Friday, January 25, 2019 2:19 PM

Two-dimensional signals & systems

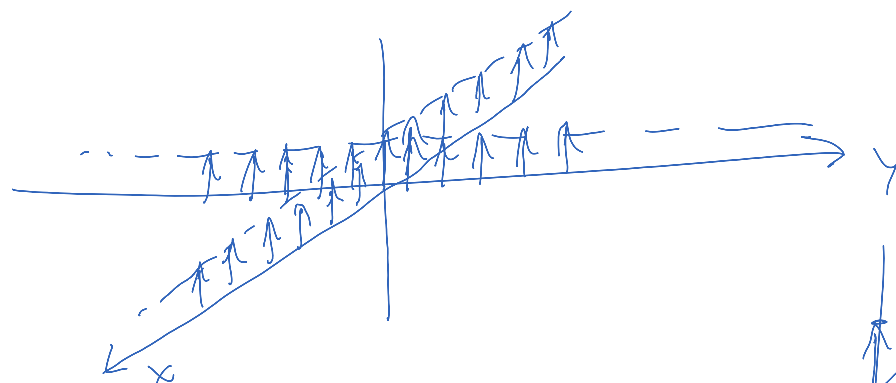
Continuous signals & systems; basic signals

Dirac delta: $\delta(x, y) = 0 \quad , \quad x, y \neq 0$

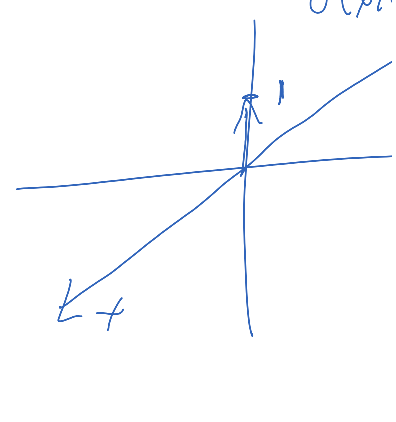
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,\delta(x-x_0, y-y_0)\,dx\,dy = f(x_0, y$$

(sifting property)

$$\delta(x,y) = \delta(x)\,\delta(y)$$



$\delta(x,$



step function:

$$u(x,y) = \begin{cases} 1 & , x, \\ 0 & , \text{ot} \end{cases}$$

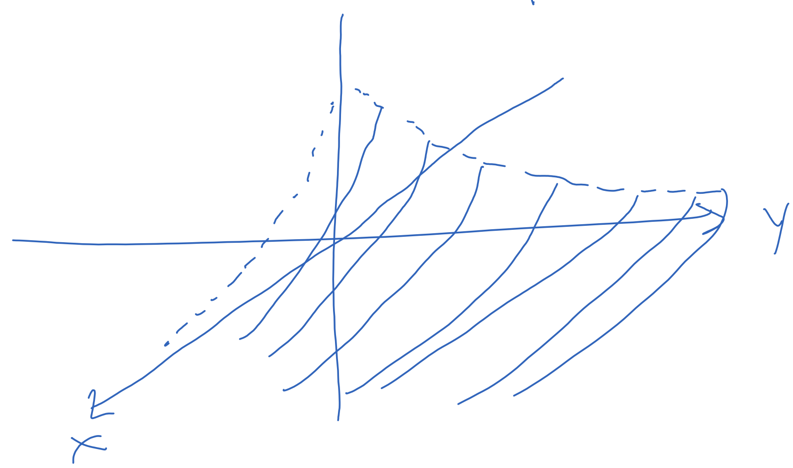
$= u(x)\, u(y)$

decaying exponential:

$$f(x,y) = \begin{cases} \exp\{-\alpha x - \beta y \\ 0 \end{cases}$$

$\alpha, \beta > 0$

$\doteq [e$



Sinusoid:

$$f(x,y) = \sin(\omega_{0x} x + \omega_{0y} y)$$

ω_{0x} = frequency along x in

ω_{0y} = " " y i

A <u>separable</u> signal is one that

factored into a product of 1-D s

$$\delta(x,y) = \delta(x)\,\delta(y)$$

$$u(x,y) = u(x)\,u(y)$$

$$\exp\{-\alpha x - \beta y\}\,u(x,y) = \left[e^{-\alpha x}u(x)\right]\left[\right.$$

$$e^{j(\omega_{ox}x + \omega_{oy}y)} = \left[e^{j\omega_{ox}x}\right.$$

Linear systems

If $T[\cdot]$ is a system, it is linear if

$$T[af(x,y) + bg(x,y)] = aT[f(x,y)] + bT[g(x,y)]$$

for all $a, b, f(x,y), g(x,y)$

- superposition

Ex: film with $T[f(x,y)] = g(x,y) = \log(1+I)$

- not linear

$$T[(1)(1)+(1)(1)] = \log(1+2) = \log 3$$

$$(1)T[1] + (1)T[1] = 2\log(1+1) = \log 4$$

$\neq$

$\Rightarrow$ not linear

Ex: optical system that blurs horizontally

$$T[f(x,y)] = \int_0^1 f(x-x', y)\, dx'$$

show that system is linear

Must show that superposition holds for all inputs.

$$T[af(x,y) + b\,g(x,y)] = \int_0^1 af(x-x', y) + b\,g(x-x', y)\, dx'$$

$$= a\int_0^1 f(x-x', y)\, dx' + b\int_0^1 g(x-x', y)\, dx'$$

$$= a\,T[f(x,y)] + b\,T[g(x,y)]$$

$\Rightarrow$ linear

shift-invariant systems

$$g(x,y) = T[f(x,y)]$$

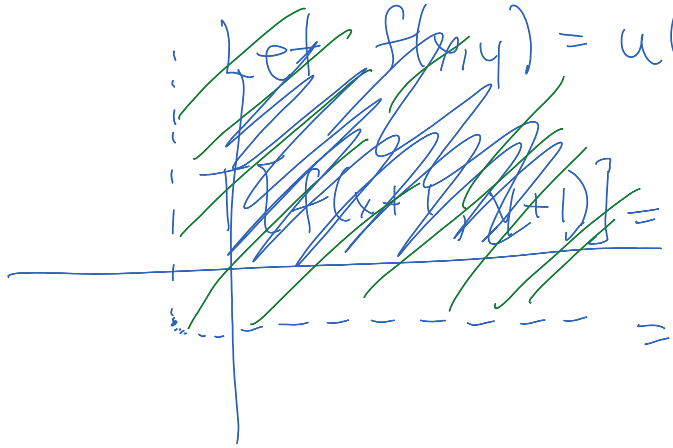
The system is SI iff

$$g(x-x_0, y-y_0) = T[f(x-x_0, y-y_0)]$$

for all shifts (x_0, y_0) and any $f(x,y)$.

