Numerical Study of a Parametrically Driven Double-Well Pendulum

John Blue *(Davidson College)*
Daniel P. Lathrop (UMD)

Summary

- We add a repulsive/unstable fixed point Gaussian to the potential energy term of a parametrically driven pendulum (to model the placement of magnet directly below the pendulum) and study the resulting motion of the new system.
- To study the dynamics of the system, we used numerical integration to solve the equations of motion, and techniques such as Poincaré sections, bifurcations diagrams, and Lyapunov exponents to analyze the resulting motion.
- Findings:
  - Up to 10 multiple coexisting attractors
  - Multiple, sometimes simultaneous period-doubling cascades leading to chaos and a small strange attractor
  - A crisis at larger driving amplitudes leading to a large strange attractor
  - Evidence of a riddled basin of attraction

Background

- The parametrically driven pendulum (a planar pendulum with a vertically oscillating pivot) is a mechanical system that exhibits chaotic motion for certain parameter values. When the driving frequency is close to twice the natural frequency, studies have shown the system exhibits stable oscillatory and rotational motion and takes a period doubling route to chaos[1][2].
- We added a repulsive Gaussian term to the potential energy term of the Lagrangian to model a magnet directly below the pendulum (see fig. 1), yielding a double well potential energy function (see fig. 2). For no driving, this changes what was previously a stable fixed point to a saddle point.
- The goal of the study was to numerically analyze the modified system, searching for different bifurcations, routes to chaos, and chaotic regions.

Equations of Motion

Suppose the pivot is oscillating between A and $-A$ with frequency ω. We add to the usual potential an additional term approximating the repulsive magnet

$$U_m = be^{-\frac{1}{2}r^2}$$

where b controls the strength of the magnet and r is the distance from the magnet to the pendulum. This yields the following equation of motion:

$$\frac{d\theta}{dt} = \dot{\theta}$$

$$\frac{d\dot{\theta}}{dt} = -k\dot{\theta} - \frac{\sin\theta}{l}\left[g - A\omega^2\cos(\omega t) + \frac{b}{m}(A\cos(\omega t) - d)e^{f(\theta,t)}\right]$$

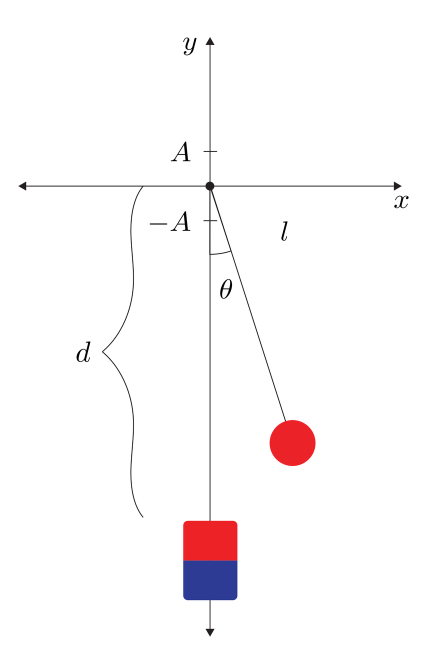

Fig 1. Diagram of a parametrically driven pendulum with external forcing.

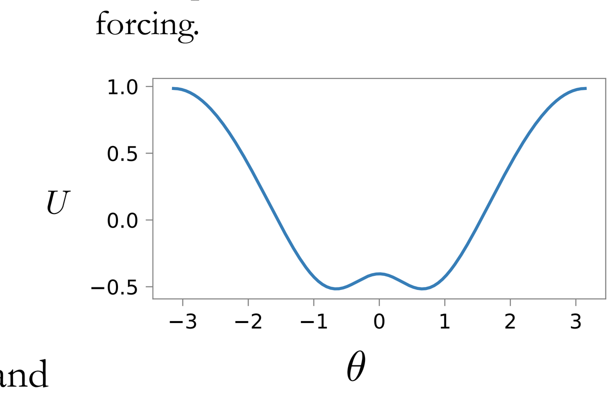

Fig 2. Potential energy function of system with A=0.01 and t = 0.

