# Financial Frictions and the Wealth Distribution

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# Motivation

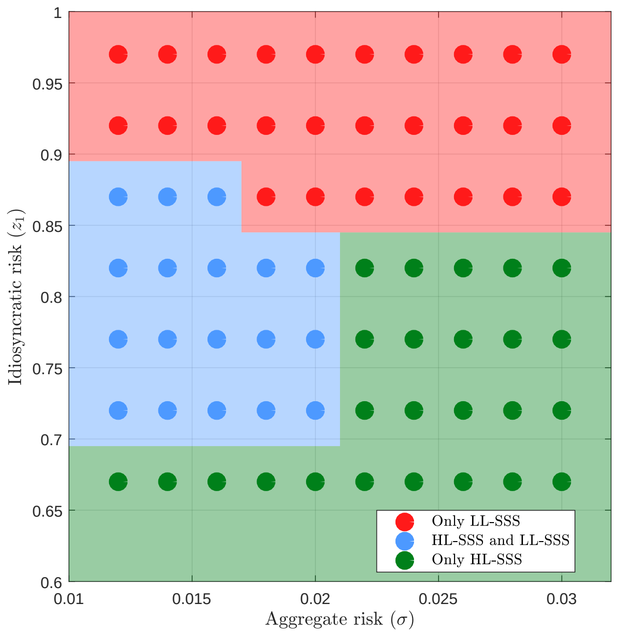
**Our goal**

How do financial frictions interact with households' wealth heterogeneity to shape aggregate dynamics?

- To investigate this question, we postulate, compute, and estimate a continuous-time model with a financial expert and a non-trivial distribution of wealth among households.
- We generate:
  1. Highly nonlinear behavior.
  2. Endogenously time-varying volatility, risk-free interest rate, and levels of leverage.
  3. Endogenous aggregate risk.

# Four main results

- Multiple stochastic steady states or SSS(s):
  - Why? Interaction of precautionary behavior by households with desire to issue debt by the financial expert.
  - Higher micro turbulence leads to higher macro volatility, more inequality, lower risk-free interest rates, and more leverage.
- Strong state-dependence on the responses of endogenous variables (GIRFs and DIRFs) to aggregate shocks.
- Long spells at different basins of attraction.
  - Multimodal and skewed ergodic distributions of endogenous variables, with endogenous time-varying volatility and aggregate risk.
- Thus, key importance of heterogeneity and breakdown of "approximate aggregation."


1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
Idiosyncratic risk ($z_1$)
0.01
0.015
0.02
0.025
0.03
Aggregate risk ($\sigma$)
Only LL-SSS
HL-SSS and LL-SSS
Only HL-SSS

# Methodological contribution

- New approach to (globally) compute and estimate with the likelihood approach HA models:

    1. Computation: we use tools from machine learning.

    2. Estimation: we use tools from inference with diffusions.

- Strong theoretical foundations and many practical advantages.

    1. Deal with a large class of arbitrary operators efficiently.

    2. Algorithm that is i) easy to code, ii) stable, iii) scalable, and iv) massively parallel.

# The firm

- Representative firm with technology:

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}$$

- Competitive input markets:

$$w_t = (1-\alpha) K_t^{\alpha} L_t^{-\alpha}$$
$$rc_t = \alpha K_t^{\alpha-1} L_t^{1-\alpha}$$

- Aggregate capital evolves:

$$\frac{dK_t}{K_t} = (\iota_t - \delta)\, dt + \sigma dZ_t$$

- Instantaneous return rate on capital $dr_t^k$:

$$dr_t^k = (rc_t - \delta)\, dt + \sigma dZ_t$$

# The expert I

- Representative expert holds capital $\widehat{K}_t$ and issues risk-free debt $\widehat{B}_t$ at rate $r_t$ to households.
- Expert can be interpreted as a financial intermediary.
- Financial friction: expert cannot issue state-contingent claims (i.e., outside equity) and must absorb all risk from capital.
- Expert's net wealth (i.e., inside equity): $\widehat{N}_t = \widehat{K}_t - \widehat{B}_t$.
- Together with market clearing, our assumptions imply that economy has a risky asset in positive net supply and a risk-free asset in zero net supply.

# The expert II

- The law of motion for expert's net wealth $\widehat{N}_t$:

$$\begin{aligned} d\widehat{N}_t &= \widehat{K}_t dr_t^k - r_t \widehat{B}_t dt - \widehat{C}_t dt \\ &= \left[ \left( r_t + \widehat{\omega}_t \left( rc_t - \delta - r_t \right) \right) \widehat{N}_t - \widehat{C}_t \right] dt + \sigma \widehat{\omega}_t \widehat{N}_t dZ_t \end{aligned}$$

  where $\widehat{\omega}_t \equiv \frac{\widehat{K}_t}{\widehat{N}_t}$ is the leverage ratio.

- The law of motion for expert's capital $\widehat{K}_t$:

$$d\widehat{K}_t = d\widehat{N}_t + d\widehat{B}_t$$

- The expert decides her consumption levels and capital holdings to solve:

$$\max_{\left\{ \widehat{C}_t, \widehat{\omega}_t \right\}_{t \geq 0}} \mathbb{E}_0 \left[ \int_0^\infty e^{-\widehat{\rho} t} \log(\widehat{C}_t) dt \right]$$

  given initial conditions and a NPG condition.

# Households I

- Continuum of infinitely-lived households with unit mass.
- Heterogeneous in wealth $a_m$ and labor supply $z_m$ for $m \in [0,1]$.
- $G_t(a,z)$: distribution of households conditional on realization of aggregate variables.
- Preferences:

$$\mathbb{E}_0 \left[ \int_0^\infty e^{-\rho t} \frac{c_t^{1-\gamma} - 1}{1-\gamma} dt \right]$$

- We could have more general Duffie and Epstein (1992) recursive preferences.
- $\rho > \widehat{\rho}$. Intuition from Aiyagari (1994) (and different from BGG class of models!).

# Households II

- $z_t$ units of labor valued at wage $w_t$.
- Labor productivity evolves stochastically following a Markov chain:
  1. $z_t \in \{z_1, z_2\}$, with $z_1 < z_2$.
  2. Ergodic mean of $z_t$ is $1$.
  3. Jump intensity from state 1 to state 2: $\lambda_1$ (reverse intensity is $\lambda_2$).
- Households save $a_t \geq 0$ in the riskless debt issued by experts with an interest rate $r_t$. Thus, their wealth follows:

$$da_t = (w_t z_t + r_t a_t - c_t)\, dt = s\left(a_t, z_t, K_t, G_t\right) dt$$

- Optimal choice: $c_t = c\left(a_t, z_t, K_t, G_t\right)$.
- Total consumption by households:

$$C_t \equiv \int c\left(a_t, z_t, K_t, G_t\right) dG_t\left(a, z\right)$$

# Market clearing

1. Total amount of labor rented by the firm is equal to labor supplied:
$$L_t = \int z dG_t = 1$$
Then, total payments to labor are given by $w_t$.
2. Total amount of debt of the expert equals the total households' savings:
$$B_t \equiv \int a dG_t \left(da, dz\right) = \widehat{B}_t$$
with law of motion $d\widehat{B}_t = dB_t = \left(w_t + r_t B_t - C_t\right) dt$.
3. The total amount of capital in this economy is owned by the expert:
$$K_t = \widehat{K}_t$$
Thus, $d\widehat{K}_t = dK_t = \left(Y_t - \delta K_t - C_t - \widehat{C}_t\right) dt + \sigma K_t dZ_t$ and $\widehat{\omega}_t = \frac{K_t}{N_t}$, where $\widehat{N}_t = N_t = K_t - B_t$.
4. Also:
$$\iota_t = \frac{Y_t - C_t - \widehat{C}_t}{K_t}$$

# Density

- The households distribution $G_t(a, z)$ has density (i.e., the Radon-Nikodym derivative) $g_t(a, z)$.

- The dynamics of this density conditional on the realization of aggregate variables are given by the Kolmogorov forward (KF) equation:

$$\frac{\partial g_{it}}{\partial t} = -\frac{\partial}{\partial a}\left(s\left(a_t, z_t, K_t, G_t\right) g_{it}(a)\right) - \lambda_i g_{it}(a) + \lambda_j g_{jt}(a), \;\; i \neq j = 1, 2$$

where $g_{it}(a) \equiv g_t(a, z_i)$, $i = 1, 2$.

