Giovanni Bellettini

Lecture Notes on Mean Curvature Flow, Barriers and Singular Perturbations

EDIZIONI
DELLA
NORMALE

12

APPUNTI

LECTURE NOTES

Giovanni Bellettini
Dipartimento di Matematica
Università di Roma Tor Vergata
via della Ricerca Scientifica 1
00133 Roma, Italia

Lecture Notes on Mean Curvature Flow, Barriers and Singular Perturbations

Giovanni Bellettini

Lecture Notes on Mean Curvature Flow, Barriers and Singular Perturbations

EDIZIONI
DELLA
NORMALE

The Publisher disclaims any responsibility arising from copyrighted figures from other sources.

ISBN 978-88-7642-428-1
ISBN 978-88-7642-429-8 (eBook)

Contents

Introduction

Mean curvature motion of a hypersurface is one of the first meaningful examples of geometric evolution problem of parabolic type and has been a subject of interest in analysis, differential geometry and geometric measure theory, starting from the pioneering book of Brakke [67]. One of the appealing aspects of mean curvature motion is that it can be viewed, at least formally, as the gradient flow of the area functional [9] and therefore it is strictly related to the theory of minimal surfaces [156, 93]. Indeed, in various situations, for instance imposing graphicality and suitable boundary conditions, the asymptotic limit, as time goes to infinity, of the evolving manifold turns out to be a minimal surface [169]. Mean curvature motion arises also in problems of applied mathematics and statistical physics [192, 256]. For instance, it emerges quite naturally in some simplified models in the theory of phase transitions: in particular, in connection with the limit, as $\epsilon \downarrow 0$, of solutions u_ϵ to the semilinear parabolic equation

$$\epsilon \frac{\partial u}{\partial t} = \epsilon \Delta u - \frac{1}{2\epsilon} u(u^2 - 1), \tag{1}$$

and to its nonlocal version [117].

The scope of this book is to introduce the reader to some elementary aspects of one-codimensional mean curvature flow embedded in $\mathbb{R}^n$ and to some relatively more advanced topics, connected to certain aspects of the theory of minimal barriers in the sense of De Giorgi and to the limits of solutions to (1).

The arguments treated here share some common features. A relevant one is the role taken by the signed distance function from the evolving hypersurface and, related to it, the idea of looking at the manifold as a set of points rather than as the image of an embedding map defined on a reference manifold. The approach using the distance function fits in a natural way in the asymptotic description of solutions to equation (1) as $\epsilon \downarrow 0$, at least for short times. The parametrization free description of the evolving manifolds is also well suited for the language of minimal barriers. It must be remarked, however, that the parametric description of the flow (which allows for treating immersions and not only embeddings) is convenient in various arguments and will be adopted from time to time.

Let us recall that a smooth time-dependent family of solid sets $E(t) \subset \mathbb{R}^n$ flows by mean curvature if the normal velocity of each point of the boundary $\partial E(t)$ equals the sum of its principal curvatures. Using the signed distance function

$$d(t, z) := \mathrm{dist}(z, E(t)) - \mathrm{dist}(z, \mathbb{R}^n \setminus E(t)),$$

this evolution of sets can be expressed, at time t, as follows:

$$\frac{\partial d}{\partial t} = \Delta d \qquad \text{on the hypersurface } \{d(t, \cdot) = 0\}. \tag{2}$$

This is the heat equation for d, valid however only on the moving manifold $\partial E(t)$, for t in a suitable time interval. Up to a few exceptions, we will confine ourselves to consider compact evolving boundaries.

Although equation (2) is invariant under the change of $E(t)$ into its complement $\mathbb{R}^n \setminus E(t)$, and therefore can be regarded as an equation for $\partial E(t)$, we are often lead to consider the evolution of the solid set rather than of its boundary. There are various motivations for adopting this point of view: for instance, it is convenient in the investigation of the asymptotic properties of solutions to (1), where the set $\{u_\epsilon(t, \cdot) < 0\}$ takes approximately the role of the phase given by the interior of $E(t)$, and $\{u_\epsilon(t, \cdot) > 0\}$ of approximately the interior of the other phase $\mathbb{R}^n \setminus E(t)$. The above mentioned point of view is also well suited for the description of the flow with the barriers method.

We have already observed that equation (2) is not the only way to describe the evolution process, which can indeed alternatively be expressed in a parametric way, or also written looking at the evolving manifolds as graphs on a fixed embedded manifold, or again as the zero level set of a function which is not, in general, a distance function. In any case, the evolution problem turns out to be locally well-posed. In the literature, the proof of the short-time existence has been given in various different ways (see, *e.g.*, [149, 138, 173] and [205]). Uniqueness is valid as well, at least for short times, and can be deduced as a consequence of one of the most important properties of the flow, namely the inclusion principle (also called comparison principle). This can be stated as follows: if two smooth solid sets $E_1 = E_1(0)$ and $E_2 = E_2(0)$ having compact boundary, initially satisfy the inclusion

$$E_1(0) \subseteq E_2(0),$$

then the two corresponding smooth mean curvature flows $E_1(t)$ and $E_2(t)$ satisfy

$$E_1(t) \subseteq E_2(t).$$

Once the existence and uniqueness of a smooth solution for short times are settled, it is natural to investigate the properties of solutions for later times. Here the situation becomes much more involved. A remarkable theorem due to Grayson [158] (see also the previous paper [149] of Gage-Hamilton for related results on convex curves, and [172] for an alternative proof of Grayson's result) shows that, when $n = 2$, the solution remains smooth up to the extinction time $t^\dagger$, and at some moment before $t^\dagger$ it becomes convex; at $t^\dagger$ it reduces to a point approaching, after rescaling, the form of a circle. If $n \geq 2$ is arbitrary, Huisken [168] proved that if the initial datum $E \subset \mathbb{R}^n$ is bounded, uniformly convex and smooth, then the solution remains uniformly convex and smooth up to the extinction, when it reduces to a point approaching the form of a sphere. Several other results are present in the literature and cannot be quoted here(1). However, it is worthwhile to recall that, as observed by Grayson in [159], a suitable bounded smooth dumbell-shaped set in $\mathbb{R}^3$ must develop a singularity when its evolution is not yet reduced to a point.

It is then apparent that the study of the profile of the solution at a singularity of the flow becomes an issue, as well as the problem of continuing the solution in a reasonable way after a singularity time. The study of singularities(2) is beyond the scope of the present book, which aims to an elementary presentation of the subject. In particular, we will not treat the regularity theory for mean curvature flow and we will not deepen the problem of the classification of singularities. For these arguments we refer the reader to the existing bibliography, in particular to the book of Ecker [126] and to the original articles of Huisken [170, 171] (see also [257, 262, 174, 175, 176, 265, 267, 269, 270, 94] and the recent book [203] and references therein).

On the other hand, we will devote some effort to discuss some specific aspects of weak solutions, trying to inspect which are the natural properties that a weak solution must share in order to naturally continue the flow after a singularity time. This subject was already investigated in the above mentioned book of Brakke [67], in a geometric measure theoretic setting. Brakke's solution is a sort of distributional (sub)solution

(1) See for instance the references in [126, 244, 203].

(2) Here by singularity we mean that, at a certain positive time, the norm of the second fundamental form blows up somewhere on the evolving manifold.

and turns out to be nonunique. Several other notions of weak solution have been proposed in the literature, from different points of views. We mention, among others[(3)],

- the level set evolution, obtained using the notion of viscosity solution in the sense of Crandall-Lions (see [137, 86, 138, 139, 96, 140, 182, 152] and references therein);
- the solution constructed using the limits of solutions to equation (1), see [102] and [184];
- the elliptic regularization of Ilmanen [185];
- the solution of Almgren-Taylor-Wang [9], which amounts in a time discretization of the problem, and looks at the flow as a limit of a sequence of prescribed mean curvature problems, as well as the approach of Luckhaus and Sturzenhecker [196] and [105, 14, 17], and its modification with the use of the signed distance function [79, 80];
- the set theoretic subsolutions of Ilmanen [183];
- the barriers method [106, 108];
- the distance solution introduced by Soner, by means of sub and supersolutions of the flow, defined via the signed distance function [248];
- the solution obtained as limit of the heat equation with threshold, see [209, 132, 37] (and also [130, 179, 161]);
- the evolution obtained as a limit of regularized higher order equations [109, 43];
- the solution constructed as the value function of a deterministic game [181] (see also [253]);
- the solution constructed in [239].

Global existence and consistency[(4)] are two desirable properties for a weak solution. Uniqueness also would be required[(5)]; however the fattening phenomenon indicates that a unique weak solution may fail to exist[(6)], at least in case of very special initial conditions. In Chapters 9, 10, 11, 13 and 14 we will be concerned with the barriers method; we will develop

(3) It is also worth mentioning that a complete comparison between some of these weak solutions is, at the moment, still incomplete (after the onset of singularities).

(4) Namely, the property that the weak solution coincides with the classical one, till the latter exists.

(5) For instance, the theory of minimal barriers, as well as the level set solution, gives an evolution which shares all these properties.

(6) The solution constructed with the barriers method, as well as the level set solution, is unique even in presence of fattening; roughly speaking, these global solutions should capture all together a lot of reasonable (weak) solutions after the singularity time. For certain weak solutions, this behaviour is pointed out in Theorem 14.4 in Chapter 14.

the theory at a certain level of generality, and then we will apply it to the case of mean curvature flow. We will also compare this notion of solution with an abstract evolution law enjoining the global existence, the consistency, the inclusion and the semigroup properties (such as, for instance, the level set solution). Inspecting the properties and the mutual relations of the other above mentioned notions of weak solution would lead us too far, and indeed this argument will not be covered here. Similarly, we will not consider other interesting generalizations of one-codimensional mean curvature flow, such as

- the mean curvature flow in presence of exterior forcings and anisotropies (see, *e.g.*, [86, 153, 56]);
- the higher codimensional mean curvature flow (see, *e.g.*, [10, 106, 19, 47, 245, 263]), and its approximation via the modification of equation (1) for vector-valued functions u taking values approximately close to a circle [20, 62];
- the mean curvature flow of partitions (see, *e.g.*, [111, 207]) and its approximation via the generalization of equation (1) for vector-valued functions u taking values approximately close to a finite set of vectors [72].

The last three chapters of the book are devoted to the asymptotic analysis, as $\epsilon \downarrow 0$, of solutions u_ϵ of equation (1). In this framework, we prove a convergence result of the zero level sets of u_ϵ to a compact mean curvature flow, under various assumptions, for instance:

- for short times, and in particular before the onset of singularities for mean curvature flow;
- for well-prepared initial data.

The main ingredients of the proof of this result are the construction of suitable sub/supersolutions to equation (1), and the use of the maximum principle. We also discuss another tool that we believe to be useful for understanding the proof of the convergence theorem, namely the expansion in powers of ϵ of solutions u_ϵ far from (outer expansion) and close to (inner expansion) their zero-level set. Even though this expansion argument remains formal, it gives various hints for the rigorous convergence proof, in particular for the shape of the sub/supersolutions of equation (1).

Generalizations of the convergence result (for instance for long times) will not be discussed here; the interested reader can consult for instance [136, 184, 249, 250].

Let us now briefly illustrate the content of the various chapters.

In Chapter 1 we study some properties of the signed distance function d from the boundary ∂E of a set $E \subset \mathbb{R}^n$. With few exceptions, ∂E will be assumed to be smooth and compact. We next recall some properties of the first and second derivatives of d, in particular the eikonal equation $|\nabla d|^2 = 1$, and the related properties of the projection map $\mathrm{pr} : z \to z - d(z)\nabla d(z)$ from points z in a suitable neighbourhood of ∂E onto ∂E. The mean curvature vector and the second fundamental form of ∂E are then introduced in the language of the distance function, taking advantage of the fact that differentiation is made in the ambient space $\mathbb{R}^n$, and next restricted to ∂E. The expansion of the eigenvalues of the Hessian $\nabla^2 d$ in a neighbourhood of ∂E is considered in Theorem 1.18. This expansion turns out to be useful for the (local in time) existence and uniqueness result of a smooth compact mean curvature flow considered in Chapter 7. The tangential calculus on ∂E, as well as the classical definitions of mean curvature vector and second fundamental form, are also recalled, and compared with the previous definitions given in terms of d.

In Chapter 2 we compute the first variation of various geometric functionals, including volume functionals, the perimeter functional, and slight generalizations of it. These computations are useful in order to give a variational meaning to mean curvature flow. Some of these calculations turn out to be useful in the derivation of the monotonicity formula in Chapter 4, and are related to some special solutions to mean curvature flow. In the last section of the chapter we make a brief remark, in the direction of looking at mean curvature flow as a gradient flow.

An introduction to mean curvature flow is treated in Chapters 3-8.

In Chapter 3 we give the definition of smooth flow and of normal velocity, using both the signed distance function and the parametrizations. Then we define smooth mean curvature flows, distinguishing the compact case from the noncompact one. These definitions, and the related notation, will be extensively used when introducing barriers. The chapter concludes with some elementary examples of smooth mean curvature flows, and some examples of special solutions, such as self-similar and translatory solutions. At the end of the chapter we compute the evolution equations satisfied by various geometric quantities (such as the derivatives of the curvature) in the case of parametric plane curves evolving by curvature.

In Chapter 4 we describe the monotonicity formula of Huisken [170]. This formula has a lot of consequences in the study of mean curvature flow, in particular for what concerns the analysis of singularities. In this book we will show only one application of the monotonicity formula, related to a gradient estimate for mean curvature evolutions of entire graphs, due to Ecker and Huisken [127].

In Chapter 5 we discuss various versions of the inclusion principle between smooth mean curvature flows. It is worthwhile to remark that at least one of the two evolving hypersurfaces under consideration is supposed to be compact. These principles have far reaching consequences, some of which will be analyzed in the subsequent chapters. Some reminds on the weak maximum and minimum principles are given in the appendix.

In Chapter 6 we characterize a smooth mean curvature flow using the evolution of its tubular neighbourhoods, which is the starting observation made by Evans and Spruck in [138] to prove short-time existence. A remark on the strong inclusion principle is briefly discussed as well. Using the formulation based on the tubular neighbourhoods, we derive the evolution equations for various geometric quantities. These differential equations are derived in more generality, and using a parametric description of the flow, in the appendix of the chapter.

In Chapter 7 we prove the short-time existence and uniqueness theorem in $\mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}$ of mean curvature flow starting from a smooth compact boundary. Here we follow the approach of Evans and Spruck [138], which fits well in the framework of the signed distance function. Some remarks concerning higher regularity of the solution are also given.

In Chapter 8 we describe the example, due to Grayson, of formation of a singularity in finite time, starting from a smooth bounded dumbell shaped subset of $\mathbb{R}^3$. The example is rather explicit, since the dumbell is supposed to be rotationally symmetric. The main tool in the proof is the inclusion principle. This example shows that, starting from a smooth compact embedded hypersurface, singularities at finite time, different from the extinction, in general cannot be avoided.

In Chapter 9 we start the study of barriers and minimal barriers. The main definitions are given for an arbitrary family $\mathfrak{F}$ of test evolutions. In this generality we discuss various properties, such as the semigroup property and the translation invariance. Particular attention is devoted to the case in which the test evolutions have compact boundary. We also start the analysis of the evolution of the complement of a set, which is then completed in Chapter 13.

Various examples illustrated in Chapter 9 show that the minimal barrier is very sensitive to modifications of the original set on subsets of zero Lebesgue measure, thus making its characterization difficult to describe.

As we shall see, it is useful to regularize the notion of minimal barrier, in order to get evolutions insensitive to the above mentioned modifications.

In Chapter 10 we introduce the inner and outer regularizations with respect to an arbitrary family $\mathfrak{F}$ of tests, and we study some of their properties. We then specialize to the case of mean curvature flow of hypersurfaces, discussing various examples.

In Chapter 11 we consider another example of singularity, called fattening, which is related to a sort of instability of the evolution, and to nonuniqueness. The example is given for compact plane curves evolving by curvature, and therefore the initial condition must necessarily be singular. For simplicity, we choose as initial condition two disjoint open discs, the boundaries of which are tangent at one point (an initial eight-shaped curve with a transverse self-intersection would give raise to fattening as well). Fattening essentially means that the barrier evolution of the curve develops, at some positive time, an interior part, thus failing in particular to remain one-dimensional. This phenomenon fits well in the language of barriers, and in the description of the evolution using the level set method.

In Chapter 12 we discuss Ilmanen's interposition lemma, which is the major tool used in Chapter 14 in order to compare the weak evolution obtained using the barriers method with other notions of generalized evolution. The proof given in this chapter follows that of Cardaliaguet in [75].

In Chapter 13 we prove the avoidance principle between the complement of two barriers. Namely, if the complements of two barriers, one of which is compact, are initially at a positive distance, then this distance is nondecreasing for subsequent times. The proof of this theorem is rather involved and utilizes Ilmanen's interposition lemma.

In Chapter 14 we compare the minimal barrier solution, and the inner and outer regularizations, with an abstract evolution of sets, that we call comparison flow. This evolution must satisfy various natural properties (such as the inclusion principle, the consistency and the semigroup properties), which are shared, for instance, by the level set solution to mean curvature flow. The main result of the chapter shows, in particular, that the inner regularization gives a lower bound for any comparison flow, while the outer regularization always provides an upper bound. A by-product of this theorem and of the known properties of the level set solution, is the connection between the minimal barriers theory and the level set solutions.

In Chapter 15 we start the study of the asymptotic behaviour of solutions to equation (1). In particular, we study some variational properties of a functional in one dimension, the minimizer γ of which turns out to be useful in the description of the solution u_ϵ of (1). This chapter can be considered as a preliminary chapter, and collects material used in the last two chapters.

In Chapter 16 we justify, by the formal method of matched asymptotic expansions, the convergence of the zero level set of solutions to equation (1) to a mean curvature flow. Although the argument is formal, as already remarked it eventually leads to a rigorous convergence proof for short times.

Using the comparison principle, and on the basis of the expression of the leading terms found in the asymptotic expansion, in Chapter 17 we prove the convergence result (Theorem 17.5). After choosing a well-prepared initial datum, the idea is to write explicit subsolutions and supersolutions to (1), which bound the solution to be close to a given smooth compact mean curvature flow. We thus obtain convergence, in Hausdorff distance, of the zero level set of the solution u_ϵ of (1) to a mean curvature flow, together with an estimate of the rate of convergence of the order ϵ^2, up to logarithmic corrections [55]. We conclude the chapter with a short list of definitions and results which are the starting point for an extension of Theorem 17.5 to a (subsequential) convergence result valid for all times [184].

Eventually, we warn the reader that we make no claim on completeness concerning the references at the end of the book.

ACKNOWLEDGEMENTS. The point of view on mean curvature flow developed in this book reflects, in some sense, some of the ideas exposed by Ennio De Giorgi in various lectures that I had the chance to follow in the earlies 1990 at the Scuola Normale Superiore di Pisa.(7) I would like to express my deepest gratitude to him.

This book represents an extended version of the content of various courses and lectures that I delivered in the last few years, in particular at the Universities of Crete, Padova, Roma Tor Vergata, Castilla La Mancha, Trieste, Regensburg, at KTH (Stockholm), SISSA (Trieste) and at the Centro De Giorgi (SNS, Pisa). My thanks go to all these institutions.

It is a great pleasure to thank my friends and colleagues Matteo Novaga and Maurizio Paolini, for the discussions that we had on several of the arguments treated in this book, and for their valuable suggestions.

(7) The organization of the material and all conclusions are full responsability of the author.

I am indebted to Alessandra Lunardi for her comments on the local well-posedness of mean curvature flow. I am also grateful to Pierre Cardaliaguet, for pointing out to me his proof [75] of Ilmanen's interposition lemma. I thank Giovanni Alberti, Nick Alikakos, Gerhard Dziuk, Annibale Magni, Carlo Mantegazza, Pietro Majer, Andrea Mondino, Errico Presutti, Reiner Schätzle and Carlo Sinestrari for various discussions and suggestions. I am thankful to Jun Sun for his careful reading and comments on a part of the book. I thank Odette Sylvia Yaptieu Djeungue, Ngoc Quoc Nguyen Thuong and Aleks Jevnikar for various comments.

I am grateful to the kind hospitality of ICTP and the Dipartimento di Matematica e Informatica of the University of Trieste, where this book was partially written. I am also grateful to the Centro De Giorgi and to the Mathematisches Forschungsinstitut Oberwolfach for the hospitality and the stimulating research environment.

Chapter 1
Signed distance from a smooth boundary

In this chapter we collect some of the main properties of the distance function from a smooth boundary. We refer the reader to [142, 147, 115, 15, 273, 116, 74, 204, 18] for a wider discussion on this subject.

Notation

Let $n \in \mathbb{N}$, $n \geq 1$. The vector space $\mathbb{R}^n$ is endowed with the Euclidean norm $|\cdot|$, induced by the Euclidean scalar product, denoted by $\langle \cdot, \cdot \rangle$. We denote by $\{e_1, \dots, e_n\}$ the canonical basis of $\mathbb{R}^n$. Vectors in $\mathbb{R}^n$ are denoted by $z = z_k e_k$ where, unless otherwise specified, we adopt the convention of summation on repeated indices. We also use the notation $z = (z_1, \dots, z_n)$.

Given $\rho > 0$ and $z \in \mathbb{R}^n$, $B_\rho(z)$ is the open ball of radius ρ centered at z, and $\mathbb{S}^{n-1} := \{z \in \mathbb{R}^n : |z| = 1\}$.
We will use the notation

$$\nabla_i = \frac{\partial}{\partial z_i}, \qquad \nabla^2_{ik} = \frac{\partial^2}{\partial z_i \partial z_k}.$$

Let $\Omega \subseteq \mathbb{R}^n$ be an open set. If $\mathrm{u} : \Omega \to \mathbb{R}$ is a function of class $\mathcal{C}^1(\Omega)$, we identify the covector field du (the differential of u) with the vector field $\nabla \mathrm{u}$ (the gradient of u) in the usual way using the Euclidean scalar product, that is, if $z \in \Omega$ then $\mathrm{du}(z)(v) = \langle \nabla \mathrm{u}(z), v \rangle$ for any vector $v \in \mathbb{R}^n$. If $\mathrm{u} \in \mathcal{C}^2(\Omega)$ then $\nabla^2 \mathrm{u} = (\nabla^2_{ij} \mathrm{u})$ is the Hessian matrix of u and $\Delta \mathrm{u} = \mathrm{div}(\nabla \mathrm{u})$ is the Laplacian of u.

Let $m \in \mathbb{N}$, $m \geq 1$, and let $\mathrm{X} = (\mathrm{X}_1, \dots, \mathrm{X}_m) : \Omega \to \mathbb{R}^m$ be a vector field of class $\mathcal{C}^1(\Omega; \mathbb{R}^m)$. If $z \in \Omega$, we denote by $\mathrm{d}X(z) : \mathbb{R}^n \to \mathbb{R}^m$ the linear map given by the differential of X at z. The $(m \times n)$ Jacobian matrix representing the differential $\mathrm{dX}(z)$ of X at z in the canonical bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ is denoted by $\nabla \mathrm{X}(z)$. If $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$, the ij-entry $(\nabla \mathrm{X}(z))_{ij}$ of $\nabla \mathrm{X}(z)$ is $\frac{\partial \mathrm{X}_i}{\partial z_j}(z) = \nabla_j X_i(z)$.

If $B \subseteq \mathbb{R}^n$ we denote by $\overline{B}$ the topological closure of B, and by $\mathrm{int}(B)$ and ∂B the topological interior part and the topological boundary of B, respectively.

Distance function

If $B \subseteq \mathbb{R}^n$ is nonempty we let

$$\mathrm{dist}(z, B) := \inf_{\zeta \in B} |\zeta - z|, \qquad z \in \mathbb{R}^n.$$

We also let $\mathrm{dist}(z, \emptyset) := +\infty$.

We recall that $\mathrm{dist}(\cdot, B)$ is Lipschitz continuous, and therefore almost everywhere differentiable, by Rademacher's theorem [16, Theorem 2.14]. Moreover $|\nabla \mathrm{dist}(\cdot, B)| \leq 1$ almost everywhere. Note also that

- $(\mathrm{dist}(z, B))^2 = \inf_{\zeta \in B} |\zeta - z|^2$;
- $\mathrm{dist}(z, B) = \mathrm{dist}(z, \overline{B})$ for any $z \in \mathbb{R}^n$, so that, from the point of view of the distance function, B could be taken closed without loss of generality;
- $\overline{B} = \{z \in \mathbb{R}^n : \mathrm{dist}(z, B) = 0\}$;
- $\mathrm{dist}(z, B) = \mathrm{dist}(z, \partial B)$ for any $z \notin \mathrm{int}(B)$.

Denoting by

$$\mathcal{P}(\mathbb{R}^n)$$

the set of all subsets of $\mathbb{R}^n$, we define the projection map

$$\mathrm{pr}(\cdot, B) : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$$

as follows: given $z \in \mathbb{R}^n$,

$$\mathrm{pr}(z, B) := \left\{x \in \overline{B} : |z - x| = \mathrm{dist}(z, B)\right\}, \tag{1.1}$$

which turns out to be a compact set.

Remark 1.1 (Uniqueness of projection). Let $B \subseteq \mathbb{R}^n$ be a nonempty closed set, and

$$z \in \mathbb{R}^n \setminus B.$$

It is possible to prove (see for instance [74, Corollary 3.4.5 (i)]) that $\mathrm{dist}(\cdot, B)$ is differentiable at z if and only if there exists a unique point of B nearest to z, namely $\mathrm{pr}(z, B)$ is a singleton,

$$\mathrm{pr}(z, B) = x \in B.$$

In this case

$$\nabla \mathrm{dist}(z, B) = \frac{z - x}{|z - x|},$$

so that

$$x = z - \mathrm{dist}(z, B)\nabla \mathrm{dist}(z, B), \tag{1.2}$$

and

$$\mathrm{pr}(\lambda z + (1 - \lambda)x, B) = x, \qquad \lambda \in (0, 1].$$

In particular, $\mathrm{dist}(\cdot, B)$ is differentiable at any point of the form $\lambda z + (1 - \lambda)x$ for $\lambda \in (0, 1]$.

For any $\rho > 0$ we write[(1)]

$$\begin{aligned} B_\rho^+ &:= \{z \in \mathbb{R}^n : \mathrm{dist}(z, B) < \rho\}, \\ B_\rho^- &:= \{z \in \mathbb{R}^n : \mathrm{dist}(z, \mathbb{R}^n \setminus B) > \rho\}. \end{aligned} \tag{1.3}$$

We have

$$B_\rho^+ = \left(\overline{B}\right)_\rho^+, \tag{1.4}$$

and, since $\overline{\mathbb{R}^n \setminus B} = \mathbb{R}^n \setminus \mathrm{int}(B)$,

$$B_\rho^- = (\mathrm{int}(B))_\rho^-, \tag{1.5}$$

as well.

Signed distance function from the boundary of a set

Given a set $B \subseteq \mathbb{R}^n$, we let

$$d(z, B) := \mathrm{dist}(z, B) - \mathrm{dist}(z, \mathbb{R}^n \setminus B), \qquad z \in \mathbb{R}^n, \tag{1.6}$$

be the signed distance function from ∂B, negative in the interior of B. Note that $d(z, \emptyset) = +\infty$ for any $z \in \mathbb{R}^n$, $d(z, \mathbb{R}^n) = -\infty$ for any $z \in \mathbb{R}^n$, and

$$d(z, B) = -d(z, \mathbb{R}^n \setminus B), \qquad z \in \mathbb{R}^n.$$

In Figure 1.1 we draw the graphs of the distance function and of the signed distance function when $B \subset \mathbb{R}$ is an interval.

---

(1) Note that $\emptyset_\rho^+ = \emptyset$ for any $\rho > 0$.