Where k is the coefficient of friction, and

$$f(\theta,t) = -\frac{1}{2}(l^2 + d^2 + A^2\cos^2(\omega t) + 2Al\cos(\omega t)\cos\theta - 2Ad\cos(\omega t) - 2dl\cos\theta)$$

Note that the equations have an odd symmetry in that if $(\theta(t), \dot{\theta}(t))$ is a solution, then so is $(-\theta(t), -\dot{\theta}(t))$.

Methods

- Numerical solving: Equations were solved using Runge-Kutta Fehlburg 7(8) with adaptive step size and the following parameter values (note that the given l value yields a natural frequency that's half the driving frequency).

$$l = \frac{g}{\pi^2} \approx 0.994,\ \omega = 2\pi,\ d = 4,\ b = 50,\ m = 0.1,\ k = 0.2$$

- Bifurcation diagram: Bifurcation diagrams show states reached by the system once it has reached its long-term behavior. Our diagrams show the angular velocity values when $\cos(\omega t) = 1$ (i.e., the pivot is at its maximum point) for pendulums with 500 different initial conditions in the range $(\theta, \dot{\theta}) \in [-\pi, \pi] \times [-3, 3]$, with A varying as the bifurcation parameter.
- Poincaré section: Poincaré sections show a single plane of the full three-dimensional subspace and are useful for visualizing periodic orbits and chaotic attractors.
- Lyapunov Exponent: Lyapunov exponents measure the rate of separation between two solutions in phase space. For a system with the Lyapunov exponent λ, and two orbits in phase space with an initial separation vector $\delta\mathbf{x_0}$, then the separation grows roughly as $|\delta\mathbf{x}| \approx e^{\lambda}|\delta\mathbf{x_0}|$.

Results

Multi-Stability

The first notable feature about the system is its ability to sustain multiple stable limit cycles. Up to 10 stable period limit cycles have been observed to exist (at A = 0.03349388), and every observed strange attractor has coexisted with stable limit cycles. Note that some individual orbits (such as the period 4) maintain the odd symmetry about the origin, whereas others (such as the period 1 orbits) break the symmetry.

Five Stable Limit Cycles at A=0.028

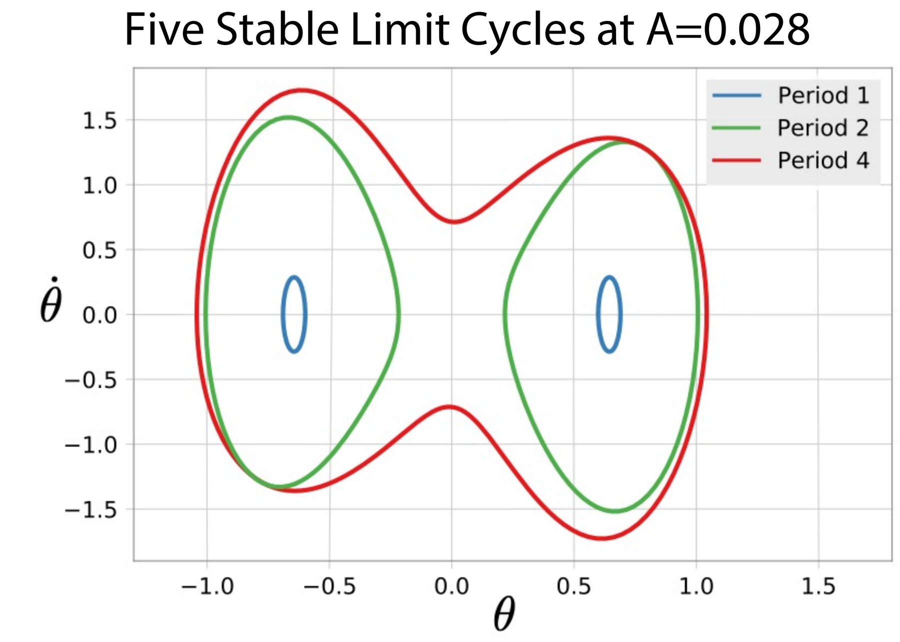

Fig 3. A 2-D projection of the 5 stable coexisting orbits in phase space.