- The density satisfies the normalization:

$$\sum_{i=1}^{2} \int_0^{\infty} g_{it}(a) da = 1$$

# Equilibrium

An equilibrium in this economy is composed by a set of prices $\left\{w_t, rc_t, r_t, r_t^k\right\}_{t\geq 0}$, quantities $\left\{K_t, N_t, B_t, \widehat{C}_t, c_{mt}\right\}_{t\geq 0}$, and a density $\left\{g_t\left(\cdot\right)\right\}_{t\geq 0}$ such that:

1. Given $w_t$, $r_t$, and $g_t$, the solution of the household $m$'s problem is $c_t = c\left(a_t, z_t, K_t, G_t\right)$.
2. Given $r_t^k$, $r_t$, and $N_t$, the solution of the expert's problem is $\widehat{C}_t$, $K_t$, and $B_t$.
3. Given $K_t$, firms maximize their profits and input prices are given by $w_t$ and $rc_t$.
4. Given $w_t$, $r_t$, and $c_t$, $g_t$ is the solution of the KF equation.
5. Given $g_t$ and $B_t$, the debt market clears.

# Characterizing the equilibrium I

- First, we proceed with the expert's problem. Because of log-utility:

$$\widehat{C}_t = \widehat{\rho} N_t$$

$$\omega_t = \widehat{\omega}_t = \frac{rc_t - \delta - r_t}{\sigma^2}$$

- We can use the equilibrium values of $rc_t$, $L_t$, and $\omega_t$ to get the wage:

$$w_t = (1-\alpha) K_t^{\alpha}$$

the rental rate of capital:

$$rc_t = \alpha K_t^{\alpha-1}$$

and the risk-free interest rate:

$$r_t = \alpha K_t^{\alpha-1} - \delta - \sigma^2 \frac{K_t}{N_t}$$

# Characterizing the equilibrium II

- Expert's net wealth evolves as:

$$dN_t = \underbrace{\left(\alpha K_t^{\alpha-1} - \delta - \widehat{\rho} - \sigma^2 \left(1 - \frac{K_t}{N_t}\right) \frac{K_t}{N_t}\right)}_{\mu_t^N(B_t, N_t)} N_t dt + \underbrace{\sigma K_t}_{\sigma_t^N(B_t, N_t)} dZ_t$$

- And debt as:

$$dB_t = \left((1-\alpha) K_t^{\alpha} + \left(\alpha K_t^{\alpha-1} - \delta - \sigma^2 \frac{K_t}{N_t}\right) B_t - C_t\right) dt$$

- Nonlinear structure of law of motion for $dN_t$ and $dB_t$.
- We need to find:

$$C_t \equiv \int c\left(a_t, z_t, K_t, G_t\right) g_t\left(a, z\right) dadz$$

$$\frac{\partial g_{it}}{\partial t} = -\frac{\partial}{\partial a}\left(s\left(a_t, z_t, K_t, G_t\right) g_{it}(a)\right) - \lambda_i g_{it}(a) + \lambda_j g_{jt}(a), \;\; i \neq j = 1, 2$$

# The DSS

- No aggregate shocks ($\sigma = 0$), but we still have idiosyncratic household shocks.
- Then:

$$r = r_t^k = rc_t - \delta = \alpha K_t^{\alpha-1} - \delta$$

  and

$$dN_t = \left(\alpha K_t^{\alpha-1} - \delta - \widehat{\rho}\right) N_t dt$$

- Since in a steady state the drift of expert's wealth must be zero, we get:

$$K = \left(\frac{\widehat{\rho} + \delta}{\alpha}\right)^{\frac{1}{\alpha-1}}$$

  and:

$$r = \widehat{\rho} < \rho$$

- The value of $N$ is given by the dispersion of the idiosyncratic shocks (no analytic expression).

# How do we find aggregate consumption?

- As in Krusell and Smith (1998), households only track a finite set of $n$ moments of $g_t(a, z)$ to form their expectations.
- No exogenous state variable (shocks to capital encoded in $K$). Instead, two endogenous states.
- For ease of exposition, we set $n = 1$. The solution can be trivially extended to the case with $n > 1$.
- More concretely, households consider a *perceived law of motion* (PLM) of aggregate debt:

$$dB_t = h\left(B_t, N_t\right) dt$$

where

$$h\left(B_t, N_t\right) = \frac{\mathbb{E}\left[dB_t | B_t, N_t\right]}{dt}$$

## A new HJB equation

- Given the PLM, the household's Hamilton-Jacobi-Bellman (HJB) equation becomes:

$$\begin{aligned}\rho V_i(a,B,N) &= \max_c \frac{c^{1-\gamma}-1}{1-\gamma} + s\frac{\partial V_i}{\partial a} + \lambda_i\left[V_j(a,B,N) - V_i(a,B,N)\right] \\ &\quad + h(B,N)\frac{\partial V_i}{\partial B} + \mu^N(B,N)\frac{\partial V_i}{\partial N} + \frac{\left[\sigma^N(B,N)\right]^2}{2}\frac{\partial^2 V_i}{\partial N^2}\end{aligned}$$

  $i \neq j = 1, 2$, and where

$$s = s(a, z, N+B, G)$$

- We solve the HJB with a first-order, implicit upwind scheme in a finite difference stencil.

- Sparse system. Why?

- Alternatives for solving the HJB? Meshfree, FEM, deep learning, ...

# An algorithm to find the PLM

1) Start with $h_0$, an initial guess for $h$.

2) Using current guess $h_n$, solve for the household consumption, $c_m$, in the HJB equation.

3) Construct a time series for $B_t$ by simulating by $J$ periods the cross-sectional distribution of households with a constant time step $\Delta t$ (starting at DSS and with a burn-in).

4) Given $B_t$, find $N_t$, $K_t$, and:

$$\widehat{\mathbf{h}} = \left\{ \widehat{h}_1, \widehat{h}_2 ..., \widehat{h}_j \equiv \frac{B_{t_j+\Delta t} - B_{t_j}}{\Delta t}, ..., \widehat{h}_J \right\}$$

5 ) Define $\mathbf{S} = \{\mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_J\}$, where $\mathbf{s}_j = \left\{ s_j^1, s_j^2 \right\} = \left\{ B_{t_j}, N_{t_j} \right\}$.

6) Use $\left(\widehat{\mathbf{h}}, \mathbf{S}\right)$ and a universal nonlinear approximator to obtain $h_{n+1}$, a new guess for $h$.

7) Iterate steps 2)-6) until $h_{n+1}$ is sufficiently close to $h_n$.

# A universal nonlinear approximator

- We approximate the PLM with a neural network (NN):

$$h(\mathbf{s};\theta) = \theta_0^1 + \sum_{q=1}^{Q} \theta_q^1 \phi\left(\theta_{0,q}^2 + \sum_{i=1}^{D} \theta_{i,q}^2 s^i\right)$$

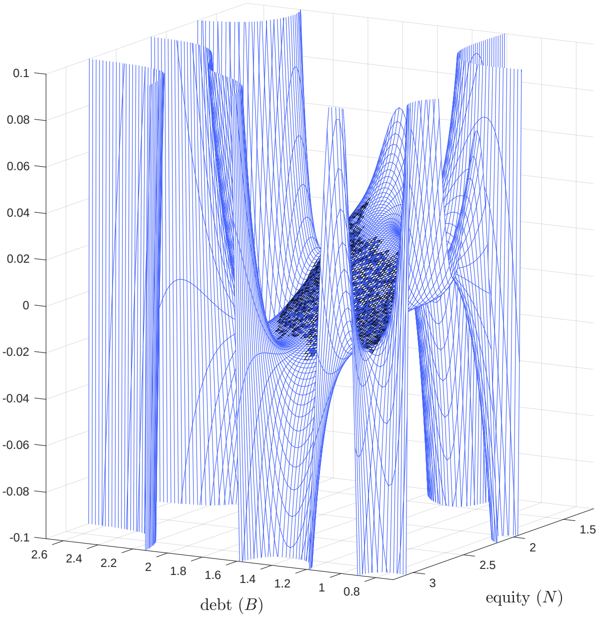
  where $Q = 16$, $D = 2$, and $\phi(x) = \log(1 + e^x)$.

