ELEC 6430 Solutions

1) $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-ax'-by'\}u(x',y')\,\delta(y-y')\,dx'dy'$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{-ax'-by'\}u(x')u(y')\,\delta(y-y')\,dy'\,dx'$$

$$= \int_{-\infty}^{\infty} \exp\{-ax'-by\}u(x')u(y)\,dx'$$

$$= e^{-by}u(y)\int_{0}^{\infty} e^{-ax'}dx'$$

$$= e^{-by}u(y)\left(\frac{-1}{a}\right)e^{-ax'}\Big|_0^{\infty} = \frac{1}{a}e^{-by}u(y)$$

2) $$F(\omega_x,\omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x-ay)\exp\{-j(\omega_x x+\omega_y y)\}\,dx\,dy$$

$$= \int_{-\infty}^{\infty} \exp\{-j(a\omega_x y+\omega_y y)\}\,dy$$

$$= \int_{-\infty}^{\infty} e^{-j(a\omega_x+\omega_y)y}\,dy$$

Note that $\mathcal{F}^{-1}\{2\pi\delta(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega)e^{j\omega x}d\omega = 1$

$$\Rightarrow \mathcal{F}\{1\} = 2\pi\delta(\omega) = \int_{-\infty}^{\infty} e^{-j\omega x}dx$$

$$\Rightarrow F(\omega_x,\omega_y) = \int_{-\infty}^{\infty} e^{-j(a\omega_x+\omega_y)y}dy = 2\pi\delta(a\omega_x+\omega_y)$$

3) Let $f_1(x,y) = f(x-x_0,y-y_0)$.

Then $$\tilde{f}_1(x,y) = \int_{-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} f_1(x-x',y-y')\,dx'dy'$$

$$= \int_{-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} f(x-x'-x_0,y-y'-y_0)\,dx'dy' = \tilde{f}(x-x_0,y-y_0)$$

$\Rightarrow$ shift-invariant

4) $\iint [a f_1(x-x', y-y') + b f_2(x-x', y-y')]\,dx'dy'$

$= a \iint f_1(x-x', y-y')\,dx'dy' + b \iint f_2(x-x', y-y')\,dx'dy'$

$= a \tilde{f}_1(x,y) + b \hat{f}_2(x,y) \Rightarrow$ linear

5) Let $\Pi_{\Delta x}(x) = \begin{cases} 1, & -\frac{\Delta x}{2} < x \leq \frac{\Delta x}{2} \\ 0, & \text{otherwise} \end{cases}$,

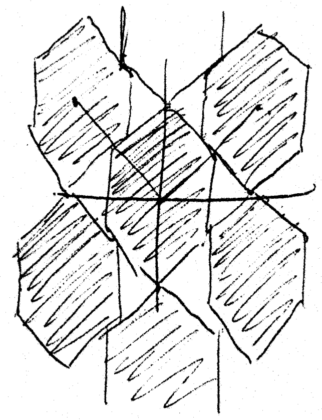
Then $f(x,y) = \Pi_{\Delta x}(x)\,\Pi_{\Delta y}(y)$, which is separable.

1) $\frac{2\pi}{\Delta x} > 2(2) \Rightarrow \Delta x < \frac{\pi}{2}$

$\frac{2\pi}{\Delta y} > 2(3) \qquad \Delta y < \frac{\pi}{3}$

Sampling diagonally has the effect of redefining the axes (and the aliasing) diagonally.

From figure, sample spacing is same in both directions.



$$\frac{2\pi}{\Delta} > 3\sqrt{2}$$

$$\Rightarrow \Delta < \pi\frac{\sqrt{2}}{3}$$

2) a) i) Let $a = b = 1$, $x_1(m,n) = x_2(m,n) = 1$

$T(ax_1(m,n)+bx_2(m,n)) = 2^2 = 4 \neq aT\{x_1(m,n)\} + bT\{x_2(m,n)\} = 1^2 + 1^2 = 2$

