SUTD 2020 10.009/10.011 2D Schrödinger's Equation Project

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Part A: Theory (Chemistry) Segment

For this assignment, we have chosen the value of the principal quantum number, $\mathbf{n = 4}$. It is assumed that we are using a **right-handed coordinate system**.

l	m	Orbital Name	$\Psi(r,\theta,\phi)$
0	0	$4s$	$\begin{aligned} R_0^4 \times Y_0^0 &= \left(\frac{1}{4}a^{-3/2}\left[1-\frac{3}{4}\left(\frac{r}{a}\right)+\frac{1}{8}\left(\frac{r}{a}\right)^2-\frac{1}{192}\left(\frac{r}{a}\right)^3\right]\exp(-\frac{r}{4a})\right)\times\left(\sqrt{\frac{1}{4\pi}}\right) \\ &= \left(\frac{1}{8\sqrt{\pi}}\right)a^{-3/2}\left[1-\frac{3}{4}\left(\frac{r}{a}\right)+\frac{1}{8}\left(\frac{r}{a}\right)^2-\frac{1}{192}\left(\frac{r}{a}\right)^3\right]\exp(-\frac{r}{4a}) \end{aligned}$
1	+1	$4p_x$	$\begin{aligned} R_1^4 \times \left(\frac{1}{\sqrt{2}}\left(Y_1^{-1}-Y_1^{+1}\right)\right) &= \left(\frac{\sqrt{5}}{16\sqrt{3}}a^{-3/2}\left(\frac{r}{a}\right)\left[1-\frac{1}{4}\left(\frac{r}{a}\right)+\frac{1}{80}\left(\frac{r}{a}\right)^2\right]\exp(-\frac{r}{4a})\right)\times \\ &\left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{\frac{3}{8\pi}}\sin\theta\exp(-i\phi)+\sqrt{\frac{3}{8\pi}}\sin\theta\exp(i\phi)\right) \\ &= \left(\frac{1}{32}\sqrt{\frac{5}{\pi}}\right)ra^{-5/2}\left[1-\frac{1}{4}\left(\frac{r}{a}\right)+\frac{1}{80}\left(\frac{r}{a}\right)^2\right]\exp(-\frac{r}{4a})\sin\theta\cos\phi \end{aligned}$
	0	$4p_z$	$\begin{aligned} R_1^4 \times Y_1^0 &= \left(\frac{\sqrt{5}}{16\sqrt{3}}a^{-3/2}\left(\frac{r}{a}\right)\left[1-\frac{1}{4}\left(\frac{r}{a}\right)+\frac{1}{80}\left(\frac{r}{a}\right)^2\right]\exp(-\frac{r}{4a})\right)\times\left(\sqrt{\frac{3}{4\pi}}\cos\theta\right) \\ &= \left(\frac{1}{32}\sqrt{\frac{5}{\pi}}\right)ra^{-5/2}\left[1-\frac{1}{4}\left(\frac{r}{a}\right)+\frac{1}{80}\left(\frac{r}{a}\right)^2\right]\exp(-\frac{r}{4a})\cos\theta \end{aligned}$
	-1	$4p_y$	$\begin{aligned} R_1^4 \times \left(\frac{i}{\sqrt{2}}\left(Y_1^{-1}+Y_1^{+1}\right)\right) &= \left(\frac{\sqrt{5}}{16\sqrt{3}}a^{-3/2}\left(\frac{r}{a}\right)\left[1-\frac{1}{4}\left(\frac{r}{a}\right)+\frac{1}{80}\left(\frac{r}{a}\right)^2\right]\exp(-\frac{r}{4a})\right)\times \\ &\left(\frac{i}{\sqrt{2}}\right)\left(\sqrt{\frac{3}{8\pi}}\sin\theta\exp(-i\phi)-\sqrt{\frac{3}{8\pi}}\sin\theta\exp(i\phi)\right) \\ &= \left(\frac{1}{32}\sqrt{\frac{5}{\pi}}\right)ra^{-5/2}\left[1-\frac{1}{4}\left(\frac{r}{a}\right)+\frac{1}{80}\left(\frac{r}{a}\right)^2\right]\exp(-\frac{r}{4a})\sin\theta\sin\phi \end{aligned}$