Coexistance of Symmetric Strange Attractors and Five Limit Cycles

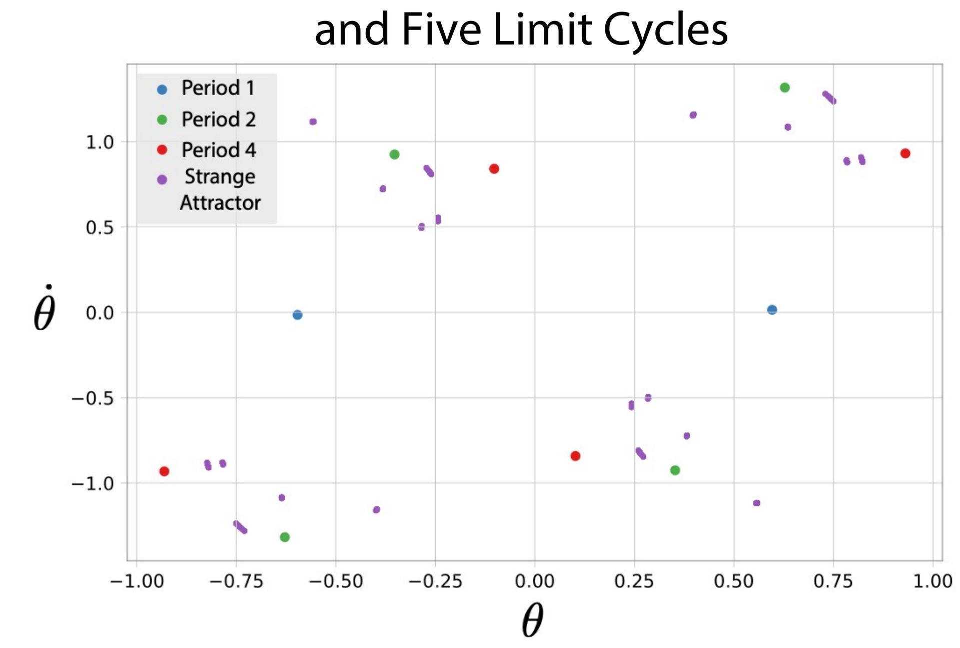

Fig 4. Poincaré section showing 5 stable limit cycles and two strange attractors. The maximum Lyapunov exponent for an orbit in a strange attractor is 0.034, indicating the motion is chaotic.

Period Doubling Cascades

Bifurcation Diagram

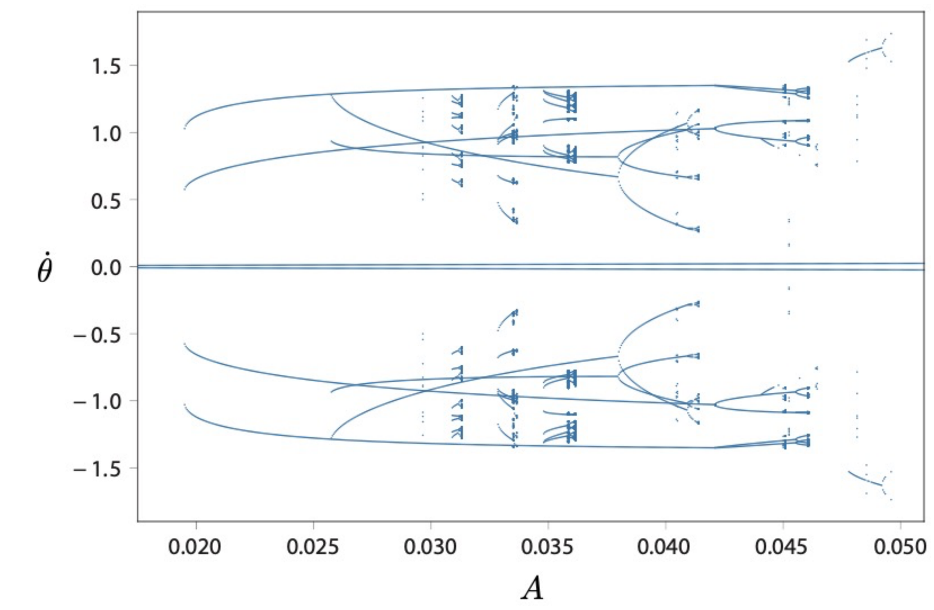

Figure 5. A bifurcation diagram of the system at low driving amplitudes. From the diagram it is possible to see saddle-node bifurcations leading to 2 period-2 orbits at roughly A=0.0195, and 1 period-4 orbit at A= 0.0258, as well as period doubling cascades starting at roughly A=0.0296, 0.0308, 0.0328, 0.0347, 0.0374, and 0.0418.

