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TRANSLATING SOLUTIONS OF NON-PARAMETRIC MEAN CURVATURE FLOWS WITH CAPILLARY-TYPE BOUNDARY VALUE PROBLEMS

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Abstract. In this note, we study the mean curvature flow and the prescribed mean curvature type equation with general capillary-type boundary condition, which is $u_\nu = -\phi(x)(1+|Du|^2)^{\frac{1-q}{2}}$ for any parameter $q > 0$. Using the maximum principle, we prove the gradient estimates for the solutions of such a class of boundary value problems. As a consequence, we obtain the corresponding existence theorem for a class of mean curvature equations. In addition, we study the related additive eigenvalue problem for general boundary value problems and describe the asymptotic behavior of the solution at infinity time. The originality of the paper lies in the range $0 < q < 1$, since there are no any related results before. For parabolic case, we generalize the result of Ma-Wang-Wei [25] to any $q > 0$. And in elliptic case, we generalize the results in [32] to any $q \geq 0$ and to any bounded smooth domain.

1. **Introduction.** In this article, we are interested in the large time behavior of solutions of the mean curvature flow on a strictly convex bounded domain with general capillary-type boundary conditions. The main originality of this paper is twofold: on the one hand, we obtain existence results for these nonlinear capillary-type problems, and on the other hand we apply the classical approaches to prove asymptotic problems.

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First, we introduce the following mean curvature flow and mean curvature equation with general capillary-type boundary conditions

$$\begin{cases} u_t = \sum_{i,j=1}^{n} (\delta_{ij} - \frac{u_i u_j}{1+|Du|^2}) u_{ij} - f(x,u) & \text{in } \Omega \times (0,\infty), \\ u_\nu = -\phi(x) v^{1-q} & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1}$$

and

$$\begin{cases} \operatorname{div}(\frac{Du}{\sqrt{1+|Du|^2}}) = f(x,u) & \text{in } \Omega\,, \\ u_\nu = -\phi(x) v^{1-q} & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, ν is an inward unit normal vector to $\partial\Omega$, $q \geq 0$, $v = \sqrt{1+|Du|^2}$, $u_i = u_{x_i}$, $u_{ij} = u_{x_i x_j}$, and $\phi(x)$ is smooth. Also $u_0(x)$ and $\phi(x)$ are smooth functions satisfying

$$u_{0,\nu} = -\phi(x)(1+|Du_0|^2)^{\frac{1-q}{2}} \text{ on } \partial\Omega.$$

In (1) and (2), when $q = 0$, it is the capillary problem and when $q = 1$, it is the Neumann boundary value problem.

The capillary problem has been studied for more than forty years. Ural'tseva [30] firstly got the boundary gradient estimates and the corresponding existence theorem on the positive gravity case. In 1976, Simon-Spruck [28] and Gerhardt [9] also obtained the existence theorem on the positive gravity case respectively. For more general quasi-linear divergence structure equation with conormal derivative boundary value problem, Lieberman [20] got the gradient estimate. They all obtained these estimates via test function technique. In 1975, Spruck [29] first used the maximum principle for positive gravity case in $\mathbb{R}^2$. Later, Korevaar [18] and Lieberman [19] obtained the gradient estimates respectively based on the maximum principle.

Lieberman [20] also proved the gradient estimates for a more general class of quasi-linear elliptic equations with the boundary condition for $q = 0$ and $q > 1$ in zero gravity case ($f_u = 0$). In the physical interpretation of the capillary problem, f_u measures gravitational effects. Recently, the third author in this article and Ma [26] got the boundary gradient estimates of mean curvature equations with Neumann problem via the maximum principle and derived an existence result in positive gravity case ($f_u \geq C > 0$) for some positive constant C. In [32], the third author gave a new maximum principle proof of boundary gradient estimates for $q > 1$ and $q = 0$ in zero gravity case. But so far no further progress has been made for $0 < q < 1$. In this paper, we mainly want to study the case $0 < q < 1$ on a strictly convex domain but our method still holds for arbitrary $q > 0$.

For the mean curvature flow, it has also been studied by many mathematicians. In [5, 13, 14], Brakke and Huisken studied the motion of the parametric surfaces by their mean curvature. Their work suggests that it's geometrically natural to consider the surfaces whose speed in the direction of their unit normal is equal to the mean curvature. In the graphical setting, it's natural to consider the parabolic equation. In [15], Huisken studied the vertical capillary problem (1) with $\phi(x) = 0$ and proved that the solution asymptotically converges to a constant function. In his paper, he used integral methods to prove a time-independent gradient bound by the

Sobolev inequality and iteration method. In $\mathbb{R}^2$, Altschuler and Wu [1] considered the prescribed contact angle problem where the boundary value condition is

$$u_\nu = -\phi(x)\sqrt{1+|Du|^2} \quad \text{on} \quad \partial\Omega \times [0,\infty).$$

They proved that if Ω is strictly convex and $|D\varphi| < \min_{\partial\Omega} \kappa$, where κ is the curvature of $\partial\Omega$, the solution of mean curvature flow converges to a surface which moves at a constant speed up to a translation. For $n \geq 2$, Guan [12] studied the more generalized mean curvature type equation with the prescribed contact angle problem as below

$$\begin{cases} u_t = \displaystyle\sum_{i,j=1}^{n} (\delta_{ij} - \frac{u_i u_j}{1+|Du|^2}) u_{ij} + \varphi(u, Du) & \text{in } \Omega \times (0,\infty), \\ u_\nu = -\phi(x)\sqrt{1+|Du|^2} & \text{on } \partial\Omega \times\ (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{3}$$

where ν is an inward unit normal vector to $\partial\Omega$. He considered the asymptotic behavior of the solutions as $t \to \infty$ for two special cases:

$$(i) \quad \varphi(u, Du) = -\mathbf{k}u\sqrt{1+|Du|^2} \ \text{ for } \mathbf{k} > 0; \quad (ii) \quad \varphi(u, Du) = n/u \ \text{ for } u > 0.$$

Guan proved that in both cases the solution asymptotically approaches the solution of the corresponding stationary equation. But in [12], $\varphi(u, Du)$ must have crucial monotonicity with respect to u. Andrews and Clutterbuck [2] studied the mean curvature equation on a convex domain with the homogeneous Neumann boundary and initial data $u_0 \in C(\overline{\Omega})$ for $q = 1$.

Recently, the third author in this note deduced the existence theorem for the mean curvature flow of graphs with general Neumann boundary condition in [33] by applying the technique in [26]. In [25], Ma, Wang and the second author adjust the auxiliary function to give a uniform gradient estimate for the mean curvature type equation and the mean curvature equation with Neumann boundary on a strictly convex domain and describe the asymptotic behavior of the corresponding mean curvature flow.

But for the general capillary-type boundary with $0 < q < 1$, there are no any related results. Therefore, in this article our goal is to solve this type of problems on a strictly convex bounded domain for mean curvature equation and mean curvature flow.

Assume f is a C^1 function,

$$f_z(x,z) \geq 0, \tag{4}$$

and for some positive constants L_1, L_2,

$$|f_x| \leq L_1, \tag{5}$$

$$|\phi|_{C^2(\overline{\Omega})} \leq L_2. \tag{6}$$

Now we state the first existence and uniqueness of solution to the equation (1) as below.

Theorem 1.1. *Assume that $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a strictly convex bounded C^3 domain. Let ν be the inward unit normal vector to $\partial\Omega$ and $q > 0$. Under the conditions (4)-(6), there exists a unique $C^{2,\sigma}(\overline{\Omega} \times [0,\infty))$ solution of (1) for some $\sigma \in (0,1)$.*

Before stating the asymptotic behavior of the mean curvature flow with the capillary-type boundary, we need to introduce the additive eigenvalue problem, which will play a vital role in the whole paper.

The additive eigenvalue problem is to be considered the stationary equation with the following boundary condition

$$\begin{cases} H(x, Du, D^2u) = \lambda & \text{in } \Omega, \\ u_\nu = -\phi\,(x)\, v^{1-q} & \text{on } \partial\Omega. \end{cases} \tag{7}$$

This problem is to seek for a pair (λ, u) such that u is a viscosity solution to the stationary problem (7). If (λ, u) is such a pair, then λ and u are called an additive eigenvalue and an additive eigenfunction of equation (7) respectively.

The additive eigenvalue problem appears in ergodic optimal control and the homogenization of Hamilton-Jacobi equations. In ergodic optimal control the additive eigenvalue λ corresponds to averaged long-run optimal costs while λ determines the effective Hamiltonian in the homogenization of Hamilton-Jacobi equations. This problem is often called the ergodic problem in the viewpoint of ergodic optimal control. The additive eigenvalue problem has been studied by so many mathematicians, such as Lions, Ishii, Barles, Giga et al. They applied the additive eigenvalue problem to the large time behavior of the Cauchy problem of Hamilton-Jacobi equations. We also refer to the literatures [3, 4, 7, 23, 24] for the asymptotic problems which treat Hamiltion-Jacobi equations under various boundary conditions including four types of boundary conditions: dynamical boundary condition, state constraint boundary condition, Dirichlet boundary condition and Neumann boundary condition. For Hamiltion-Jacobi equation, the regularity of the solutions is generally no more than Lipschitz continuity and the strong maximum principle can not be expected there. More details can be found in [3, 4, 17, 22] and their references therein.

