Constructing inverse diagrams in (internal models of) HoTT

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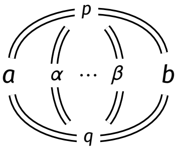
Background

In plain HoTT, all types A are ∞-groupoids.

- Objects are elements $a : A$
- $\hom(x, y)$ for n-cells x and y are iterated identity types



1-cells $\quad p, q : \hom(a, b) \equiv (a =_A b)$,

2-cells $\quad \alpha, \beta : \hom(p, q) \equiv (p =_{a=b} q)$,

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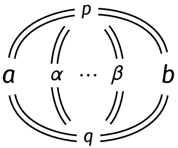

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$$
\begin{tikzcd}
a \arrow[rr, bend left=60, equal, "p"{name=P}] \arrow[rr, bend right=60, equal, "q"'{name=Q}] & & b
\arrow[from=P, to=Q, phantom, "\alpha \ \cdots \ \beta"]
\end{tikzcd}
$$

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Question

How do we talk about $(\infty, 1)$-categories in plain homotopy type theory

…in a way that exploits HoTT's inherent higher categorical structure?

Simplicial objects in type theory?

Some models of $(\infty, 1)$-categories start with simplicial objects in some $\mathcal{C}$ ($= \mathit{Set}, \hat{\Delta}, \ldots$)
$\Longrightarrow$ Look for

1. a category $\mathcal{C}$ of type theoretic data +
2. a construction *defined in HoTT* that can *externally* be seen to give simplicial objects in $\mathcal{C}$.

Straightforward first try for (1): universe type $\mathcal{U}$ is a 1-category

- Objects: closed $\mathcal{U}$-small types
- $\mathrm{hom}(A, B) :=$ function type $A \longrightarrow B$

Might call $\mathcal{U}$-valued Δ-presheaves *simplicial types*.

Can we achieve (2)? What remains is to define $\mathcal{U}$-valued Δ-presheaves in HoTT.

Semisimplicial types

First, simplify by forgetting degeneracy maps: ask for the data of $\mathcal{U}$-valued Δ_+-presheaves, aka *semisimplicial types*.

Standard encoding of a Δ_+-presheaf $\mathcal{S}$ in $\mathcal{U}$:

$$
\begin{aligned}
& A_0 : \mathcal{U}, \quad A_1 : A_0 \to A_0 \to \mathcal{U}, \\
& A_2 : (x, y, z : A_0) \to A_1(x, y) \to A_1(x, z) \to A_1(y, z) \to \mathcal{U}, \\
& A_3 : (x, y, z, w : A_0) \to \\
& \qquad (e_{x,y} : A_1(x, y)) \to \cdots \to (e_{z,w} : A_1(z, w)) \to \\
& \qquad (f_{x,y,z} : A_2(x, y, z, e_{x,y}, e_{x,z}, e_{y,z})) \to \cdots \to (f_{y,z,w} : A_2(y, \ldots, e_{z,w})) \to \mathcal{U}, \\
& \ldots
\end{aligned}
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$\mathcal{S}_n$ is the *total space* of A_n. Face maps are given by projecting out subtuples.

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$$A_0 : \mathcal{U}, \quad \mathcal{S}_0 = A_0, \quad A_1 : A_0 \to A_0 \to \mathcal{U}, \quad \mathcal{S}_1 = (x, y : A_0) \times A_1(x, y)$$

$$A_2 : (x, y, z : A_0) \to A_1(x, y) \to A_1(x, z) \to A_1(y, z) \to \mathcal{U},$$

$$\mathcal{S}_2 = (x, y, z : A_0) \times (e_{x,y} : A_1(x, y)) \times (e_{x,z} : A_1(x, z)) \times (e_{y,z} : A_1(y, z)) \times A_2(x, y, z, e_{x,y}, e_{x,z}, e_{y,z}),$$

$$\begin{aligned} A_3 :\ & (x, y, z, w : A_0) \to \\ & (e_{x,y} : A_1(x, y)) \to \cdots \to (e_{z,w} : A_1(z, w)) \to \\ & (f_{x,y,z} : A_2(x, y, z, e_{x,y}, e_{x,z}, e_{y,z})) \to \cdots \to (f_{y,z,w} : A_2(y, \ldots, e_{z,w})) \to \mathcal{U}, \quad \ldots \end{aligned}$$

$\mathcal{S}_n$ is the *total space* of A_n. Face maps are given by projecting out subtuples.