Ex: $g(x,y) = T[f(x,y)] = f(x,y)\,u(x,y)$

Let $f(x,y) = u(x,y)$ Let $(x_0, y_0) = (-1, -$



$= f(x+1, y+1)\,u(x,y)$

$= u(x+1, y+1)\,u(x,y)$

$= u(x+1, y+1)\,u(x,y)$

$= u(x,y)$

$g(x+1, y+1) = u(x+1, y+1)$

$\neq u(x,y) \Rightarrow$ not SI

Linear shift-invariant systems

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')\,\delta(x-x', y-y')\,dx'dy'$$

For linear system,

$$g(x,y) = T[f(x,y)]$$

$$= T\left[\iint f(x',y')\,\delta(x-x', y-y')\,dx'dy'\right]$$

$$= \iint f(x',y')\,T[\delta(x-x', y-y')]\,dx'dy'$$

If we define $h(x,y) = T[\delta(x,y)]$, we have

$$h(x-x', y-y') = T[\delta(x-x', y-y')] \text{ for all } x', y'$$

for a shift-invariant system

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x', y')\, h(x-x', y-y')\, dx'\, dy'$$

$$= f(x,y) * h(x,y) \quad : \text{ 2-D convolution}$$

$h(x,y)$ is the impulse response

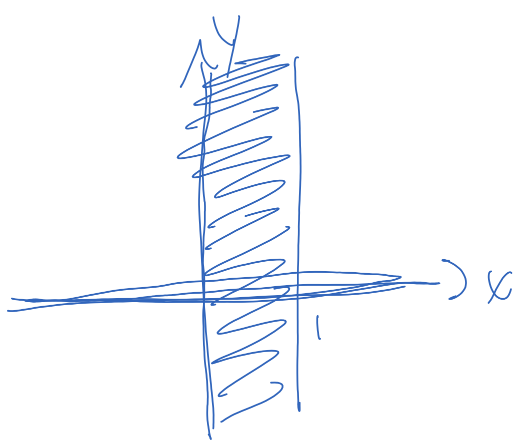
Ex: horizontal blur

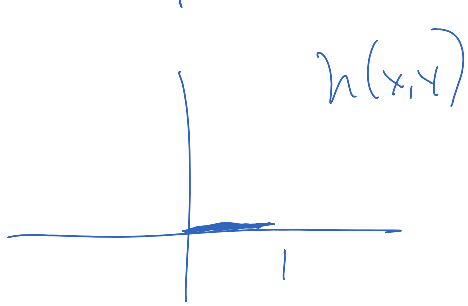
$$h(x,y) = \int_0^1 \delta(x-x', y)\, dx'$$

$$= \int_0^1 \delta(x-x')\, \delta(y)\, dx'$$

$$= \delta(y') \int_0^1 \delta(x-x')\, dx'$$

$$= \delta(y)\,[u(x) - u(x-1)]$$

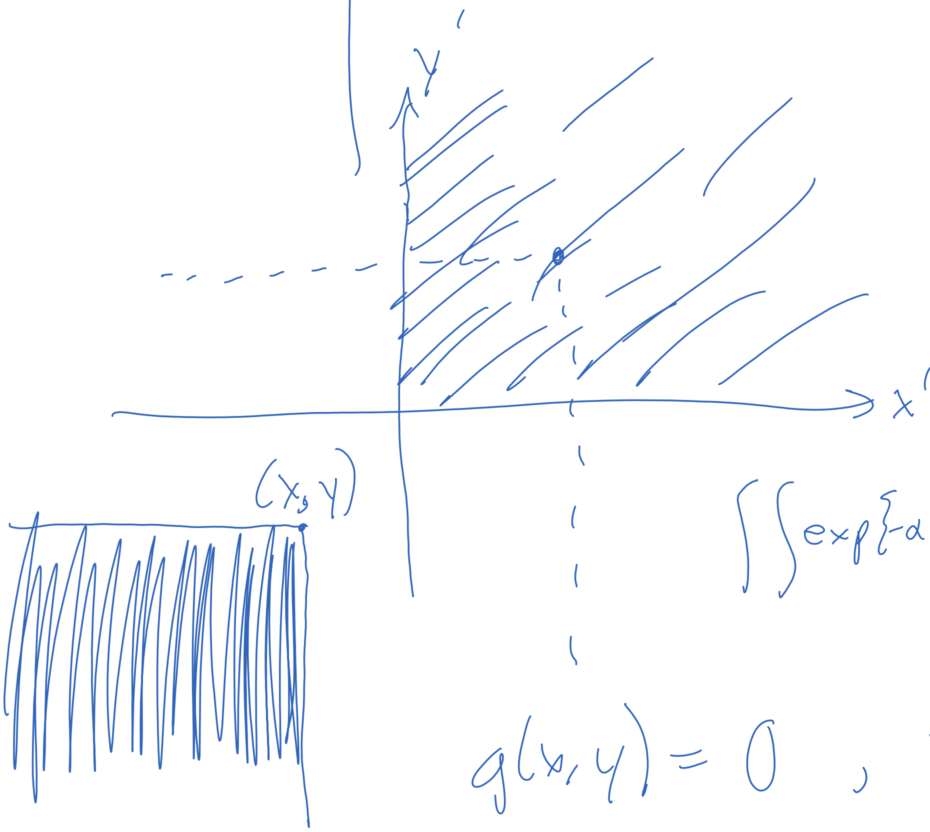




Read 4.3 - 4.5

HW #1 - to be posted

Ex: $\exp\{-\alpha x - \beta y\} u(x,y) * u(x,y)$



$\iint \exp\{-\alpha x' - \beta y'\} u(x', y') \, u(x - x', y -$

$g(x,y) = 0, \quad x < 0 \text{ or } y < 0$

$$g(x,y) = \int_0^y \int_0^x \exp\{-\alpha x' - \beta y'\} \, dx' \, dy'$$

$$= \int^y e^{-\beta y'} dy' \int^x e^{-\alpha x'} dx'$$

$$= -\frac{1}{\beta} e^{-\beta y'} \Big|_0^y \cdot -\frac{1}{\alpha} e^{-\alpha x'} \Big|_0^x$$

$$= \frac{1}{\alpha\beta}\left[e^{-\beta y} - 1\right]\left[e^{-\alpha x} - 1\right]$$

$$g(x,y) = \frac{1}{\alpha\beta}\left[1 - e^{-\alpha x}\right]\left[1 - e^{-\beta y}\right] u(x,y)$$

Fourier transform

$$F(\omega_x, \omega_y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) \exp\{-j(\omega_x x + \omega_y y)\} dx\, dy$$

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega_x, \omega_y) \exp\{j(\omega_x x + \omega_y y)\} d\omega_x\, d\omega_y$$

$$f(x,y) \longleftrightarrow F(\omega_x, \omega_y)$$

$$\mathcal{F}\{f(x,y)\} = F(\omega_x, \omega_y)$$

Fourier transform properties

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - x_0, y - y_0) \exp\{-j(\omega_x x + \omega_y y)\} dx\, dy$$

$$= \exp\{-j(\omega_x x_0 + \omega_y y_0)\}$$

Linearity

$$a f(x,y) + b g(x,y) \longleftrightarrow a F(\omega_x,\omega_y) + b G(\omega_x,\omega_y)$$

Convolution

$$f(x,y) * g(x,y) \longleftrightarrow F(\omega_x,\omega_y)\, G(\omega_x,\omega_y)$$

Multiplication

$$f(x,y)\, g(x,y) \longleftrightarrow \frac{1}{4\pi^2} F(\omega_x,\omega_y) * G(\omega_x,\omega_y)$$