- $\theta$ is selected as:

$$\theta^* = \arg\min_{\theta} \frac{1}{2} \sum_{j=1}^{J} \left\| h(\mathbf{s}_j;\theta) - \widehat{h}_j \right\|^2$$

- Easy to code, stable, and good extrapolation properties.

- You can flush the algorithm to a GPU, a TPU, a FPGA, or a AI accelerator instead of a standard CPU.


0.1
0.08
0.06
0.04
0.02
0
-0.02
-0.04
-0.06
-0.08
-0.1
2.6
2.4
2.2
2
1.8
1.6
1.4
1.2
1
0.8
debt $(B)$
1.5
2
2.5
3
equity $(N)$

# Two classic (yet remarkable) results

**Universal approximation theorem: Hornik, Stinchcombe, and White (1989)**

A neural network with at least one hidden layer can approximate any Borel measurable function mapping finite-dimensional spaces to any desired degree of accuracy.

- Assume, as well, that we are dealing with the class of functions for which the Fourier transform of their gradient is integrable.

**Breaking the curse of dimensionality: Barron (1993)**

A one-layer NN achieves integrated square errors of order $\mathcal{O}(1/Q)$, where $Q$ is the number of nodes. In comparison, for series approximations, the integrated square error is of order $\mathcal{O}(1/(Q^{2/D}))$ where $D$ is the dimensions of the function to be approximated.

- We actually rely on more general theorems by Leshno et al. (1993) and Bach (2017).

# Estimation with aggregate variables I

- $D+1$ observations of $Y_t$ at fixed time intervals $[0, \Delta, 2\Delta, .., D\Delta]$:

$$Y_0^D = \{Y_0, Y_\Delta, Y_{2\Delta}, ..., Y_D\}.$$

- More general case: sequential Monte Carlo approximation to the Kushner-Stratonovich equation (Fernández-Villaverde and Rubio Ramírez, 2007).

- We are interested in estimating a vector of structural parameters $\Psi$.

- Likelihood:

$$\mathcal{L}_D\left(Y_0^D|\Psi\right) = \prod_{d=1}^{D} p_Y\left(Y_{d\Delta}|Y_{(d-1)\Delta};\Psi\right),$$

  where

$$p_Y\left(Y_{d\Delta}|Y_{(d-1)\Delta};\Psi\right) = \int f_{d\Delta}(Y_{d\Delta}, B)dB.$$

  given a density, $f_{d\Delta}(Y_{d\Delta}, B)$, implied by the solution of the model.

# Estimation with aggregate variables II

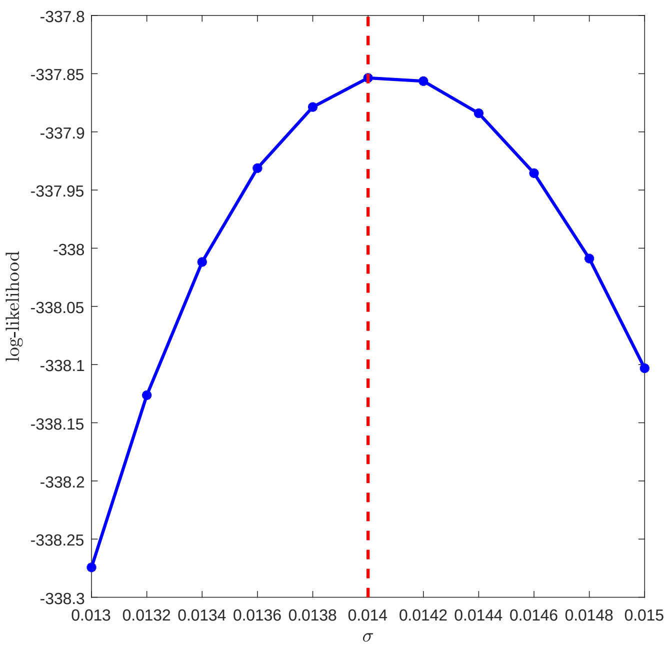
- After finding the diffusion for $Y_t$, $f_t^d(Y,B)$ follows the Kolmogorov forward (KF) equation in the interval $[(d-1)\Delta, d\Delta]$:

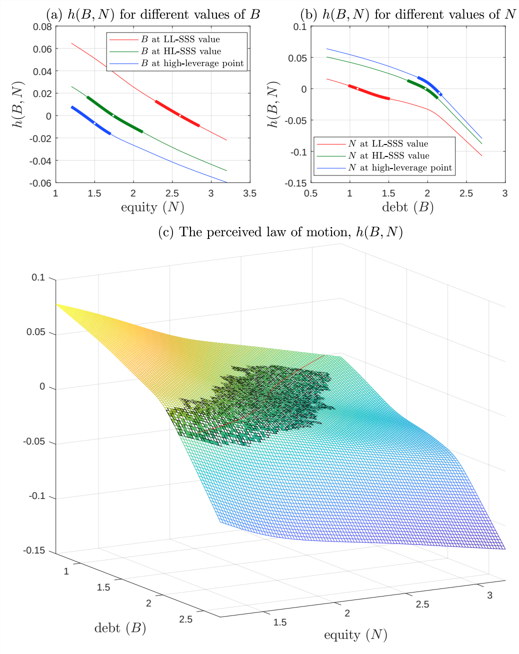
$$
\begin{aligned}
\frac{\partial f_t}{\partial t} &= -\frac{\partial}{\partial Y}\left[\mu^Y(Y,B) f_t(Y,B)\right] - \frac{\partial}{\partial B}\left[h(B, Y^{\frac{1}{\alpha}} - B) f_t^d(Y,B)\right] \\
&\quad + \frac{1}{2}\frac{\partial^2}{\partial Y^2}\left[\left(\sigma^Y(Y)\right)^2 f_t(Y,B)\right]
\end{aligned}
$$

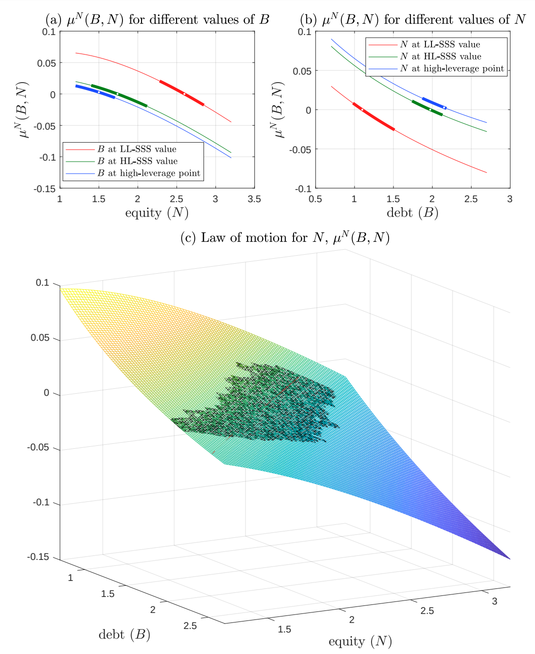
- The operator in the KF equation is the adjoint of the infinitesimal generator of the HJB.
- Thus, the solution of the KF equation amounts to transposing and inverting a sparse matrix that has already been computed.
- Our approach provides a highly efficient way of evaluating the likelihood once the model is solved.
- Conveniently, retraining of the neural network is easy for new parameter values.

