

# HOT Compilation Notes

Rahul Manne  
rmanne@andrew.cmu.edu

## Disclaimer/README

These are only reference notes, and by no means fully capture what is taught in class.

Notes for 170131 (on substitution) are extremely incoherent so I did not include them by default.

There may be errors, feel free to report them to me.

## 1 Compiler Structure

SML  
 $\xrightarrow{\text{elaborate}}$  IL-Module  
 $\xrightarrow{\text{phase-splitting}}$  IL-Direct  
 $\xrightarrow{\text{cps conversion}}$  IL-CPS  
 $\xrightarrow{\text{closure conversion}}$  IL-Closure  
 $\xrightarrow{\text{hoisting}}$  IL-Hoist  
 $\xrightarrow{\text{allocation}}$  IL-Alloc  
 $\xrightarrow{\text{code-generation}}$  C

## 2 Introduction to the $F\omega$ type system

### 2.1 Context for Judgements

### 2.2 Grammar

$$\begin{aligned} k &::= \text{Type} \mid k \rightarrow k \\ c &::= \alpha \mid c \rightarrow c \mid \forall \alpha : k . c \mid c \ c \\ e &::= x \mid \lambda x : c . e \mid e \ e \mid \Lambda \alpha : k . e \mid e[c] \end{aligned}$$

Kind  $k$ , Type Constructor  $c$ , and Term  $e$ .

Note: Type is often referred to with just “T” (by Crary), for simplicity.

### 2.3 Context for Judgements

$$\Gamma ::= \epsilon \mid \Gamma, x : \tau \mid \Gamma, \alpha : k \tag{1}$$

Note: For simplicity, whenever a new  $\alpha$  appears in the context, we implicitly ensure that  $\alpha$  is not already in  $\Gamma$ .

### 2.4 $\Gamma \vdash c : k$

$$\begin{array}{c} \frac{\Gamma(\alpha) = k}{\Gamma \vdash \alpha : k} \quad \frac{\Gamma \vdash \tau : T \quad \Gamma \vdash \tau_2 : T}{\Gamma \vdash \tau_1 \rightarrow \tau_2 : T} \quad \frac{\Gamma, \alpha : k \vdash \tau : T}{\Gamma \vdash \forall \alpha : k . \tau} \quad \frac{\Gamma, \alpha : k \vdash c : k'}{\Gamma \vdash \lambda \alpha : k . c : k \rightarrow k'} \\[10pt] \frac{\Gamma \vdash c_1 : k \rightarrow k' \quad \Gamma \vdash c_2 : k}{\Gamma \vdash c_1 \ c_2 : k'} \end{array}$$

### 2.5 $\Gamma \vdash e : \tau$

$$\begin{array}{c} \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \quad \frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau . e : \tau \rightarrow \tau'} \quad \frac{\Gamma \vdash e_1 : \tau \rightarrow \tau' \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 \ e_2 : \tau'} \\[10pt] \frac{\Gamma, \alpha : k \vdash e : \tau}{\Gamma \vdash \Lambda \alpha : k . e : \forall \alpha : k . \tau} \quad \frac{\Gamma \vdash e : \forall \alpha : k . e \quad \Gamma \vdash c : k}{\Gamma \vdash e[c] : [c / \alpha]e} \quad \frac{\Gamma \vdash e : \tau \quad \Gamma \vdash \tau \equiv \tau' : T}{\Gamma \vdash e : \tau'} \end{array}$$

## 2.6 $\Gamma \vdash c \equiv c : k$

Definitional Equivalence.

$$\frac{\Gamma \vdash c : k}{\Gamma \vdash c \equiv c : k} \quad \frac{\Gamma \vdash c \equiv c' : k}{\Gamma \vdash c' \equiv c : k} \quad \frac{\Gamma \vdash c_1 \equiv c_2 : k \quad \Gamma \vdash c_2 \equiv c_3 : k}{\Gamma \vdash c_1 \equiv c_3 : k}$$

The above are identity, reflexivity, and transitivity respectively.