Semisimplicial types

Some observations:

- The type of each A_n depends on $A_0, \ldots, A_{n-1}$.
- For given *fixed* n, can define the type of tuples $(A_0, \ldots, A_n)$, e.g. fixing $n = 2$,

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record SST₂ : Type₁ where
  A₀ : Type₀
  A₁ : A₀ → A₀ → Type₀
  A₂ : (x y z : A₀) → A₁ x y → A₁ x z → A₁ y z → Type₀
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Define *in HoTT* a function $\mathrm{SST} : \mathbb{N} \longrightarrow \mathcal{U}^+$ so that $\mathrm{SST}(n)$ is the type of sequences $(A_0, \ldots, A_n)$?

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More generally, is the type of semisimplicial types definable in (plain) HoTT?

Obstruction: coherence problem because equality in HoTT is structure, not property.

Semisimplicial types

Difficulty: haven't managed to internalize the matching objects of semisimplicial types.

- For nice enough $\mathcal{C}$, can construct "Reedy fibrant" $\mathcal{C}$-valued diagrams indexed by inverse $\mathcal{I}$.
- Construction by well founded induction, using certain limits—the *matching objects*—at each stage.
- Matching objects give a functor *M* from (a subcategory of) *CoSv*$(\mathcal{I})$ to $\mathcal{C}$.
- Coherence problem arises from failure of *M* to be strict for $\mathcal{C} = \mathcal{U}$.

Inverse diagrams in internal models of HoTT

Current work:

- Formulate models of type theory inside HoTT.
- Construct semisimplicial types and more general inverse diagrams in the model.

Control the height of the tower of coherence conditions by truncating the internal model.

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Goals:

- Investigate, inside HoTT, minimal models in which coherence issues arise.
- Determine minimal sufficient conditions for the model—and by extension, type theory—to support semisimplicial types.
- Develop constructions to test the theory of higher models of type theory.
- Bonus 🎁—provide main part of proof relating open problems in HoTT.

Technical outline

- Internal model: Categories with families
- Diagrams:
  1. The index categories we use
  2. Matching objects
  3. Constructing diagrams in internal CwFs

Categories with families

Common categorical model of type theory:

Definition

A *category with families* is a category Con together with

- a choice of terminal object $\mathbb{1} \in Con$
- $Ty : Con^{op} \longrightarrow Set$
- $Tm : \big(el(Ty)\big)^{op} \longrightarrow Set$
- For every $(\Gamma, A) \in el(Ty)$, a choice of terminal object in

$$el_{Con/\Gamma}\Big[Tm\big(dom(\cdot), Ty(\cdot)(A)\big)\Big].$$

In particular, have context extension $\Gamma \triangleright A$ and substitution on types $A[\sigma]$ and terms $a[\sigma]$.

Countable locally finite inverse categories

For a category $\mathcal{I}$, define $j \prec i$ iff there's a morphism $j \leftarrow i$.
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Definition

An inverse category $\mathcal{I}$ is *countable and locally finite* if

1. there is $\#: Ob(\mathcal{I}) \cong \mathbb{N}$ such that $\#j < \#i$ whenever $j \prec i$,
2. for $i, j \in Ob(\mathcal{I})$, $\hom(i, j)$ is finite and totally ordered,
3. $\hom(i, i) \cong Fin(1)$ for all i.

Write $\mathrm{idx}: \hom(i, j) \cong Fin(|\hom(i, j)|)$ for the canonical order isomorphism.

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Examples: Δ_+ (also $\square_+$, Ω_+)

We will refer to objects $i \in Ob(\mathcal{I})$ as natural numbers.

0 is always $\prec$-minimal.

Matching objects

Let $\mathcal{I}$ be inverse and $i \in Ob(\mathcal{I})$.

$$\begin{aligned} \mathcal{I}_{\prec i},\ \mathcal{I}_{\preceq i} &\quad \text{— full subcategories on objects } j \prec i \text{ and } j \preceq i\text{, resp.} \\ {}^{i/\!\!/}\mathcal{I} &\quad \text{— full subcategory on } Ob({}^{i/}\mathcal{I}) - \{id_i\}. \end{aligned}$$

The codomain forgetful functor U projects from ${}^{i/\!\!/}\mathcal{I}$ to $\mathcal{I}_{\prec i}$.