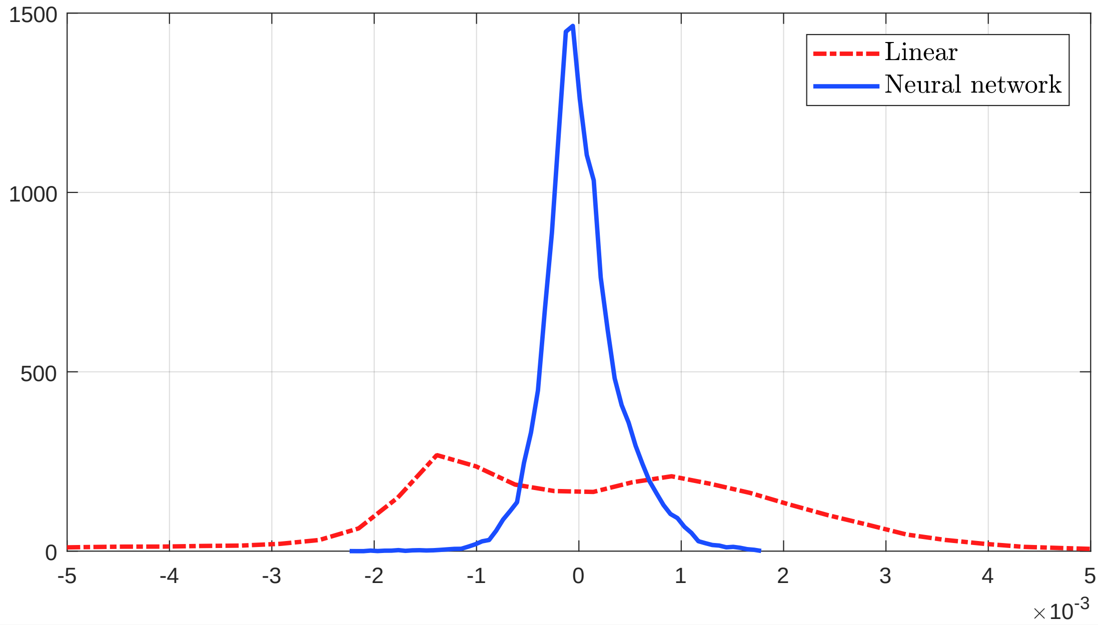
# Parametrization

| Parameter | Value | Description | Source/Target |
|---|---|---|---|
| $\alpha$ | 0.35 | capital share | standard |
| $\delta$ | 0.1 | yearly capital depreciation | standard |
| $\gamma$ | 2 | risk aversion | standard |
| $\rho$ | 0.05 | households' discount rate | standard |
| $\lambda_1$ | 0.986 | transition rate u.-to-e. | monthly job finding rate of 0.3 |
| $\lambda_2$ | 0.052 | transition rate e.-to-u. | unemployment rate 5 percent |
| $y_1$ | 0.72 | income in unemployment state | Hall and Milgrom (2008) |
| $y_2$ | 1.015 | income in employment state | $\mathbb{E}(y) = 1$ |
| $\widehat{\rho}$ | 0.0497 | experts' discount rate | $K/N = 2$ |


-337.8
-337.85
-337.9
-337.95
-338
-338.05
-338.1
-338.15
-338.2
-338.25
-338.3
log-likelihood
0.013 0.0132 0.0134 0.0136 0.0138 0.014 0.0142 0.0144 0.0146 0.0148 0.015
σ


(a) $h(B,N)$ for different values of $B$
$B$ at LL-SSS value
$B$ at HL-SSS value
$B$ at high-leverage point
$h(B,N)$
equity $(N)$
(b) $h(B,N)$ for different values of $N$
$N$ at LL-SSS value
$N$ at HL-SSS value
$N$ at high-leverage point
$h(B,N)$
debt $(B)$
(c) The perceived law of motion, $h(B,N)$
debt $(B)$
equity $(N)$


(a) $\mu^N(B,N)$ for different values of $B$
$\mu^N(B,N)$
equity $(N)$
$B$ at LL-SSS value
$B$ at HL-SSS value
$B$ at high-leverage point
(b) $\mu^N(B,N)$ for different values of $N$
$\mu^N(B,N)$
debt $(B)$
$N$ at LL-SSS value
$N$ at HL-SSS value
$N$ at high-leverage point
(c) Law of motion for $N$, $\mu^N(B,N)$
debt $(B)$
equity $(N)$


1500
1000
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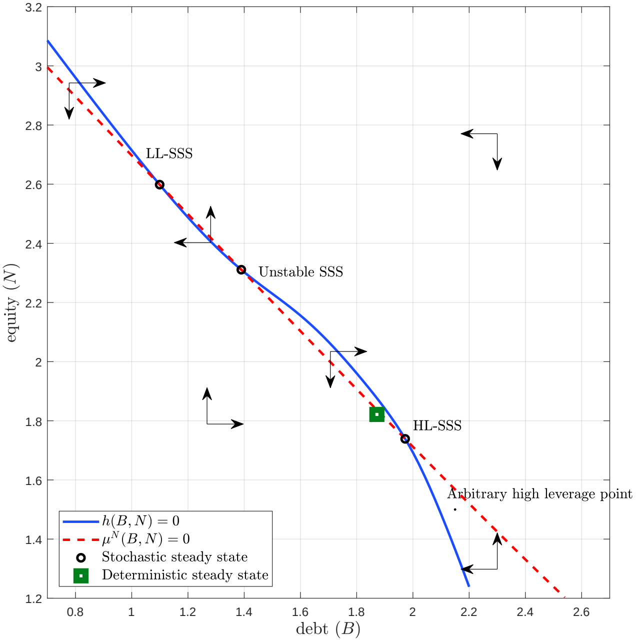
1
2
3
4
5
×10^-3
Linear
Neural network



LL-SSS
Unstable SSS
HL-SSS
Arbitrary high leverage point
$h(B,N)=0$
$\mu^N(B,N)=0$
Stochastic steady state
Deterministic steady state
equity $(N)$
debt $(B)$
3.2
3
2.8
2.6
2.4
2.2
2
1.8
1.6
1.4
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1
1.2
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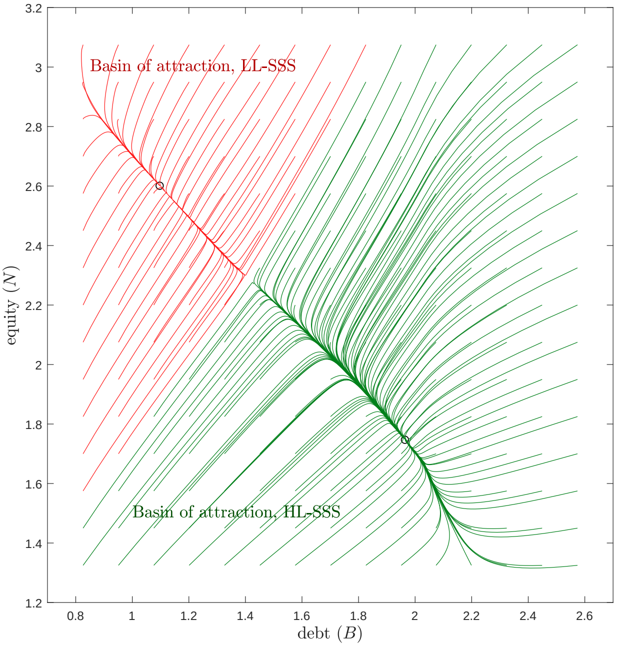
2
2.2
2.4
2.6


Basin of attraction, LL-SSS
Basin of attraction, HL-SSS
equity ($N$)
debt ($B$)
3.2
3
2.8
2.6
2.4
2.2
2
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1.6
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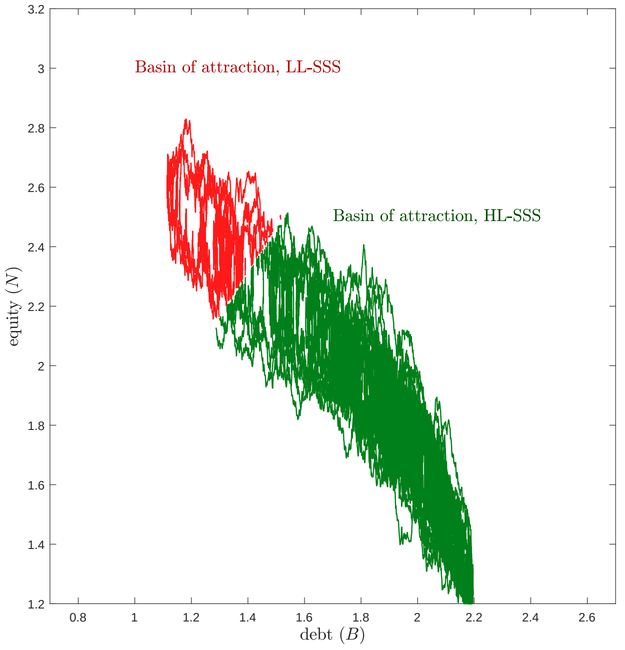