The following are “compatibility” rules.

$$\frac{\Gamma \vdash c_1 \equiv c'_1 : k \quad \Gamma \vdash c_2 \equiv c'_2 : k}{\Gamma \vdash c_1 c_2 \equiv c'_1 c'_2 : k} \quad \frac{\Gamma, \alpha : k_1 \vdash c \equiv c' : k_2}{\Gamma \vdash \lambda \alpha : k_1 . c \equiv \lambda \alpha : k_1 . c' : k_1 \rightarrow k_2}$$

$$\frac{\Gamma \vdash \tau_1 \equiv \tau'_1 : k \quad \Gamma \vdash \tau_2 \equiv \tau'_2 : k}{\Gamma \vdash \tau_1 \rightarrow \tau_2 \equiv \tau'_1 \rightarrow \tau'_2 : T} \quad \frac{\Gamma, \alpha : k \vdash \tau \equiv \tau' : T}{\Gamma \vdash \forall \alpha : k . \tau \equiv \forall \alpha : k . \tau' : T}$$

congruence = compatible equivalence relation

The following are the rules for beta equivalence and extensionality:

$$\frac{\Gamma \vdash c_2 : k \quad \Gamma, \alpha : k \vdash c_1 : k'}{\Gamma \vdash (\lambda \alpha : k . c_1) c_2 \equiv [c_2 / \alpha] c_1 : k'}$$

$$\frac{\Gamma, \alpha : k_1 \vdash c \equiv c' : k_2 \quad \Gamma \vdash c : k_1 \rightarrow k_2 \quad \Gamma \vdash c' : k_1 \rightarrow k_2}{\Gamma \vdash c \equiv c' : k_1 \rightarrow k_2}$$

## 2.7 Extending $F\omega$

Note: This helps in the understanding of sml's module system

Grammar:

$$k ::= \dots \mid k \times k$$

$$c ::= \dots \mid \langle c, c \rangle \mid \pi_1 c \mid \pi_2 c$$

New Judgements:

$$\frac{\Gamma \vdash c_1 : k_2 \quad \Gamma \vdash c_2 : k_2}{\Gamma \vdash \langle c_1, c_2 \rangle : k_1 \times k_2} \quad \frac{\Gamma \vdash c : k_1 \times k_2}{\Gamma \vdash \pi_i c : k_i} \quad \frac{\Gamma \vdash c_1 \equiv c'_1 : k_1 \quad \Gamma \vdash c_2 \equiv c'_2 : k_2}{\Gamma \vdash \langle c_1, c_2 \rangle \equiv \langle c'_1, c'_2 \rangle : k_1 \times k_2}$$

$$\frac{\Gamma \vdash c \equiv c' : k_1 \times k_2}{\Gamma \vdash \pi_i c \equiv \pi_i c' : k_i} \quad \frac{\Gamma \vdash c_1 : k_1 \quad \Gamma \vdash c_2 : k_2}{\Gamma \vdash \pi_i \langle c_1, c_2 \rangle \equiv c_i : k_i}$$

$$\frac{\Gamma \vdash \pi_1 c \equiv \pi_1 c' : k_1 \quad \Gamma \vdash \pi_2 c \equiv \pi_2 c' : k_2}{\Gamma \vdash c \equiv c' : k_1 \times k_2}$$

### 3 Algorithmic Equivalence in the $F\omega$ Type System

#### 3.1 Normalize-and-Compare

Note: We don't use this.

$\lambda\alpha : k . c_1 \ c_2 \xrightarrow{\beta} [c_2 / \alpha]c_1$   
 $\pi_i \langle c_1, c_2 \rangle \xrightarrow{\beta} c_i$   
+ some  $\eta$  reduction rules

According to some equivalence theorem, they will have normal forms and those normal forms will be equal if they are equivalent.

#### 3.2 Grammar and Properties

Paths:

$p ::= \alpha \mid p \ c \mid \pi_1 \ p \mid \pi_2 \ p$

Weak-Head Normal Form:

$n ::= p \mid c_1 \rightarrow c_2 \mid \forall\alpha : k . c.$

Regularity:

If  $\vdash \Gamma \text{ ok}$  and  $\Gamma \vdash c_1 \equiv c_2 : k$ , then  $\Gamma \vdash c_1 : k$  and  $\Gamma \vdash c_2 : k$ .

If  $\vdash \Gamma \text{ ok}$  and  $\Gamma \vdash c : k$ , then  $\Gamma \vdash k : \text{kind}$ .

Soundness:

If  $\vdash \Gamma \text{ ok}$  and  $\Gamma \vdash c_1, c_2 : k$  and  $\Gamma \vdash c_1 \Leftrightarrow c_2 : k$ , then  $\vdash c_1 \equiv c_2 : k$ .

