HOT Compilation Notes

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Disclaimer/README

These are only reference notes, and by no means fully capture what is taught in class.

Notes for 170131 (on substitution) are extremely incoherent so I did not include them by default.

There may be errors, feel free to report them to me.

1 Compiler Structure

SML
$\xrightarrow{\text{elaborate}}$ IL-Module
$\xrightarrow{\text{phase-splitting}}$ IL-Direct
$\xrightarrow{\text{cps conversion}}$ IL-CPS
$\xrightarrow{\text{closure conversion}}$ IL-Closure
$\xrightarrow{\text{hoisting}}$ IL-Hoist
$\xrightarrow{\text{allocation}}$ IL-Alloc
$\xrightarrow{\text{code-generation}}$ C

2 Introduction to the $F\omega$ type system

2.1 Grammar

$$
\begin{aligned}
k &::= \text{Type} \mid k \to k \\
c &::= \alpha \mid c \to c \mid \forall \alpha : k \,.\, c \mid c\ c \\
e &::= x \mid \lambda x : c \,.\, e \mid e\ e \mid \Lambda \alpha : k \,.\, e \mid e[c]
\end{aligned}
$$

Kind k, Type Constructor c, and Term e.

Note: Type is often referred to with just "T" (by Crary), for simplicity.

2.2 Context for Judgements

$$\Gamma ::= \epsilon \mid \Gamma, x : \tau \mid \Gamma, \alpha : k \tag{1}$$

Note: For simplicity, whenever a new α appears in the context, we implicitly ensure that α is not already in Γ.

2.3 $\Gamma \vdash c : k$

$$\frac{\Gamma(\alpha) = k}{\Gamma \vdash \alpha : k} \qquad \frac{\Gamma \vdash \tau : T \qquad \Gamma \vdash \tau_2 : T}{\Gamma \vdash \tau_1 \to \tau_2 : T} \qquad \frac{\Gamma, \alpha : k \vdash \tau : T}{\Gamma \vdash \forall \alpha : k \,.\, \tau} \qquad \frac{\Gamma, \alpha : k \vdash c : k'}{\Gamma \vdash \lambda \alpha : k \,.\, c : k \to k'}$$

$$\frac{\Gamma \vdash c_1 : k \to k' \qquad \Gamma \vdash c_2 : k}{\Gamma \vdash c_1\ c_2 : k'}$$

2.4 $\Gamma \vdash e : \tau$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau \,.\, e : \tau \to \tau'} \qquad \frac{\Gamma \vdash e_1 : \tau \to \tau' \qquad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1\ e_2 : \tau'}$$

$$\frac{\Gamma, \alpha : k \vdash e : \tau}{\Gamma \vdash \Lambda \alpha : k \,.\, e : \forall \alpha : k \,.\, \tau} \qquad \frac{\Gamma \vdash e : \forall \alpha : k \,.\, e \qquad \Gamma \vdash c : k}{\Gamma \vdash e[c] : [c \,/\, \alpha] e} \qquad \frac{\Gamma \vdash e : \tau \qquad \Gamma \vdash \tau \equiv \tau' : T}{\Gamma \vdash e : \tau'}$$

2.5 $\Gamma \vdash c \equiv c : k$

Definitional Equivalence.

$$\frac{\Gamma \vdash c : k}{\Gamma \vdash c \equiv c : k} \qquad \frac{\Gamma \vdash c \equiv c' : k}{\Gamma \vdash c' \equiv c : k} \qquad \frac{\Gamma \vdash c_1 \equiv c_2 : k \qquad \Gamma \vdash c_2 \equiv c_2 : k}{\Gamma \vdash c_1 \equiv c_3 : k}$$

The above are identity, reflexivity, and transitivity respectively.

The following are "compatibility" rules.

$$\frac{\Gamma \vdash c_1 \equiv c_1' : k \qquad \Gamma \vdash c_2 \equiv c_2' : k}{\Gamma \vdash c_1 c_2 \equiv c_1' c_2' : k} \qquad \frac{\Gamma, \alpha : k_1 \vdash c \equiv c' : k_2}{\Gamma \vdash \lambda\alpha : k_1 \,.\, c \equiv \lambda\alpha : k_1 \,.\, c' : k_1 \to k_2}$$

$$\frac{\Gamma \vdash \tau_1 \equiv \tau_1' : k \qquad \Gamma \vdash \tau_2 \equiv \tau_2' : k}{\Gamma \vdash \tau_1 \to \tau_2 \equiv \tau_1' \to \tau_2' : T} \qquad \frac{\Gamma, \alpha : k \vdash \tau \equiv \tau' : T}{\Gamma \vdash \forall\alpha : k \,.\, \tau \equiv \forall\alpha : k \,.\, \tau' : T}$$

congruence = compatible equivalence relation
The following are the rules for beta equivalence and extensionality:

$$\frac{\Gamma \vdash c_2 : k \qquad \Gamma, \alpha : k \vdash c_1 : k'}{\Gamma \vdash (\lambda\alpha : k \,.\, c_1)\ c_2 \equiv [c_2 \,/\, \alpha]c_1 : k'}$$

$$\frac{\Gamma, \alpha : k_1 \vdash c\ \alpha \equiv c'\ \alpha : k_2 \qquad \Gamma \vdash c : k_1 \to k_2 \qquad \Gamma \vdash c' : k_1 \to k_2}{\Gamma \vdash c \equiv c' : k_1 \to k_2}$$

2.6 Extending $F\omega$

Note: This helps in the understanding of sml's module system

Grammar:

$$k ::= \ldots \mid k \times k$$
$$c ::= \ldots \mid \langle c, c \rangle \mid \pi_1 c \mid \pi_2 c$$