2	+2	$4d_{x^2-y^2}$	$\begin{aligned} R_2^4 \times \left(\frac{1}{\sqrt{2}}\left(Y_2^{-2}+Y_2^{+2}\right)\right) &= \left(\frac{1}{64\sqrt{5}}a^{-3/2}\left(\frac{r}{a}\right)^2\left[1-\frac{1}{12}\left(\frac{r}{a}\right)\right]\exp(-\frac{r}{4a})\right)\times \\ &\left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{\frac{15}{32\pi}}\sin^2\theta\exp(-i2\phi)+\sqrt{\frac{15}{32\pi}}\sin^2\theta\exp(i2\phi)\right) \\ &= \left(\frac{1}{256}\sqrt{\frac{3}{\pi}}\right)r^2a^{-7/2}\left[1-\frac{1}{12}\left(\frac{r}{a}\right)\right]\exp(-\frac{r}{4a})\sin^2\theta\left[\cos^2\phi-\sin^2\phi\right] \end{aligned}$
	+1	$4d_{xz}$	$\begin{aligned} R_2^4 \times \left(\frac{1}{\sqrt{2}}\left(Y_2^{-1}-Y_2^{+1}\right)\right) &= \left(\frac{1}{64\sqrt{5}}a^{-3/2}\left(\frac{r}{a}\right)^2\left[1-\frac{1}{12}\left(\frac{r}{a}\right)\right]\exp(-\frac{r}{4a})\right)\times \\ &\left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta\exp(-i\phi)+\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta\exp(i\phi)\right) \\ &= \left(\frac{1}{128}\sqrt{\frac{3}{\pi}}\right)r^2a^{-7/2}\left[1-\frac{1}{12}\left(\frac{r}{a}\right)\right]\exp(-\frac{r}{4a})\sin\theta\cos\phi\cos\theta \end{aligned}$
	0	$4d_{z^2}$	$\begin{aligned} R_2^4 \times Y_2^0 &= \left(\frac{1}{64\sqrt{5}}a^{-3/2}\left(\frac{r}{a}\right)^2\left[1-\frac{1}{12}\left(\frac{r}{a}\right)\right]\exp(-\frac{r}{4a})\right)\times\left(\sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta-1\right)\right) \\ &= \left(\frac{1}{256\sqrt{\pi}}\right)r^2a^{-7/2}\left[1-\frac{1}{12}\left(\frac{r}{a}\right)\right]\exp(-\frac{r}{4a})\left(3\cos^2\theta-1\right) \end{aligned}$
	-1	$4d_{yz}$	$\begin{aligned} R_2^4 \times \left(\frac{i}{\sqrt{2}}\left(Y_2^{-1}+Y_2^{+1}\right)\right) &= \left(\frac{1}{64\sqrt{5}}a^{-3/2}\left(\frac{r}{a}\right)^2\left[1-\frac{1}{12}\left(\frac{r}{a}\right)\right]\exp(-\frac{r}{4a})\right)\times \\ &\left(\frac{i}{\sqrt{2}}\right)\left(\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta\exp(-i\phi)-\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta\exp(i\phi)\right) \\ &= \left(\frac{1}{128}\sqrt{\frac{3}{\pi}}\right)r^2a^{-7/2}\left[1-\frac{1}{12}\left(\frac{r}{a}\right)\right]\exp(-\frac{r}{4a})\sin\theta\sin\phi\cos\theta \end{aligned}$
	-2	$4d_{xy}$	$\begin{aligned} R_2^4 \times \left(\frac{i}{\sqrt{2}}\left(Y_2^{-2}-Y_2^{+2}\right)\right) &= \left(\frac{1}{64\sqrt{5}}a^{-3/2}\left(\frac{r}{a}\right)^2\left[1-\frac{1}{12}\left(\frac{r}{a}\right)\right]\exp(-\frac{r}{4a})\right)\times \\ &\left(\frac{i}{\sqrt{2}}\right)\left(\sqrt{\frac{15}{32\pi}}\sin^2\theta\exp(-i2\phi)-\sqrt{\frac{15}{32\pi}}\sin^2\theta\exp(i2\phi)\right) \\ &= \left(\frac{1}{128}\sqrt{\frac{3}{\pi}}\right)r^2a^{-7/2}\left[1-\frac{1}{12}\left(\frac{r}{a}\right)\right]\exp(-\frac{r}{4a})\sin^2\theta\sin\phi\cos\phi \end{aligned}$