Next we give our second result.

Theorem 1.2. *Let Ω be a strictly convex bounded domain in $\mathbb{R}^n (n \geq 2)$ with smooth boundary and $q > 0$ be a constant. For $\phi(x), \varphi(x) \in C^\infty(\overline{\Omega})$, there exists a unique $\lambda \in \mathbb{R}$ and $u \in C^\infty(\overline{\Omega})$ solving*

$$\begin{cases} \displaystyle\sum_{i,j=1}^{n} (\delta_{ij} - \frac{u_i u_j}{1+|Du|^2}) u_{ij} = \lambda + \varphi(x) & \text{in } \Omega, \\ u_\nu = -\phi(x) v^{1-q} & \text{on } \partial\Omega, \end{cases} \tag{8}$$

where ν is an inward unit normal vector to $\partial\Omega$.

Moreover, the solution u is unique up to a constant.

Then, following the argument in [25, 27], we have the third result as follows.

Theorem 1.3. *Let Ω be a strictly convex bounded domain in $\mathbb{R}^n (n \geq 2)$ with smooth boundary and $q > 0$ be a constant. Suppose $\phi(x)$ and $\varphi(x)$ are $C^\infty(\overline{\Omega})$. Let u_1 and u_2 be any two solutions to the equation*

$$\begin{cases} u_t = \displaystyle\sum_{i,j=1}^{n} (\delta_{ij} - \frac{u_i u_j}{1+|Du|^2}) u_{ij} - \varphi(x) & \textit{in } \Omega \times (0, \infty), \\ u_\nu = -\phi(x) v^{1-q} & \textit{on } \partial\Omega \times \, (0, \infty), \\ u(x, 0) = u_0(x) & \textit{on } \Omega, \end{cases} \tag{9}$$

with smooth initial data $u_{0,1}$ *and* $u_{0,2}$. *Set* $u = u_1 - u_2$. *Then* $\lim_{t\to\infty} |u|_{C^\infty(\overline{\Omega})} = 0$. *In particular, the solution* $u(x,t)$ *of the problem (9) satisfies* $\lim_{t\to\infty} |u(x,t) - \lambda t - \omega|_{C^\infty(\overline{\Omega})} = 0$, *where* (λ, ω) *is the solution to (8).*

Throughout this paper, we denote

$$a^{ij} = \delta_{ij} - \frac{u_i u_j}{1+|Du|^2}. \tag{10}$$

Remark 1. When $\phi(x) = \varphi(x) = 0$, Theorem 1.3 implies that $u(x,t)$ converges to a constant as $t \to \infty$. This is Huisken's result in [15].

Next we will state another main existence and uniqueness theorem of capillary-type boundary value problem.

Theorem 1.4. *Let* Ω *be a strictly convex bounded domain in* $\mathbb{R}^n (n \geq 2)$ *with smooth boundary and* $q > 0$ *be a constant. For any* $\phi \in C^\infty(\overline{\Omega})$ *and* $\boldsymbol{k} > 0$, *there exists a unique function* $u \in C^\infty(\overline{\Omega})$ *solving*

$$\begin{cases} \operatorname{div}(\dfrac{Du}{\sqrt{1+|Du|^2}}) = \boldsymbol{k} u & \text{in } \Omega\,, \\ u_\nu = -\phi(x) v^{1-q} & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where ν *is an inward unit normal vector to* $\partial\Omega$.

Remark 2. In [26], the third author in this note and Ma proved that for any bounded domain Ω with smooth boundary,

$$\begin{cases} \operatorname{div}(\dfrac{Du}{\sqrt{1+|Du|^2}}) = u & \text{in } \Omega, \\ u_\nu = \phi(x) & \text{on } \partial\Omega, \end{cases} \tag{12}$$

has a unique solution. To get the existence, they need to get C^0 estimate first and then obtain C^1 estimate with C^0 estimate. Here we will directly compute the C^1 estimate without the C^0 estimate but depending on the convexity of the domain.

Actually, we can generalize Theorem 1.4 to any $q \geq 0$ and any bounded smooth domain Ω by adjusting the auxiliary function in [32] , which will be given a remark in the last section of the paper .

In addition, we give the existence of the additive eigenvalue problem for some mean curvature equations.

Theorem 1.5. *Let* Ω *be a strictly convex bounded domain in* $\mathbb{R}^n (n \geq 2)$ *with smooth boundary and* $q > 0$ *be a constant. For any* $\phi \in C^\infty(\overline{\Omega})$, *there exists a unique* $\lambda \in \mathbb{R}$ *and a function* $u \in C^\infty(\overline{\Omega})$ *solving*

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = \lambda & \text{in } \Omega\,, \\ u_\nu = -\phi(x) v^{1-q} & \text{on } \partial\Omega, \end{cases} \tag{13}$$

where ν *is an inward unit normal vector to* $\partial\Omega$.

Moreover, the solution u *is unique up to a constant.*

Remark 3. Integrating two sides in Ω in the formula (13), we obtain

$$\lambda = \frac{\int_{\partial\Omega} \frac{\phi(x)}{(1+|Du|^2)^{q/2}} ds}{|\Omega|}.$$

It is obvious that for $\phi(x) = 0$, $\lambda = 0$.

Remark 4. Consider the constant mean curvature equation with the prescribed contact angle boundary value condition

$$\begin{cases} \operatorname{div}(\frac{Du}{\sqrt{1+|Du|^2}}) = \lambda & \text{in } \Omega, \\ u_\nu = \cos\theta_0\sqrt{1+|Du|^2} & \text{on } \partial\Omega, \end{cases} \tag{14}$$

where ν is an inward unit normal vector to $\partial\Omega$. Even if the domain Ω is a strictly convex bounded domain in $\mathbb{R}^n$ with smooth boundary, θ_0 is a constant and it satisfies the compatibility condition $\lambda|\Omega| = -\cos\theta_o|\partial\Omega|$, the existence of solution for the equation (14) is a very delicate problem. For $n = 2$, Giusti [10] got an existence theorem when the boundary curvature of the domain Ω denoted by k, satisfying $k < \frac{|\partial\Omega|}{|\Omega|}$ at each point. For more information, one can see the related papers [6], [10] and the book [8].

To help the readers know the method for Theorem 1.2 and Theorem 1.5, we introduce the strategy. Firstly, we give the uniform C^1 estimate (independent of ε) for the solution to mean curvature type equations

$$\begin{cases} H(x, Du, D^2u) = \varepsilon u & \text{in } \Omega, \\ u_\nu = -\phi\,(x)\,v^{1-q} & \text{on } \partial\Omega. \end{cases} \tag{15}$$

Then by the maximum principle, we can give the C^0 uniform estimate for εu. By the Schauder theory, we get uniform high order estimates. Letting $\varepsilon \to 0$, Theorem 1.2 and Theorem 1.5 have been proved.

In this paper, in order to simplify the proof of the theorems, we write $O(z)$ as an expression that there exists a constant $C > 0$ such that $|O(z)| \leq Cz$ and C is not related to $\mathbf{k}$, where $\mathbf{k}$ is in (11).

When Ω is a strictly convex smooth domain, there exists a defining function h for Ω such that $\{h_{ij}\}_{n\times n} \geq k_0\{\delta_{ij}\}_{n\times n}$ in Ω for $k_0 > 0$, $h_\nu = -1$ and $|Dh| = 1$ on $\partial\Omega$. Actually, as h is convex, we know $\sup_\Omega |Dh|^2 \leq 1$. In the remaining paragraphs, we will always denote h as the defining function for the strictly convex domain.

In this paper, we proceed as below. In Section 2, in order to prove Theorem 1.1 for mean curvature flow, we shall first give a priori C^0 and C^1 estimates by choosing a new auxiliary function. In Section 3, similarly, we shall show the C^0 estimate and uniform C^1 estimate of mean curvature equation with capillary-type problem so that we can prove Theorem 1.4. In Section 4, adopting the classical method, we will prove the existence of solution of the additive eigenvalue problem, and then will prove Theorem 1.2 and Theorem 1.5. In Section 5, making use of the additive eigenvalue problem, we will describe the long time asymptotic behavior of the corresponding mean curvature flow and finish the proof of Theorem 1.3. Finally, we give some remarks and the comparison with the results in [32] in section 6.

2. **Mean curvature flow with general Capillary-type boundary value problem.** In this section, we will consider the parabolic equation

$$\begin{cases} u_t = \sum_{i,j=1}^{n} a^{ij}u_{ij} - f(x,u) & \text{in } \Omega\times[0,T], \\ u_\nu = -\phi(x)v^{1-q} & \text{on } \partial\Omega\times[0,T], \\ u(x,0) = u_0(x) & \text{on } \Omega, \end{cases} \tag{16}$$

where Ω is a strictly convex bounded domain in $\mathbb{R}^n$ with C^3 boundary, ν is an inward unit normal vector to $\partial\Omega$, f satisfy (4) (5) and a^{ij} has been denoted as (10).

