + b) = f(x + b)f(x - b) = g(x)\,.$$
This fact and $P_1(x) = x^2 - r_1^2|e_1(\alpha)|^2$ prove that P_j is a polynomial in x^2 for $j = 1, \ldots, n$.

The second and the third statements follow from the following two facts. First, let the zeros of $f(x)$ be $\alpha_1, \ldots, \alpha_m$. Then the zeros of the polynomial (4.2) are $\alpha_1 \pm \sqrt{a}, \ldots, \alpha_m \pm \sqrt{a}$. Second, let the zeros of P_n be $\pm\beta_1, \ldots, \pm\beta_{2^{n-1}}$ $(0 \le \beta_j)$. Then the zeros of P_{n+1} are $\beta_1^2, \ldots, \beta_{2^{n-1}}^2$. □

Using Proposition 4.1, we can carry out the computation on C_m in Step 3 as follows. Substitute $\varphi_m(s)$ for α in P_{n+1} and examine whether each real zero of
$$P_{n+1}\left(\sum_{j=1}^{n} c_j e_j\big(\varphi_m(s)\big) \sum_{j=1}^{n} \overline{c_j}\,\overline{e_j}\big(\overline{\varphi_m}(s)\big)\right) \tag{4.3}$$

in the interval S_m satisfies the condition

$$\left|\sum_{j=1}^{n} c_j e_j\big(\varphi_m(s)\big)\right| = \sum_{j=1}^{n} r_j \big|e_j\big(\varphi_m(s)\big)\big| \,. \tag{4.4}$$

There exists a zero in S_m that satisfies the condition (4.4) if and only if F has a pseudozero on C_m. We can use interval computations to examine this condition because we can decide which zeros of (4.3) are also zeros of $e_j(\varphi_m(s))$ and, as specified in the second statement of Proposition 4.1, for every zero σ of (4.3) the left-hand side of (4.4)

$$\left|\sum_{j=1}^{n} c_j e_j\big(\varphi_m(\sigma)\big)\right|$$

is equal to one of

$$\pm r_1 \big|e_1\big(\varphi_m(\sigma)\big)\big| \pm \cdots \pm r_n \big|e_n\big(\varphi_m(\sigma)\big)\big| \,.$$

4.2. Special case ($e_j = x^{j-1}$)

When $e_j = x^{j-1}$ the computations are much easier than they are in the general case.

From Theorem 3.4, $\alpha \in \mathbb{C}$ is a pseudozero of

$$F = \langle c_1, r_1\rangle \cdot 1 + \langle c_2, r_2\rangle x + \cdots + \langle c_n, r_n\rangle x^{n-1}$$

if and only if the following inequality holds:

$$|c_1 + c_2\alpha + \cdots + c_n\alpha^{n-1}| \le r_1 + r_2|\alpha| + \cdots + r_n|\alpha|^{n-1} \,. \tag{4.5}$$

Since the both sides of (4.5) are nonnegative, (4.5) is equivalent to

$$|c_1 + c_2\alpha + \cdots + c_n\alpha^{n-1}|^2 \le \Big(r_1 + r_2|\alpha| + \cdots + r_n|\alpha|^{n-1}\Big)^2 \,. \tag{4.6}$$

The left-hand side of (4.6) can be written as

$$\Big(c_1 + c_2\alpha + \cdots + c_n\alpha^{n-1}\Big)\Big(\overline{c_1} + \overline{c_2}\,\overline{\alpha} + \cdots + \overline{c_n}\,\overline{\alpha}^{n-1}\Big) = g(\alpha, \overline{\alpha}) \,,$$

where $g \in \mathbb{Q}[x, y]$. And we can write the right-hand side of (4.6) as

$$h_1(|\alpha|^2) + h_2(|\alpha|^2)|\alpha| \,,$$

where h_1, $h_2 \in \mathbb{Q}[x]$ and all of the coefficients of h_1 and h_2 are nonnegative. (Note that $r_j \ge 0$.) Since $|\alpha|^2 = \alpha\overline{\alpha}$, we have $h_1(|\alpha|^2) = h_1(\alpha\overline{\alpha})$ and $h_2(|\alpha|^2) = h_2(\alpha\overline{\alpha})$.