Basin of attraction, LL-SSS
Basin of attraction, HL-SSS
equity $(N)$
debt $(B)$
3.2
3
2.8
2.6
2.4
2.2
2
1.8
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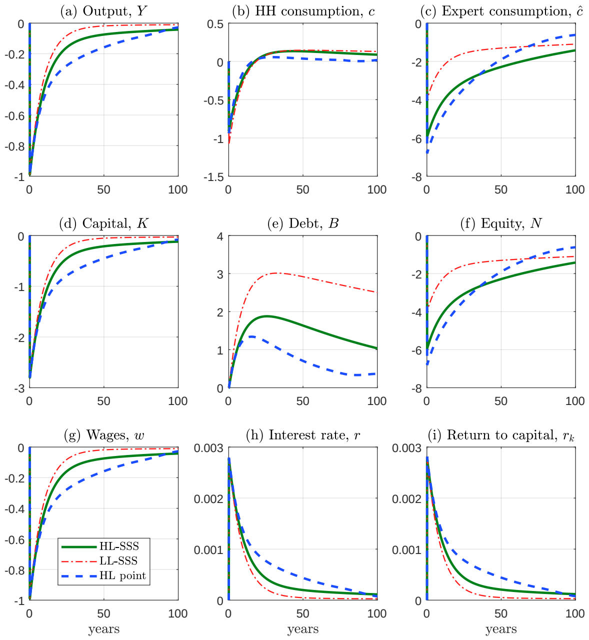
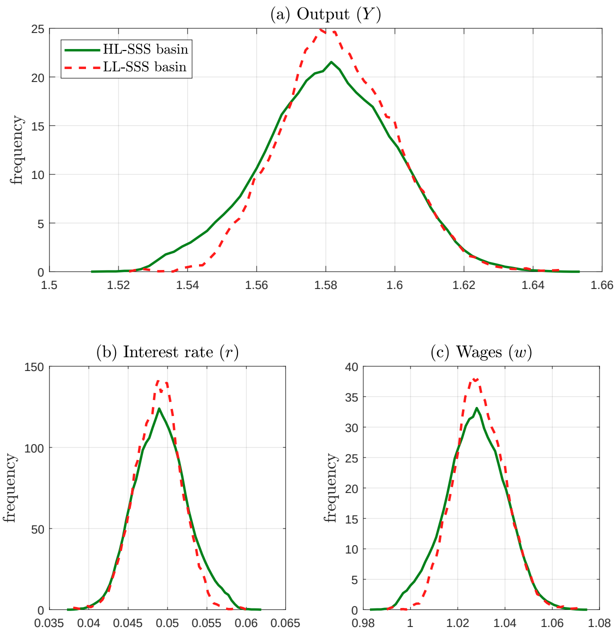
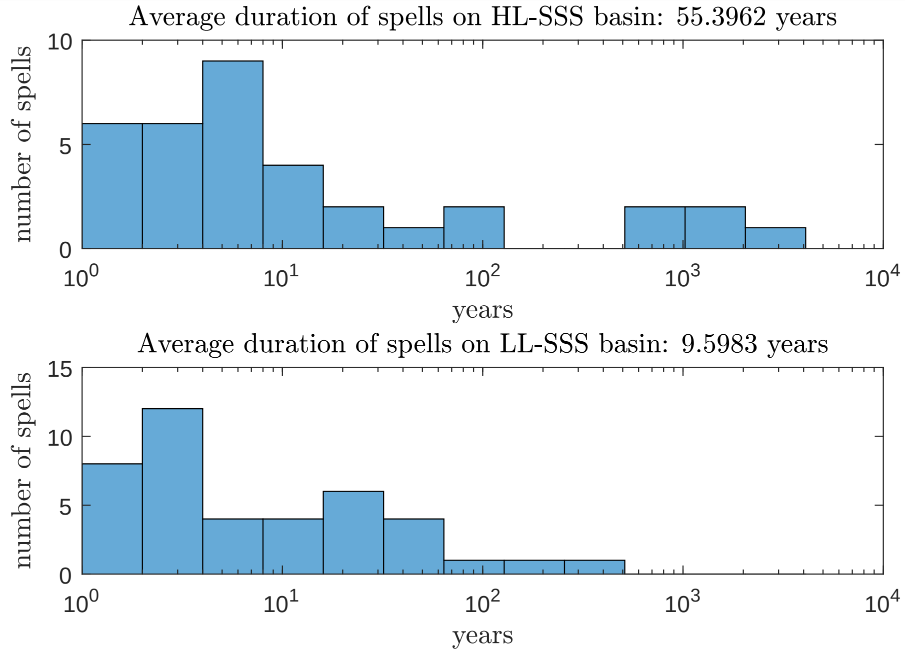
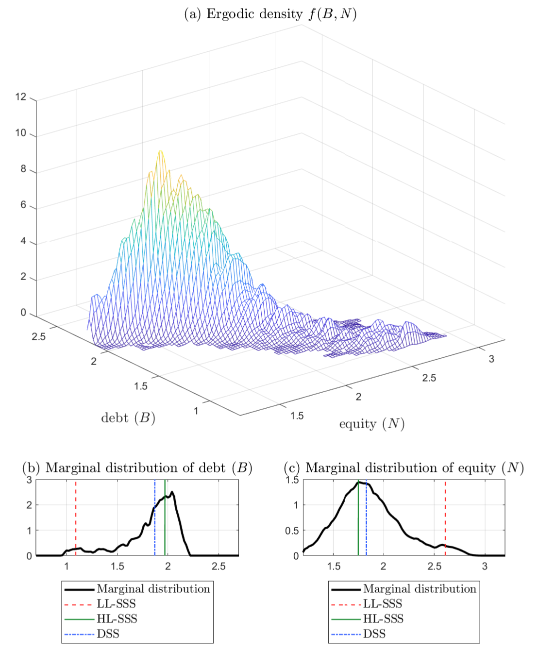

(a) Output, $Y$
(b) HH consumption, $c$
(c) Expert consumption, $\hat{c}$
(d) Capital, $K$
(e) Debt, $B$
(f) Equity, $N$
(g) Wages, $w$
(h) Interest rate, $r$
(i) Return to capital, $r_k$
HL-SSS
LL-SSS
HL point
years

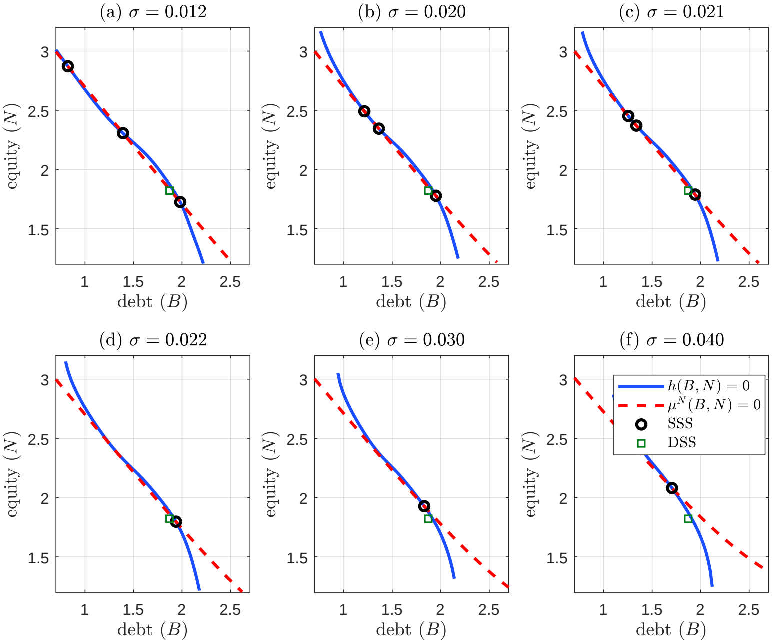
| | Mean | Standard deviation | Skewness | Kurtosis |
|---|---|---|---|---|
| $Y^{\text{basin } HL}$ | 1.5807 | 0.0193 | -0.0831 | 2.8750 |
| $Y^{\text{basin } LL}$ | 1.5835 | 0.0166 | 0.16417 | 3.1228 |
| $r^{\text{basin } HL}$ | 4.92 | 0.3360 | 0.1725 | 2.8967 |
| $r^{\text{basin } LL}$ | 4.88 | 0.2896 | -0.0730 | 3.0905 |
| $w^{\text{basin } HL}$ | 1.0274 | 0.0125 | -0.0831 | 2.875 |
| $w^{\text{basin } LL}$ | 1.0293 | 0.0108 | 0.1642 | 3.1228 |

