

# Generalized linear Models

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## Generalized linear Models

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### Definition

Normal  
distribution;  
the linear  
model

Binomial  
distribution;  
the logistic  
regression  
model

Poisson  
log-linear  
model

Other  
examples

Limited Range  
of observations

Definition

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A generalized linear model is characterized by the following 3 points:

- 1 The distribution is a distribution from **the exponential family**.
- 2 There is a linear part of the model called the **linear predictor**.
- 3 There is a function that relates the mean of the distribution with the linear part. This function is called **the link function**.

Let's, for the ease of illustration, take the linear part as  $\alpha + \beta x$  where  $x$  is an group indicator variable. Some special cases of a generalized linear model are

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# Normal distribution; the linear model

# Normal distribution: the linear model

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This distribution is used for continuous data. The probability of observing an observation  $y_i$

$$f(y_i) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left[ -\frac{(y_i - \mu)^2}{2\sigma^2} \right]$$

and so the log-likelihood is:

$$l(\mu, \sigma) = \sum_{i=1}^n \left[ \ln(\sigma\sqrt{2\pi}) - \frac{(y_i - \mu)^2}{2\sigma^2} \right]$$

The mean of this distribution is  $\mu$  (population mean) and the variance is  $\sigma^2$ .

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The link function is the identity:  $\mu = \alpha + \beta x$ .

$\alpha$  is the mean of the group not indicated and  $\beta$  is the difference in group means.

This is the case because the linear model is for the mean.

As can be seen from the formula for  $l(\mu, \sigma)$  the log-likelihood is maximized for  $\mu$  if  $\sum(\frac{(y_i - \mu)^2}{2\sigma^2})$  is minimized.

This is equivalent to the least-squares method.

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Binomial distribution; the logistic regression  
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# Binomial distribution; the logistic regression model

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The mean of this distribution is  $\pi$  (population fraction) and the variance is  $\pi(1 - \pi)$ . The log-likelihood is:

$$l(\pi) = \ln[L(\pi)] = \sum_{i=1}^{98} [y_i \ln(\pi) + (1 - y_i) \ln(1 - \pi)]$$

The link function is the log-odds (logit):  $\log\left(\frac{\pi}{1-\pi}\right) = \alpha + \beta x$ .

$\alpha$  is the log-odds of the group not indicated and  $\beta$  is the difference in log-odds between the groups ( which is the log odds-ratio). This is the case because the linear model is for the log-odds.



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# Poisson log-linear model

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The mean of this distribution is  $\mu$  (the population mean of the counts) and the variance is  $\mu$ . The log-likelihood is:

$$l(\mu) = \sum_{i=1}^{24} [-\mu + y_i \ln(\mu) - \ln(y_i!)]$$

The link function is the log:  $\log(\mu) = \alpha + \beta x$ .

$\alpha$  log of the mean of the group not indicated and  $\beta$  is the difference in log-means between the groups ( which is the log-ratio of the means).

This is the case because the linear model is for the log of the means.

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# Other examples

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Other examples of generalized models are:

the exponential distribution with a link function  $\frac{1}{\mu} = \alpha + \beta x$ ;

The gamma distribution with a link function  $\frac{1}{\mu} = \alpha + \beta x$  and

the inverse-gauss distribution with a link function  $\frac{1}{\mu^2} = \alpha + \beta x$

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# Limited Range of observations

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From generalized linear models one can observe that in many cases, as soon as the range of the observations is limited (e.g. there are only zero's and one's or the observations are larger than zero ) then:

- 1 the mean and the variance are related
- 2 the link function is not the identity

The last point means that the difference between groups is not a difference in means but a difference in transformed means.

This also shows that if the range of the observations is limited, the normal distribution is not appropriate except as an approximation.