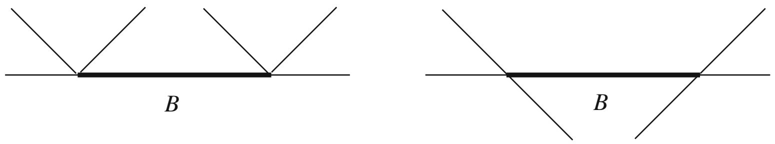

Figure 1.1. Graph of the distance function from ∂B when B (in bold) is an interval (left). On the right, graph of the signed distance function from ∂B negative in the interior of B.

1.1. Smoothness and eikonal equation

In this section we define smoothness of a boundary using the signed distance function. We also recall some of the main properties of the gradient of this function, in particular the eikonal equation in a neighbourhood of its zero level set.

Definition 1.2 (Smoothness). Let $E \subset \mathbb{R}^n$. We say that ∂E has smooth boundary, and we write $\partial E \in \mathcal{C}^\infty$, if there exists an open set U containing ∂E such that

$$d(\cdot, E) \in \mathcal{C}^\infty(\mathrm{U}).$$

When $\partial E \in \mathcal{C}^\infty$ is compact we can choose, if necessary, U of the form $\mathrm{U} = (\partial E)^+_\rho$ for some $\rho > 0$; that is, U is the ρ-tubular (open) neighbourhood of ∂E.

Remark 1.3. It is possible to show[2] that $\partial E \in \mathcal{C}^\infty$ if and only if ∂E is an $(n-1)$-dimensional manifold of class $\mathcal{C}^\infty$.

Notation. When there is no ambiguity in the choice of the set E, for simplicity we use the notation

$$\mathrm{pr}(\cdot) = \mathrm{pr}(\cdot, \partial E),$$

and

$$d(\cdot) = d(\cdot, E).$$

Moreover, as far as possible, we use the symbols z for a generic point in $\mathbb{R}^n$ and x for a point on ∂E.

Example 1.4 (Sphere, I). Let $\rho > 0$ and $E = \overline{B_\rho(z_0)}$. Then

$$d(z) = |z - z_0| - \rho,$$

[2] See, *e.g.*, [15, Theorem 2 statement (i) and Theorem 9]. In a similar way to Definition 1.2, if $k \geq 2$ is an integer, we write $\partial E \in \mathcal{C}^k$ if there exists an open set U containing ∂E such that $d \in \mathcal{C}^k(\mathrm{U})$. Then $\partial E \in \mathcal{C}^k$ if and only if ∂E is an $(n-1)$-dimensional manifold of class $\mathcal{C}^k$ (see [114, Section 5.4], [115, Theorems 5.1, 5.2] and [273, Section 11, Proposition 13.8]).

and for $z \in \mathrm{U} = \{z \in \mathbb{R}^n : z \neq z_0\}$ we have

$$\nabla d(z) = \frac{z - z_0}{|z - z_0|}.$$

When $\partial E \in \mathcal{C}^\infty$, we denote by $T_x(\partial E) \subset \mathbb{R}^n$ the tangent space to ∂E at $x \in \partial E$, and by $N_x(\partial E) \subset \mathbb{R}^n$ the normal line to ∂E at x. Occasionally, to simplify notation we write

$$\Sigma = \partial E.$$

The proof of the following theorem can be found, for instance, in [15] (see also [156, Appendix B]).

Theorem 1.5 (Eikonal equation and projection). *Let $\partial E \in \mathcal{C}^\infty$ and let $\mathrm{U} \subset \mathbb{R}^n$ be a neighbourhood of ∂E so that $d \in \mathcal{C}^\infty(\mathrm{U})$. Then*

(i) *d satisfies the eikonal equation in* U:

$$|\nabla d(z)|^2 = 1, \qquad z \in \mathrm{U}, \tag{1.7}$$

so that, in particular, ∇d coincides, on ∂E, with the unit normal vector field to ∂E pointing toward the interior of $\mathbb{R}^n \setminus E$.

(ii) *For any $z \in \mathrm{U}$ the set* $\mathrm{pr}(z)$ *is a singleton; precisely*

$$\mathrm{pr}(z) = z - d(z)\nabla d(z). \tag{1.8}$$

Moreover

$$\nabla d(z) = \nabla d(\mathrm{pr}(z)). \tag{1.9}$$

From equality (1.8) it follows that the ij-component of the Jacobian matrix of the map pr equals

$$\nabla_j \mathrm{pr}_i = \mathrm{Id}_{ij} - \nabla_i d \nabla_j d - d\nabla^2_{ij} d = \nabla_i \mathrm{pr}_j \qquad \text{in } \mathrm{U}, \tag{1.10}$$

where Id_{ij} is equal to one if $i = j$ and equal to zero if $i \neq j$. In particular, denoting by

$$P_{T_x(\partial E)} : \mathbb{R}^n \to \mathbb{R}^n$$

the linear map given by the orthogonal projection onto the tangent space $T_x(\partial E)$, we have

$$\mathrm{dpr}(x) = P_{T_x(\partial E)}, \qquad x \in \partial E. \tag{1.11}$$

The following operator (see [210, 208, 156, 211]) applies to functions defined on the whole of U; it can be used to define the tangential derivatives (see Section 1.4). For notational simplicity we omit the dependence of δ on ∂E.

Definition 1.6 (The operator δ). Let $\partial E \in \mathcal{C}^\infty$ and let $\mathrm{U} \subset \mathbb{R}^n$ be a neighbourhood of ∂E so that $d \in \mathcal{C}^\infty(\mathrm{U})$. Let $\mathrm{u} \in \mathcal{C}^\infty(\mathrm{U})$. We define

$$\delta \mathrm{u} := \nabla \mathrm{u} - \langle \nabla \mathrm{u}, \nabla d \rangle \nabla d = (\delta_1 \mathrm{u}, \ldots, \delta_n \mathrm{u}) \qquad \text{in U;} \tag{1.12}$$

namely, for any $i \in \{1, \ldots, n\}$,

$$\delta_i \mathrm{u} = \nabla_i \mathrm{u} - \nabla_k \mathrm{u} \nabla_k d \nabla_i d \quad \text{in U.}$$

We also set

$$|\delta \mathrm{u}|^2 := \sum_{i=1}^n |\delta_i \mathrm{u}|^2 \quad \text{in U.}$$

For a vector field $\mathrm{X} = (\mathrm{X}_1, \ldots, \mathrm{X}_n) \in \mathcal{C}^\infty(\mathrm{U}; \mathbb{R}^n)$, the δ-divergence of X is defined by

$$\delta_h \mathrm{X}_h := \nabla_h \mathrm{X}_h - \nabla_i d \; \nabla_j d \; \nabla_i \mathrm{X}_j \qquad \text{in U,} \tag{1.13}$$

and the δ-laplacian of u is defined by

$$\delta_h \delta_h \mathrm{u} \qquad \text{in U.}$$

It directly follows from the definition that

$$\langle \delta \mathrm{u}, \nabla d \rangle = 0 \qquad \text{in U.}$$

In particular, the restriction of $\delta \mathrm{u}$ to ∂E is tangential.
It is possible to prove the following commutation rule, valid at all points of U:

$$\delta_h \delta_k - \delta_k \delta_h = (\nabla_h d \; \delta_k \nabla_i d - \nabla_k d \; \delta_h \nabla_i d)\, \delta_i, \quad h, k \in \{1, \ldots, n\}, \tag{1.14}$$

see, *e.g.*, [208, page 7].

Example 1.7 (δ-derivative of the coordinate functions). The orthogonal projection onto the tangent space to ∂E can be obtained by δ-differentiating the identity map. Indeed, for any $k \in \{1, \ldots, n\}$, consider the k-th coordinate function $\pi_k : \mathbb{R}^n \to \mathbb{R}$, defined by

$$\pi_k(z) := \langle z, e_k \rangle = z_k, \qquad z \in \mathbb{R}^n.$$

We can write the identity map $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\mathrm{id} = (\pi_1, \ldots, \pi_n) \qquad \text{in } \mathbb{R}^n. \tag{1.15}$$

Then, for each $i, j \in \{1, \ldots, n\}$, we have

$$\delta_i \pi_j = \mathrm{Id}_{ij} - \nabla_i d \nabla_j d \qquad \text{in U,} \tag{1.16}$$

and in particular

$$\delta_i \pi_j(x) = P_{T_x(\partial E)_{ij}}, \qquad x \in \partial E.$$

1.2. Mean curvature vector and second fundamental form

Let us now define the mean curvature and the second fundamental form of ∂E using the signed distance function d.

Let $\partial E \in \mathcal{C}^\infty$ and let U be a neighbourhood of ∂E so that $d \in \mathcal{C}^\infty(\mathrm{U})$. Differentiating the eikonal equation (1.7) with respect to z_i it follows that

$$\nabla^2_{ik} d \, \nabla_k d = 0 \qquad \text{in U}, \quad i \in \{1, \ldots, n\}, \tag{1.17}$$

or equivalently

$$\nabla d(z) \in \ker \left(\nabla^2 d(z)\right), \qquad z \in \mathrm{U}. \tag{1.18}$$

Hence $\nabla d(z)$ is a zero eigenvector of the symmetric matrix $\nabla^2 d(z)$. Therefore, given $z \in \mathrm{U}$, it is possible to choose an orthonormal basis of $\mathbb{R}^n$ which diagonalizes $\nabla^2 d(z)$ and the last vector of which is $\nabla d(z)$.

From (1.17) it follows that

$$\nabla^2_{ij} d = \delta_i \nabla_j d \qquad \text{in U}, \tag{1.19}$$

and in particular the matrix $(\delta_i \nabla_j d)_{ij}$ is symmetric and

$$\Delta d = \delta_h \nabla_h d \qquad \text{in U}. \tag{1.20}$$

Definition 1.8 (Mean curvature). Let $\partial E \in \mathcal{C}^\infty$ and $x \in \partial E$. The mean curvature vector of ∂E at x is defined as

$$\Delta d(x) \nabla d(x).$$

The mean curvature of ∂E at x is defined as

$$\Delta d(x).$$

The mean curvature vector is invariant under the substitution of E with $\mathbb{R}^n \setminus E$ in (1.6), while the mean curvature[3] changes sign. Moreover Δd is positive for a smooth uniformly convex set E (compare Example 1.13 below), so that in this case $\Delta d \nabla d$ points toward the interior of $\mathbb{R}^n \setminus E$.

Example 1.9 (Intervals). Let $n=1$, $E \subset \mathbb{R}$ be a finite union of nonempty disjoint closed intervals. Then d is linear around the boundary points of the intervals, and hence $\Delta d = 0$ on ∂E (see Figure 1.1).

(3) As a consequence of (1.22), adopting Definition 1.8 implies that the mean curvature is the sum of the principal curvatures of ∂E, and not (as is usual in differential geometry) the sum of the principal curvatures of ∂E divided by $(n-1)$.

Definition 1.10 (Second fundamental form). Let $\partial E \in \mathcal{C}^\infty$ and $x \in \partial E$. The second fundamental form of ∂E at x is given by the symmetric bilinear map taking $\mathbb{R}^n \times \mathbb{R}^n$ into $\mathbb{R}^n$ defined as follows:

$$\langle \nabla^2 d(x) v, w \rangle \nabla d(x), \qquad (v, w) \in \mathbb{R}^n \times \mathbb{R}^n.$$

In the canonical bases, the components of the second fundamental form at x are therefore

$$\nabla^2_{ij} d(x) \nabla_k d(x), \qquad i, j \in \{1, \dots, n\}, \; k \in \{1, \dots, n\}. \tag{1.21}$$

In particular, the k-th component of the mean curvature vector is obtained by taking the trace in (1.21) with respect to the indices i and j.

Remark 1.11. Let $x \in \partial E$ and $v, w \in \mathbb{R}^n$. Write

$$v = v^\top + v^\perp, \qquad w = w^\top + w^\perp,$$

where

$$v^\top := P_{T_x(\partial E)} v, \qquad w^\top := P_{T_x(\partial E)} w$$

are the tangential components of v and w respectively, and

$$v^\perp := v - v^\top, \qquad w^\perp := w - w^\top$$

are the normal components. Recalling (1.18), we have

$$\begin{aligned} \langle \nabla^2 d(x) v, w \rangle &= \langle \nabla^2 d(x) v^\top, w \rangle = \langle v^\top, \nabla^2 d(x) w \rangle \\ &= \langle v^\top, \nabla^2 d(x) w^\top \rangle = \langle \nabla^2 d(x) v^\top, w^\top \rangle. \end{aligned}$$

Therefore, the second fundamental form at x is determined by the symmetric bilinear map taking $T_x(\partial E) \times T_x(\partial E)$ into $N_x(\partial E)$, and given by

$$\langle \nabla^2 d(x) \tau_1, \tau_2 \rangle \nabla d(x), \qquad \tau_1, \tau_2 \in T_x(\partial E).$$

We denote by

$$\kappa_1(x), \dots, \kappa_n(x)$$

the eigenvalues of $\nabla^2 d(x)$. If we take $\nabla d(x)$ as the last eigenvector, from (1.18) we have $\kappa_n(x) = 0$, and

$$\Delta d(x) = \sum_{i=1}^{n-1} \kappa_i(x). \tag{1.22}$$

The numbers $\kappa_1(x), \dots, \kappa_{n-1}(x)$ are usually called the principal curvatures of ∂E at x.

We define

$$|\nabla^2 d|^2 := \nabla^2_{ij} d \; \nabla^2_{ji} d = \mathrm{tr}\left(\nabla^2 d \; \nabla^2 d\right).$$

Then

$$|\nabla^2 d(x)|^2 = \sum_{i=1}^{n-1} (\kappa_i(x))^2.$$

Example 1.12 (Embedded plane curves). Let $n = 2$, $\partial E \in \mathcal{C}^\infty$, and $i, j \in \{1, 2\}$. Then

$$\nabla^2_{ij} d = \Delta d (\nabla d^R)_i (\nabla d^R)_j \qquad \text{on } \partial E,$$

where ∇d^R is the $\pi/2$-counterclockwise rotation of ∇d[(4)].

Example 1.13 (Sphere, II). In the case of the sphere considered in Example 1.4 we have, for $z \neq z_0$,

$$\Delta d(z) = \frac{n-1}{|z - z_0|}$$

and

$$\nabla^2_{ij} d(z) = \frac{1}{|z - z_0|} \left(\mathrm{Id}_{ij} - \frac{(z_i - z_{0i})}{|z - z_0|} \frac{(z_j - z_{0j})}{|z - z_0|} \right),$$

therefore

$$|\nabla^2 d(z)|^2 = \frac{n-1}{|z - z_0|^2}.$$

Observe that, assuming for simplicity $z_0 = 0$,

$$\begin{aligned} \delta \Delta d(z) &= (n-1) \delta \frac{1}{|z|} \\ &= -(n-1) \frac{z}{|z|^3} + (n-1) \langle \frac{z}{|z|^3}, \frac{z}{|z|} \rangle \frac{z}{|z|} = 0, \end{aligned} \qquad z \in \mathrm{U} = \mathbb{R}^n \setminus \{0\}.$$

Remark 1.14 (Squared distance function). The mean curvature vector can be expressed by differentiating three times the squared distance function. Precisely, let

$$\eta := d^2/2.$$

(4) Let us check that $\nabla^2_{11} d = \Delta d ((\nabla d^R)_1)^2$, *i.e.*, $\nabla^2_{11} d = (\nabla^2_{11} d + \nabla^2_{22} d)(\nabla_2 d)^2$. From (1.17) we have $\nabla^2_{11} d \, \nabla_1 d + \nabla^2_{12} d \, \nabla_2 d = 0 = \nabla^2_{21} d \, \nabla_1 d + \nabla^2_{22} d \, \nabla_2 d$. Hence $(\nabla^2_{11} d)(1 - (\nabla_2 d)^2) = \nabla^2_{11} d (\nabla_1 d)^2 = \nabla^2_{22} d (\nabla_2 d)^2$. Let us now check that $\nabla^2_{12} d = \Delta d (\nabla_1 d)^R (\nabla_2 d)^R$, *i.e.*, $\nabla^2_{12} d = -\Delta d \, \nabla_1 d \nabla_2 d$. From (1.17) it follows that $-\Delta d \, \nabla_1 d \, \nabla_2 d = \nabla^2_{12} d (\nabla_2 d)^2 + \nabla^2_{12} d (\nabla_1 d)^2 = \nabla^2_{12} d$.

Then $\eta \in C^\infty(\mathrm{U})$, $\eta = 0$ on ∂E, and

$$\nabla \eta = d\nabla d = 0 \qquad \text{on } \partial E.$$

Moreover, if $i, j \in \{1, \dots, n\}$, we have

$$\nabla^2_{ij}\eta = d\nabla^2_{ij}d + \nabla_i d\nabla_j d \qquad \text{in U},$$

so that

$$\nabla^2_{ij}\eta = \nabla_i d\nabla_j d \qquad \text{on } \partial E.$$

Hence, denoting by

$$P_{N_x(\partial E)} := \mathrm{id} - P_{T_x(\partial E)}$$

the orthogonal projection onto the normal line $N_x(\partial E)$, we have

$$\nabla^2\eta(x) = P_{N_x(\partial E)}.$$

Finally, for $i, j, k \in \{1, \dots, n\}$, we have

$$\nabla^3_{ijk}\eta = d\ \nabla^3_{ijk}d + \nabla_i d\ \nabla^2_{jk}d + \nabla_j d\ \nabla^2_{ik}d + \nabla_k d\ \nabla^2_{ij}d \qquad \text{in U}, \quad (1.23)$$

where we use the notation $\nabla^3_{ijk} = \dfrac{\partial^3}{\partial z_i \partial z_j \partial z_k}$. Therefore, recalling (1.18), we find[(5)]

$$\Delta\nabla\eta = \Delta d\nabla d \qquad \text{on } \partial E. \qquad (1.24)$$

Remark 1.15. (Second derivatives of the projection and of the coordinate functions). Consistently with (1.11), the mean curvature vector can be expressed by differentiating the projection. Indeed, differentiating (1.10) with respect to z_k, we obtain

$$\nabla^2_{ki}\mathrm{pr}_j = -\nabla_i d\ \nabla^2_{jk}d - \nabla_j d\ \nabla^2_{ik}d - \nabla_k d\ \nabla^2_{ij}d - d\ \nabla^3_{ijk}d \qquad \text{in U}.$$

---

(5) The smoothness and the second fundamental form of a manifold with codimension $k \in \{1, \dots, n\}$ can be defined using the squared distance function, as observed for instance in [106], [108] (see also [19, 18, 129, 44], [15, Section 4] and [202]). In particular, if $\Gamma \subset \mathbb{R}^n$ is a k-dimensional embedded compact connected manifold of class $\mathcal{C}^\infty$ without boundary, and $\eta^\Gamma(\cdot) := \frac{1}{2}\mathrm{dist}(\cdot, \Gamma)^2$, then there exists $\rho > 0$ such that, setting $\mathrm{U} := \Gamma^+_\rho$, we have

- $\eta^\Gamma \in \mathcal{C}^\infty(\mathrm{U})$,
- $|\nabla\eta^\Gamma|^2 = 2\eta^\Gamma$ in U,
- if $z \in U$ the point $z - \nabla\eta^\Gamma(z)$ belongs to Γ and is the unique solution of $\min\{|x - z| : x \in \Gamma\}$.

Moreover, for any $x \in \Gamma$, the matrix $\nabla^2\eta^\Gamma(x)$ is the orthogonal projection onto the normal space to Γ at x (and therefore its rank is k), and $-\Delta\nabla\eta^\Gamma$ is, on Γ, the mean curvature vector $\mathbf{H}_\Gamma$ as defined in differential geometry for instance using local parametrizations.

In particular, using (1.17),

$$\Delta \mathrm{pr}(x) = -\Delta d(x) \nabla d(x), \qquad x \in \partial E.$$

The mean curvature vector can be expressed, as well, by twice δ-differentiating the identity map. Indeed, from (1.16) and (1.19) it follows

$$\delta_h \delta_i \pi_j = -\nabla_i d\ \nabla_{jh} d - \nabla_j d\ \nabla_{ih} d \qquad \text{in U.}$$

Therefore, using (1.17),

$$\delta_h \delta_h \mathrm{id} = -\Delta d \nabla d \qquad \text{in U.}$$

It is often useful to express the mean curvature vector looking at ∂E as a level set of a smooth function (which is not necessarily the signed distance function) with nonvanishing gradient.

Example 1.16 (Mean curvature, level sets and graphs). Let $\partial E \in \mathcal{C}^\infty$. If $\mathrm{u} \in \mathcal{C}^\infty(\mathbb{R}^n)$ is a function so that

$$E = \{\mathrm{u} \leq 0\}, \qquad \partial E = \{\mathrm{u} = 0\},$$

and with

$$\nabla \mathrm{u} \text{ never vanishing on } \{\mathrm{u} = 0\},$$

then

$$\nabla d = \frac{\nabla \mathrm{u}}{|\nabla \mathrm{u}|} \qquad \text{on } \{\mathrm{u} = 0\},$$

and

$$P_{T_x(\{\mathrm{u}=0\})_{lm}} = \mathrm{Id}_{lm} - \frac{\nabla_l \mathrm{u}(x)}{|\nabla \mathrm{u}(x)|} \frac{\nabla_m \mathrm{u}(x)}{|\nabla \mathrm{u}(x)|}, \qquad l, m \in \{1, \ldots, n\}, \qquad x \in \{\mathrm{u} = 0\}.$$

The (ij, k)-component (1.21) of the second fundamental form of ∂E at $x \in \partial E$ is

$$\left(P_{T_x(\{\mathrm{u}=0\})} \frac{\nabla^2 \mathrm{u}(x)}{|\nabla \mathrm{u}(x)|} P_{T_x(\{\mathrm{u}=0\})} \right)_{ij} \frac{\nabla_k \mathrm{u}(x)}{|\nabla \mathrm{u}(x)|}. \tag{1.25}$$

The mean curvature vector of $\{\mathrm{u} = 0\}$ equals

$$\mathrm{div} \left(\frac{\nabla \mathrm{u}}{|\nabla \mathrm{u}|} \right) \frac{\nabla \mathrm{u}}{|\nabla \mathrm{u}|}, \tag{1.26}$$

and its mean curvature equals[(6)]

$$\operatorname{div}\left(\frac{\nabla \mathrm{u}}{|\nabla \mathrm{u}|}\right) = \frac{\Delta \mathrm{u}|\nabla \mathrm{u}|^2 - \nabla_i \mathrm{u} \nabla_j \mathrm{u} \nabla^2_{ij} \mathrm{u}}{|\nabla \mathrm{u}|^3}. \tag{1.27}$$

If $\{\mathrm{u} = 0\}$, in a neighbourhood Ω of one of its points, is the graph of a smooth function v defined in an open set $O \subset \mathbb{R}^{n-1}$, namely if, inside Ω, we can write

$$\mathrm{u}(s, z_n) = v(s) - z_n,$$

so that

$$\{\mathrm{u} = 0\} \cap \Omega = \{(s, z_n) \in \Omega \times \mathbb{R} : z_n = v(s)\},$$

and

$$\{\mathrm{u} \leq 0\} \cap \Omega = \{(s, z_n) \in \Omega \times \mathbb{R} : z_n \geq v(s)\}, \tag{1.28}$$

then the unit normal vector field[(7)] to the graph of v pointing toward $\{\mathrm{u} > 0\} \cap \Omega$ at $x = (s, v(s))$ is given by

$$\frac{(\nabla v(s), -1)}{\sqrt{1 + |\nabla v(s)|^2}}, \tag{1.29}$$

where the symbol ∇ in (1.29) denotes the gradient with respect to s, and

$$P_{T_x(\mathrm{graph}(v))_{lm}}$$
$$= \mathrm{Id}_{lm} - \frac{1}{1 + |\nabla v(s)|^2} \begin{cases} \nabla_l v(s) \nabla_m v(s) & l, m \in \{1, \ldots, n-1\}, \\ -\nabla_l v(s) & l \in \{1, \ldots, n-1\}, m = n, \\ -\nabla_m v(s) & l = n, m \in \{1, \ldots, n-1\}, \\ 1 & l = m = n. \end{cases}$$

(6) Note that, if $x \in \{\mathrm{u} = 0\}$,

$$\begin{aligned} \operatorname{div}\left(\frac{\nabla \mathrm{u}}{|\nabla \mathrm{u}|}\right)(x) =& \operatorname{tr}\left(P_{T_x(\{\mathrm{u}=0\})} \frac{\nabla^2 \mathrm{u}(x)}{|\nabla u(x)|} P_{T_x(\{\mathrm{u}=0\})}\right) \\ =& \operatorname{tr}\left(P_{T_x(\{\mathrm{u}=0\})} \frac{\nabla^2 \mathrm{u}(x)}{|\nabla u(x)|}\right), \end{aligned}$$

where the last equality is a consequence of $P_{T_x(\{\mathrm{u}=0\})_{ij}} \frac{\nabla^2_{jk} \mathrm{u}(x)}{|\nabla \mathrm{u}(x|} \frac{\nabla_i \mathrm{u}(x)}{|\nabla \mathrm{u}(x|} \frac{\nabla_k \mathrm{u}(x)}{|\nabla \mathrm{u}(x|} = 0.$

(7) The right-hand side of (1.29) gives an extension of the unit normal vector field on the whole of $\Omega \times \mathbb{R}$ which, at the point $(s, z_n) \in \Omega \times \mathbb{R}$, takes the same value as at the point $(s, v(s))$, namely it is constant along the last direction e_n.

Moreover, the mean curvature vector at $x = (s, v(s))$ equals

$$\operatorname{div}\left(\frac{\nabla v(s)}{\sqrt{1+|\nabla v(s)|^2}}\right)\frac{(\nabla v(s), -1)}{\sqrt{1+|\nabla v(s)|^2}}, \tag{1.30}$$

where the symbol div in (1.30) denotes the divergence with respect to s. In particular, if $s \in O$ is such that $\nabla v(s) = 0$, then the mean curvature of the graph of v at $x = (s, v(s))$ equals

$$\Delta v(s).$$

Example 1.17 (Surfaces of revolution). Let $v \in \mathcal{C}^\infty(\mathbb{R})$ be a function which never vanishes, and set

$$E := \left\{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_2^2 + z_3^2 \le (v(z_1))^2\right\}.$$

The set E is a solid of revolution, the boundary of which is the graph of v, rotated around the z_1-axis. Let us represent ∂E as

$$\partial E = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \mathrm{u}(x_1, x_2, x_3) = 0\},$$

where

$$\mathrm{u}(z) := \frac{1}{2}\Big[z_2^2 + z_3^2 - (v(z_1))^2\Big], \qquad z = (z_1, z_2, z_3) \in \mathbb{R}^3,$$

so that

$$|\nabla \mathrm{u}|^2 = v^2\left(1 + (v')^2\right) > 0 \qquad \text{on } \partial E. \tag{1.31}$$

Direct computations give

$$\nabla^2 \mathrm{u} = \operatorname{diag}\Big(-vv'' - (v')^2, 1, 1\Big) \qquad \text{in } \mathbb{R}^3. \tag{1.32}$$

Thus, using (1.27),

$$\begin{aligned}\operatorname{div}\left(\frac{\nabla \mathrm{u}}{|\nabla \mathrm{u}|}\right) = &\left(v^2(v')^2 + z_2^2 + z_3^2\right)^{-3/2}\Big\{\left(-vv'' - (v')^2 + 2\right)\left(v^2(v')^2 + z_2^2 + z_3^2\right)\\ &- \Big[-v^2(v')^2\left(vv'' + (v')^2\right) + z_2^2 + z_3^2\Big]\Big\}, \quad z \in \mathbb{R}^3.\end{aligned}$$

Hence, if $x = (x_1, x_2, x_3)$ is a point of ∂E, so that $(v(x_1))^2 = x_2^2 + x_3^2$, the mean curvature of ∂E at x equals

$$\begin{aligned}&\left(v^2(v')^2 + v^2\right)^{-3/2} v^2\left[-vv'' + (v')^2 + 1\right]\\ &= \frac{1}{(1+(v')^2)^{1/2}}\left(\frac{-v''}{1+(v')^2} + \frac{1}{v}\right)\\ &= \left(-\frac{v'}{(1+(v')^2)^{1/2}}\right)' + \frac{1}{v(1+(v')^2)^{1/2}},\end{aligned} \tag{1.33}$$

where the right-hand side is evaluated at x_1.