We first give C^0 and C^1 estimate for this problem (16) and then get the existence of the solution to the mean curvature flow by the standard theory.

Lemma 2.1. *Let u be the solution to (16). Then we have*

$$\sup_{\overline{\Omega}\times[0,T]}|u_t| = \sup_{\overline{\Omega}\times\{0\}}|u_t|,$$

and

$$\max_{\overline{\Omega}\times[0,T]}|u(x,t)-u_0(x)| \le C_1 T, \tag{17}$$

where C_1 is a constant independent of T.

Proof. Following the method in [1, Lemma2.2], it suffices to prove the following: for any fixed $T>0$, if u_t admits a positive local maximum at some point $(x_0,t_0)\in \overline{\Omega}\times[0,T]$, that is

$$u_t(x_0,t_0) = \max_{\overline{\Omega}\times[0,T]} u_t > 0,$$

then $t_0=0$.

We argue by contradiction. Thus we assume $t_0>0$. It is easy to calculate that u_t satisfies the equation

$$\frac{d}{dt}u_t = \sum_{i,j=1}^n a^{ij}(u_t)_{ij} - \frac{2}{v}\sum_{i,j=1}^n a^{ij}v_i(u_t)_j - f_u u_t. \tag{18}$$

Then from $f_u\ge 0$ and the parabolic maximum principle to obtain that u_t attains its maximum on $\partial\Omega\times[0,T]$ or on $\Omega\times\{t=0\}$. Hence $x_0\in\partial\Omega$.

On the other hand, differentiating the boundary condition $u_\nu v^{q-1} = -\phi(x)$ with respect to t, we have

$$(u_\nu)_t v^{q-1} + (q-1)u_\nu v^{q-2}v_t = 0,$$

and it yields

$$\begin{aligned}(u_t)_\nu\left[v^{q-1}+(q-1)v^{q-3}u_\nu^2\right] =& -(q-1)v^{q-2}\frac{(|D'u|^2)_t}{2v}u_\nu\\ =& -(q-1)v^{q-3}u_\nu\sum_{i,j=1}^n C^{ij}u_{it}u_j,\end{aligned} \tag{19}$$

where $C^{ij}=\delta_{ij}-\nu^i\nu^j$, $|D'u|^2=\sum_{i,j=1}^n C^{ij}u_iu_j$ and $v^2 = 1+|D'u|^2+u_\nu^2$.

Note that $v^{q-1}+(q-1)v^{q-3}u_\nu^2 > 0$ and $\sum_{i=1}^n C^{ij}u_{it} = \sum_{i=1}^n C^{ij}u_{ti}$ is the tangential part vector of Du_t on the boundary. The right side of equation (19) is zero at (x_0,t_0). It's a contradiction to the Hopf's lemma. Hence it follows that

$$\max_{\overline{\Omega}\times[0,T]}|u(x,t)-u_0(x)| \le C_1 T, \tag{20}$$

where C_1 is a constant which is independent of T. ☐

For interior gradient estimate for mean curvature equation with zero gravity, the auxiliary function with main term like $\log\log|Du|^2$ works in Wang's paper [31]. For global gradient estimate for oblique boundary problem, $|Du|^2$ is not suitable to deal with the boundary. In [20], Lieberman used the auxiliary function with the main term like $|D'u|^2$ to deal with the oblique boundary for $q = 0$ and $q > 1$. After that, in [26] , Ma and Xu used the auxiliary function with main term like $|D(u + \phi d)|^2$, where d is the distance function to the bounary and solved the equation for $q = 1$. In this paper, we are inspired by the auxiliary function $|v + \phi u_\nu|$ in Gerhardt's paper [9], where Gerhardt used the integral methods to solve the prescribed contact boundary problem. To solve the boundary problem for $0 < q < 1$, we will introduce the auxiliary function with the main term as $v^{q+1} - (q+1)\sum_{l=1}^n \phi(x)u_l h_l$. Actually, this main term also works for $q \geq 1$ even for a non-strict domain. As we want to study the asymptotic behavior of the mean curvature flow, it's necessary to get the uniform gradient estimate with respect to time. Next we will apply the maximum principle to prove the gradient estimates for Theorem 1.1 by constructing a new auxiliary function.

Theorem 2.2. *Assume that $\Omega \subset \mathbb{R}^n$ is a strictly convex bounded C^3 domain, $n \geq 2$, and u is the solution to (16). Let ν be an inward unit normal vector to $\partial\Omega$. Suppose $f(x, z) \in C^1(\overline{\Omega} \times \mathbb{R})$, and $\phi(x) \in C^2(\overline{\Omega})$ are given functions respectively which satisfy the conditions (4)-(6). Then there exists a positive constant M_1 such that for any fixed $T > 0$,*

$$\sup_{\overline{\Omega}\times[0,T]} |Du| \leq M_1(n, \Omega, L_1, L_2).$$

Here M_1 is independent of T and $|u|_{C^0}$.

Proof. Let

$$\Phi(x,t) = \log\left[v^{q+1} - (q+1)\sum_{l=1}^n \phi(x)u_l h_l\right] + g(h) := \log w + g(h),$$

where

$$w = v^{q+1} - (q+1)\sum_{l=1}^n \phi(x)u_l h_l, \quad g(h) = \beta h$$

and β is to be determined later.

Suppose that $\Phi(x,t)$ attains its maximum at the point $(x_0, t_0) \in \overline{\Omega} \times [0, T]$. Now we divide two cases to prove Theorem 2.2. In the following proof, all computations will be done at (x_0, t_0). If $t_0 = 0$, then we get the Theorem.

Case1: $x_0 \in \partial\Omega$.

If Φ attains its maximum at x_0 on $\partial\Omega$, then we have

$$0 \geq \Phi_\nu(x_0, t_0) = \frac{w_\nu}{w} - \beta = \frac{q+1}{w}[v^q v_\nu - \sum_{l=1}^n (u_l h_l \phi)_\nu] - \beta. \tag{21}$$

Choose the coordinates in $\mathbb{R}^n$ such that the positive x_n-axis is the interior normal direction to $\partial\Omega$ at x_0. More specifically, u_n denote the unit inner normal derivative and u_i, $1 \leq i \leq n-1$ denote the $n-1$ tangential derivatives of u on the boundary. D denotes the derivative in $\mathbb{R}^n$ and ∇ denotes the tangential derivative on the boundary. We also denote $\nabla_i(u_n) := u_{ni}$ for $1 \leq i \leq n-1$. So $|D'u|^2 = \sum_{k=1}^{n-1} u_k^2$. By the Gauss-Weingarten formula, we get

$$D_{kn}u = u_{nk} + \tilde{\kappa}_{kj}u_j,$$

where $\tilde{\kappa}_{kj}$ is the curvature matrix of the boundary.

As

$$
\begin{aligned}
(v^2)_\nu = (v^2)_n =& 2\sum_{k=1}^{n-1} u_k D_n u_k + 2u_n u_{nn} \\
=& 2\sum_{k=1}^{n-1} u_k (u_{nk} + \sum_{i=1}^{n-1} u_i \tilde{\kappa}_{ik}) - 2\phi v^{1-q} u_{nn},
\end{aligned}
$$

it follows that

$$
v_n = \frac{1}{v}\sum_{k=1}^{n-1} u_k (u_{nk} + \sum_{i=1}^{n-1} u_i \tilde{\kappa}_{ik}) - \phi v^{-q} u_{nn}. \tag{22}
$$

By (21) and (22), we obtain

$$
\begin{aligned}
0 \geq& \Phi_\nu(x_0,t_0) = \frac{q+1}{w}[v^q v_n - \sum_{l=1}^{n}(\phi u_l h_l)_n] - \beta \\
=& \frac{q+1}{w} v^{q-1} \sum_{k=1}^{n-1} u_k (u_{nk} + \sum_{i=1}^{n-1} u_i \tilde{\kappa}_{ik}) + \frac{q+1}{w}(\phi_n u_n - \phi \sum_{l=1}^{n} u_l h_{ln}) - \beta,
\end{aligned} \tag{23}
$$

where the second equality holds because we cancelled the term u_{nn} due to $h_n = -1$ and $h_i = 0$ for $i = 1, \cdots, n-1$ on $\partial\Omega$.