l	m	Orbital Name	$\Psi(r,\theta,\phi)$
	$+3$	$4f_{x\left(x^2-3y^2\right)}$	$R_3^4 \times \left(\frac{1}{\sqrt{2}}\left(Y_3^{-3} - Y_3^{+3}\right)\right) = \left(\frac{1}{768\sqrt{35}}a^{-3/2}\left(\frac{r}{a}\right)^3 \exp(-\frac{r}{4a})\right) \times$ $\left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{\frac{35}{64\pi}}\sin^3\theta\exp(-i3\phi) + \sqrt{\frac{35}{64\pi}}\sin^3\theta\exp(i3\phi)\right)$ $= \left(\frac{1}{3072}\sqrt{\frac{1}{2\pi}}\right) r^3 a^{-9/2}\exp(-\frac{r}{4a})\sin^3\theta\left[4\cos^3\phi - 3\cos\phi\right]$
	$+2$	$4f_{z\left(x^2-y^2\right)}$	$R_3^4 \times \left(\frac{1}{\sqrt{2}}\left(Y_3^{-2} + Y_3^{+2}\right)\right) = \left(\frac{1}{768\sqrt{35}}a^{-3/2}\left(\frac{r}{a}\right)^3 \exp(-\frac{r}{4a})\right) \times$ $\left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{\frac{105}{32\pi}}\cos\theta\sin^2\theta\exp(-i2\phi) + \sqrt{\frac{105}{32\pi}}\cos\theta\sin^2\theta\exp(i2\phi)\right)$ $= \left(\frac{1}{3072}\sqrt{\frac{3}{\pi}}\right) r^3 a^{-9/2}\exp(-\frac{r}{4a})\cos\theta\sin^2\theta\left[\cos^2\phi - \sin^2\phi\right]$
	$+1$	$4f_{xz^2}$	$R_3^4 \times \left(\frac{1}{\sqrt{2}}\left(Y_3^{-1} - Y_3^{+1}\right)\right) = \left(\frac{1}{768\sqrt{35}}a^{-3/2}\left(\frac{r}{a}\right)^3 \exp(-\frac{r}{4a})\right) \times$ $\left(\frac{1}{\sqrt{2}}\right)\left(\sqrt{\frac{21}{64\pi}}\sin\theta\left(5\cos^2\theta - 1\right)\exp(-i\phi) + \sqrt{\frac{21}{64\pi}}\sin\theta\left(5\cos^2\theta - 1\right)\exp(i\phi)\right)$ $= \left(\frac{1}{3072}\sqrt{\frac{3}{10\pi}}\right) r^3 a^{-9/2}\exp(-\frac{r}{4a})\sin\theta\cos\phi\left[5\cos^2\theta - 1\right]$