1.3. Expansion of the Hessian of the signed distance

The next result (see, *e.g.*, [15]) describes the expansion of the eigenvalues of $\nabla^2 d$ on the whole of U.

Theorem 1.18 (Expansion of $\nabla^2 d$). *Let $\partial E \in \mathcal{C}^\infty$ and let* U *be a neighbourhood of ∂E so that $d \in \mathcal{C}^\infty(\mathrm{U})$. Let $z \in \mathrm{U}$ and let*

$$x := \mathrm{pr}(z)$$

be the point of ∂E nearest to z. Fix an orthonormal basis $\{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ in which $\nabla^2 d(x)$ is diagonal, such that $v_n = \nabla d(x)$, and

$$\nabla^2 d(x) v_i = \kappa_i(x) v_i, \qquad i \in \{1, \dots, n\}.$$

Then

- $v_n \in \ker(\nabla^2 d(z))$,
- *the basis $\{v_1, \dots, v_n\}$ diagonalizes $\nabla^2 d(z)$,*
- *if we denote by $\mu_i(z)$ the eigenvalue of $\nabla^2 d(z)$ corresponding to v_i for $i \in \{1, \dots, n\}$, then*

$$\mu_i(z) = \frac{\kappa_i(x)}{1 + d(z)\kappa_i(x)}. \tag{1.34}$$

Proof. Differentiating (1.17) with respect to z_j, we deduce

$$\nabla^3_{ijk} d \; \nabla_k d = -\nabla^2_{jk} d \; \nabla^2_{ik} d \qquad \text{in U}, \quad i, j \in \{1, \dots, n\}. \tag{1.35}$$

Define

$$\mathrm{II}(\lambda) := \nabla^2 d \left(x + \lambda \nabla d(x)\right)$$

for $\lambda \in \mathbb{R}$, $|\lambda|$ small enough in such a way that $x + \lambda \nabla d(x) \in \mathrm{U}$. Fix $i, j \in \{1, \dots, n\}$, and consider the ij-th entry $\mathrm{II}_{ij}(\lambda)$ of $\mathrm{II}(\lambda)$. Then, using (1.9) and (1.35), we get

$$\begin{aligned} \mathrm{II}'_{ij}(\lambda) =& \nabla^3_{ijk} d(x + \lambda \nabla d(x)) \nabla_k d(x) \\ =& \nabla^3_{ijk} d(x + \lambda \nabla d(x)) \nabla_k d(x + \lambda \nabla d(x)) \\ =& -\nabla^2_{jk} d(x + \lambda \nabla d(x)) \; \nabla^2_{ik} d(x + \lambda \nabla d(x)) \\ =& -(\mathrm{II}^2(\lambda))_{ij}. \end{aligned}$$

Hence the matrix-valued function II satisfies the differential equation[8]

$$\mathrm{II}' = -\mathrm{II}^2 \tag{1.36}$$

in its domain.

[8] See also [160, page 15].

Observe that

$$\mathrm{II}(0) = \sum_{l=1}^{n} \kappa_l(x) v_l \otimes v_l, \tag{1.37}$$

where if $v, w \in \mathbb{R}^n$, we set

$$(v \otimes w)_{ij} := v_i w_j, \qquad i, j \in \{1, \ldots, n\}.$$

The solution of system (1.36) with the initial condition (1.37) is given by

$$\mathrm{II}(\lambda) = \sum_{l=1}^{n} \frac{\kappa_l(x)}{1 + \lambda\kappa_l(x)} v_l \otimes v_l.$$

From this expression of $\mathrm{II}(\lambda)$, all assertions of the theorem follow. □

Note that if $z \in \mathrm{U}$ is such that $d(z) = \lambda$, then the principal curvatures of $\{d = \lambda\}$ at z are given by $\mu_i(z)$, for $i \in \{1, \ldots, n\}$.

Example 1.19. Let $\rho > 0$, $E = \overline{B_\rho(z_0)}$ and $z \in \mathbb{R}^n \setminus E$. Then $\kappa_i(x) = \frac{1}{\rho} > 0$ for $x = \mathrm{pr}(z)$ and $i \in \{1, \ldots, n-1\}$. Hence $d(z) > 0$ and (1.34) imply

$$\mu_i(z) < \kappa_i(x).$$

On the other hand, if $z \in \mathrm{int}(E) \setminus \{z_0\}$, then $d(z) < 0$, and

$$\mu_i(z) > \kappa_i(x).$$

Remark 1.20. In the statement of Theorem 1.18 all points z in the neighbourhood U satisfy, in particular, $1+d(z)\kappa_i(x) > 0$ for any $i \in \{1, \ldots, n\}$. This condition is satisfied if, for instance, we require

$$|d(z)| < \left(\max\left\{\langle\nabla^2 d(x)\xi, \xi\rangle : x \in \partial E, \xi \in \mathbb{S}^{n-1}\right\}\right)^{-1}. \tag{1.38}$$

The right-hand side of (1.38) does not identify the size of a tubular neighbourhood where d is differentiable. Indeed, there are examples of sets E with $\partial E \in \mathcal{C}^\infty$ and compact, for which, if $\overline{\rho}$ denotes the right-hand side of (1.38), then there exists $z \in (\partial E)^+_{\overline{\rho}}$ such that $\mathrm{pr}(z)$ is not a singleton[(9)].

[(9)] To exhibit one example, for instance when ∂E is an embedded plane curve, it is enough to let two almost flat portions of the curve, distant in the parametrization, sufficiently close to each other in $\mathbb{R}^2$.

Remark 1.21. From (1.34) we deduce that, for any $i \in \{1, \dots, n\}$,

$$\kappa_i(x) = \frac{\mu_i(z)}{1 - d(z)\mu_i(z)}, \qquad x = \mathrm{pr}(z), \tag{1.39}$$

hence

$$\Delta d(x) = \mathrm{tr}\Big(\nabla^2 d(z)\big(\mathrm{Id} - d(z)\nabla^2 d(z)\big)^{-1}\Big), \qquad x = \mathrm{pr}(z). \tag{1.40}$$

This formula will be used in the proof of the short-time existence of a smooth compact mean curvature flow (see Chapter 7).

Remark 1.22. Again from Theorem 1.18, we deduce the following facts.

- For any $i \in \{1, \dots, n\}$ we have $\dfrac{\mu_i(z)}{1 - d(z)\mu_i(z)} \leq \mu_i(z)$ in $\mathrm{U} \cap E$, hence

$$\sum_{i=1}^{n} \frac{\mu_i(z)}{1 - d(z)\mu_i(z)} \leq \sum_{i=1}^{n} \mu_i(z) \qquad \text{in } \mathrm{U} \cap \{d \leq 0\}.$$

  Similarly $\dfrac{\mu_i(z)}{1 - d(z)\mu_i(z)} \geq \mu_i(z)$ in $\mathrm{U} \cap (\mathbb{R}^n \setminus E)$, hence

$$\sum_{i=1}^{n} \frac{\mu_i(z)}{1 - d(z)\mu_i(z)} \geq \sum_{i=1}^{n} \mu_i(z) \qquad \text{in } \mathrm{U} \cap \{d \geq 0\}.$$

- Since, for $r \in \mathbb{R}$, $(1 + \lambda r)^{-1} = 1 - \lambda r + \mathcal{O}(\lambda^2)$ for $\lambda \in \mathbb{R}$ with $1 + \lambda r > 0$, we can expand the Laplacian of d out of ∂E in terms of its values on ∂E, and obtain from (1.34) the following formula:

$$\begin{aligned} \Delta d(z) =& \Delta d(x) - d(z) \sum_{i=1}^{n-1} \big(\kappa_i(x)\big)^2 + \mathcal{O}\big(d(z)^2\big) \\ =& \Delta d(x) - d(z)|\nabla^2 d(x)|^2 + \mathcal{O}\big(d(z)^2\big), \qquad x = \mathrm{pr}(z). \end{aligned} \tag{1.41}$$

This expansion will be useful in Chapter 16 and 17.

1.4. Tangential derivatives

In this section we recall how to define the tangential derivatives for functions and vector fields defined on $\partial E = \Sigma \in \mathcal{C}^\infty$.

When we write $u \in \mathcal{C}^\infty(\partial E)$ (respectively $X \in \mathcal{C}^\infty(\partial E; \mathbb{R}^n)$) we mean that u (respectively X) admits a smooth extension defined on an open neighbourhood of ∂E[10].

Definition 1.23 (Canonical extension of a function). Let $u \in \mathcal{C}^\infty(\partial E)$. The function

$$\overline{u} : \mathrm{U} \to \mathbb{R}, \qquad \overline{u} := u \circ \mathrm{pr} \tag{1.42}$$

is called the canonical extension of u in U.

If $\mathrm{u} \in \mathcal{C}^\infty(\mathrm{U})$, by $\overline{\mathrm{u}} : \mathrm{U} \to \mathbb{R}$ we mean the canonical extension of $\mathrm{u}_{|\partial E}$.

Example 1.24 (Canonical extension of the identity). Let $k \in \{1, \ldots, n\}$ and π_k be as in Example1.7. Then

$$\overline{\pi}_k(z) = z_k - d(z)\nabla_k d(z), \qquad z \in \mathrm{U}.$$

Example 1.25 (Canonical extension of Δd). From (1.40) it follows that

$$\overline{\Delta d}(z) = \mathrm{tr}\Big(\nabla^2 d(z)\big(\mathrm{Id} - d(z)\nabla^2 d(z)\big)^{-1}\Big), \qquad z \in \mathrm{U}. \tag{1.43}$$

Definition 1.26 (Tangential gradient). Let $u \in \mathcal{C}^\infty(\partial E)$. The vector field $\delta\overline{u}$ restricted to ∂E is called the tangential gradient of u on ∂E, and is denoted by

$$\nabla^\Sigma u.$$

It is not difficult to show that the tangential gradient of u coincides on ∂E with the δ operator applied to any smooth extension of u in U[11]: indeed, if u_1 and u_2 are two smooth extensions of u in U, then $\partial E \subseteq \{\mathrm{u}_1 - \mathrm{u}_2 = 0\}$, so that $\nabla(\mathrm{u}_1 - \mathrm{u}_2) = \langle \nabla(\mathrm{u}_1 - \mathrm{u}_2), \nabla d\rangle \nabla d$ on ∂E. Hence $\nabla^\Sigma(\mathrm{u}_1 - \mathrm{u}_2) = 0$ on ∂E.

The canonical extension $\overline{u}$ satisfies the following property:

$$\langle \nabla\overline{u}, \nabla d\rangle = 0 \qquad \text{in U}, \tag{1.44}$$

so that

$$\nabla\overline{u} = \delta\overline{u} \qquad \text{in U}.$$

(10) For clarity of exposition, in this chapter we have tried to distinguish a function defined on ∂E from a function defined in a neighbourhood U of ∂E, using two different symbols (u and u respectively). If no confusion is possible, this distinction of symbols will not be used in the other chapters.

(11) This justifies the notation $\nabla^\Sigma u$, which does not involve any extension of u in U.

Definition 1.27 (Canonical extension of a vector field). Let $X = (X_1, \ldots, X_n) \in \mathcal{C}^\infty(\partial E; \mathbb{R}^n)$ be a smooth vector field. The vector field

$$\overline{X} : \mathrm{U} \to \mathbb{R}^n, \qquad \overline{X} := X \circ \mathrm{pr} = (X_1 \circ \mathrm{pr}, \ldots, X_n \circ \mathrm{pr})$$

is called the canonical extension of X in U.

Definition 1.28 (Tangential divergence). Let $X = (X_1, \ldots, X_n) \in \mathcal{C}^\infty(\partial E; \mathbb{R}^n)$ be a smooth vector field. The scalar quantity $\mathrm{div}\overline{X}$ restricted to ∂E is called the tangential divergence of X on ∂E, and is denoted by

$$\mathrm{div}^\Sigma X.$$

Note that

$$\mathrm{div}^\Sigma X = \langle e_h, \nabla^\Sigma X_h \rangle = \delta_h \overline{X}_h \qquad \text{on } \partial E,$$

and if $X^e = (X_1^e, \ldots, X_n^e) : \mathrm{U} \to \mathbb{R}$ is any smooth extension of X on U, we have

$$\mathrm{div}^\Sigma X = \delta_h X_h^e \qquad \text{on } \partial E.$$

Example 1.29. From (1.9) it follows that the canonical extension of the unit normal vector field to ∂E pointing toward the interior of $\mathbb{R}^n \setminus E$ is given by

$$\nabla d(z), \qquad z \in \mathrm{U}.$$

Hence the definition of tangential divergence implies that

$$\Delta d = \mathrm{div}^\Sigma \nabla d \qquad \text{on } \partial E. \tag{1.45}$$

Therefore the mean curvature vector of ∂E can be written as

$$\mathrm{div}^\Sigma \nabla d \; \nabla d.$$

Definition 1.30 (Tangential Laplacian). Let $u \in \mathcal{C}^\infty(\partial E)$. The tangential Laplacian $\Delta^\Sigma u$ of u on ∂E is defined as

$$\Delta^\Sigma u := \mathrm{div}^\Sigma \left(\nabla^\Sigma u\right) \qquad \text{on } \partial E.$$

The tangential Laplacian can be computed using the canonical extension as follows (see, *e.g.*, [115, Chapter 4, Section 1.3], [247, Proposition 2.68]).

Lemma 1.31. *We have*

$$\Delta^\Sigma u = \Delta \overline{u} \qquad \text{on } \partial E. \tag{1.46}$$

Proof. We claim that, for $i \in \{1, \dots, n\}$, we have

$$\nabla_i \overline{u} = \left(\mathrm{Id}_{ij} - d\nabla^2_{ij} d\right) \overline{\nabla^\Sigma_j u} \qquad \text{in U.} \tag{1.47}$$

Indeed, let $z \in$ U and set $x := \mathrm{pr}(z)$. Taking into account formula (1.44), which says that $\nabla \overline{u}(x)$ is orthogonal to $\nabla d(x) = \nabla d(z)$, and differentiating the equality

$$\overline{u}(z) = \overline{u}(\mathrm{pr}(z)), \qquad z \in \mathrm{U},$$

we find

$$\nabla_i \overline{u}(z) = \left(\mathrm{Id}_{ij} - d(z)\nabla^2_{ij} d(z)\right) \nabla_j \overline{u}(x), \qquad z \in \mathrm{U}. \tag{1.48}$$

Since $\overline{\nabla^\Sigma u}(z) = \nabla^\Sigma u(x) = \nabla \overline{u}(x)$, from (1.48) claim (1.47) follows.

Differentiating (1.47) and taking into account (1.18) it follows

$$\Delta \overline{u} + d \,\mathrm{div} \left(\nabla^2 d \; \overline{\nabla^\Sigma u}\right) = \mathrm{div}\left(\overline{\nabla^\Sigma u}\right) \qquad \text{in U.}$$

From this equality and Definition 1.28, assertion (1.46) follows. □

Example 1.32 (Tangential Laplacian of the identity). The mean curvature vector can be obtained by differentiating tangentially the identity map (1.15). Indeed, if we apply (1.46) to $u = \pi_k$ for a given index $k \in \{1, \dots, n\}$, we find, using equation (1.17),

$$\Delta^\Sigma \pi_k = \Delta \overline{\pi_k} = -\Delta d \nabla_k d \qquad \text{on } \partial E,$$

and hence

$$\Delta^\Sigma \mathrm{id} = -\Delta d \nabla d \qquad \text{on } \partial E,$$

or also

$$\delta_h \delta_h \mathrm{id} = -\Delta d \nabla d \qquad \text{on } \partial E.$$

1.5. Appendix: second fundamental form using parametrizations

Throughout the book we will occasionally use a parametric description of the flowing manifolds. Therefore, in this appendix, we recall the definition of mean curvature vector and second fundamental form of a parametric hypersurface. We refer the reader to [255, 123, 150] for these arguments.

Notation. We denote by $\mathcal{S} \subset \mathbb{R}^n$ an $(n-1)$-dimensional oriented connected $\mathcal{C}^\infty$ submanifold without boundary.

Definition 1.33 (Immersions and embeddings). Let $\varphi \in \mathcal{C}^\infty(\mathcal{S}; \mathbb{R}^n)$. We say that φ is an immersion of $\mathcal{S}$ in $\mathbb{R}^n$, and we write

$$\varphi \in \mathrm{Imm}(\mathcal{S}; \mathbb{R}^n),$$

if φ is proper and, for any $s \in \mathcal{S}$, the differential $d\varphi(s)$ is injective. If in addition φ is a homeomorphism(12) between $\mathcal{S}$ and $\varphi(\mathcal{S})$, then we say that φ is an embedding of $\mathcal{S}$ in $\mathbb{R}^n$, and we write

$$\varphi \in \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n).$$

Unless a few exceptions, in this book we will be concerned with a compact submanifold $\mathcal{S}$(13).

Let $\varphi \in \mathcal{C}^\infty(\mathcal{S}; \mathbb{R}^n)$. Considering local coordinates around $s \in \mathcal{S}$, given by a parametrization c taking some open neighbourhood O of an $(n-1)$-dimensional Euclidean vector space into $\mathcal{S}$, we write

$$\widehat{\varphi} := \varphi \circ \mathrm{c} : O \to \mathbb{R}^n.$$

Let $\varphi \in \mathrm{Imm}(\mathcal{S}; \mathbb{R}^n)$, and let $s_1, \ldots, s_{n-1}$ be local coordinates on $\mathcal{S}$ around $s \in \mathcal{S}$, so that $\widehat{\varphi}$ is the expression of φ in the local system of coordinates $s_1, \ldots, s_{n-1}$. We denote by $g = (g_{\alpha\beta})$ the Riemannian metric tensor induced on $\mathcal{S}$ by the scalar product of $\mathbb{R}^n$, that is,

$$g(s) = g_{\alpha\beta}(s) := \langle \frac{\partial \widehat{\varphi}}{\partial s_\alpha}(s_1, \ldots, s_{n-1}), \frac{\partial \widehat{\varphi}}{\partial s_\beta}(s_1, \ldots, s_{n-1}) \rangle, \quad s = \mathrm{c}(s_1, \ldots, s_{n-1}). \tag{1.49}$$

By $g^{\alpha\beta}(s)$ we denote the $\alpha\beta$-component of the inverse matrix of $(g_{\alpha\beta}(s))$. Since $\varphi \in \mathrm{Imm}(\mathcal{S}; \mathbb{R}^n)$ is locally an embedding, we can choose, for $\sigma \in \mathcal{S}$ belonging to a suitable neighbourhood of a point $s \in \mathcal{S}$, a smooth unit normal vector field to the corresponding image point $\varphi(\sigma)$, that will be denoted by $\nu(\sigma)$.

If $\alpha, \beta \in \{1, \ldots, n-1\}$ we define

$$h_{\alpha\beta}(s) := \langle \nu(s), \frac{\partial^2 \widehat{\varphi}}{\partial s_\alpha \partial s_\beta}(s_1, \ldots, s_{n-1}) \rangle, \qquad s = \mathrm{c}(s_1, \ldots, s_{n-1}), \tag{1.50}$$

(12) The set $\varphi(\mathcal{S}) \subset \mathbb{R}^n$ is considered endowed with the induced topology; see [157] for more.

(13) In this case the assumption that φ is proper is automatically satisfied.

and[14]

$$\begin{aligned} A(s) &:= \big(h_{\alpha\beta}(s)\ \nu(s)\big), \\ \mathrm{H}(s) &:= g^{\alpha\beta}(s) h_{\alpha\beta}(s), \qquad\qquad s = \mathrm{c}(s_1, \dots, s_{n-1}). \\ \mathbf{H}(s) &:= \mathrm{H}(s)\nu(s), \end{aligned} \tag{1.51}$$

Observe that $h_{\alpha\beta}\nu$ and $\mathbf{H}$ are invariant under a change of sign of ν, while $h_{\alpha\beta}$ and H change sign.

Remark 1.34. Let $u \in \mathcal{C}^\infty(\mathcal{S})$, and set locally $\mathcal{U} := u \circ \mathrm{c}$. Define

$$\mathrm{grad}\, u = (\mathrm{grad}\, u)^\gamma \frac{\partial \widehat{\varphi}}{\partial s_\gamma},$$

where

$$(\mathrm{grad}\, u)^\alpha := g^{\alpha\beta} \frac{\partial \mathcal{U}}{\partial s_\beta}, \qquad \alpha \in \{1, \dots, n-1\}.$$

Set also

$$\Delta_g u := \frac{1}{\sqrt{\det(g_{\alpha\beta})}} \frac{\partial}{\partial s_\alpha} \left(\sqrt{\det(g_{\alpha\beta})}\ (\mathrm{grad}\, u)^\alpha \right),$$

which can be written as

$$\Delta_g u = \frac{\partial (\mathrm{grad}\, u)^\alpha}{\partial s_\alpha} + \Gamma^\beta_{\alpha\beta} (\mathrm{grad}\, u)^\alpha,$$

where $\Gamma^\beta_{\alpha\beta} := \frac{1}{\sqrt{\det(g_{\alpha\beta})}} \frac{\partial}{\partial s_\alpha} \sqrt{\det(g_{\alpha\beta})}$.
Then it turns out (see, *e.g.*, [204, page 4]) that

$$\mathbf{H} = \Delta_g \varphi = (\Delta_g \varphi_1, \dots, \Delta_g \varphi_n),$$

where $g_{\alpha\beta}$ and $g^{\alpha\beta}$ are evaluated at $s = \mathrm{c}(s_1, \dots, s_{n-1})$, and $\widehat{\varphi}$ is evaluated at $(s_1, \dots, s_{n-1})$.
Convention. If $\varphi \in \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n)$ and $\partial E = \varphi(\mathcal{S})$, we will choose ν so that

$$-\nu(s) = \nabla d(x), \qquad x = \varphi(s). \tag{1.52}$$

It is then possible to prove[15] that

$$\begin{aligned} \mathbf{H}(s) &= -\,\Delta d(x) \nabla d(x), \\ \mathrm{H}(s) &= \Delta d(x), \end{aligned} \qquad x = \varphi(s). \tag{1.53}$$

[14] In formula (1.51) the symbol A denotes the second fundamental form as usually defined in differential geometry. The notation on the right-hand side means that the $\alpha\beta$-component of $A(s)$ is given by $h_{\alpha\beta}(s)\nu(s)$.

[15] See, *e.g.*, [19, 18, 15].

Example 1.35. Let $n = 2$ and $\partial E = \gamma(\mathbb{S}^1)$, where $\mathbb{S}^1 \subset \mathbb{R}^2$ is the unit circle and $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ is a smooth embedding. Let $s \in \mathbb{S}^1$; the (mean) curvature vector of ∂E at $x = \gamma(s)$ is given by

$$\frac{1}{|\gamma'(s)|^2}\left(\gamma''(s) - \langle \gamma''(s), \frac{\gamma'(s)}{|\gamma'(s)|}\rangle \frac{\gamma'(s)}{|\gamma'(s)|}\right).$$

Chapter 2
First variations

In this chapter we compute the first variation of some integral functionals. The result of these computations shows, in particular, the connection between mean curvature flow and the first variation of area.

Notation. We let $V \in \mathcal{C}^\infty(\mathbb{R}^n; \mathbb{R}^n)$ be a compactly supported vector field, and we define $\Psi \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^n; \mathbb{R}^n)$ as

$$\Psi(\lambda, z) := z + \lambda V(z), \qquad \lambda \in \mathbb{R}, \ z \in \mathbb{R}^n.$$

We also let

$$\Psi_\lambda(z) := \Psi(\lambda, z), \qquad \lambda \in \mathbb{R}, \ z \in \mathbb{R}^n. \tag{2.1}$$

Given a set $E \subset \mathbb{R}^n$ with $\partial E \in \mathcal{C}^\infty$, we define

$$E_\lambda := \Psi_\lambda(E).$$

For $|\lambda|$ small enough the map Ψ_λ is a diffeomorphism of $\mathbb{R}^n$ of class $\mathcal{C}^\infty$ smoothly varying with λ, and is a compact perturbation of the identity. Therefore E_λ differs from E only in a compact subset of $\mathbb{R}^n$[(1)].

We need to recall the change of variables formula[(2)] in $\mathbb{R}^n$.

Theorem 2.1 (Change of variables). *Let $\vartheta : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz injective map. Let $B \subseteq \mathbb{R}^n$ be a Borel set and $g \in L^1(\mathbb{R}^n)$. Then*

$$\int_{\vartheta(B)} g(y)\, dy = \int_B g(\vartheta(z)) |\det(\nabla\vartheta(z))|\, dz.$$

(1) The computations of this chapter still hold if one starts from a map $\Psi \in \mathcal{C}^\infty(\mathbb{R}\times\mathbb{R}^n; \mathbb{R}^n)$, defines Ψ_λ as in (2.1), assumes that $\Psi_0 = \mathrm{id}$ and that $\Psi_\lambda - \mathrm{id}$ has compact support for any $\lambda \in \mathbb{R}$, and then defines $V := \frac{\partial \Psi_\lambda}{\partial \lambda}_{|\lambda=0}$.

(2) See, *e.g.*, [16].

2.1. First variation of volume integrals

We begin by considering the first variation of volume integrals.

Proposition 2.2. *Let* $\partial E \in \mathcal{C}^\infty$, $g \in \mathrm{Lip}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ *and*

$$\mathcal{G}(E) := \int_E g \, dz.$$

Then

$$\left(\frac{d}{d\lambda}\mathcal{G}(E_\lambda)\right)_{|\lambda=0} = \int_{\partial E} g \, \langle \nabla d, V \rangle \, d\mathcal{H}^{n-1}. \tag{2.2}$$

Proof. The change of variables formula implies that

$$\int_{E_\lambda} g \, dy = \int_E g(\Psi_\lambda) |\det(\nabla \Psi_\lambda)| \, dz,$$

for $|\lambda|$ sufficiently small. Therefore

$$\begin{aligned} \frac{d}{d\lambda} \int_{E_\lambda} g \, dy &= \int_E \frac{\partial}{\partial \lambda} \Big(g(\Psi_\lambda) |\det(\nabla \Psi_\lambda)| \Big) \, dz \\ &= \int_E \bigg(\langle \nabla g(\Psi_\lambda), \frac{\partial \Psi_\lambda}{\partial \lambda} \rangle |\det(\nabla \Psi_\lambda)| \\ &\qquad + g(\Psi_\lambda) \frac{\partial}{\partial \lambda} |\det(\nabla \Psi_\lambda)| \bigg) \, dz. \end{aligned} \tag{2.3}$$

Now, we recall that, if M is an $(n \times n)$-matrix and if $\mathrm{tr}(M)$ denotes its trace, then[3]

$$\det(\mathrm{Id} + \lambda M) = 1 + \lambda \mathrm{tr}(M) + \mathfrak{o}(\lambda).$$

Hence

$$\left(\frac{\partial}{\partial \lambda} \det(\nabla \Psi_\lambda)\right)_{|\lambda=0} = \mathrm{div} V \qquad \text{in } \mathbb{R}^n.$$

Since $\det(\nabla \Psi_\lambda)_{|\lambda=0} = 1$, it follows that

$$\frac{\partial}{\partial \lambda} \, |\det(\nabla \Psi_\lambda)|_{|\lambda=0} = \mathrm{div} V \qquad \text{in } \mathbb{R}^n. \tag{2.4}$$

Using (2.4), from (2.3) we deduce

$$\left(\frac{d}{d\lambda}\mathcal{G}(E_\lambda)\right)_{|\lambda=0} = \int_E \big(\langle \nabla g, V \rangle + g \, \mathrm{div} V \big) \, dz = \int_E \mathrm{div}(gV) \, dz,$$

and (2.2) follows from the divergence theorem. □

(3) See [242, page 50].