Differentiating the boundary condition for $i = 1, \cdots, n-1$,

$$
u_{ni} = -(1-q)v^{-q} v_i \phi - v^{1-q}\phi_i \quad \text{on } \partial\Omega. \tag{24}
$$

As $\Phi_i(x_0,t_0) = 0$ for $i = 1, \cdots, n-1$, with the property of the defining function, we have

$$
0 = \Phi_i(x_0,t_0) = \frac{q+1}{w}(v^q v_i + \phi_i u_n + \phi u_{ni} - \phi \sum_{l=1}^{n} u_l h_{li}). \tag{25}
$$

It holds that for $i = 1, \cdots, n-1$

$$
v_i(x_0,t_0) = \frac{1}{v^q}(-\phi_i u_n - \phi u_{ni} + \phi \sum_{l=1}^{n} u_l h_{li}). \tag{26}
$$

By (24) and (26), we have for $i = 1, \cdots, n-1$

$$
u_{ni}(x_0,t_0) = \frac{1}{1+(q-1)v^{-2q}\phi^2}\big[\phi(q-1)v^{-2q}(-\phi_i u_n + \phi \sum_{l=1}^{n} u_l h_{li}) - v^{1-q}\phi_i\big].
$$

Note here we always assume that v is very large. Combining with (23), we obtain

$$
\begin{aligned}
0 \geq& \Phi_\nu(x_0,t_0) \\
=& \frac{q+1}{w} v^{q-1} \sum_{k=1}^{n-1} u_k (u_{nk} + \sum_{i=1}^{n-1} u_i \tilde{\kappa}_{ik}) + \frac{q+1}{w}[\phi_n u_n - \phi \sum_{l=1}^{n} u_l h_{ln}] - \beta \\
=& \frac{q+1}{w} \cdot \frac{v^{q-1}}{1+(q-1)v^{-2q}\phi^2} \sum_{k=1}^{n-1} u_k \big[\phi(q-1)v^{-2q}(\phi \sum_{l=1}^{n} u_l h_{lk} - \phi_k u_n) - v^{1-q}\phi_k\big] \\
&+ \frac{q+1}{w} v^{q-1} \sum_{k,i=1}^{n-1} u_k u_i \tilde{\kappa}_{ik} + \frac{q+1}{w}[\phi_n u_n - \phi \sum_{l=1}^{n} u_l h_{ln}] - \beta.
\end{aligned} \tag{27}
$$

If $u_n(x_0,t_0) = 0$, then

$$
|D'u|^2 \geq u_n^2 \text{ at } (x_0,t_0).
$$

If $u_n(x_0,t_0)$ is not zero, then we argue as below. As $u_n=-\phi v^{1-q}$ on $\partial\Omega$, it implies that

$$|u_n|^2\le C(1+|D'u|^2+u_n^2)^{1-q}.$$

By $q>0$,

$$|u_n|^{2q}\le C(2+\frac{|D'u|^2}{u_n^2})^{1-q}.$$

For $0<q<1$, we consider that if $|u_n|>(100C)^{\frac{1}{2q}}$, then it holds that $\frac{|D'u|^2}{u_n^2}>100^{\frac{1}{1-q}}-2>1$ and

$$|D'u|^2\ge u_n^2.$$

If $|u_n|\le(100C)^{\frac{1}{2q}}$, as it is assumed that v is very large, then one gets

$$|D'u|^2\ge u_n^2.$$

For $q\ge1$, we have $|u_n|\le C$, as we assume v is very large, it holds that

$$|D'u|^2\ge u_n^2.$$

Above all,

$$|D'u|^2\ge u_n^2\quad\text{at }(x_0,t_0).$$

As we always assume v is very large, it holds that on $\partial\Omega$

$$\frac{3}{2}v^{q+1}\ge w=v^{q+1}-(q+1)\phi(x)u_n\ge\frac{1}{2}v^{q+1}\quad\text{for}\quad q>0.\tag{28}$$

Therefore, by (27) and (28), it yields that

$$0\ge\Phi_\nu(x_0,t_0)\ge\frac{1+q}{3}k_1-Cv^{-q}-\beta,\tag{29}$$

where κ_1 is the smallest principle curvature of the boundary. So, taking $0<\beta<\frac{1}{4}\kappa_1$, we get

$$|Du|^2(x_0,t_0)\le C.$$

Case2: $x_0\in\Omega$.

It is easy to see that the first and the second derivatives of Φ are as follows

$$0\le\Phi_t(x_0,t_0)=\frac{q+1}{w}\sum_{k=1}^n[v^{q-1}u_k-\phi(x_0)h_k]u_{tk},\tag{30}$$

$$\begin{aligned}0=\Phi_i(x_0,t_0)&=\frac{w_i}{w}+\beta h_i\\&=\frac{q+1}{w}\big[v^qv_i-\phi\sum_{l=1}^n h_lu_{li}-\sum_{l=1}^n u_l(\phi_ih_l+\phi h_{li})\big]+\beta h_i,\end{aligned}\tag{31}$$

and

$$\Phi_{ij}=\frac{w_{ij}}{w}-\frac{w_iw_j}{w^2}+\beta h_{ij}=\frac{w_{ij}}{w}+\beta h_{ij}-\beta^2h_ih_j.\tag{32}$$

It holds that

$$\begin{aligned}0\ge&\sum_{i,j=1}^n a^{ij}\Phi_{ij}(x_0,t_0)-\Phi_t(x_0,t_0)\\=&\Big\{\frac{1}{w}\sum_{i,j=1}^n a^{ij}w_{ij}-\frac{q+1}{w}\sum_{k=1}^n[v^{q-1}u_k-\phi h_k]u_{tk}\Big\}+\sum_{i,j=1}^n a^{ij}(\beta h_{ij}-\beta^2h_ih_j)\\:=&\bar I_1+\bar I_2.\end{aligned}\tag{33}$$

By rotating the axes, we assume that

$$u_1(x_0,t_0) = |Du|(x_0,t_0) > 0, \quad u_i(x_0,t_0) = 0 \ \ (i = 2,3,\cdots,n),$$

and the matrix

$$\{u_{ij}(x_0,t_0)\}_{2\le i,j\le n} \text{ is diagonal.}$$

Therefore

$$a^{11} = \frac{1}{v^2}, \ a^{ij} = 0 \text{ for } i \ne j, \ \text{ and } a^{ii} = 1 \text{ for } i \ge 2. \tag{34}$$

For $i = 1,\cdots,n$, it holds that

$$v_i(x_0,t_0) = \frac{1}{v}\sum_{k=1}^{n} u_k u_{ki} = \frac{u_1 u_{1i}}{v}. \tag{35}$$

By $\Phi_i(x_0,t_0) = 0$ and (35), we have for $i \ge 1$,

$$(q+1)u_1 v^{q-1} u_{1i} - (q+1)\phi \sum_{l=1}^{n} h_l u_{li} = -\beta h_i w + (q+1)(\phi_i h_1 + \phi h_{1i})u_1. \tag{36}$$

For $i \ge 2$, it follows that

$$\begin{aligned} u_{1i} &= \frac{1}{(q+1)(u_1 v^{q-1} - \phi h_1)}\big[-\beta h_i w + (q+1)(\phi_i h_1 + \phi h_{1i})u_1 + (q+1)\phi h_i u_{ii}\big] \\ &= -\frac{\beta}{q+1} h_i v + O(v^{1-q}) + O(1) + \frac{\phi h_i u_{ii}}{u_1 v^{q-1} - \phi h_1} \\ &= -\frac{\beta}{q+1} h_i v + O(v^{1-q}) + O(1) + O(v^{-q})u_{ii}. \end{aligned} \tag{37}$$

We write $O(z)$ as an expression that there exists a constant $C > 0$ such that $|O(z)| \le Cz$ and C is not related to α, where α is determined later. And we point out that α will be taken as a small constant. We emphasis this notation again and use it to make the expression clearly and simply.

For $i = 1$, by (36) and (37), it follows that

$$\begin{aligned} u_{11} =& \frac{1}{(q+1)(u_1 v^{q-1} - \phi h_1)}[-\beta h_1 w + (q+1)(\phi_1 h_1 + \phi h_{11})u_1 + (q+1)\phi \sum_{l=2}^{n} h_l u_{1l}] \\ =& -\frac{\beta}{q+1} h_1 v + O(v^{1-q}) + O(1) + \sum_{l=2}^{n} O(v^{-2q})u_{ll}. \end{aligned} \tag{38}$$

Since we have denoted

$$a^{ij} = \delta_{ij} - \frac{u_i u_j}{1+|Du|^2},$$

the first equation in (16) can be rewritten

$$\sum_{i,j=1}^{n} a^{ij} u_{ij} - f = u_t. \tag{39}$$

Differentiating (39) to x_l for $l \ge 1$, we have

$$\sum_{i,j=1}^{n} a^{ij} u_{ijl} - u_{tl} = -\sum_{i,j,m=1}^{n} a^{ij}_{p_m} u_{ml} u_{ij} + f_{x_l} + f_u u_l, \tag{40}$$

and

$$a^{ij}_{p_m} = \frac{2u_i u_j u_m}{v^4} - \frac{\delta_{im} u_j + \delta_{jm} u_i}{v^2}. \tag{41}$$