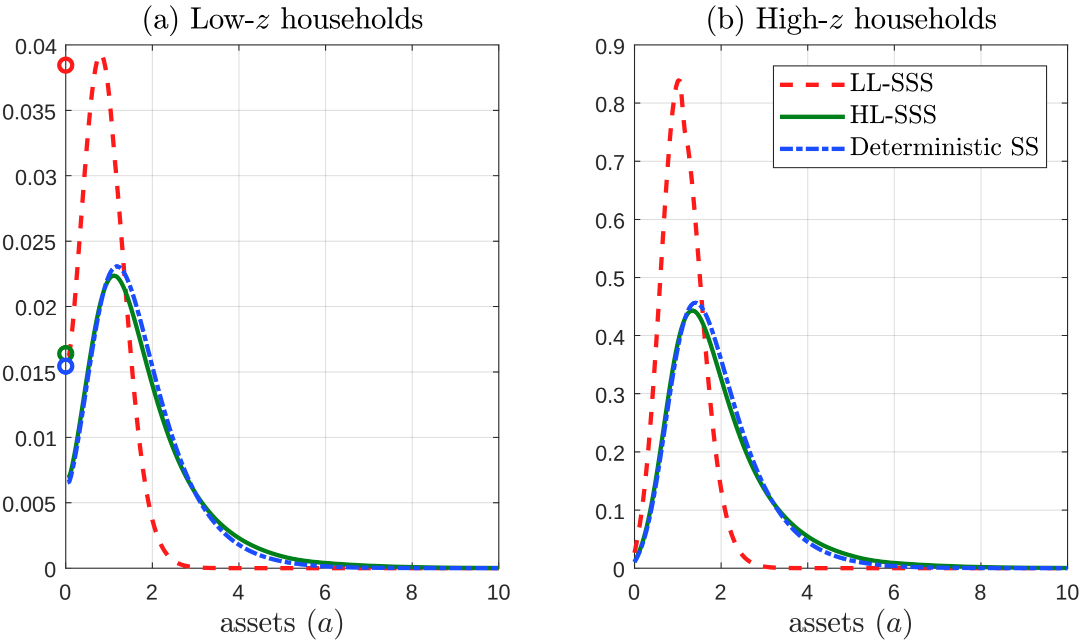
**Table 1:** Moments conditional on basin of attraction.


(a) Output ($Y$)
HL-SSS basin
LL-SSS basin
frequency
0
5
10
15
20
25
1.5
1.52
1.54
1.56
1.58
1.6
1.62
1.64
1.66
(b) Interest rate ($r$)
frequency
0
50
100
150
0.035
0.04
0.045
0.05
0.055
0.06
0.065
(c) Wages ($w$)
frequency
0
5
10
15
20
25
30
35
40
0.98
1
1.02
1.04
1.06
1.08

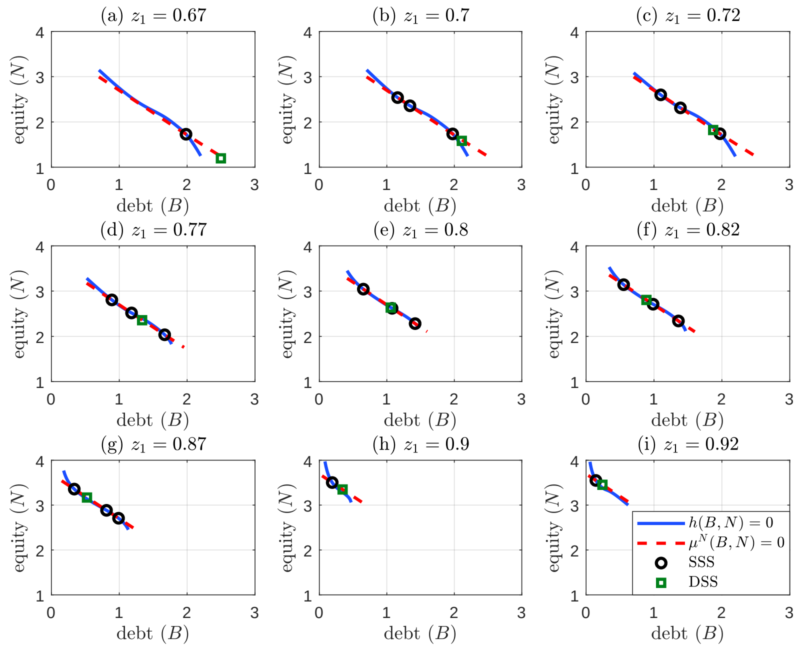

Average duration of spells on HL-SSS basin: 55.3962 years
number of spells
10
5
0
$10^0$
$10^1$
$10^2$
$10^3$
$10^4$
years
Average duration of spells on LL-SSS basin: 9.5983 years
number of spells
15
10
5
0
$10^0$
$10^1$
$10^2$
$10^3$
$10^4$
years

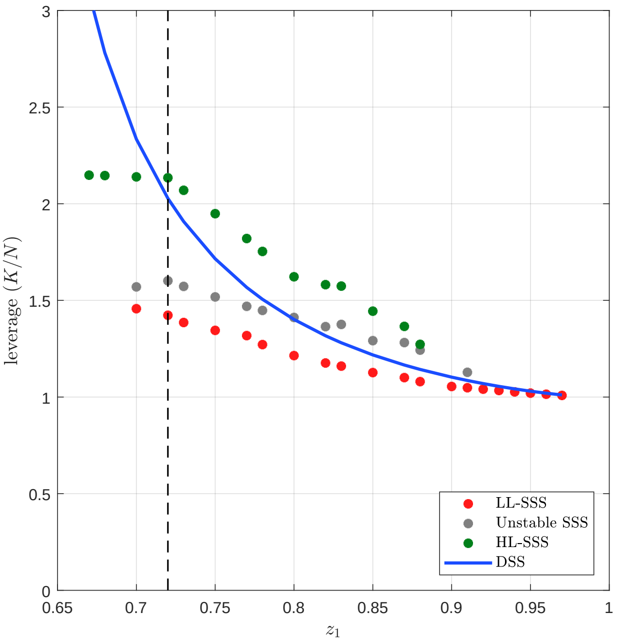

(a) Ergodic density $f(B, N)$
12
10
8
6
4
2
0
2.5
2
1.5
1
debt $(B)$
1.5
2
2.5
3
equity $(N)$
(b) Marginal distribution of debt $(B)$
3
2
1
0
1
1.5
2
2.5
Marginal distribution
LL-SSS
HL-SSS
DSS
(c) Marginal distribution of equity $(N)$
1.5
1
0.5
0
1.5
2
2.5
3
Marginal distribution
LL-SSS
HL-SSS
DSS