2.2. First variation of the perimeter

Let $\partial E \in \mathcal{C}^\infty$ be compact. Denote by $P(E)$ the perimeter of E in $\mathbb{R}^n$ (4).

In this section we compute the first variation of the perimeter functional, with respect to smooth compact perturbations of the identity. To this purpose, we preliminarily recall the coarea formula(5).

Theorem 2.3 (Coarea formula). *Let* $u \in \mathrm{Lip}\,(\mathbb{R}^n)$ *be such that* ess-inf $|\nabla u| > 0$, *let* $g \in L^1(\mathbb{R}^n)$ *and* $\mu \in \mathbb{R}$. *Then*

$$\int_{\{u>\mu\}} g\, dz = \int_{\mu}^{+\infty} \left(\int_{\{u=\sigma\}} \frac{g}{|\nabla u|}\, d\mathcal{H}^{n-1} \right) d\sigma. \tag{2.5}$$

Remembering the definition of the operator δ given in Definition 1.6 of Chapter 1, a preliminary step in the computation of the first variation of the perimeter(6) is the following.

(4) The perimeter $P(E, \mathbb{R}^n)$ of E in $\mathbb{R}^n$, denoted for simplicity by $P(E)$, is defined (see [98, 99], and also [212] and [29]) as

$$P(E) := \lim_{t \to 0^+} \int_{\mathbb{R}^n} \left| \nabla_y \int_E \rho(t, y-z)\, dz \right| dy,$$

where $\rho(t,\zeta) := \frac{1}{(4\pi t)^{\frac{n}{2}}} e^{-\frac{|\zeta|^2}{4t}}$ for $(t,\zeta) \in (0,+\infty) \times \mathbb{R}^n$. Alternatively, and in the general case when $\Omega \subseteq \mathbb{R}^n$ is an open set, the perimeter $P(E,\Omega)$ of E in Ω is defined by

$$P(E,\Omega) := \sup \left\{ \int_{\Omega \cap E} \mathrm{div} X\, dz : X \in \mathcal{C}^1_c(\Omega; \mathbb{R}^n), \sum_{i=1}^n (X_i(z))^2 \le 1 \right\}$$

(see, *e.g.*, [156] and [16] and references therein). A Borel set E is said of finite perimeter in Ω if $P(E,\Omega) < +\infty$; it is said of locally finite perimeter if $P(E, A) < +\infty$ for any bounded open set $A \subset \mathbb{R}^n$. It is well known that $P(E) = P(\mathbb{R}^n \setminus E)$ and that, if for instance E is an open set with smooth compact boundary, then

$$P(E,\Omega) = P(\overline{E},\Omega) = \mathcal{H}^{n-1}(\Omega \cap \partial E),$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. Namely, given $S \subseteq \mathbb{R}^n$,

$$\mathcal{H}^{n-1}(S) = c_n \lim_{\rho \downarrow 0} \inf \left\{ \sum_{i=1}^{+\infty} (\mathrm{diam}(S_i))^{n-1} : S \subseteq \bigcup_{i=1}^{+\infty} S_i,\ \mathrm{diam}(S_i) < \rho \right\},$$

where c_n is an explicit constant depending only on the dimension, and $\mathrm{diam}(B) := \sup\{|z-\zeta| : z, \zeta \in B\}$ for any $B \subseteq \mathbb{R}^n$. The functional $P(\cdot)$ is sometimes called the area functional, where the word area refers to $P(E)$, namely to the area of ∂E.

(5) See, *e.g.*, [135, Proposition 3, page 118].

(6) Theorem 2.4 below is the classical first variation of area: it is proved in [156] with a slightly different method. See also [242], and [198] for the first variation of $P(\cdot,\Omega)$ with respect to perturbations compactly supported in Ω.

Theorem 2.4 (First variation of the perimeter, I). *Let $\partial E \in \mathcal{C}^\infty$ be compact. Then*

$$\left(\frac{d}{d\lambda}P(E_\lambda)\right)_{|\lambda=0} = \int_{\partial E} \delta_h V_h \, d\mathcal{H}^{n-1}. \tag{2.6}$$

Proof. Without loss of generality, we assume that E is closed. Let us choose a function $u \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that

$$E = \{u \leq 0\}, \qquad \text{and} \qquad \nabla u \neq 0 \text{ on } \{u = 0\} = \partial E.$$

For any $\lambda \in \mathbb{R}$, with $|\lambda|$ small enough so that Ψ_λ is a diffeomorphism of $\mathbb{R}^n$, let us define[(7)] the function $v_\lambda : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$v_\lambda(\Psi_\lambda(z)) := u(z), \qquad z \in \mathbb{R}^n, \tag{2.7}$$

i.e., $v_\lambda(y) = u(\Psi_\lambda^{-1}(y))$ for any $y \in \mathbb{R}^n$. Then

$$E_\lambda = \{v_\lambda \leq 0\},$$

and

$$\nabla v_\lambda \neq 0 \text{ on } \{v_\lambda = 0\} = \partial E_\lambda.$$

We claim that

$$P(E_\lambda) = \int_{\partial E} \frac{|\nabla v_\lambda(\Psi_\lambda)||\det(\nabla\Psi_\lambda)|}{|\nabla u|} \, d\mathcal{H}^{n-1}. \tag{2.8}$$

Observing that $\Psi_\lambda\big(\{|u| < \rho\}\big) = \{|v_\lambda| < \rho\}$ for $\rho > 0$ small enough, from the change of variables formula it follows that

$$\int_{\{|v_\lambda|<\rho\}} |\nabla v_\lambda| \, dy = \int_{\{|u|<\rho\}} |\nabla v_\lambda(\Psi_\lambda)||\det(\nabla\Psi_\lambda)| \, dz. \tag{2.9}$$

Moreover, using the coarea formula (2.5) we have

$$\int_{\{|v_\lambda|<\rho\}} |\nabla v_\lambda| \, dy = \int_{-\rho}^{\rho} \mathcal{H}^{n-1}\left(\{v_\lambda = \sigma\}\right) \, d\sigma. \tag{2.10}$$

By (2.9), (2.10) and the smoothness of ∂E_λ it follows that

$$\begin{aligned} &\lim_{\rho\downarrow 0} \frac{1}{2\rho} \int_{\{|u|<\rho\}} |\nabla v_\lambda(\Psi_\lambda)||\det(\nabla\Psi_\lambda)| \, dz \\ &= \lim_{\rho\downarrow 0} \frac{1}{2\rho} \int_{-\rho}^{\rho} \mathcal{H}^{n-1}\left(\{v_\lambda = \sigma\}\right) \, d\sigma \\ &= \mathcal{H}^{n-1}(\partial E_\lambda) = P(E_\lambda). \end{aligned} \tag{2.11}$$

(7) This kind of variation is rather common, for instance it is used in continuum mechanics as well, see for instance [151, page 153].

On the other hand, using again the coarea formula and the properties of u, it follows that

$$\begin{aligned}&\lim_{\rho\downarrow 0}\frac{1}{2\rho}\int_{\{|u|<\rho\}}|\nabla v_\lambda(\Psi_\lambda)||\det(\nabla\Psi_\lambda)|\,dz\\ =&\int_{\partial E}\frac{|\nabla v_\lambda(\Psi_\lambda)||\det(\nabla\Psi_\lambda)|}{|\nabla u|}\,d\mathcal{H}^{n-1}.\end{aligned}\tag{2.12}$$

From (2.11) and (2.12) our claim (2.8) follows.

Differentiating (2.7) with respect to z_j we deduce

$$\frac{\partial u}{\partial z_j}(z)=\frac{\partial v_\lambda}{\partial y_i}(\Psi_\lambda(z))\left(\mathrm{Id}_{ij}+\lambda\frac{\partial V_i}{\partial z_j}(z)\right),\qquad z\in\mathbb{R}^n.\tag{2.13}$$

In particular, $\nabla v_\lambda(\Psi_\lambda(z))=\nabla u(z)$ if $z\in\mathbb{R}^n$ and $\lambda=0$, so that

$$(\nabla v_\lambda(\Psi_\lambda))_{|\lambda=0}=\nabla u\qquad\text{on }\partial E.\tag{2.14}$$

From (2.13) and (2.14) it follows that

$$\left(\frac{\partial}{\partial\lambda}\nabla_i v_\lambda(\Psi_\lambda(z))\right)_{|\lambda=0}=-\nabla_j u(z)\nabla_j V_i(z),\qquad z\in\mathbb{R}^n.$$

Hence

$$\left(\frac{\partial}{\partial\lambda}|\nabla v_\lambda(\Psi_\lambda)|\right)_{|\lambda=0}=-\frac{\nabla_i u}{|\nabla u|}\nabla_j u\nabla_j V_i.\tag{2.15}$$

Using (2.4), (2.14) and (2.15) we get

$$\begin{aligned}\frac{\partial}{\partial\lambda}\left(\frac{|\nabla v_\lambda(\Psi_\lambda)||\det(\nabla\Psi_\lambda)|}{|\nabla u|}\right)_{|\lambda=0}&=\mathrm{div}V+\frac{1}{|\nabla u|}\left(\frac{\partial}{\partial\lambda}|\nabla v_\lambda(\Psi_\lambda)|\right)_{|\lambda=0}\\&=\mathrm{div}V-\frac{\nabla_i u}{|\nabla u|}\frac{\nabla_j u}{|\nabla u|}\nabla_j V_i.\end{aligned}\tag{2.16}$$

Then (2.6) follows from (2.8), (2.16) and the definition (1.13) of $\delta_h V_h$. □

Remark 2.5. Let $X\in\mathcal{C}^\infty(\partial E;\mathbb{R}^n)$ be a compactly supported vector field, and split X as

$$X=X^\top+X^\perp,$$

where

$$X^\top:=X-\langle X,\nabla d\rangle\nabla d$$

is the orthogonal projection of X onto the tangent space to ∂E, and

$$X^\perp = \xi \nabla d, \qquad \xi := \langle X, \nabla d\rangle,$$

is the normal component of X. Recalling the definitions of tangential gradient and of tangential divergence given in Section 1.4, and setting as usual $\Sigma = \partial E$, we have

$$\begin{aligned} \mathrm{div}^\Sigma X^\perp &= \mathrm{div}^\Sigma(\xi\nabla d) = \langle \nabla^\Sigma \xi, \nabla d\rangle + \xi \mathrm{div}^\Sigma \nabla d \\ &= \xi \mathrm{div}^\Sigma \nabla d = \xi \Delta d \qquad \text{on } \partial E, \end{aligned} \tag{2.17}$$

where the last equality follows from (1.45). Therefore

$$\mathrm{div}^\Sigma X = \mathrm{div}^\Sigma (X^\top + X^\perp) = \mathrm{div}^\Sigma X^\top + \Delta d \langle X, \nabla d\rangle \qquad \text{on } \partial E.$$

Note that, from (2.6) and (2.17), it follows that, if

$$V = V^\perp \qquad \text{on } \partial E, \tag{2.18}$$

then

$$\begin{aligned} \left(\frac{d}{d\lambda} P(E_\lambda, \Omega)\right)_{|\lambda=0} &= \int_{\partial E} \delta_h V_h \, d\mathcal{H}^{n-1} \\ = \int_{\Omega\cap\partial E} \mathrm{div}^\Sigma V^\perp \, d\mathcal{H}^{n-1} &= \int_{\partial E} \Delta d \langle \nabla d, V\rangle \, d\mathcal{H}^{n-1}. \end{aligned} \tag{2.19}$$

However, formula (2.19) is valid also without assuming (2.18). To give an intuitive reason, observe that the map

$$V \to \left(\frac{d}{d\lambda} P(E_\lambda)\right)_{|\lambda=0}$$

is linear. Moreover, the set obtained by varying ∂E according to the tangential part of V, is an $\mathfrak{o}(\lambda)$ displacement from ∂E, and the corresponding first variation vanishes. Therefore, one expects that

$$\int_{\partial E} \mathrm{div}^\Sigma V \, d\mathcal{H}^{n-1} = \int_{\partial E} \mathrm{div}^\Sigma V^\perp \, d\mathcal{H}^{n-1}.$$

Indeed, the following classical integration by parts theorem on the manifold ∂E holds (see, *e.g.*, [242, 243, 16]).

Theorem 2.6 (Integration by parts). *Let $\partial E \in \mathcal{C}^\infty$ be compact, and let $X \in \mathcal{C}^\infty(\partial E; \mathbb{R}^n)$ be a smooth vector field. Then*

$$\int_{\partial E} \mathrm{div}^\Sigma X \, d\mathcal{H}^{n-1} = \int_{\partial E} \Delta d \, \langle \nabla d, X\rangle \, d\mathcal{H}^{n-1}. \tag{2.20}$$

Proof. Using the coarea formula and (1.7) we have, for $\rho > 0$ small enough,

$$\int_{(\partial E)^+_\rho} \operatorname{div}\overline{X}\, dz = \int_{-\rho}^{\rho} \int_{\{d=\sigma\}} \operatorname{div}\overline{X}\, d\mathcal{H}^{n-1} d\sigma,$$

where $\overline{X}$ is the canonical extension of X (see Definition 1.27).

Recalling from Definition 1.28 that $\operatorname{div}^\Sigma X = \operatorname{div}\overline{X}$ on ∂E, from the previous equality and the smoothness of ∂E it follows that

$$\int_{\partial E} \operatorname{div}^\Sigma X\, d\mathcal{H}^{n-1} = \lim_{\rho \downarrow 0} \frac{1}{2\rho} \int_{(\partial E)^+_\rho} \operatorname{div}\overline{X}\, dz.$$

Using the divergence theorem we then have

$$\begin{aligned} \int_{\partial E} \operatorname{div}^\Sigma X\, d\mathcal{H}^{n-1} &= \lim_{\rho \downarrow 0} \frac{1}{2\rho} \int_{\partial\left((\partial E)^+_\rho\right)} \langle \mathrm{n}^\rho, \overline{X}\rangle\, d\mathcal{H}^{n-1} \\ &= \lim_{\rho \downarrow 0} \frac{1}{2\rho} \left(\int_{\Gamma^i_\rho} \langle -\nabla d, \overline{X}\rangle\, d\mathcal{H}^{n-1} + \int_{\Gamma^e_\rho} \langle \nabla d, \overline{X}\rangle\, d\mathcal{H}^{n-1} \right), \end{aligned} \tag{2.21}$$

where n^ρ is the outward unit normal to $\partial\left((\partial E)^+_\rho\right)$, and

$$\Gamma^i_\rho := E \cap \partial\left((\partial E)^+_\rho\right), \qquad \Gamma^e_\rho := (\mathbb{R}^n \setminus E) \cap \partial\left((\partial E)^+_\rho\right).$$

With arguments similar to those used in the proof of (2.8)[8] we get, using (2.13) and the fact that $\nabla d(\mathrm{Id} - \rho\nabla^2 d)^{-1} = \nabla d$ on ∂E,

$$\int_{\Gamma^i_\rho} \langle -\nabla d, \overline{X}\rangle\, d\mathcal{H}^{n-1} = \int_{\partial E} \langle -\nabla d, X\rangle |\det(\mathrm{Id} - \rho\nabla^2 d)|\, d\mathcal{H}^{n-1}.$$

Similarly[9],

$$\int_{\Gamma^e_\rho} \langle \nabla d, \overline{X}\rangle\, d\mathcal{H}^{n-1} = \int_{\partial E} \langle \nabla d, X\rangle |\det(\mathrm{Id} + \rho\nabla^2 d)|\, d\mathcal{H}^{n-1}.$$

Therefore

$$\begin{aligned} &\frac{1}{2\rho} \left(\int_{\Gamma^i_\rho} \langle -\nabla d, \overline{X}\rangle\, d\mathcal{H}^{n-1} + \int_{\Gamma^e_\rho} \langle \nabla d, \overline{X}\rangle\, d\mathcal{H}^{n-1} \right) \\ &= \int_{\partial E} \langle \nabla d, X\rangle \left[\frac{|\det(\mathrm{Id} + \rho\nabla^2 d)| - |\det(\mathrm{Id} - \rho\nabla^2 d)|}{2\rho} \right] d\mathcal{H}^{n-1}. \end{aligned} \tag{2.22}$$

---

(8) Applied with the choices $u = d$, $\lambda = \rho > 0$, $V = \nabla d$ in a neighbourhood of ∂E and smoothly extended outside keeping the compactness of the support, and $\Psi_\rho = z - \rho V$.

(9) Take in this case $\Psi_\rho = z + \rho V$.

The assertion then follows from (2.21), (2.22), and the relations

$$\begin{aligned}&\lim_{\rho\downarrow 0}\frac{|\det(\mathrm{Id}+\rho\nabla^2 d)|-|\det(\mathrm{Id}-\rho\nabla^2 d)|}{2\rho}\\ =&\lim_{\rho\downarrow 0}\frac{|1+\rho\Delta d|-|1-\rho\Delta d|}{2\rho}=\Delta d \quad \text{on} \quad \partial E.\end{aligned}$$

□

From Theorem 2.6 it follows that, if $\phi \in \mathcal{C}^\infty(\partial E)$, then

$$\begin{aligned}\int_{\partial E}\phi \,\mathrm{div}^\Sigma X \, d\mathcal{H}^{n-1} = &-\int_{\partial E}\langle\nabla^\Sigma\phi, X\rangle \, d\mathcal{H}^{n-1}\\ &+\int_{\partial E}\phi \,\Delta d\,\langle\nabla d, X\rangle \, d\mathcal{H}^{n-1},\end{aligned}$$

and, if $\psi \in \mathcal{C}^\infty(\partial E)$, also

$$\begin{aligned}\int_{\partial E}\phi \,\Delta^\Sigma\psi \, d\mathcal{H}^{n-1} &= \int_{\partial E}\psi \,\Delta^\Sigma\phi \, d\mathcal{H}^{n-1}\\ &= -\int_{\partial E}\langle\nabla^\Sigma\phi, \nabla^\Sigma\psi\rangle \, d\mathcal{H}^{n-1}.\end{aligned} \tag{2.23}$$

The first variation formula for the perimeter reads as follows.

Theorem 2.7 (First variation of the perimeter, II). *Let $\partial E \in \mathcal{C}^\infty$ be compact. Then*

$$\left(\frac{d}{d\lambda}P(E_\lambda)\right)_{|\lambda=0} = \int_{\partial E}\Delta d\,\langle\nabla d, V\rangle \, d\mathcal{H}^{n-1}. \tag{2.24}$$

Proof. This follows from (2.6) and (2.20). □

We remark once more that what enters in formula (2.24) is only the component of V along the normal direction to ∂E.

Remark 2.8. Formula (2.24) allows us to look at mean curvature flow, at least formally, as the “gradient flow” of the perimeter functional, see Section 2.3 below.

Corollary 2.9. *Let $\partial E \in \mathcal{C}^\infty$ be compact and $g \in \mathrm{Lip}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$. Define*

$$\mathcal{L}(E) := P(E) + \int_E g \, dz. \tag{2.25}$$

Then

$$\left(\frac{d}{d\lambda}\mathcal{L}(E_\lambda)\right)_{|\lambda=0} = \int_{\partial E}(\Delta d + g)\langle\nabla d, V\rangle \, d\mathcal{H}^{n-1}.$$

Proof. This is a consequence of Proposition 2.2 and Theorem 2.7. □

From Corollary 2.9 it follows that sufficiently smooth critical points E of the functional $\mathcal{L}$ satisfy the so-called prescribed mean curvature equation(10):

$$\Delta d = -g \qquad \text{on } \partial E. \tag{2.26}$$

Example 2.10. As a particular case of (2.25), if $F \subseteq \mathbb{R}^n$ is an open set with compact Lipschitz boundary and $\tau > 0$ is given, we can consider the functional

$$\mathcal{A}_\tau(E) := P(E) + \frac{1}{\tau}\int_{(F\setminus E)\cup(E\setminus F)} \operatorname{dist}(z, \partial F)\, dz,$$

where $\partial E \in \mathcal{C}^\infty$ is compact. Notice that, if $z \in F \setminus E$ then $\operatorname{dist}(z, \partial F) = -d(z, F)$, and, if $z \in E \setminus F$, then $\operatorname{dist}(z, \partial F) = d(z, F)$. Hence

$$\mathcal{A}_\tau(E) = P(E) + \frac{1}{\tau}\int_E d(z, F)\, dz + C,$$

where

$$C := -\frac{1}{\tau}\int_F d(z, F)\, dz$$

does not depend on E. Then equation (2.26) reads as

$$\Delta d(x) = -\frac{d(x, F)}{\tau}, \qquad x \in \partial E. \tag{2.27}$$

The functional $\mathcal{A}_\tau$, defined on finite perimeter sets, has been considered by Almgren, Taylor and Wang in [9], as a starting point for the definition of a notion of weak solution to mean curvature flow(11) (see also [196, 105, 14]).

Notation. Let $a(\lambda, z)$ be a smooth real function, defined for λ in a neighbourhood N of 0 and for z in a neighbourhood U of ∂E, and summable on ∂E. We set in $N \times \mathrm{U}$

$$\nabla_i a := \frac{\partial a}{\partial z_i} \quad \text{for } i \in \{1, \ldots, n\}, \qquad \nabla a := (\nabla_1 a, \ldots, \nabla_n a),$$

and recalling the operator δ in Definition 1.6, we define

$$\delta a := \nabla a - \langle \nabla a, \nabla d\rangle \nabla d.$$

(10) See, *e.g.*, [155].

(11) Indeed, the right hand side of (2.27) can be considered as a sort of approximation of the inner normal velocity.

The next result can be considered as a starting point for the definition of a notion of weak solution to mean curvature flow in a distributional sense; see [67].

Corollary 2.11. *We have*

$$\begin{aligned} \left(\frac{d}{d\lambda}\int_{\partial E_\lambda} a \, d\mathcal{H}^{n-1}\right)_{|\lambda=0} &= \int_{\partial E}\left(\frac{da}{d\lambda} + a\,\delta_h V_h\right) d\mathcal{H}^{n-1} \\ = \int_{\partial E}\left(\frac{\partial a}{\partial\lambda} + a\,\Delta d\,\langle\nabla d, V^\perp\rangle + \langle\nabla a - \delta a, V^\perp\rangle\right) &d\mathcal{H}^{n-1}, \end{aligned} \tag{2.28}$$

where

$$\frac{da}{d\lambda} := \frac{\partial a}{\partial\lambda} + \langle\nabla a, V\rangle. \tag{2.29}$$

Proof. Following the notation and the proof of formula (2.8) we have, for $|\lambda|$ small enough,

$$\int_{\partial E_\lambda} a(\lambda, y)\, d\mathcal{H}^{n-1} = \int_{\partial E} a(\lambda, \Psi_\lambda)\frac{|\nabla v_\lambda(\Psi_\lambda)||\det(\nabla\Psi_\lambda))|}{|\nabla u|}\, d\mathcal{H}^{n-1}.$$

Then

$$\begin{aligned} &\left(\frac{d}{d\lambda}\int_{\partial E_\lambda} a\, d\mathcal{H}^{n-1}\right)_{|\lambda=0} \\ &= \int_{\partial E}\left\{\frac{da}{d\lambda}_{|\lambda=0} + \left(a(\lambda, \Psi_\lambda)\,\frac{\partial}{\partial\lambda}\frac{|\nabla v_\lambda(\Psi_\lambda)||\det(\nabla\Psi_\lambda))|}{|\nabla u|}\right)_{|\lambda=0}\right\} d\mathcal{H}^{n-1}, \end{aligned}$$

and the first equality of (2.28) follows as in the proof of Theorem 2.7.

On the other hand, on ∂E we have

$$\begin{aligned} a\,\mathrm{div}^\Sigma V + \langle\nabla a, V\rangle &= \mathrm{div}^\Sigma(aV) + \langle\nabla a - \delta a, V\rangle \\ &= \mathrm{div}^\Sigma(aV) + \langle\nabla a - \delta a, V^\perp\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} &\int_{\partial E}\left(\frac{da}{d\lambda} + a\,\delta_h V_h\right) d\mathcal{H}^{n-1} \\ &= \int_{\partial E}\left(\frac{\partial a}{\partial\lambda} + \mathrm{div}^\Sigma(aV) + \langle\nabla a - \delta a, V^\perp\rangle\right) d\mathcal{H}^{n-1}. \end{aligned}$$

Recalling from Theorem 2.6 that

$$\begin{aligned} \int_{\partial E}\mathrm{div}^\Sigma(aV)\, d\mathcal{H}^{n-1} &= \int_{\partial E}\mathrm{div}^\Sigma(aV^\perp)\, d\mathcal{H}^{n-1} \\ &= \int_{\partial E} a\,\Delta d\,\langle\nabla d, V^\perp\rangle\, d\mathcal{H}^{n-1}, \end{aligned}$$

the second equality in (2.28) follows. $\square$

It is clear that $\langle \nabla d, V^\perp \rangle = \langle \nabla d, V \rangle$ and $\langle \nabla a - \delta a, V^\perp \rangle = \langle \nabla a, V^\perp \rangle$ in formula (2.28).

As we shall see in Chapter 3, the next example is related to self-similar solutions to mean curvature flow[12].

Example 2.12 (Gaussian densities). Let $T > 0$ and let $\partial E \in \mathcal{C}^\infty$ be compact. The first variation of the functional

$$S^-(E) := \int_{\partial E} e^{-\frac{|x|^2}{4T}} \, d\mathcal{H}^{n-1} \tag{2.30}$$

is given by

$$\begin{aligned} &\left(\frac{d}{d\lambda} S^-(E_\lambda)\right)_{|\lambda=0} \\ &= \int_{\partial E} \langle \nabla d(x), V(x) \rangle \left(\Delta d(x) - \frac{1}{2T} \langle \nabla d(x), x \rangle \right) e^{-\frac{|x|^2}{4T}} \, d\mathcal{H}^{n-1}. \end{aligned} \tag{2.31}$$

Indeed, this formula follows from (2.28), setting

$$a(z, \lambda) = a(z) := e^{-\frac{|z|^2}{4T}},$$

and observing that

$$\begin{aligned} \nabla a(x) - \delta a(x) &= \langle \nabla d(x), \nabla a(x) \rangle \nabla d(x) \\ &= -\frac{1}{2T} e^{-\frac{|x|^2}{4T}} \langle \nabla d(x), x \rangle \nabla d(x), \qquad x \in \partial E. \end{aligned}$$

Similarly, the first variation of the functional

$$S^+(E) := \int_{\partial E} e^{\frac{|x|^2}{4T}} \, d\mathcal{H}^{n-1} \tag{2.32}$$

is given by

$$\begin{aligned} &\left(\frac{d}{d\lambda} S^+(E_\lambda)\right)_{|\lambda=0} \\ &= \int_{\partial E} \langle \nabla d(x), V(x) \rangle \left(\Delta d(x) + \frac{1}{2T} \langle \nabla d(x), x \rangle \right) e^{\frac{|x|^2}{4T}} \, d\mathcal{H}^{n-1}. \end{aligned} \tag{2.33}$$

(12) The expressions in round parentheses in formula (2.31) and (2.33) are related to the Ornstein-Uhlenbeck operator [64] applied to d. The functional (2.30) is sometimes called Gaussian perimeter.

2.3. Direction of maximal slope of the perimeter

Our purpose in this short section is not to investigate the possible notions of derivative of the perimeter functional. We limit ourselves to the following observation which, formally at least, shows that the direction of maximal slope of the perimeter is given by the mean curvature vector, properly normalized[(13)].