Therefore we can write (4.6) as

$$g(\alpha, \overline{\alpha}) \le h_1(\alpha\overline{\alpha}) + h_2(\alpha\overline{\alpha})\,|\alpha| \,. \tag{4.7}$$

Since $h_2(\alpha\overline{\alpha})|\alpha|$ is nonnegative, (4.7) is equivalent to

$$g(\alpha, \overline{\alpha}) - h_1(\alpha\overline{\alpha}) \le 0 \tag{4.8}$$

or

$$\big(g(\alpha, \overline{\alpha}) - h_1(\alpha\overline{\alpha})\big)^2 - h_2(\alpha\overline{\alpha})^2\alpha\overline{\alpha} \le 0 \,. \tag{4.9}$$

By putting $\alpha = \varphi_m(s)$, we can rewrite (4.8) and (4.9) into the inequalities among rational functions in s.

$$g\bigl(\varphi_m(s), \overline{\varphi_m}(s)\bigr) - h_1\bigl(\varphi_m(s)\overline{\varphi_m}(s)\bigr) \leq 0\,, \tag{4.10}$$

$$\Bigl(g\bigl(\varphi_m(s), \overline{\varphi_m}(s)\bigr) - h_1\bigl(\varphi_m(s)\overline{\varphi_m}(s)\bigr)\Bigr)^2 - h_2\bigl(\varphi_m(s)\overline{\varphi_m}(s)\bigr)^2 \varphi_m(s)\overline{\varphi_m}(s) \leq 0\,. \tag{4.11}$$

Therefore the computation for Steps 2 and 3 in Algorithm 1 is as follows.

Step 2. Examine whether α satisfies (4.8) or (4.9). If it does, then return "yes."

Step 3. Examine that the left-hand side of either (4.10) or (4.11) has zero in some S_m $(m = 1, \ldots, M)$.
If it has, then return "yes."
Otherwise return "no."

The total degrees of the polynomials in (4.8) and (4.9) with respect to α and $\overline{\alpha}$ are at most $4(n-1)$ while the degree of P_{n+1} in Proposition 4.1 is 2^{n-1} in the general case.

5. Examples

In this section we show three examples.

Example 1. For the following interval polynomial F and complex domain D with boundary C, we decide whether there exists a pseudozero in D.

F is

$$\langle 1, 0.1\rangle x^2 + \langle -0.65, 0.05\rangle x + \langle 1, 0.1\rangle \cdot 1\,.$$

D is a rectangle whose vertexes are

$$R_1 = 0.3 + 0.8i\,, \qquad R_2 = 0.32 + 0.8i\,,$$
$$R_3 = 0.32 + 1.2i\,, \qquad R_4 = 0.3 + 1.2i$$

(see Figure 1) and C is a union of C_1, C_2, C_3 and C_4, where edges C_j are represented as follows.

$$\begin{aligned} &C_1 : s + 0.8i && (0.3 \leq s \leq 0.32)\,,\\ &C_2 : 0.32 + is && (0.8 \leq s \leq 1.2)\,,\\ &C_3 : s + 1.2i && (0.3 \leq s \leq 0.32)\,,\\ &C_4 : 0.3 + is && (0.8 \leq s \leq 1.2)\,. \end{aligned}$$

In Step 1 of Algorithm 1 we take $f = x^2 - 0.65x + 1 \in F$. Using the Sturm algorithm we find that f has no zero on C. Then we examine whether f has a zero in D by using the argument principle and find that f has no zero in D.

In Step 2, the inequality does not hold at the lower right vertex R_2.