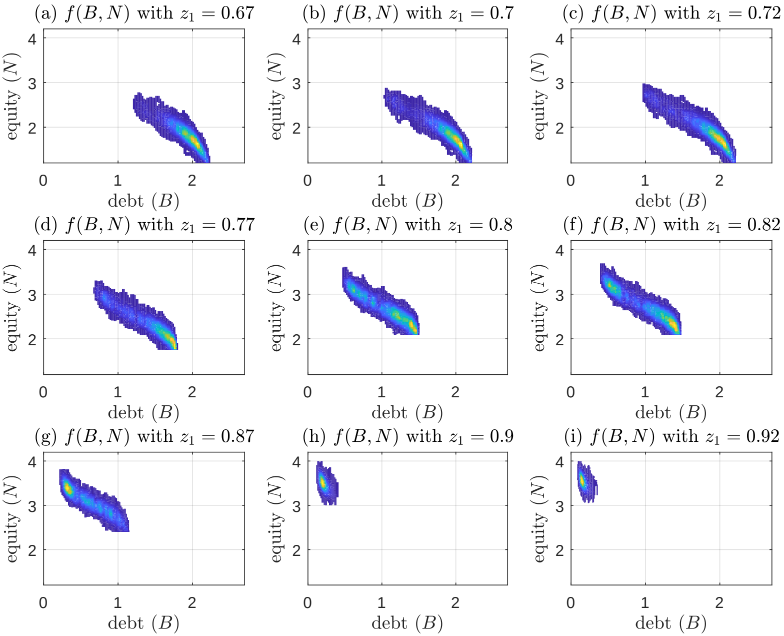

(a) $\sigma = 0.012$
(b) $\sigma = 0.020$
(c) $\sigma = 0.021$
(d) $\sigma = 0.022$
(e) $\sigma = 0.030$
(f) $\sigma = 0.040$
equity $(N)$
debt $(B)$
$h(B, N) = 0$
$\mu^N(B, N) = 0$
SSS
DSS



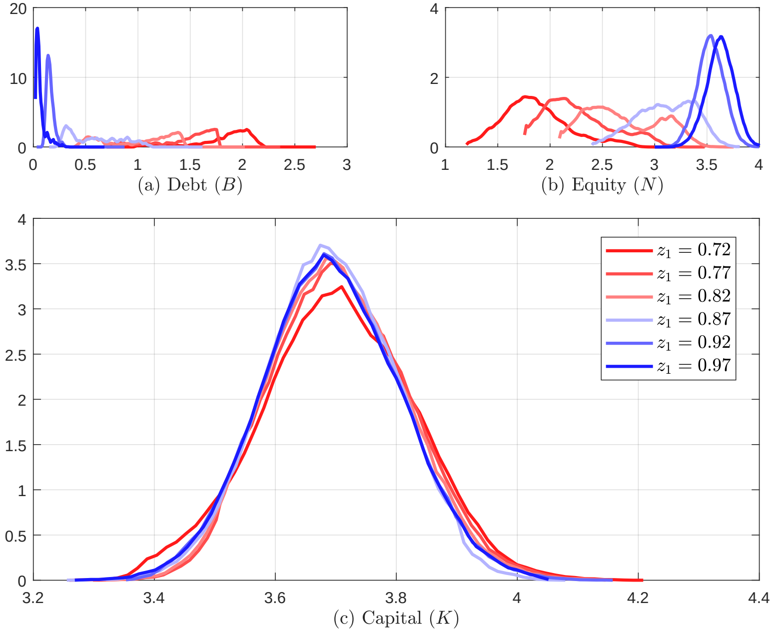
(a) Low-$z$ households
(b) High-$z$ households
LL-SSS
HL-SSS
Deterministic SS
assets $(a)$
assets $(a)$
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
2
4
6
8
10

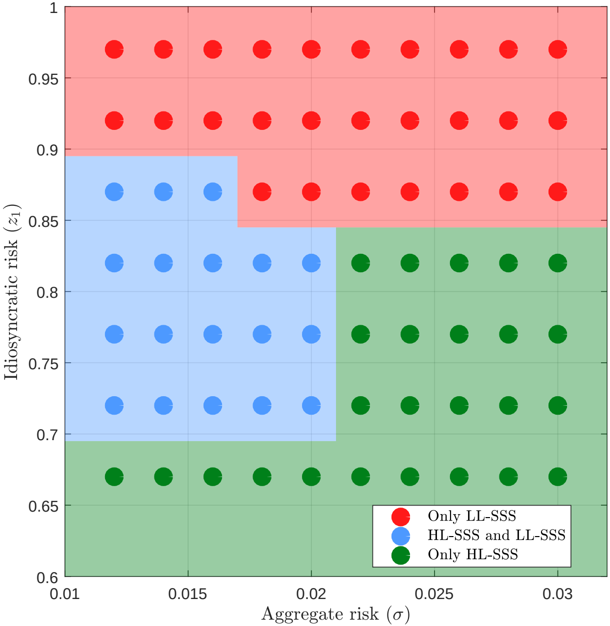

(a) $z_1 = 0.67$
(b) $z_1 = 0.7$
(c) $z_1 = 0.72$
(d) $z_1 = 0.77$
(e) $z_1 = 0.8$
(f) $z_1 = 0.82$
(g) $z_1 = 0.87$
(h) $z_1 = 0.9$
(i) $z_1 = 0.92$
equity $(N)$
debt $(B)$
$h(B,N)=0$
$\mu^N(B,N)=0$
SSS
DSS

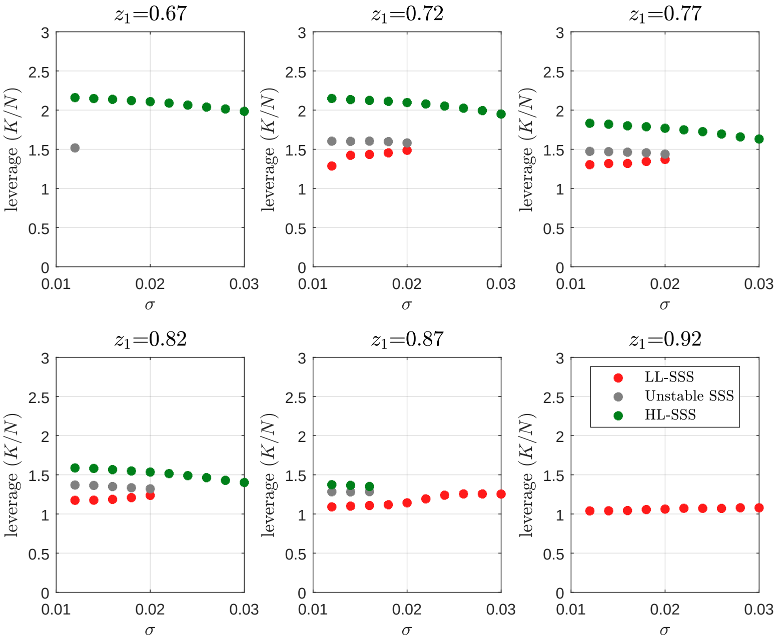

3
2.5
2
1.5
1
0.5
0
0.65
0.7
0.75
0.8
0.85
0.9
0.95
1
leverage $(K/N)$
$z_1$
LL-SSS
Unstable SSS
HL-SSS
DSS



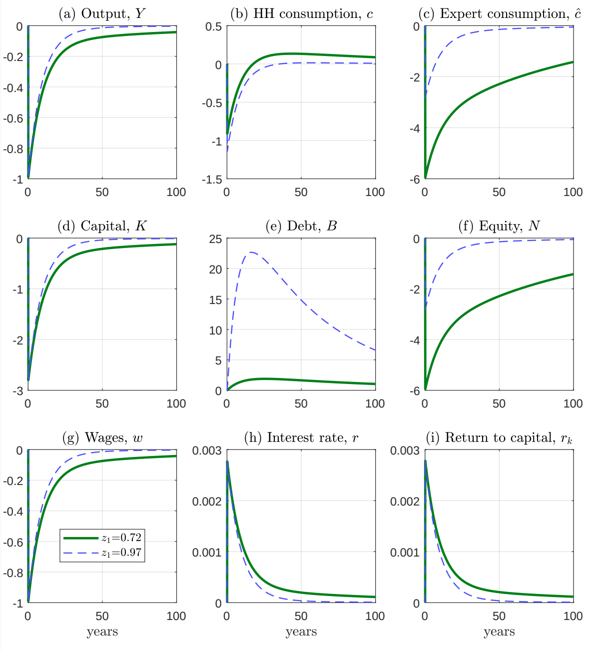
(a) $f(B,N)$ with $z_1 = 0.67$
(b) $f(B,N)$ with $z_1 = 0.7$
(c) $f(B,N)$ with $z_1 = 0.72$
(d) $f(B,N)$ with $z_1 = 0.77$
(e) $f(B,N)$ with $z_1 = 0.8$
(f) $f(B,N)$ with $z_1 = 0.82$
(g) $f(B,N)$ with $z_1 = 0.87$
(h) $f(B,N)$ with $z_1 = 0.9$
(i) $f(B,N)$ with $z_1 = 0.92$
equity $(N)$
debt $(B)$
0
1
2
3
4


