# Time-optimal generation of unitary matrices using a finite control set

Clarice D. Aiello

May 30, 2013

This quantum control example is motivated by the experimental need to synthesize unitary matrices in $SU(2)$ in optimal time, given an explicit and finite control set generating the whole space, and an admissible error.

A general unitary $U_s$ ('state') $\in SU(2)$ is represented up to a global phase by the complex numbers $a$ and $b$,

$$U_s = \begin{pmatrix} a & b \\ -b^\star & a^\star \end{pmatrix} , \tag{1}$$

with $|a|^2 + |b|^2 = 1$.

The objective is to steer $U_s$ from the initial unitary state represented by the identity matrix $\mathbb{1}$ to a given $U_g$ ('goal') via a sequence $U_{seq}$ of $n$ time-optimal controls $U_{c_m}(t_m) \in \mathsf{U}$, applied each for a time $t_m$,

$$U_g \sim U_{seq} = U_{c_n}(t_n) \cdot U_{c_{n-1}}(t_{n-1}) \cdot U_{c_{n-2}}(t_{n-2}) \cdot \ldots \cdot U_{c_2}(t_2) \cdot U_{c_1}(t_1) \cdot \mathbb{1} \ . \tag{2}$$

The control sequence is such that the total rotation angle, $\sum_{m=1}^{n} t_m$ , is minimal. Moreover, the fitness function that checks whether $U_{seq}$ is close enough to $U_g$ is called 'trace fidelity' and given by $\phi(U_{seq}, U_g) \equiv |\mathrm{Tr}(U_{seq}^\dagger U_g)|/2$. Experimentally acceptable fidelities are on the order of $\phi(U_{seq}, U_g) \sim 99\%$.

The available controls $U_{c_m}(t_m)$ in the set $\mathsf{U} = \{Z(\pm t_m), W(\pm t_m)\}$ are unitaries inducing rotations of angle $\pm t_m$ around two non-parallel axes. $\mathsf{U}$ can be shown to generate any $U_s \in SU(2)$.

Explicitly, the elements of $\mathbf{U}$ are:

$$Z(t) = \begin{pmatrix} e^{-\frac{it}{2}} & 0 \\ 0 & e^{\frac{it}{2}} \end{pmatrix} , \tag{3}$$

$$W(t) = \begin{pmatrix} \cos(t/2) - i\cos\alpha\sin(t/2) & -\sin(t/2)\sin\alpha \\ \sin(t/2)\sin\alpha & \cos(t/2) + i\cos\alpha\sin(t/2) \end{pmatrix} , \tag{4}$$

with a given $\alpha \in ]0, \pi[$. Experimentally, angular rotations cannot be arbitrary; an angular resolution of $\sim \pi/10$ is considered almost challenging.