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Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition

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Abstract. In this work we study the behaviour of compact, smooth, immersed manifolds with boundary which move under the mean curvature flow in Euclidian space. We thereby prescribe the Neumann boundary condition in a purely geometric manner by requiring a vertical contact angle between the unit normal fields of the immersions and a given, smooth hypersurface Σ. We deduce a very sharp local gradient bound depending only on the curvature of the immersions and Σ. Combining this with a short time existence result, we obtain the existence of a unique solution to any given smooth initial and boundary data. This solution either exists for any $t > 0$ or on a maximal finite time interval $[0, T]$ such that the curvature explodes as $t \to T$.

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1 Introduction and main results

The behaviour of hypersurfaces which move under various geometric flows has been studied intensively during the last years. One major topic within this research is devoted to the evolution of compact, convex hypersurfaces *without boundary*. Here solutions to a given convex initial hypersurface will typically exist within a maximal finite time interval $[0, T]$ such that the curvature of the immersions explodes as $t \to T$. Results concerning this topic are due to Andrews [1], Huisken [8], [9], Tso [18] and others.

Another point of interest is the behaviour of compact surfaces *with boundary* which move under mean-curvature-like flows. Here in general the hypersurface is described as graph of a scalar function $w : \Omega \times [0, T] \longrightarrow \mathbb{R}$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain, and boundary conditions are prescribed on $\partial\Omega \times [0, T]$; thus geometrically the hypersurface moves in an orthogonal cylinder. Both Dirichlet- as well as Neumann boundary conditions have been studied; results concerning the Dirichlet problem are due to Huisken [10], Stone [15], Oliker-Ural'ceva [12], [13], and others. In the Neumann case a solution will typically converge to a hypersurface which moves by translation; here important results have been established by Altschuler, Wu [2], Huisken [10], Guan [4], and others.

Thus the evolution of compact manifolds has been studied extensively in the boundaryless case and for boundary values prescribed on orthogonal cylinders. The scope of this work is to relate these two fields by studying the boundary value problem for given Neumann data on *arbitrary smooth hypersurfaces* in $\mathbb{R}^{n+1}, n \geq 1$: Establishing a sharp local gradient bound on the hypersurfaces, we prove existence and regularity results for arbitrary smooth support and initial hypersurfaces. In a future work [17], we will give a complete classification of possible singularities in a special case.

The results presented here are part of the results of the author's Dissertation [16], the research upon which has been done during the years 1992-1994. The author is especially grateful to G. Huisken for his excellent tutorship during this time.

1.1 The problem

Throughout this work, $\boldsymbol{\Sigma}$ denotes a smooth hypersurface in $\mathbb{R}^{n+1}$, $n \geq 1$, and M^n a compact, smooth, orientable n-manifold with compact, smooth boundary ∂M^n. Furthermore, $\boldsymbol{F}_0 : M^n \longrightarrow \mathbb{R}^{n+1}$ is a smooth immersion with $M_0 := \boldsymbol{F}_0(M^n)$ and

$$
(1.1) \qquad \begin{aligned} \partial M_0 \equiv \boldsymbol{F}_0(\partial M^n) &= M_0 \cap \boldsymbol{\Sigma}, \\ \langle \boldsymbol{\nu}_0, \boldsymbol{\mu} \circ \boldsymbol{F}_0 \rangle(x) &= 0 \quad \forall x \in \partial M^n, \end{aligned}
$$

where $\boldsymbol{\nu}_0$ and $\boldsymbol{\mu}$ are unit normal fields to M_0 and $\boldsymbol{\Sigma}$ respectively. We then define

Definition 1.1. *(Mean curvature flow with Neumann boundary condition) Let* $\boldsymbol{F} : M^n \times [0, T_C) \longrightarrow \mathbb{R}^{n+1}$ *be a family of smooth immersions.* $\boldsymbol{F}$ *is said to move under the mean curvature flow with homogenious Neumann boundary condition, if*

$$
(1.2) \qquad \left\{ \begin{array}{rcll} \frac{d}{dt}\boldsymbol{F}(x,t) &=& \boldsymbol{H}(x,t) & \forall (x,t) \in M^n \times [0, T_C), \\ \boldsymbol{F}(.,0) &=& \boldsymbol{F}_0, & \\ \boldsymbol{F}(\partial M^n, t) &\subset& \boldsymbol{\Sigma} & \forall (x,t) \in \partial M^n \times [0, T_C), \\ \langle \boldsymbol{\nu}, \boldsymbol{\mu} \circ \boldsymbol{F} \rangle(x,t) &=& 0 & \forall (x,t) \in \partial M^n \times [0, T_C), \end{array} \right.
$$

where $\boldsymbol{H}$ *denotes the mean curvature vector of the immersions.*

Remark 1.2. In the sequel we assume that either $\boldsymbol{\Sigma}$ has no boundary or the hypersurfaces M_t "keep away" from $\partial\boldsymbol{\Sigma}$, i.e.

$$
\exists \varepsilon > 0 \text{ such that } \operatorname{dist}_{\Sigma}(\boldsymbol{F}(x,t), \partial\boldsymbol{\Sigma}) \geq \varepsilon \quad \forall (x,t) \in \partial M^n \times [0, T_C).
$$

This can be guaranteed à priori by barrier constructions, cf. Sect. 4. Furthermore, we can often show that the solutions of equation 1.2 are contained in a compact domain $\mathscr{K} \subset \mathbb{R}^{n+1}$. In many cases, this implies that $\mathscr{K} \cap \Sigma$ is also compact, hence the second fundamental form ${}^{\Sigma}\!A$ of $\mathscr{K} \cap \Sigma$ will be uniformly bounded. For the sake of technical simplicity we therefore require $|{}^{\Sigma}\!A|$ to be uniformly bounded globally.

Given a solution $\boldsymbol{F}$ of equation 1.2, we can locally define an exterior normal field $\boldsymbol{\mu}$ of Σ with respect to $\boldsymbol{F}$: Let $c : (-s, 0] \longrightarrow M_t$ be a smooth curve with $c(0) = \boldsymbol{F}(x_0, t) \in \partial M_t$, and choose $\boldsymbol{\mu}$ such that $\langle c'(0), \boldsymbol{\mu} \circ \boldsymbol{F}(x_0, t)\rangle \geq 0$. Thus, if Σ is the boundary of a domain $\mathscr{G} \subset \mathbb{R}^{n+1}$ and $M_t \subset \bar{\mathscr{G}}$ (i.e. the surface moves *in* the domain), $\boldsymbol{\mu}$ and the "standard" exterior unit normal field of $\mathscr{G}$ will be parallel; if $M_t \subset \mathbb{R}^{n+1} \backslash \mathscr{G}$, they will be antiparallel. In the sequel, by "exterior" we will always refer to a given solution as described above.

1.2 Main result

We prove the following general existence result:

Theorem 1.3. *Let Σ be as in Remark 1.2, and let M_0 fulfil equations 1.1. Then there exists a unique solution to equation 1.2 on a maximal time interval $[0, T_C)$. This solution is smooth for $t > 0$ and in the class $C^{2+\alpha, 1+\alpha/2}$ (with arbitrary $0 < \alpha < 1$) for $t \geq 0$. Moreover, if $T_C < \infty$, then*

$$sup\{|A|^2(x,t) : x \in M^n\} \to \infty$$

as $t \to T_C$, where $A(.,t)$ denotes the second fundamental form of the hypersurfaces M_t.

The crucial step hereby is to establish a precise local gradient estimate under the assumption of bounded curvature; to this end we generalize a method of Ecker and Huisken [3]. Furthermore, we discuss maximum principles and the "C^0-behaviour" of hypersurfaces moving under the mean curvature flow equation 1.2.

1.3 Basic notation and conventions

In the sequel, $g_{ij} = g_{ij}(x,t)$ will denote the induced metric on M^n with connection Γ_{ij}^k, $A = (h_{ij})$ the second fundamental form and $H = g^{ij} h_{ij}$ the mean curvature of M_t.

To distinguish quantities on M^n and M_t from those on Σ, we use indices ${}^{\Sigma}, {}_{,\Sigma}$: Thus, g_{ij}^{Σ} is the induced metric on Σ, ${}^{\Sigma}\!A$ the second fundamental form and so on.

We will use the Einstein convention for summation of a pair of co- and contravariant indices; here the range of greek indices is from 1 to $n+1$, whereas capital arabic indices range from 1 to $n-1$. Thus greek indices are used in $\mathbb{R}^{n+1}$, capital arabic indices on the boundaries ∂M^n and ∂M_t; the usual small arabics are reserved for calculations on M^n, M_t, Σ.

$D_\alpha = \frac{\partial}{\partial x^\alpha}$ and ${}^0\nabla_\alpha$ will denote the coordinate and covariant derivative in $\mathbb{R}^{n+1}$; $\langle .,. \rangle$ is the scalar product in $\mathbb{R}^{n+1}$, and $\|.\|$ is the norm of vectors (in $\mathbb{R}^{n+1}$ as well as on M^n). Vector fields in $\mathbb{R}^{n+1}$ and on TM^n will be denoted by $\boldsymbol{\mu}, \boldsymbol{\nu}, ...$ and $\mu, \nu, ...$, respectively.

2 Short time existence

The first step in order to obtain an existence result for equation 1.2 is to prove that there exists a unique solution at least for a short time. This can be done by transforming the given problem into an equivalent initial-boundary-value problem for a *scalar* function and using standard results of the parabolic theory. Ecker and Huisken have applied this method to the case of manifolds without boundary [3]; by using appropriate coordinates which account for the fact that the boundary can be curved, it is possible to adapt this technique to the given problem 1.2. As the calculations are tedious but in general straightforward, we restrict ourselves to presenting the major ideas and leave the details to the reader.

We begin by defining *generalized Gaussian coordinates*: Let $M_0 = \boldsymbol{F}_0(M^n)$ be a smooth immersion of M^n, and let $x = (x^1, ..., x^n)$ be coordinates on M^n. In a tubular neighbourhood $\mathscr{U} \subset \mathbb{R}^{n+1}$ of M_0 we can define a smooth vector field ξ with the following properties:

$$\boldsymbol{\xi}|_{M_0} = \boldsymbol{\nu}_0\ , \quad \boldsymbol{\xi}|_{\Sigma\cap\mathscr{U}} \in T\Sigma\ , \quad \|\boldsymbol{\xi}\| \equiv 1\ .$$

The flux lines $\boldsymbol{\Phi} = \boldsymbol{\Phi}(x, .)$ to the corresponding flow are perpendicular to M_0 and tangential to Σ; hence for any given point $\boldsymbol{p} = \boldsymbol{\Phi}(x, s) \in \mathscr{U}$ we can define $x^{n+1}(\boldsymbol{p}) :=$ length of the flux line through $\boldsymbol{p}$ between $\boldsymbol{p}$ and the intersection point $\boldsymbol{p}_0 = \boldsymbol{\Phi}(x, 0)$ on M_0, and $(x^1, ..., x^n)(\boldsymbol{p}) :=$ coordinates of $\boldsymbol{p}_0$ on M^n. These coordinates will be referred to as *generalized Gaussian coordinates* - note that with respect to these coordinates the part $\Sigma\cap\mathscr{U}$ appears as part of an "orthogonal cylinder".