20
10
0
0
0.5
1
1.5
2
2.5
3
(a) Debt ($B$)
4
2
0
1
1.5
2
2.5
3
3.5
4
(b) Equity ($N$)
4
3.5
3
2.5
2
1.5
1
0.5
0
3.2
3.4
3.6
3.8
4
4.2
4.4
(c) Capital ($K$)
$z_1 = 0.72$
$z_1 = 0.77$
$z_1 = 0.82$
$z_1 = 0.87$
$z_1 = 0.92$
$z_1 = 0.97$

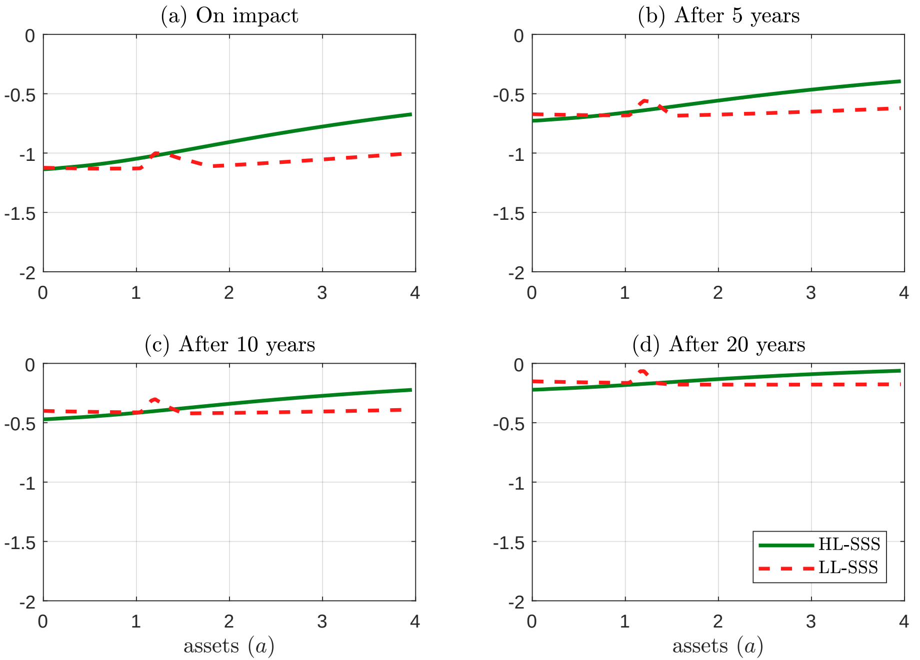

Idiosyncratic risk ($z_1$)
Aggregate risk ($\sigma$)
1
0.95
0.9
0.85
0.8
0.75
0.7
0.65
0.6
0.01
0.015
0.02
0.025
0.03
Only LL-SSS
HL-SSS and LL-SSS
Only HL-SSS

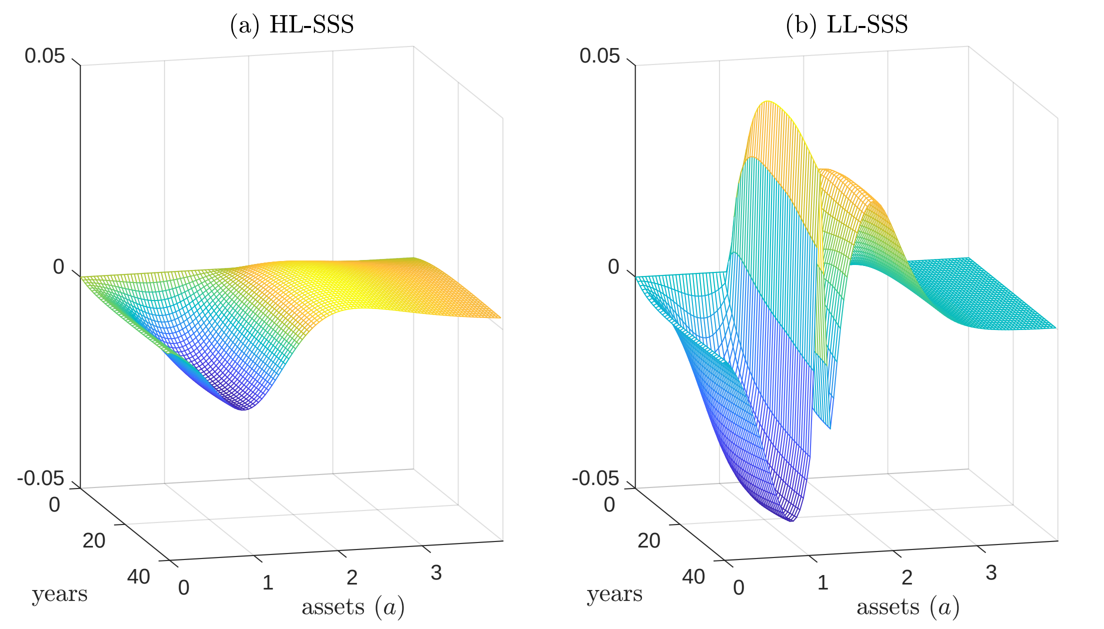


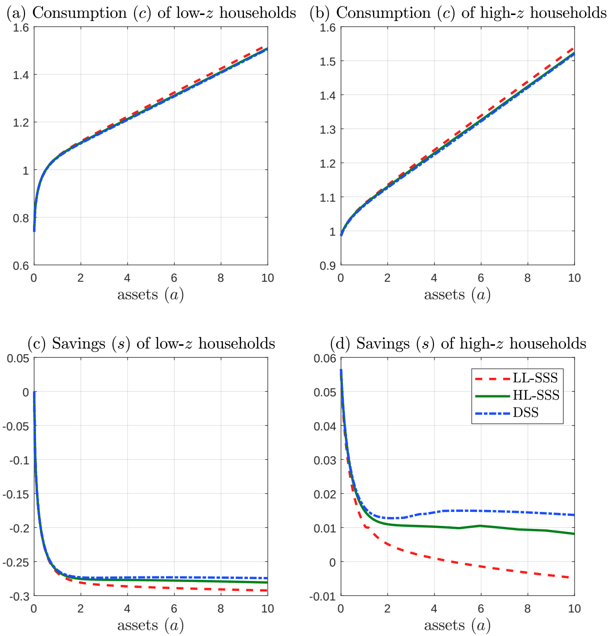
$z_1$=0.67
$z_1$=0.72
$z_1$=0.77
$z_1$=0.82
$z_1$=0.87
$z_1$=0.92
leverage $(K/N)$
$\sigma$
0
0.5
1
1.5
2
2.5
3
0.01
0.02
0.03
LL-SSS
Unstable SSS
HL-SSS



(a) Output, $Y$
(b) HH consumption, $c$
(c) Expert consumption, $\hat{c}$
(d) Capital, $K$
(e) Debt, $B$
(f) Equity, $N$
(g) Wages, $w$
(h) Interest rate, $r$
(i) Return to capital, $r_k$
$z_1$=0.72
$z_1$=0.97
years


(a) On impact
(b) After 5 years
(c) After 10 years
(d) After 20 years
0
-0.5
-1
-1.5
-2
0 1 2 3 4
assets ($a$)
HL-SSS
LL-SSS


(a) HL-SSS
(b) LL-SSS
0.05
0
-0.05
0
20
40
years
0
1
2
3
assets $(a)$


(a) Consumption ($c$) of low-$z$ households
(b) Consumption ($c$) of high-$z$ households
(c) Savings ($s$) of low-$z$ households
(d) Savings ($s$) of high-$z$ households
assets ($a$)
assets ($a$)
assets ($a$)
assets ($a$)
LL-SSS
HL-SSS
DSS

# Concluding remarks

- We have shown how a continuous-time model with a non-trivial distribution of wealth among households and financial frictions can be built, computed, and estimated.
- Four important economic lessons:
    1. Multiplicity of SSS(s).
    2. State-dependence of GIRFs and DIRFs.
    3. Long spells at different basins of attraction.
    4. Importance of household heterogeneity.
- Many avenues for extension.