Problem 1) $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = \begin{pmatrix} -11 & 9 \\ 9 & -11 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \vec{y}' = A\vec{y}$, $\vec{y}_0 = \begin{pmatrix} 1.0 \\ 1.2 \end{pmatrix}$

Find eigen vals & eigen vectors of A

$\det(A-\lambda I) = 0$

$= \begin{vmatrix} -11-\lambda & 9 \\ 9 & -11-\lambda \end{vmatrix} = (-11-\lambda)(-11-\lambda) - 81$

$= 121 + 22\lambda + \lambda^2 - 81$

$= \lambda^2 + 22\lambda + 40$

$= (\lambda+20)(\lambda+2)$ ← by inspection.

$\Rightarrow \lambda_1 = -20, \ \lambda_2 = -2$

$\vec{v}_i: A\vec{v}_i = \lambda\vec{v}_i$

$\Rightarrow (A - I\lambda_i)\vec{v}_i = \vec{0}$

$\Rightarrow (A - I\lambda_1)\vec{v}_1 = \vec{0}$

$= \begin{pmatrix} 9 & 9 \\ 9 & 9 \end{pmatrix}\begin{pmatrix} v_{1,1} \\ v_{1,2} \end{pmatrix} = \begin{pmatrix} 9v_{1,1} + 9v_{1,2} \\ 9v_{1,1} + 9v_{1,2} \end{pmatrix} \Rightarrow v_{1,1} = -v_{1,2}$

Let $v_{1,1} = 1 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\vec{v}_2 = (A - I\lambda_2)\vec{v}_2 = \vec{0}$

$= \begin{pmatrix} -9 & 9 \\ 9 & -9 \end{pmatrix}\begin{pmatrix} v_{2,1} \\ v_{2,2} \end{pmatrix} = \begin{pmatrix} -9v_{2,1} + 9v_{2,2} \\ 9v_{2,1} - 9v_{2,2} \end{pmatrix} \Rightarrow v_{2,1} = v_{2,2}$

Let $v_{2,1} = 1 \Rightarrow \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{v}_1$ + $\vec{v}_2$ are L.I, A is diagonalizable since $P = [\vec{v}_1 \ \vec{v}_2]$ is invertable

a)

the solution to $\vec{y}$ is then $\vec{y}(t) = \alpha_i(0)e^{\lambda_i t}\vec{v}_i$

$\vec{y}_0 = \begin{pmatrix} 1.0 \\ 1.2 \end{pmatrix} = \alpha_1(0)e^0\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \alpha_2(0)e^0\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha_1(0) + \alpha_2(0) \\ -\alpha_1(0) + \alpha_2(0) \end{pmatrix}$

$\alpha_1(0) = 1.0 - \alpha_2(0) \Rightarrow 1.2 = -1.0 + \alpha_2(0) + \alpha_2(0) \Rightarrow 2.2 = 2\alpha_2(0) \Rightarrow \alpha_2(0) = 1.1$

$\Rightarrow \alpha_1(0) = -0.1$

$\vec{y} = -0.1e^{-20t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1.1e^{-2t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \boxed{y_1 = -0.1e^{-20t} + 1.1e^{-2t}}$

Forward Euler: $\vec{u}_{n+1} = \vec{u}_n + h\vec{f}(\vec{u}_n) = \vec{u}_n + hA\vec{u}_n = (I + hA)\vec{u}_n \Rightarrow \underline{\vec{u}_n = (I+hA)^n\vec{u}_0}$

since A is diagonalizable, $(I+hA)^n\vec{v}_i = (1+h\lambda_i)^n\vec{v}_i$

and $\vec{u}_0 = \vec{y}_0 = \alpha_1(0)\vec{v}_1 + \alpha_2(0)\vec{v}_2$

$\vec{u}_n = \alpha_1(0)(1+h\lambda_1)^n\vec{v}_1 + \alpha_2(0)(1+h\lambda_2)^n\vec{v}_2 = -0.1(1-20h)^n\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1.1(1-2h)^n\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow \boxed{u_{n,1} = -0.1(1-20h)^n + 1.1(1-2h)^n}$

cont →

Problem 1 cont.