Separability

$$F_y(\omega_x, y) = \int_{-\infty}^{\infty} f(x,y)\, e^{-j\omega_x x}\, dx$$

$$F(\omega_x,\omega_y) = \int_{-\infty}^{\infty} F_y(\omega_x, y)\, e^{-j\omega_y y}\, dy$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(x,y)\, e^{-j\omega_x x}\, dx\right] e^{-j\omega_y y}\, dy$$

$$= \int\int f(x,y) \exp\{-j(\omega_x x + \omega_y y)\}\, dx\,dy$$

FT of separable signals

$$f(x,y) = f_X(x) f_Y(y)$$

$$F(\omega_x, \omega_y) = F_X(\omega_x) F_Y(\omega_y)$$

- FT is also separable

Shifts

$$f(x-x_0, y-y_0) \longleftrightarrow \exp\{-j(\omega_x x_0 + \omega_y y_0)\} F(\omega_x, \omega_y)$$

$$\exp\{j(\omega_{x_0} x + \omega_{y_0} y)\} f(x,y) \longleftrightarrow F(\omega_x - \omega_{x_0}, \omega_y - \omega_{y_0})$$

(modulation)

Scaling

$$f(ax, by) \longleftrightarrow \frac{1}{|ab|} F\left(\frac{\omega_x}{a}, \frac{\omega_y}{b}\right)$$

Parseval's theorem

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2\, dx\,dy = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |F(\omega_x, \omega_y)|^2\, d\omega_x\, d\omega_y$$

~~Spatial derivatives~~

$$\cancel{\frac{\partial f(x,y)}{\partial x}} \longleftrightarrow -j\omega_x F(\omega_x, \omega_y)$$

$$\frac{\partial f(x,y)}{\partial y} \longleftrightarrow -j\omega_y F(\omega_x, \omega_y)$$

Laplacian:

$$\frac{\partial^2 f(x,y)}{\partial x^2} + \frac{\partial^2 f(x,y)}{\partial y^2} \longleftrightarrow -(\omega_x^2 + \omega_y^2) F(\omega_x, \omega_y)$$

Ex: $f(x,y) = \begin{cases} 1, & 0 \le x \le a,\ 0 \le y \le b \\ 0, & \text{otherwise} \end{cases}$

$$F(\omega_x, \omega_y) = \int_0^b \int_0^a \exp\{-j(\omega_x x + \omega_y y)\}\, dx\, dy$$

$$= \left[\int_0^b e^{-j\omega_y y}\, dy\right]\left[\int_0^a e^{-j\omega_x x}\, dx\right]$$

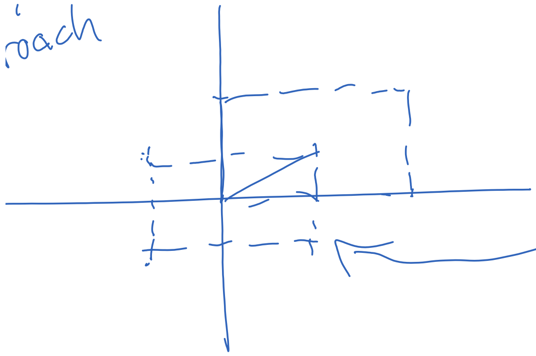
$$= -\frac{1}{j\omega_y} e^{-j\omega_y y}\Big|_0^b \cdot -\frac{1}{j\omega_x} e^{-j\omega_x x}\Big|_0^a$$

$$= -\frac{1}{\omega_x \omega_y}\left(1 - e^{-j\omega_x a}\right)\left(1 - e^{-j\omega_y b}\right)$$

$$= -\frac{1}{\omega_x \omega_y} e^{-j\frac{\omega_x a}{2}}\left(e^{j\frac{\omega_x a}{2}} - e^{-j\frac{\omega_x a}{2}}\right) \cdots$$

$$= ab \exp\left\{-j\left(\frac{a\omega_x}{2} + \frac{b\omega_y}{2}\right)\right\} \frac{\sin\frac{a\omega_x}{2}}{\frac{a\omega_x}{2}} \cdot \frac{\sin\frac{b\omega_y}{2}}{\frac{b\omega_y}{2}}$$

roach



2-D centered pulse
$\Rightarrow$ separable
$\Rightarrow$ FT is prod. of sinc functions plus linear phase

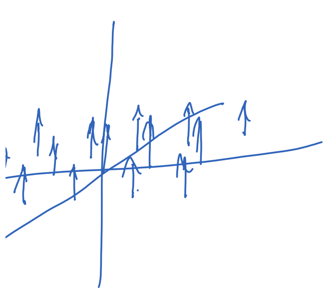
: IFT of $\delta(\omega_x - \omega_{x0}, \omega_y - \omega_{y0})$

$$x,y) = \frac{1}{4\pi^2}\int\int \delta(\omega_x - \omega_{x0}, \omega_y - \omega_{y0}) \exp\{j(\omega_x x + \omega_y y)\}\, d\omega_x\, d\omega_y$$

$$= \frac{1}{4\pi^2} \exp\{j(\omega_{x0} x + \omega_{y0} y)\}$$

$$1 \longleftrightarrow 4\pi^2 \delta(\omega_x, \omega_y)$$

$$f(x,y) = \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$$



$$= \sum_m \sum_n \delta(x - m\Delta x)\, \delta(y - n\Delta y)$$

$$= \left[\sum_m \delta(x - m\Delta x)\right]\left[\sum_n \delta(y - n\Delta y)\right]$$

$$= \left[\sum_m a_m \exp\left\{j\frac{2\pi m}{\Delta x}x\right\}\right]\left[\sum_n b_n \exp\left\{j\frac{2\pi n}{\Delta y}y\right\}\right]$$

$$a_m = \frac{1}{\Delta x}\int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} \delta(x)\exp\left\{-j\frac{2\pi m}{\Delta x}x\right\}dx$$

$$= \frac{1}{\Delta x} \quad \text{for all } m$$

$$b_n = \frac{1}{\Delta y} \quad \text{for all } n$$

$$(x,y) = \left[\sum_m \frac{1}{\Delta x}\exp\left\{j\frac{2\pi m}{\Delta x}x\right\}\right]\left[\sum_n \frac{1}{\Delta y}\exp\left\{j\frac{2\pi n}{\Delta y}y\right\}\right]$$

$$= \frac{1}{\Delta x\Delta y}\sum_m\sum_n \exp\left\{j2\pi\left(\frac{m}{\Delta x}x + \frac{n}{\Delta y}y\right)\right\}$$

$$(\omega_x,\omega_y) = \frac{4\pi^2}{\Delta x\Delta y}\sum_m\sum_n \delta\left(\omega_x - m\frac{2\pi}{\Delta x}, \omega_y - n\frac{2\pi}{\Delta y}\right)$$

$$f(x,y) = \cos(\omega_{x_0}x + \omega_{y_0}y)$$

$$= \frac{1}{2}\exp\{j(\omega_{x_0}x + \omega_{y_0}y)\}$$

$$+ \frac{1}{2}\exp(-j(\omega_{x_0}x + \omega_{y_0}y))$$

$$F(\omega_x,\omega_y) = \frac{4\pi^2}{2}\delta(\omega_x - \omega_{x_0}, \omega_y - \omega_{y_0})$$

$$+ \frac{4\pi^2}{2}\delta(\omega_x + \omega_{x_0}, \omega_y + \omega_{y_0})$$

$$f(x,y) = \cos(\omega_{x_0} x + \omega_{y_0} y) * \left[\delta(x-x_0, y-y_0) + \delta(x+x_0, y+y_0)\right]$$

$$= \cos(\omega_{x_0}(x-x_0) + \omega_{y_0}(\cancel{x}y - y_0)) + \cos(\omega_{x_0}(x+x_0) + \omega_{y_0}(y+y_0))$$

ing conv. thm.:

$$g(x,y) = \delta(x-x_0, y-y_0) + \delta(x+x_0, y+y_0)$$

$$(\omega_x, \omega_y) = \exp\{-j(\omega_x x_0 + \omega_y y_0)\} + \exp\{j(\omega_x x_0 + \omega_y y_0)\}$$

$$= 2\cos(\omega_x x_0 + \omega_y y_0)$$

$$(\omega_x, \omega_y) = \left[\tfrac{1}{2}\delta(\omega_x - \omega_{x_0}, \omega_y - \omega_{y_0}) + \tfrac{1}{2}\delta(\omega_x + \omega_{x_0}, \omega_y + \omega_{y_0})\right] \times 2\cos(\omega_x x_0 + \omega_y y_0)$$

