Problem 2
*Friends and Steps*

Four friends of varying heights all begin snooping around a dilapidated abandoned house. After exploring the main level for a while and not finding anything of interest, Jackson suddenly exclaims “Come look at this.” There, behind the book shelf, was a hidden stairway leading deep into the ground.

Excited by this find, the friends begin to descend the dusty staircase. Because of their different heights, they all naturally take the stairs at a different rate. Jackson goes down two steps at a time, Bruce three steps, Jessica four steps and Katie five steps at a time.

The house's cellar proves to be the perfect place to establish a new club the friends have wanted to create for some time. After exploring the cellar for some length and discussing all the future meetings they will have, the friends begin to leave. Before they leave, Katie notices a curious pattern that occurs on the stairs. The only steps that have footprints from all four of them are the first and last ones. Also, because of the unique way in which they descended the stairs some steps only have one footprint on them.

They decide to use this unique pattern as the basis for a challenge problem to determine if new people are fit for their club which they call Phi. How many steps must the staircase contain? How many steps have exactly one footprint on them?

Solve this and the club membership will be yours.

Hint: Try figuring out the least common multiple. This will give you the number of steps. From there find the numbers that are divisible by one of the step amounts but not by any other. The mathematical term for this is relatively prime.

## Solution:

Let's call the top step step 1, and number the steps counting downward. Then Jackson will step on all the even-numbered steps, Bruce will step on all the steps which are numbered with a multiple of three, Jessica on all the multiples of four, and Katie on all the steps that are multiples of 5.

First we will see how many steps there are. Since all four people will step on the last step, the number of the last step must be a multiple of 2, 3, 4, and 5. 3*4*5 = 60 is the smallest such number (the least common multiple of 2, 3, 4, and 5), so the last step is number 60.

Now we will count the steps that only Jackson steps on. These will be given by even numbers which are not multiples of 3, 4, or 5. We an write such a number as 2*n with n between 1 and 30, and relatively prime to 30 (so that n is not divisible by 2, 3, or 5). There are 8 numbers below thirty relatively prime to 30, so Jackson steps on eight steps that nobody else steps on. (We could also just count which of the even-numbered steps have numbers not divisible by 3, 4, or 5. They are steps 2, 14, 22, 26, 34, 38, 46, 58, and there are 8 of them.)

Now let's count the steps that only Bruce steps on. He steps on multiples of three which are not multiples of 2, 4, or 5. The steps he steps on are given by 3*n where n is between 1 and 20 and n is relatively prime to 20 (so that n is not divisible by 2, 4, or 5). There are 8 such numbers, so he steps on 8 steps that nobody else steps on. (Steps 3, 9, 21, 27, 33, 39, 51, 57).

Every step that Jessica steps on is an even-numbered step that Bruce also steps on, so Jessica steps on 0 steps that nobody else steps on.

Katie steps on steps numbered 5*n with n between 1 and 12, and n relatively prime to 12 (so that n is not divisible by 2, 3, or 4). There are 4 numbers relatively prime to twelve below twelve, so Bruce steps on 4 steps that nobody else steps on.

Thus there are 20 steps that are stepped on by only one person. Notice that we counted the numbers relatively prime to a certain number and less than the number several times. There is a special name for this - it is called the Euler phi function. Basically phi(n) = the number of numbers between 1 and n relatively prime to n. It is a very important function in number theory (and in solving fun number problems like this one).

Motivation from this problem from mathforum.org. Information on Phi from Dr. Darren.