Bifurcation Diagram

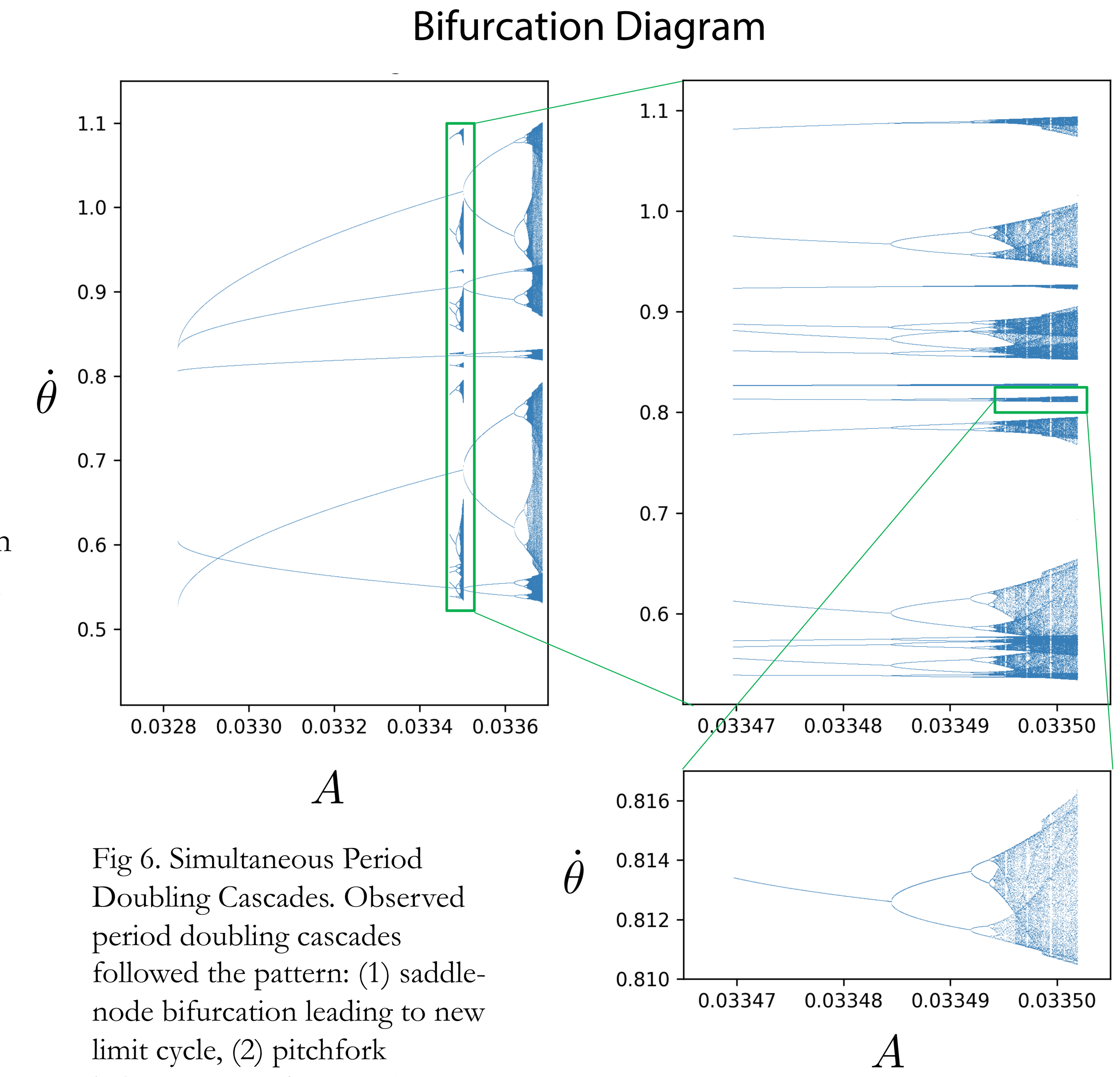

Fig 6. Simultaneous Period Doubling Cascades. Observed period doubling cascades followed the pattern: (1) saddle-node bifurcation leading to new limit cycle, (2) pitchfork bifurcation yielding two limit cycles, and (3) two period doubling cascades leading to a "small" strange attractor.