New Judgements:

$$\frac{\Gamma \vdash c_1 : k_2 \qquad \Gamma \vdash c_2 : k_2}{\Gamma \vdash \langle c_1, c_2 \rangle : k_1 \times k_2} \qquad \frac{\Gamma \vdash c : k_1 \times k_2}{\Gamma \vdash \pi_i\ c : k_1} \qquad \frac{\Gamma \vdash c_1 \equiv c_1' : k_1 \qquad \Gamma \vdash c_2 \equiv c_2' : k_2}{\Gamma \vdash \langle c_1, c_2 \rangle \equiv \langle c_1', c_2' \rangle : k_1 \times k_2}$$

$$\frac{\Gamma \vdash c \equiv c' : k_1 \times k_2}{\Gamma \vdash \pi_i\ c \equiv \pi_i\ c' : k_i} \qquad \frac{\Gamma \vdash c_1 : k_1 \qquad \Gamma \vdash c_2 : k_2}{\Gamma \vdash \pi_i \langle c_1, c_2 \rangle \equiv c_i : k_i}$$

$$\frac{\Gamma \vdash \pi_1\ c \equiv \pi_1\ c' : k_1 \qquad \Gamma \vdash \pi_2\ c \equiv \pi_2\ c' : k_2}{\Gamma \vdash c \equiv c' : k_1 \times k_2}$$

3 Algorithmic Equivalence in the $F\omega$ Type System

3.1 Normalize-and-Compare

Note: We don't use this.

$\lambda\alpha : k \,.\, c_1 \; c_2 \xrightarrow{\beta} [c_2 \,/\, \alpha]c_1$
$\pi_i\langle c_1, c_2\rangle \xrightarrow{\beta} c_i$
+ some η reduction rules

According to some equivalence theorem, they will have normal forms and those normal forms will be equal if they are equivalent.

3.2 Grammar and Properties

Paths:
$p ::= \alpha \mid p \; c \mid \pi_1 \; p \mid \pi_2 \; p$

Weak-Head Normal Form:
$n ::= p \mid c_1 \to c_2 \mid \forall\alpha : k \,.\, c.$

Regularity:
If $\vdash \Gamma\,\mathrm{ok}$ and $\Gamma \vdash c_1 \equiv c_2 : k$, then $\Gamma \vdash c_1 : k$ and $\Gamma \vdash c_2 : k$.
If $\vdash \Gamma\,\mathrm{ok}$ and $\Gamma \vdash c : k$, then $\Gamma \vdash k : \mathrm{kind}$.

Soundness:
If $\vdash \Gamma\,\mathrm{ok}$ and $\Gamma \vdash c_1, c_2 : k$ and $\Gamma \vdash c_1 \Leftrightarrow c_2 : k$, then $\vdash c_1 \equiv c_2 : k$.

Completeness:
If $\vdash \Gamma\,\mathrm{ok}$ and $\Gamma \vdash c_1 \equiv c_2 : k$, then $\Gamma \vdash c_1 \Leftrightarrow c_2 : k$.

$$\frac{}{\vdash \epsilon\,\mathrm{ok}} \qquad \frac{\vdash \Gamma\,\mathrm{ok} \qquad \Gamma \vdash k : \mathrm{kind}}{\vdash \Gamma, \alpha : k\,\mathrm{ok}} \qquad \frac{\vdash \Gamma\,\mathrm{ok} \qquad \Gamma \vdash \tau : \mathrm{Type}}{\vdash \Gamma, x : \tau\,\mathrm{ok}}$$

3.3 Algorithmic Constructor Equivalence

Form: $\overset{+}{\Gamma} \vdash \overset{+}{c_1} \Leftrightarrow \overset{+}{c_2} : \overset{+}{k}$

Note: $\overset{+}{x}$ indicates that x is an input.

$$\frac{\Gamma, \alpha : k_1 \vdash c\ \alpha \Leftrightarrow c'\ \alpha : k_2}{\Gamma \vdash c \Leftrightarrow c' : k_1 \to k_2} \qquad \frac{\Gamma \vdash \pi_1\ c \leftrightarrow \pi_1\ c' : k_1 \qquad \Gamma \vdash \pi_2\ c \leftrightarrow \pi_2\ c' : k_2}{\Gamma \vdash c \Leftrightarrow c' : k_1 \times k_2}$$

$$\frac{c_1 \Downarrow c_1' \qquad c_2 \Downarrow c_2' \qquad \Gamma \vdash c_1' \leftrightarrow c_2' : \mathrm{Type}}{\Gamma \vdash c_1 \Leftrightarrow c_2 : \mathrm{Type}}$$

3.4 Algorithmic Path Equivalence

Form: $\overset{+}{\Gamma} \vdash \overset{+}{c_1} \Leftrightarrow \overset{+}{c_2} : \overset{-}{k}$

Note: $\overset{-}{x}$ indicates that x is an output.

$$\frac{\Gamma(\alpha) = k}{\Gamma \vdash \alpha \leftrightarrow \alpha : k} \qquad \frac{\Gamma \vdash p \leftrightarrow p' : k_1 \to k_2 \qquad \Gamma \vdash c \Leftrightarrow c' : k_1}{\Gamma \vdash p\ c \leftrightarrow p'\ c' : k_1} \qquad \frac{\Gamma \vdash p \leftrightarrow p' : k_1 \times k_2}{\Gamma \vdash \pi_i\ p \leftrightarrow \pi_i\ p' : k_i}$$

$$\frac{\Gamma \vdash c_1 \Leftrightarrow c_1' : T \qquad \Gamma \vdash c_1 \Leftrightarrow c_2' : T}{\Gamma \vdash c_1 \to c_2 \leftrightarrow c_1' \to c_2' : T} \qquad \frac{\Gamma, \alpha : k \vdash c \Leftrightarrow c' : T}{\Gamma \vdash \forall \alpha : k\ .\ c \leftrightarrow \forall \alpha : k\ .\ c' : T}$$

3.5 Weak-Head Normalization

Form: $\overset{+}{c} \Downarrow \overset{-}{n}$

$$\frac{c \rightsquigarrow c' \qquad c' \Downarrow c''}{c \Downarrow c''} \qquad \frac{c \not\rightsquigarrow}{c \Downarrow c}$$