Now we define a scalar initial-boundary-value problem:

$$(2.1)\qquad \begin{cases} \frac{\partial w}{\partial t}(x,t) &= (-v\tilde{H})(x,t) \quad \forall (x,t) \in M^n \times [0,\delta]\ , \\ w(.,0) &= 0\ , \\ (\mu^k D_k w) &= \mu^{n+1} \qquad\qquad \forall (x,t) \in \partial M^n \times [0,\delta]\ . \end{cases}$$

Here $v(x) := \langle \tilde{\boldsymbol{\nu}}, \boldsymbol{\xi}\rangle^{-1}(x, w(x))$, where $\tilde{\boldsymbol{\nu}}$ is the upper unit normal field to $\tilde{M}_t := \tilde{\boldsymbol{F}}(M^n, t) := \text{graph}(w(., t))$, and $\tilde{H}$ denotes the mean curvature of $\tilde{M}_t$. It can easily be seen that equation 2.1 is a *parabolic* equation with a *transversal* Neumann boundary condition. Hence, applying Hamilton's results concerning the existence of solutions to *linear* BVPs on manifolds ([5], Chapters III and IV), we obtain a $C^\infty(M^n \times (0,\delta]) \cap C^0(M^n \times [0,\delta])$-solution of the linearized problem (Note that, as $\langle \boldsymbol{\nu}_0, \boldsymbol{\mu}\circ\boldsymbol{F}_0\rangle(x) = 0 \quad \forall x \in \partial M^n$, the required compatibility condition is fulfilled). By the standard regularity theory for linear parabolic equations (e.g. [11], chapter IV) it follows that this solution actually is of the class $C^{2+\alpha,1+\alpha/2}$ for any $\alpha \in (0,1)$. Now the existence of a unique solution in $C^{2+\alpha,1+\alpha/2}(M^n \times [0,\delta])$ as usual is a consequence of the implicit function Theorem.

Next we show that equations 1.2 and 2.1 are equivalent in the following sense: To any solution M_t of 1.2 there exists a unique solution $\tilde{M}_t$ to 2.1 and vice versa such that M_t and $\tilde{M}_t$ are equal as subsets of $\mathbb{R}^{n+1}$. We follow the ideas of Ecker and Huisken and define $\varphi : M^n \times [0,\delta] \longrightarrow M^n$ by the following system of ordinary differential equations:

$$
\left\{ \begin{array}{rcl} \frac{d}{dt}\varphi(x,t) & = & -(D_x\tilde{F})^{-1} \cdot (\frac{\partial}{\partial t}\tilde{F})^T\,(\varphi(x,t),t)\,, \\ \varphi(x,0) & = & x\,. \end{array} \right. \tag{2.2}
$$

Here, $\boldsymbol{v}^T := \boldsymbol{v} - \langle \boldsymbol{v}, \tilde{\boldsymbol{\nu}} \rangle \tilde{\boldsymbol{\nu}}$ denotes the tangential component of a vector. In view of the work of Ecker and Huisken [3], the only thing which remains to check is that - at least for a short time - φ is a diffeomorphism on M^n. To this end, it suffices to prove that φ is tangential to ∂M^n at the boundary, i.e.

$$x \in \partial M^n \Rightarrow \varphi(x,t) \in \partial M^n \quad \forall t \in [0,\delta].$$

But this is a straightforward consequence of our coordinate choice: At the boundary, we have $\langle \boldsymbol{\xi}, \boldsymbol{\mu} \rangle = 0$ which implies $\langle \frac{\partial}{\partial t}\tilde{\boldsymbol{F}}, \boldsymbol{\mu} \rangle = 0$. The scalar boundary condition yields $\langle \tilde{\boldsymbol{\nu}}, \boldsymbol{\mu} \rangle = 0$; combining these two relations and using equation 2.2 we obtain $\langle (D_x\tilde{\boldsymbol{F}}) \cdot (\frac{d\varphi}{dt}), \boldsymbol{\mu} \rangle = 0$. This means that $(D_x\tilde{\boldsymbol{F}}) \cdot (\frac{d\varphi}{dt})$ is tangential to $\boldsymbol{\Sigma}$; thus $\frac{d\varphi}{dt}$ must be tangential to ∂M^n, for transversality is conserved by the isomorphism $(D_x\tilde{\boldsymbol{F}})$. Recalling the existence of a unique solution to equation 2.1 for a short time, we have thus proved:

Theorem 2.1. *(Short time existence) For any $\alpha \in (0,1)$, equation 1.2 admits a unique solution*

$$\boldsymbol{F} \in C^\infty(M^n \times (0,\delta]) \cap C^{2+\alpha,1+\alpha/2}(M^n \times [0,\delta]),$$

where $\delta > 0$ is a small constant. $\square$

3 Maximum principles

Here we provide maximum principles for scalar functions and symmetric tensors on manifolds which obey a parabolic equation with a Neumann boundary condition. The results of this section are crucial for the following and thus, despite the fact that the main ideas behind the proofs are well known, we will at least highlight the most important points.

First, let $\tilde{g}$ be a time independent metric on M^n with connection $\tilde{\nabla}$. We then consider the following linear parabolic differential operator for scalar functions:

$$L(\varphi) := \frac{\partial}{\partial t}\varphi - a^{ij}(x,t) \cdot \tilde{\nabla}_i \tilde{\nabla}_j \varphi - b^i(x,t) \cdot \tilde{\nabla}_i \varphi - c(x,t) \cdot \varphi\,,$$

with $a^{ij}, b^i, c \in L^\infty(M^n \times [0,T])$. Furthermore, let $f : M^n \times [0,T] \longrightarrow \mathbb{R}$ be a function with the following properties:

$$\begin{aligned} f(0,x,t) &\geq 0\,, \\ |f(s,x,t)-f(0,x,t)| &\leq K(x,t)\cdot|s| \quad \text{for} \quad |s|\leq\rho\,; \end{aligned}$$

here $\rho > 0$ is a (small) positive constant and $K \in L^\infty(M^n \times [0,T])$.

Finally, define a vectorfield v on $\partial M^n \times [0,T]$ with $v \in TM^n$ and $\tilde{g}(v,\mu) > 0$, where μ denotes the outward pointing unit normal field of ∂M^n in M^n (i.e. v is transversal to the boundary and outward pointing).

Theorem 3.1. *(Weak maximum principle for scalar functions) Let L, f, v be as above. Let $\varphi : M^n \times [0,T] \longrightarrow \mathbb{R}$ be a continuous function which is of the class $C^{2,1}$ in the neighbourhood of all points $(x,t) \in M^n \times [0,T]$ with $|\varphi(x,t)| < \varepsilon$ ($\varepsilon > 0$ a small constant) and fulfils the following conditions:*

$$\begin{array}{llcll} (i) & \varphi(.,0) & \geq & 0\,, & \\ (ii) & L(\varphi)(x,t) & \geq & 0 & \forall (x,t) \in M^n\times(0,T] \text{ with } |\varphi(x,t)|<\rho, \\ (iii) & \tilde{\nabla}_v\,\varphi(x,t) & = & f(\varphi,x,t) & \forall (x,t) \in \partial M^n\times(0,T] \text{ with } |\varphi(x,t)|<\rho. \end{array}$$

Then $\varphi(x,t) \geq 0 \quad \forall (x,t) \in M^n \times [0,T]$.

Proof. First, let $\hat{\varphi}$ be a continuous function which is of class $C^{2,1}$ in the neighbourhood of each point (x,t) where $-\rho/10 < \hat{\varphi}(x,t) < \rho$. It can easily be seen that the conditions

$$\begin{array}{llcll} (i) & \hat{\varphi}(.,0) & > & 0\,, & \\ (ii) & L(\hat{\varphi})(x,t) & > & 0 & \forall (x,t) \in M^n\times(0,T] \text{ with } \hat{\varphi}(x,t)=0\,, \\ (iii) & \tilde{\nabla}_v\,\hat{\varphi}(x,t) & > & 0 & \forall (x,t) \in \partial M^n\times(0,T] \text{ with } \hat{\varphi}(x,t)=0 \end{array}$$

then imply $\hat{\varphi} > 0$ on $M^n \times [0,T]$.

Now let Λ_δ be a small neighbourhood of ∂M^n in M^n and define

$$\hat{\varphi}(x,t) = \begin{cases} \varphi(x,t) + \varepsilon\, e^{\alpha t}\cdot\Psi(d(x)) & \text{on } \Lambda_\delta\,, \\ \varphi(x,t) + \varepsilon\, e^{\alpha t}\cdot 1/2 & \text{on } M^n\backslash\Lambda_\delta\,, \end{cases}$$

where

$$\Psi(s) := C\cdot(s-\delta)^4 + 1/2\,.$$

Here $\delta > 0, C > 0$ and $\alpha > 0$ are constants which will be chosen such that (for $\varepsilon > 0$ sufficiently small) $\hat{\varphi}$ fulfils the above conditions: The boundary condition (iii) can be guaranteed for sufficiently small $\delta > 0$ and sufficiently large $C > 0$ (depending on $\|K\|_\infty$ and $\Delta := \min_{\partial M^n} \tilde{g}(v,\mu) > 0$). (ii) then follows for sufficiently large α, depending on C, δ and the data of the differential operator. The calculations are straightforward and therefore left out. Hence we have $\hat{\varphi} > 0$ for all $\varepsilon > 0$, and $\varepsilon \to 0$ yields the result. □

Corollary 3.2. *(Strong maximum principle for scalar functions) Under the assumptions of Theorem 3.1 let $\varphi(.,0) \not\equiv 0$. Then*

$$\varphi(x,t) > 0 \quad \mathit{on} \quad M^n\times(0,T) \cup \overset{\circ}{M^n}\times\{T\}\,.$$

Proof. [14], chapter III. □

Next, we turn to symmetric tensors M_{ij} on $M^n \times [0,T]$; here we adapt a Theorem due to Hamilton [6] to the situation of compact manifolds with boundary. To this end we let $N_{ij} := p(M_{ij}, g_{ij})$ be a polynomial in M_{ij} which is obtained by contracting M_{ij} with the metric g_{ij}; here, g_{ij} may depend on time. N_{ij} is said to have the *zero eigenvector condition*, if $N_{ij}v^iv^j \geq 0$ for all zero eigenvectors of M_{ij}. We then define a parabolic operator

$$L(M_{ij}) := \tfrac{\partial}{\partial t}M_{ij} - \Delta M_{ij} - u^k\nabla_k M_{ij} - N_{ij} \ ,$$

where Δ is the Laplace-Beltrami operator on M^n with respect to the metric g. If $\mu = \mu(x,t)$ denotes the outward pointing unit normal field of ∂M^n in M^n, we obtain:

Theorem 3.3. *(Weak maximum principle for symmetric tensors) Let L be as above. Let M_{ij} and $N_{ij} := p(M_{ij}, g_{ij})$ be continuous symmetric tensors on $M^n \times [0,T]$, where M_{ij} is of the class $C^{2,1}$ for $t > 0$ and the zero eigenvector condition holds for N_{ij}. Furthermore, let the vector fields u^k, w^k be bounded on $M^n \times [0,T]$ and $\partial M^n \times [0,T]$ respectively with $g(u,\mu) \geq 0$, $g(w,\mu) > 0$ on the boundary. Suppose the following conditions are fulfilled ($\delta > 0$ being an arbitrary, small constant):*

(i)	$M_{ij}(.,0)$	$\geq 0,$	
(ii)	$L(M_{ij})$	≥ 0	*on* $M^n \times (0,T]$,
(iii)	$(\nabla_w M_{ij})v^iv^j(x,t)$	≥ 0	$\forall (x,t) \in \partial M^n \times (0,T]$, $\forall$ *minimal eigenvectors* v *of* M_{ij} *with eigenvalue* $\in [-\delta, 0]$.