Riddled Basin of Attraction and Large Strange Attractor

Basin of Attraction
A=0.09

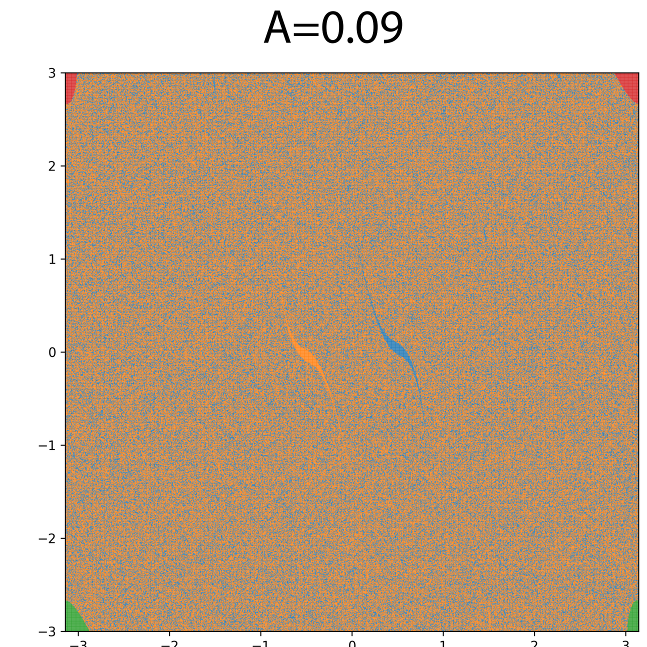

Fig 7. Basin of attraction. 1000 x 1000 points plotted. Orange and blue points end up in oscillatory period 1 motion; green and red points end up in rotational period 1 motion. A "riddled" basin of attraction is one where each point in an attractor's basin is arbitrarily close to a point in another attractor's basin.

Strange Attractor
A=0.0975

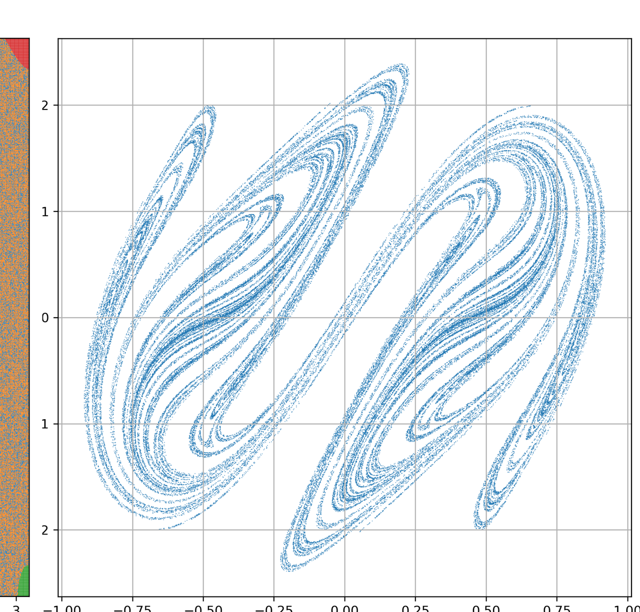

Fig 8. Poincaré section showing the strange attractor present at A=0.0975. 100,000 points plotted. The Lyapunov exponents are 0.588, -0.788, and 0. The attractor maintains the odd symmetry present in the equations of motions.

Conclusions

The standard parametrically driven pendulum has served as a simple mechanical model for studying chaos. With the addition of a repulsive Gaussian to the potential energy term, our system exhibits new behavior including many coexisting stable limit cycles and many (sometimes simultaneous) period-doubling cascades leading to chaos. Possible future research could include an analysis of this system for a different set of parameter values, higher driving amplitudes, and studying other other well-known systems after changing fixed points to saddle points.

References

[1] R. W. Leven, B. Pompe, C. Wilke, and B. P. Koch, Physica D **16**, (1985).
[2] J. McLaughlin, Journal of Statistical Physics **24**, (1981).

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