3	0	$4f_{z^3}$	$R_3^4 \times Y_3^0 = \left(\frac{1}{768\sqrt{35}}a^{-3/2}\left(\frac{r}{a}\right)^3 \exp(-\frac{r}{4a})\right) \times \left(\sqrt{\frac{7}{16\pi}}\left(5\cos^3\theta - 3\cos\theta\right)\right)$ $= \left(\frac{1}{3072}\sqrt{\frac{1}{5\pi}}\right) r^3 a^{-9/2}\exp(-\frac{r}{4a})\left[5\cos^3\theta - 3\cos\theta\right]$
	-1	$4f_{yz^2}$	$R_3^4 \times \left(\frac{i}{\sqrt{2}}\left(Y_3^{-1} + Y_3^{+1}\right)\right) = \left(\frac{1}{768\sqrt{35}}a^{-3/2}\left(\frac{r}{a}\right)^3 \exp(-\frac{r}{4a})\right) \times$ $\left(\frac{i}{\sqrt{2}}\right)\left(\sqrt{\frac{21}{64\pi}}\sin\theta\left(5\cos^2\theta - 1\right)\exp(-i\phi) - \sqrt{\frac{21}{64\pi}}\sin\theta\left(5\cos^2\theta - 1\right)\exp(i\phi)\right)$ $= \left(\frac{1}{3072}\sqrt{\frac{3}{10\pi}}\right) r^3 a^{-9/2}\exp(-\frac{r}{4a})\sin\theta\sin\phi\left[5\cos^2\theta - 1\right]$
	-2	$4f_{xyz}$	$R_3^4 \times \left(\frac{i}{\sqrt{2}}\left(Y_3^{-2} - Y_3^{+2}\right)\right) = \left(\frac{1}{768\sqrt{35}}a^{-3/2}\left(\frac{r}{a}\right)^3 \exp(-\frac{r}{4a})\right) \times$ $\left(\frac{i}{\sqrt{2}}\right)\left(\sqrt{\frac{105}{32\pi}}\cos\theta\sin^2\theta\exp(-i2\phi) - \sqrt{\frac{105}{32\pi}}\cos\theta\sin^2\theta\exp(i2\phi)\right)$ $= \left(\frac{1}{1536}\sqrt{\frac{3}{\pi}}\right) r^3 a^{-9/2}\exp(-\frac{r}{4a})\sin^2\theta\sin\phi\cos\phi\cos\theta$
	-3	$4f_{y\left(3x^2-y^2\right)}$	$R_3^4 \times \left(\frac{i}{\sqrt{2}}\left(Y_3^{-3} + Y_3^{+3}\right)\right) = \left(\frac{1}{768\sqrt{35}}a^{-3/2}\left(\frac{r}{a}\right)^3 \exp(-\frac{r}{4a})\right) \times$ $\left(\frac{i}{\sqrt{2}}\right)\left(\sqrt{\frac{35}{64\pi}}\sin^3\theta\exp(-i3\phi) - \sqrt{\frac{35}{64\pi}}\sin^3\theta\exp(i3\phi)\right)$ $= \left(\frac{1}{3072}\sqrt{\frac{1}{2\pi}}\right) r^3 a^{-9/2}\exp(-\frac{r}{4a})\sin^3\theta\left[3\sin\phi - 4\sin^3\phi\right]$