Assume that $\partial E \in \mathcal{C}^\infty$ is compact. Define

$$\mathcal{X} := \left\{ X \in \mathcal{C}^1(\partial E; \mathbb{R}^n) : \int_{\partial E} |X|^2 \, d\mathcal{H}^{n-1} = 1 \right\}.$$

Given $X \in \mathcal{C}^1(\partial E; \mathbb{R}^n)$, denote by X^e a compactly supported extension of X on the whole of $\mathbb{R}^n$ which is smooth in a neighbourhood U of ∂E, where U is such that $d \in \mathcal{C}^\infty(\mathrm{U})$. Define $\Psi_\lambda^X : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\Psi_\lambda^X(z) := z + \lambda X^e(z), \qquad z \in \mathbb{R}^n.$$

As a consequence of Theorem 2.7, we have that $\left(\frac{d}{d\lambda} P(\Psi_\lambda^X(E))\right)_{|\lambda=0}$ is independent of the particular extension X^e of X.

Proposition 2.13. *The solution to*

$$\sup \left\{ \left(\frac{d}{d\lambda} P\left(\Psi_\lambda^X(E)\right)\right)_{|\lambda=0} : X \in \mathcal{X} \right\}$$

is given by the vector field $X_{\max} \in \mathcal{X}$ defined as

$$X_{\max}(x) := \frac{\Delta d(x) \nabla d(x)}{\left(\int_{\partial E} (\Delta d)^2 \, d\mathcal{H}^{n-1}\right)^{1/2}}, \qquad x \in \partial E. \tag{2.34}$$

Proof. Given $X \in \mathcal{X}$, from Theorem 2.7 and the Schwarz's inequality

(13) We recall that the gradient $\nabla u : \mathbb{R}^n \to \mathbb{R}^n$ of a function $u \in \mathcal{C}^1(\mathbb{R}^n)$ at a point $z \in \mathbb{R}^n$ can be characterized as follows. For any $e \in \mathbb{S}^{n-1}$ denote by $\nabla_e u : \mathbb{R}^n \to \mathbb{R}$ the derivative of u along the direction e. If $e_{\max} = e_{\max}(z) \in \mathbb{S}^{n-1}$ is a solution of $\max\left\{\nabla_e u(z) : e \in \mathbb{S}^{n-1}\right\}$, then $\nabla u(z) = \nabla_{e_{\max}} u(z)\, e_{\max}$.

we have

$$
\begin{aligned}
\left(\frac{d}{d\lambda} P\left(\Psi_\lambda^X(E)\right)\right)_{|\lambda=0} &= \int_{\partial E} \Delta d \langle \nabla d, X\rangle \, d\mathcal{H}^{n-1} \\
&\leq \left(\int_{\partial E} (\Delta d)^2 \, d\mathcal{H}^{n-1}\right)^{1/2} \left(\int_{\partial E} \langle \nabla d, X\rangle^2 \, d\mathcal{H}^{n-1}\right)^{1/2} \\
&\leq \left(\int_{\partial E} (\Delta d)^2 \, d\mathcal{H}^{n-1}\right)^{1/2} \left(\int_{\partial E} |X|^2 \, d\mathcal{H}^{n-1}\right)^{1/2} \\
&= \left(\int_{\partial E} (\Delta d)^2 \, d\mathcal{H}^{n-1}\right)^{1/2}.
\end{aligned}
$$

On the other hand, the vector field in (2.34) belongs to $\mathcal{X}$ and is the only vector field in $\mathcal{X}$ which realizes the equalities in place of the inequalities in the above formula. $\square$

Chapter 3
Smooth flows

In this chapter we define smooth mean curvature flows, and we illustrate some examples and some special solutions.

We begin with the definition of smooth flow, expressed with the signed distance function. Recall that $\mathcal{P}(\mathbb{R}^n)$ denotes the set of all subsets of $\mathbb{R}^n$.

Definition 3.1 (Smooth flow). We say that f is a smooth flow if

- there exist $a, b \in \mathbb{R}$, $a < b$, such that $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$,
- for any $t \in [a, b]$ the set $f(t)$ is closed,
- if we denote[(1)] by

$$\begin{aligned} d(t, z) &:= d(z, f(t)) \\ &= \operatorname{dist}(z, f(t)) - \operatorname{dist}(z, \mathbb{R}^n \setminus f(t)), \quad (t, z) \in [a, b] \times \mathbb{R}^n, \end{aligned} \tag{3.1}$$

  the signed distance function negative inside the interior of $f(t)$, then
  - for any $t \in [a, b]$ there exists an open set $\mathrm{A}_t \subseteq \mathbb{R}^n$ containing $\partial f(t)$ such that, setting

$$Q := \bigcup_{t \in [a,b]} (\{t\} \times \mathrm{A}_t),$$

    we have[(2)]

$$d \in \mathcal{C}^\infty(Q). \tag{3.2}$$

Notice, in particular, that $\partial f(t) \in \mathcal{C}^\infty$ for any $t \in [a, b]$. If in addition $\partial f(t)$ is compact, we can take A_t to be a tubular neighbourhood of $\partial f(t)$. More precisely, we give the following definition.

---

(1) Even if here the variable t occurs, for simplicity of notation and unless otherwise specified, we use the same symbol d used in formula (1.6).

(2) Inclusion (3.2) means that there exists an open set containing Q where d is of class $\mathcal{C}^\infty$.

Definition 3.2 (Smooth compact flow). We say that f is a smooth compact flow if

- there exist $a, b \in \mathbb{R}, a < b$, such that $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$,
- for any $\in [a, b]$ the set $f(t)$ is closed and $\partial f(t)$ is compact,
- there exists $\sigma > 0$ such that the function d defined in (3.1) satisfies

$$d \in \mathcal{C}^\infty(Q_\sigma),$$

where $Q_\sigma := \{(t, z) \in [a, b] \times \mathbb{R}^n : |d(t, z)| < \sigma\}$.

Notation. If we need to specify the dependence on f of the signed distance function defined in (3.1), we use the symbol d_f in place of d. Moreover, the set Q_σ in Definition 3.2 will be often denoted by Q[3].

Observe that $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ is a smooth flow (respectively a smooth compact flow) if and only if the map

$$f^c(t) := \overline{\mathbb{R}^n \setminus f(t)}, \qquad t \in [a, b], \tag{3.3}$$

is a smooth flow (respectively a smooth compact flow).

Definition 3.3. Let $E \subset \mathbb{R}^n$ be a closed set with $\partial E \in \mathcal{C}^\infty$, and let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth flow. If $f(a) = E$ we say[4] that f starts from E at time a.

Keeping the same notation used in Definition 3.1, we now give the definition of normal velocity vector. We denote, as usual, by $\nabla = (\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n})$ the gradient with respect to the space variables.

Definition 3.4 (Normal velocity vector and outer normal velocity). Let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth flow and let $t \in [a, b]$. The normal velocity vector of the flow at $x \in \partial f(t)$ is defined as

$$-\frac{\partial d}{\partial t}(t, x)\, \nabla d(t, x). \tag{3.4}$$

---

[3] A condition slightly more restrictive than (3.2) consists in requiring the existence of an open set A containing $\partial f(t)$ for any $t \in [a, b]$, such that $d \in \mathcal{C}^\infty([a, b] \times A)$. This latter condition will be adopted in the definition of the class $\mathcal{F}$ in Chapter 9 (Definition 9.4); adopting (3.2) in Definition 9.4 would not give any difference in the corresponding theory of barriers.

[4] With a slight abuse of language, we occasionally and equivalently say that f starts from ∂E at time a.

The outer normal velocity of the flow at $x \in \partial f(t)$ is defined[(5)] as

$$-\frac{\partial d}{\partial t}(t, x).$$

The normal velocity vector is invariant if we substitute $d = d_f$ in (3.4) with d_{f^c}, while the outer normal velocity changes sign.

Note also that the normal velocity vector can be written in terms of the squared distance function

$$\eta := \frac{1}{2}d^2. \tag{3.5}$$

Indeed

$$-\frac{\partial d}{\partial t}\,\nabla d = -\frac{\partial}{\partial t}\nabla\eta \qquad \text{on } \partial f(t),\ t \in [a, b].$$

Since in the sequel of the book we will occasionally need a parametric description of the flowing manifolds, we recall the definition of smooth parametric flow, and of normal velocity in a parametric setting.

Definition 3.5 (Parametric smooth flow). Let $\mathcal{S} \subset \mathbb{R}^n$ be a smooth $(n-1)$-dimensional embedded oriented connected manifold without boundary, let $a, b \in \mathbb{R}$ be with $a < b$, and let $\varphi \in \mathcal{C}^\infty([a, b] \times \mathcal{S}; \mathbb{R}^n)$. We write

$$\varphi \in \mathcal{X}([a, b]; \mathrm{Imm}(\mathcal{S}; \mathbb{R}^n))$$

if, for any $t \in [a, b]$, $\varphi(t, \cdot)$ is proper, and denoting by $\mathrm{d}\varphi(t, \cdot)$ the differential of φ with respect to s, we have that $\mathrm{d}\varphi(t, s)$ is injective for any $s \in \mathcal{S}$.

If, in addition, for any $t \in [a, b]$ the map $\varphi(t, \cdot)$ is a homeomorphism between $\mathcal{S}$ and $\varphi(t, \mathcal{S})$, then we write

$$\varphi \in \mathcal{X}([a, b]; \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n)).$$

In this book we will be essentially concerned with the case of smooth compact mean curvature flows. The only noncompact cases that will be considered are:

- when $\varphi(t, \mathcal{S})$ is a smooth entire graph on $\mathbb{R}^{n-1}$,
- the evolution of a cylinder (Example 3.22),

(5) More generally, let $e \in \mathbb{S}^{n-1}$ be a unit vector. The velocity vector of the flow at $x \in \partial f(t)$ along e is defined as $-\langle \nabla d(t, x), e\rangle^{-1}\ \frac{\partial d}{\partial t}(t, x)\ e$, and has the property that its orthogonal projection onto the normal space to $\partial f(t)$ at x is the normal velocity vector at x. The velocity of the flow at $x \in \partial f(t)$ along e is defined as $-\langle \nabla d(t, x), e\rangle^{-1}\ \dfrac{\partial d}{\partial t}(t, x)$. Note also (compare with Lemma 6.1) that, given $t \in [a, b)$ and $x \in \partial f(t)$, the vector $\xi := \frac{\partial d}{\partial t}(t, x)\nabla d(t, x)$ has the following property: if $\epsilon > 0$ is such that $t + \epsilon < b$, then for any $\tau \in (0, \epsilon)$ there exists a unique $x_\tau \in \partial f(t + \tau)$ such that $\mathrm{dist}(x, \partial f(t + \tau)) = |x_\tau - x|$, and $\xi = \lim_{\tau \downarrow 0} \frac{x_\tau - x}{\tau}$.

- the translating solutions (Definition 3.26),
- the evolution of certain solids of revolutions (the flow $I_v(t)$ considered in Chapter 8) under suitable periodicity assumptions,
- Example 9.19 (see also Section 10.1).

Remark 3.6 (Flows and parametrizations). Assume that $\mathcal{S}$ is compact, and let $\varphi \in \mathcal{X}([a, b]; \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n))$. It is possible to prove[(6)] that, if we denote by $f(t)$ the closure of one connected component of $\mathbb{R}^n \setminus \varphi(t, \mathcal{S})$, then $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ is a smooth compact flow. Conversely, if $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ is a smooth flow and $\partial f(t)$ is connected, then there exists[(7)] a smooth $(n-1)$-dimensional embedded oriented connected manifold $\mathcal{S}$ without boundary (for instance, $\mathcal{S} = \partial f(a)$) and a map $\varphi \in \mathcal{X}([a, b]; \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n))$ such that $\partial f(t) = \varphi(t, \mathcal{S})$ for any $t \in [a, b]$. Hence, for any $t \in [a, b]$, the map $\varphi(t, \cdot)$ is a smooth embedding of the manifold $\mathcal{S}$ in $\mathbb{R}^n$, and $\partial f(t)$ is the image of the embedding; in addition, φ depends smoothly on the variable t.

We now give the definition of normal velocity vector using the map φ. Recall that, if $\varphi \in \mathcal{X}([a, b]; \mathrm{Imm}(\mathcal{S}; \mathbb{R}^n))$, locally we can always choose a smooth unit vector field

$$\nu(t, \cdot)$$

normal to $\varphi(t, \mathcal{S})$.

Convention. If $\varphi \in \mathcal{X}([a, b]; \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n))$ and $\partial f(t) = \varphi(t, \mathcal{S})$ for any $t \in [a, b]$, we will choose $\nu(t, \cdot)$ so that

$$-\nu(t, s) = \nabla d(t, x), \qquad x = \varphi(t, s). \tag{3.6}$$

This is consistent with the choice made in formula (1.52).

Definition 3.7 (Normal velocity vector using parametrizations). Let $\varphi \in \mathcal{X}([a, b]; \mathrm{Imm}(\mathcal{S}; \mathbb{R}^n))$. We define the vector field $\mathbf{V} : [a, b] \times \mathcal{S} \to \mathbb{R}^n$ as

$$\mathbf{V}(t, s) := \langle \nu(t, s), \frac{\partial \varphi}{\partial t}(t, s) \rangle \, \nu(t, s), \qquad (t, s) \in [a, b] \times \mathcal{S}. \tag{3.7}$$

Now, let us see which are the relations between Definition 3.4 and Definition 3.7.

(6) See [15, page 18].

(7) See [15, page 22].

Proposition 3.8. *Let $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth flow, and let $\varphi \in \mathcal{X}([a,b]; \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n))$ be as in Remark* 3.6. *For any $t \in [a,b]$ and any $s \in \mathcal{S}$ we have*

$$-\frac{\partial d}{\partial t}(t,x)\, \nabla d(t,x) = \mathbf{V}(t,s), \qquad x := \varphi(t,s) \in \partial f(t). \tag{3.8}$$

Proof. We know that

$$d(t, \varphi(t,s)) = 0, \qquad t \in [a,b],\ s \in \mathcal{S}. \tag{3.9}$$

Hence, differentiating (3.9) with respect to t and using that $x = \varphi(t,s)$, we get

$$\frac{\partial d}{\partial t}(t,x) + \langle \nabla d(t,x), \frac{\partial \varphi}{\partial t}(t,s)\rangle = 0. \tag{3.10}$$

Then (3.8) follows from (3.7) and (3.6). □

Note that at $x = \varphi(t,s)$ we have

$$\frac{\partial d}{\partial t}(t,x) = \langle \nu(t,s), \frac{\partial \varphi}{\partial t}(t,s)\rangle =: \mathrm{V}(t,s). \tag{3.11}$$

Remark 3.9 (Normal parametrization). It is possible to show(8) that, among the maps $\varphi \in \mathcal{X}([a,b]; \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n))$ parametrizing the smooth compact flow f, there exists one, called normal parametrization, such that $\frac{\partial \varphi}{\partial t}(t,s)$ belongs to the normal line $N_x(\partial f(t))$ to $\partial f(t)$ at $x = \varphi(t,s)$, and hence $\mathbf{V}(t,s) = \frac{\partial \varphi}{\partial t}(t,s)$ for any $(t,s) \in [a,b] \times \mathcal{S}$. Such a parametrization is unique, once the parametrization at the initial time a is fixed.

Remark 3.10 (Normal velocity using the level sets). The normal velocity vector can also be expressed as follows. Let $u : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ be a continuous function of class $\mathcal{C}^\infty$ in $[a,b] \times \mathrm{A}$, where $\mathrm{A} \subset \mathbb{R}^n$ is an open set containing $\bigcup_{t \in [a,b]} \{u(t,\cdot) = 0\}$, and such that

$$u^2 + |\nabla u|^2 > 0 \qquad \text{in } [a,b] \times \mathrm{A}. \tag{3.12}$$

Then the map $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ defined as

$$f(t) := \{z \in \mathbb{R}^n : u(t,z) \le 0\}, \qquad t \in [a,b], \tag{3.13}$$

is a smooth flow, and

$$\partial f(t) = \{z \in \mathbb{R}^n : u(t,z) = 0\}, \qquad t \in [a,b].$$

(8) See [15, Theorem 8].

The normal velocity vector equals

$$-\frac{\frac{\partial u}{\partial t}}{|\nabla u|}\frac{\nabla u}{|\nabla u|} \qquad \text{on } \partial f(t). \tag{3.14}$$

If, in addition, we split $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$ and we assume that there exist an open set $O \subseteq \mathbb{R}^{n-1}$ and a function $v \in \mathcal{C}^\infty([a,b] \times O)$ such that

$$u(t,s,z_n) := v(t,s) - z_n, \qquad (t,s,z_n) \in [a,b] \times \mathrm{A},$$

we can parametrize the flow as

$$(t,s) \in [a,b] \times O \to \varphi(t,s) := (s, v(t,s)) \in \mathrm{A}.$$

Therefore $\dfrac{\partial \varphi}{\partial t} = (0, \dfrac{\partial v}{\partial t})$, and the normal velocity vector can be written as

$$\frac{\frac{\partial v}{\partial t}}{1+|\nabla v|^2}(-\nabla v, 1),$$

where ∇v is the gradient of v with respect to s.

3.1. Smooth mean curvature flows

We are now in a position to define the classical mean curvature flow using the signed distance function d defined in (3.1).

Definition 3.11 (Smooth mean curvature flow). We say that f is a smooth mean curvature flow if

- there exist $a, b \in \mathbb{R}, a < b$, such that $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ is a smooth flow;
- the following equation holds:

$$\frac{\partial d}{\partial t}(t,x)\nabla d(t,x) = \Delta d(t,x)\nabla d(t,x), \quad t \in [a,b],\ x \in \partial f(t). \tag{3.15}$$

Definition 3.12 (Smooth compact mean curvature flow). We say that f is a smooth compact mean curvature flow if

- there exist $a, b \in \mathbb{R}, a < b$, such that $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ is a smooth compact flow;
- equation (3.15) holds.

Remark 3.13. If $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ is a smooth (respectively smooth compact) mean curvature flow, then the map $f^c : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ defined in (3.3) is a smooth (respectively smooth compact) mean curvature flow.

Remark 3.14. Let η be the squared distance function defined in (3.5). Recalling the expression (1.24) of the mean curvature vector, we have that (3.15) can be written equivalently[(9)] as

$$\frac{\partial}{\partial t}\nabla\eta(t,x) = \Delta\nabla\eta(t,x), \qquad t \in [a,b],\ x \in \partial f(t).$$

Remark 3.15. Recalling that $|\nabla d(t,z)|^2 = 1$ for any $(t,z) \in Q$, the equation in (3.15) is equivalent to

$$\frac{\partial d}{\partial t}(t,x) = \Delta d(t,x), \qquad t \in [a,b],\ x \in \partial f(t), \tag{3.16}$$

that is to the system

$$\begin{cases} \dfrac{\partial d}{\partial t}(t,x) = \Delta d(t,x), \\ \\ d(t,x) = 0, \end{cases} \qquad t \in [a,b].$$

It is sometimes useful to consider subsolutions and supersolutions to mean curvature flow. Accordingly, we say that f is a smooth subsolution (respectively smooth compact subsolution) to mean curvature flow if

- there exist $a, b \in \mathbb{R}, a < b$, such that $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ is a smooth flow (respectively smooth compact flow);

(9) The squared distance function can be used to define smooth compact mean curvature flow in codimension $k \geq 1$. Indeed, in [108] the following definition is given: f is a smooth compact k-codimensional mean curvature flow if

- there exist $a, b \in \mathbb{R}, a < b$, such that $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$;
- the set $\{(t,z) : t \in [a,b], z \in f(t)\}$ is compact;
- letting
$$\eta^f(t,z) := \frac{1}{2}\mathrm{dist}(z, f(t))^2, \qquad t \in [a,b],\ z \in \mathbb{R}^n,$$
there exists an open set $A \subset \mathbb{R}^n$ containing $\cup_{t\in[a,b]} f(t)$ such that $\eta^f \in \mathcal{C}^\infty\big([a,b] \times A\big)$ and rank $\left(\nabla^2\eta^f(t,x)\right) = n-k$ for any $t \in [a,b]$ and any $x \in f(t)$;
- the following system holds:
$$\frac{\partial}{\partial t}\nabla\eta^f(t,x) = \Delta\nabla\eta^f(t,x), \qquad t \in [a,b],\ x \in f(t)$$
(when $k = 1$, $f(t)$ takes here the role of $\partial f(t)$ in Definition 3.12).

We refer to [15] for more details. Mean curvature flow in arbitrary codimension has been the subject of various paper, beside the book of Brakke [67] (see, *e.g.*, [10, 12, 185, 19, 20, 47, 263, 264, 245]).

- the following inequality holds:

$$\frac{\partial d}{\partial t}(t,x) - \Delta d(t,x) \geq 0 \qquad t \in [a,b],\ x \in \partial f(t). \tag{3.17}$$

If in (3.17) we replace the inequality $\geq$ with $\leq$, we say that f is a smooth supersolution (respectively smooth compact supersolution) to mean curvature flow.

Note that inequality (3.17) allows the flow to shrink more quickly than mean curvature flow, since $-\frac{\partial d}{\partial t}$ is the outer normal velocity.

Remark 3.16. Let $\theta \in \mathcal{C}^1_c(\mathbb{R} \times \mathbb{R}^n)$, and let $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth compact mean curvature flow. From the first variation computation in Corollary 2.11 applied with

$$V = -\Delta d \nabla d$$

which is the velocity field of $\partial f(\cdot)$, it follows

$$\frac{d}{dt}\int_{\partial f(t)} \theta \, d\mathcal{H}^{n-1} = \int_{\partial f(t)} \left(\frac{d\theta}{dt} - \theta\, \delta_h(\Delta d \nabla_h d) \right) d\mathcal{H}^{n-1}.$$

Since on $\partial f(t)$ we have

$$\delta_h \Delta d\ \nabla_h d = \langle \nabla d, \delta \Delta d \rangle = 0$$

and $\delta_h \nabla_h d = \Delta d$ (see formula (1.20)), it follows, for $t \in [a,b]$,

$$\frac{d}{dt}\int_{\partial f(t)} \theta \, d\mathcal{H}^{n-1} = \int_{\partial f(t)} \left(\frac{\partial\theta}{\partial t} - \Delta d \langle \nabla d, \nabla\theta \rangle - \theta (\Delta d)^2 \right) d\mathcal{H}^{n-1}. \tag{3.18}$$

This formula, considering θ as a test function, is at the core of Brakke's weak definition of mean curvature flow, see [67].

Remark 3.17 (Evolutions of perimeter and enclosed volume). Let f : $[a,b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth compact mean curvature flow. Let $P(f(t))$ be the perimeter of $f(t)$ for any $t \in [a,b]$. Then, as a consequence of the first variation computation (see (2.24)) and of the evolution equation (3.15), we have[(10)]

$$\frac{d}{dt} P(f(t)) = -\int_{\partial f(t)} (\Delta d(t,\cdot))^2 \, d\mathcal{H}^{n-1}, \qquad t \in [a,b], \tag{3.19}$$

(10) Formula (3.19) is the particular case of (3.18) for $\theta \equiv 1$ on $\cup_{t \in [a,b]} (\{t\} \times \partial f(t))$.

which can also be written as

$$P\big(f(\tau_1)\big) - P\big(f(\tau_2)\big) = \frac{1}{2}\int_{\tau_1}^{\tau_2}\int_{\partial f(t)} \left(\frac{\partial d}{\partial t}\right)^2 d\mathcal{H}^{n-1}\, dt + \frac{1}{2}\int_{\tau_1}^{\tau_2}\int_{\partial f(t)} (\Delta d)^2\, d\mathcal{H}^{n-1}\, dt\,,$$

for any $\tau_1, \tau_2 \in [a, b]$ with $\tau_1 < \tau_2$.

In particular,

$$\text{the function } \ t \in [a,b] \to P(f(t)) \ \text{ is decreasing.}$$

Moreover, if $f(t)$ is bounded, and denoting by $|f(t)|$ the Lebsegue measure of $f(t)$ for any $t \in [a, b]$, we have that, as a consequence of the first variation computation (2.2) for volume integrals and of (3.15),

$$\frac{d}{dt}|f(t)| = -\int_{\partial f(t)} \Delta d(t,\cdot)\, d\mathcal{H}^{n-1}, \qquad t \in [a,b]. \tag{3.20}$$

In particular, if $n = 2$ and $\partial f(t)$ is connected, then $\frac{d}{dt}|f(t)| = -2\pi$ for any $t \in [a, b]$.

Smooth mean curvature flow for immersed hypersurfaces can be defined using parametrizations(11). Consistently with the notation that we have introduced in Chapter 1, given a map $\varphi \in \mathcal{X}([a, b]; \mathrm{Imm}(\mathcal{S}; \mathbb{R}^n))$ and $t \in [a, b]$, we let

$$h_{\alpha\beta}(t,s), \qquad A(t,s), \qquad t \in [a,b],\ s \in \mathcal{S}, \tag{3.21}$$

be defined as in (1.50) and (1.51) respectively, where $\varphi(s)$ is replaced by $\varphi(t, s)$, and similarly we let

$$\mathrm{H}(t,s) \qquad \text{and} \qquad \mathbf{H}(t,s), \qquad t \in [a,b],\ s \in \mathcal{S} \tag{3.22}$$

be defined as in (1.51).

Definition 3.18 (Parametric smooth mean curvature flow). Assume that $\varphi \in \mathcal{X}([a, b]; \mathrm{Imm}(\mathcal{S}; \mathbb{R}^n))$. We say that φ is a parametric smooth mean curvature flow if

$$\mathbf{V}(t,s) = \mathrm{H}(t,s)\nu(t,s), \qquad t \in [a,b],\ s \in \mathcal{S}. \tag{3.23}$$

If $\varphi \in \mathcal{X}([a, b]; \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n))$ we say that φ is a parametric smooth embedded mean curvature flow.

(11) The parametric approach is largely used (see, *e.g.*, [168, 149, 158, 22, 23, 126, 203]).

If $\varphi_0 \in \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n)$ is given and the parametric smooth mean curvature flow satisfies $\varphi(a, \cdot) = \varphi_0(\cdot)$, then we say that φ starts from φ_0 at time a.

Note that when $\varphi \in \mathcal{X}([a, b]; \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n))$, system (3.23) is equivalent to (3.15), thanks to formulas (1.53) and (3.8).

Remark 3.19. The map $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ is a smooth compact mean curvature flow with $\partial f(t)$ connected for any $t \in [a, b]$ if and only if $\mathcal{S}$ is compact, $\varphi \in \mathcal{X}([a, b]; \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n))$ is a parametric smooth mean curvature flow, and $\varphi(t, \mathcal{S}) = \partial f(t)$ for any $t \in [a, b]$(12).

It will be convenient to introduce a convention for maps defined on non-compact intervals and taking values in $\mathcal{P}(\mathbb{R}^n)$. Let $J \subseteq \mathbb{R}$ be an interval. A map $f : J \to \mathcal{P}(\mathbb{R}^n)$ is a smooth (compact) mean curvature flow if the restriction of f to any compact interval contained in J is a smooth (compact) mean curvature flow.

3.2. Examples

In this section we give some examples of smooth mean curvature flows.

Example 3.20 (Hypersurfaces with zero mean curvature). Assume that $E \subset \mathbb{R}^n$ is such that $\partial E \in \mathcal{C}^\infty$ has zero mean curvature. Then, given $T > 0$, the map $f : [0, T] \to \mathcal{P}(\mathbb{R}^n)$ defined as

$$f(t) := \overline{E}, \qquad t \in [0, T],$$

is a smooth mean curvature flow starting from $\overline{E}$. Hence smooth boundaries with vanishing mean curvature, such as hyperplanes, are stationary solutions to mean curvature flow.

Example 3.21 (Sphere). Let $R_0 > 0$ and $z_0 \in \mathbb{R}^n$. A smooth compact mean curvature flow starting from the closed ball $\overline{B_{R_0}(z_0)}$ is given by $f : [0, t^\dagger) \to \mathcal{P}(\mathbb{R}^n)$ with

$$t^\dagger := \frac{R_0^2}{2(n-1)},$$

and

$$f(t) = \overline{B_{R(t)}(z_0)}, \qquad t \in [0, t^\dagger),$$

where(13)

$$R(t) = \sqrt{R_0^2 - 2(n-1)t}, \qquad t \in [0, t^\dagger). \tag{3.24}$$

(12) This result extends to a smooth mean curvature flow.