3.6 Weak-Head Reduction

Form: $\overset{+}{c} \rightsquigarrow \overset{-}{c'}$

$$\frac{}{(\lambda \alpha : k\ .\ c_1)\ c_2 \rightsquigarrow [c_2\ /\ \alpha] c_1} \qquad \frac{}{\pi_i \langle c_1, c_2 \rangle \rightsquigarrow c_i} \qquad \frac{c_1 \rightsquigarrow c_1'}{c_1\ c_2 \rightsquigarrow c_1'\ c_2} \qquad \frac{c \rightsquigarrow c'}{\pi_i c \rightsquigarrow \pi_i c'}$$

3.7 Kind Synthesis and Checking

Form: $\overset{+}{\Gamma} \vdash \overset{+}{c} \Rightarrow \overset{-}{k}$ and $\overset{+}{\Gamma} \vdash \overset{+}{c} \Leftarrow \overset{+}{k}$

$$\frac{\Gamma(\alpha) = k}{\Gamma \vdash \alpha \Rightarrow k} \qquad \frac{\Gamma, \alpha : k \vdash c \Rightarrow k'}{\Gamma \vdash \lambda\alpha : k \,.\, c \Rightarrow k \to k'} \qquad \frac{\Gamma \vdash c_1 \Rightarrow k \to k' \qquad \Gamma \vdash c_2 \Leftarrow k}{\Gamma \vdash c_1\ c_2 \Rightarrow k'}$$

$$\frac{\Gamma \vdash c_1 \Rightarrow k_1 \qquad \Gamma \vdash c_2 \Rightarrow k_2}{\Gamma \vdash \langle c_1, c_2 \rangle \Rightarrow k_1 \times k_2} \qquad \frac{\Gamma \vdash c \Rightarrow k_1 \times k_2}{\Gamma \vdash \pi_i\ c \Rightarrow k_1} \qquad \frac{\Gamma \vdash c_1 \Leftarrow T \qquad \Gamma \vdash c_2 \Leftarrow T}{\Gamma \vdash c_1 \to c_2 \Rightarrow T}$$

$$\frac{\Gamma, \alpha : k \vdash c \Leftarrow T}{\Gamma \vdash \forall\alpha : k \,.\, c \Rightarrow T} \qquad \frac{\Gamma \vdash c \Rightarrow k}{\Gamma \vdash c \Leftarrow k}$$

3.8 Type Checking and Synthesis

Form: $\overset{+}{\Gamma} \vdash \overset{+}{e} \Rightarrow \overset{-}{c}$ and $\overset{+}{\Gamma} \vdash \overset{+}{e} \Leftarrow \overset{+}{c}$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x \Rightarrow \tau} \qquad \frac{\Gamma \vdash \tau \Leftarrow T \qquad \Gamma, x : \tau \vdash e \Rightarrow \tau'}{\Gamma \vdash \lambda x : \tau \,.\, e \Rightarrow \tau \to \tau'} \qquad \frac{\Gamma \vdash e_1 \Rightarrow \tau_1 \qquad \tau_1 \Downarrow \tau \to \tau' \qquad \Gamma \vdash e_2 \Leftarrow \tau}{\Gamma \vdash e_1\ e_2 \Rightarrow \tau'}$$

$$\frac{\Gamma, \alpha : k \vdash e \Rightarrow \tau}{\Gamma \vdash \Lambda\alpha :.\ e \Rightarrow \forall\alpha : k \,.\, \tau} \qquad \frac{\Gamma \vdash e \Rightarrow \tau \qquad \tau \Downarrow \forall\alpha : k \,.\, \tau' \qquad \Gamma \vdash c \Leftarrow k}{\Gamma \vdash e[c] \Rightarrow [c \,/\, \alpha]\tau'}$$

$$\frac{\Gamma \vdash e \Rightarrow \tau' \qquad \Gamma \vdash \tau \Leftrightarrow \tau' : T}{\Gamma \vdash e \Leftarrow \tau}$$

4 Singleton Kinds

```
sig
  type t
  type 'a u
  type ('a,'b) v
  type w = int
  type w' = w
  .
  .
  .
end
```

To represent this in type our type system, $t : \text{Type}$
$u : \text{Type} \to \text{Type}$
$v : \text{Type} \to \text{Type} \to \text{Type}$
(or $v : \text{Type} \times \text{Type} \to \text{Type}$)
$w : S(\int)$
$w' : S(w)$

4.1 Grammar and Judgements (Attempt 1)

Grammar:

$k ::= \text{Type} \mid k \to k \mid k \times k \mid S(c)$
$c ::= \ldots$

Judgements:

$$\frac{}{\Gamma \vdash c : S(c)} \qquad \frac{\Gamma \vdash c : S(c)}{\Gamma \vdash c \equiv c' : \text{Type}} \qquad \frac{\Gamma \vdash c : \text{Type}}{\Gamma \vdash S(c) : \text{kind}}$$

Signature for `list`.

```
sig
  .
  .
  .
  type 'a s = 'a list
  type 'a t
end
```

So we have $t : \text{Type} \to \text{Type}$.
How do we represent 'a s? Is $s : \text{Type} \to S(\alpha)$? But then what's α.

4.2 Dependent Kinds (Grammar)

$k ::= \text{Type} \mid \Pi\alpha : k \,.\, k \mid \Sigma\alpha : k \,.\, k \mid S(c)$
$c ::= \ldots$

Note: Π is also known as "dependent product"

Σ is also known as "dependent sum" (but also sometimes as "dependent product").

To avoid confusion, we name Π "dependent function (spaces)".

Now, we have $s : \Pi\alpha : \text{Type} \,.\, S(list\,\alpha)$.