Therefore for $l \geq 1$, (40) becomes

$$\sum_{i,j=1}^{n} a^{ij} u_{ijl} - u_{tl} = \sum_{i,j,m=1}^{n} \frac{2}{v^2} a^{ij} u_{lj} u_{im} u_m + f_{x_l} + f_u u_l. \tag{42}$$

To deal with $\bar{I}_1$, we denote $\bar{I}_1 = \frac{1}{w} I_1$ and handle I_1. By the coordinate, in fact,

$$I_1 = \sum_{i,j=1}^{n} a^{ij} w_{ij} - (q+1)\big[v^{q-1} u_1 u_{t1} - \phi \sum_{l=1}^{n} u_{tl} h_l\big]. \tag{43}$$

By direct computations, it holds that

$$w_i =(q+1) v^q v_i - (q+1)\phi \sum_{l=1}^{n} h_l u_{li} - (q+1) \sum_{l=1}^{n} u_l (\phi_i h_l + \phi h_{li}), \tag{44}$$

$$\begin{aligned}
w_{ij} =&(q+1) v^{q-1} \Big(\sum_{k=1}^{n} u_k u_{kij} + \sum_{k=1}^{n} u_{ki} u_{kj} - \frac{1}{v^2} \sum_{k=1}^{n} u_k u_{ki} \sum_{l=1}^{n} u_l u_{lj} \Big) \\
&+ (q+1) q v^{q-1} v_i v_j - (q+1)\phi \sum_{l=1}^{n} h_l u_{lij} - (q+1) \sum_{l=1}^{n} (\phi_j h_l + \phi h_{lj}) u_{li} \\
&- (q+1) \sum_{l=1}^{n} (\phi_i h_l + \phi h_{li}) u_{lj} - (q+1) \sum_{l=1}^{n} u_l (\phi_i h_{lj} + \phi_{ij} h_l + \phi_j h_{li} + \phi h_{lij}) \\
=&(q+1)(v^{q-1} u_1 u_{1ij} - \phi \sum_{l=1}^{n} h_l u_{lij}) + (q+1) v^{q-1} \Big(\sum_{k=1}^{n} u_{ki} u_{kj} - \frac{u_1^2}{v^2} u_{1i} u_{1j} \Big) \\
&+ (q+1) q v^{q-1} v_i v_j - (q+1) \sum_{l=1}^{n} [(\phi_j h_l + \phi h_{lj}) u_{li} + (\phi_i h_l + \phi h_{li}) u_{lj}] \\
&- (q+1)(\phi_i h_{1j} + \phi_j h_{1i} + \phi_{ij} h_1 + \phi h_{1ij}) u_1.
\end{aligned} \tag{45}$$

Based on (42) and (45), we rewrite I_1 in (43) as

$$I_1 :=I_{11} + I_{12} + I_{13}, \tag{46}$$

where

$$\begin{aligned}
I_{11} =&(q+1) v^{q-1} u_1 \Big(\frac{2u_1}{v^2} \sum_{i,j=1}^{n} a^{ij} u_{1i} u_{1j} + f_{x_1} + f_u u_1 \Big) \\
&- (q+1)\phi \sum_{l=1}^{n} h_l \Big(\frac{2u_1}{v^2} \sum_{i,j=1}^{n} a^{ij} u_{lj} u_{1i} + f_{x_l} + f_u u_l \Big) \\
&+ (q+1) v^{q-1} \sum_{i,j=1}^{n} a^{ij} \Big(\sum_{k=1}^{n} u_{ki} u_{kj} - \frac{u_1^2}{v^2} u_{1i} u_{1j} \Big),
\end{aligned} \tag{47}$$

$$I_{12} =(q+1) \sum_{i,j=1}^{n} a^{ij} \big(q v^{q-1} v_i v_j - 2\phi_i \sum_{l=1}^{n} h_l u_{lj} - 2\phi \sum_{l=1}^{n} h_{lj} u_{li} \big), \tag{48}$$

and

$$I_{13} = -(q+1) u_1 \sum_{i,j=1}^{n} a^{ij} (2\phi_i h_{1j} + h_1 \phi_{ij} + \phi h_{1ij}). \tag{49}$$

Firstly, from (37) and (38), it is not difficult to obtain

$$I_{12}+I_{13}\geq -Cv-\sum_{i=2}^{n}C|u_{ii}|. \tag{50}$$

Next we deal with I_{11}. By $f_u\geq 0$ and $|f_x|\leq L_1$, it yields that

$$\begin{aligned}I_{11}\geq&(q+1)\big(v^{q-1}u_1-\phi h_1\big)\frac{2u_1}{v^2}\sum_{i,j=1}^{n}a^{ij}u_{1i}u_{1j}-(q+1)\phi\sum_{l=2}^{n}h_l\sum_{i,j=1}^{n}\frac{2u_1}{v^2}a^{ij}u_{1i}u_{lj}\\&+(q+1)v^{q-1}\sum_{i,j=1}^{n}a^{ij}\big(\sum_{k=1}^{n}u_{ki}u_{kj}-\frac{u_1^2}{v^2}u_{1i}u_{1j}\big)-Cv^q\\:=&J_1+J_2+J_3-Cv^q.\end{aligned} \tag{51}$$

For J_1, we get that

$$J_1=\frac{2(q+1)u_1}{v^2}\big(v^{q-1}u_1-\phi h_1\big)\big(\frac{u_{11}^2}{v^2}+\sum_{i=2}^{n}u_{1i}^2\big). \tag{52}$$

For J_2, by Cauchy inequality, it holds that

$$\begin{aligned}J_2=&-\frac{2(q+1)\phi u_1}{v^2}\sum_{k=2}^{n}h_k\big(\frac{1}{v^2}u_{11}u_{1k}+\sum_{i=2}^{n}u_{1i}u_{ik}\big)\\=&-\frac{2(q+1)\phi u_1}{v^2}\sum_{k=2}^{n}h_k\big(\frac{1}{v^2}u_{11}u_{1k}+u_{1k}u_{kk}\big)\\\geq&-Cv^{-3}u_{11}^2-Cv^{-1}\sum_{i=2}^{n}u_{1i}^2-Cv^{-1}\sum_{i=2}^{n}u_{ii}^2.\end{aligned} \tag{53}$$

For J_3, we have

$$\begin{aligned}J_3=&(q+1)v^{q-1}\sum_{i=1}^{n}a^{ii}\big(\frac{u_{1i}^2}{v^2}+\sum_{k=2}^{n}u_{ki}^2\big)\\=&(q+1)v^{q-3}\big(\frac{u_{11}^2}{v^2}+\sum_{i=2}^{n}u_{1i}^2\big)+(q+1)v^{q-3}\big(\sum_{i=2}^{n}u_{1i}^2+v^2\sum_{i=2}^{n}u_{ii}^2\big).\end{aligned} \tag{54}$$

By (52)-(54) and $q>0$, we obtain

$$\begin{aligned}I_{11}\geq&\big[\frac{2(q+1)u_1}{v^4}\big(v^{q-1}u_1-\phi h_1\big)-Cv^{-3}\big]u_{11}^2\\&+\big[\frac{2(q+1)u_1}{v^2}(v^{q-1}u_1-\phi h_1)-Cv^{-1}\big]\sum_{i=2}^{n}u_{1i}^2\\&+\big[(q+1)v^{q-1}-Cv^{-1}\big]\sum_{i=2}^{n}u_{ii}^2-Cv^q\\\geq&v^{q-3}u_{11}^2+v^{q-1}\sum_{i=2}^{n}u_{1i}^2+v^{q-1}\sum_{i=2}^{n}u_{ii}^2-Cv^q.\end{aligned} \tag{55}$$

Putting I_{11} in (55), $I_{12}+I_{13}$ in (50) into (46), and combining $\bar{I}_1 = \frac{1}{w}I_1$, it follows that

$$\begin{aligned}\bar{I}_1 \geq& \frac{1}{w}\big(v^{q-3}u_{11}^2 + v^{q-1}\sum_{i=2}^n u_{1i}^2 + v^{q-1}\sum_{i=2}^n u_{ii}^2 - Cv^q - Cv - \sum_{i=2}^n C|u_{ii}|\big)\\ \geq& - Cv^{-q} - Cv^{-1} - Cv^{-2q},\end{aligned} \tag{56}$$

since it holds that $w > \frac{1}{2}v^{q+1}$ and

$$v^{q-1}\sum_{i=2}^n u_{1i}^2 - \sum_{i=2}^n C|u_{ii}| \geq -Cv^{1-q}.$$

Because

$$\bar{I}_2 = \beta(\frac{h_{11}}{v^2} + \sum_{i=2}^n h_{ii}) - \beta^2(\frac{h_1^2}{v^2} + \sum_{i=2}^n h_i^2), \tag{57}$$

the formula (33) changes to

$$\begin{aligned}0 \geq& \sum_{i,j=1}^n a^{ij}\Phi_{ij} - \Phi_t\\ \geq& - Cv^{-q} - Cv^{-1} + \beta k_0\frac{n+(n-1){u_1}^2}{v^2} - \frac{\beta^2}{v^2}[{h_1}^2 + (1+{u_1}^2)\sum_{i=2}^n {h_i}^2].\end{aligned} \tag{58}$$

Taking $0 < \beta < \min\{\frac{k_0(n-1)}{4}, \frac{1}{8\kappa_1}\}$, $|Du|(x_0,t_0) \leq C$.