<u>ampling</u>

$$s(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y)$$

Sampled signal:

$$f_s(x,y) = f(x,y)\, s(x,y)$$

$$= f(x,y) \sum_m \sum_n \delta(x - m\Delta x, y - n\Delta y)$$

$$= \sum_m \sum_n f(x,y)\,\delta(x - m\Delta x, y - n\Delta y)$$

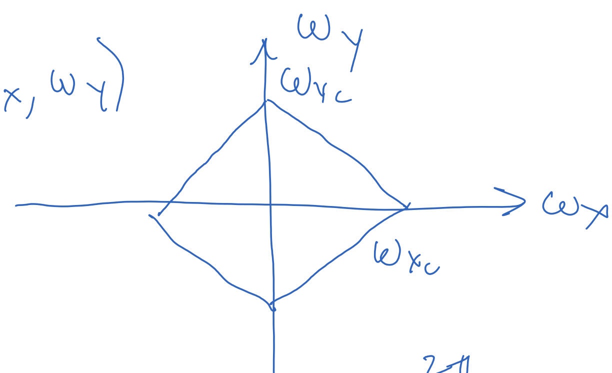
$$= \sum_m \sum_n f(m\Delta x, n\Delta y)\,\delta(x - m\Delta x, y - n\Delta y)$$

$$\omega_x, \omega_y) = \frac{1}{4\pi^2} F(\omega_x, \omega_y) * S(\omega_x, \omega_y)$$

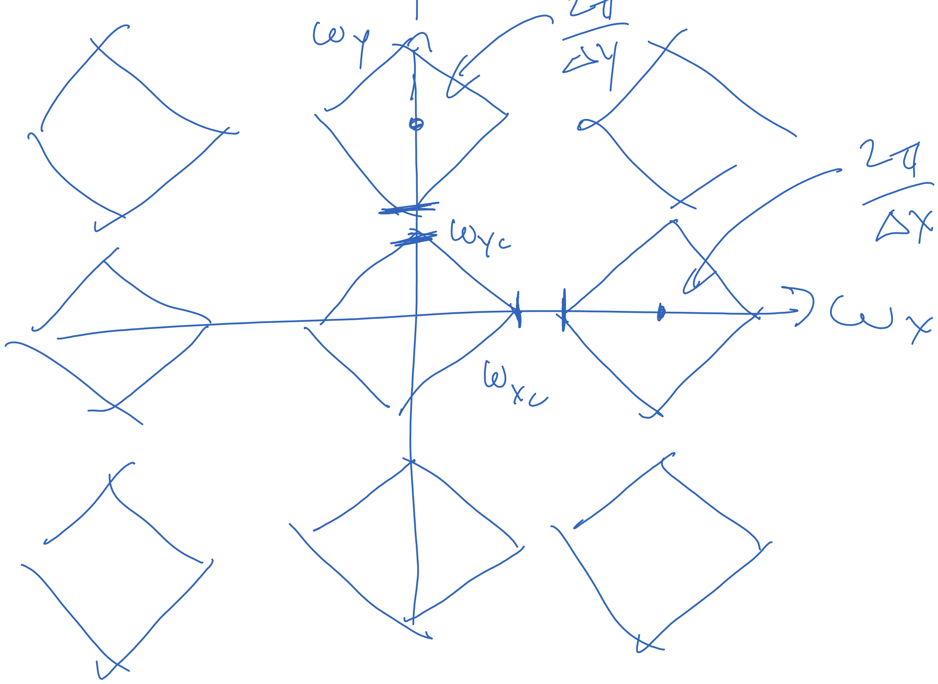
$$= \frac{1}{4\pi^2} F(\omega_x, \omega_y) * \frac{4\pi^2}{\Delta x \Delta y} \sum_m \sum_n \delta\left(\omega_x - m\frac{2\pi}{\Delta x}, \omega_y - n\frac{2\pi}{\Delta y}\right)$$

$$= \frac{1}{\Delta x \Delta y} \sum_m \sum_n F\left(\omega_x - m\frac{2\pi}{\Delta y}, \omega_y - n\frac{2\pi}{\Delta y}\right)$$

$F(\omega_x, \omega_y)$



$\omega_y)$



$$\frac{2\pi}{\Delta x} - \omega_{xc} > \omega_{xc}$$

$$\boxed{\begin{aligned} \frac{2\pi}{\Delta x} &> 2\omega_{xc} \\ \frac{2\pi}{\Delta y} &> 2\omega_{yc} \end{aligned}}$$

can recover $F(\omega_x, \omega_y)$ if the copies
't overlap. If Δx and Δy become too
ge, then copies overlap. The original copy
then no longer be recovered.

$\Rightarrow$ this is called spatial aliasing

mpling theorem

A bandlimited image $f(x, y)$ sampled on a uniform rectangular grid with spacing $\Delta x, \Delta y$ can be recovered from the sample values $f(m\Delta x, n\Delta y)$ if

$$\frac{2\pi}{\Delta x} > 2\omega_{xc}$$

and

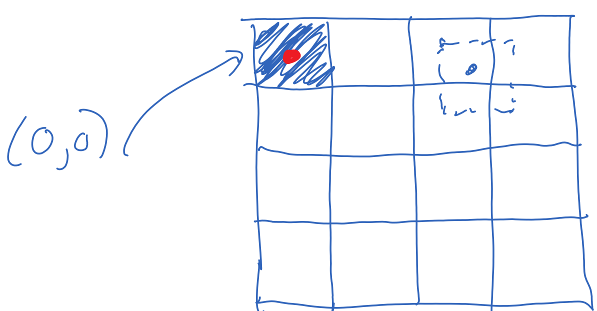
$$\frac{2\pi}{\Delta y} > 2\omega_{yc}$$

l 4.6 - 4.7

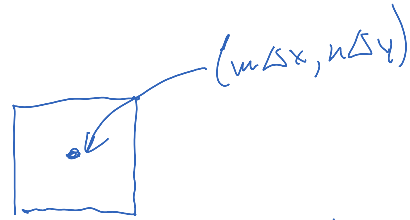
ct 2 due Fri.

deal sampling

more realistic sampling model
presents the sampling process as
tegrating intensity over rectangular
tches.