New judgements we need to be able to make:
$\Gamma \vdash k : \text{kind}$
$\Gamma \vdash k \equiv k' : kind$
$\Gamma \vdash k \leq k'$
$\Gamma \vdash c : k$
$\Gamma \vdash c \equiv c' : k$
$\Gamma \vdash e : \tau$

Note: $S(\int) \leq \text{Type}$

4.3 $\Gamma \vdash k : \text{kind}$

$$\frac{}{\Gamma \vdash \text{Type} : \text{kind}} \qquad \frac{\Gamma \vdash c : \text{Type}}{\Gamma \vdash S(c) : \text{kind}} \qquad \frac{\Gamma \vdash k_1 : \text{kind} \qquad \Gamma, \alpha : k_1 \vdash k_2 : \text{kind}}{\Gamma \vdash \Pi\alpha : k_1 \,.\, k_2 : \text{kind}}$$

$$\frac{\Gamma \vdash k_1 : \text{kind} \qquad \Gamma, \alpha : k_1 \vdash k_2 : \text{kind}}{\Gamma \vdash \Sigma\alpha : k_1 \,.\, k_2 : \text{kind}}$$

4.4 $\Gamma \vdash k \equiv k' : \text{kind}$

$$\frac{\Gamma \vdash k : \text{kind}}{\Gamma \vdash k \equiv k : \text{kind}} \qquad \frac{\Gamma \vdash k_1 \equiv k_2 : \text{kind}}{\Gamma \vdash k_2 \equiv k_1 : \text{kind}} \qquad \frac{\Gamma \vdash k_1 \equiv k_2 : \text{kind} \qquad \Gamma \vdash k_2 \equiv k_2 : \text{kind}}{\Gamma \vdash k_1 \equiv k_2 : \text{kind}}$$

$$\frac{\Gamma \vdash c \equiv c' : \text{Type}}{\Gamma \vdash S(c) \equiv S(c') : \text{kind}} \qquad \frac{\Gamma \vdash k_1 \equiv k_1' : \text{kind} \qquad \Gamma, \alpha : k_1 \vdash k_2 \equiv k_2' : \text{kind}}{\Gamma \vdash \Pi\alpha : k_1 \,.\, k_2 \equiv \Pi\alpha : k_1' \,.\, k_2' : \text{kind}}$$

$$\frac{\Gamma \vdash k_1 \equiv k_1' : \text{kind} \qquad \Gamma, \alpha : k_1 \vdash k_2 \equiv k_2' : \text{kind}}{\Gamma \vdash \Sigma\alpha : k_1 \,.\, k_2 \equiv \Sigma\alpha : k_1' \,.\, k_2' : \text{kind}}$$

Note: for the latter two, keep $\Pi\alpha : k_1 \,.\, k_2 \stackrel{?}{\equiv} \Pi\alpha' : k_1' \,.\, k_2'$ in mind

4.5 $\Gamma \vdash \alpha : k$

$$\frac{\Gamma(\alpha) = k}{\Gamma \vdash \alpha : k} \qquad \frac{\Gamma \vdash c_1 : \text{Type} \qquad \Gamma \vdash c_2 : \text{Type}}{\Gamma \vdash c_1 \to c_2 : \text{Type}} \qquad \frac{\Gamma \vdash k : kid \qquad \Gamma, \alpha : k \vdash c : \text{Type}}{\Gamma \vdash \forall \alpha : k \,.\, c : \text{Type}}$$

$$\frac{\Gamma \vdash k_1 : \text{kind} \qquad \Gamma, \alpha : k_1 \vdash c : k_2}{\Gamma \vdash \lambda \alpha : k_1 \,.\, c : \Pi \alpha : k_1 \,.\, k_2} \qquad \frac{\Gamma \vdash c_1 : \Pi \alpha : k \,.\, k' \qquad \Gamma \vdash c_2 : k}{\Gamma \vdash c_1 \; c_2 : [c_1 \,/\, \alpha] k'}$$

$$\frac{\Gamma \vdash c_1 : k_2 \qquad \Gamma \vdash c_2 : [c_1 \,/\, \alpha] k_2 \qquad \Gamma, \alpha : k_1 \vdash k_2 : \text{kind}}{\Gamma \vdash \langle c_1, c_2 \rangle : \Sigma \alpha : k_2 \,.\, k_2} \qquad \frac{\Gamma \vdash c : \Sigma \alpha : k_1 \,.\, k_2}{\Gamma \vdash \pi_1 c : k_1}$$

$$\frac{\Gamma \vdash c : \Sigma \alpha : k_1 \,.\, k_2}{\Gamma \vdash \pi_2 c : [\pi_1 c \,/\, \alpha] k_2} \qquad \frac{\Gamma \vdash c : k \qquad \Gamma \vdash k \leq k'}{\Gamma \vdash c : k'}$$

Additional Judgements If $\vdash \Gamma$ ok and $\Gamma \vdash c : k$, then $\Gamma \vdash k : \text{kind}$.
If $\vdash \Gamma$ ok and $\Gamma \vdash k_1 \equiv k_2$, then $\Gamma \vdash k_1, k_2$ kind.
If $\vdash \Gamma$ ok and $\Gamma \vdash k_1 \leq k_2$, then $\Gamma \vdash k_1, k_2$ kind.

$$\frac{\Gamma \vdash c : \text{Type}}{\Gamma \vdash c : S(c)}$$

Sub-kinding:

$$\frac{\Gamma \vdash k \equiv k' : \text{kind}}{\Gamma \vdash k \leq k'} \qquad \frac{\Gamma \vdash k_1 \leq k_2 \qquad \Gamma \vdash k_2 \leq k_3}{\Gamma \vdash k_1 \leq k_3} \qquad \frac{\Gamma \vdash c : \text{Type}}{\Gamma \vdash S(c) \leq \text{Type}} \qquad \frac{\Gamma \vdash c \equiv c' : \text{Type}}{\Gamma \vdash S(c) \leq S(c')}$$

$$\frac{\Gamma \vdash k_1' \leq k_1 \qquad \Gamma, \alpha : k_1' \vdash k_2 \leq k_2' \qquad \Gamma, \alpha : k_1 \vdash k_2 : \text{kind}}{\Gamma \vdash \Pi \alpha : k_1 \,.\, k_2 \leq \Pi \alpha : k_1' \,.\, k_2'}$$

$$\frac{\Gamma \vdash k_1 \leq k_1' \qquad \Gamma, \alpha : k_1 \vdash k_2 \leq k_2' \qquad \Gamma, \alpha : k_1' \vdash k_2' : \text{kind}}{\Gamma \vdash \Sigma \alpha : k_1 \,.\, k_2 \leq \Sigma \alpha : k_1' \,.\, k_2'}$$

Note: Something about contravariance for 1st condition. Π contravariant the same way arrow is contravariant. Covariance for 2nd condition. This is for Π.