Above all, we know

$$\sup_{\overline{\Omega}\times[0,T]} \log\big[v^{q+1} - (q+1)\sum_{l=1}^n \phi(x)u_l h_l\big] + g(h) \leq C.$$

As we assume v are large, it holds that

$$\sup_{\overline{\Omega}\times[0,T]} |Du| \leq C.$$

□

With the gradient estimates of the solution to the parabolic equation in (1), it becomes uniformly parabolic equation for any fixed $T > 0$. By the standard theory, we obtain the long time existence of solutions.

Theorem 2.3 (Theorem 1.1)**.** *Assume that $\Omega \subset \mathbb{R}^n$ is a strictly convex bounded C^3 domain for $n \geq 2$. Let ν be the inward unit normal vector to $\partial\Omega$. Under the conditions (4)-(6), there exists a unique solution $u \in C^{2,\sigma}(\overline{\Omega}\times[0,\infty))$ to the problem (1) for some $\sigma \in (0,1)$.*

Using the same auxiliary function without t and almost the same computation, we can also obtain

Theorem 2.4. *Let Ω be a strictly convex bounded domain in $\mathbb{R}^n$ with smooth boundary and suppose $q > 0$ is a constant and $\boldsymbol{k} \geq 0$. Let $\phi \in C^2(\overline{\Omega}), \varphi \in C^1(\overline{\Omega})$ and $u \in C^3(\overline{\Omega})$ satisfy*

$$\begin{cases}\sum_{i,j=1}^n (\delta_{ij} - \frac{u_i u_j}{1+|Du|^2})u_{ij} = \boldsymbol{k}u + \varphi(x) & in\ \Omega\,,\\ u_\nu = -\phi(x)v^{1-q} & on\ \partial\Omega,\end{cases} \tag{59}$$

where ν is an inward unit normal vector to $\partial\Omega$. Then we have

$$\sup_{\overline{\Omega}} |Du| \leq M_1,$$

where M_1 is independent of $\boldsymbol{k}$ and depends on Ω, $|\phi|_{C^2(\overline{\Omega})}, |\varphi|_{C^1(\bar{\Omega})}$.

3. **Uniformly global C^0 and C^1 estimate for the proof of Theorem 1.4.** In this section, we give C^0 estimate for the solution to the classical mean curvature equation (60) and then we will use the same auxiliary function in Theorem 2.2 to give the C^1 estimate for the solution to equation (60) with more delicate computations.

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = \mathbf{k}uv^{\alpha} + \varphi(x) & \text{in } \Omega\,, \\ u_\nu = -\phi(x)v^{1-q} & \text{on } \partial\Omega, \end{cases} \tag{60}$$

where $\mathbf{k} > 0$, $q > 0$ and $\alpha \in \mathbb{R}$.

Lemma 3.1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $\partial\Omega \in C^3$, $n \geq 2$. Assume $u \in C^2(\overline{\Omega}) \bigcap C^3(\Omega)$ is the solution to (60) with $q > 0$ and $\boldsymbol{k} > 0$. There exists a constant $L > 0$ such that*

$$\sup_{\overline{\Omega}} |\boldsymbol{k}u| \leq L,$$

where L depends on Ω, $|\varphi|_{C^0(\overline{\Omega})}$ and $|\phi|_{C^0(\overline{\Omega})}$.

Proof. We denote the solution to equation (2) by u_k. We give the C^0 estimate for the solution. Let g be a smooth function on $\overline{\Omega}$ satisfying $(\sqrt{1+|Dg|^2})^{q-1}D_\nu g < -\sup_{\overline{\Omega}}|\phi(x)|$ on $\partial\Omega$ for $q > 0$.

Let ζ be a point where $g - u_k$ achieves its minimum. If $\zeta \in \partial\Omega$, then $D_T g(\zeta) = D_T u_k(\zeta)$ and $D_\nu g(\zeta) \geq D_\nu u_k(\zeta)$, where T denotes the tangent vector to $\partial\Omega$.

For $0 < q \leq 1$, we know that

$$\frac{p}{(\sqrt{1+a^2+p^2})^{1-q}}$$

is monotonously increasing with respect to p for fixed a.

$$(\sqrt{1+|Dg|^2})^{q-1}D_\nu g \geq (\sqrt{1+|Du|^2})^{q-1}D_\nu u(\zeta) = -\phi(\zeta). \tag{61}$$

It's a contradiction to the definition of g.

For $q > 1$, as $D_\nu g < 0$ on $\partial\Omega$, it holds that $|Dg|(\zeta) \leq |Du_k|(\zeta)$ and $D_\nu u_k < 0$. Therefore

$$\begin{aligned} (\sqrt{1+|Dg|^2})^{q-1}D_\nu g(\zeta) &\geq (\sqrt{1+|Dg|^2})^{q-1}D_\nu u(\zeta) \\ &\geq (\sqrt{1+|Du|^2})^{q-1}D_\nu u(\zeta) = -\phi(\zeta). \end{aligned} \tag{62}$$

It's contradicted to g's definition. So $\zeta \in \Omega$, then $Dg(\zeta) = Du_k(\zeta)$ and $D^2g(\zeta) \geq D^2u_k(\zeta)$. Therefore, there exists a constant $c = c(g)$ such that

$$c \geq \sum_{i,j=1}^{n} a_{ij}(Dg)g_{ij}(\zeta) \geq \sum_{i,j=1}^{n} a_{ij}(Du_k)(u_k)_{ij}(\zeta) = \mathbf{k}u_k(\zeta) + \varphi(\zeta),$$

where

$$a_{ij}(Du) = (\delta_{ij} - \frac{u_i u_j}{1+|Du|^2})\frac{1}{(\sqrt{1+|Du|^2})^{\alpha+1}}.$$

So, combining with $g(x) - u_k(x) \geq g(\zeta) - u_k(\zeta)$ for $x \in \Omega$, we see that

$$\mathbf{k}u_k(x) \leq \mathbf{k}g(x) - \mathbf{k}g(\zeta) + c.$$

By the similar method, we have the lower bound for $\mathbf{k}u_k(x)$. It follows that

$$\sup_{\overline{\Omega}} |\mathbf{k}u_k| \leq L, \text{ for some } L > 0 \tag{63}$$

and therefore $\sup_{\overline{\Omega}} |u_k| \leq L/\mathbf{k}$. □

With similar proof to Lemma 3.1 , we know

Corollary 1. *Let Ω be a bounded domain in $\mathbb{R}^n$ and $\partial\Omega \in C^3$, $n \geq 2$. Assume $u \in C^2(\overline{\Omega}) \bigcap C^3(\Omega)$ is the solution to (59) with $q > 0$ and $\boldsymbol{k} > 0$. There exists a constant $L > 0$ such that*

$$\sup_{\overline{\Omega}} |\boldsymbol{k}u| \leq L$$

where L depends on Ω and $|\phi|_{C^0(\overline{\Omega})}, |\varphi|_{C^0(\overline{\Omega})}$.

Next we will prove the uniform gradient estimate for equation (2).

Theorem 3.2. *Let Ω be a strictly convex bounded domain in $\mathbb{R}^n$ with C^3 boundary and suppose $q > 0$ is a constant, $\boldsymbol{k} \geq 0$ and $\alpha \in \mathbb{R}$.. Suppose $\phi(x) \in C^2(\overline{\Omega})$ and $u \in C^2(\overline{\Omega}) \bigcap C^3(\Omega)$ is a solution to*

$$\begin{cases} \operatorname{div}\big(\dfrac{Du}{\sqrt{1+|Du|^2}}\big) = \boldsymbol{k}uv^{\alpha} & \text{in } \Omega\,, \\ u_\nu = -\phi(x)v^{1-q} & \text{on } \partial\Omega. \end{cases} \tag{64}$$

Then there exists a positive constant $C > 0$ such that

$$\sup_{\overline{\Omega}} |Du| \leq C,$$

where C is independent of $\boldsymbol{k}$ and depends on Ω, $|\phi|_{C^2(\overline{\Omega})}$.

Proof. We choose the auxiliary function

$$\tilde{\Phi}(x) = \log\big[v^{q+1} - (q+1)\sum_{l=1}^{n} u_l h_l \phi(x)\big] + g(h) := \log w + g(h),$$

which is similar to the proof in Theorem 2.2. Here we omit the detailed proof. □

4. **Proof of Theorem 1.4 and Theorem 1.5.** In this section, we will give the proof for the existence of solution to equation (60) and the corresponding additive eigenvalue problem, which follows the argument in [22, 25] and includes Theorem 1.4 and Theorem 1.5.