$$f'(m\Delta x, n\Delta y) = \int_{-\frac{\Delta y}{2}}^{\frac{\Delta y}{2}} \int_{-\frac{\Delta x}{2}}^{\frac{\Delta x}{2}} f(m\Delta x - x', n\Delta y - y')\, dx'dy'$$



$$\Pi(x,y) = \begin{cases} 1, & -\frac{\Delta x}{2} < x \le \frac{\Delta x}{2}, \quad -\frac{\Delta y}{2} < y \le \frac{\Delta y}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\Delta x, n\Delta y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Pi(x', y')\, f(m\Delta x - x', n\Delta y - y')\, dx'dy'$$

$$\int^{\infty}$$

$$x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Pi(x', y') f(x - x', y - y') \, dx' dy'$$

$$= f(x, y) * \Pi(x, y)$$

original scene

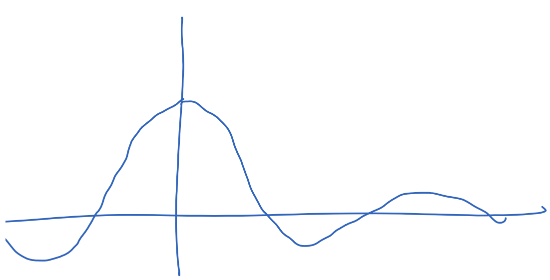
$$_i(x, y) = f_i(x, y) \, s(x, y)$$

$$= \sum_m \sum_n f_i(m\Delta x, n\Delta y) \, \delta(x - m\Delta x, y - n\Delta y)$$

⇒ filtering followed by ideal sampling

$$_i(\omega_x, \omega_y) = F(\omega_x, \omega_y) \, \mathcal{F}\{\Pi(x, y)\}$$

$$= F(\omega_x, \omega_y) \, \frac{\sin \frac{\Delta x}{2}\omega_x}{\frac{\Delta x}{2}\omega_x} \cdot \frac{\sin \frac{\Delta y}{2}\omega_y}{\frac{\Delta y}{2}\omega_y}$$



* We're sampling the filtered image instead of the original

* filtered signal has higher frequencies suppressed

- aliasing will be reduced but not eliminated
- image will be slightly blurred

lay/reconstruction

DSP, reconstruction is done in concept by eating an impulse train from a sequence, en lowpass filtering. This is implemented ing electronics.

image processing, the display is the filter. The HVS is also a filter.) Samples are jected as tiny rectangular patches (LCD display), aussian spots (CRT), etc.

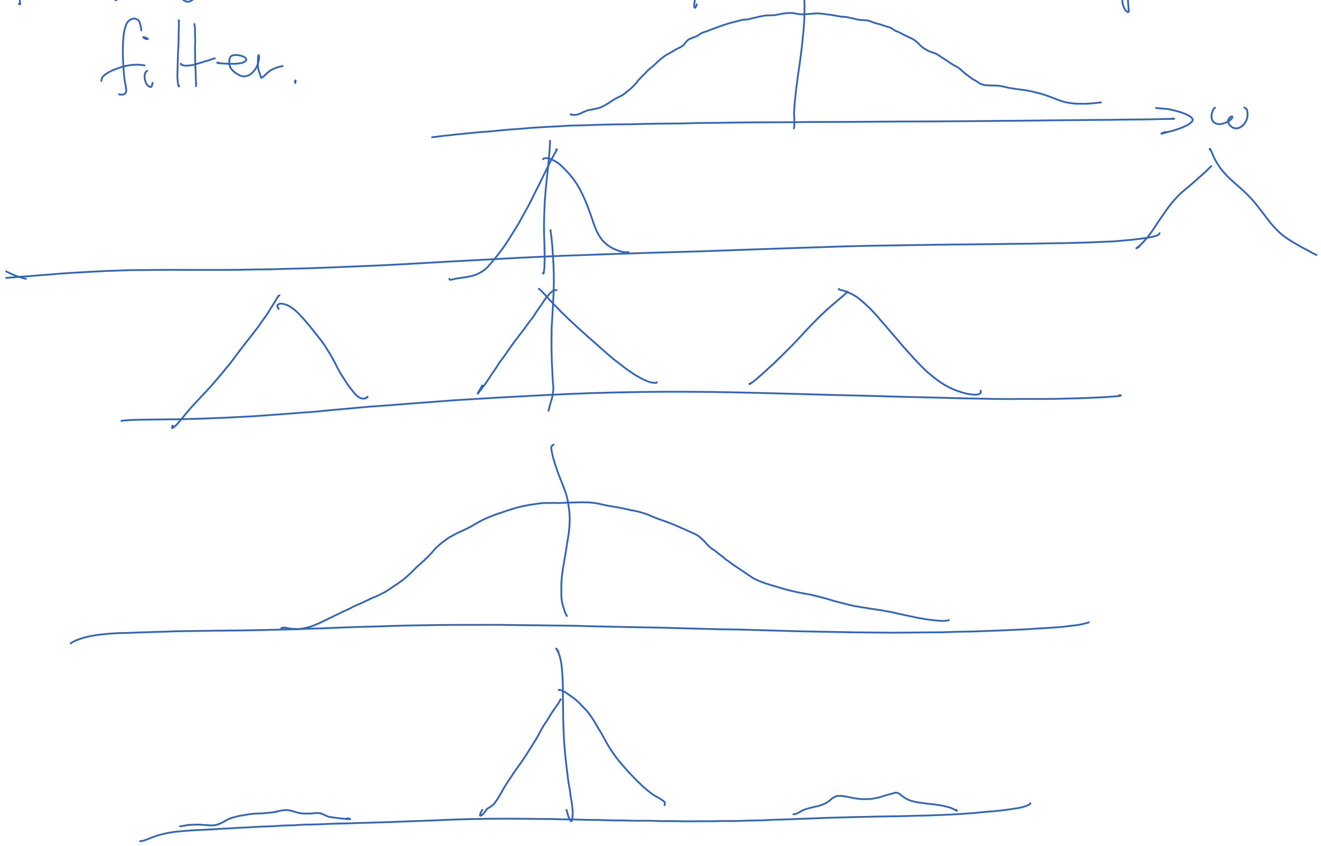
optical display "filter" can be modeled as —

$_s(x,y) \rightarrow \boxed{p(x,y)} \rightarrow f_s(x,y) * p(x,y)$ reduces/removes spectral replicas

$F_s(\omega_x, \omega_y)\, P(\omega_x, \omega_y)$

periodically replicated

Think of human visual system as a lowpass filter.