$$\lim_{h\to 0} u_{1,n} = \lim_{h\to 0}\left(-0.1(1-20h)^{\frac{t}{h}} + 1.1(1-2h)^{\frac{t}{h}}\right)$$

$$= \lim_{h\to 0}\left[e^{\ln(-0.1(1-20h)^{t/h})} + e^{\ln(1.1(1-2h)^{t/h})}\right]$$

exp is continuous $\Rightarrow e^{\lim_{h\to 0}\left[\ln(-0.1) + t\frac{\ln(1-20h)}{h}\right]} + e^{\lim_{h\to 0}\left[\ln(1.1) + t\frac{\ln(1-2h)}{h}\right]}$

$\frac{1}{1-20h}\cdot 20$

L'hopital $\Rightarrow e^{\lim_{h\to 0}\left[\ln(-0.1) + t\frac{1}{1-20h}\cdot -20 \cdot \frac{1}{1}\right]} + e^{\lim_{h\to 0}\left[\ln(1.1) + t\frac{1}{1-2h}\cdot -2\cdot\frac{1}{1}\right]}$

$$= e^{\ln(-0.1) - 20t} + e^{\ln(1.1) - 2t} = \underline{-0.1e^{-20t} + 1.1e^{-2t} = y_1(t)}$$

<u>Forward Euler converges as $h\to 0$</u>

<u>Backward Euler:</u> $\vec{u}_{n+1} = \vec{u}_n + h\vec{f}(\vec{u}_{n+1}) \Rightarrow \vec{u}_n = (I-hA)\vec{u}_{n+1} \Rightarrow \vec{u}_{n+1} = (I-hA)^{-1}\vec{u}_n$

$\Rightarrow \vec{u}_n = (I-hA)^{-n}\vec{u}_0$

A is diagonalizable, $(I-hA)^{-n}\vec{v}_i = (1-h\lambda_i)^{-n}\vec{v}_i$

$$\vec{u}_n = \alpha_1(0)(1-h\lambda_1)^{-n}\vec{v}_1 + \alpha_2(0)(1-h\lambda_2)^{-n}\vec{v}_2 = -0.1(1+20h)^{-n}\begin{pmatrix}1\\-1\end{pmatrix} + 1.1(1+2h)^{-n}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\Rightarrow \underline{u_{n,1} = -0.1(1+20h)^{-n} + 1.1(1+2h)^{-n}}$$

$$\lim_{h\to 0} u_{n,1} = \lim_{h\to 0}\left[-0.1(1+20h)^{-t/h} + 1.1(1+2h)^{-t/h}\right]$$

$$= \lim_{h\to 0}\left[e^{\ln(-0.1(1+20h)^{-t/h})} + e^{\ln(1.1(1+2h)^{-t/h})}\right]$$

$$\Rightarrow e^{\lim_{h\to 0}\left[\ln(-0.1) - t\frac{\ln(1+20h)}{h}\right]} + e^{\lim_{h\to 0}\left[\ln(1.1) - t\frac{\ln(1+2h)}{h}\right]}$$

$$\Rightarrow e^{\lim_{h\to 0}\left[\ln(-0.1) - t\frac{1}{1+20h}\cdot 20\cdot\frac{1}{1}\right]} + e^{\lim_{h\to 0}\left[\ln(1.1) - t\frac{1}{1+20h}\cdot 20\cdot\frac{1}{1}\right]}$$

$$= e^{\ln(-0.1) - 20t} + e^{\ln(1.1) - 2t} = \underline{-0.1e^{-20t} + 1.1e^{-2t} = y_1(t)}$$

<u>Backward Euler converges as $h\to 0$</u>

b) Stability:

Forward Euler: $|1+h\lambda_i| \le 1 \Rightarrow |1-20h| \le 1$ & $|1-2h| \le 1$

$-1 \le 1-20h \le 1$ & $-1 \le 1-2h \le 1$

$-1 \le 1-20h$ & $-1 \le 1-2h$ $\quad (h>0)$

$+2 \ge +20h$ & $+2 \ge +2h$

$h \le 0.1$ & $h \le 1.0$

So $h = 0.10$, $h = 0.09$ are abs. stable

Backward Euler: $|1-h\lambda_i| \ge 1 \Rightarrow |1+20h| \ge 1$ & $|1+2h| \ge 1$