Part C: In-Depth Theory Segment

(a) We have Equation 1 as

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Psi}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2\Psi}{\partial\phi^2} = (E-V)\left(-\frac{2mr^2}{\hbar^2}\right)\Psi \tag{1}$$

To separate Equation 1 into its radial and angular components, substitute $\Psi(r,\theta,\phi) = R(r)\cdot Y(\theta,\phi)$ into Equation 1 to obtain

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}(R\cdot Y)\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}(R\cdot Y)\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}(R\cdot Y) = (E-V)\left(-\frac{2mr^2}{\hbar^2}\right)(R\cdot Y)$$

Factorising out the respective R and Y, we could reduce the radial partial derivative to an ordinary derivative and we would get

$$Y\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{R}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = (E-V)\left(-\frac{2mr^2}{\hbar^2}\right)(R\cdot Y)$$

Divide both sides by $R\cdot Y$ to obtain

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = (E-V)\left(-\frac{2mr^2}{\hbar^2}\right)$$

Since $-V = \frac{e^2}{4\pi\varepsilon_0 r}$, we could rearrange the equation to become

$$\left[\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2mr^2}{\hbar^2}\left(E + \frac{e^2}{4\pi\varepsilon_0 r}\right)\right] + \left[\frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2}\right] = 0$$

Therefore, we have successfully separated Equation 1 into two portions, radial and angular, whereby

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2mr^2}{\hbar^2}\left(E + \frac{e^2}{4\pi\varepsilon_0 r}\right) = l\,(l+1)$$
$$\frac{1}{Y\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{1}{Y\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} = -l\,(l+1)$$

(b) We have $\mathbf{n = 4}$.

l	m	$P_l(x)$	$P_l^m(x)$	$Y_l^m(\theta,\phi)$
0	0	$\frac{1}{2^0 0!}\left(\frac{\partial}{\partial x}\right)^0\left(x^2-1\right)^0 = 1$	$\left(1-x^2\right)^0\left(\frac{\partial}{\partial x}\right)^0 P_0(x) = 1$	$\sqrt{\frac{1}{4\pi}\frac{0!}{0!}}e^0 P_0^0(\cos\theta) = \sqrt{\frac{1}{4\pi}}$
1	+1	$\frac{1}{2^1 1!}\left(\frac{\partial}{\partial x}\right)^1\left(x^2-1\right)^1 = x$	$\left(1-x^2\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial x}\right)^1 P_1(x)$ $= \sqrt{1-x^2}$	$-\sqrt{\frac{3}{4\pi}\frac{0!}{2!}}e^{i\phi}P_1^{+1}(\cos\theta)$ $= -\sqrt{\frac{3}{8\pi}}e^{i\phi}\sqrt{1-\cos^2\theta}$ $= -\sqrt{\frac{3}{8\pi}}\sin\theta e^{i\phi}$