In Step 3, we first examine the left-side rational functions in (4.10) and (4.11) on C_1. They are polynomials

$$0.99s^4 - 1.3s^3 + 3.6672s^2 - 2.132s + 0.371504$$

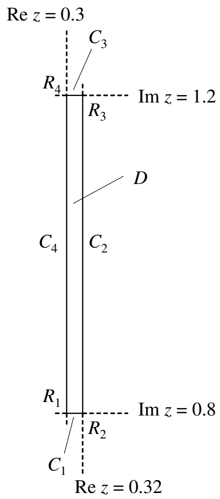

FIGURE 1. Domain D in Example 1.

and

$$0.9801s^8 - 2.574s^7 + 8.95096s^6 - 13.7561s^5 + 19.7267s^4 - 16.6029s^3 + 7.2697s^2 - 1.58409s + 0.137843\,.$$

Neither polynomial has a zero in $[0.3, 0.32)$. That is, there is no pseudozero on C_1. We next examine the left-side rational functions in (4.10) and (4.11) on C_2. They are polynomials

$$0.99s^4 - 1.81325s^2 + 0.787543$$

and

$$0.9801s^8 - 3.59033s^6 + 4.84697s^4 - 2.85616s^2 + 0.620211\,.$$

The first polynomial has two zeros in $[0.8, 1.2)$. That is, there is a pseudozero on C_2.

Example 2. For the following interval polynomial F and complex domain $E(r)$ with boundary $C(r)$, where $r = 1.07$ and 1.06, we decide whether there exists a pseudozero in $E(r)$.

F is

$$\langle 0, 0.08\rangle e_1(x) + \langle -1-i, 0.09\rangle e_2(x) + \langle -2, 0.1\rangle e_3(x) + \langle -1+i, 0.11\rangle e_4(x)\,,$$

where

$$e_1(x) = \frac{1}{4}(x-i)(x+1)(x+i)\,,$$
$$e_2(x) = \frac{i}{4}(x-1)(x+1)(x+i)\,,$$

$$e_3(x) = -\frac{1}{4}(x-1)(x-i)(x+i)\,,$$
$$e_4(x) = -\frac{i}{4}(x-1)(x-i)(x+1)\,.$$

Polynomials $e_j(x)$ are the basis polynomials of the Lagrange interpolation at four points $a_1 = 1$, $a_2 = i$, $a_3 = -1$ and $a_4 = -i$. That is, $e_j(a_k) = \delta_{jk}$ holds, where δ_{jk} is the Kronecker delta. $E(r)$ is the outside of the open disk centered at the origin and with radius r. The boundary $C(r)$ is the union of $C_1(r)$ and $\{r\}$, where $C_1(r)$ is represented by

$$\varphi_r : \mathbb{R} \to C_1(r)\,, \qquad s \mapsto \frac{r(s-i)}{s+i}\,.$$

We can apply our algorithm to the unbounded domains $E(r)$ because all the polynomials in F are of the same degree.

In Step 1 of Algorithm 1 we take $f = x^3 - 1 \in F$, the "center polynomial." It is clear that f has no zero in $E(r)$ for $1 < r$.

In Step 2, the point $r \in C(r)$ is not a pseudozero of F, for either $r = 1.07$ or 1.06.

In Step 3, we first describe the examination of the existence of a pseudozero on $C(1.07)$. The numerator of

$$P_5\Big(\big(\varphi_{1.07}(s)^3 - 1\big)\big(\overline{\varphi_{1.07}}(s)^3 - 1\big)\Big)$$

is of degree 48 and has no real zero. Therefore F has no pseudozero in $E(1.07)$. We next describe the examination of $C(1.06)$. The numerator of

$$P_5\Big(\big(\varphi_{1.06}(s)^3 - 1\big)\big(\overline{\varphi_{1.06}}(s)^3 - 1\big)\Big)$$

is of degree 48 and has two real zeros

$$0.57158251\cdots\,, \qquad 0.58145954\cdots$$

both of which satisfy the equality (4.4). Therefore F has pseudozeros on $C(1.06) \subset E(1.06)$.

Example 3. Let F be the interval polynomial in Example 2 and $D(r)$ be the closed disk centered at the origin and with radius r. The boundary $C(r)$ of $D(r)$ is the union of $C_1(r)$ and $\{r\}$ as described in Example 2. We decide whether there exists a pseudozero of F in $D(0.94)$ and $D(0.95)$.

In Step 1 of Algorithm 1 we take the same polynomial $f = x^3 - 1 \in F$ in Example 2. It is clear that f has no zero in $D(r)$ for $r < 1$.

In Step 2, the point $r \in C(r)$ is not a pseudozero of F, for either $r = 0.94$ or 0.95.