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Definition

Let $i \in Ob(\mathcal{I})$ and $\mathcal{D} : \mathcal{I}_{\prec i} \longrightarrow \mathcal{C}$. The *matching object* M_i of $\mathcal{D}$ is the limit

$$\lim_{{}^{i/\!\!/}\mathcal{I}} (\mathcal{D} \circ U).$$

Matching objects

Definition

Let $i \in Ob(\mathcal{I})$ and $\mathcal{D} : \mathcal{I}_{<i} \longrightarrow \mathcal{C}$. The *matching object* M_i of $\mathcal{D}$ is given by $M_i = \lim_{i /\!\!/ \mathcal{I}} (\mathcal{D} \circ U)$.

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2. Assume $\mathcal{D} : \mathcal{I}_{\prec i} \longrightarrow \mathcal{C}$. To extend $\mathcal{D}$ to $\mathcal{I}_{\preceq i}$, simply give $\mathcal{D}_i \in \mathcal{C}$ and an $f : \mathcal{D}_i \longrightarrow M_i$.

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Remark

When $\mathcal{C} = \mathcal{U}^+$, giving $(\mathcal{D}_i, f)$ is equivalent to giving a morphism $A_i : M_i \longrightarrow \mathcal{U}$.

If M_i were to exist in $\mathcal{U}$, would get the components for semisimplicial types in the case $\mathcal{I} = \Delta_+^{op}$.

Refining M_i with linear cosieves

Definition

For $h < i \in \mathcal{I}$ and $t \leq |\hom(i, h)|$, define the *linear cosieve of shape* (i, h, t) by

$$S_{i,h,t} := \left(\bigcup_{k<h} \hom(i, k) \right) \cup \left\{ f \in \hom(i, h) \mid \operatorname{idx}(f) < t \right\}.$$

Define ${}^{i,h,t/}\mathcal{I}$ to be the full subcategory of ${}^{i/}\mathcal{I}$ on $S_{i,h,t}$.

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Get the following filtration of ${}^{i/\!/}\mathcal{I}$:

$$\varnothing = {}^{i,0,0/}\mathcal{I} \hookrightarrow {}^{i,0,1/}\mathcal{I} \hookrightarrow \cdots \hookrightarrow {}^{i,h,|\mathrm{hom}(i,h)|/}\mathcal{I} = {}^{i,h+1,0/}\mathcal{I} \hookrightarrow \cdots$$
$$\hookrightarrow {}^{i,i-1,|\mathrm{hom}(i,i-1)|/}\mathcal{I} = {}^{i/\!/}\mathcal{I}$$

Computing matching objects

From

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$$\hookrightarrow {}^{i,i-1,|\hom(i,i-1)|/}\mathcal{I} = {}^{i/\!\!/}\mathcal{I}$$

we will recursively compute a sequence of *partial matching objects*

$$\mathbb{1} = M_{i,0,0} \rightsquigarrow M_{i,0,1} \rightsquigarrow \cdots \rightsquigarrow M_{i,h,|\hom(i,h)|} = M_{i,h+1,0} \rightsquigarrow \cdots$$
$$\rightsquigarrow M_{i,i-1,|\hom(i,i-1)|} = M_i,$$

where $M_{i,h,t} \approx \lim_{{}^{i,h,t/\!\!/}\mathcal{I}}(\mathscr{D} \circ U)$.

Constructing diagrams in internal CwFs

From now on,

- Take $\mathcal{C} = \mathit{Con}$ of an internal CwF equipped with Π-types and a universe type V
- Assume $\mathcal{I}$ to be inverse, countable and locally finite (for intuition, take $\mathcal{I} = \Delta_+^{op}$)
- Work in HoTT (informally)

  Note: Categorical terms will still have the HoTT equivalent of their usual meanings.

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Warning: Actual construction is a large mutually recursive definition with seven main components, formalized in Agda for precision.

This talk: main ideas for key components.

Constructing diagrams in internal CwFs

"Main" component $SCT : \mathbb{N} \longrightarrow Con$.

$$\begin{aligned} SCT(0) &:\equiv \mathbb{1} \\ SCT(1) &:\equiv SCT(0) \triangleright V \\ SCT(n+1) &:\equiv SCT(n) \triangleright \Pi^{*}_{n,(n,n-1,|\mathrm{hom}(n,n-1)|)} V \end{aligned}$$

where

- $\Pi^{*}_{n,(i,h,t)} : Ty(M_{n,(i,h,t)}) \longrightarrow Ty(SCT(n))$ is a HoTT function.