1	0	$\frac{1}{2^1 1!}\left(\frac{\partial}{\partial x}\right)^1\left(x^2-1\right)^1 = x$	$\left(1-x^2\right)^0\left(\frac{\partial}{\partial x}\right)^0 P_1(x) = x$	$\sqrt{\frac{3}{4\pi}\frac{1!}{1!}}e^0 P_1^0(\cos\theta) = \sqrt{\frac{3}{4\pi}}\cos\theta$
1	−1	$\frac{1}{2^1 1!}\left(\frac{\partial}{\partial x}\right)^1\left(x^2-1\right)^1 = x$	$\left(1-x^2\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial x}\right)^1 P_1(x)$ $= \sqrt{1-x^2}$	$\sqrt{\frac{3}{4\pi}\frac{0!}{2!}}e^{-i\phi}P_1^{-1}(\cos\theta)$ $= \sqrt{\frac{3}{8\pi}}e^{-i\phi}\sqrt{1-\cos^2\theta}$ $= \sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}$
2	+2	$\frac{1}{2^2 2!}\left(\frac{\partial}{\partial x}\right)^2\left(x^2-1\right)^2 = \frac{1}{2}\left(3x^2-1\right)$	$\left(1-x^2\right)^1\left(\frac{\partial}{\partial x}\right)^2 P_2(x)$ $= 3\left(1-x^2\right)$	$\sqrt{\frac{5}{4\pi}\frac{0!}{4!}}e^{2i\phi}P_2^{+2}(\cos\theta)$ $= \sqrt{\frac{5}{96\pi}}e^{2i\phi}3\left(1-\cos^2\theta\right)$ $= \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{2i\phi}$
2	+1	$\frac{1}{2^2 2!}\left(\frac{\partial}{\partial x}\right)^2\left(x^2-1\right)^2 = \frac{1}{2}\left(3x^2-1\right)$	$\left(1-x^2\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial x}\right)^1 P_2(x)$ $= 3x\sqrt{1-x^2}$	$-\sqrt{\frac{5}{4\pi}\frac{1!}{3!}}e^{i\phi}P_2^{+1}(\cos\theta)$ $= -\sqrt{\frac{5}{24\pi}}e^{i\phi}3\cos\theta\sqrt{1-\cos^2\theta}$ $= -\sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta e^{i\phi}$
2	0	$\frac{1}{2^2 2!}\left(\frac{\partial}{\partial x}\right)^2\left(x^2-1\right)^2 = \frac{1}{2}\left(3x^2-1\right)$	$\left(1-x^2\right)^0\left(\frac{\partial}{\partial x}\right)^0 P_2(x)$ $= \frac{3}{2}x^2 - \frac{1}{2}$	$\sqrt{\frac{5}{4\pi}\frac{2!}{2!}}e^0 P_2^0(\cos\theta)$ $= \sqrt{\frac{5}{16\pi}}\left(3\cos^2\theta-1\right)$
2	−1	$\frac{1}{2^2 2!}\left(\frac{\partial}{\partial x}\right)^2\left(x^2-1\right)^2 = \frac{1}{2}\left(3x^2-1\right)$	$\left(1-x^2\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial x}\right)^1 P_2(x)$ $= 3x\sqrt{1-x^2}$	$\sqrt{\frac{5}{4\pi}\frac{1!}{3!}}e^{-i\phi}P_2^{+1}(\cos\theta)$ $= \sqrt{\frac{5}{24\pi}}e^{-i\phi}3\cos\theta\sqrt{1-\cos^2\theta}$ $= \sqrt{\frac{15}{8\pi}}\cos\theta\sin\theta e^{-i\phi}$
2	−2	$\frac{1}{2^2 2!}\left(\frac{\partial}{\partial x}\right)^2\left(x^2-1\right)^2 = \frac{1}{2}\left(3x^2-1\right)$	$\left(1-x^2\right)^1\left(\frac{\partial}{\partial x}\right)^2 P_2(x)$ $= 3\left(1-x^2\right)$	$\sqrt{\frac{5}{4\pi}\frac{0!}{4!}}e^{-2i\phi}P_2^{+2}(\cos\theta)$ $= \sqrt{\frac{5}{96\pi}}e^{-2i\phi}3\left(1-\cos^2\theta\right)$ $= \sqrt{\frac{15}{32\pi}}\sin^2\theta e^{-2i\phi}$