In Step 3, we first describe the examination of $C(0.94)$. The numerator of

$$P_5\Big(\big(\varphi_{0.94}(s)^3 - 1\big)\big(\overline{\varphi_{0.94}}(s)^3 - 1\big)\Big)$$

is of degree 48 and has no real zero. Therefore F has no pseudozero in $D(0.94)$. We next describe the examination of $C(0.95)$. The numerator of

$$P_5\Big(big(\varphi_{0.95}(s)^3 - 1\big)\big(\overline{\varphi_{0.95}}(s)^3 - 1\big)\Big)$$

is of degree 48 and has four real zeros

$$-0.58890429\cdots, \quad -0.56393838\cdots, \quad 0.55768361\cdots, \quad 0.59564616\cdots$$

all of which satisfy the equality (4.4). Therefore F has pseudozeros on $C(0.95) \subset D(0.95)$.

Combining the results of Examples 2 and 3, we can conclude that all of the pseudozeros of F lie in the annulus $\{\, z \in \mathbb{C} \mid 0.94 < |z| < 1.07 \,\}$ and that we cannot replace 0.94 by 0.95 or 1.07 by 1.06.

6. Conclusion

We have proposed an algorithm that decide whether a complex interval polynomial has a pseudozero in a prescribed complex domain.

Using numerical computations for efficiency is one of our future directions. Another is to consider the following type of problem for a given interval polynomial F when we have more than one domain: Does there exist a polynomial in F that has zeros in every domain?

Appendix

Theorem 3.3 (Rouché's theorem) is proved as follows.

Proof. The following inequality holds for $0 \le t_1 \le t_2 \le 1$:

$$|f(z) + t_2 g(z)| \le |f(z) + t_1 g(z)| + |t_2 - t_1||g(z)|\,.$$

Let $m(t)$ be $\min_{z\in C}\{|f(z) + tg(z)|\}$ and M be $\max_{z\in C}\{|g(z)|\}$. Then

$$m(t_2) \le |f(z) + t_1 g(z)| + |t_2 - t_1|M\,.$$

Therefore the following inequality holds:

$$m(t_2) \le m(t_1) + |t_2 - t_1|M\,.$$

When we interchange t_1 and t_2, the resulting inequality also holds. Therefore

$$|m(t_2) - m(t_1)| \le |t_2 - t_1|M\,.$$

This inequality implies that $m(t)$ is continuous in the interval $0 \le t \le 1$. From the hypothesis, $m(t) > 0$ holds for any t in $[0,1]$. Therefore, $m = \min_{0\le t\le 1}\{|m(t)|\}$ should be positive. Now we denote the length of C by L and the maximum of $|f(z)g'(z) - f'(z)g(z))|$ on C by G. Let $N(t)$ be

$$\frac{1}{2\pi i}\int_C \frac{f'(z) + tg'(z)}{f(z) + tg(z)}dz\,,$$

which is the number of zeros of $f(z) + tg(z)$ inside C with multiplicity counted. Then we have

$$|N(t_2) - N(t_1)| = \frac{1}{2\pi} \left| \int_C \frac{(t_1 - t_2)\big(f(z)g'(z) - f'(z)g(z)\big)}{\big(f(z) + t_1 g(z)\big)\big(f(z) + t_2 g(z)\big)} dz \right|$$
$$\leq \frac{|t_1 - t_2| GL}{2\pi m^2},$$

which implies that $N(t)$ is continuous on the interval $0 \leq t \leq 1$. Therefore the equality $N(0) = N(1)$ holds because $N(t)$ is a nonnegative integer for any t. □

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Hiroshi Sekigawa
NTT Communication Science Laboratories
Nippon Telegraph and Telephone Corporation
3-1 Morinosato-Wakamiya, Atsugi-shi
Kanagawa, 243-0198
Japan
e-mail: `sekigawa@theory.brl.ntt.co.jp`

Kiyoshi Shirayanagi
Department of Mathematical Sciences
Tokai University
1117 Kitakaname, Hiratsuka-shi
Kanagawa, 259-1292
Japan
e-mail: `shirayan@ss.u-tokai.ac.jp`

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