(13) Note that the right-hand side of (3.24) is well defined for any $t \in (-\infty, t^\dagger]$.

Indeed

$$d(t,z) = |z - z_0| - R(t), \qquad (t,z) \in [0,t^\dagger) \times \mathbb{R}^n,$$

hence $d \in \mathcal{C}^\infty([0,t^\dagger) \times \mathrm{A})$ where $\mathrm{A} = \mathbb{R}^n \setminus \{z_0\}$. If $(t,z) \in [0,t^\dagger) \times \mathrm{A}$,

$$\frac{\partial d}{\partial t}(t,z) = -\dot{R}(t), \tag{3.25}$$

$$\nabla d(t,z) = \frac{z - z_0}{|z - z_0|},$$

and for $i, j \in \{1, \ldots, n\}$,

$$\nabla^2_{ij} d(t,z) = \frac{1}{|z - z_0|}\left(\mathrm{Id}_{ij} - \frac{(z_i - z_{0i})}{|z - z_0|}\frac{(z_j - z_{0j})}{|z - z_0|}\right), \tag{3.26}$$

and

$$\Delta d(t,z) = \frac{n-1}{|z - z_0|}. \tag{3.27}$$

Therefore the evoution law (3.16) becomes

$$\dot{R}(t) = -\frac{n-1}{R(t)}, \qquad t \in [0,t^\dagger).$$

Coupled with $R(0) = R_0$, the solution is given in (3.24). Observe that

$$B_{R(t)}(z_0) = \sqrt{1 - \frac{t}{t^\dagger}}\, B_{R_0}(z_0), \qquad t \in [0,t^\dagger),$$

and

$$t^\dagger = \frac{1}{2 \max\limits_{x \in \partial B_{R_0}(z_0)} |\nabla^2 d(0,x)|^2}.$$

Uniqueness of such a smooth compact mean curvature flow will be a consequence of the results of Chapter 5.

Example 3.22 (Cylinder). Let $n \geq 3$, $R_0 > 0$, and let

$$C_{R_0} := \{z = (\widehat{z}, z_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |\widehat{z}| \leq R_0\},$$

where $|\widehat{z}|$ is the Euclidean norm of $\widehat{z} \in \mathbb{R}^{n-1}$. A smooth mean curvature flow starting from C_{R_0} is given by $f : [0,t^\dagger) \to \mathcal{P}(\mathbb{R}^n)$ with

$$t^\dagger := \frac{R_0^2}{2(n-2)},$$

and

$$f(t) = C_{R(t)} = \{(\widehat{z}, z_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : |\widehat{z}| \leq R(t)\}, \qquad t \in [0,t^\dagger),$$

where

$$R(t) = \sqrt{R_0^2 - 2(n-2)t}, \qquad t \in [0, t^\dagger).$$

Observe that

$$C_{R(t)} = \sqrt{1 - \frac{t}{t^\dagger}} C_{R_0}, \qquad t \in [0, t^\dagger).$$

In this case

$$d(t, z) = |\widehat{z}| - R(t), \qquad (t, z) \in [0, t^\dagger) \times \mathbb{R}^n,$$

and $d \in \mathcal{C}^\infty([0, t^\dagger) \times \mathrm{A})$, where $\mathrm{A} = \mathbb{R}^n \setminus \{z_n = 0\}$.

Definition 3.23 (Self-similar evolutions). Let $E \subset \mathbb{R}^n$ be with $\partial E \in \mathcal{C}^\infty$, $\mathrm{I} \subseteq \mathbb{R}$ be a nonempty open interval and let $\alpha \in \mathcal{C}^\infty(\mathrm{I}; (0, +\infty))$. We say that $f : \mathrm{I} \to \mathcal{P}(\mathbb{R}^n)$ is a smooth self-similar evolution given by (E, α) if

$$f(t) = \alpha(t)\overline{E}, \qquad t \in \mathrm{I}. \tag{3.28}$$

Note that for any $\lambda > 0$ the pair $(\lambda E, \frac{\alpha}{\lambda})$ gives raise to the same self-similar evolution as the pair (E, α).

The following proposition describes a class of special solutions to mean curvature flow[14].

Proposition 3.24 (Contracting and expanding self-similar solutions). *Let $f : \mathrm{I} \to \mathcal{P}(\mathbb{R}^n)$ be a smooth self-similar evolution given by the pair (E,α). If f is a smooth mean curvature flow then one of the following three conditions hold: setting $d(\cdot) := d(\cdot, E)$,*

(i) *there exist $t_0 \in \mathbb{R}$ and $T > 0$ such that*

$$\mathrm{I} \subseteq (-\infty, t_0), \qquad \alpha(t) = \sqrt{\frac{t_0}{T}}\sqrt{1 - \frac{t}{t_0}}, \quad t \in \mathrm{I},$$

and[15],[16]

$$\Delta d(x) = \frac{1}{2T}\langle \nabla d(x), x\rangle, \qquad x \in \partial E; \tag{3.29}$$

[14] See, *e.g.*, [186, 219, 95].

[15] A partial classification of solutions to (3.29) can be found in [171].

[16] Note that a nonsmooth cone with zero mean curvature and vertex at the origin, satisfies (3.29) at any of its points $x \neq 0$.

(ii) $\mathrm{I} \subseteq \mathbb{R}$, $\alpha'(t) = 0$ *for any* $t \in \mathrm{I}$, *and*

$$\Delta d(x) = 0, \qquad x \in \partial E;$$

(iii) *there exist* $t_0 \in \mathbb{R}$ *and* $T > 0$ *such that*

$$\mathrm{I} \subseteq (t_0, +\infty), \qquad \alpha(t) = \sqrt{\frac{t_0}{T}}\sqrt{\frac{t}{t_0} - 1}, \quad t \in \mathrm{I},$$

and

$$\Delta d(x) = -\frac{1}{2T}\langle \nabla d(x), x\rangle, \qquad x \in \partial E. \tag{3.30}$$

Conversely, let $\partial E \in \mathcal{C}^\infty$ *and* α *be such that one of the conditions* (i)-(iii) *holds. Define* $f : \mathrm{I} \to \mathcal{P}(\mathbb{R}^n)$ *as in* (3.28). *Then* f *is a smooth mean curvature flow.*

The map f in (i) is called a self-similar contracting mean curvature flow, while the map f in (iii) is called is a self-similar expanding mean curvature flow.

Proof. Assume that f in (3.28) is a smooth mean curvature flow. Let $z \in \mathbb{R}^n$, and recall that $\alpha(t) > 0$ for any $t \in \mathrm{I}$. We have

$$\begin{aligned}\operatorname{dist}(z, f(t)) &= \inf_{y \in f(t)} |y - z| = \alpha(t) \inf_{y/\alpha(t) \in E} |y/\alpha(t) - z/\alpha(t)| \\ &= \alpha(t)\operatorname{dist}(z/\alpha(t), E).\end{aligned}$$

Similarly,

$$\operatorname{dist}(z, \mathbb{R}^n \setminus f(t)) = \alpha(t)\operatorname{dist}(z/\alpha(t), \mathbb{R}^n \setminus E).$$

Hence, if d_f is the signed distance function from $\partial f(t)$ defined in (3.1) and d is, as in the statement, the signed distance function from ∂E defined in (1.6), we have

$$d_f(t, z) = \alpha(t) d(z/\alpha(t)), \qquad (t, z) \in \mathrm{I} \times \mathbb{R}^n.$$

Then we compute for $(t, z) \in Q$:

$$\nabla d_f(t, z) = \nabla d(z/\alpha(t)), \qquad \Delta d_f(t, z) = \frac{1}{\alpha(t)}\Delta d(z/\alpha(t)), \tag{3.31}$$

$$\frac{\partial d_f}{\partial t}(t, z) = \alpha'(t) d(z/\alpha(t)) - \frac{\alpha'(t)}{\alpha(t)}\langle \nabla d(z/\alpha(t)), z\rangle, \tag{3.32}$$

where $'$ denotes differentiation with respect to t. Since

$$\partial f(t)=\{x \in \mathbb{R}^n : d_f(t,x) = 0\} = \alpha(t)\partial E = \{x \in \mathbb{R}^n : d(x/\alpha(t))=0\},$$

from (3.32) we deduce

$$\frac{\partial d_f}{\partial t}(t,x) = -\alpha'(t)\langle \nabla d(x/\alpha(t)), \frac{x}{\alpha(t)}\rangle, \qquad x \in \partial f(t). \tag{3.33}$$

Using (3.31) and (3.33), it follows that equation (3.16) expressing mean curvature flow of $\partial f(t)$ becomes an equation for the function d on ∂E, which reads as

$$-\alpha'(t)\langle \nabla d(x/\alpha(t)), x/\alpha(t)\rangle = \frac{1}{\alpha(t)}\Delta d(x/\alpha(t)), \qquad x/\alpha(t) \in \partial E,$$

i.e.,

$$\Delta d(x) = -\alpha'(t)\alpha(t)\langle \nabla d(x), x\rangle, \qquad x \in \partial E.$$

Since the left-hand side does not depend on t, we deduce

$$\alpha'(t)\alpha(t) \equiv c \in \mathbb{R}, \qquad t \in \mathrm{I}.$$

Now, we distinguish the three cases: $c < 0, c = 0$ and $c > 0$.
If $c < 0$, writing $c = -\frac{1}{2T}$ for $T > 0$, we have $\alpha(t) = \sqrt{-c}\sqrt{2(t_0 - t)}$ for any $t \in \mathrm{I} \subseteq (-\infty, t_0)$, and (i) follows.
If $c = 0$ then (ii) immediately follows.
If $c > 0$, writing $c = \frac{1}{2T}$ for $T > 0$, we have $\alpha(t) = \sqrt{c}\sqrt{2(t - t_0)}$ for any $t \in \mathrm{I} \subseteq (t_0, +\infty)$, and (iii) follows.

Conversely, let $\partial E \in \mathcal{C}^\infty$, let α be as (i), and assume that (3.29) holds for some $T > 0$. Repeating the previous computations in reverse order, one checks that the map f in (3.28) is a smooth mean curvature flow in I. Similar reasonings apply to cases (ii) and (iii). □

Remark 3.25. In view of Example 2.12, if we assume that $\partial E \in \mathcal{C}^\infty$, then equation (3.29) expresses the stationarity condition of ∂E for the functional

$$S^-(E) = \int_{\partial E} e^{-\frac{|x|^2}{4T}}\, d\mathcal{H}^{n-1}$$

in (2.30), and (3.30) expresses the stationarity condition of ∂E for the functional

$$S^+(E) = \int_{\partial E} e^{\frac{|x|^2}{4T}}\, d\mathcal{H}^{n-1}$$

in (2.32).

The mean curvature evolutions of a sphere and of a cylinder described in Examples 3.21 and 3.22 are examples of self-similar contracting mean curvature flows (see also [26] and [87]). Another class of special solutions to mean curvature flow is given by translatory solutions(17).

Definition 3.26 (Translatory solutions). We say that $f : \mathbb{R} \to \mathcal{P}(\mathbb{R}^n)$ is a translatory evolution if there exist a set $E \subset \mathbb{R}^n$ with $\partial E \in C^\infty$ and a vector $v \in \mathbb{R}^n$, such that

$$f(t) = \overline{E} + tv, \qquad t \in \mathbb{R}. \tag{3.34}$$

For a translatory solution f, we have

$$d_f(t,z) = d(z - tv), \qquad (t,z) \in \mathbb{R} \times \mathbb{R}^n,$$

where $d(\cdot) = d(\cdot, E)$. Hence f is a translatory smooth mean curvature evolution if and only if

$$\Delta d(x) = -\langle \nabla d(x), v \rangle, \qquad x \in \partial E. \tag{3.35}$$

Note that equation (3.35) expresses the stationarity condition of ∂E for the functional

$$\int_{\partial E} e^{-\langle v,x \rangle} \, d\mathcal{H}^{n-1}.$$

Example 3.27 (The level set equation). Let $u : [a,b] \times \mathbb{R}^n \to \mathbb{R}$ and $\mathrm{A} \subset \mathbb{R}^n$ be as in Remark 3.10, and define $f : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ as in (3.13). Then, recalling the expression (1.26) of the mean curvature vector using the level sets, and the expression (3.14) of the normal velocity vector, we have that f is a smooth mean curvature flow provided

$$\frac{\frac{\partial u}{\partial t}}{|\nabla u|} \frac{\nabla u}{|\nabla u|} = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) \frac{\nabla u}{|\nabla u|} \qquad \text{on } \{u = 0\}. \tag{3.36}$$

Note that this partial differential equation is invariant under the transformation $u \to \lambda u$, where $\lambda \in \mathbb{R} \setminus \{0\}$. Note also that if u is a solution to (3.36) which is smooth in a time-space open region around one of its level sets $\{u(t,\cdot) = \lambda\}$, then the equation (3.36) impose this level set to smoothly flow by mean curvature(18).

---

(17) See, *e.g.*, [21] and references therein.

(18) The partial differential equation (3.36) is degenerate where the gradient ∇u vanishes. This equation, considered in $[0, +\infty) \times \mathbb{R}^n$, has been extensively studied, in particular in the framewok of viscosity solutions, especially when $\{u(t,\cdot) = 0\}$ is compact (see, *e.g.*, [137, 139, 140, 86, 153]). It has been studied also under convexity assumptions of the initial set, and in this case it turns out that $u(t,x) = t - w(x)$ for a suitable function w, see [137] for the details.

Since we are assuming the nondegeneracy condition (3.12), equation (3.36) can be rewritten in scalar form as

$$|\nabla u|^2 \left(\frac{\partial u}{\partial t} - \Delta u \right) = -\nabla_i u \nabla_j u \, \nabla^2_{ij} u. \tag{3.37}$$

If $|\nabla u|^2 = 1$ in a time-space neighbourhood of $\{u = 0\}$, then equation (3.37) reduces to

$$\frac{\partial u}{\partial t} = \Delta u.$$

Example 3.28 (Mean curvature flow of a graph). Following a notation similar to that of Remark 3.10, if there exists a function $v \in \mathcal{C}^\infty([a, b] \times O)$ such that

$$u(t, s, z_n) := v(t, s) - z_n, \qquad (t, s, z_n) \in [a, b] \times \mathrm{A},$$

at the points of the graph of v we have from (1.30) that the mean curvature vector equals

$$\operatorname{div} \left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} \right) \frac{(\nabla v, -1)}{\sqrt{1 + |\nabla v|^2}}.$$

The smooth mean curvature flow of the graph of v is therefore expressed by the equation[(19)]

$$\frac{\partial v}{\partial t} = \sqrt{1 + |\nabla v|^2} \operatorname{div} \left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} \right) \qquad \text{in } [a, b] \times O. \tag{3.38}$$

We will encounter this equation in Chapter 4. Observe that the velocity of the flow in the direction e_n is given by

$$-\frac{\frac{\partial u}{\partial t}}{|\nabla u|} \langle \frac{\nabla u}{|\nabla u|}, e_n \rangle^{-1} = \frac{\partial v}{\partial t}.$$

---

(19) The quasilinear parabolic equation (3.38) was studied by Ecker and Huisken in [127] and [128]. They proved, in particular, the following result ([128, Theorem 5.1]).

Theorem 3.29. *Let* $v_0 \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^{n-1})$. *Then there exists a function* $v \in \mathcal{C}^\infty((0, +\infty) \times \mathbb{R}^{n-1}) \cap \mathcal{C}([0, +\infty) \times \mathbb{R}^{n-1})$ *solving* (3.38) *in* $(0, +\infty) \times \mathbb{R}^{n-1}$ *and such that* $\lim_{t \to 0^+} v(\cdot, t) = v_0(\cdot)$ *in* $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^{n-1})$.

See also [27], [238, Section 4], [239, 91, 92, 34] and [36] for further results. Uniqueness for smooth solutions of (3.38) when $n = 2$ and the initial datum v_0 is merely continuous in $\mathbb{R}$, is proved in [35, Theorem 4.2] (see also [88]). Equation (3.38) resembles the parabolic partial differential equation $\frac{\partial v}{\partial t} = \operatorname{div} \left(\frac{\nabla v}{\sqrt{1+|\nabla v|^2}} \right)$, where the velocity in direction e_n equals the mean curvature, which has been studied, among other places, in [125].

Remark 3.30. If $n = 2$, equation (3.38) takes the form

$$\frac{\partial v}{\partial t} = \psi(v_s)_s,$$

where

$$\psi(p) = \operatorname{arctg}(p), \qquad p \in \mathbb{R}.$$

Example 3.31 (Grim reaper). If we look for special solutions to equation (3.38) of the form

$$v(t, s) = t + h(s)$$

for some smooth function $h : O \to \mathbb{R}$, we have to impose

$$\sqrt{1 + |\nabla h|^2}\, \operatorname{div}\left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}\right) = 1 \qquad \text{in } O. \tag{3.39}$$

If we assume $n = 2$ then (3.39) reduces to the following ordinary differential equation:

$$\frac{h''}{1 + h'^2} = (\psi(h'))' = 1 \qquad \text{in } O. \tag{3.40}$$

A solution to (3.40) is given by $O = (-\frac{\pi}{2}, \frac{\pi}{2})$ and

$$h(s) = -\log(\cos s), \qquad s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

The corresponding solution

$$v(t, s) = t - \log(\cos s), \qquad t \in [0, +\infty),\ s \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$

is called grim reaper, and is an example of translating solution to mean curvature flow, with the choice $E = \{(s, z_2) \in \mathbb{R}^2 : z_2 \geq -\log(\cos s), s \in (-\pi/2, \pi/2)\}$ and $v = (0, 1)$ in (3.34).

Example 3.32 (Solids of revolutions). Let $v_0 \in C^\infty(\mathbb{R}; (0, +\infty))$ and let $E \subset \mathbb{R}^3$ be the solid of revolution having as boundary the rotated graph of v, as in Example 1.17. A smooth mean curvature flow starting from E is given[20] by $f : [0, T] \to \mathcal{P}(\mathbb{R}^3)$, with

$$f(t) = \left\{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_2^2 + z_3^2 \leq (v(t, z_1))^2\right\}, \qquad t \in [0, T],$$

[20] Mean curvature flow of rotationally symmetric surfaces has been studied in [124, 252, 11, 31].

for some $T > 0$ sufficiently small, where $v \in C^\infty([0,T] \times \mathbb{R}; (0,+\infty))$ is a solution to

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{v''}{1+(v')^2} - \dfrac{1}{v}, \\ v(0,\cdot) = v_0(\cdot). \end{cases} \tag{3.41}$$

Indeed, to obtain (3.41) starting from (3.36), it is enough to recall the expression (1.33) of the mean curvature of a surface of revolution, and to use the equality

$$\frac{\frac{\partial u}{\partial t}}{|\nabla u|} = \frac{-v\frac{\partial v}{\partial t}}{(v^2(v')^2 + z_2^2 + z_3^2)^{1/2}},$$

where

$$u(t,z) := \frac{1}{2}(z_2^2 + z_3^2 - (v(t,z_3))^2).$$

Example 3.33 (Parametric plane curves). Smooth embedded closed plane curves evolving by (mean) curvature are examples of smooth compact (mean) curvature flows. It is worthwhile to observe that, in this case, it is convenient to use a parametric description of the flow(21). This is particularly useful when we look for the evolution equations satisfied by the curvature and by its derivatives, as we want to show in this example. Let $\mathbb{S}^1 \subset \mathbb{R}^2$ be the unit circle, and let

$$\gamma \in \mathcal{X}([0,T]; \mathrm{Imm}(\mathbb{S}^1; \mathbb{R}^2))$$

be a smooth parametric curvature flow(22), where $\gamma(t)(\sigma) = \gamma(t,\sigma)$ for any $(t,\sigma) \in [0,T] \times \mathbb{S}^1$. Hence

$$\partial_t \gamma = \mathrm{H}\,\nu \qquad \text{in } [0,T] \times \mathbb{S}^1, \tag{3.42}$$

where

$$\partial_t := \frac{\partial}{\partial t}.$$

(21) Let $\gamma_0 : \mathbb{S}^1 \mapsto \mathbb{R}^2$ be the initial smooth embedding. When $\gamma_0(\mathbb{S}^1)$ is convex (*i.e.*, it encloses a convex set), Gage-Hamilton showed in [149] that $\gamma(t,\cdot)$ remains convex (actually, it becomes strictly convex), evolves smoothly up to the extinction time, when it disappears, converging (under a suitable rescaling) to a circle. Grayson showed in [158] that, even when $\gamma_0(\mathbb{S}^1)$ is not convex, its curvature evolution $\gamma(t,\cdot)$ is smooth and $\gamma(t,\mathbb{S}^1)$ becomes convex (and eventually it shrinks to a point). A different proof of the same result has been given by Huisken in [172], see also [200]. Geometric evolutions of curves on a surface have been studied for instance by Angenent in [22] and [23]. See also [24], and the books [272] and [203].

(22) If $\gamma(0,\cdot)$ is an embedding, then also $\gamma(t,\cdot)$ is an embedding, since smooth parametric mean curvature flow preserves embeddedness (see Theorem 5.10).

Let s be the arclength parameter, so that, if ∂_{s} denotes the derivative with respect to s, we have

$$\partial_{\mathrm{s}} = \frac{\partial_\sigma}{|\gamma_\sigma|},$$

where

$$\partial_\sigma = \frac{\partial}{\partial \sigma}.$$

The arclength parameter s ranges between zero and the length of $\gamma(t, \cdot)$; therefore, it depends on t. Indeed, the following commutation rule holds:

$$\partial_t \partial_{\mathrm{s}} = \partial_{\mathrm{s}} \partial_t + \mathrm{H}^2 \partial_{\mathrm{s}}. \tag{3.43}$$

To show (3.43), we note that

$$\frac{\partial_t \partial_\sigma}{|\gamma_\sigma|} = \frac{\partial_\sigma}{|\gamma_\sigma|} \partial_t = \partial_{\mathrm{s}} \partial_t.$$

Hence, if we let $\tau(t) := \gamma_{\mathrm{s}}(t)$,

$$\begin{aligned}
\partial_t \partial_{\mathrm{s}} = \partial_t \left(\frac{\partial_\sigma}{|\gamma_\sigma|} \right) &= \frac{\partial_t \partial_\sigma}{|\gamma_\sigma|} - \frac{\langle \gamma_\sigma, \partial_t \gamma_\sigma \rangle \partial_\sigma}{|\gamma_\sigma|^3} = \frac{\partial_\sigma}{|\gamma_\sigma|} \partial_t - \langle \frac{\gamma_\sigma}{|\gamma_\sigma|}, \frac{\partial_t \gamma_\sigma}{|\gamma_\sigma|} \rangle \frac{\partial_\sigma}{|\gamma_\sigma|} \\
&= \partial_{\mathrm{s}} \partial_t - \langle \tau, \frac{\partial_t \partial_\sigma \gamma}{|\gamma_\sigma|} \rangle \partial_{\mathrm{s}} = \partial_{\mathrm{s}} \partial_t - \langle \tau, \frac{\partial_\sigma \partial_t \gamma}{|\gamma_\sigma|} \rangle \partial_{\mathrm{s}} \\
&= \partial_{\mathrm{s}} \partial_t - \langle \tau, \partial_{\mathrm{s}} \partial_t \gamma \rangle \partial_{\mathrm{s}}.
\end{aligned}$$

From the Frenet-Serret formulas[(23)] and the evolution equation (3.42), we have

$$\partial_{\mathrm{s}} \partial_t \gamma = (\partial_{\mathrm{s}} \mathrm{H}) \nu - \mathrm{H}^2 \tau,$$

and therefore the commutation rule (3.43) follows.

The evolution equations for the normal ν and for the curvature H read as follows:

$$\partial_t \nu = -(\partial_{\mathrm{s}} \mathrm{H}) \tau, \tag{3.44}$$

$$\partial_t \mathrm{H} = \partial_{\mathrm{s}}^2 \mathrm{H} + \mathrm{H}^3. \tag{3.45}$$

Indeed,

$$\begin{aligned}
\partial_t \tau &= \partial_t \partial_{\mathrm{s}} \gamma = \partial_{\mathrm{s}} \partial_t \gamma + \mathrm{H}^2 \partial_{\mathrm{s}} \gamma \\
&= \partial_{\mathrm{s}} (\mathrm{H} \nu) + \mathrm{H}^2 \tau = (\partial_{\mathrm{s}} \mathrm{H}) \nu + \mathrm{H} \partial_{\mathrm{s}} \nu + \mathrm{H}^2 \tau = (\partial_{\mathrm{s}} \mathrm{H}) \nu,
\end{aligned}$$

(23) That is, $\partial_{\mathrm{s}} \tau = \mathrm{H} \nu$, $\partial_{\mathrm{s}} \nu = -\mathrm{H} \tau$.

and (3.44) follows. Moreover, we have $\langle \partial_s \tau, \partial_t \nu \rangle = \mathrm{H}\langle \nu, \partial_t \nu \rangle = 0$, since $|\nu|^2 = 1$. Therefore

$$\partial_t \mathrm{H} = \partial_t \langle \partial_s \tau, \nu \rangle = \langle \partial_t \partial_s \tau, \nu \rangle .$$

Using the commutation rule (3.43) and the evolution equation (3.42), we have

$$\begin{aligned}
\partial_t \mathrm{H} =& \langle \partial_t \partial_s \tau, \nu \rangle = \langle \partial_s \partial_t \partial_s \gamma, \nu \rangle + \mathrm{H}^2 \langle \partial_s \tau, \nu \rangle \\
=& \langle \partial_s^2 \partial_t \gamma, \nu \rangle + \langle \partial_s \left[\mathrm{H}^2 \partial_s \gamma \right], \nu \rangle + \mathrm{H}^3 \\
=& \langle \partial_s^2 (\mathrm{H}\nu), \nu \rangle + \mathrm{H}^2 \langle \partial_s^2 \gamma, \nu \rangle + \mathrm{H}^3 \\
=& \partial_s^2 \mathrm{H} + \mathrm{H} \langle \partial_s^2 \nu, \nu \rangle + 2\mathrm{H}^3 = \partial_s^2 \mathrm{H} + \mathrm{H}^3,
\end{aligned}$$

which is (3.45).

As we shall see in the sequel, also the equations for H^2 and for all derivatives of H are of importance: the equation for H^2 follows directly from (3.45), and we have

$$\partial_t (\mathrm{H}^2) = 2\mathrm{H} \left(\partial_s^2 \mathrm{H} + \mathrm{H}^3 \right) = \partial_s^2 (\mathrm{H}^2) - 2(\partial_s \mathrm{H})^2 + 2\mathrm{H}^4. \tag{3.46}$$

In order to compute the evolution equation for the derivatives of the curvature, we need to introduce some notation. We use the symbol $\mathfrak{q}^r(\partial_s^l \mathrm{H})$ to denote a polynomial in the curvature and its derivatives up to the order $l \in \mathbb{N}$, such that every of its monomials is of the form

$$\prod_{i=1}^{N} \partial_s^{j_i} \mathrm{H} \qquad \text{with } 0 \le j_i \le l \text{ and } N \ge 1$$

with

$$r = \sum_{i=1}^{N} (j_i + 1).$$

For instance, if $j \in \mathbb{N}$ is given,

$$\mathrm{H}^2 \partial_s^j \mathrm{H}$$

can be expressed as

$$\mathfrak{q}^{j+3}(\partial_s^j \mathrm{H}).$$

Indeed, in this case we have only one monomial, $N = 3$, $j_1 = j_2 = 0$, $j_3 = j$, so that $r = 1 + 1 + (j + 1) = j + 3$. In a similar manner, $\partial_s(\mathrm{H}^2 \partial_s^j \mathrm{H})$ can be expressed as $\mathfrak{q}^{j+4}(\partial_s^{j+1} \mathrm{H})$.