$\Rightarrow$ not linear

If $y(m,n) = T\{x(m,n)\}$

ii) $x_1(m,n) = x(m-m_0, n-n_0)$, $T\{x_1(m,n)\} = (x(m-m_0, n-n_0))^2 = y(m-m_0, n-n_0) \Rightarrow$ SI

b) i) $T\{ax_1(m,n) + bx_2(m,n)\} = [ax_1(m,n) + bx_2(m,n)] + [ax_1(m,n-1) + bx_2(m,n-1)]$

$= a[x_1(m,n) + x_1(m,n-1)] + b[x_2(m,n) + x_2(m,n-1)]$

$= aT\{x_1(m,n)\} + bT\{x_2(m,n)\} \Rightarrow$ linear

ii) $x_1(m,n) = x(m-m_0, n-n_0)$, $y(m,n) = T\{x(m,n)\}$

$y_1(m,n) = T\{x_1(m,n)\} = x_1(m,n) + x_1(m,n-1) = x(m-m_0, n-n_0) + x(m-m_0, n-1-n_0)$

$= y(m-m_0, n-n_0) \Rightarrow$ SI

c) i) $T\{ax_1(m,n) + bx_2(m,n)\} = ax_1(m,n) + bx_2(m,n)$

$+ e^n[ax_1(m,n-1) + bx_2(m,n-1)]$

$= a[x_1(m,n) + e^n x_1(m,n-1)] + b[x_2(m,n) + e^n x_2(m,n)] = aT\{x_1(m,n)\} + bT\{x_2(m,n)\}$

$\Rightarrow$ linear

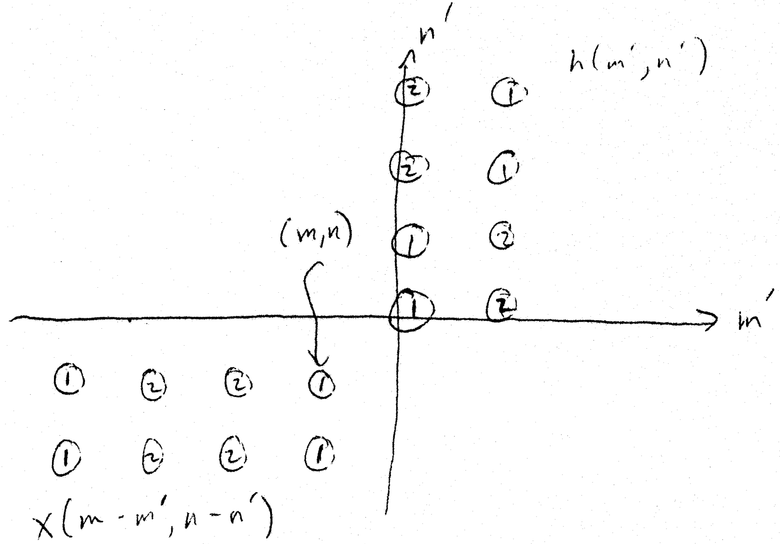
ii) $y(m,n) = T\{x(m,n)\} = x(m,n) + e^n x(m,n-1)$

$T\{x(m-m_0, n-n_0)\} = x(m-m_0, n-n_0) + e^n x(m-m_0, n-n_0-1)$

$y(m-m_0, n-n_0) = x(m-m_0, n-n_0) + e^{n-n_0} x(m-m_0, n-n_0-1) \neq T\{x(m-m_0, n-n_0)\}$

$\Rightarrow$ not SI

3)



2	5	6	4	1
4	10	12	8	2
3	9	12	9	3
2	8	12	10	4
1	4	6	5	2

ELEC 6430 — Solutions

1) a) $X(\omega) = \sum_{n=-\infty}^{\infty} \left(\delta(n) + \delta(n-2)\right) e^{-j\omega n} = 1 + e^{-j2\omega}$

b) $X(\omega_m, \omega_n) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[\delta(m,n) + \delta(m-2, n-1)\right] e^{-j(\omega_m m + \omega_n n)} = 1 + e^{-j(2\omega_m + \omega_n)}$