l	m	$P_l(x)$	$P_l^m(x)$	$Y_l^m(\theta,\phi)$
3	$+3$	$\frac{1}{2^3 3!}\left(\frac{\partial}{\partial x}\right)^3\left(x^2-1\right)^3=\frac{1}{2}x\left(5x^2-3\right)$	$\left(1-x^2\right)^{\frac{3}{2}}\left(\frac{\partial}{\partial x}\right)^3 P_3(x)$ $=15\left(1-x^2\right)^{\frac{3}{2}}$	$-\sqrt{\frac{7}{4\pi}\frac{0!}{6!}}e^{3i\phi}P_3^{+3}(\cos\theta)$ $=-\sqrt{\frac{7}{2880\pi}}e^{3i\phi}15\left(1-\cos^2\theta\right)^{\frac{3}{2}}$ $=-\sqrt{\frac{35}{64\pi}}\sin^3\theta e^{3i\phi}$
	$+2$	$\frac{1}{2^3 3!}\left(\frac{\partial}{\partial x}\right)^3\left(x^2-1\right)^3=\frac{1}{2}x\left(5x^2-3\right)$	$\left(1-x^2\right)^1\left(\frac{\partial}{\partial x}\right)^2 P_3(x)$ $=15x\left(1-x^2\right)$	$\sqrt{\frac{7}{4\pi}\frac{1!}{5!}}e^{2i\phi}P_3^{+2}(\cos\theta)$ $=\sqrt{\frac{7}{480\pi}}e^{2i\phi}15\cos\theta\left(1-\cos^2\theta\right)$ $=\sqrt{\frac{105}{32\pi}}\cos\theta\sin^2\theta e^{2i\phi}$
	$+1$	$\frac{1}{2^3 3!}\left(\frac{\partial}{\partial x}\right)^3\left(x^2-1\right)^3=\frac{1}{2}x\left(5x^2-3\right)$	$\left(1-x^2\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial x}\right)^1 P_3(x)$ $=\frac{3}{2}\left(5x^2-1\right)\sqrt{1-x^2}$	$-\sqrt{\frac{7}{4\pi}\frac{2!}{4!}}e^{i\phi}P_3^{+1}(\cos\theta)$ $=-\sqrt{\frac{7}{48\pi}}e^{i\phi}\left(\frac{3}{2}\right)\left(5\cos^2\theta-1\right)\sqrt{1-\cos^2\theta}$ $=-\sqrt{\frac{21}{64\pi}}\sin\theta\left(5\cos^2\theta-1\right)e^{i\phi}$
	0	$\frac{1}{2^3 3!}\left(\frac{\partial}{\partial x}\right)^3\left(x^2-1\right)^3=\frac{1}{2}x\left(5x^2-3\right)$	$\frac{1}{2}x\left(5x^2-3\right)$	$\sqrt{\frac{7}{4\pi}\frac{3!}{3!}}e^0P_3^0(\cos\theta)$ $=\sqrt{\frac{7}{16\pi}}\left(5\cos^3\theta-3\cos\theta\right)$
	-1	$\frac{1}{2^3 3!}\left(\frac{\partial}{\partial x}\right)^3\left(x^2-1\right)^3=\frac{1}{2}x\left(5x^2-3\right)$	$\left(1-x^2\right)^{\frac{1}{2}}\left(\frac{\partial}{\partial x}\right)^1 P_3(x)$ $=\frac{3}{2}\left(5x^2-1\right)\sqrt{1-x^2}$	$\sqrt{\frac{7}{4\pi}\frac{2!}{4!}}e^{-i\phi}P_3^{+1}(\cos\theta)$ $=\sqrt{\frac{7}{48\pi}}e^{-i\phi}\left(\frac{3}{2}\right)\left(5\cos^2\theta-1\right)\sqrt{1-\cos^2\theta}$ $=\sqrt{\frac{21}{64\pi}}\sin\theta\left(5\cos^2\theta-1\right)e^{-i\phi}$
	-2	$\frac{1}{2^3 3!}\left(\frac{\partial}{\partial x}\right)^3\left(x^2-1\right)^3=\frac{1}{2}x\left(5x^2-3\right)$	$\left(1-x^2\right)^1\left(\frac{\partial}{\partial x}\right)^2 P_3(x)$ $=15x\left(1-x^2\right)$	$\sqrt{\frac{7}{4\pi}\frac{1!}{5!}}e^{-2i\phi}P_3^{+2}(\cos\theta)$ $=\sqrt{\frac{7}{480\pi}}e^{-2i\phi}15\cos\theta\left(1-\cos^2\theta\right)$ $=\sqrt{\frac{105}{32\pi}}\cos\theta\sin^2\theta e^{-2i\phi}$
	-3	$\frac{1}{2^3 3!}\left(\frac{\partial}{\partial x}\right)^3\left(x^2-1\right)^3=\frac{1}{2}x\left(5x^2-3\right)$	$\left(1-x^2\right)^{\frac{3}{2}}\left(\frac{\partial}{\partial x}\right)^3 P_3(x)$ $=15\left(1-x^2\right)^{\frac{3}{2}}$	$\sqrt{\frac{7}{4\pi}\frac{0!}{6!}}e^{-3i\phi}P_3^{+3}(\cos\theta)$ $=\sqrt{\frac{7}{2880\pi}}e^{-3i\phi}15\left(1-\cos^2\theta\right)^{\frac{3}{2}}$ $=\sqrt{\frac{35}{64\pi}}\sin^3\theta e^{-3i\phi}$