Then, the evolution equation for the derivatives of H reads as follows. Let $j \in \mathbb{N}$; then

$$\partial_t \partial_s^j \mathrm{H} = \partial_s^{j+2} \mathrm{H} + \mathfrak{q}^{j+3}(\partial_s^j \mathrm{H}). \tag{3.47}$$

To prove (3.47), we argue by induction on j. The case $j = 0$ is equation (3.45). Suppose now that (3.47) holds for $j-1$; using the commutation rule (3.43) and the inductive hypothesis, we get

$$\begin{aligned}\partial_t \partial_s^j \mathrm{H} &= \partial_s \partial_t \partial_s^{j-1}\mathrm{H} + \mathrm{H}^2 \partial_s^j \mathrm{H} \\ &= \partial_s \left[\partial_s^{j+1}\mathrm{H} + \mathfrak{q}^{j+2}(\partial_s^{j-1}\mathrm{H})\right] + \mathfrak{q}^{j+3}(\partial_s^j \mathrm{H}),\end{aligned}$$

where we express $\mathrm{H}^2\partial_s^j\mathrm{H}$ as $\mathfrak{q}^{j+3}(\partial_s^j\mathrm{H})$. Hence, observing that

$$\partial_s\left[\mathfrak{q}^{j+2}(\partial_s^{j-1}\mathrm{H})\right] = \mathfrak{q}^{j+3}(\partial_s^j\mathrm{H}),$$

we deduce

$$\partial_t \partial_s^j \mathrm{H} = \partial_s^{j+2}\mathrm{H} + \mathfrak{q}^{j+3}(\partial_s^j\mathrm{H}),$$

which gives the inductive step.

3.3. Extensions and tangential derivatives

In this short section we extend some of the definitions given in Chapter 1 to a smooth flow. These definitions will be useful especially in Chapters 4 and 6.

Let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth flow, and let d and Q be as in Definition 3.1. We set

$$\mathrm{pr}(t, z) := z - d(t, z)\nabla d(t, z), \qquad (t, z) \in Q.$$

For a given $t \in [a, b]$, it follows from Theorem 1.5 that $\mathrm{pr}(t, z)$ is the unique projection point of z on $\partial f(t)$.
Let $u \in \mathcal{C}^\infty(Q)$. We set

$$\overline{u}(t, z) := u(t, \mathrm{pr}(t, z)), \qquad (t, z) \in Q, \tag{3.48}$$

see formula (1.42)[(24)].

In a similar way to (1.12), we set also

$$\delta u := \nabla u - \langle \nabla d, \nabla u\rangle \nabla d = (\delta_1 u, \ldots, \delta_n u) \qquad \text{in } Q. \tag{3.49}$$

Notice that

$$\delta \overline{u} = \nabla \overline{u} \qquad \text{in } Q. \tag{3.50}$$

(24) It is clear that $\overline{u}(t, \cdot)$ coincides with the canonical extension of the restriction of $u(t, \cdot)$ to $\partial f(t)$ (see Definition 1.23).

Indeed, since $\nabla_i \mathrm{pr}_j = \mathrm{Id}_{ij} - \nabla_i d \nabla_j d - d\nabla_{ij} d$ (see (1.10)), we have, using (1.17),

$$\begin{aligned}\langle \nabla d, \nabla \overline{u}\,\rangle =& \nabla_i d \nabla_j u \nabla_i \mathrm{pr}_j \\ =& \nabla_i d \nabla_j u \left(\mathrm{Id}_{ij} - \nabla_i d \nabla_j d - d\nabla_{ij} d\right) \\ =& \nabla_i d \nabla_j u \big(\mathrm{Id}_{ij} - \nabla_i d \nabla_j d\big) = 0 \qquad \text{in } Q.\end{aligned} \tag{3.51}$$

We conclude the chapter with the definition of the derivative $\frac{d}{dt}$, which is sometimes called material (or convective) derivative. This concept has been already used in Chapter 2; see formula (2.29).

Definition 3.34 (The derivative d/dt**).** Let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth flow. Let $u \in \mathcal{C}^1(Q)$. We define

$$\frac{du}{dt} := \frac{\partial u}{\partial t} - \langle \frac{\partial d}{\partial t} \nabla d, \nabla u\rangle \qquad \text{in } Q. \tag{3.52}$$

Remark 3.35. From (3.51) it follows that

$$\frac{d\overline{u}}{dt} = \frac{\partial \overline{u}}{\partial t} \qquad \text{in } Q.$$

Chapter 4
Huisken's monotonicity formula

In this chapter we prove Huisken's monotonicity formula [170][(1), (2)], which describes how the perimeter of a smooth hypersurface flowing by mean curvature changes when weighted with a suitable heat kernel. Here we limit ourselves to describe only one application of this formula; several other applications and a much wider discussion (in particular related to the study of singularities of mean curvature flow) can be found in [170, 171, 126] and [203].

In what follows, for a smooth mean curvature flow $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$, the function d and the set Q are given as in Definition 3.1. We will need also the definition of the δ-operator given in (3.49), and of the (convective) derivative d/dt introduced in Definition 3.34.

Lemma 4.1. *Let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth compact mean curvature flow. Let $\theta \in \mathcal{C}^\infty(Q)$. Then, for any $t \in [a, b]$, we have*

$$\begin{aligned} &\frac{d}{dt}\int_{\partial f(t)} \theta \, d\mathcal{H}^{n-1} \\ &= \int_{\partial f(t)} \left(\frac{d\theta}{dt} - \theta \, (\Delta d)^2\right) d\mathcal{H}^{n-1} \qquad (4.1) \\ &= \int_{\partial f(t)} \frac{\partial \theta}{\partial t} \, d\mathcal{H}^{n-1} + \mathcal{B}(f(t), \theta), \end{aligned}$$

(1) In [67, Section 6.8] certain heat-type kernels were integrated over a mean curvature flow.

(2) For other types of monotonicity formulas, such as monotonicity formulas for minimal (or even critical) manifolds we refer the reader to [100, formula (54)], [143, Section 5.4], [8], [67, Section 4.17] and [242, 208]. For monotonicity formulas for other kinds of partial differential equations see, *e.g.*, [258, 177] and [16].

where

$$\mathcal{B}(f(t), \theta) := -\int_{\partial f(t)} \Delta d \, \langle \nabla d, \nabla \theta \rangle \, d\mathcal{H}^{n-1} - \int_{\partial f(t)} \theta \, (\Delta d)^2 \, d\mathcal{H}^{n-1}.$$

Proof. This follows from formula (3.18), recalling from (3.52) that

$$\frac{d\theta}{dt} = \frac{\partial \theta}{\partial t} - \Delta d \langle \nabla d, \nabla \theta \rangle. \tag{4.2}$$

□

Remark 4.2. Using the integration by parts formula in Theorem 2.6 it follows that

$$\int_{\partial f(t)} \delta_h \nabla_h \theta \, d\mathcal{H}^{n-1} - \int_{\partial f(t)} \Delta d \, \langle \nabla d, \nabla \theta \rangle \, d\mathcal{H}^{n-1} = 0, \qquad t \in [a, b].$$

Therefore, assuming

$$\theta > 0 \qquad \text{in } Q,$$

we deduce an alternative expression of the right-hand side of (4.1), namely

$$\begin{aligned} \frac{d}{dt} \int_{\partial f(t)} \theta \, d\mathcal{H}^{n-1} = & -\int_{\partial f(t)} \theta \left(\Delta d + \frac{1}{\theta} \langle \nabla d, \nabla \theta \rangle \right)^2 \, d\mathcal{H}^{n-1} \\ & + \int_{\partial f(t)} \left(\frac{\partial \theta}{\partial t} + \frac{1}{\theta} \langle \nabla d, \nabla \theta \rangle^2 + \delta_h \nabla_h \theta \right) d\mathcal{H}^{n-1}. \end{aligned} \tag{4.3}$$

Definition 4.3 (The function $\rho_{(t_0,z_0)}$). Let $t_0 \in \mathbb{R}$ and $z_0 \in \mathbb{R}^n$. We set

$$\rho_{(t_0,z_0)}(t, z) := \frac{e^{-\frac{|z-z_0|^2}{4(t_0-t)}}}{(4\pi(t_0 - t))^{\frac{n-1}{2}}}, \qquad t < t_0, \; z \in \mathbb{R}^n. \tag{4.4}$$

Note that if we restrict the right-hand side of (4.4) to points z_0, z into an $(n-1)$-dimensional subspace of $\mathbb{R}^n$, then $\rho_{(t_0,z_0)}$ solves

$$\frac{\partial \rho_{(t_0,z_0)}}{\partial t} + \sum_{i=1}^{n-1} \frac{\partial^2 \rho_{(t_0,z_0)}}{\partial^2 z_i} = 0.$$

The proof of the next theorem is based on the equation (4.7) satisfied by $\rho_{(t_0,z_0)}$.

Theorem 4.4 (Monotonicity formula). *Let*

$$f : [a, b] \subset (-\infty, t_0) \to \mathcal{P}(\mathbb{R}^n)$$

be a smooth compact mean curvature flow. Then, for any $t \in [a, b]$, we have

$$\begin{aligned} &\frac{d}{dt}\int_{\partial f(t)} \rho_{(t_0,z_0)}\, d\mathcal{H}^{n-1} \\ &= -\int_{\partial f(t)} \left(\Delta d + \frac{\langle \nabla d, \nabla \rho_{(t_0,z_0)}\rangle}{\rho_{(t_0,z_0)}}\right)^2 \rho_{(t_0,z_0)}\, d\mathcal{H}^{n-1}. \end{aligned} \tag{4.5}$$

In particular,

$$\frac{d}{dt}\int_{\partial f(t)} \rho_{(t_0,z_0)}\, d\mathcal{H}^{n-1} \leq 0, \qquad t \in [a, b]. \tag{4.6}$$

Proof. Since $\rho_{(t_0,z_0)}$ is positive, we can use formula (4.3) with the choice $\theta = \rho_{(t_0,z_0)}$. Then equality (4.5) will follow if we prove that

$$\frac{\partial \rho_{(t_0,z_0)}}{\partial t} + \frac{1}{\rho_{(t_0,z_0)}}\langle \nabla d, \nabla \rho_{(t_0,z_0)}\rangle^2 + \delta_h \nabla_h \rho_{(t_0,z_0)} = 0 \qquad \text{on } \partial f(t). \tag{4.7}$$

Assume, for simplicity of notation, that $t_0 = 0$ and $z_0 = 0$, and set $\rho := \rho_{(t_0,z_0)}$, so that

$$\rho(t, z) = \frac{e^{\frac{|z|^2}{4t}}}{(-4\pi t)^{\frac{n-1}{2}}}, \qquad t < 0,\ z \in \mathbb{R}^n.$$

For $(t, z) \in (-\infty, 0) \times \mathbb{R}^n$ we have

$$\begin{aligned} \frac{\partial \rho}{\partial t}(t, z) &= -\left(\frac{|z|^2}{4t^2} + \frac{n-1}{2t}\right)\rho(t, z), \\ \nabla \rho(t, z) &= \frac{\rho(t, z)}{2t} z, \\ \nabla^2_{ij}\rho(t, z) &= \left(\frac{1}{4t^2}\, z_i z_j + \frac{1}{2t}\, \mathrm{Id}_{ij}\right)\rho(t, z), \qquad i, j \in \{1, \ldots, n\}. \end{aligned} \tag{4.8}$$

Hence, for $(t, z) \in Q$, we have

$$\begin{aligned} \delta_h \nabla_h \rho(t, z) &= \nabla^2_{hh}\rho(t, z) - \nabla_i d(t, z)\nabla_h d(t, x)\nabla^2_{ih}\rho(t, z) \\ &= \left(\frac{|z|^2}{4t^2} + \frac{n}{2t} - \frac{\langle \nabla d(t, z), z\rangle^2}{4t^2} - \frac{1}{2t}\right)\rho(t, z) \\ &= \left(\frac{|z|^2}{4t^2} - \frac{\langle \nabla d(t, z), z\rangle^2}{4t^2} + \frac{n-1}{2t}\right)\rho(t, z). \end{aligned} \tag{4.9}$$

Since

$$\frac{1}{\rho(t,z)}\langle \nabla d(t,z), \nabla \rho(t,z)\rangle^2 = \frac{1}{4t^2}\langle \nabla d(t,z), z\rangle^2 \rho(t,z),$$

from (4.8) and (4.9) we deduce (4.7). □

Note that we can rewrite formula (4.5) as follows:

$$\begin{aligned}
&\frac{d}{dt}\int_{\partial f(t)} \rho_{(t_0,z_0)}(t,x)\, d\mathcal{H}^{n-1}(x) \\
=&-\int_{\partial f(t)}\left(\Delta d(t,x) - \frac{\langle \nabla d(t,x), x-z_0\rangle}{2(t_0-t)}\right)^2 \rho_{(t_0,z_0)}(t,x) d\mathcal{H}^{n-1}(x) \le 0,
\end{aligned}$$

which is directly related to Example 2.12.

A more general weighted monotonicity formula is proved in [127][(3)]. This formula reads as follows.

Theorem 4.5 (Weighted monotonicity formula). *Let $\rho_{(t_0,z_0)}$ and f be as in Theorem* 4.4. *Let $\theta \in \mathcal{C}^\infty(Q)$. Then, for any $t \in [a,b]$, we have*

$$\begin{aligned}
&\frac{d}{dt}\int_{\partial f(t)} \theta \rho_{(t_0,z_0)}\, d\mathcal{H}^{n-1} \\
=&-\int_{\partial f(t)} \theta\left(\Delta d + \frac{\langle \nabla d, \nabla \rho_{(t_0,z_0)}\rangle}{\rho_{(t_0,z_0)}}\right)^2 \rho_{(t_0,z_0)}\, d\mathcal{H}^{n-1} \\
&+\int_{\partial f(t)}\left(\frac{d\theta}{dt} - \delta_h\delta_h\theta\right) \rho_{(t_0,z_0)}\, d\mathcal{H}^{n-1}.
\end{aligned} \tag{4.10}$$

In particular, if θ satisfies

$$\theta \ge 0 \qquad \text{in } Q,$$

and

$$\frac{d\theta}{dt} - \delta_h\delta_h\theta \le 0 \qquad \text{on } \partial f(t),\ t \in [a,b],$$

then

$$\frac{d}{dt}\int_{\partial f(t)} \theta\rho_{(t_0,z_0)}\, d\mathcal{H}^{n-1} \le 0, \qquad t \in [a,b].$$

(3) Even for unbounded boundaries, provided all involved integrals are finite.

Proof. Set $\rho = \rho_{(t_0, z_0)}$. Recalling (4.1), we have

$$\frac{d}{dt}\int_{\partial f(t)} \theta\rho \, d\mathcal{H}^{n-1} = \int_{\partial f(t)} \left(\rho\frac{d\theta}{dt} + \theta\frac{d\rho}{dt} - \theta\rho\,(\Delta d)^2\right) d\mathcal{H}^{n-1}.$$

Adding and subtracting the term

$$\int_{\partial f(t)} \rho\, \delta_h\delta_h\theta \, d\mathcal{H}^{n-1}$$

and using the tangential integration by parts formula (2.23), we can write

$$\int_{\partial f(t)} \rho\frac{d\theta}{dt}\, d\mathcal{H}^{n-1} = \int_{\partial f(t)} \left[\rho\left(\frac{d\theta}{dt} - \delta_h\delta_h\theta\right) + \theta\delta_h\delta_h\rho\right] d\mathcal{H}^{n-1}.$$

Hence

$$\begin{aligned}\frac{d}{dt}\int_{\partial E(t)} \theta\,\rho \, d\mathcal{H}^{n-1} &= \int_{\partial f(t)} \theta\left(\frac{d\rho}{dt} + \delta_h\delta_h\rho - \rho\,(\Delta d)^2\right) d\mathcal{H}^{n-1}\\ &\quad + \int_{\partial f(t)}\left(\frac{d\theta}{dt} - \delta_h\delta_h\theta\right)\rho\, d\mathcal{H}^{n-1}.\end{aligned}$$

Therefore, to prove (4.10), it is sufficient to show that

$$\begin{aligned}&\frac{\partial\rho}{\partial t} - \Delta d\,\langle\nabla d, \nabla\rho\rangle + \delta_h\delta_h\rho - \rho\,(\Delta d)^2\\ &= -\rho\left(\Delta d + \frac{1}{\rho}\langle\nabla d, \nabla\rho\rangle\right)^2 \qquad \text{on } \partial f(t).\end{aligned} \tag{4.11}$$

Let us compute

$$\frac{\partial\rho}{\partial t} - \Delta d\,\langle\nabla d, \nabla\rho\rangle + \delta_h\delta_h\rho.$$

Using (4.7) and the equality

$$\delta_h\nabla_h\rho = \delta_h\delta_h\rho + \Delta d\langle\nabla d, \nabla\rho\rangle,$$

which follows by δ_h-differentiating the relation[(4)] $\delta_h\rho = \nabla_h\rho -$

[(4)] From (1.20) we have

$$\delta_h\delta_h\rho = \delta_h\nabla_h\rho - \delta_h\langle\nabla d, \nabla\rho\rangle\nabla_h d - \langle\nabla d, \nabla\rho\rangle\delta_h\nabla_h d = \delta_h\nabla_h\rho - \Delta d\langle\nabla d, \nabla\rho\rangle.$$

$\langle \nabla d, \nabla \rho \rangle \nabla_h d$, we have

$$
\begin{aligned}
&\frac{\partial \rho}{\partial t} - \Delta d \, \langle \nabla d, \nabla \rho \rangle + \delta_h \delta_h \rho \\
=& - \frac{1}{\rho} \langle \nabla d, \nabla \rho \rangle^2 - \delta_h \nabla_h \rho - \Delta d \langle \nabla d, \nabla \rho \rangle + \delta_h \delta_h \rho \\
=& - 2 \Delta d \, \langle \nabla d, \nabla \rho \rangle - \frac{1}{\rho} \langle \nabla d, \nabla \rho \rangle^2,
\end{aligned}
$$

and (4.11) follows. □

Remark 4.6. Theorem 4.4 still holds in the special case when $\partial f(a) = \partial E$ is a smooth hypersurface with zero mean curvature (hence which stays still; see Example 3.20), and provided all integrals involved converge. We have in this case, for $(t_0, z_0) \in \mathbb{R} \times \mathbb{R}^n$,

$$
\begin{aligned}
&\frac{d}{dt} \int_{\partial E} \rho_{(t_0,z_0)}(t, x) \, d\mathcal{H}^{n-1}(x) \\
&= - \int_{\partial E} \langle \nabla d(x), \nabla \log \rho_{(t_0,z_0)}(t, x) \rangle^2 \rho_{(t_0,z_0)}(t, x) \, d\mathcal{H}^{n-1}(x).
\end{aligned}
$$

4.1. The gradient estimate of Ecker and Huisken

In this section, closely following [211], we show an application of the monotonicity formula: we derive a maximum principle for the gradient, for the quasilinear parabolic equation (3.38) expressing the mean curvature flow of a graph over $\mathbb{R}^{n-1}$. We refer to the original paper [127] of Ecker–Huisken for all details, and to the book [126].

Given a smooth function $v = v(t, s) : (0, T) \times \mathbb{R}^{n-1} \to \mathbb{R}$, we denote here by ∇v the gradient of v with respect to s.

Theorem 4.7 (The gradient estimate). *Let* $v_0 \in \mathrm{Lip}(\mathbb{R}^{n-1})$ *and assume that there exist* $T > 0$ *and a function* $v \in \mathcal{C}^\infty((0, T] \times \mathbb{R}^{n-1})$ *with the following properties:*

- *the map* $f : (0, T] \to \mathcal{P}(\mathbb{R}^n)$ *defined by*

$$
f(t) := \{(s, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y \le v(t, s)\}, \qquad t \in [0, T].
$$

is a smooth mean curvature flow[5], *hence* v *satisfies the equation*

$$
\frac{\partial v}{\partial t} = \sqrt{1 + |\nabla v|^2} \, \mathrm{div} \left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} \right) \qquad \text{in } (0, T) \times \mathbb{R}^{n-1},
$$

[5] Replace $[a, b]$ with $(0, T]$ in Definition 3.1.

- $\lim_{t\downarrow 0} v(t,s) = v_0(s)$ *for any* $s \in \mathbb{R}^{n-1}$.

Then

$$\sup_{s\in\mathbb{R}^{n-1}} |\nabla v(t,s)| \le \sup_{s\in\mathbb{R}^{n-1}} |\nabla v_0(s)|, \qquad t \in (0,T). \tag{4.12}$$

Proof. Let us consider the signed distance function d and the set Q in a similar way to Definition 3.1.

We are interested in the last component $\nabla_n d(t,x)$ of the unit normal to

$$\partial f(t) = \mathrm{graph}(v(t,\cdot))$$

at $x = (s, v(t,s))$, which has the expression

$$\frac{1}{\sqrt{1+|\nabla v(t,s)|^2}}.$$

Define[(6)]

$$w(t,z) := \frac{1}{\nabla_n d(t,z)} > 0, \qquad (t,z) \in Q,$$

and, for almost every $s \in \mathbb{R}^{n-1}$, if $x = (s, v_0(s))$, set $w(0,x) := \sqrt{1+|\nabla v_0(s)|^2} > 0$.

The thesis (4.12) is equivalent to

$$\sup_{(t,s)\in(0,T)\times\mathbb{R}^{n-1}} w(t,s,v(t,s)) = \sup_{t=0,\ s\in\mathbb{R}^{n-1}} w(t,s,v(t,s)).$$

Let $p > 2$ and define

$$\theta := \alpha(w) \ge 0,$$

where, for any $\sigma \in \mathbb{R}$,

$$\alpha(\sigma) := \Big[\max\big(\sigma, \|w(0,\cdot,v(0,\cdot))\|_{L^\infty(\mathbb{R}^{n-1})}\big) - \|w(0,\cdot,v(0,\cdot))\|_{L^\infty(\mathbb{R}^{n-1})}\Big]^p.$$

The function α has the following properties:

- it is nondecreasing,
- it is convex and nonnegative,
- $\alpha(\sigma) = 0$ for $\sigma \le \|w(0,\cdot,v(0,\cdot))\|_{L^\infty(\mathbb{R}^{n-1})}$.

(6) In minimal surface theory, the gradient estimate for a $\mathcal{C}^2$ solution of the minimal surface equation [101] is obtained by exploiting a differential inequality involving w (see *e.g.*, [156]). In our context, the differential inequality that we need is (4.20) below.

Moreover, the choice $p > 2$ implies that

- $\alpha \in \mathcal{C}^2(\mathbb{R})$.

Formula (4.12) is equivalent to prove that

$$\theta \equiv 0 \qquad \text{in } (0, T) \times \mathbb{R}^n.$$

To prove that θ is identically zero, it is enough to show that

$$\int_{\partial f(t)} \theta\rho \, d\mathcal{H}^{n-1} = 0, \qquad t \in (0, T), \tag{4.13}$$

where $\rho = \rho_{(t_0,z_0)}$, for some $t_0 > T$ and $z_0 \in \mathbb{R}^n$.

Since $\theta = 0$ almost everywhere on $\partial f(0)$, we have $\int_{\partial f(0)} \theta\rho \, d\mathcal{H}^{n-1} = 0$. Therefore, in order to prove (4.13) it is sufficient to show that

$$\frac{d}{dt} \int_{\partial f(t)} \theta\rho \, d\mathcal{H}^{n-1} \leq 0, \qquad t \in (0, T). \tag{4.14}$$

It is possible to prove that the weighted monotonicity formula (4.10) is still valid when $\partial f(t)$ is unbounded (namely when f is merely a smooth mean curvature flow), provided all involved integrals converge. Theorem 4.5 then ensures that (4.14) follows, once we prove that

$$\frac{d\theta}{dt} - \delta_h \delta_h \theta \leq 0 \qquad \text{on } \partial f(t), \ t \in (0, T). \tag{4.15}$$

Observe that

$$\delta w = -w^2 \, \delta \nabla_n d \qquad \text{in } Q. \tag{4.16}$$

The evolution equation satisfied by $\nabla_n d$ reads as follows[(7)]:

$$\frac{d}{dt} \nabla_n d = \delta_n \Delta d \qquad \text{on } \partial f(t), \ t \in (0, T). \tag{4.17}$$

Moreover

$$\frac{dw}{dt} = -w^2 \frac{d}{dt} \nabla_n d = -w^2 \delta_n \Delta d \qquad \text{on } \partial f(t), \ t \in (0, T). \tag{4.18}$$

[(7)] We will obtain the evolution equation for ∇d in Lemma 6.5. In any case, this equation can be found as follows. Write $f(t) = \{u(t, \cdot) \leq 0\}$, with $\partial f(t) = \{u(t, \cdot) = 0\}$, for a smooth function u with nonvanishing gradient in a neighbourhood of $\{u = 0\}$. For any $j \in \{1, \ldots, n\}$ we have, using the expression (3.14) of the normal velocity, and the evolution equation (3.36),

$$\frac{d}{dt} \frac{\nabla_j u}{|\nabla u|} = \frac{\partial}{\partial t} \frac{\nabla_j u}{|\nabla u|} - H \langle \frac{\nabla u}{|\nabla u|}, \nabla \frac{\nabla_j u}{|\nabla u|} \rangle.$$

Now, let us compute $\delta_n \Delta d$. We have in Q, using (1.19) and the commutation rule (1.14)[8],

$$\delta_n \Delta d = \delta_n \delta_h \nabla_h d = \delta_h \delta_n \nabla_h d + (\nabla_n d \; \delta_h \nabla_i d - \nabla_h d \; \delta_n \nabla_i d) \, \delta_i \nabla_h d.$$

Since $\nabla_h d \; \delta_i \nabla_h d = \nabla_h d \nabla^2_{ih} d = 0$, it follows that

$$\delta_n \Delta d = \delta_h \delta_n \nabla_h d + \nabla_n d \; \delta_h \nabla_i d \; \delta_i \nabla_h d = \delta_h \delta_h \nabla_n d + \nabla_n d \; |\nabla^2 d|^2.$$

Hence

$$w^2 \delta_n \Delta d = w^2 \delta_h \delta_h \nabla_n d + w |\nabla^2 d|^2 \qquad \text{on } \partial f(t). \tag{4.19}$$

From (4.18) and (4.19) we get

$$\frac{dw}{dt} = -w^2 \delta_h \delta_h \nabla_n d - w |\nabla^2 d|^2 \qquad \text{on } \partial f(t).$$

Observe now that

$$-w^2 \delta_h \delta_h \nabla_n d = \delta_h \delta_h w - 2 w^{-1} |\delta w|^2.$$

Indeed, using (4.16) twice,

$$\begin{aligned} \delta_h \delta_h w = \delta_h (-w^2 \delta_h \nabla_n d) &= -w^2 \delta_h \delta_h \nabla_n d - 2w \; \delta_h w \; \delta_h \nabla_n d \\ &= -w^2 \delta_h \delta_h \nabla_n d + 2 w^3 \delta_h \nabla_n d \; \delta_h \nabla_n d \\ &= -w^2 \delta_h \delta_h \nabla_n d + 2 w^{-1} |\delta w|^2. \end{aligned}$$

A direct computation gives

$$\begin{aligned} \frac{\partial}{\partial t} \frac{\nabla_j u}{|\nabla u|} =& \frac{\nabla_j \frac{\partial u}{\partial t}}{|\nabla u|} - \frac{\nabla_j u}{|\nabla u|^2} \langle \frac{\nabla u}{|\nabla u|}, \nabla \frac{\partial u}{\partial t} \rangle = \frac{\nabla_j (|\nabla u| H)}{|\nabla u|} - \frac{\nabla_j u}{|\nabla u|^2} \langle \frac{\nabla u}{|\nabla u|}, \nabla (|\nabla u| H) \rangle \\ =& H \langle \nabla_j \nabla u, \frac{\nabla u}{|\nabla u|^2} \rangle + \nabla_j H - H \frac{\nabla_j u}{|\nabla u|} \langle \frac{\nabla^2 u}{|\nabla u} \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \rangle - \frac{\nabla_j u}{|\nabla u|} \langle \frac{\nabla u}{|\nabla u|}, \nabla H \rangle \end{aligned}$$

and

$$-H \langle \frac{\nabla u}{|\nabla u|}, \nabla \frac{\nabla_j u}{|\nabla u|} \rangle = -H \langle \nabla_j \nabla u, \frac{\nabla u}{|\nabla u|^2} \rangle + H \frac{\nabla_j u}{|\nabla u|} \langle \frac{\nabla^2 u}{|\nabla u} \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|} \rangle.$$

Therefore

$$\frac{d}{dt} \frac{\nabla_j u}{|\nabla u|} = \nabla_j H - \frac{\nabla_j u}{|\nabla u|} \langle \frac{\nabla u}{|\nabla u|}, \nabla H \rangle$$

which, for $j = n$, is (4.17).