Theorem 4.1. *Let Ω be a strictly convex bounded domain in $\mathbb{R}^n (n \geq 2)$ with smooth boundary, $q > 0$ be a constant and $\alpha \in \mathbb{R}$. For any $\phi \in C^\infty(\overline{\Omega})$ and $\boldsymbol{k} > 0$, there exists a unique function $u \in C^\infty(\overline{\Omega})$ solving (64).*

Proof. First, we give the proof of existence. Setting

$$G(x,p) := v^{q-1}\langle \nu, p\rangle + \phi(x),$$

where $p = (p_1, \cdots, p_n)$. Then we have

$$G_{p_l} := \frac{\partial G}{\partial p_l} = (q-1)p_l v^{q-3} \sum_{k=1}^{n} p_k \nu_k + v^{q-1}\nu_l$$

and

$$\sum_{l=1}^{n} G_{p_l}\nu_l = v^{q-1}[1+(q-1)\frac{|\sum_{l=1}^{n} p_l\nu_l|^2}{v^2}],$$

where $\nu = (\nu_1, \cdots, \nu_n)$ is an inner normal vector on the boundary and $v = \sqrt{1+|p|^2}$.

For $q > 0$, it holds that

$$\sum_{l=1}^{n} G_{p_l}\nu_l > 0.$$

According to the paper by Lieberman and Trudinger [21], this capillary-type boundary value problem belongs to oblique boundary value problems. In order to make use of the continuity method to prove the existence for such oblique boundary value problems, we consider a family of capillary-type boundary value problems

$$\mathtt{div}(\frac{Du}{\sqrt{1+|Du|^2}}) = \mathbf{k}uv^{\alpha} \quad \text{in } \Omega, \tag{65}$$

$$\frac{\partial u}{\partial \gamma} = -\tau\phi(x)v^{1-q} \quad \text{on } \partial\Omega, \tag{66}$$

where $\tau \in [0, 1]$.

For $\tau = 0$, $u = 0$ is the unique solution. And we need to find the solution for $\tau = 1$.

By the standard elliptic regularity theory or as in the reference [21], if we can get the a priori estimates for the $C^2(\overline{\Omega})$ solution of the equations (65) and (66)

$$\sup_{\Omega} |u| \le \tilde{K}_1, \tag{67}$$

$$\sup_{\Omega} |Du| \le \tilde{K}_2, \tag{68}$$

where $\tilde{K}_1, \tilde{K}_2$ are independent of τ. Then we can get the existence. As we have obtained the global C^0 and C^1 a priori estimates for $\tau = 1$, by the continuity method, the existence of such problem has been proved.

Next we prove the uniqueness. Assume there exist two different solutions u_1 and u_2 for (11). Let $\bar{w} = u_1 - u_2$ and $\bar{w}$ satisfies

$$\begin{cases} \sum_{i,j=1}^{n} \tilde{a}^{ij}\bar{w}_{ij} + \sum_{i=1}^{n} b_i\bar{w}_i = \mathbf{k}(u_1 - u_2) & \text{in } \Omega, \\ \sum_{l=1}^{n} \bar{G}_{p_l}\bar{w}_l = 0 & \text{on } \partial\Omega, \end{cases} \tag{69}$$

where $\tilde{a}^{ij} = \frac{\tilde{a}^{ij}(Du_1)}{(\sqrt{1+|Du_1|^2})^{3+\alpha}}$,

$$b_i = (u_2)_{kl}\int_0^1 \tilde{a}^{kl}_{p_i}(\eta Du_1 + (1-\eta)Du_2)d\eta$$

and

$$\bar{G}_{p_l} = \int_0^1 \frac{\partial G(x, tDu_1 + (1-t)Du_2)}{\partial p_l}dt.$$

Because $\sum_{l=1}^{n} G_{p_l}\nu_l > 0$ and $\mathbf{k} > 0$, we know $\bar{w} = 0$ and therefore $u_1 = u_2$ by the maximum principle and Hopf's lemma. This completes the proof. $\square$

By Theorem 4.1, we get Theorem 1.4 with $\alpha = 0$.

Theorem 4.2. *Let Ω be a strictly convex bounded domain in $\mathbb{R}^n(n \geq 2)$ with smooth boundary, $q > 0$ be a constant and $\alpha \in \mathbb{R}$. For any $\phi \in C^\infty(\overline{\Omega})$, there exists a unique $\lambda \in \mathbb{R}$ and a function $u \in C^\infty(\overline{\Omega})$ solving*

$$\begin{cases} \operatorname{div}\left(\dfrac{Du}{\sqrt{1+|Du|^2}}\right) = \lambda v^\alpha & \text{in } \Omega\,, \\ u_\nu = -\phi(x)v^{1-q} & \text{on } \partial\Omega, \end{cases} \tag{70}$$

where ν is an inward unit normal vector to $\partial\Omega$.

Moreover, the solution u is unique up to a constant. Here λ is called the additive eigenvalue.

Proof. We denote by u_ε the solution to equation (11) with $\mathbf{k} = \varepsilon$. We know that for each ε, the solution to equation (11) exists, which is achieved in Section 3.

Based on Lemma 3.1, we get

$$\sup_\Omega |\varepsilon u_\varepsilon| \leq C.$$

Consider $w_\varepsilon = u_\varepsilon - \frac{\int_\Omega u_\varepsilon}{|\Omega|}$. Then w_ε satisfies

$$\begin{cases} (\sqrt{1+|Dw_\varepsilon|^2})^{-\alpha}\operatorname{div}(\dfrac{Dw_\varepsilon}{\sqrt{1+|Dw_\varepsilon|^2}}) = \varepsilon w_\varepsilon + \varepsilon\dfrac{\int_\Omega u_\varepsilon}{|\Omega|} & \text{in } \Omega, \\ (\sqrt{1+|Dw_\varepsilon|^2})^{q-1}D_\nu w_\varepsilon = -\phi(x) & \text{on } \partial\Omega, \end{cases} \tag{71}$$

where ν is an inward unit normal vector to $\partial\Omega$.

As

$$\sup_{\overline{\Omega}} |Dw_\varepsilon| = \sup_{\overline{\Omega}} |Du_\varepsilon| \leq C_0,$$

it holds that $|w_\varepsilon| \leq C$. Also we have $|\varepsilon u_\varepsilon| \leq C$, and $\varepsilon|\frac{\int_\Omega u_\varepsilon}{|\Omega|}| \leq C$. By Schauder theory in [21], we know $|w_\varepsilon|_{C^{2,\alpha}(\overline{\Omega})} \leq C$ for some $\alpha \in (0,1)$. Taking $\varepsilon \to 0$, we have $w_\varepsilon \to w$ in $C^{2,\alpha'}$ for some $\alpha' < \alpha$ and $\varepsilon w_\varepsilon + \varepsilon\frac{\int_\Omega u_\varepsilon}{|\Omega|} \to \lambda$, where (λ, w) satisfies equation (70). Actually, uniform higher order derivatives for w_ε also follows from the Schauder theory.

Let $\hat{w} = u_1 - u_2$, it is obvious that $\hat{w}$ satisfies

$$\begin{cases} \displaystyle\sum_{i,j=1}^n \tilde{a}^{ij}\hat{w}_{ij} + \sum_{i=1}^n b_i\hat{w}_i = \lambda_1 - \lambda_2 \leq 0 & \text{in } \Omega, \\ \displaystyle\sum_{l=1}^n \bar{G}_{p_l}w_l = 0 & \text{on } \partial\Omega, \end{cases} \tag{72}$$

where $\tilde{a}^{ij} = \frac{a^{ij}(Du_1)}{(\sqrt{1+|Du_1|^2})^{1+\alpha}}$,

$$b_i = (u_2)_{kl}\int_0^1 \tilde{a}^{kl}_{p_i}(\eta Du_1 + (1-\eta)Du_2)d\eta$$

and

$$\bar{G}_{p_l} = \int_0^1 \frac{\partial G(x, tDu_1 + (1-t)Du_2)}{\partial p_l}dt.$$

Note that $\sum_{l=1}^n \bar{G}_{p_l}\nu_l > 0$. By the Hopf's lemma and the strong maximum principle, w must be a constant. Consequently, $\lambda_1 = \lambda_2$. This completes the proof. $\square$

Let $\alpha = 0$, we get Theorem 1.5 by Theorem 4.2. Actually, with the same proof, we can get Theorem 1.2 and we skip the proof.

5. **Asymptotic behavior of the mean curvature flow with oblique boundary.** In this section, by making use of the additive eigenvalue problem of (8) and following the argument in [25, 27], we will study the asymptotic behavior of the mean curvature flow with capillary-type boundary in Theorem 1.3.