2) $X(k,\ell) = \sum_{m=0}^{3} \sum_{n=0}^{3} \left[\delta(m,n) + \delta(m-2, n-1)\right] e^{-j\left(\frac{2\pi}{4}[km + \ell n]\right)} = 1 + e^{-j(\pi k + \frac{\pi}{2}\ell)}$,

$0 \le k \le 3$
$0 \le \ell \le 3$

3) From the previous HW, the linear convolution is:

$$\begin{array}{|ccccc}
2 & 5 & 6 & 4 & 1 \\
4 & 10 & 12 & 8 & 2 \\
3 & 9 & 12 & 9 & 3 \\
2 & 8 & 12 & 10 & 4 \\
1 & 4 & 6 & 5 & 2 \\
\hline
\end{array}$$

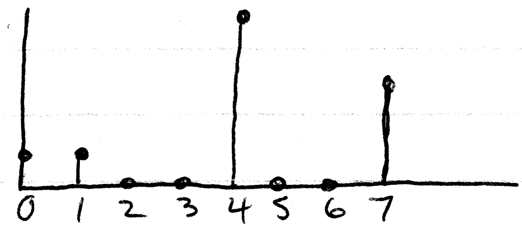
The circular convolution is a 4x4 window of the above, periodically replicated (aliased) on a 4x4 period:

$$\begin{array}{cc|ccccc|cc}
\cdots & 1 & 2 & 5 & 6 & 4 & 1 & 2 & \cdots \\
\cdots & 2 & 4 & 10 & 12 & 8 & 2 & 4 & \cdots \\
\cdots & 3 & 3 & 9 & 12 & 9 & 3 & 3 & \cdots \\
\cdots & 4 & 2 & 8 & 12 & 10 & 4 & 2 & \cdots \\
\cdots & 2 & 1 & 4 & 6 & 5 & 2 & 1 & \cdots \\
\hline
 & 1 & 2 & 5 & 6 & 4 & 1 & &
\end{array}
\Rightarrow
\begin{array}{|cccc}
6 & 10 & 12 & 8 \\
6 & 9 & 12 & 9 \\
6 & 8 & 12 & 10 \\
6 & 9 & 12 & 9 \\
\hline
\end{array}$$

4) We have 2×4, $4\times2 \Rightarrow (4+2-1)\times(2+4-1) = 5\times5$, which is the same size as the linear convolution above.

1) $L = 8$

j	0	1	2	3	4	5	6	7
$A_j = \Sigma \frac{1}{2}h(x_i) + h(x_{i-1}) - \frac{1}{2}h(x_0)$	0	10	15	15	15	15	40	80
$A_j(L-1)/A_{L-1}$	0	$7/8$	$\frac{21}{16}$	$\frac{21}{16}$	$\frac{21}{16}$	$\frac{21}{16}$	$\frac{7}{2}$	7
round	0	1	1	1	1	1	4	7



2) We know from class that the density of the negative is given by $D_1 = \log\{(T_0 I_0)^\gamma e^k\}$.
If we shine a light I_s through this negative to expose a second negative with light I_1, we have

$$D_1 = \log \frac{I_s}{I_1} = \log\{(T_0 I_0)^\gamma e^k\} \Rightarrow I_1 = \frac{I_s}{(T_0 I_0)^\gamma e^k}.$$

The second negative also follows the curve

$$D = \log(T_1 I_1)^\gamma e^k$$

Shining light I_{s_2} through this developed negative, we measure I_m!

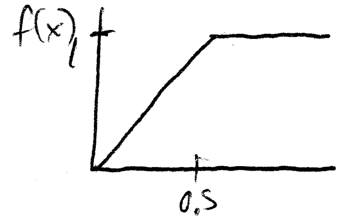
$$D = \log \frac{I_{s_2}}{I_m} = \log (T_1 I_1)^\gamma e^k \Rightarrow I_m = \frac{I_{s_2}}{(T_1 I_1)^\gamma e^k}$$

Substituting for I_1 above,

$$I_m = \frac{I_{s_2}}{\left(T_1 \frac{I_s}{(T_0 I_0)^\gamma e^k}\right)^\gamma e^k} = \frac{I_{s_2} T_0^{\gamma^2}}{(T_1 I_s)^\gamma e^{k(1-\gamma)}} I_0^{\gamma^2}$$

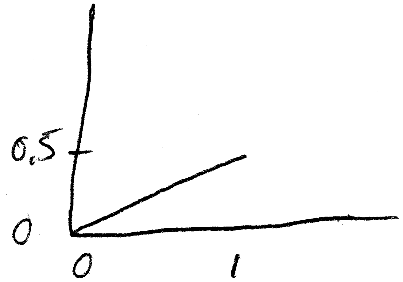
From this, we see that I_m + I_0 move together.
If $\gamma = 1$, $I_m = \alpha I_0$.