Then $M_{ij} \geq 0$ *on* $M^n \times [0,T]$.

Proof. We let $\hat{M}_{ij} := M_{ij} + \varepsilon e^{\alpha t} g_{ij}$, where $0 < \varepsilon < \delta e^{-\alpha T}$ and $\alpha > 0$ will be chosen later, and show that $\hat{M}_{ij} > 0$ on $M^n \times [0,T]$. The Theorem then follows by $\varepsilon \to 0$.

Suppose there exists a first zero eigenvector v of $\hat{M}_{ij}$ at a point $(x_0,t_0) \in M^n \times (0,T]$. Extend v to a local vector field in a neighbourhood of (x_0,t_0) such that

$$\nabla v(x_0,t_0) = 0 \ , \quad \tfrac{\partial}{\partial t}v(x_0,t_0) = 0 \ ,$$

and define a scalar function f by

$$f(x,t) := (\hat{M}_{ij}v^iv^j)(x,t).$$

Similarly as in [6], for sufficiently large values of α, f fulfils the inequality

$$(*) \qquad \tfrac{\partial f}{\partial t}(x_0,t_0) > (\Delta f + u^k\,\nabla_k f)(x_0,t_0) \ ,$$

and f has a first zero at (x_0,t_0). Now x_0 cannot be an *interior* point, for then $(*)$ yields $\frac{\partial f}{\partial t}(x_0,t_0) > 0$ in contradiction to the minimality of t_0. Hence $x_0 \in \partial M^n$, and $f(x,t_0) > 0$ in the interior, i.e. $f(.,t_0)$ has a local minimum at x_0. Thus for all *tangential* directions $\tau \in T_{x_0}\partial M^n$ we have $0 = \nabla_\tau f(x_0, t_0)$, and by assumption (iii) and $g(w, \mu) \geq 0$ we conclude

$$(**) \qquad \nabla_\mu M_{ij}(x_0, t_0) v^i v^j \geq 0.$$

Since $g(u, \mu) \geq 0$, this implies $u^k \nabla_k M_{ij}(x_0, t_0) v^i v^j \geq 0$ and from $(*)$ we get

$$\tfrac{\partial f}{\partial t}(x_0, t_0) > \Delta f(x_0, t_0)\,.$$

Now $\frac{\partial f}{\partial t}(x_0, t_0) \leq 0$ by the minimality of t_0, and so we have $\Delta f(., t_0) < 0$ even in a small neighbourhood of x_0 on M^n. Therefore $\nabla_\mu f(x_0, t_0) < 0$ by the elliptic Hopf Lemma and in contradiction to $(**)$. □

In applications, the following Lemma will be useful:

Lemma 3.4. *Under the assumptions of Theorem 3.3 suppose that for* $(x, t) \in \partial M^n \times [0, T]$ *the tensor* M_{ij} *decomposes, i.e.*

$$M_{I\mu}(x, t) = M_{\mu I}(x, t) = 0 \quad \forall\, 1 \leq I \leq (n-1).$$

Then condition (iii) *of Theorem 3.3 can be replaced by*

$$\begin{array}{llll} (iii') & (\nabla_w M_{ij}) v^i v^j (x, t) & \geq 0 & \forall \text{ minimal eigenvectors } v \in T_x\partial M^n \\ & & & \text{of } M_{ij} \text{ with eigenvalue} \in [-\delta, 0]\,, \\ (iii)'' & (\nabla_w M_{\mu\mu})(x, t) & \geq 0 & \text{if } M_{\mu\mu} \text{ min. and } -\delta \leq M_{\mu\mu}(x, t) \leq 0\,. \end{array}$$

Proof. If at a boundary point $v := v^i \partial_i$ denotes a minimal eigenvector then, due to the block form of $M_{ij}(x, t)$, the vectors $v' := v^I \partial_I \in T_x\partial M^n$ and $v'' := v^n \mu$ are also minimal eigenvectors; at least one of these two vectors is nonzero, and so working with *this* vector instead of v and conditions $(iii)', (iii)''$ instead of (iii) in the proof of Theorem 3.3, we obtain the desired result. □

4 Barriers

In this section, we study the "C^0-behaviour" of hypersurfaces moving under mean-curvature-like flows. To this end, let $\boldsymbol{F} : M^n \times [0, T] \longrightarrow \mathbb{R}^{n+1}$ be a solution to equation 1.2; furthermore, let M_B^n be a compact, orientable manifold with boundary, and

$$\boldsymbol{F}_B : M_B^n \times [0, T] \longrightarrow \mathbb{R}^{n+1}, \quad B_t := \boldsymbol{F}_B(M_B^n, t)$$

a family of smooth imbeddings with $\partial B_t \subset \boldsymbol{\Sigma}\ \forall t$ such that B_t and $\boldsymbol{\Sigma}$ bound a domain $\mathscr{G}_t \subset \mathbb{R}^{n+1}$. Assume that $\boldsymbol{F}_B$ obeys the following initial-boundary-value problem:

$$(4.1) \qquad \left\{ \begin{array}{rcll} \langle \frac{d}{dt}\boldsymbol{F}_B, \boldsymbol{\nu}_B\rangle(x, t) & \geq & -H_B(x, t) & \forall (x, t) \in M_B^n \times (0, T]\,, \\ \boldsymbol{F}_B(., 0) & = & \boldsymbol{F}_{B,0}\,, & \\ \partial B_t & \subset & \boldsymbol{\Sigma}\,, & \\ \langle \boldsymbol{\nu}_B, \boldsymbol{\mu}\rangle & \leq & 0 \quad \text{on} & \partial M_B^n \times [0, T]\,. \end{array} \right.$$

Here ν_B denotes the exterior unit normal field to B_t with respect to the domain $\mathcal{G}_t$. Thus the equation means that the normal component of the speed which determines the motion of the B_t is not greater than it would be under the mean curvature flow. B_t is called a *barrier* for the M_t, if $M_0 \subset \bar{\mathcal{G}}_0 \backslash B_0$ implies $M_t \subset \bar{\mathcal{G}}_t \backslash B_t \quad \forall t \in [0, T)$.

The main result of this section is

Theorem 4.1. *Let F_B be a family of smooth imbeddings as above, and let $M_0 \subset \bar{\mathcal{G}}_0 \backslash B_0$. Then B_t is a barrier for M_t.*

Before proving and discussing this Theorem in detail, we have to do some preparatory work. We thereby leave the tedious, but nevertheless elementary calculations to the reader.

In the sequel, let $t_0 > 0$ be the first time at which there exists $p_0 \in M_{t_0} \cap B_{t_0}$. Then clearly

$$T_{p_0} M_{t_0} = T_{p_0} B_{t_0} ,$$

for a transverse intersection of M_{t_0} and B_{t_0} would contradict the minimality of t_0. (By the boundary conditions for M_t and B_t, this holds equally if $p_0 \in \partial M_{t_0}$.)

We introduce generalized Gaussian coordinates in a tubular neighbourhood $\mathcal{U}_\varepsilon$ of M_{t_0} as in Sect. 2. If we thereby choose ξ and ν such that $\nu(p_0) = \xi(p_0) = \nu_B(p_0)$, we have $x^{n+1}(p) < 0$ if and only if $p \in \mathcal{G}$. Now, cutting off the hypersurfaces B_t at the boundary of $\mathcal{U}_\varepsilon$ and choosing $\delta > 0$ sufficiently small, we obtain graph representations $u, w : M^n \times [t_-, t_+] \longrightarrow \mathbb{R}$ of the cutted B_t and of M_t on the time interval $[t_-, t_+] := [t_0 - \delta, t_0 + \delta] \cap [0, T]$. To be more precise, we let $u(x,t)$ be the x^{n+1}-coordinate of the intersection point of the flux line through $F(x, t_0)$ on B_t, if such an intersection point exists in $\mathcal{U}_\varepsilon$; otherwise, we set $u(x,t) := \pm\varepsilon$, choosing the sign such that u is a Lipschitz function.

Next, we define $\varphi := u - w$ and observe that $\varphi > 0$ on $[t_-, t_0)$. Moreover, choosing $\delta > 0$ such that $M_t \subset \mathcal{U}_{\varepsilon/10}$ on $[t_-, t_+]$, the above construction implies that φ is smooth in a neighbourhood of any $(x_1, t_1) \in M^n \times [t_-, t_+]$ with $|\varphi(x_1, t_1)| < \varepsilon/2$.

As in Sect. 2, the geometric boundary condition $\langle \nu_B, \mu \rangle \leq 0$ implies the scalar boundary condition $\mu^k D_k u \geq \mu^{n+1}$ (in the neighbourhood of a boundary point (x_1, t_1) with $|\varphi(x_1, t_1)| < \varepsilon/2$), where $\mu^\alpha = \mu^\alpha(x, t, u)$ are the components of μ with respect to the chosen coodinates. Analogously, $\mu^k D_k w = \mu^{n+1}$ with $\mu^\alpha = \mu^\alpha(x, t, w)$, and hence

$$\begin{aligned}
0 &\leq \mu^k(x,t,u) D_k u - \mu^k(x,t,w) D_k w - \mu^{n+1}(x,t,u) + \mu^{n+1}(x,t,w) \\
&= \int_0^1 \tfrac{d}{ds} \left(\mu^k(x,t,su + (1-s)w) \cdot (sD_k u + (1-s) D_k w) \right) ds \\
&- \int_0^1 \tfrac{d}{ds} \left(\mu^{n+1}(x,t,su + (1-s)w) \right) ds \\
&= \left[\int_0^1 \mu^k(x,t,.)\, ds \right] \cdot D_k \varphi
\end{aligned}$$

$$+ \quad \left[\int_0^1 \left(D_z \mu^k(x,t,.) \cdot (sD_k u + (1-s)D_k w) - D_z \mu^{n+1}(x,t,.) \right) ds \right] \cdot \varphi$$
$$=: \quad v^k(x,t) \cdot D_k \varphi + \Psi(x,t) \cdot \varphi \,.$$

Hereby the functions v^k and $f(\varphi, x, t) := -\Psi(x,t) \cdot \varphi$ are smooth, and, by the mean value Theorem,

$$v^k(x,t) \;=\; \int_0^1 \mu^k(x,t,su + (1-s)w)\, ds \;=\; \mu^k(x,t,\sigma)\,, \quad |\sigma| < \varepsilon$$

near (x_1, t_1). By the compactness of ∂M^n, v^k will therefore be a *tranversal* vector field for sufficiently small $\varepsilon > 0$. Thus we have proved:

Lemma 4.2. *Let* $(x_1, t_1) \in \partial M^n \times [t_-, t_+]$ *with* $|\varphi(x_1, t_1)| < \varepsilon/4$. *Then in a neighbourhood of* (x_1, t_1), *the boundary condition*

$$D_v \varphi(x,t) = f(\varphi, x, t)$$

is fulfilled for φ. *Hereby,* f *is a smooth function with*

$$|f(s,x,t) - f(0,x,t| \le const \cdot |s|\,, \quad f(0,x,t) \ge 0\,,$$

and $v \in TM^n$ *is a smooth, outward pointing vector field, i.e.* $g(v, \mu) > 0$. □

4.1 Proof of the theorem and further results

Proof of Theorem 4.1. Let $t_0 \in (0, T]$ be as above; we have to show that $t_0 = T$. Suppose $t_0 < T$, and choose $\delta > 0$, $\varepsilon > 0$ such that u, w are graph representations of B_t, M_t as before. Then, if $\varphi = u - w$, the choice of t_0 implies $\varphi(.,t) > 0$ on $[t_-, t_0)$, and $\varphi(x_0, t_0) = 0$ for some $x_0 \in M^n$. As in Sect. 2, we have parabolic differential equations for u and w:

$$\begin{aligned} \tfrac{\partial u}{\partial t} \;&\ge\; -v_B \cdot H_B \;=\; \alpha^{ij}(x,t,u,Du) D_i D_j u + \gamma(x,t,u,Du)\,, \\ \tfrac{\partial w}{\partial t} \;&=\; -v \cdot H \;=\; \alpha^{ij}(x,t,w,Dw) D_i D_j w + \gamma(x,t,w,Dw)\,. \end{aligned}$$

By standard methods, this implies a *linear* parabolic differential equation for $\varphi = u - w$:

$$\tfrac{\partial \varphi}{\partial t} \ge a^{ij} D_i D_j \varphi + b^i D_i \varphi + c\varphi\,,$$

with $a^{ij} = a^{ij}(x,t)$, $b^i = b^i(x,t)$, $c = c(x,t)$. Using Lemma 4.2, we see that φ fulfils the conditions of Theorem 3.1 (*weak* maximum principle); hence $\varphi(.,t) \ge 0$ on $[t_-, t_+]$. By the *strong* maximum principle (Theorem 3.2), $\varphi(.,t) > 0$ on $[t_-, t_+)$ in contradiction to $\varphi(x_0, t_0) = 0$. □

Remark 4.3. If $M_0 \cap B_0 \neq \emptyset$ and $M_0 \not\equiv B_0$, the strong maximum principle yields that M_t and B_t separate immediately, i.e. $M_t \cap B_t = \emptyset\ \forall t > 0$.

Theorem 4.1 implies the case of "static" barriers: Let $\frac{d}{dt}F_B \equiv 0$ and $B := F_{B,0}(M_B^n)$. Furthermore, let $H_B \geq 0$. Then F_B fulfils equation 4.1, and Theorem 4.1 yields

Corollary 4.4. *("Static" Barriers) If $H_B \geq 0$ and $\langle \nu_B, \mu \rangle \leq 0$ on the boundary, B is a barrier for M_t.* □

Here we have not made any assumptions on M_0; instead, we have required $H_B \geq 0$ for the mean curvature of the barrier. This condition can be weakened if we impose a condition on the curvatures of M_0 and Σ: If $H(.,0) \geq C > 0$ and Σ is assumed to be convex with respect to the outward unit normal field, it is sufficient to require $H_B \geq -C$ for the mean curvature of B. The proof is very similar to the above techniques; in particular, it can be shown [17] that due to the convexity of Σ the relation $H \geq C$ is preserved for $t > 0$.

We remark that the above results can be generalized in a straightforward manner to domains which are bounded by Σ and *several* barriers.

Corollary 4.5. *(Imbeddings) Let ${}^{\Sigma}H \geq 0$ with respect to the outward unit normal field μ. Then M_0 being an imbedding implies that all M_t are imbeddings as well.*

Proof. We must show that M_t cannot selfintersect. Suppose t_0 is the first time where a selfintersection occurs at points $p = F(x_0, t_0)$, $q = F(y_0, t_0)$. By Theorem 4.1 we can exclude the cases $x_0, y_0 \in M^n \backslash \partial M^n$ and $x_0, y_0 \in \partial M^n$; thus it remains to rule out the case $x_0 \in M^n \backslash \partial M^n$, $y_0 \in \partial M^n$. But as ${}^{\Sigma}H \geq 0$, the hypersurface Σ is a barrier to M_t (according to Corollary 4.4), and hence

$$x \notin \partial M^n \quad \Leftrightarrow \quad F(x,0) \notin \Sigma \quad \Leftrightarrow \quad F(x,t) \notin \Sigma \;\; \forall t < T\,.$$

Thus $F(x_0, t_0) \neq F(y_0, t_0)$ by the choice of x_0 and y_0. □

We remark that the condition ${}^{\Sigma}H \geq 0$ is required only to prevent M_t from intersecting Σ.

5 Distance of hypersurfaces

In Sect. 4 we considered families B_t of hypersurfaces such that a solution of equation 1.2 could never intersect the B_t. Now we are interested in discussing the behaviour of the *distance* between *two* solutions. To this end, let M_1^n, M_2^n be compact, smooth, orientable manifolds. The main Theorem of this section is

Theorem 5.1. *Let $\mathcal{G}$ be a convex domain in $\mathbb{R}^{n+1}$ with smooth boundary $\Sigma = \partial\mathcal{G}$. Let $F_i : M_i^n \times [0,T] \longrightarrow \mathbb{R}^{n+1}$ be two families of smooth immersions moving in $\bar{\mathcal{G}}$ under the mean curvature flow equation 1.2; defining $M_i(t) := F_i(M_i^n, t)$, let $M_1(0) \cap M_2(0) = \emptyset$. Then the function $d(t) := dist_{n+1}(M_1(t), M_2(t))$ is monotonically increasing.*

As in the last section, the main idea of the proof is to consider graph representations of $M_i(t)$ in the neighbourhood of "critical points", i.e. points $\boldsymbol{p}_i \in M_i(t)$ with $d(t) = \|\boldsymbol{p}_1 - \boldsymbol{p}_2\|$ and applying the maximum principles of Sect. 3. Again, we first have to do some preparatory work: Throughout this section, let $\mathcal{G} \subset \mathbb{R}^{n+1}$ be a convex domain with smooth boundary $\boldsymbol{\Sigma}$ and exterior unit normal field $\boldsymbol{\mu}$. Given $\boldsymbol{p},\boldsymbol{q} \in \bar{\mathcal{G}}$, $\boldsymbol{p} \neq \boldsymbol{q}$, define

$$\omega_{pq} := \frac{\boldsymbol{p}-\boldsymbol{q}}{\|\boldsymbol{p}-\boldsymbol{q}\|} .$$

The following Lemma is a simple consequence of the convexity of $\mathcal{G}$:

Lemma 5.2. *(Convexity of $\mathcal{G}$)*
(i) *Let $\boldsymbol{p} \in \boldsymbol{\Sigma}$, $\boldsymbol{q} \in \bar{\mathcal{G}}$, $\boldsymbol{p} \neq \boldsymbol{q}$. Then* $\langle \boldsymbol{\mu}(\boldsymbol{p}), \omega_{pq} \rangle \geq 0$.
(ii) *Let $\boldsymbol{p},\boldsymbol{q} \in \boldsymbol{\Sigma}$, $\boldsymbol{p} \neq \boldsymbol{q}$,* $\langle \boldsymbol{\mu}(\boldsymbol{p}), \omega_{pq} \rangle = \langle \boldsymbol{\mu}(\boldsymbol{q}), \omega_{pq} \rangle = 0$.
Then $\langle \boldsymbol{\mu}(\boldsymbol{p}), \boldsymbol{\mu}(\boldsymbol{q}) \rangle = 1$, and $\boldsymbol{p},\boldsymbol{q}$ belong to the same hyperplane. □

We now consider smooth immersions $\boldsymbol{F}_i$ of M_i^n $(i = 1,2)$ in $\mathbb{R}^{n+1}$ with unit normal fields $\boldsymbol{\nu}_i$. Let $M_i := \boldsymbol{F}_i(M_i^n) \subset \bar{\mathcal{G}}$, $M_1 \cap M_2 = \emptyset$, $\partial M_i \subset \boldsymbol{\Sigma}$ and $\langle \boldsymbol{\nu}_i, \boldsymbol{\mu} \rangle = 0$ on ∂M_i. Moreover, let $\boldsymbol{p} := \boldsymbol{F}_1(x_1)$ and $\boldsymbol{q} := \boldsymbol{F}_2(\tilde{x}_2)$ such that $d := \|\boldsymbol{p} - \boldsymbol{q}\| = \mathrm{dist}_{n+1}(M_1, M_2)$. We then introduce Cartesian coordinates $\{y^1, ..., y^{n+1}\}$ in $\mathbb{R}^{n+1}$ such that $y(\boldsymbol{p}) = (0, ..., 0, d/2)$, $y(\boldsymbol{q}) = (0, ..., 0, -d/2)$. In a neighbourhood of $\boldsymbol{p},\boldsymbol{q}$, the hypersurfaces M_i can be written as graphs of smooth functions u_i with $u_1(0) = d/2$, $u_2(0) = -d/2$ and $Du_1(0) = Du_2(0) = 0$. This can be seen as follows: By the choice of $\boldsymbol{p},\boldsymbol{q}$, the function

$$f : M_1^n \longrightarrow \mathbb{R},\ f(x) := \|\boldsymbol{F}_1(x) - \boldsymbol{q}\|^2$$

has a global minimum in x_1. Now, if $\boldsymbol{p} \notin \boldsymbol{\Sigma}$, we have $\nabla f(x_1) = 0$ and thereby $Du_1(0) = 0$. If $\boldsymbol{p} \in \boldsymbol{\Sigma}$, we have $\langle \nabla f(x_1), \boldsymbol{w} \rangle = 0$ for all *tangential* directions $\boldsymbol{w} \in T\partial M_1(\boldsymbol{p})$, and

$$0 \geq \langle \nabla f(x_1), \boldsymbol{\mu} \rangle = 2 \langle \boldsymbol{F}_1(x_1) - \boldsymbol{q}, \boldsymbol{\mu} \rangle = 2 \langle \boldsymbol{p} - \boldsymbol{q}, \boldsymbol{\mu} \rangle .$$

Using Lemma 5.2(i), this implies $\langle \nabla f(x_1), \boldsymbol{\mu} \rangle = 0$ and thus $Du_1(0) = 0$ as before.