Completeness:

If  $\vdash \Gamma \text{ ok}$  and  $\Gamma \vdash c_1 \equiv c_2 : k$ , then  $\Gamma \vdash c_1 \Leftrightarrow c_2 : k$ .

$$\frac{}{\vdash \epsilon \text{ ok}} \qquad \frac{\vdash \Gamma \text{ ok} \quad \Gamma \vdash k : \text{kind}}{\vdash \Gamma, \alpha : k \text{ ok}} \qquad \frac{\vdash \Gamma \text{ ok} \quad \Gamma \vdash \tau : \text{Type}}{\vdash \Gamma, x : \tau \text{ ok}}$$

### 3.3 Algorithmic Constructor Equivalence

Form:  $\Gamma \vdash c_1^+ \Leftrightarrow c_2^+ : k$

Note:  $\overset{+}{x}$  indicates that  $x$  is an input.

$$\frac{\Gamma, \alpha : k_1 \vdash c \ \alpha \Leftrightarrow c' \ \alpha : k_2}{\Gamma \vdash c \Leftrightarrow c' : k_1 \rightarrow k_2} \quad \frac{\Gamma \vdash \pi_1 \ c \Leftrightarrow \pi_1 \ c' : k_1 \quad \Gamma \vdash \pi_2 \ c \Leftrightarrow \pi_2 \ c' : k_2}{\Gamma \vdash c \Leftrightarrow c' : k_1 \times k_2}$$

$$\frac{c_1 \Downarrow c'_1 \quad c_2 \Downarrow c'_2 \quad \Gamma \vdash c'_1 \Leftrightarrow c'_2 : \text{Type}}{\Gamma \vdash c_1 \Leftrightarrow c_2 : \text{Type}}$$

### 3.4 Algorithmic Path Equivalence

Form:  $\Gamma \vdash c_1^+ \Leftrightarrow c_2^+ : \bar{k}$

Note:  $\bar{x}$  indicates that  $x$  is an output.

$$\frac{\Gamma(\alpha) = k}{\Gamma \vdash \alpha \Leftrightarrow \alpha : k} \quad \frac{\Gamma \vdash p \Leftrightarrow p' : k_1 \rightarrow k_2 \quad \Gamma \vdash c \Leftrightarrow c' : k_1}{\Gamma \vdash p \ c \Leftrightarrow p' \ c' : k_1} \quad \frac{\Gamma \vdash p \Leftrightarrow p' : k_1 \times k_2}{\Gamma \vdash \pi_i \ p \Leftrightarrow \pi_i \ p' : k_i}$$

$$\frac{\Gamma \vdash c_1 \Leftrightarrow c'_1 : T \quad \Gamma \vdash c_1 \Leftrightarrow c'_2 : T}{\Gamma \vdash c_1 \rightarrow c_2 \Leftrightarrow c'_1 \rightarrow c'_2 : T} \quad \frac{\Gamma, \alpha : k \vdash c \Leftrightarrow c' : T}{\Gamma \vdash \forall \alpha : k . c \Leftrightarrow \forall \alpha : k . c' : T}$$

### 3.5 Weak-Head Normalization

Form:  $\overset{+}{c} \Downarrow \bar{n}$

$$\frac{c \rightsquigarrow c' \quad c' \Downarrow c''}{c \Downarrow c''} \quad \frac{c \not\rightsquigarrow}{c \Downarrow c}$$

### 3.6 Weak-Head Reduction

Form:  $\overset{+}{c} \rightsquigarrow \bar{c}'$

$$\frac{}{(\lambda \alpha : k . c_1) \ c_2 \rightsquigarrow [c_2 / \alpha] c_1} \quad \frac{}{\pi_i \langle c_1, c_2 \rangle \rightsquigarrow c_i} \quad \frac{c_1 \rightsquigarrow c'_1}{c_1 \ c_2 \rightsquigarrow c'_1 \ c_2} \quad \frac{c \rightsquigarrow c'}{\pi_i c \rightsquigarrow \pi_i c'}$$

### 3.7 Kind Synthesis and Checking

Form:  $\Gamma \vdash \overset{+}{c} \Rightarrow \bar{k}$  and  $\Gamma \vdash \overset{+}{c} \Leftarrow \overset{+}{k}$