4.6 $\Gamma \vdash c \equiv c : k$

$$\frac{\Gamma \vdash c : k}{\Gamma \vdash c \equiv c : k} \qquad \frac{\Gamma \vdash c_1 \equiv c_2 : k}{\Gamma \vdash c_2 \equiv c_1 : k} \qquad \frac{\Gamma \vdash c_1 \equiv c_2 : k \qquad \Gamma \vdash c_2 \equiv c_3 : k}{\Gamma \vdash c_1 \equiv c_3 : k}$$

$$\frac{\Gamma \vdash c_2 : k \qquad \Gamma, \alpha : k \vdash c_1 : k'}{\Gamma \vdash (\lambda\alpha : k \,.\, c_1)\ c_2 \equiv [c_2 \,/\, \alpha]c_1 : [c_2 \,/\, \alpha]k'} \qquad \frac{\Gamma \vdash c_1 : k_1 \qquad \Gamma \vdash c_2 : k_2}{\Gamma \vdash \pi_i\langle c_1, c_2\rangle \equiv c_i : k_i} \qquad \frac{\Gamma \vdash c : S(c')}{\Gamma \vdash c \equiv c' : S(c')}$$

$$\frac{\Gamma \vdash c_1 \equiv c_2 : k \qquad \Gamma \vdash k \leq k'}{\Gamma \vdash c_1 \equiv c_2 : k'}\star \qquad \frac{\Gamma \vdash c \equiv c' : \mathrm{Type}}{\Gamma \vdash c \equiv c' : S(c)}\star$$

$$\frac{\Gamma \vdash k_1 \equiv k_1' : \mathrm{kind} \qquad \Gamma, \alpha : k_1 \vdash c \equiv c' : k_2}{\Gamma \vdash \lambda\alpha : k_1 \,.\, c \equiv \lambda\alpha : k_1' \,.\, c' : \Pi\alpha : k_1 \,.\, k_2} \qquad \frac{\Gamma \vdash c_1 \equiv c_1' : \Pi\alpha : k \,.\, k' \qquad \Gamma \vdash c_2 \equiv c_2' : k}{\Gamma \vdash c_1\ c_2 \equiv c_1'\ c_2' : [c_2 \,/\, \alpha]k'}$$

5 Sub-Typing

$\tau \leq \tau'$ means you can use a τ wherever a τ' is expected.

$$\frac{\Gamma \vdash e : \tau \qquad \tau \leq \tau'}{\Gamma \vdash e : \tau'} \qquad \frac{\tau_1 \leq \tau_1' \qquad \tau_2 \leq \tau_2'}{\Gamma \vdash \tau \times \tau_2 \leq \tau_1' \times \tau_2'} \qquad \frac{\tau_1' \leq \tau_1 \qquad \tau_2 \leq \tau_2'}{\tau_1 \to \tau_2 \leq \tau_1' \to \tau_2'}$$

Note 1: This is a case of covariance on both sides.
Note 2: This is contravariant on the left and covariant on the right.

$\mathcal{N} \leq \mathcal{R}$.
Assume we have $f : \mathcal{N} \to \mathcal{N}$.
$f \not: \mathcal{R} \to \mathcal{R}$.

Assume we have $f : \mathcal{R} \to \mathcal{R}$.
Contravariance: $f : \mathcal{N} \to \mathcal{R}$.

$$\frac{\tau \equiv \tau' : \text{Type}}{\text{ref}(\tau) \leq \text{ref}(\tau')}$$

ref(τ) is neither covariant nor contravariant. Called "invariant". (Poorly named, but it's what's used in literature.)

$$\frac{\Gamma \vdash k_1' \leq k_1 \qquad \Gamma, \alpha : k_1' \vdash k_2 \leq k_2' \qquad \Gamma, \alpha : k_1 \vdash k_2 : \mathit{kind}}{\Gamma \vdash \Pi\alpha : k_1 \,.\, k_2 \leq \Pi\alpha : k_1' \,.\, k_2'}$$

$$\frac{\Gamma \vdash k_1' \leq k_1 \qquad \Gamma, \alpha : k_1' \vdash k_2 \leq k_2' \qquad \Gamma, \alpha : k_1 \vdash k_2 : \text{kind}}{\Gamma \vdash \Sigma\alpha : k_1 \,.\, k_2 \leq \Sigma\alpha : k_1' \,.\, k_2'} \qquad \frac{\Gamma \vdash c : S(c')}{\Gamma \vdash c \equiv c' : S(c')}$$

$$\frac{\Gamma \vdash c : S(c')}{\Gamma \vdash c \equiv c' : \text{Type}} \qquad \frac{\Gamma \vdash c \equiv c' : \text{Type}}{\Gamma \vdash c : c' : S(c)}$$

Note 1: Sound, but not what we want.

More compatiblity rules.

$$\frac{\Gamma \vdash c_1 \equiv c_1' : k_1 \qquad \Gamma \vdash c_2 \equiv c_2' : [c_1 \,/\, \alpha]k_2 \qquad \Gamma, \alpha : k_1 \vdash k_2 : \text{kind}}{\Gamma \vdash \langle c_1, c_2 \rangle \equiv \langle c_1', c_2' \rangle : \Sigma\alpha : k_1 \,.\, k_2} \qquad \frac{\Gamma \vdash c \equiv c' : \Sigma\alpha : k_1 \,.\, k_2}{\Gamma \vdash \pi_1 c \equiv \pi_1 c' : k_1}$$

$$\frac{\Gamma \vdash c \equiv c' : \Sigma\alpha : k_1 \,.\, k_2}{\Gamma \vdash \pi_2 c \equiv \pi_2 c' : [\pi_1 c \,/\, \alpha]k_2} \qquad \frac{\Gamma \vdash c_1 \equiv c_1' : \text{Type} \qquad \Gamma \vdash c_2 \equiv c_2' : \text{Type}}{\Gamma \vdash c_1 \to c_2 \equiv c_1' \to c_2' : \text{Type}}$$

$$\frac{\Gamma \vdash k \equiv k' : \text{kind} \qquad \Gamma, \alpha : k \vdash c \equiv c' : \text{Type}}{\Gamma \vdash \forall\alpha : k \,.\, c \equiv \forall\alpha : k' \,.\, c' : \text{Type}}$$