Lemma 5.3. *The matrix $(D_iD_j(u_1 - u_2))(0)$ is positive semidefinite.*

Proof. Let $u_i : \Omega_i \longrightarrow \mathbb{R}$ with $0 \in \overline{\Omega_i}$, and let $\varphi := u_1 - u_2$ on $\Omega_1 \cap \Omega_2$. If $\boldsymbol{p},\boldsymbol{q} \notin \boldsymbol{\Sigma}$, 0 is in the *interior* of Ω_i, and the result follows by Taylor expansion of φ. The same argument applies if $\boldsymbol{p} \notin \boldsymbol{\Sigma}$ and $\boldsymbol{q} \in \boldsymbol{\Sigma}$ (or vice versa), or if $\boldsymbol{p},\boldsymbol{q} \in \boldsymbol{\Sigma}$ and $T_0\partial\Omega_1 = T_0\partial\Omega_2$. Finally the case $\boldsymbol{p},\boldsymbol{q} \in \boldsymbol{\Sigma}$ and $T_0\partial\Omega_1 \neq T_0\partial\Omega_2$ can be ruled out as, according to Lemma 5.2(ii), we have $\langle \boldsymbol{\mu}(\boldsymbol{p}), \boldsymbol{\mu}(\boldsymbol{q}) \rangle = 1$ if $\boldsymbol{p},\boldsymbol{q} \in \boldsymbol{\Sigma}$. □

Finally, we state a Lemma due to Hamilton [7]:

Lemma 5.4. *Let $g : M^n \times [0,T] \longrightarrow \mathbb{R}$ be a smooth function, and define $f(t) := \min_{M^n} g(x,t)$. Then f is a Lipschitz function, and*

$$\frac{df}{dt}(t) \geq \min\{\frac{\partial g}{\partial t}(\hat{x},t)\ :\ \hat{x} \in M^n \textit{ such that } g(\hat{x},t) = \min_{x \in M^n} g(x,t)\} .$$

5.1 Proof of the theorem and further results

Proof of Theorem 5.1. Let

$$g : M_1^n \times M_2^n \times [0,T] \longrightarrow \mathbb{R}\ ,\ g(x,\tilde{x},t) := \|\boldsymbol{F}_1(x,t) - \boldsymbol{F}_2(\tilde{x},t)\|^2\ ,$$

then $d(t) = \min_{M_1^n \times M_2^n} g(x,\tilde{x},t)$, and, using the evolution equation 1.2, we have

$$\tfrac{d}{dt} g(x,\tilde{x},t) = 2\,\langle \boldsymbol{F}_1(x,t) - \boldsymbol{F}_2(\tilde{x},t), \boldsymbol{H}_1(x,t) - \boldsymbol{H}_2(\tilde{x},t)\rangle\ .$$

Now let $t_0 \in [0,T)$, $\boldsymbol{p} = \boldsymbol{F}_1(x_1,t_0)$, $\boldsymbol{q} = \boldsymbol{F}_2(\tilde{x}_2,t_0)$ be such that $d(t_0) = \|\boldsymbol{p} - \boldsymbol{q}\| > 0$ and use the Cartesian coordinates defined above to conclude

$$\begin{aligned} \boldsymbol{H}_1(x_1,t_0) - \boldsymbol{H}_2(\tilde{x}_2,t_0) &= \Delta(u_1 - u_2)(0,t_0) \cdot \omega_{pq}\ , \\ \boldsymbol{F}_1(x_1,t_0) - \boldsymbol{F}_2(\tilde{x}_2,t_0) &= d(t_0) \cdot \omega_{pq}\ . \end{aligned}$$

Lemma 5.3 yields $\Delta(u_1 - u_2)(0,t_0) \geq 0$, and thus $\frac{d}{dt} g(x_1,\tilde{x}_2,t_0) \geq 0$. The result follows from Lemma 5.4. □

Let us, for a moment, return to the study of barriers: $\mathscr{G}$ and $\boldsymbol{\Sigma}$ being defined as above, we let $B := \boldsymbol{F}_B(M_B^n) \subset \bar{\mathscr{G}}$ be a smooth imbedding with unit normal field $\boldsymbol{\nu}_B$, $\partial B \subset \boldsymbol{\Sigma}$ and $\langle \boldsymbol{\nu}_B, \boldsymbol{\mu}\rangle = 0$ on ∂B. We assume that $\mathscr{G}$ is divided by B into two disjoint subdomains $\mathscr{G}_1, \mathscr{G}_2$, and that $\boldsymbol{\nu}_B$ is the exterior normal field to $\mathscr{G}_1$. In this situation, we have

Theorem 5.5. *Let $\boldsymbol{F}$ be a family of smooth immersions evolving under the mean curvature flow equation 1.2, and $M_0 \subset \bar{\mathscr{G}}_1$. Let $H_B \geq 0$ with respect to $\boldsymbol{\nu}_B$. Then the function $d(t) := dist_{n+1}(M_t, B)$ is monotonically increasing.*

Proof. Defining $g(x,\tilde{x},t) := \|\boldsymbol{F}_B(x) - \boldsymbol{F}(\tilde{x},t)\|^2$ and $\boldsymbol{p}, \boldsymbol{q}, x_1, \tilde{x}_2$ etc. as before, we have

$$\tfrac{d}{dt} g(x_1,\tilde{x}_2,t_0) = 2\,d(t_0)\,\langle \omega_{pq}, 0 - \boldsymbol{H}(\tilde{x}_2,t_0)\rangle\ .$$

Using $H_B \geq 0$ and $\omega_{pq} = \boldsymbol{\nu}_B(x_1)$,

$$\tfrac{d}{dt} g(x_1,\tilde{x}_2,t_0) \geq 2\,d(t_0)\,\langle \omega_{pq}, \boldsymbol{H}_B(x_1,t_0) - \boldsymbol{H}(\tilde{x}_2,t_0)\rangle,$$

and, proceeding as in the proof of Theorem 5.1, we obtain the result. □

There is a variety of configurations which can be studied using essentially the same techniques as presented here; for example, it is possible to consider imbedded hypersurfaces moving under

$$\langle \tfrac{d}{dt}\boldsymbol{F}, \boldsymbol{\nu}\rangle(x,t) \geq -H(x,t), \quad \langle \boldsymbol{\nu}, \boldsymbol{\mu}\rangle = 0 \ \text{ on } \partial M^n\ .$$

Here we must be careful with defining *exterior* normal fields, for the evolution inequality is not invariant under the transformation $\boldsymbol{\nu} \longrightarrow -\boldsymbol{\nu}$. However, if we assume $\mathscr{G}$ to be a domain bounded by $\boldsymbol{\Sigma}$ and $M_1(0), M_2(0)$ and define $\boldsymbol{\nu}_i$ as exterior normal fields to $\mathscr{G}$, we can conclude that the distance between $M_1(t)$ and $M_2(t)$ once more monotonically increases. A proof of this fact is given in [16].

6 A local gradient bound

In Sect. 2, we saw that locally equation 1.2 is equivalent to the scalar problem 2.1. Now, by a precise control of the gradient Dw, the latter equation will be (locally) *uniformly* parabolic; from there, by standard results of the linear theory the existence Theorem 1.3 follows. In order to establish the local gradient bound, we assume that the curvature of the M_t already is under control on an arbitrary closed time interval $[0, T] \subset [0, T_C)$. We caution the reader to remark that the crucial point is to obtain a local gradient estimate which is *uniform on some sufficiently large time interval*; this is certainly *not* a straightforward consequence of the underlying curvature bound.

Assumption 6.1 *Throughout this section, let $\boldsymbol{F} : M^n \times [0, T] \longrightarrow \mathbb{R}^{n+1}$ be a solution to equation 1.2, and let the curvature of M_t be uniformly bounded on $[0, T]$ as follows:*

$$|g^{ij}\, h_{jk}|(x, t) \leq (2n^2 \rho_0)^{-1} \quad \forall 1 \leq i, k \leq n\,, \quad \forall (x, t) \in M^n \times [0, T]\,,$$

where ρ_0 is a positive constant. Furthermore, let $\boldsymbol{\Sigma}$ be smooth with a global curvature bound, i.e. there exists $\rho_\Sigma > 0$ such that

$$|{}^{\Sigma}g^{ij}\, h^{\Sigma}_{jk}|(\boldsymbol{p}) \leq (2n^2 \rho_\Sigma)^{-1} \quad \forall 1 \leq i, k \leq n\,, \quad \forall \boldsymbol{p} \in \boldsymbol{\Sigma}\,.$$

Finally, let $\boldsymbol{\Sigma} \cap B^{n+1}_{\rho_\Sigma}(\boldsymbol{p})$ be connected for all $\boldsymbol{p} \in \boldsymbol{\Sigma}$.

We emphasize that in order to obtain the local gradient bound on a small region in spacetime, it is sufficient to require a *local* curvature bound to the M_t on a (slightly larger) region; however, the global curvature bound somewhat simplifies the notation. The global curvature bound on $\boldsymbol{\Sigma}$ is a very slight restriction, as was discussed in Remark 1.2. Finally, the connectedness of $\boldsymbol{\Sigma} \cap B^{n+1}_{\rho_\Sigma}(\boldsymbol{p})$ is included for technical reasons; in general, this requirement can be fulfilled by choosing ρ_Σ sufficiently small, i.e. working in sufficiently small domains.

Now let us fix some notation: By the *standard graph representation (SGR) w of M_t* near a given point $\boldsymbol{p}_0 := \boldsymbol{F}(x_0, t_0)$ we mean the local graph representation around $\boldsymbol{p}_0$ with respect to a Cartesian coordinate system in $\mathbb{R}^{n+1}$ such that $\boldsymbol{p}_0 = (0, w(0, t_0)) = (0, 0)$ and $\boldsymbol{\nu}(x_0, t_0) =: \boldsymbol{\omega} = \boldsymbol{e}_{n+1}$ (i.e. $Dw(0, t_0) = 0$). Furthermore, given $\boldsymbol{\omega} \in \mathbb{R}^{n+1}$, we define

$$u : M^n \times [0, T] \longrightarrow \mathbb{R}\,, \; u(x, t) := \langle \boldsymbol{F}(x, t), \boldsymbol{\omega} \rangle\,;$$

thus $u(x, t)$ describes the height of a point $\boldsymbol{F}(x, .)$ over the hyperplane $\mathscr{H}$ through $\boldsymbol{F}(x_0, t_0)$, whereas $w(y, .)$ is the height of a point with fixed coordinates on $\mathscr{H}$.

$\mathscr{Z}_r(\boldsymbol{p}) := B^n_r(\boldsymbol{p}) \times (-\infty, +\infty) \subset \mathbb{R}^{n+1}$ denotes the space cylinder around $\boldsymbol{p}$ with axis $\boldsymbol{\omega}$; analogously $\mathscr{Z}_{r,C}(\boldsymbol{p}) := B^n_r(\boldsymbol{p}) \times (-C\, r, +C\, r)$.

$\boldsymbol{\Sigma}^{\Delta}_{t_0} := \{\boldsymbol{F}(x, t) : x \in \partial M^n,\; |t - t_0| \leq \Delta\}$ is the subset of $\boldsymbol{\Sigma}$ which coincides with the boundaries ∂M_t in the time interval $[t_0 - \Delta, t_0 + \Delta]$.

6.1 Hypersurfaces with bounded curvature

In order to get the right scaling behaviour of the desired estimates, we first have to state some facts concerning hypersurfaces $M := \boldsymbol{F}(M^n) \subset \mathbb{R}^{n+1}$ with a curvature bound

$$|g^{ij}\, h_{jk}|(x) \leq (2\,n^2\rho)^{-1} \quad \forall 1 \leq i,k \leq n\,, \quad \forall x \in M^n$$

as in Assumption 6.1.