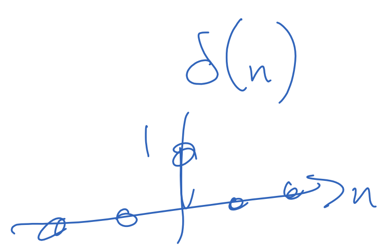
sic signals

delta (impulse, unit sample)

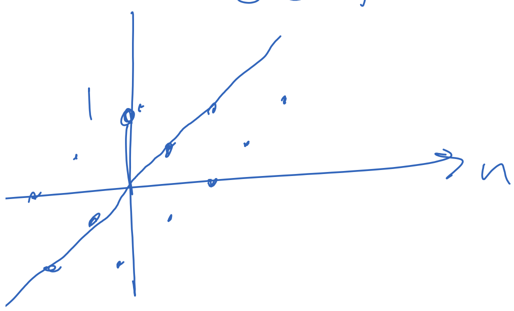
Kronecker delta, not Dirac

$$\delta(m,n) = \begin{cases} 1, & m=n=0 \\ 0, & \text{otherwise} \end{cases}$$

$$= \delta(m)\,\delta(n)$$



where $\delta(n) = \begin{cases} 1 & , n=0 \\ 0 & , n \neq 0 \end{cases}$

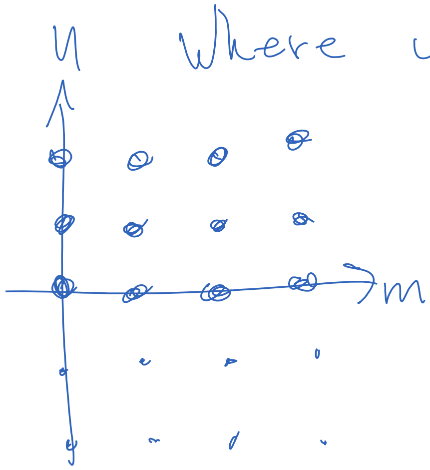


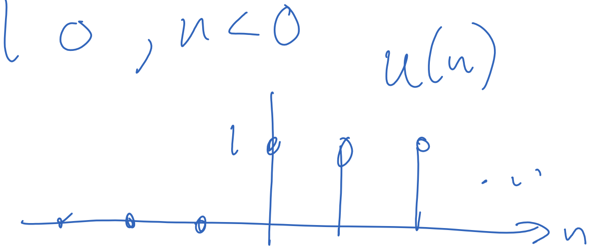
ep

$u(m,n) = \begin{cases} 1 & , m,n \geq 0 \\ 0 & , \text{otherwise} \end{cases}$

$= u(m)\,u(n)$

where $u(n) = \begin{cases} 1 & , n \geq 0 \\ 0 & , n < 0 \end{cases}$





d 4.8 - 4.9

posted

onential

$f(m,n) = \exp\{-\alpha m - \beta n\}\, u(m,n)$

$-\alpha m \qquad -\beta n$

$$= [e \quad u(m)][e \quad u(n)]$$

$\alpha, \beta \geq 0$ to prevent blowing up

.soid

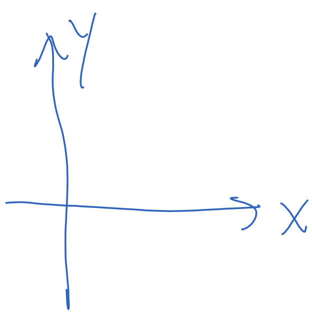
$$f(m,n) = f(x,y)\Big|_{(x,y)=(m\Delta x,\, n\Delta y)}$$

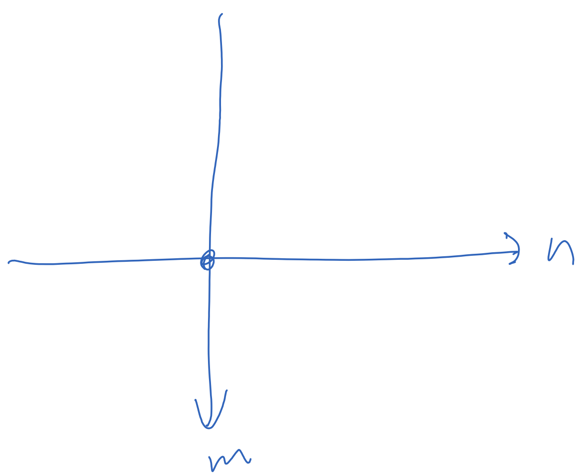
$$= \sin(\underbrace{\omega_{0x}\Delta x}\, m + \underbrace{\omega_{0y}\Delta y}\, n)$$

$$= \sin(\omega_{0m} m + \omega_{0n} n)$$

e

~s





l systems

$$f(m,n) \longrightarrow \boxed{\begin{matrix}\text{2-D}\\ \text{system}\\ T[\cdot]\end{matrix}} \longrightarrow g(m,n)$$

Integer
index values

-y

$f(m,n) + b\,g(m,n)] = aT[f(m,n)] + bT[g(m,n)]$

for all $a, b, f(m,n), g(m,n)$

nvariance

$[f(m-k, n-\ell)] = g(m-k, n-\ell)$

for all $k, \ell, f(m,n)$

(k, ℓ must be integers)

- can be represented by a convolution sum

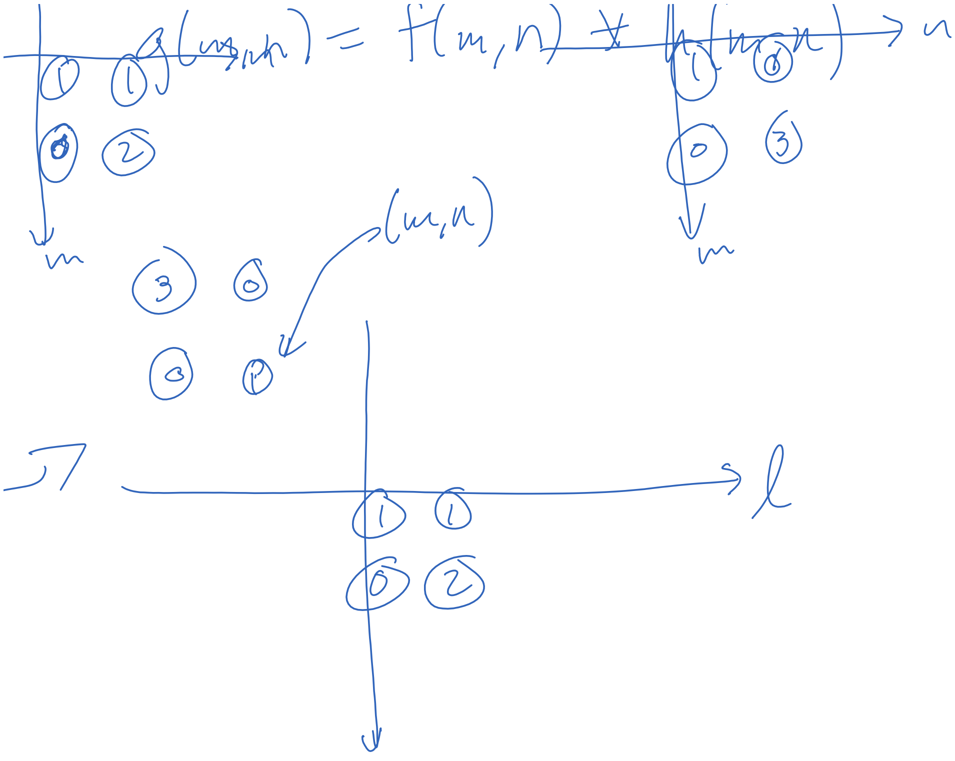
$$f(m,n) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} f(k,\ell)\,\delta(m-k, n-\ell)$$