$$\begin{array}{c}
\frac{\Gamma(\alpha) = k}{\Gamma \vdash \alpha \Rightarrow k} \quad \frac{\Gamma, \alpha : k \vdash c \Rightarrow k'}{\Gamma \vdash \lambda \alpha : k . c \Rightarrow k \rightarrow k'} \quad \frac{\Gamma \vdash c_1 \Rightarrow k \rightarrow k' \quad \Gamma \vdash c_2 \Leftarrow k}{\Gamma \vdash c_1 \ c_2 \Rightarrow k'} \\
\\
\frac{\Gamma \vdash c_1 \Rightarrow k_1 \quad \Gamma \vdash c_2 \Rightarrow k_2}{\Gamma \vdash \langle c_1, c_2 \rangle \Rightarrow k_1 \times k_2} \quad \frac{\Gamma \vdash c \Rightarrow k_1 \times k_2}{\Gamma \vdash \pi_i \ c \Rightarrow k_1} \quad \frac{\Gamma \vdash c_1 \Leftarrow T \quad \Gamma \vdash c_2 \Leftarrow T}{\Gamma \vdash c_1 \rightarrow c_2 \Rightarrow T} \\
\\
\frac{\Gamma, \alpha : k \vdash c \Leftarrow T}{\Gamma \vdash \forall \alpha : k . c \Rightarrow T} \quad \frac{\Gamma \vdash c \Rightarrow k}{\Gamma \vdash c \Leftarrow k}
\end{array}$$

### 3.8 Type Checking and Synthesis

Form:  $\Gamma \vdash \overset{+}{e} \Rightarrow \bar{c}$  and  $\Gamma \vdash \overset{+}{e} \Leftarrow \overset{+}{c}$

$$\begin{array}{c}
\frac{\Gamma(x) = \tau}{\Gamma \vdash x \Rightarrow \tau} \quad \frac{\Gamma \vdash \tau \Leftarrow T \quad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x : \tau . e \Rightarrow \tau \rightarrow \tau'} \quad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \quad \tau_1 \Downarrow \tau \rightarrow \tau' \quad \Gamma \vdash e_2 \Leftarrow \tau}{\Gamma \vdash e_1 \ e_2 \Rightarrow \tau'} \\
\\
\frac{\Gamma, \alpha : k \vdash e \Rightarrow \tau}{\Gamma \vdash \Lambda \alpha : k . e \Rightarrow \forall \alpha : k . \tau} \quad \frac{\Gamma \vdash e \Rightarrow \tau \quad \tau \Downarrow \forall \alpha : k . \tau' \quad \Gamma \vdash c \Leftarrow k}{\Gamma \vdash e[c] \Rightarrow [c / \alpha] \tau'} \\
\\
\frac{\Gamma \vdash e \Rightarrow \tau' \quad \Gamma \vdash \tau \Leftarrow \tau' : T}{\Gamma \vdash e \Leftarrow \tau}
\end{array}$$

## 4 Singleton Kinds

```
sig
  type t
  type 'a u
  type ('a, 'b) v
  type w = int
  type w' = w
  .
  .
  .
end
```

To represent this in type our type system,  $t : \text{Type}$   
 $u : \text{Type} \rightarrow \text{Type}$   
 $v : \text{Type} \rightarrow \text{Type} \rightarrow \text{Type}$   
 (or  $v : \text{Type} \times \text{Type} \rightarrow \text{Type}$ )  
 $w : S(f)$   
 $w' : S(w)$

### 4.1 Grammar and Judgements (Attempt 1)

Grammar:

$k ::= \text{Type} \mid k \rightarrow k \mid k \times k \mid S(c)$   
 $c ::= \dots$

Judgements:

$$\frac{}{\Gamma \vdash c : S(c)} \qquad \frac{\Gamma \vdash c : S(c)}{\Gamma \vdash c \equiv c' : \text{Type}} \qquad \frac{\Gamma \vdash c : \text{Type}}{\Gamma \vdash S(c) : \text{kind}}$$

Signature for `list`.

```
sig
  .
  .
  .
  type 'a s = 'a list
  type 'a t
end
```

So we have  $t : \text{Type} \rightarrow \text{Type}$ .

How do we represent 'a s? Is  $s : \text{Type} \rightarrow S(\alpha)$ ? But then what's  $\alpha$ .

## 4.2 Dependent Kinds (Grammar)

$k ::= \text{Type} \mid \Pi\alpha : k . k \mid \Sigma\alpha : k . k \mid S(c)$

$c ::= \dots$

Note:  $\Pi$  is also known as “dependent product”

$\Sigma$  is also known as “dependent sum” (but also sometimes as “dependent product”).

To avoid confusion, we name  $\Pi$  “dependent function (spaces)”.

Now, we have  $s : \Pi\alpha : \text{Type} . S(\text{list } \alpha)$ .