Rules for extentionality.

$$\frac{\Gamma, \alpha : k_1 \vdash c\ \alpha \equiv c'\ \alpha : k_2 \qquad \Gamma \vdash c : \Pi alpha : k_1\ .\ k_2' \qquad \Gamma \vdash c' : \Pi alpha : k_1\ .\ k_2''}{\Gamma \vdash c \equiv c' : \Pi alpha : k_1\ .\ k_2}$$

$$\frac{\Gamma, \alpha : k_1 \vdash c\ \alpha \equiv c'\ \alpha : k_2 \qquad \Gamma \vdash c \equiv c' : \Pi alpha : k_1\ .\ k_2'}{\Gamma \vdash c \equiv c' : \Pi\alpha : k_1\ .\ k_2}$$

$$\frac{\Gamma \vdash \pi_1 c \equiv \pi_1 c' : k_1 \qquad \Gamma \vdash \pi_2 c \equiv \pi_2 c' : [\pi_1 c \,/\, \alpha]k_2 \qquad \Gamma, \alpha : k_1 \vdash k_2 : \text{kind}}{\Gamma \vdash c \equiv c' : \Sigma alpha : k_1\ .\ k_2}$$

Note 1: We only need this for proofs (regularity). We can safely ignore this.

We have no way of dealing with $S(c : k)$. So instead of redefining everything, treat it as a macro following the following rules:
$S(c : \text{Type}) = S(c)$
$S(c : \Pi\alpha : k_1\ .\ k_2) = \Pi\alpha : k_1\ .\ S(c\ \alpha : k_2)$
$S(c : S(c')) = S(c)$ (note here, $c \equiv c'$, so we can use either, but it's easier for us to use c)
$S(c : \Pi\alpha : k_1\ .\ k_2) = \Sigma\alpha : S(\pi_1 c : k_1)\ .\ S(\pi_2 c : k_2)$
OR $S(\pi_1 c : k_1) \times S(\pi_2 c : [\pi_1 c \,/\, \alpha]k_2)$
We use the 2nd because it's nicer when not working without theory. The first is more theoretic, the second is syntactic.

1. If $\Gamma \vdash c : k$, then $\Gamma \vdash c : S(c : k)$
2. If $\Gamma \vdash c : S(c' : k)$, then $\Gamma \vdash c \equiv c' : k$

But the first doesn't hold. So let's make it hold. Add "declarative" rules:

$$\frac{\Gamma \vdash k_1 : \text{kind} \qquad \Gamma, \alpha : k_1 \vdash c\ \alpha : k_2}{\Gamma \vdash c : \Pi\alpha : k_1\ .\ k_2}$$

$$\frac{\Gamma \vdash \pi_1 c : k_2 \qquad \Gamma \vdash \pi_1 c : [\pi_1 c \,/\, \alpha]k_2 \qquad \Gamma, \alpha : k_1 \vdash k_2 : \text{kind}}{\Gamma \vdash c : \Sigma\alpha : k_1\ .\ k_2}$$

Notes on definitional equivalence:
$\alpha : \text{Type} \vdash \alpha \not\equiv \text{int} : \text{Type}$
$\alpha : S(\text{int}) \vdash \alpha \equiv \text{int} : \text{Type}$
$\vdash \lambda\alpha : \text{Type}\ .\ \alpha \not\equiv \lambda\alpha : \text{Type}\ .\ \text{int} : \text{Type} \to \text{Type}$
$\vdash \lambda\alpha : \text{Type}\ .\ \alpha \not\equiv \lambda\alpha : \text{Type}\ .\ \text{int} : S(\text{int}) \to \text{Type}$
$\beta : (\text{Type} \to \text{Type}) \to \text{Type} \vdash \beta(\lambda\alpha : \text{Type}\ .\ \alpha \not\equiv \beta(\lambda\alpha : \text{Type}\ .\ \text{int} : \text{Type}$
$\beta : (S(\text{int}) \to \text{Type}) \to \text{Type} \vdash \beta(\lambda\alpha : \text{Type}\ .\ \alpha \equiv \beta(\lambda\alpha : \text{Type}\ .\ \text{int} : \text{Type}$
$\text{Type} \to \text{Type} \leq S(\text{int}) \to \text{Type}$

5.1 Algorithm for Equivalence Checking

$$\frac{\Gamma, \alpha : k_1 \vdash c\ \alpha \Leftrightarrow c'\ \alpha : k_2}{\Gamma \vdash c \Leftrightarrow c' : \Pi\alpha : k_1\ .\ k_2} \qquad \frac{\Gamma \vdash \pi_1 c \Leftrightarrow \pi_2 c' : k_1 \Gamma \vdash \pi_2 c \Leftrightarrow \pi_2 c' : [\pi_1 c\ /\ \alpha]k_2}{\Gamma \vdash c \Leftrightarrow c' : \Sigma\alpha : k_1\ .\ k_2}$$

$$\frac{\Gamma \vdash c_1 \Downarrow c_1' \qquad \Gamma \vdash c_2 \Downarrow c_2' \qquad \Gamma \vdash c_1' \leftrightarrow c_2' : \mathrm{Type}}{\Gamma \vdash c_1 \Leftrightarrow c_2 : \mathrm{Type}} \qquad \frac{\Gamma \vdash c \rightsquigarrow c' \qquad \Gamma \vdash c' \Downarrow c''}{\Gamma \vdash c \Downarrow c''} \qquad \frac{\Gamma \vdash c \not\rightsquigarrow}{\Gamma \vdash c \Downarrow c}$$