3)



f(x) is not invertible, since the mapping from one back to original gray level is ambiguous.

One option is



which assigns all values of "1" to 0.5

4) The simplest solution is an affine map of the form

$$s = mr + b$$

$$0 = m(50) + b \quad \Rightarrow \quad b = -50m$$

$$255 = m(120) + b \qquad 255 = 120m - 50m$$

$$\Rightarrow m = \frac{255}{70} = 3.64$$

$$b = \frac{-255}{70}(50) = -182$$

$$s = 3.64r - 182$$

Problem Set

1. A video inspection system operating at 30 frames/s is used to inspect widgets for uneven texture. The noise variance in each frame is determined to be a factor of 10 too high. Image averaging can be used to average frames and bring down the noise variance. Assuming that the noise is uncorrelated from frame to frame and has zero mean, how long will each widget need to remain stationary under the camera to make the noise level acceptable?

   Solution:

   Suppose the noise variance for one image is given by σ_n^2. Then the noise variance obtained by averaging K images is given by $\frac{1}{K}\sigma_n^2$. We want the ratio of the former to the latter to be no greater than 10:

   $$\begin{aligned} 10 &= \frac{\sigma_n^2}{\frac{1}{K}\sigma_n^2} \\ &= K \end{aligned}$$

   Thus, the time required is $K\left(\frac{1}{30}\right) = \frac{1}{3}$ s.

2. Show that subtracting a fraction of the Laplacian boosts high frequencies.

   Solution:

   Find the Fourier transform of the discrete Laplacian:

   $$h(m,n) = \delta(m,n+1) + \delta(m,n-1) + \delta(m+1,n) + \delta(m-1,n) - 4\delta(m,n)$$

   $$\begin{aligned} H(\omega_m,\omega_n) &= e^{-j\omega_n} + e^{j\omega_n} + e^{-j\omega_m} + e^{j\omega_m} - 4 \\ &= 2\cos\omega_n + 2\cos\omega_m - 4 \end{aligned}$$

   $$\begin{aligned} Y(\omega_m,\omega_n) &= X(\omega_m,\omega_n) + \lambda H(\omega_m,\omega_n)X(\omega_m,\omega_n) \\ &= [1 - 4\lambda + 2\lambda(\cos\omega_n + \cos\omega_m)]X(\omega_m,\omega_n) \end{aligned}$$

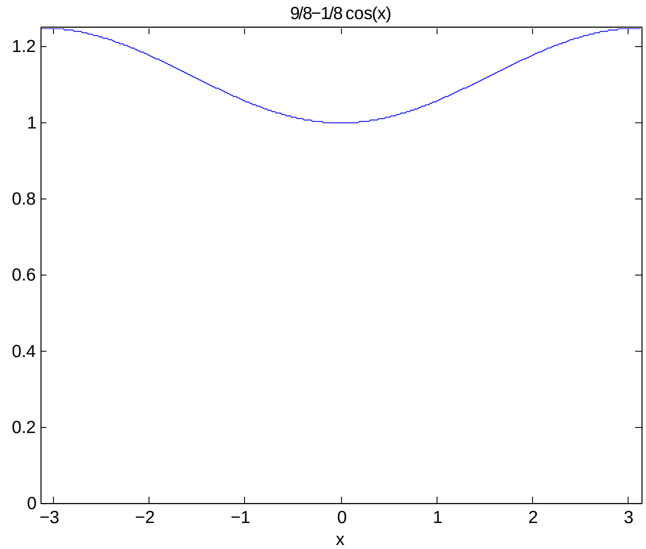
   To add higher frequencies, we must use negative λ. Let $\lambda = -\frac{1}{16}$.