(c) Let $p=2l+1$ and $q-p=n-l-1$, where $\mathbf{n=4}$.

l	$L_q(x)$	$L_{q-p}^p(x)$	$R_l^n(r)$
0	$e^x\left(\frac{\partial}{\partial x}\right)^4\left(e^{-x}x^4\right)$ $=x^4-16x^3+72x^2$ $-96x+24$	$(-1)^1\left(\frac{\partial}{\partial x}\right)^1 L_4(x)$ $=-4\left(x^3-12x^2+36x-24\right)$	$\sqrt{\left(\frac{2}{4a}\right)^3\frac{3!}{8(4!)^3}}e^{-\frac{r}{4a}}\left(\frac{2r}{4a}\right)^0 L_3^1\left(\frac{2r}{4a}\right)$ $=\frac{1}{4}a^{-\frac{3}{2}}\left[1-\frac{3}{4}\left(\frac{r}{a}\right)+\frac{1}{8}\left(\frac{r}{a}\right)^2-\frac{1}{192}\left(\frac{r}{a}\right)^3\right]e^{-\frac{r}{4a}}$
1	$e^x\left(\frac{\partial}{\partial x}\right)^5\left(e^{-x}x^5\right)$ $=-x^5+25x^4-200x^3$ $+600x^2-600x+120$	$(-1)^3\left(\frac{\partial}{\partial x}\right)^3 L_5(x)$ $=60\left(x^2-10x+20\right)$	$\sqrt{\left(\frac{2}{4a}\right)^3\frac{2!}{8(5!)^3}}e^{-\frac{r}{4a}}\left(\frac{2r}{4a}\right)^1 L_2^3\left(\frac{2r}{4a}\right)$ $=\frac{\sqrt{5}}{16\sqrt{3}}a^{-\frac{3}{2}}\left(\frac{r}{a}\right)\left[1-\frac{1}{4}\left(\frac{r}{a}\right)+\frac{1}{80}\left(\frac{r}{a}\right)^2\right]e^{-\frac{r}{4a}}$
2	$e^x\left(\frac{\partial}{\partial x}\right)^6\left(e^{-x}x^6\right)$ $=x^6-36x^5+450x^4-2400x^3$ $+5400x^2-4320x+720$	$(-1)^5\left(\frac{\partial}{\partial x}\right)^5 L_6(x)$ $=-720\left(x-6\right)$	$\sqrt{\left(\frac{2}{4a}\right)^3\frac{1!}{8(6!)^3}}e^{-\frac{r}{4a}}\left(\frac{2r}{4a}\right)^2 L_1^5\left(\frac{2r}{4a}\right)$ $=\frac{1}{64\sqrt{5}}a^{-\frac{3}{2}}\left(\frac{r}{a}\right)^2\left[1-\frac{1}{12}\left(\frac{r}{a}\right)\right]e^{-\frac{r}{4a}}$
3	$e^x\left(\frac{\partial}{\partial x}\right)^7\left(e^{-x}x^7\right)$ $=-x^7+49x^6-882x^5+7350x^4$ $-29400x^3+52920x^2-35280x$ $+5040$	$(-1)^7\left(\frac{\partial}{\partial x}\right)^7 L_7(x)$ $=5040$	$\sqrt{\left(\frac{2}{4a}\right)^3\frac{0!}{8(7!)^3}}e^{-\frac{r}{4a}}\left(\frac{2r}{4a}\right)^3 L_0^7\left(\frac{2r}{4a}\right)$ $=\frac{1}{768\sqrt{35}}a^{-\frac{3}{2}}\left(\frac{r}{a}\right)^3 e^{-\frac{r}{4a}}$