New judgements we need to be able to make:

$\Gamma \vdash k : \text{kind}$

$\Gamma \vdash k \equiv k' : \text{kind}$

$\Gamma \vdash k \leq k'$

$\Gamma \vdash c : k$

$\Gamma \vdash c \equiv c' : k$

$\Gamma \vdash e : \tau$

Note:  $S(f) \leq \text{Type}$

### 4.3 $\Gamma \vdash k : \text{kind}$

$$\frac{}{\Gamma \vdash \text{Type} : \text{kind}} \quad \frac{\Gamma \vdash c : \text{Type}}{\Gamma \vdash S(c) : \text{kind}} \quad \frac{\Gamma \vdash k_1 : \text{kind} \quad \Gamma, \alpha : k_1 \vdash k_2 : \text{kind}}{\Gamma \vdash \Pi\alpha : k_1 . k_2 : \text{kind}}$$

$$\frac{\Gamma \vdash k_1 : \text{kind} \quad \Gamma, \alpha : k_1 \vdash k_2 : \text{kind}}{\Gamma \vdash \Sigma\alpha : k_1 . k_2 : \text{kind}}$$

### 4.4 $\Gamma \vdash k \equiv k' : \text{kind}$

$$\frac{\Gamma \vdash k : \text{kind}}{\Gamma \vdash k \equiv k : \text{kind}} \quad \frac{\Gamma \vdash k_1 \equiv k_2 : \text{kind}}{\Gamma \vdash k_2 \equiv k_1 : \text{kind}} \quad \frac{\Gamma \vdash k_1 \equiv k_2 : \text{kind} \quad \Gamma \vdash k_2 \equiv k_2 : \text{kind}}{\Gamma \vdash k_1 \equiv k_2 : \text{kind}}$$

$$\frac{\Gamma \vdash c \equiv c' : \text{Type}}{\Gamma \vdash S(c) \equiv S(c') : \text{kind}} \quad \frac{\Gamma \vdash k_1 \equiv k'_1 : \text{kind} \quad \Gamma, \alpha : k_1 \vdash k_2 \equiv k'_2 : \text{kind}}{\Gamma \vdash \Pi\alpha : k_1 . k_2 \equiv \Pi\alpha : k'_1 . k'_2 : \text{kind}}$$

$$\frac{\Gamma \vdash k_1 \equiv k'_1 : \text{kind} \quad \Gamma, \alpha : k_1 \vdash k_2 \equiv k'_2 : \text{kind}}{\Gamma \vdash \Sigma\alpha : k_1 . k_2 \equiv \Sigma\alpha : k'_1 . k'_2 : \text{kind}}$$

Note: for the latter two, keep  $\Pi\alpha : k_1 . k_2 \stackrel{?}{\equiv} \Pi\alpha' : k'_1 . k'_2$  in mind



#### 4.5 $\Gamma \vdash \alpha : k$

$$\begin{array}{c}
\frac{\Gamma(\alpha) = k}{\Gamma \vdash \alpha : k} \quad \frac{\Gamma \vdash c_1 : \text{Type} \quad \Gamma \vdash c_2 : \text{Type}}{\Gamma \vdash c_1 \rightarrow c_2 : \text{Type}} \quad \frac{\Gamma \vdash k : \text{kind} \quad \Gamma, \alpha : k \vdash c : \text{Type}}{\Gamma \vdash \forall \alpha : k . c : \text{Type}} \\
\\
\frac{\Gamma \vdash k_1 : \text{kind} \quad \Gamma, \alpha : k_1 \vdash c : k_2}{\Gamma \vdash \lambda \alpha : k_1 . c : \Pi \alpha : k_1 . k_2} \quad \frac{\Gamma \vdash c_1 : \Pi \alpha : k . k' \quad \Gamma \vdash c_2 : k}{\Gamma \vdash c_1 c_2 : [c_1 / \alpha]k'} \\
\\
\frac{\Gamma \vdash c_1 : k_2 \quad \Gamma \vdash c_2 : [c_1 / \alpha]k_2 \quad \Gamma, \alpha : k_1 \vdash k_2 : \text{kind}}{\Gamma \vdash \langle c_1, c_2 \rangle : \Sigma \alpha : k_2 . k_2} \quad \frac{\Gamma \vdash c : \Sigma \alpha : k_1 . k_2}{\Gamma \vdash \pi_1 c : k_1} \\
\\
\frac{\Gamma \vdash c : \Sigma \alpha : k_1 . k_2}{\Gamma \vdash \pi_2 c : [\pi_1 c / \alpha]k_2} \quad \frac{\Gamma \vdash c : k \quad \Gamma \vdash k \leq k'}{\Gamma \vdash c : k'}
\end{array}$$

Additional Judgements If  $\vdash \Gamma$  ok and  $\Gamma \vdash c : k$ , then  $\Gamma \vdash k : \text{kind}$ .