$$\frac{}{\Gamma \vdash (\lambda\alpha : k\ .\ c_1)\ c_2 \rightsquigarrow [c_2\ /\ \alpha]c_1} \qquad \frac{\Gamma \vdash c_1 \rightsquigarrow c_1'}{\Gamma \vdash c_1\ c_2 \rightsquigarrow c_1'\ c_2} \qquad \frac{}{\Gamma \vdash \pi_i\langle c_1, c_2\rangle \rightsquigarrow c_i}$$

$$\frac{\Gamma \vdash c \rightsquigarrow c'}{\Gamma \vdash pi_i c \rightsquigarrow pi_i c'} \qquad \frac{\Gamma \vdash p \uparrow S(c)}{\Gamma \vdash p} \qquad \frac{\Gamma(\alpha) = k}{\Gamma \vdash \alpha \uparrow k} \qquad \frac{\Gamma \vdash p \uparrow \Pi\alpha : k_1\ .\ k_2}{\Gamma \vdash p\ c \uparrow [c\ /\ \alpha]k_2}$$

$$\frac{\Gamma \vdash p \uparrow \Sigma\alpha : k_1\ .\ k_2}{\Gamma \vdash \pi_1 p \uparrow k_1} \qquad \frac{\Gamma \vdash p \uparrow \Sigma\alpha : k_1\ .\ k_2}{\Gamma \vdash \pi_2 p \uparrow [\pi_1 p\ /\ \alpha]k_2}$$

$$\frac{\Gamma \vdash p \uparrow S(c)}{\Gamma \vdash p \rightsquigarrow c}$$

Example:

$$\dfrac{\dots \vdash (\lambda\alpha : \text{Type}\alpha\alpha \rightsquigarrow \alpha \qquad \dfrac{\dfrac{\alpha : S(\text{int}) \vdash \alpha \uparrow S(\text{int})}{\alpha : S(\text{int}) \vdash \alpha \rightsquigarrow \text{int}} \qquad \dfrac{\dots \vdash \text{int} \not\rightsquigarrow}{\dots \vdash \text{int} \Downarrow \text{int}}}{\dots \vdash \alpha \Downarrow \text{int}}}{\alpha : S(\text{int}) \vdash (\lambda\alpha : \text{Type} . \alpha)\alpha \Downarrow} \qquad \dfrac{}{\alpha : S(\text{int}) \vdash (\lambda\alpha : \text{Type}\text{int})\alpha \Downarrow \text{int}}$$

$$\dfrac{\dfrac{\dfrac{}{\alpha : S(\text{int}) \vdash \text{int} \leftrightarrow \text{int} : \text{Type}}}{\alpha : S(\text{int}) \vdash (\lambda\alpha : \text{Type} . \alpha)\alpha \Leftrightarrow (\lambda\alpha : \text{Type} . \text{int})\alpha \Leftrightarrow}}{\vdash \lambda\alpha : \text{Type} . \alpha \Leftrightarrow \lambda\alpha : \text{Type} . \text{int} : S(\text{int}) \rightarrow \text{Type}}$$

One final rule:

$$\frac{}{\Gamma \vdash c_1 \Leftrightarrow c_2 : S(c)}$$

The precondition is that both c_1 and c_2 belong to $S(c)$, meaning they are equivalent to c and by transitivity, equivalent to each other.

Some rules that we will never use:

$$\frac{}{\Gamma \vdash c_1 \rightarrow c_2 \uparrow \text{Type}} \qquad \frac{}{\Gamma \vdash \forall\alpha : k . c \uparrow \text{Type}}$$

Structural rules:

$$\frac{\Gamma(\alpha) = k}{\Gamma \vdash \alpha \leftrightarrow \alpha : k} \qquad \frac{\Gamma \vdash p \leftrightarrow p' : \Pi\alpha : k_1 . k_2 \qquad \Gamma \vdash c \Leftrightarrow c' : k_1}{\Gamma \vdash p\ c \leftrightarrow p'\ c' : [c\ /\ \alpha]k_2} \qquad \frac{\Gamma \vdash p \leftrightarrow p' : \Sigma\alpha : k_1 . k_2}{\Gamma \vdash \pi_1 p \leftrightarrow \pi_1 p' : k_1}$$

$$\frac{\Gamma \vdash p \leftrightarrow p' : \Sigma\alpha : k_1 . k_2}{\Gamma \vdash \pi_1 p \leftrightarrow \pi_1 p' : [\pi_1 p\ /\ \alpha]k_2} \qquad \frac{\Gamma \vdash c_1 \Leftrightarrow c_1' : \text{Type} \qquad \Gamma \vdash c_2 \Leftrightarrow c_2' : \text{Type}}{\Gamma \vdash c_1 \rightarrow c_2 \leftrightarrow c_1' \rightarrow c_2' : \text{Type}}$$

$$\frac{\Gamma \vdash k \Leftrightarrow k' : \text{kind} \qquad \Gamma, \alpha : k \vdash c \Leftrightarrow c' : \text{Type}}{\Gamma \vdash \forall\alpha : k . c \leftrightarrow \forall\alpha : k' . c' : \text{Type}}$$

If $\Gamma \vdash c \leftrightarrow c' : k$ then $\Gamma \vdash c \uparrow k$ also $\exists k' . \Gamma \vdash c' \uparrow k'$ and $\Gamma \vdash k \equiv k' : \text{kind}$

Structural comparison:

$$\frac{}{\Gamma \vdash \text{Type} \Leftrightarrow \text{Type} : \text{kind}} \quad \frac{\Gamma \vdash c \Leftrightarrow c' : \text{Type}}{\Gamma \vdash S(c) \Leftrightarrow S(c') : \text{kind}}$$

$$\frac{\Gamma \vdash k_1 \Leftrightarrow k_1' : \text{kind} \qquad \Gamma, \alpha : k_1 \vdash k_2 \Leftrightarrow k_2' : \text{kind}}{\Gamma \vdash \Pi\alpha : k_1 \,.\, k_2 \Leftrightarrow \Pi\alpha : k_1' \,.\, k_2'}$$

$$\frac{\Gamma \vdash k_1 \Leftrightarrow k_1' : \text{kind} \qquad \Gamma, \alpha : k_1 \vdash k_2 \Leftrightarrow k_2' : \text{kind}}{\Gamma \vdash \Sigma\alpha : k_1 \,.\, k_2 \Leftrightarrow \Sigma\alpha : k_1' \,.\, k_2'}$$