Lemma 6.2. *Let $\boldsymbol{p}_0 = \boldsymbol{F}(x_0, t_0)$ and $w : \mathbb{R}^n \ni \mathscr{U} \longrightarrow \mathbb{R}$ be the SGR of $\boldsymbol{F}(M^n)$ around $\boldsymbol{p}_0$. Let $\boldsymbol{p} = \boldsymbol{F}(x) = (y, w(y))$ with $\|y\| =: r < \rho$ and $\{s\, y : 0 \leq s \leq 1\} \subset \mathscr{U}$. Finally, define $v(x) = \langle \boldsymbol{\nu}(x), \boldsymbol{\omega}\rangle^{-1} = \sqrt{1 + \|Dw\|^2}(y)$. Then*

$$v(x) < 1 + (\tfrac{r}{2\rho})^2\,, \quad \langle \boldsymbol{\nu}(x), \boldsymbol{\omega}\rangle > 1 - (\tfrac{r}{2\rho})^2\,, \quad |\langle \boldsymbol{F}(x) - \boldsymbol{F}(x_0), \boldsymbol{\omega}\rangle| < \tfrac{r^2}{\rho}\,.$$

Proof (sketch). From the curvature bound we get $|h_{ik}| \leq \frac{1}{2n\rho}(1 + \|Dw\|^2)$ and thereby

$$|D_i D_k w| \leq (2\,n\rho)^{-1} \sqrt{1 + \|Dw\|^2}^{\,3}\,.$$

Now let $e := y/\|y\|$; then, by assumtion, the function $\sigma \longmapsto D_i w(\sigma\, e)$ is defined on $[0, 1]$. From Hölder's inequality and the above equation, we obtain

$$|D_i w(s\, e)| \leq \sqrt{\tfrac{r}{n}}\, \tfrac{1}{2\rho} \sqrt{\int_0^s (1 + \|Dw\|^2)^3 d\sigma}$$

and by summation

$$(1 + \|Dw\|^2)(s\, e) \leq 1 + \tfrac{r}{4\rho^2} \int_0^s (1 + \|Dw\|^2)^3(\sigma\, e)\, d\sigma\,.$$

Solving this ordinary integral equation for $1 + \|Dw\|^2$ and using Taylor expansions, the result follows. □

Note that the Lemma implies vice versa, that the *SGR* of M will be defined *on the entire ball* $B^n_r(0)$, if the $\boldsymbol{p}_0$-component of $\mathscr{Z}_r(\boldsymbol{p}_0) \cap M$ does not have any boundary points in the interior of $\mathscr{Z}_r(\boldsymbol{p}_0)$. Thus, applying the Lemma to $\boldsymbol{\Sigma}$ and recalling Remark 1.2, we arrive at the following

Lemma 6.3. *Let $r \leq \rho_\Sigma$, $\boldsymbol{p}_0 = \boldsymbol{F}(x_0, t_0) \in \boldsymbol{\Sigma}$, $\boldsymbol{\gamma} := \boldsymbol{\mu}(\boldsymbol{p}_0)$ and $v_\Sigma := \langle \boldsymbol{\mu}, \boldsymbol{\gamma}\rangle^{-1}$. Then for any $\boldsymbol{p} \in \boldsymbol{\Sigma} \cap \mathscr{Z}_r(\boldsymbol{p}_0)$, we have*

$$v_\Sigma(\boldsymbol{p}) < 1 + (\tfrac{r}{2\rho_\Sigma})^2\,, \quad \langle \boldsymbol{\mu}(\boldsymbol{p}), \boldsymbol{\gamma}\rangle > 1 - (\tfrac{r}{2\rho_\Sigma})^2\,, \quad |\langle \boldsymbol{p} - \boldsymbol{p}_0, \boldsymbol{\gamma}\rangle| < \tfrac{r^2}{\rho_\Sigma}\,. \quad \square$$

In order to obtain an analogous estimate for the hypersurfaces M_t, we have to be more careful: If $\boldsymbol{p}_0 \in M_{t_0} \backslash \partial M_{t_0}$ we can proceed exactly as before, obtaining a graph representation on an entire ball of radius $\rho_1 := \min\{\rho/4, d/2\}$, where $d := \mathrm{dist}_{n+1}(\boldsymbol{p}_0, \boldsymbol{\Sigma})$. If $\boldsymbol{p}_0 \in \partial M_{t_0}$, due to the curvature of $\boldsymbol{\Sigma}$, the *SGR* of M_{t_0} will in general *not* be defined along straight lines $\sigma \mapsto \sigma\, e$ as in the proof of Lemma 6.2. However, the curvature of $\boldsymbol{\Sigma}$ being bounded, we have a graph representation for $\boldsymbol{\Sigma}$ if $r \leq \rho_\Sigma$, and it turns out that we can work with curves of length $\leq 3\,r$ to connect various points in the domain of the *SGR* of M_{t_0}. Thus we obtain

Lemma 6.4. *Let* $\boldsymbol{p}_0 = \boldsymbol{F}(x_0, t_0)$, $d := dist_{n+1}(\boldsymbol{p}_0, \partial M_{t_0})$,

$$\rho_1 := \begin{cases} \min\{\rho_0/3\,,\, \rho_\Sigma\} & \text{if } \boldsymbol{p}_0 \in \boldsymbol{\Sigma}\,, \\ \min\{\rho_0/4\,,\, d/2\} & \text{if } \boldsymbol{p}_0 \notin \boldsymbol{\Sigma}\,, \end{cases}$$

and $r \leq \rho_1$. *Then for any* $\boldsymbol{F}(x, t_0) \in \mathscr{Z}_r(\boldsymbol{p}_0)$ *we have:*

$$v(x, t_0) < 1 + (\tfrac{r}{\rho_0})^2\,, \quad \langle \boldsymbol{\nu}(x, t_0), \boldsymbol{\omega}\rangle > 1 - (\tfrac{r}{\rho_0})^2\,, \quad |u(x, t_0) - u(x_0, t_0)| < \tfrac{2\,r^2}{\rho_0}\,.$$

□

Certainly we could have chosen $\rho_1 := \delta \cdot \min\{\rho_0, \rho_\Sigma\}$ ($\delta > 0$ being sufficiently small) for both interior points and points near the boundary, but as we want to derive the inner and boundary gradient estimates independently, the above definition is somewhat simpler from the technical point of view.

As a further consequence of the evolution equation 1.2 and the curvature bound, we can control the behaviour of points $\boldsymbol{F}(x_0, t)$ for fixed $x_0 \in M^n$: A short calculation yields for any $t_1, t_2 \in [0, T]$:

$$(6.1) \qquad (\|\boldsymbol{F}\|^2 - u^2)(x, t_2) \leq \left(\tfrac{1}{2\,\rho_0}|t_2 - t_1| + \sqrt{\|\boldsymbol{F}\|^2 - u^2}\,(x, t_1)\right)^2\,.$$

Combining this equation with the above Lemmata, we can estimate the distance of points $\boldsymbol{F}(x, t)$, $\boldsymbol{F}(x_0, t_0)$, if $\boldsymbol{F}(x, t_0) \subset \mathscr{Z}_r(\boldsymbol{p}_0)$. These points will leave the cylinder $\mathscr{Z}_r(\boldsymbol{p}_0)$ after some time; on the other hand, $\mathscr{Z}_r(\boldsymbol{p}_0)$ contains points $\boldsymbol{F}(x, t)$ which didn't start in the cylinder for $t = t_0$. Now we are interested in controlling the behaviour of $M_t \cap \mathscr{Z}_r(\boldsymbol{p}_0)$ for a short time $|t - t_0| \leq \delta$. Combining the above results, we easily see that this can be done on a *smaller* cylinder $\mathscr{Z}_{r_2}(\boldsymbol{p}_0) \subset \mathscr{Z}_r(\boldsymbol{p}_0)$ as follows:

Lemma 6.5. *Let* $r_2 \leq \frac{\rho_1}{2}$ *and* $\delta \leq 2\,\rho_0\, r_2$. *Then for any* $(x, t) \in M^n \times [0, T]$ *with* $\boldsymbol{F}(x, t) \in \mathscr{Z}_{r_2}(\boldsymbol{p}_0)$ *and* $|t - t_0| \leq \delta$ *the following estimate holds:*

$$|u(x, t) - u(x_0, t_0)| < \tfrac{1}{2\,\rho_0}\,(|t - t_0| + 16\, r_2^2)\,. \quad \square$$

As a last consequence of the curvature bound on the M_t we give an estimate for the distance of a point p from $\Sigma_{t_0}^{\Delta}$. The proof follows directly from equation 1.2.

$$(6.2) \qquad \mathrm{dist}_{n+1}(\boldsymbol{p}, \Sigma_{t_0}^{\Delta}) \;\geq\; \mathrm{dist}_{n+1}(\boldsymbol{p}, \partial M_{t_0}) - \tfrac{1}{2\,\rho_0}\,\Delta\,.$$

6.2 The interior estimate

We first consider $\boldsymbol{p}_0 = \boldsymbol{F}(x_0, t_0)$ with $x_0 \in M^n \backslash \partial M^n$. As before, we thereby choose Cartesian coordinates in $\mathbb{R}^{n+1}$ with origin at $\boldsymbol{p}_0$ and $\boldsymbol{\omega} := \boldsymbol{e}_{n+1} := \boldsymbol{\nu}(x_0, t_0)$. Using ideas of Ecker and Huisken [3], we define

$$M_{r,\delta} := \sup\{|u(x,t)| : \boldsymbol{F}(x,t) \in \mathscr{Z}_r(\boldsymbol{p}_0)\,,\, 0 \le t - t_0 \le \delta\}\,,$$

$$\varphi(x,t) := \begin{cases} \left(\frac{r}{2\beta}(u - M_{r,\delta}) + r^2 - (\|\boldsymbol{F}\|^2 - u^2)\right)_+ (x,t)\,, & \boldsymbol{F}(x,t) \in \mathscr{Z}_r(\boldsymbol{p}_0)\,, \\ 0\,, & \boldsymbol{F}(x,t) \notin \mathscr{Z}_r(\boldsymbol{p}_0)\,, \end{cases}$$

$$\eta(x,t) := \mathrm{e}^{\lambda\,\varphi(x,t)} - 1\,,$$

where $r > 0\,, \delta > 0\,, \beta > 0$ and $\lambda > 0$ will be chosen later. φ and η are Lipschitz functions for $0 \le t - t_0 \le \delta$, and $\operatorname{supp}(\varphi) = \operatorname{supp}(\eta) \subset \mathscr{Z}_r(\boldsymbol{p}_0)$. For the sequel, we fix the following notation:

$$\begin{aligned} \sup_{\mathscr{Z}_r} v(.,t_0) &:= \sup\{v(\xi,t_0) : \boldsymbol{F}(\xi,t_0) \in \mathscr{Z}_r(\boldsymbol{p}_0)\}\,, \\ \sup_{\mathscr{Z}_r,\,\delta} \eta(.,.) &:= \sup\{\eta(\xi,\tau) : \boldsymbol{F}(\xi,\tau) \in \mathscr{Z}_r(\boldsymbol{p}_0), 0 \le t - t_0 \le \delta\}\,. \end{aligned}$$