   $$H(\omega_m,\omega_n) = \frac{5}{4} - \frac{1}{8}(\cos\omega_n + \cos\omega_m)$$

   $\omega_m = 0 \ \Rightarrow$

   $$Y(0,\omega_n) = \frac{9}{8} - \frac{1}{8}(\cos\omega_n)]X(0,\omega_n)$$

   The frequency response looks like:


9/8−1/8 cos(x)
1.2
1
0.8
0.6
0.4
0.2
0
−3
−2
−1
0
1
2
3
x

Problem Set 5

1. Consider the following corresponding points:

input	output
(0,0)	(5,5)
(1,6)	(7,10)
(4,4)	(10,9)
(8,3)	(14,7)
(9,5)	(15,9)
(3,5)	(9,10)

Find the least-squares fit for 1) translation and 2) affine mapping for these point pairs.

Solution

1. Translation:

$$x_i = x_o + x_t$$

$$y_i = y_o + y_t$$

or

$$x_i - x_o = x_t$$

$$y_i - y_o = y_t$$

$$\begin{array}{ccc} \begin{bmatrix} 0-5 \\ 1-7 \\ 4-10 \\ 8-14 \\ 9-15 \\ 3-9 \end{bmatrix} & = & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} x_t \\ x_d & = & U x_t \end{array}$$

$$x_t = (U^T U)^{-1} U^T x_d = -5.8333$$

Likewise, for y_t:

$$y_t = (U^T U)^{-1} U^T x_d = -4.5000$$

Note that this is just an average value for the displacement, since U only has one column.

2. Affine mapping:

$$x_i = a_0 + a_1 x_o + a_2 y_0$$
$$y_i = b_0 + b_1 x_o + b_2 y_0$$

$$\begin{array}{ccc} \begin{bmatrix} 0 \\ 1 \\ 4 \\ 8 \\ 9 \\ 3 \end{bmatrix} & = & \begin{bmatrix} 1 & 5 & 5 \\ 1 & 7 & 10 \\ 1 & 10 & 9 \\ 1 & 14 & 7 \\ 1 & 15 & 9 \\ 1 & 9 & 10 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \\ v & = & Ua \end{array}$$

$$a = (U^T U)^{-1} U^T v = \begin{bmatrix} -4.0702 \\ 0.9502 \\ -0.1518 \end{bmatrix}$$

Likewise, for b:

$$b = (U^T U)^{-1} U^T w = \begin{bmatrix} -5.4345 \\ 0.0769 \\ 1.0199 \end{bmatrix}$$

Thus,

$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \begin{bmatrix} 0.9502 & -0.1518 \\ 0.0769 & 1.0199 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix} + \begin{bmatrix} -4.0702 \\ -5.4345 \end{bmatrix}$$

2. Find the continuous impulse response for bilinear interpolation. (Assume a single unit sample value at the origin and zero elsewhere, and find the response.)

Solution

For the first quadrant, we have

$$f(x, y) = f(0, 0)(1 - x)(1 - y) + f(0, 1)(1 - x)y + f(1, 0)(1 - y)x + f(1, 1)xy$$

Let $f(0, 0) = 1$, $f(1, 0) = f(0, 1) = f(1, 1) = 0$. Then substituting to obtain the impulse response,

$$h(x, y) = (1 - x)(1 - y)$$

We simply rotate this for each quadrant to obtain:

$$\begin{aligned} h(x, y) &= \begin{cases} (1 - x)(1 - y), & 0 \le x \le 1, 0 \le y \le 1 \\ (1 + x)(1 - y), & -1 \le x \le 0, 0 \le y \le 1 \\ (1 - x)(1 + y), & 0 \le x \le 1, -1 \le y \le 0 \\ (1 + x)(1 + y), & -1 \le x \le 0, -1 \le y \le 0 \end{cases} \\ &= (1 - |x|)(1 - |y|), \quad |x| \le 1, |y| \le 1 \end{aligned}$$