Let (λ, ω) be the solution for equation (8) and let $\widetilde{\omega}(x,t) = \omega + \lambda t$. It's easy to check that $\widetilde{\omega}$ solves the parabolic problem

$$\begin{cases} u_t = \sum_{i,j=1}^{n} (\delta_{ij} - \frac{u_i u_j}{1+|Du|^2}) u_{ij} - \varphi(x) & \text{in } \Omega \times (0,\infty), \\ u_\nu = -\phi(x) v^{1-q} & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = \omega(x) & \text{on } \Omega. \end{cases} \tag{73}$$

Corollary 2. *For a solution $u = u(x,t)$ to equation (9), there exists a constant $C_3 > 0$ such that*

$$|u(x,t) - \lambda t| \leq C_3.$$

Proof. Let $z(x,t) = u(x,t) - \widetilde{w}(x,t)$ and it satisfies the following equation

$$\begin{cases} z_t = \sum_{i,j=1}^{n} \widetilde{a}^{ij} z_{ij} + \sum_{i=1}^{n} b_i z_i & \text{in } \Omega \times (0,\infty), \\ \sum_{l=1}^{n} \bar{G}_{p_l} z_l = 0 & \text{on } \partial\Omega \times (0,\infty)\ , \\ z(x,0) = u_0(x) - w(x) & \text{on } \Omega, \end{cases} \tag{74}$$

where $\widetilde{a}^{ij} = a^{ij}(Du)$,

$$b_i = (\widetilde{w})_{kl} \int_0^1 a_{kl,p_i}(\eta Du + (1-\eta) D\widetilde{w}) d\eta,$$

and

$$\bar{G}_{p_l} = \int_0^1 \frac{\partial G(x, tDu + (1-t)Dw)}{\partial p_l} dt.$$

By the maximum principle, z achieves its maximum and minimum on $\Omega \times \{0\}$.

Therefore,

$$\sup_{\Omega \times (0,\infty)} |u - \lambda t| \leq \sup_{\Omega} |w| + \sup_{\Omega} |u_0 - w|.$$

$\square$

Theorem 5.1 (Theorem 1.3)**.** *Let u_1 and u_2 be any two solutions to equation (9) with initial data $u_{0,1}$ and $u_{0,2}$. Let $u = u_1 - u_2$. Then u converges to a constant function in $C^\infty(\bar{\Omega})$ as $t \to \infty$. In particular, the solution $u(x,t)$ to equation (9) converges to $\lambda t + \omega$ in $C^\infty(\bar{\Omega})$ as $t \to \infty$, where (λ, ω) is the solution to (8).*

Proof. As the proof of Corollary 2, u satisfies

$$\begin{cases} z_t = \sum_{i,j=1}^{n} \widetilde{a}^{ij} z_{ij} + \sum_{i=1}^{n} b_i z_i & \text{in } \Omega \times (0,\infty), \\ \sum_{l=1}^{n} \bar{G}_{p_l} z_l = 0 & \text{on } \partial\Omega \times (0,\infty), \\ z(x,0) = u_{0,1}(x) - u_{0,2}(x) & \text{on } \Omega, \end{cases} \tag{75}$$

where $\widetilde{a}^{ij} = a^{ij}(Du_1)$,

$$b_i = (u_2)_{kl} \int_0^1 a_{kl,p_i}(\eta Du_1 + (1-\eta)Du_2) d\eta,$$

and

$$\bar{G}_{p_l} = \int_0^1 \frac{\partial G(x, tDu_1 + (1-t)Du_2)}{\partial p_l} dt.$$

Let $osc(u)(t) = \max_\Omega u(x,t) - \min_\Omega u(x,t)$. By the strong maximum principle, $osc(u)(t)$ is a strictly decreasing function unless u is a constant.

We next claim that $\lim_{t\to\infty} osc(u)(t) = 0$. Then

$$\lim_{t\to\infty} \max_\Omega u = \lim_{t\to\infty} \min_\Omega u = c_0 \text{ for some } c_0.$$

That is $\lim_{t\to\infty} |u - c_0| = 0$.

Otherwise we have $\lim_{t\to\infty} osc(u)(t) = \delta$ for some $\delta > 0$.

Given a sequence $t_n \to \infty$, we define

$$u_{1,n}(\cdot,t) = u_1(\cdot, t+t_n) - \lambda t_n$$

and

$$u_{2,n}(\cdot,t) = u_2(\cdot, t+t_n) - \lambda t_n.$$

According to Corollary 2, we know $|u_{1,n} - \lambda t| \le C$, $|u_{2,n} - \lambda t| \le C$. By uniform C^1 estimate for mean curvature flow in Theorem 2.2, we know $u_{1,n}(\cdot,t)$ and $u_{2,n}(\cdot,t)$ are locally (in time) C^k uniformly bounded with respect to n for any $\mathbf{k}$ by Schauder theory in [21]. So there exists a subsequence (still denoted by t_n) such that $u_{1,n}(\cdot,t)$ and $u_{2,n}(\cdot,t)$ converge locally uniformly in any C^k to $u_1^*(\cdot,t)$ and $u_2^*(\cdot,t)$ respectively. That is

$$u_1^*(\cdot,t) = \lim_{n\to\infty} u_{1,n}(\cdot,t),\ u_2^*(\cdot,t) = \lim_{n\to\infty} u_{2,n}(\cdot,t).$$

Let $u^* = u_1^* - u_2^*$,

$$\begin{aligned} osc(u^*) &= osc(u_1^* - u_2^*) \\ &= \lim_{n\to\infty} osc(u_1(x,t+t_n) - \lambda t_n - u_2(x,t+t_n) + \lambda t_n) \\ &= \lim_{n\to\infty} osc(u_1(x,t+t_n) - u_2(x,t+t_n)) \\ &= \lim_{n\to\infty} osc(u)(t+t_n) = \delta, \end{aligned} \tag{76}$$

where the second equality holds because of the uniform convergence of $u_{1,n}(\cdot,t)$ and $u_{2,n}(\cdot,t)$.

But u^* satisfies the parabolic equation

$$\begin{cases} z_t = \sum_{i,j=1}^{n} \widetilde{a}^{ij} z_{ij} + \sum_{i=1}^{n} b_i z_i & \text{in } \Omega \times (-\infty, \infty), \\ \sum_{l=1}^{n} \bar{G}_{p_l} z_l = 0 & \text{on } \partial\Omega \times (-\infty, \infty), \end{cases} \tag{77}$$

where $\widetilde{a}^{ij} = a^{ij}(Du_1^*)$,

$$b_i = (u_2^*)_{kl} \int_0^1 a_{kl,p_i}(\eta Du_1^* + (1-\eta)Du_2^*) d\eta$$

and

$$\bar{G}_{p_l} = \int_0^1 \frac{\partial G(x, tDu_1^* + (1-t)Du_2^*)}{\partial p_l} dt.$$

By the strong maximum principle and the Hopf's lemma, we know u^* is a constant. This makes a contradiction to $osc(u^*) = \delta$.

With high regularity estimates of u by the classical Schauder theory, Theorem 5.2 is proved by the inequality in [11, 16]

$$|u|_{k,\alpha} \le C(|u|_{k_1,\alpha_1})^t (|u|_{k_2,\alpha_2})^{1-t},$$

where $0 < t < 1$, $k + \alpha = t(k_1 + \alpha_1) + (1-t)(k_2 + \alpha_2)$. □

6. **some remarks.** For the following oblique boundary value problem of prescribed mean curvature equations

$$\mathtt{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = f(x,u) \quad \text{in} \quad \Omega, \tag{78}$$

$$v^{q-1}\frac{\partial u}{\partial \gamma} + \psi(x,u) = 0 \quad \text{on} \quad \partial\Omega, \tag{79}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $n \ge 2$, $\partial\Omega$ is C^3, γ is the inward unit normal to $\partial\Omega$ and $q \ge 0$, $v = \sqrt{1+|Du|^2}$.

Remark 5. For the problem (78)-(79), after slightly modifying the auxiliary function $P(x) = \log |D'u|^2 e^{h(u)+g(d)}$ in [32] to $\tilde{P}(x) = |D'u|^2 e^{A_0 d + B_0 d^2}$, where A_0 and B_0 are constants and large enough, we can generalize the results in [32] to any constant $q \ge 0$.

When $f = ku, k > 0$ and $\psi = \psi(x)$, following the similar method in Spruck [29], we can obtain the C^0 estimate. Combining our C^1 estimate and using the continuity method, we get the existence and uniqueness of the solution. Hence Theorem 1.4 can be generalized to any $q \ge 0$ and any bounded smooth domain Ω.

Remark 6. We should point out an error on page 1724 in [32] i.e. the inequality (39) cannot be directly derived from the inequality (38). We can correct it using the slightly modified function $\tilde{P}(x) = |D'u|^2 e^{A_0 d + B_0 d^2}$ to get $c^{11}(x_0)$ having the positive lower bound in [32].

Remark 7. For the problem (78)-(79), $\psi = \psi(x,z)$, when $0 \le q < 1$, we need $\psi_z \le 0$; when $q \ge 1$, we need $|\psi|_{C^2}$ is bounded or $\psi_z \le 0$.

Remark 8. Since we can cover the results in [32] and the proof is similar to that in [32], we omit the details here. Note that in elliptic case the domain Ω is not required to be convex.

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