If  $\vdash \Gamma$  ok and  $\Gamma \vdash k_1 \equiv k_2$ , then  $\Gamma \vdash k_1, k_2$  kind.

If  $\vdash \Gamma$  ok and  $\Gamma \vdash k_1 \leq k_2$ , then  $\Gamma \vdash k_1, k_2$  kind.

$$\frac{\Gamma \vdash c : \text{Type}}{\Gamma \vdash c : S(c)}$$

Sub-kinding:

$$\begin{array}{c}
\frac{\Gamma \vdash k \equiv k' : \text{kind}}{\Gamma \vdash k \leq k'} \quad \frac{\Gamma \vdash k_1 \leq k_2 \quad \Gamma \vdash k_2 \leq k_3}{\Gamma \vdash k_1 \leq k_3} \quad \frac{\Gamma \vdash c : \text{Type}}{\Gamma \vdash S(c) \leq \text{Type}} \quad \frac{\Gamma \vdash c \equiv c' : \text{Type}}{\Gamma \vdash S(c) \leq S(c')} \\
\\
\frac{\Gamma \vdash k'_1 \leq k_1 \quad \Gamma, \alpha : k'_1 \vdash k_2 \leq k'_2 \quad \Gamma, \alpha : k_1 \vdash k_2 : \text{kind}}{\Gamma \vdash \Pi \alpha : k_1 . k_2 \leq \Pi \alpha : k'_1 . k'_2} \\
\\
\frac{\Gamma \vdash k_1 \leq k'_1 \quad \Gamma, \alpha : k_1 \vdash k_2 \leq k'_2 \quad \Gamma, \alpha : k'_1 \vdash k'_2 : \text{kind}}{\Gamma \vdash \Sigma \alpha : k_1 . k_2 \leq \Sigma \alpha : k'_1 . k'_2}
\end{array}$$

Note: Something about contravariance for 1st condition.  $\Pi$  contravariant the same way arrow is contravariant. Covariance for 2nd condition. This is for  $\Pi$ .

**4.6**  $\Gamma \vdash c \equiv c : k$

$$\begin{array}{c}
\frac{\Gamma \vdash c : k}{\Gamma \vdash c \equiv c : k} \quad \frac{\Gamma \vdash c_1 \equiv c_2 : k}{\Gamma \vdash c_2 \equiv c_1 : k} \quad \frac{\Gamma \vdash c_1 \equiv c_2 : k \quad \Gamma \vdash c_2 \equiv c_3 : k}{\Gamma \vdash c_1 \equiv c_3 : k} \\
\\
\frac{\Gamma \vdash c_2 : k \quad \Gamma, \alpha : k \vdash c_1 : k'}{\Gamma \vdash (\lambda \alpha : k . c_1) c_2 \equiv [c_2 / \alpha] c_1 : [c_2 / \alpha] k'} \quad \frac{\Gamma \vdash c_1 : k_1 \quad \Gamma \vdash c_2 : k_2}{\Gamma \vdash \pi_i \langle c_1, c_2 \rangle \equiv c_i : k_i} \quad \frac{\Gamma \vdash c : S(c')}{\Gamma \vdash c \equiv c' : S(c')} \\
\\
\frac{\Gamma \vdash c_1 \equiv c_2 : k \quad \Gamma \vdash k \leq k'}{\Gamma \vdash c_1 \equiv c_2 : k'} \star \quad \frac{\Gamma \vdash c \equiv c' : \text{Type}}{\Gamma \vdash c \equiv c' : S(c)} \star \\
\\
\frac{\Gamma \vdash k_1 \equiv k'_1 : \text{kind} \quad \Gamma, \alpha : k_1 \vdash c \equiv c' : k_2}{\Gamma \vdash \lambda \alpha : k_1 . c \equiv \lambda \alpha : k'_1 . c' : \Pi \alpha : k_1 . k_2} \quad \frac{\Gamma \vdash c_1 \equiv c'_1 : \Pi \alpha : k . k' \quad \Gamma \vdash c_2 \equiv c'_2 : k}{\Gamma \vdash c_1 c_2 \equiv c'_1 c'_2 : [c_2 / \alpha] k'}
\end{array}$$