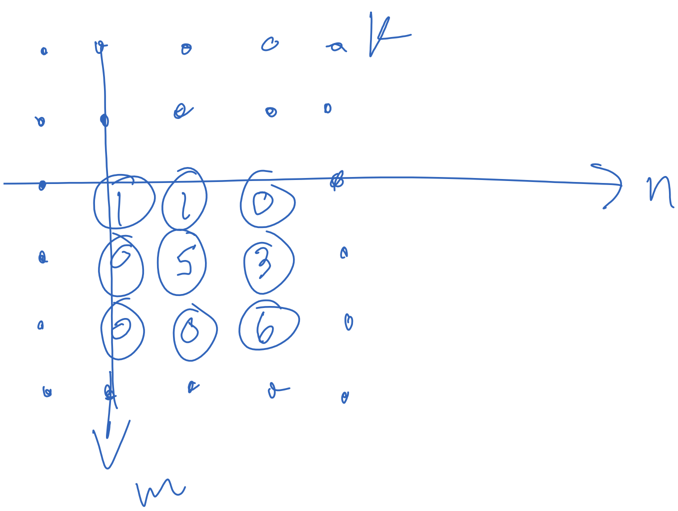
$$g(m,n) = \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} f(k,\ell)\,h(m-k, n-\ell)$$

where $h(m,n) = T[\delta(m,n)]$

is the impulse response $h(m,n)$

$g(m,n) = f(m,n) * h(m,n)$





ransform of 2-D sequence

? $x(n) = X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$

ω is in rad/sample

$\omega n) = \sum_{n=-\infty}^{\infty} x(m,n) e^{-j\omega_n n}$: 1-D FT of $x(m,n)$ along row m

$$n) = \sum_{m=-\infty}^{\infty} X_n(m; \omega_n) e^{-j\omega_m m} e^{-j(\omega_m m + \omega_n n)}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x(m,n) e$$

IFT of 2-D seq

$$\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} X(\omega_m, \omega_n) e^{j(\omega_m m + \omega_n n)} d\omega_m d\omega_n$$

$$= X(\omega_m - 2\pi, \omega_n) = X(\omega_m, \omega_n - 2\pi) = X(\omega_m - 2\pi, \omega_n - 2\pi)$$

impulse response gives frequency

: of system

$$H(\omega_m, \omega_n) = \mathcal{F}\{h(m,n)\}$$

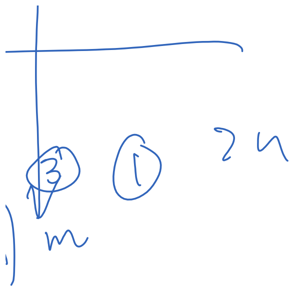
$$?(\omega_m, \omega_n) = F(\omega_m, \omega_n) H(\omega_m, \omega_n)$$

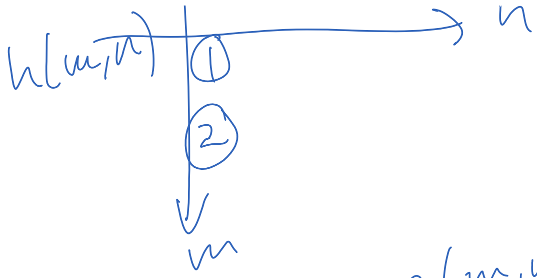
- convolution theorem
2

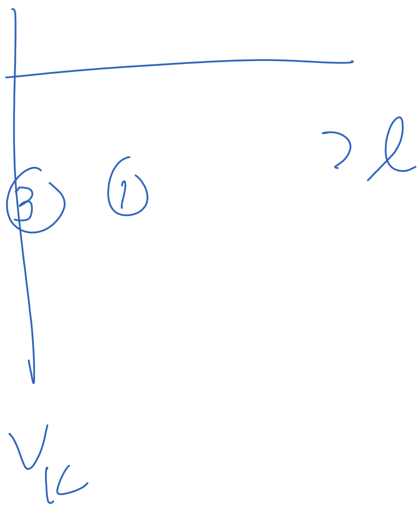
$$) * h(m,n) \longrightarrow F(\omega_m, \omega_n) H(\omega_m, \omega_n)$$

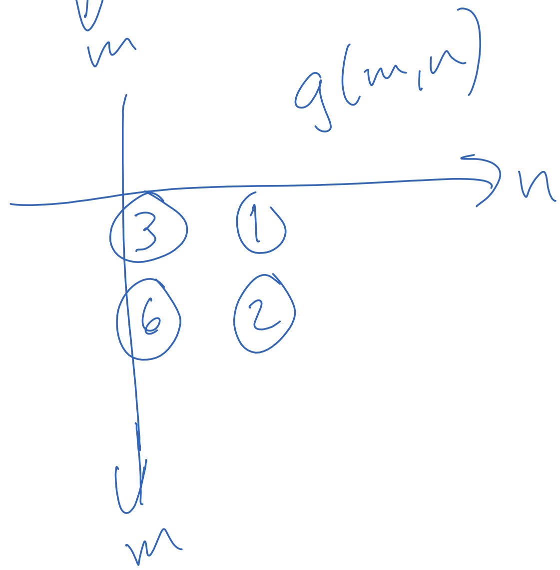
$$f(m,n) = 3\delta(m,n) + \delta(m, n-1)$$

$$(m,n) = \delta(m,n) + 2\delta(m-1, n)$$









$$f(m-k, n-l)e^{-j(\omega_m m + \omega_n n)}$$

$$(\omega_m k + \omega_n l)$$

$$F(\omega_m, \omega_n) = 3 + e^{-j\omega_n}$$

$$H(\omega_m, \omega_n) = 1 + 2e^{-j\omega_m}$$

$${}_m, \omega_n) = F(\omega_m, \omega_n) H(\omega_m, \omega_n)$$

$$= \left(3 + e^{-j\omega_n}\right)\left(1 + 2e^{-j\omega_m}\right)$$

$$= 3 + e^{-j\omega_n} + 6e^{-j\omega_m} + 2e^{-j(\omega_m + \omega_n)}$$

$$3\delta(m,n) + \delta(m,n-1) + 6\delta(m-1,n) + 2\delta(m-1,n-1)$$

: of discrete-space FT:

e – can decompose into two 1-D FTs

$f(m-k, n-l) \longleftrightarrow F(\omega_m, \omega_n)\, e^{-j(\omega_m k + \omega_n l)}$

ion $f(m,n)\,g(m,n) \longleftrightarrow \frac{1}{4\pi^2} F(\omega_m, \omega_n) * G(\omega_m, \omega_n)$

ll

ution thm.

al's theorem

$$\sum_{=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} |f(m,n)|^2 = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} |F(\omega_m, \omega_n)|^2 \, d\omega_m \, d\omega_n$$

i function and can't be stored in a

. Integrals can't be calculated perfectly

omputer

ete Fourier transform (DFT)

l image,

$) = F(\omega_m, \omega_n)\Big|_{(\omega_m, \omega_n) = \left(\frac{2\pi k}{M}, \frac{2\pi l}{N}\right)}$

$$0 \le k \le M - j\left(\frac{2\pi km}{, M} 0 \le l \frac{2\pi ln}{\le N-}\right)\}$$

$$= \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) \exp \quad \text{- DFT}$$

$$,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} F(k,l) \exp\left\{j\left(\frac{2\pi km}{M} + \frac{2\pi ln}{N}\right)\right\}$$

s:

r

val's theorem

$$\sum \sum |f(m,n)|^2 = \frac{1}{MN} \sum \sum |F(k,l)|^2$$

$$\sum \sum |f(m,n) - g(m,n)|^2 = \frac{1}{MN} \sum \sum |F(k,l) - G(k,l)|^2$$

rability - Can take 1-D DFT of rows
d then 1-D DFTs of columns (or reverse)