$\Gamma \vdash k \leq k'$

$$\frac{}{\Gamma \vdash \text{Type} \leq \text{Type}} \qquad \frac{}{\Gamma \vdash S(c) \leq \text{Type}} \qquad \frac{\Gamma \vdash c \Leftrightarrow c' : \text{Type}}{\Gamma \vdash S(c) \leq S(c')}$$

$$\frac{\Gamma \vdash k_1' \leq k_1 \qquad \Gamma, \alpha : k_1' \vdash k_2 \leq k_2'}{\Gamma \vdash \Pi\alpha : k_1 \,.\, k_2 \leq \Pi\alpha : k_1' \,.\, k_2'} \qquad \frac{\Gamma \vdash k_1 \leq k_1' \qquad \Gamma, \alpha : k_1 \vdash k_2 \leq k_2'}{\Gamma \vdash \Sigma\alpha : k_1 \,.\, k_2 \leq \Sigma\alpha : k_1' \,.\, k_2'}$$

$\Gamma \vdash k \Leftarrow \text{kind}$

$$\frac{}{\Gamma \vdash \text{Type} \Leftarrow \text{kind}} \qquad \frac{\Gamma \vdash c \Leftarrow \text{Type}}{\Gamma \vdash S(c) \Leftarrow \text{kind}} \qquad \frac{\Gamma \vdash k_1 \Leftarrow \text{kind} \qquad \Gamma, \alpha : k_1 \vdash k_2 \Leftarrow \text{kind}}{\Gamma \vdash \Pi\alpha : k_1 \,.\, k_2 \Leftarrow \text{kind}}$$

Suppose $\vdash \Gamma\,\text{ok}$. Then:

Soundness

- If $\Gamma \vdash c_1, c_2 : k$ and $\Gamma \vdash c_1 \Leftrightarrow c_2 : k$ then $\Gamma \vdash c_1 \equiv c_2 : k$
- If $\Gamma \vdash k_1, k_2 : \text{kind}$ and $\Gamma \vdash k_1 \Leftrightarrow k_2 : \text{kind}$ then $\Gamma \vdash k_1 \equiv k_2 : \text{kind}$
- If $\Gamma \vdash k_1, k_2 : \text{kind}$ and $\Gamma \vdash k_1 \leq k_2$ then $\Gamma \vdash k_1 \leq k_2$
- If $\Gamma \vdash k \Leftarrow \text{kind}$ then $\Gamma \vdash k : \text{kind}$
- If $\Gamma \vdash c \Rightarrow k$ then $\Gamma \vdash c : k$

Completeness

- If $\Gamma \vdash c_1 \equiv c_2 : k$ then $\Gamma \vdash c_1 \Leftrightarrow c_2 : k$
- If $\Gamma \vdash k_1 \equiv k_2 : \text{kind}$ then $\Gamma \vdash k_1 \Leftrightarrow k_2 : \text{kind}$
- If $\Gamma \vdash k_1 \leq k_2$ then $\Gamma \vdash k_1 \leq k_2$
- If $\Gamma \vdash k : \text{kind}$ then $\Gamma \vdash k \Leftarrow \text{kind}$
- If $\Gamma \vdash c : k$ then $\Gamma \vdash c \Rightarrow k'$ and $\Gamma \vdash k' \leq S(c : k)$

TODO: principle type
TODO: principle kind is a subkind of every other kind

Checking principle...
$\Gamma \vdash c \Rightarrow k$

$$\frac{\Gamma \vdash c \Rightarrow k' \qquad \Gamma \vdash k' \leq k}{\Gamma \vdash c \Leftarrow k}$$

$$\frac{\Gamma(\alpha) = k}{\Gamma \vdash \alpha \Rightarrow S(\alpha : k)} \qquad \frac{\Gamma \vdash k \Leftarrow \text{kind} \qquad \Gamma, \alpha : k \vdash c \Rightarrow k'}{\Gamma \vdash \lambda\alpha : k \,.\, c \Rightarrow \Pi\alpha : k \,.\, k'} \qquad \frac{\Gamma \vdash c_1 \Rightarrow \Pi\alpha : k \,.\, k' \qquad \Gamma \vdash c_2 \Leftarrow k}{\Gamma \vdash c_1\ c_2 \Rightarrow [c_2 \,/\, \alpha]k'}$$

$$\frac{\Gamma \vdash c_1 \Rightarrow k_1 \qquad \Gamma \vdash c_2 \Rightarrow k_2}{\Gamma \vdash \langle c_1, c_2\rangle \Rightarrow k_1 \times k_2} \qquad \frac{\Gamma \vdash c \Rightarrow \Sigma\alpha : k_1 \,.\, k_2}{\Gamma \vdash \pi_1 c \Rightarrow k_1} \qquad \frac{\Gamma \vdash c \Rightarrow \Sigma\alpha : k_1 \,.\, k_2}{\Gamma \vdash \pi_2 c \Rightarrow [\pi_1 c \,/\, \alpha]k_2}$$

$$\frac{\Gamma \vdash c_1 \Leftarrow \text{Type} \qquad \Gamma \vdash c_2 \Leftarrow \text{Type}}{\Gamma \vdash c_1 \to c_2 \Rightarrow S(c_1 \to c_2)} \qquad \frac{\Gamma \vdash k \Leftarrow \text{kind} \qquad \Gamma, \alpha : k \vdash c \Leftarrow \text{Type}}{\Gamma \vdash \forall\alpha : k \,.\, c \Rightarrow S(\forall\alpha : k \,.\, c)}$$