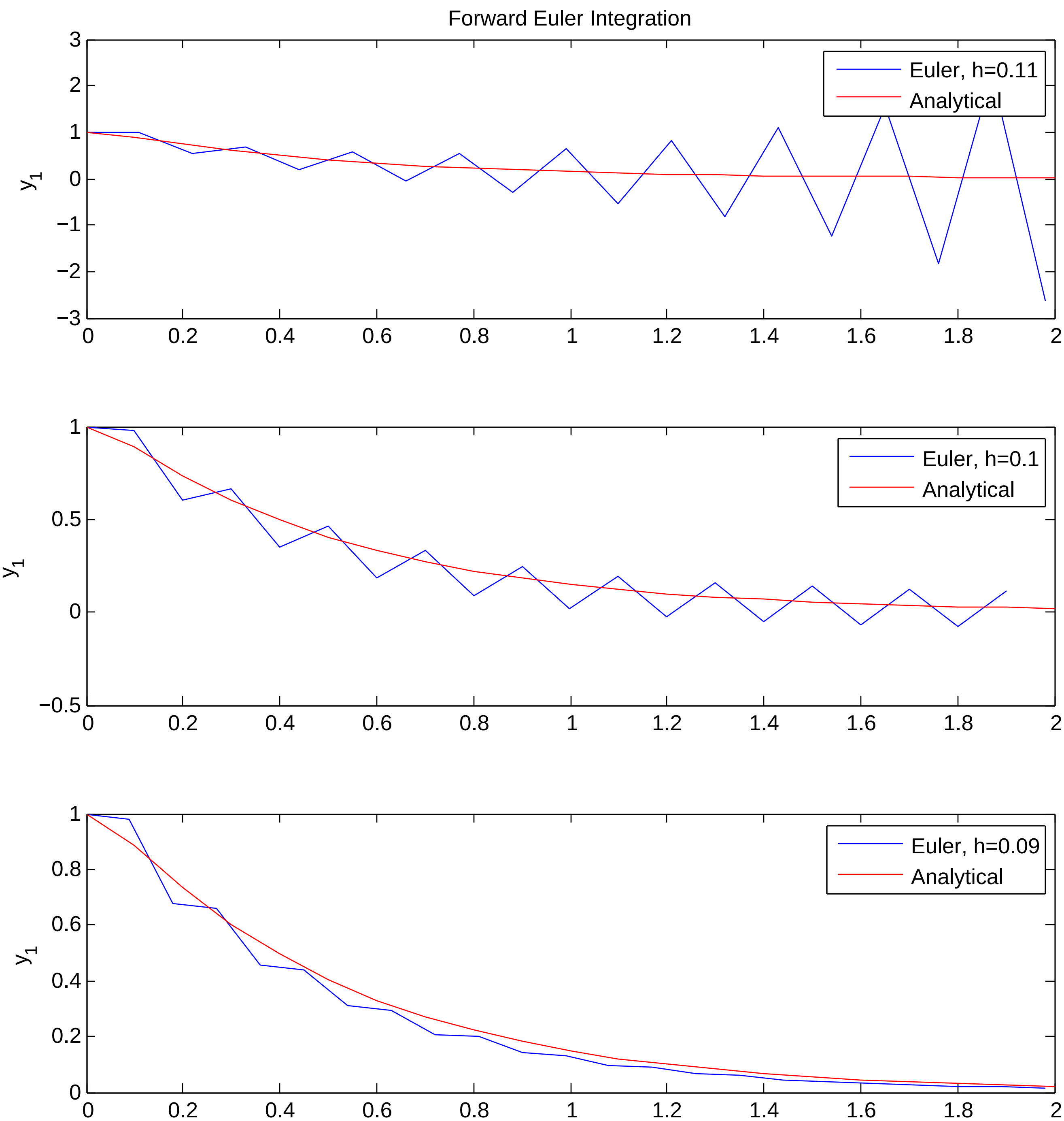
$1+20h \ge 1$ & $1+2h \ge 1$ $\quad (h>0)$