seq. is separable, then DFT is separable.
$n,n) = f_m(m) f_n(n))$

iodicity

$$k+M, l+N) = F(k, l+N) = F(k+M, l) = F(k,l)$$

$n+M, n+N) = f(m, n+N) = f(m+M, n) = f(m,n)$

cular shift

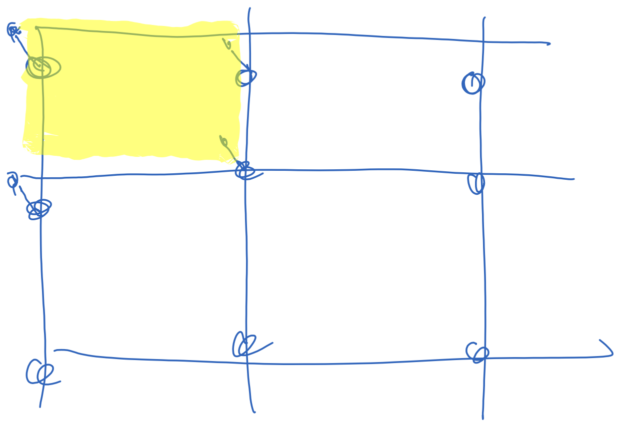
$_0 n - n_0) \overset{?}{\longleftrightarrow} F(k,\ell)\exp\left\{-j\left(\frac{2\pi k m_0}{M} + \frac{2\pi \ell n_0}{N}\right)\right\}$

$, ((n-n_0))_N)$

$((m))_M = m \bmod M$

ct of $\exp\left\{-j\left(\frac{2\pi k m_0}{M} + \frac{2\pi \ell n_0}{N}\right)\right\}$

$, n-n_0)$



$((n-n_0))_N)$

convolution

$(m,n) \overset{?}{\longleftrightarrow} \boxed{F(k,\ell)H(k,\ell)}$

$) = \sum_{k=0}^{M-1}\sum_{\ell=0}^{N-1} f(k,\ell)\, h(((m-k))_M, ((n-\ell))_N)$

e a linear convolution of $f(m,n)$

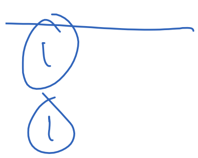
th a periodic version of $h(m,n)$

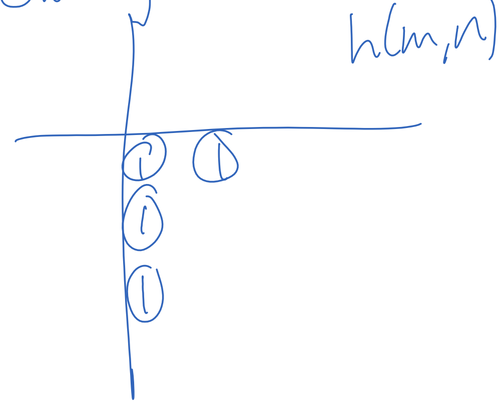
Size is $g(m,n)$? $M \times N$

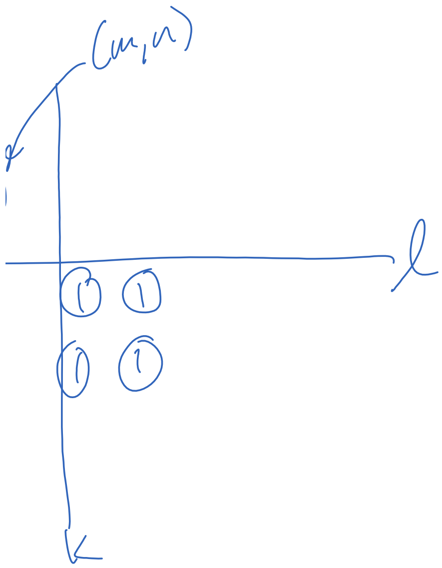
size is $f * h$? $(M_f + M_h - 1) \times (N_f + N_h - 1)$

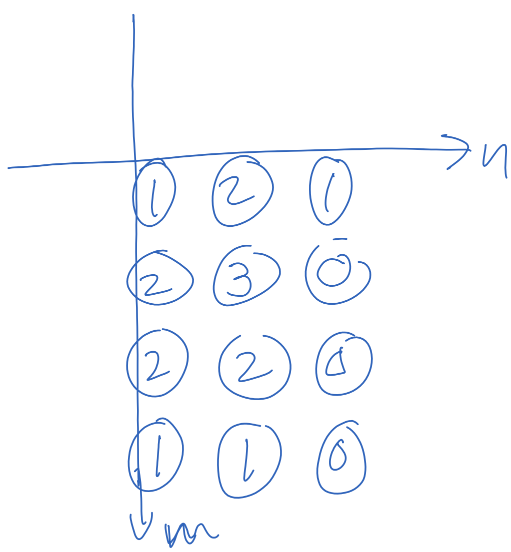
plement circular convolution by
tic extension of one of the signals

nplement by linear convolution
ed by periodic extension of the
t, then windowing

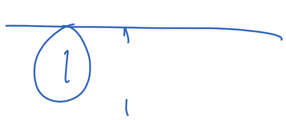








linear conv.



)(o) |
)(o) | Circ. conv.

convolution w/ DFTs:

et $h(m,n)$ be $M_h \times N_h$

$f(m,n)$ be $M_f \times N_f$

pad $h(m,n)$ and $f(m,n)$ to be

$\geq (M_f + M_h - 1) \times (N_f + N_h - 1)$

DFTs of padded seq's.

iply DFTs pointwise

$H .* F$

e IDFT of result

Fourier transform (FFT)

ient DFT implementation

ructed by decomposing DFT into
um of small DFTs
- requires $M^2 N^2$ multiplies
- requires $MN \log_2 MN$ multiplies

)24 x 1024,

DFT = 10^{12} mults.

FFT = 10^{7}

(1 day vs. 1 sec)

$$\sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) \exp\left\{-j\left(\frac{2\pi km}{M} + \frac{2\pi ln}{N}\right)\right\}$$

$$\sum_{m=0}^{M-1} \underbrace{\left\{\sum_{n=0}^{N-1} f(m,n) \exp\left\{-j\frac{2\pi ln}{N}\right\}\right\}}_{\text{1-D DFT}} \exp\left\{-j\frac{2\pi km}{M}\right\}$$

⇒ row-column decomposition

1-D FFT of all rows

Then 1-D FFT of columns of result

th DFT requires N^2 mults

es

row-col w/ DFTs

$+ \; N \cdot M^2 = NM(M+N)$

ompare to $M^2 N^2$

1-D FFTs

(each requires $\frac{1}{2} N \log_2 N$ mults)

or R-C w/ FFTs

$$\frac{1}{2} MN \log_2 N + \frac{1}{2} NM \log_2 M$$

$$= \frac{1}{2} MN \log_2 MN$$

$l \times 1024$,

direct DFT $= 2^{40} \approx 10^{12}$ mults

R-C DFT $= 2^{31} \approx 2 \times 10^{9}$ "

R-C FFT $= 10 \times 2^{20} \approx 10^{7}$

adix FFT

) divide-and-conquer strategy)

DFT is divided into successively
aller 2-D DFTs

ultiplies is $\frac{3}{8} N^2 \log_2 N^2$

ompared to $\frac{1}{2} N^2 \log_2 N^2$ for R-C FFT

* more complex than R-C FFT