Now let $(x_1, t_1)\,, x_1 \in M^n \backslash \partial M^n\,,\ 0 \le t_1 - t_0 \le \delta$ be a point where $\eta\, v$ attains a new maximum. Proceeding as in [3], we obtain

$$2n \ge \lambda\, r^2 \left(\tfrac{1}{4\beta^2}(1 - v^{-2}) - \tfrac{2}{\beta} v^{-1}\right)$$

at this point. Hence, defining $\lambda := \frac{8\,n\,\alpha\,\beta^2}{r^2}$ ($\alpha > 1$ to be chosen), we conclude

$$v(x_1, t_1) < \tfrac{\alpha}{\alpha - 1}\,(8\,\beta + 1)\,.$$

This in turn yields

$$(\eta\, v)(x_1, t_1) < \sup_{\mathscr{Z}_r,\,\delta} \eta(.,.) \cdot \max\left\{ \tfrac{\alpha}{\alpha-1}\,(8\,\beta + 1)\,,\ (1 + \hat{\varepsilon}) \sup_{\mathscr{Z}_r} v(.,t_0) \right\}\,,$$

where $\hat{\varepsilon} > 0$ is an arbitrary constant. Now this inequality can only be broken at a boundary point $x_2 \in \partial M^n$; hence an interior bound for $\eta\, v$ can be obtained by choosing $r > 0$ and $\delta > 0$ such that the $\boldsymbol{p}_0$-components of $M_t \cap \mathscr{Z}_r(\boldsymbol{p}_0)$ do not have any boundary points for $0 \le t - t_0 \le \delta$. Using Lemma 6.5 and equation 6.2, by a short calculation we see that $r \le \min\{\rho_0/8, d/8\}$ and $\delta \le 2\,\rho_0\, r$ are sufficiently small ($d = \operatorname{dist}_{n+1}(\boldsymbol{p}_0, \partial M_{t_0})$). Hence, from

$$\sup_{\mathscr{Z}_r,\,\delta} \eta(.,.) \le \mathrm{e}^{8\,n\,\alpha\,\beta^2} - 1\,, \tag{6.3}$$

we conclude

Lemma 6.6. *Let $\boldsymbol{p}_0 = \boldsymbol{F}(x_0, t_0)$, $x_0 \in M^n \backslash \partial M^n$, $d := dist_{n+1}(\boldsymbol{p}_0, \partial M_{t_0}) > 0$ and $\alpha > 1$, $\beta > 0$, $\hat{\varepsilon} > 0$, $r \leq \min\{\frac{\rho_0}{8}, \frac{d}{8}\}$, $\delta \leq 2\rho_0 r$. Then for any (x,t) with $\boldsymbol{F}(x,t) \in \mathscr{Z}_r(\boldsymbol{p}_0)$ and $0 \leq t - t_0 \leq \delta$, the following estimate holds:*

$$(\eta v)(x,t) < (e^{8n\alpha\beta^2} - 1) \cdot \max\left\{ \tfrac{\alpha}{\alpha-1}(8\beta+1)\,,\ (1+\hat{\varepsilon}) \sup_{\mathscr{Z}_r} v(.,t_0) \right\} . \quad \square$$

Lemma 6.6 immediately yields an estimate for v by bounding η from below. Certainly, this has to be done on a smaller cylinder $\mathscr{Z}_{r_1}(\boldsymbol{p}_0)$, since $\eta \equiv 0$ on $\partial\mathscr{Z}_r(\boldsymbol{p}_0)$. The crucial point is to determine the scaling behaviour of β, and it turns out that a good choice is

$$\beta := \tfrac{1}{2}\sqrt{\tfrac{r}{\rho_0}} .$$

If we then let $\kappa \in (0,1)$, $r \leq (\frac{\kappa}{32})^2 \rho_0$, $r_1 \leq \sqrt{\frac{\kappa}{4}}\, r$ and $\delta \leq 8r^2$, we estimate

$$r^2 \ \geq\ \varphi(x,t)\ \geq\ (1-\kappa) r^2$$

in the cylinder $\mathscr{Z}_{r_1}(\boldsymbol{p}_0)$ and for $0 \leq t - t_0 \leq \delta$. This in turn implies the desired gradient estimate in the interior case on a time interval which is comparable to ρ_0^2, d^2:

Theorem 6.7. *Let $\boldsymbol{p}_0 = \boldsymbol{F}(x_0, t_0)$, $d = dist_{n+1}(\boldsymbol{p}_0, \partial M_{t_0}) > 0$. Then*

$$\forall \varepsilon > 0 \quad \exists\, C_1 > 0,\ C_2 > 0,\ C_3 > 0,\ C_4 > 0 \ \textit{ depending only on } \ n, \varepsilon$$

such that

$$v(x,t) < 1 + \varepsilon \quad \forall (x,t) \textit{ with } \boldsymbol{F}(x,t) \in \mathscr{Z}_{r_1}(\boldsymbol{p}_0)\,,\ 0 \leq t - t_0 \leq \delta\,;$$

hereby $r_1 = \min\{C_1\rho_0, C_2 d\}$ and $\delta = \min\{C_3\rho_0^2, C_4 d^2\}$.

Proof. Choose

$$r := \tfrac{1}{n}\min\left\{\sqrt{\tfrac{\kappa}{4}}\,d,\ (\tfrac{\kappa}{32})^2\rho_0\right\}\,,\ r_1 := \sqrt{\tfrac{\kappa}{4}}\,r\,,\ \delta \leq \min\left\{8r^2, 2\rho_0 r\right\} .$$

Furthermore, let $\alpha := 1 + \frac{1}{\kappa}$ and $\hat{\varepsilon} := \kappa$. Then

$$(1+\hat{\varepsilon}) \sup_{\mathscr{Z}_r} v(.,t_0) \leq (1+\kappa)^2 \qquad \text{(Lemma 6.4) and } \ \tfrac{\alpha}{\alpha-1}(1+8\beta) \leq (1+\kappa)^2 .$$

Moreover, for (x,t) as in the Theorem, $\eta(x,t) > e^{8n\alpha\beta^2(1-\kappa)} - 1$. Now we apply the Taylor expansion

$$\tfrac{e^z - 1}{e^{z(1-\kappa)} - 1} \leq 1 + 2\kappa + 2z\,, \quad z \in (0, \tfrac{1}{32})\,, \quad \kappa \in (0, \tfrac{1}{2}]$$

to the estimate Lemma 6.6 and conclude

$$v(x,t) < 1 + 10\kappa .$$

The result follows if we choose $\kappa \leq \min\{\frac{1}{2}, \frac{\varepsilon}{10}\}$. $\square$

Remark 6.8. At a first glance, the estimate Theorem 6.7 depends on the point (x_0, t_0), since r, r_1, δ depend on the distance of $\boldsymbol{F}(x_0, t_0)$ from $\boldsymbol{\Sigma}$. However, for all $(\tilde{x}, \tilde{t})$ with $\mathrm{dist}_{n+1}(\boldsymbol{F}(\tilde{x}, \tilde{t}), \boldsymbol{\Sigma}) \geq d_0 > 0$, we obtain the same estimate, since r, r_1, δ depend only on the minimal distance of all $\boldsymbol{F}(\tilde{x}, \tilde{t})$ from $\boldsymbol{\Sigma}$.

Remark 6.9. It is clear that Theorem 6.7 immediately implies a local gradient estimate for the *SGR* of M_t in the vicinity of (x_0, t_0): Given $\varepsilon > 0$, there exist $\hat{r}_1 > 0$ and $\hat{\delta} > 0$ such that the *SGR* of the M_t is defined on $B_{r_1}(0) \times [t_0, t_0 + \hat{\delta}]$ and fulfils the estimate

$$\sqrt{1 + \|Dw\|^2(y, t)} < 1 + \varepsilon \; ;$$

hereby $\hat{r}_1, \hat{\delta}$ depend only on ε, ρ_0, d and have the "natural" scaling behaviour $\hat{r}_1 \sim \rho_0, \hat{\delta} \sim \rho_0^2$. Thus using Lemma 6.5, we obtain a local C^1-estimate $\|w\|_{C^1} \leq C_0$ on $B^n_{\hat{r}_1} \times [t_0, t_0 + \hat{\delta}]$.

6.3 The boundary estimate

We now turn our attention to boundary points. Here we have to rule out *boundary maxima* at least for sufficiently large values of ηv. This can be done if we guarantee that $\langle \nabla(\eta v), \boldsymbol{\mu} \rangle < 0$ for large values of ηv. Now the cutoff-function η which we used in the interior case essentially is a paraboloid centered at $\boldsymbol{p}_0$; therefore it does not produce a sufficiently large term $\langle \nabla \eta, \boldsymbol{\mu} \rangle < 0$ that could compensate the (possibly large and positive) term $\langle \nabla v, \boldsymbol{\mu} \rangle$ in the vicinity of $\boldsymbol{p}_0$. The idea is to center the paraboloid not at $\boldsymbol{p}_0$ but at a nearby point $\boldsymbol{p}_m \notin \boldsymbol{\Sigma}$ such that η posesses a sufficiently large gradient near $\boldsymbol{p}_0$.

We choose Cartesian coordinates in $\mathbb{R}^{n+1}$ with origin at $\boldsymbol{p}_0 = \boldsymbol{F}(x_0, t_0)$ and set $\boldsymbol{\omega} := \boldsymbol{\nu}(x_0, t_0)$, $\boldsymbol{\gamma} := \boldsymbol{\mu}(\boldsymbol{p}_0)$ and

$$\boldsymbol{p}_m := \boldsymbol{p}_0 - \tfrac{1}{K} r \boldsymbol{\gamma}, \quad K > 1, \quad r \leq \tfrac{\rho_\Sigma}{4} .$$

Thus $\mathscr{Z}_{r,1}(\boldsymbol{p}_m) \subset B^{n+1}_{\rho_\Sigma}(\boldsymbol{p}_0)$, and hence Lemma 6.3 holds on $\boldsymbol{\Sigma} \cap \mathscr{Z}_{r,1}(\boldsymbol{p}_m)$, and this set is connected due to Assumption 6.1. Now, similar to the interior case, we set (with $\mathscr{Z}_r := \mathscr{Z}_r(\boldsymbol{p}_m)$)

$$M_{r,\delta} := \sup\{|u(x,t)| : \boldsymbol{F}(x,t) \in \mathscr{Z}_r \, , \, 0 \leq t - t_0 \leq \delta\} ,$$

$$\varphi(x,t) := \begin{cases} \left(\frac{r}{2\beta}(u - M_{r,\delta}) + r^2 - (\|\boldsymbol{F} - \boldsymbol{p}_m\|^2 - u^2)\right)_+ (x,t), & \boldsymbol{F}(x,t) \in \mathscr{Z}_r, \\ 0, & \boldsymbol{F}(x,t) \notin \mathscr{Z}_r, \end{cases}$$

$$\eta(x,t) := \mathrm{e}^{\lambda \varphi(x,t)} - 1 ,$$

where $0 < r < \frac{\rho_\Sigma}{4}$, $\delta > 0$ and $\alpha > 1$ will be chosen later and

$$\beta := \tfrac{1}{2} \sqrt{\tfrac{r}{\rho_0}} , \quad \lambda := \tfrac{8 n \alpha \beta^2}{r^2} = \tfrac{2 n \alpha}{\rho_0 r} .$$

Now we estimate the normal component of $\nabla(\eta v)$ as follows:

Lemma 6.10. *Let $K \geq 8$, $r \leq \frac{1}{64}\min\{\rho_0, \frac{\rho_0}{K^2}, \frac{\rho_0}{n\,\alpha}, \frac{\rho_\Sigma}{K}, \frac{\rho_\Sigma^2}{K^2\rho_0}\}$ and $\delta \leq r^2$. Then for any $(x,t) \in M^n \times [0,T]$ with $\boldsymbol{F}(x,t) \in \boldsymbol{\Sigma} \cap \mathscr{Z}_r(\boldsymbol{p}_m)$, $\varphi(x,t) > 0$ and $0 \leq t - t_0 \leq \delta$, the following estimate holds:*

$$\langle \nabla(\eta\, v), \boldsymbol{\mu}\rangle < \tfrac{1}{2\rho_0}\, v\, \left((\eta\, v) - \tfrac{4\,n\,\alpha}{K}\right) .$$

Proof. We compute

$$(*) \qquad \langle \nabla\varphi, \boldsymbol{\mu}\rangle \;\leq\; |\langle \boldsymbol{\mu}, \boldsymbol{\omega}\rangle| \left(\tfrac{r}{2\beta} + 2 M_{r,\delta}\right) - 2\,\langle \boldsymbol{F} - \boldsymbol{p}_m, \boldsymbol{\mu}\rangle .$$

By the choice of r, δ, we can apply Lemma 6.5 on the cylinder $\mathscr{Z}_{2r}(\boldsymbol{p}_0)$; this yields $|u(x,t)| \leq \frac{3}{4} r$ and thus

$$M_{r,\delta} < r .$$

Now a lengthy but nevertheless straightforward calculation involving Lemma 6.3 and the Cauchy-Schwarz-inequality yields

$$\langle \boldsymbol{F} - \boldsymbol{p}_m, \boldsymbol{\mu}\rangle > \tfrac{3}{4}\tfrac{r}{K}, \quad \text{and} \quad |\langle \boldsymbol{\mu}, \boldsymbol{\omega}\rangle|^2 < 2(\tfrac{r}{\rho_\Sigma})^2 .$$

Substituting these expressions into equation (*), we obtain

$$(**) \qquad \langle \nabla\varphi, \boldsymbol{\mu}\rangle < -\tfrac{r}{K} .$$

Now we compute $\langle \nabla v, \boldsymbol{\mu}\rangle = -v^2 A(\boldsymbol{\omega}^T, \boldsymbol{\mu})$; using Assumption 6.1 (bounded curvature), this yields

$$(***) \qquad |\langle \nabla v, \boldsymbol{\mu}\rangle| \;\leq\; v^2\, |A|\, \|\boldsymbol{\omega}^T\| \;\leq\; v^2 \tfrac{1}{2\rho_0} .$$

Finally,

$$\langle \nabla(\eta\, v), \boldsymbol{\mu}\rangle = v\,\langle \nabla\eta, \boldsymbol{\mu}\rangle + \eta\,\langle \nabla v, \boldsymbol{\mu}\rangle = \lambda\, v\,(\eta + 1)\,\langle \nabla\varphi, \boldsymbol{\mu}\rangle + \eta\,\langle \nabla v, \boldsymbol{\mu}\rangle ,$$

and the result follows using equations (**), (***). □

Lemma 6.11. *Let $r > 0$, $\delta > 0$ be as in Lemma 6.10 and $\alpha \geq 2$, $\hat{\varepsilon} \leq 1$. Then for any $(x,t) \in M^n \times [0,T]$ with $\boldsymbol{F}(x,t) \in \mathscr{Z}_r(\boldsymbol{p}_m)$, $0 \leq t - t_0 \leq \delta$, the following estimate holds:*

$$(\eta\, v)(x,t) < \sup_{\mathscr{Z}_r,\,\delta} \eta(.,.) \cdot \max\left\{ \tfrac{\alpha}{\alpha-1}\,(8\,\beta + 1), (1 + \hat{\varepsilon}) \sup_{\mathscr{Z}_r} v(., t_0) \right\} .$$

Proof. Exactly as in the interior case, this estimate is valid at least until equality holds at a boundary point $\boldsymbol{p} = \boldsymbol{F}(x_2, t_2)$, $x_2 \in \partial M^n$, $0 \leq t_2 - t_0 \leq \delta$. At this point, $\eta\, v$ must have a maximum, hence we have

$$(*) \qquad \langle \nabla(\eta\, v), \boldsymbol{\mu}\rangle(x_2, t_2) \geq 0 .$$

Now, by Lemma 6.4, $\sup_{\mathscr{Z}_r} v(., t_0) < 2$; furthermore a Taylor expansion of equation 6.3 yields $\sup_{\mathscr{Z}_r,\,\delta} \eta(.,.) \leq 4\,n\,\alpha \frac{r}{\rho_0}$, and thus we have

$$(\eta\, v)(x_2, t_2) < 16\,n\,\alpha\,\tfrac{r}{\rho_0} .$$

Lemma 6.10 now implies $\langle \nabla(\eta\, v), \boldsymbol{\mu}\rangle(\boldsymbol{p}) < 0$ in contradiction to (*). □

We proceed as in the interior case by first estimating η from below on a smaller cylinder $\mathscr{Z}_{r_1}(\boldsymbol{p}_0) \subset \mathscr{Z}_r(\boldsymbol{p}_m)$ and then Taylor expanding the quotient of the estimates for η from above (on $\mathscr{Z}_r(\boldsymbol{p}_m)$) and from below (on $\mathscr{Z}_{r_1}(\boldsymbol{p}_0)$). This can be done similarly as before, so we directly state the results:

Lemma 6.12. *Let* $r \leq \frac{1}{64}\min\{\rho_0\,,\,\frac{\rho_0}{K^2}\,,\,\frac{\rho_0}{n\,\alpha}\,,\,\frac{\rho_\Sigma}{K}\,,\,\frac{\rho_\Sigma^2}{K^2\rho_0}\}\,,\ \delta \leq r^2\,,\ r_1 \leq \frac{r}{K}$ *and* $\alpha \geq 2\,,\ K \geq 8\,,\ \hat{\varepsilon} \leq 1$. *Then for any* $(x,t) \in M^n \times [0,T]$ *with* $\boldsymbol{F}(x,t) \in \mathscr{Z}_{r_1}(\boldsymbol{p}_0)$ *and* $0 \leq t - t_0 \leq \delta$ *the following estimate holds:*

$$v(x,t) < \left(1 + \tfrac{6}{K} + 4\,n\,\alpha\,\tfrac{r}{\rho_0}\right) \cdot \max\left\{\ \tfrac{\alpha}{\alpha-1}\,(8\,\beta+1)\,,\ (1+\hat{\varepsilon})\,\sup_{\mathscr{Z}_r} v(.,t_0)\ \right\}\ . \quad \square$$

Theorem 6.13. *Let* $\boldsymbol{p}_0 = \boldsymbol{F}(x_0,t_0) \in \partial M_{t_0}$. *Then*

$$\forall\,\varepsilon > 0 \quad \exists\, C_i > 0\,,\ 1 \leq i \leq 6\,,\ \textit{depending only on}\ \ n, \varepsilon$$

such that

$$v(x,t) < 1 + \varepsilon \quad \forall (x,t)\ \textit{with}\ \ \boldsymbol{F}(x,t) \in \mathscr{Z}_{r_1}(\boldsymbol{p}_0)\ ,\ \ 0 \leq t - t_0 \leq \delta\,;$$

hereby $r_1 \;=\; \min\{C_1\,\rho_0\,,\ C_2\,\rho_\Sigma\,,\ C_3\,\frac{\rho_\Sigma^2}{\rho_0}\}$ *and* $\delta \;=\; \min\{C_4\,\rho_0^2\,,\ C_5\,\rho_\Sigma^2\,,\ C_6\,\frac{\rho_\Sigma^4}{\rho_0^2}\}$. $\square$

Thus again we have a gradient estimate on a time interval which is comparable to $\rho_\Sigma^2\,,\ \rho_0^2$. The Theorem immediately provides an estimate for the gradient of the *SGR* near the boundary point $\boldsymbol{p}_0$; but, as $\boldsymbol{\Sigma}$ in general will be curved, this graph representation (which was defined with respect to *Cartesian coordinates in* $\mathbb{R}^{n+1}$) will be defined on a subset $\cup_{t=t_0}^{t_0+\delta}\ \mathscr{U}_t \times \{t\}$ and *not* on a product structure $\mathscr{U} \times [t_0, t_0{+}\delta]$. A graph representation on a product structure can be obtained by working with generalized Gaussian coordinates as in Sect. 2. The gradient bound with respect to the latter graph representation is equivalent to estimating

$$\tilde{v}(x,t) := \langle \boldsymbol{\nu}, \boldsymbol{\xi}\circ\boldsymbol{F}\rangle^{-1}(x,t)$$

from above. Now a connection between the above estimate for $v = \langle \boldsymbol{\nu}, \boldsymbol{\omega}\rangle^{-1}$ and $\tilde{v}$ can easily be established by controlling $\langle \boldsymbol{\xi}, \boldsymbol{\omega}\rangle$. This is a simple exercise of analytic geometry, and it turns out that Theorem 6.13 exactly carries over to $\tilde{v}$. Thus we remark:

Remark 6.14. Theorems 6.7 and 6.13 yield a gradient bound for appropriate local graph representations $w : \mathscr{U}_{\hat{r}} \times [t_0, t_0{+}\delta] \longrightarrow \mathbb{R}$ of the M_t around a given point $\boldsymbol{F}(x_0,t_0)$ which is independent of $\boldsymbol{F}(x_0,t_0)$. Hence, by Lemma 6.5, we obtain a C^1-estimate for w on a domain $\mathscr{U}_{\hat{r}} \times [t_0, t_0{+}\hat{\delta}]$, whose "dimensions" have the natural scaling behaviour in ρ_0, i.e. $\hat{r} \sim \rho_0\,,\ \hat{\delta} \sim \rho_0^2$.

6.4 Long time existence of solutions

Proof of Theorem 1.3. According to Theorem 2.1, we have a solution to equation 1.2 on a *maximal* time interval $[0, T_C)$. The only thing that remains to check is that, if

$$(*) \qquad \sup\{\, |A|^2(x,t) \,:\, x \in M^n \,,\, t \in [0, T_C)\,\} \;\leq\; \tfrac{1}{2n^2\rho_0} \;<\; \infty \,,$$

we can extend the solution to a larger time interval $[0, T_C + \delta)$. To this end, we start by defining the limit surface

$$\boldsymbol{F}(x, T_C) := \lim_{t \to T_C} \boldsymbol{F}(x,t)\,, \quad M_{T_C} := \boldsymbol{F}(M^n, T_C)\,,$$

which is possible due to equation $(*)$ and Lemma 6.5. By the gradient estimates Theorems 6.7, 6.13, Remark 6.14 and $(*)$ we see that M_{T_C} can be covered by local graph representations of $C^{1+\alpha,(1+\alpha)/2}$-functions. As in chapter 2, each of these functions fulfils a uniformly parabolic differential equation on its domain of definition; on the boundary, in addition we have a transversal Neumann condition. Hence from linear parabolic theory we conclude that M_{T_C} must actually be a *smooth* hypersurface. Now we can apply the short time existence Theorem 2.1 with M_{T_C} as initial hypersurface and extend the given solution to a solution on $[0, T_C + \delta)$. $\square$

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