$20h \ge 0$ & $2h \ge 0$

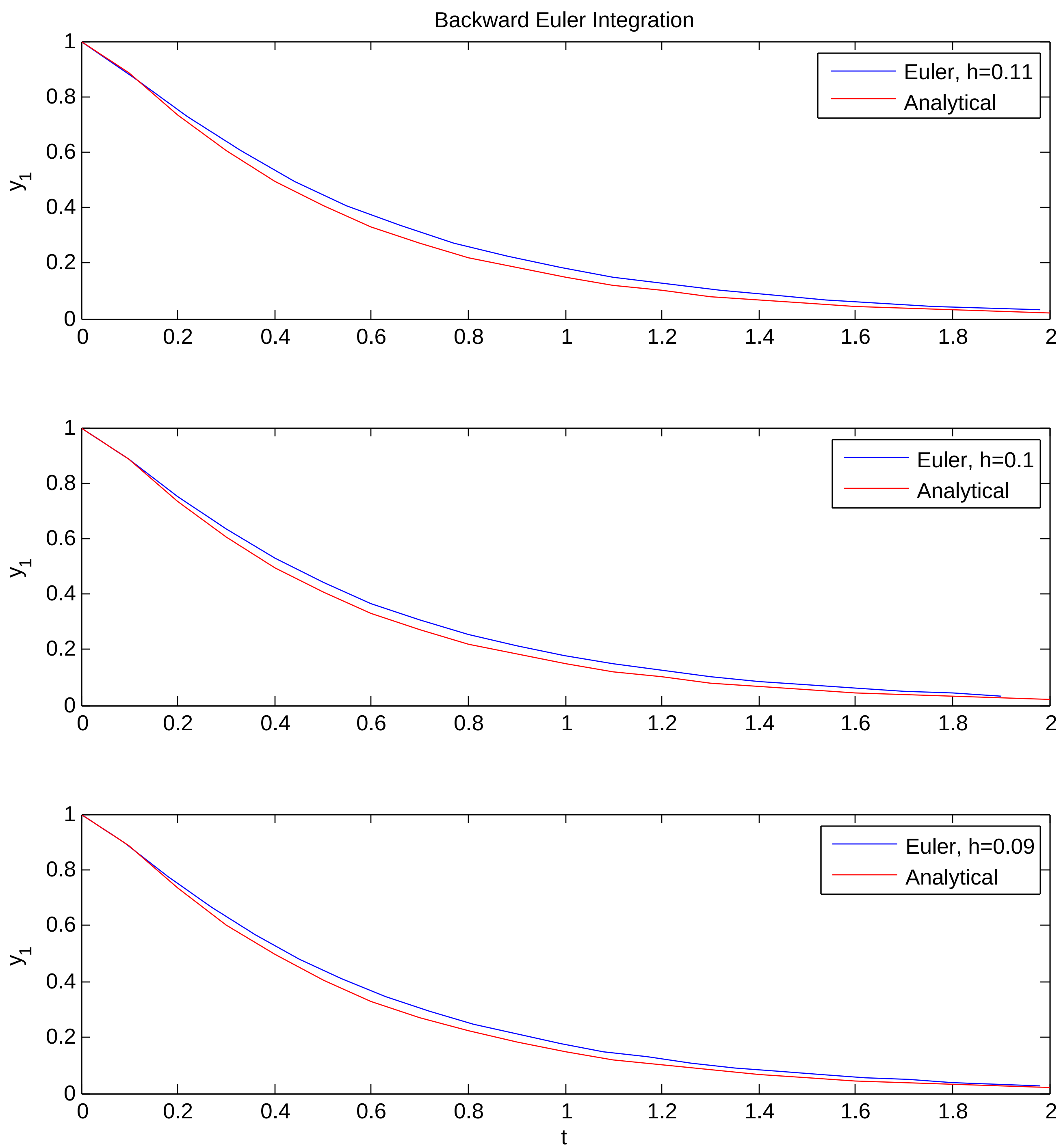
$h \ge 0$ & $h \ge 0$

So $h = 0.11, 0.10, 0.09$ are abs stable.