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- $M_{n,(i,h,t)} : Con$ is the context $SCT(n)$ extended with a telescope of components of the *(i,h,t)-partial matching object.*

  e.g. for $\mathcal{I} = \Delta_{+}^{op}$,

$$M_{n,(1,0,2)} \equiv SCT(n) \triangleright A_0 \triangleright A_0$$

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  $$M_{n,(1,0,2)} \equiv SCT(n) \triangleright A_0 \triangleright A_0$$
- $\Pi^*_{n,(i,h,t)}$ iteratedly applies the isomorphism $Ty(\Gamma \triangleright A) \cong Ty(\Gamma)$ given by Π-introduction.
  e.g. for $\mathcal{I} = \Delta_+^{op}$,
  $$\Pi^*_{n,(1,0,2)} V \;\equiv\; \Pi^*_{n,(1,0,1)}(\Pi_{A_0} V) \;\equiv\; \Pi_{A_0}\Pi_{A_0} V$$

Constructing diagrams in internal CwFs

From earlier: want to recursively compute partial matching objects $M_{i,h,t}$

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For technical reasons, also index over n.

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For technical reasons, also index over n.
First two cases easy:

$$\begin{aligned} M_{n,(i,0,0)} &:\equiv SCT(n), \\ M_{n,(i,h+1,0)} &:\equiv M_{n,(i,h,|\hom(i,h)|)}. \end{aligned}$$

Computing matching objects

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Definition

Let S be a cosieve under i in $\mathcal{I}$, and $f \in \hom(i, j)$.
The *restriction* $(S \cdot f)$ *of* S *along* f is the cosieve under j given by

$$S \cdot f := \left\{\alpha : j \longrightarrow k \mid k \in Ob(I),\ \alpha \circ f \in S\right\}.$$

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Definition

A countable and locally finite inverse $\mathcal{I}$ is *well oriented* if for all $f \in \mathrm{hom}(x,y)$ and $g, h \in \mathrm{hom}(y,z)$,

$$g < h \implies g \circ f \leq h \circ f.$$

Examples: Δ_+, $\square_+$ (Ω_+?...)

Partial matching object as functor

Lemma

In a well oriented inverse category, the restriction of a linear cosieve $S_{i,h,t}$ along any $f \in \hom(i,j)$ is a linear cosieve.

$$S_{i,h,t} \xrightarrow{f} S \cdot f = S_{j,h',t'}$$

Thus linear cosieves organize into a full subcategory $\mathit{LCoSv}(\mathcal{I})$ of $\mathit{CoSv}(\mathcal{I})$.

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Key Idea

View partial matching objects as the object part of a weak functorial action $LCoSv(\mathcal{I}) \longrightarrow Con$, and simultaneously define the action on morphisms

$$\vec{M}_{n,(i,h,t)}(f) : Sub(M_{n,(i,h,t)}\ ,\ M_{n,(i,h,t)\cdot f})$$

(definition omitted in this talk)

Constructing diagrams in internal CwFs

Now we can define

$$M_{n,(i,h,t+1)} :\equiv M_{n,(i,h,t)} \triangleright A_h\left[\vec{M}_{n,(i,h,t)}(\bar{t})\right]$$

where

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- A_h is constructed earlier (since $h < i$) by another component of the mutually recursive definition

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Lemma

Let $\mathcal{I}$ be well oriented, $S_{i,h,t}$ be a linear sieve, $f \in \mathrm{hom}(i,j)$ and $j \leq h$. Then

$$S_{i,h,t} \cdot f = S_{j,j-1,|\mathrm{hom}(j,j-1)|}.$$

Constructing diagrams in internal CwFs

Summary:

- Have $SCT : \mathbb{N} \longrightarrow Con$.
  When $\mathcal{I} = \Delta_{+}^{op}$, gives the components of semisimplicial types in the internal CwF.

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- Have $SCT : \mathbb{N} \longrightarrow Con$.
  When $\mathcal{I} = \Delta_+^{op}$, gives the components of semisimplicial types in the internal CwF.
- Goes via a large mutually recursive definition, with all components very closely intertwined.

Constructing diagrams in internal CwFs

Elided in this talk:

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- Dealing with explicit weakenings of the internal CwF.

Coda—Relating open problems

Open Question "Does HoTT interpret itself?"

Define a type *Syn* encoding the syntax of HoTT, plus interpretation function

$$[\![\cdot]\!] : \mathit{Syn} \to \mathcal{U}$$

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Equivalently, find a notion of "model of type theory" such that

1. The syntax is initial, and
2. The "standard model" given by a universe type is an instance?

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$\Longrightarrow$ Just precompose with *SST* to get

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Thanks!