In the following figures, one can see the unstable case diverge. Forward-Euler $h=0.1$ does not decay, but does not diverge. The other cases all decay like the analytic solution.



Forward Euler Integration
Euler, h=0.11
Analytical
y_1
Euler, h=0.1
Analytical
y_1
Euler, h=0.09
Analytical
y_1



Backward Euler Integration
Euler, h=0.11
Analytical
Euler, h=0.1
Analytical
Euler, h=0.09
Analytical
y_1
t
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Problem 21 $y'' + 2\alpha y' + \omega^2 y = 0$

$\Rightarrow y'' = -2\alpha y' - \omega^2 y$

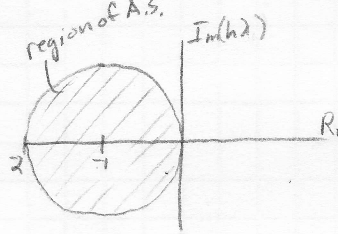
let $\vec{y} = \begin{pmatrix} y \\ y' \end{pmatrix} \Rightarrow \vec{y}\,' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -2\alpha \end{pmatrix} \vec{y} = A\vec{y}$

Eigenvalues, Eigenvectors of A:

$\det(A - \lambda I) = 0 = \begin{vmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda - 2\alpha \end{vmatrix} = \lambda^2 + 2\alpha\lambda + \omega^2$

$\lambda = \dfrac{-2\alpha \pm \sqrt{4\alpha^2 - 4\cdot\omega^2}}{2} = -\alpha \pm \sqrt{\alpha^2 - \omega^2}$

a) $\alpha = 2,\ \omega = 1 \Rightarrow \lambda = -2 \pm \sqrt{4-1} = -2 \pm \sqrt{3}$



$\left.\begin{array}{l} |1 + Re(h\lambda)| \le 1 \\ \sqrt{Re(\lambda h)^2 + Im(\lambda h)^2} \le 1 \end{array}\right\}$ For absolute stability.

$\lambda_+ : |1 + (-2+\sqrt{3})h| \Rightarrow -1 \le 1 + (-2+\sqrt{3})h \Rightarrow -2 \le (-2+\sqrt{3})h$

$\Rightarrow h_+ \le \dfrac{2}{2-\sqrt{3}}$

$\lambda_- : |1 - (2+\sqrt{3})h| \Rightarrow -1 \le 1 - (2+\sqrt{3})h \Rightarrow -2 \le -(2+\sqrt{3})h$

$\Rightarrow h_- \le \dfrac{2}{2+\sqrt{3}}$

$\max(h_-) < \max(h_+)$, so $\boxed{h \le \dfrac{2}{2+\sqrt{3}}}$ to maintain A.S.

b) $\alpha = 1,\ \omega = 2 \Rightarrow \lambda = -1 \pm \sqrt{1-4} = -1 \pm \sqrt{3}\, i$

$\lambda_+ : |1 + (-1+\sqrt{3})h| \le 1 \Rightarrow -1 \le 1 - (1-\sqrt{3})h \Rightarrow -2 \le -(1-\sqrt{3})h$

$\Rightarrow h \le \dfrac{2}{1-\sqrt{3}}$

$|\sqrt{1h^2 + 3h^2}| \le 1 \Rightarrow |\sqrt{4h^2}| \le 1 \Rightarrow 2h \le 1 \Rightarrow h \le 1/2$

$\lambda_- : |1 - (1+\sqrt{3})h| \le 1 \Rightarrow -1 \le 1 - (1+\sqrt{3})h \Rightarrow -2 \le -(1+\sqrt{3})h$

$\Rightarrow h \le \dfrac{2}{1+\sqrt{3}}$

$|\sqrt{1h^2 + 3h^2}| \le 1 \Rightarrow h \le 1/2$

For absolute stability, $\boxed{h \le 1/2}$