[8] Formula (4.19) can be proved without using (1.14), by means of Lemma 1.31. Indeed, following the notation of Chapter 6 (see formula (6.8)) and (1.9), we have

$$\left. \begin{aligned} \delta_n \overline{\Delta d} = \nabla_n \overline{\Delta d} =& \nabla_n \big(\nabla^2_{ij} d \; G_{ji} \big) = \nabla^3_{ijn} d \; \mathrm{Id}_{ij} + \nabla^2_{ij} d \; \nabla_n d \; \nabla^2_{ij} d + \mathcal{O}(d) \\ =& \Delta \nabla_n d + \nabla_n d |\nabla^2 d|^2 + \mathcal{O}(d) = \Delta \overline{\nabla_n d} + \nabla_n d |\nabla^2 d|^2 + \mathcal{O}(d) \end{aligned} \right. \qquad \text{in } Q,$$

and (4.19) follows from Lemma 1.31.

Hence

$$\frac{dw}{dt} - \delta_h \delta_h w = -w|\nabla^2 d|^2 - 2w^{-1}|\delta w|^2 \leq 0 \qquad \text{on } \partial f(t). \tag{4.20}$$

We then have, using the convexity and the monotonicity of α, and (4.20),

$$\begin{aligned} \frac{d\theta}{dt} - \delta_h \delta_h \theta &= \alpha'(w) \left(\frac{dw}{dt} - \delta_h \delta_h w \right) - \alpha''(w)|\delta w|^2 \\ &\leq \alpha'(w) \left(\frac{dw}{dt} - \delta_h \delta_h w \right) \leq 0 \qquad \text{on } \partial f(t), \end{aligned}$$

which shows (4.15) □

Once (4.12) is established, the authors of [127] are able to prove the following global in time existence result for mean curvature evolution of graphs.

Theorem 4.8. *Let* $v_0 : \mathbb{R}^{n-1} \to \mathbb{R}$ *be a Lipschitz function. Then there exists a smooth function* $v : (0, +\infty) \times \mathbb{R}^{n-1} \to \mathbb{R}$ *satisfying*

$$\frac{\partial v}{\partial t} = \sqrt{1 + |\nabla v|^2} \, \mathrm{div} \left(\frac{\nabla v}{\sqrt{1 + |\nabla v|^2}} \right)$$

in $(0, +\infty) \times \mathbb{R}^{n-1}$, *such that* $\lim_{t \downarrow 0} v(t, s) = v_0(s)$ *for any* $s \in \mathbb{R}^{n-1}$, *and*

$$\sup_{s \in \mathbb{R}^{n-1}} |\nabla v(t, s)| \leq \sup_{s \in \mathbb{R}^{n-1}} |\nabla v_0(s)|, \qquad t \in (0, +\infty).$$

Chapter 5
Inclusion principle

In this chapter we prove the inclusion principle between two smooth compact mean curvature flows. This principle is one of the most useful tools in the analysis of mean curvature flow and indeed, as we shall see, it has a lot of consequences.

We begin with the following lemma, which compares the mean curvature of two smooth boundaries which are tangent at a point, under the assumption of a local inclusion between the corresponding sets.

Lemma 5.1 (Comparison of mean curvatures at a contact point). *Let $\partial E_1, \partial E_2 \in \mathcal{C}^\infty$ and set*

$$d_i(z) := d(z, E_i) = \mathrm{dist}(z, E_i) - \mathrm{dist}(z, \mathbb{R}^n \setminus E_i), \qquad z \in \mathbb{R}^n,\ i \in \{1, 2\}.$$

Suppose that there exist $x \in \mathbb{R}^n$ and $\rho > 0$ with the following properties:

$$x \in \partial E_1 \cap \partial E_2, \qquad E_1 \cap B_\rho(x) \subseteq E_2 \cap B_\rho(x).$$

Then

$$\Delta d_1(x) \geq \Delta d_2(x). \tag{5.1}$$

Proof. Since the Laplacian is invariant under translations and rotations, we can assume that x is the origin of the coordinates, and that the unit normal vector to ∂E_1 at x pointing toward $\mathbb{R}^n \setminus E_1$ (which coincides with the unit normal vector to ∂E_2 at x pointing toward $\mathbb{R}^n \setminus E_2$) equals $-e_n$. Furthermore, we can suppose that for a suitable $\sigma \in (0, \rho)$,

$$B_\sigma(x) \cap \partial E_1 = \mathrm{graph}(v_1), \qquad B_\sigma(x) \cap \partial E_2 = \mathrm{graph}(v_2),$$

where v_1 and v_2 are functions of class $\mathcal{C}^\infty$ defined on a ball $B'_\sigma \subset \mathbb{R}^{n-1} = \mathrm{span}\{e_1, \dots, e_{n-1}\}$ of radius σ, and $v_1 \geq v_2$ in B'_σ. Then $v_1 - v_2$ has a local minimum at 0. Hence

$$\nabla v_1(0) = \nabla v_2(0) = 0,$$

so that the mean curvature $\Delta d_i(x)$ of ∂E_i at x equals[(1)] $\Delta v_i(0)$ for $i \in \{1, 2\}$. Since 0 is a local minimum of $v_1 - v_2$, we have also

$$\Delta v_1(0) \geq \Delta v_2(0),$$

and (5.1) follows. □

We now record a useful lemma which, roughly speaking, allows us to suitably interchange the operations of derivative and minimum, or of derivative and maximum[(2)].

Lemma 5.2 (Derivative of a minimum/maximum). *Let $h \geq 1$ be an integer number, let $\mathfrak{M}$ be a h-dimensional smooth compact manifold without boundary, and let $a, b \in \mathbb{R}$, $a < b$. Let $\mathfrak{u} \in \mathcal{C}^1([a, b] \times \mathfrak{M})$. Define, for any $t \in [a, b]$,*

$$\begin{aligned} u_{\min}(t) &:= \min_{p \in \mathfrak{M}} \mathfrak{u}(t, p), \qquad & P^{\mathfrak{u}}_{\min}(t) &:= \{m \in \mathfrak{M} : \mathfrak{u}(t, m) = u_{\min}(t)\}, \\ u_{\max}(t) &:= \max_{p \in \mathfrak{M}} \mathfrak{u}(t, p), \qquad & P^{\mathfrak{u}}_{\max}(t) &:= \{m \in \mathfrak{M} : \mathfrak{u}(t, m) = u_{\max}(t)\}. \end{aligned}$$

Then

$$u_{\min}, u_{\max} \in \mathrm{Lip}([a, b]),$$

and for any $t \in [a, b)$

$$\lim_{\tau \downarrow 0} \frac{1}{\tau}\Big(u_{\min}(t + \tau) - u_{\min}(t)\Big) = \min\left\{\frac{\partial \mathfrak{u}}{\partial t}(t, m) : m \in P^{\mathfrak{u}}_{\min}(t)\right\}, \tag{5.2}$$

$$\lim_{\tau \downarrow 0} \frac{1}{\tau}\Big(u_{\max}(t + \tau) - u_{\max}(t)\Big) = \max\left\{\frac{\partial \mathfrak{u}}{\partial t}(t, m) : m \in P^{\mathfrak{u}}_{\max}(t)\right\}. \tag{5.3}$$

Proof. For any $t \in [a, b)$, $m \in P^{u}_{\min}(t)$, and $\tau > 0$ small enough so that $t + \tau \leq b$, we have, using a Taylor expansion,

$$\begin{aligned} u_{\min}(t + \tau) \leq \mathfrak{u}(t + \tau, m) &= \mathfrak{u}(t, m) + \tau \frac{\partial \mathfrak{u}}{\partial t}(t, m) + \mathfrak{o}(\tau) \\ &= u_{\min}(t) + \tau \frac{\partial \mathfrak{u}}{\partial t}(t, m) + \mathfrak{o}(\tau). \end{aligned} \tag{5.4}$$

(1) See Example 1.16.

(2) Similar lemmas appear in [164, Lemma 3.5], [32, Chapter 2, Proposition 2.13], [74, Theorem 3.4.4], [89, Lemma 10.29], [90, Lemma B.40, page 531] (see also [203]).

Since $\tau > 0$, inequality (5.4) can be rewritten as

$$\frac{1}{\tau}\Big(u_{\min}(t+\tau) - u_{\min}(t)\Big) \le \frac{\partial \mathfrak{u}}{\partial t}(t,m) + \mathfrak{o}(1).$$

Therefore

$$\limsup_{\tau \downarrow 0} \frac{1}{\tau}\left(u_{\min}(t+\tau) - u_{\min}(t)\right) \le \frac{\partial \mathfrak{u}}{\partial t}(t,m).$$

Since this inequality is valid for any $m \in P^{\mathfrak{u}}_{\min}(t)$, we deduce

$$\limsup_{\tau \downarrow 0} \frac{1}{\tau}\Big(u_{\min}(t+\tau) - u_{\min}(t)\Big) \le \min\left\{\frac{\partial \mathfrak{u}}{\partial t}(t,m) : m \in P^{\mathfrak{u}}_{\min}(t)\right\}. \quad (5.5)$$

Notice that, arguing in a similar way to the proof of (5.4), it also follows that, for $t \in (a,b)$ and for any $\sigma \in \mathbb{R}$ with $|\sigma| > 0$ small enough so that $t + \sigma \in (a,b)$,

$$u_{\min}(t+\sigma) \le u_{\min}(t) + \sigma \frac{\partial \mathfrak{u}}{\partial t}(\delta, m)$$

for a suitable $\delta \in [\min(t, t+\sigma), \max(t, t+\sigma)]$. Hence

$$u_{\min}(t+\sigma) \le u_{\min}(t) + |\sigma| L,$$

where

$$L := \left\| \frac{\partial \mathfrak{u}}{\partial t} \right\|_{L^\infty([a,b]\times\mathfrak{M})}.$$

It follows that

$$\frac{u_{\min}(t+\sigma) - u_{\min}(t)}{|\sigma|} \le L,$$

and this, exchanging also the role of $t+\sigma$ and t, implies that $u_{\min} \in \mathrm{Lip}([a,b])$; in particular, $u_{\min}$ is almost everywhere differentiable in $[a,b]$.

Now, let us show that

$$\liminf_{\tau \downarrow 0} \frac{1}{\tau}\Big(u_{\min}(t+\tau) - u_{\min}(t)\Big) \ge \min\left\{\frac{\partial \mathfrak{u}}{\partial t}(t,m) : m \in P^{\mathfrak{u}}_{\min}(t)\right\}. \quad (5.6)$$

Fix $t \in [a,b)$, and let $\epsilon > 0$. Define

$$P_\epsilon(t) := \Big\{q \in \mathfrak{M} : \mathfrak{u}(t,q) < u_{\min}(t) + \epsilon\Big\},$$

which is a neighbourhood of $P^{\mathfrak{u}}_{\min}(t)$. Let $\tau > 0$ be small enough so that $t + \tau \le b$. For any $q \in P_\epsilon(t)$ we have, using a Taylor expansion,

$$\begin{aligned} \mathfrak{u}(t+\tau, q) &= \mathfrak{u}(t, q) + \tau \frac{\partial \mathfrak{u}}{\partial t}(t, q) + \mathfrak{o}(\tau) \\ &\ge u_{\min}(t) + \tau \frac{\partial \mathfrak{u}}{\partial t}(t, q) + \mathfrak{o}(\tau) \\ &\ge u_{\min}(t) + \tau \inf_{\mathrm{q} \in P_\epsilon(t)} \frac{\partial \mathfrak{u}}{\partial t}(t, \mathrm{q}) + \mathfrak{o}(\tau). \end{aligned} \tag{5.7}$$

On the other hand, if $q \in \mathfrak{M} \setminus P_\epsilon(t)$, we have $\mathfrak{u}(t, q) \ge u_{\min}(t) + \epsilon$. Therefore

$$\begin{aligned} \mathfrak{u}(t+\tau, q) &= \mathfrak{u}(t, q) + \tau \frac{\partial \mathfrak{u}}{\partial t}(t, q) + \mathfrak{o}(\tau) \\ &\ge u_{\min}(t) + \epsilon + \tau \frac{\partial \mathfrak{u}}{\partial t}(t, q) + \mathfrak{o}(\tau) \\ &\ge u_{\min}(t) + \epsilon - \tau L + \mathfrak{o}(\tau). \end{aligned} \tag{5.8}$$

Collecting together (5.7) and (5.8), we have

$$\begin{cases} \mathfrak{u}(t+\tau, q) \ge u_{\min}(t) + \tau \inf_{\mathrm{q} \in P_\epsilon(t)} \frac{\partial \mathfrak{u}}{\partial t}(t, \mathrm{q}) + \mathfrak{o}(\tau), & q \in P_\epsilon(t), \\ \mathfrak{u}(t+\tau, q) \ge u_{\min}(t) + \epsilon - \tau L + \mathfrak{o}(\tau), & q \in \mathfrak{M} \setminus P_\epsilon(t). \end{cases} \tag{5.9}$$

On the other hand, for $\tau \in \left(0, \dfrac{\epsilon}{2L}\right)$ we have

$$u_{\min}(t) + \epsilon - \tau L \ge u_{\min}(t) + \tau L \ge u_{\min}(t) + \tau \inf_{\mathrm{q} \in P_\epsilon(t)} \frac{\partial \mathfrak{u}}{\partial t}(t, \mathrm{q}). \tag{5.10}$$

Combining (5.9) and (5.10) we deduce, for $\tau \in \left(0, \dfrac{\epsilon}{2L}\right)$ such that $t + \tau \le b$,

$$\mathfrak{u}(t+\tau, p) \ge u_{\min}(t) + \tau \inf_{\mathrm{q} \in P_\epsilon(t)} \frac{\partial \mathfrak{u}}{\partial t}(t, \mathrm{q}) + \mathfrak{o}(\tau), \qquad p \in \mathfrak{M}. \tag{5.11}$$

From (5.11) it follows, for the same values of t and τ,

$$u_{\min}(t+\tau) \ge u_{\min}(t) + \tau \inf_{\mathrm{q} \in P_\epsilon(t)} \frac{\partial \mathfrak{u}}{\partial t}(t, \mathrm{q}) + \mathfrak{o}(\tau).$$

Hence

$$\liminf_{\tau \downarrow 0} \frac{1}{\tau} \left(u_{\min}(t+\tau) - u_{\min}(t)\right) \ge \inf_{\mathrm{q} \in P_\epsilon(t)} \frac{\partial \mathfrak{u}}{\partial t}(t, \mathrm{q}). \tag{5.12}$$

Since $\epsilon > 0$ is arbitrary in (5.12), we infer

$$\liminf_{\tau \downarrow 0} \frac{1}{\tau} \left(u_{\min}(t+\tau) - u_{\min}(t) \right) \geq \sup_{\epsilon > 0} \inf_{\mathrm{q} \in P_\epsilon(t)} \frac{\partial \mathfrak{u}}{\partial t}(t, \mathrm{q}). \tag{5.13}$$

Observe now that(3)

$$\sup_{\epsilon > 0} \inf_{\mathrm{q} \in P_\epsilon(t)} \frac{\partial \mathfrak{u}}{\partial t}(t, \mathrm{q}) = \min_{m \in P^{\mathfrak{u}}_{\min}(t)} \frac{\partial \mathfrak{u}}{\partial t}(t, m). \tag{5.14}$$

Formula (5.6) then follows from (5.13) and (5.14), and formula (5.2) is a consequence of (5.5) and (5.6).

The assertions of the lemma concerning $u_{\max}$ follow setting $\mathfrak{v} := -\mathfrak{u}$, so that $u_{\max} = -v_{\min}$, where $v_{\min}(t) := \min\limits_{p \in \mathfrak{M}} \mathfrak{v}$; then $P^{\mathfrak{u}}_{\max}(t) = P^{\mathfrak{v}}_{\min}(t)$ and, using also (5.2),

$$\begin{aligned}
\lim_{\tau \downarrow 0} \frac{1}{\tau} \Big(u_{\max}(t+\tau) - u_{\max}(t) \Big) &= - \lim_{\tau \downarrow 0} \frac{1}{\tau} \Big(v_{\min}(t+\tau) - v_{\min}(t) \Big) \\
&= - \min \left\{ \frac{\partial \mathfrak{v}}{\partial t}(t, m) : m \in P^{\mathfrak{v}}_{\min}(t) \right\} \\
&= \max \left\{ \frac{\partial \mathfrak{u}}{\partial t}(t, m) : m \in P^{\mathfrak{u}}_{\max}(t) \right\}.
\end{aligned}$$

This concludes the proof(4). $\square$

The following example shows that the right limit $\tau \downarrow 0$ in (5.2) cannot be replaced by the full limit $\tau \to 0$. The argument is local, so all quantities must properly modified out of a compact subset of $\mathbb{R}^2$.

(3) Denote by $\mathcal{L}$ and $\mathcal{R}$ the left-hand side and the right-hand side of (5.14), respectively. Since $P_\epsilon(t) \supseteq P^{\mathfrak{u}}_{\min}(t)$ for any $\epsilon > 0$, we have $\mathcal{L} \leq \mathcal{R}$. In addition, $\mathcal{L}$ cannot be strictly smaller than $\mathcal{R}$. Indeed, assume by contradiction that $\mathcal{L} < \mathcal{R}$, and take $\delta > 0$ so that $\mathcal{L} \leq \mathcal{R} - \delta$. Then for any $\epsilon > 0$ we have $\inf_{\mathrm{q} \in P_\epsilon(t)} \frac{\partial \mathfrak{u}}{\partial t}(t, \mathrm{q}) \leq \mathcal{R} - \delta$, hence there exists $q_\epsilon \in P_\epsilon(t)$ such that $\frac{\partial \mathfrak{u}}{\partial t}(t, q_\epsilon) \leq \mathcal{R} - \delta + |\mathrm{o}(\epsilon)|$. Since $\mathfrak{M}$ is compact, we can extract a subsequence $(q_{\epsilon'})$ so that $q_{\epsilon'} \to q \in \mathfrak{M}$ as $\epsilon' \downarrow 0$. But $0 < \epsilon_1 \leq \epsilon_2$ implies $P_{\epsilon_1}(t) \subseteq P_{\epsilon_2}(t)$. Therefore $q \in \overline{\cap_{\epsilon'} P_{\epsilon'}(t)} = \cap_{\epsilon'} P_{\epsilon'}(t) = P^{\mathfrak{u}}_{\min}(t)$. Then the continuity of $\frac{\partial \mathfrak{u}}{\partial t}$ implies $\lim_{\epsilon' \downarrow 0} \frac{\partial \mathfrak{u}}{\partial t}(t, q_{\epsilon'}) = \frac{\partial \mathfrak{u}}{\partial t}(t, q) \leq \mathcal{R} - \delta$, a contradiction. It follows $\mathcal{L} = \mathcal{R}$.

(4) With a similar proof, one can show that

$$\lim_{\tau \uparrow 0} \frac{1}{\tau} \Big(u_{\min}(t+\tau) - u_{\min}(t) \Big) = \max \left\{ \frac{\partial \mathfrak{u}}{\partial t}(t, m) : m \in P^{\mathfrak{u}}_{\min}(t) \right\},$$

and

$$\lim_{\tau \uparrow 0} \frac{1}{\tau} \Big(u_{\max}(t+\tau) - u_{\max}(t) \Big) = \min \left\{ \frac{\partial \mathfrak{u}}{\partial t}(t, m) : m \in P^{\mathfrak{u}}_{\max}(t) \right\}.$$

Example 5.3 (Nondifferentiability of $u_{\min}$). Let $\mathfrak{M}$ be a smooth simple closed curve in $\left\{(z_1, z_2) \in \mathbb{R}^2 : z_2 \leq 0\right\}$ and containing the interval $[-2, 2]$ of the horizontal coordinate axis. Let $f : [a, b] \to \mathcal{P}(\mathbb{R}^2)$ be a smooth compact flow, the boundary of which is locally the graph of a smooth function $v(t, \cdot)$ over $[-2, 2]$ and depicted in Figure 5.1: the three pictures correspond to subsequent times $t \in [a, \frac{a+b}{2})$, $t = \frac{a+b}{2}$, and $t \in (\frac{a+b}{2}, b]$ respectively.

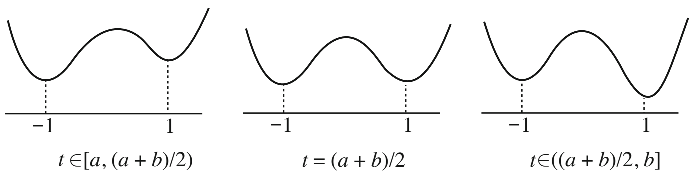

Figure 5.1. Example 5.3: a case where the function $u_{\min}$ is not differentiable.

The set $f(t)$ is locally the epigraph of $v(t, \cdot)$. Suppose that

$$
\begin{aligned}
&v(t, -1) = 1, && t \in [a, b],\\
&v(t, 1) > 1, && t \in \left[a, \frac{a+b}{2}\right),\\
&v\left(\frac{a+b}{2}, 1\right) = 1, &&\\
&\frac{\partial v}{\partial t}\left(\frac{a+b}{2}, 1\right) < 0, &&
\end{aligned}
$$

Define

$$
\mathfrak{u}(t, z) := \operatorname{dist}(z, \partial f(t)), \qquad t \in [a, b], \quad z \in \mathfrak{M}.
$$

Then the function $u_{\min}$ is not differentiable at $t = \frac{a+b}{2}$, since

$$
\lim_{\tau \downarrow 0} \frac{u_{\min}(t+\tau) - u_{\min}(t)}{\tau} < 0, \qquad \lim_{\tau \uparrow 0} \frac{u_{\min}(t+\tau) - u_{\min}(t)}{\tau} = 0.
$$

The inclusion principle in its simplest form reads as follows[(5)].

(5) To have a flavor of this result, apply the inclusion principle to the following case: $\partial f_1(a)$ and $\partial f_2(a)$ are two linked disjoint tori in $\mathbb{R}^3$, $f_1(a)$ is one solid torus, and $f_2(a)$ is the complement of the other solid torus.

Theorem 5.4 (Monotonicity of the distance). *Let $f_1, f_2 : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ be two smooth compact mean curvature flows, and assume that $\partial f_1(a)$ and $\partial f_2(a)$ are connected. Define the function*

$$\mathrm{d}(t) := \mathrm{dist}(f_1(t), \mathbb{R}^n \setminus f_2(t)), \qquad t \in [a,b]. \tag{5.15}$$

Suppose that

$$\mathrm{d}(a) > 0. \tag{5.16}$$

Then

$$\text{for any } t \in [a,b) \text{ there exists } \lim_{\tau \downarrow 0} \frac{1}{\tau}\Big(\mathrm{d}(t+\tau) - \mathrm{d}(t)\Big) \in [0, +\infty),$$

and d *is nondecreasing in* $[a,b]$.

Proof. One way to prove this theorem is to use a parametric description of the flows. Since f_1 and f_2 are smooth compact flows, it follows (see Remark 3.6) that there exist two $(n-1)$-dimensional smooth compact embedded connected submanifolds $\mathcal{S}_1 \subset \mathbb{R}^n, \mathcal{S}_2 \subset \mathbb{R}^n$ without boundary and there exist $\varphi_i \in \mathcal{X}([a,b]; \mathrm{Emb}(\mathcal{S}_i; \mathbb{R}^n)$ such that $\varphi_i(\mathcal{S}_i, t) = \partial f_i(t)$ for any $t \in [a,b]$ and $i \in \{1,2\}$.

Let

$$\mathfrak{M} := \mathcal{S}_1 \times \mathcal{S}_2 \subset \mathbb{R}^{2n},$$

and define the function $\mathfrak{u} : [a,b] \times \mathfrak{M} \to [0,+\infty)$ as

$$\mathfrak{u}(t,s,\hat{s}) := |\varphi_2(t,\hat{s}) - \varphi_1(t,s)|, \qquad (t,s,\hat{s}) \in [a,b] \times \mathfrak{M}. \tag{5.17}$$

Observe that

$$\mathrm{d}(t) = \min\left\{\mathfrak{u}(t,s,\hat{s}) : (s,\hat{s}) \in \mathfrak{M}\right\}, \qquad t \in [a,b].$$

Define

$$\theta := \inf\{t \in [a,b] : \mathrm{d}(t) = 0\}.$$

Thanks to the smoothness of f_1, f_2 and to assumption (5.16), we have $\theta > a$. Hence $\mathrm{d}(t) > 0$ in $[a,\theta)$, and therefore the function $\mathfrak{u}$ is smooth in $[a,\theta) \times \mathfrak{M}$. Thus we can apply Lemma 5.2, and we deduce

$$\begin{aligned}
&\lim_{\tau \downarrow 0} \frac{\mathrm{d}(t+\tau) - \mathrm{d}(t)}{\tau} \\
&= \min\left\{\frac{\partial \mathfrak{u}}{\partial t}(t,s,\hat{s}) : (s,\hat{s}) \in \mathfrak{M},\ \mathfrak{u}(t,s,\hat{s}) = \mathrm{d}(t)\right\} \qquad (5.18)\\
&= \min\left\{\langle \frac{\varphi_2(t,\hat{s}) - \varphi_1(t,s)}{|\varphi_2(t,\hat{s}) - \varphi_1(t,s)|}, \frac{\partial \varphi_2}{\partial t}(t,\hat{s}) - \frac{\partial \varphi_1}{\partial t}(t,s)\rangle : \right. \\
&\qquad\qquad \left. (s,\hat{s}) \in \mathfrak{M},\ \mathfrak{u}(t,s,\hat{s}) = \mathrm{d}(t)\right\},
\end{aligned}$$

for any $t \in [a, \theta)$. We now claim that

$$\lim_{\tau \downarrow 0} \frac{1}{\tau}(\mathrm{d}(t+\tau) - \mathrm{d}(t)) \geq 0, \qquad t \in [a, \theta). \tag{5.19}$$

For $t \in [a, \theta)$, let $(s_1(t), s_2(t)) \in \mathfrak{M}$ be such that

$$\frac{\partial \mathfrak{u}}{\partial t}(t, s_1(t), s_2(t)) = \min \left\{ \frac{\partial \mathfrak{u}}{\partial t}(t, s, \hat{s}) : (s, \hat{s}) \in \mathfrak{M}, \, \mathfrak{u}(t, s, \hat{s}) = \mathrm{d}(t) \right\}, \tag{5.20}$$

and set

$$x_1(t) := \varphi_1(t, s_1(t)) \in \partial f_1(t), \qquad x_2(t) := \varphi_2(t, s_2(t)) \in \partial f_2(t).$$

Let also $d_i(\cdot, t) := d(\cdot, f_i(t))$ be the signed distance function from $\partial f_i(t)$ negative in the interior of $f_i(t)$, for $i \in \{1, 2\}$. Note that the relations

$$\mathfrak{u}(t, s_1(t), s_2(t)) = |x_2(t) - x_1(t)| = \mathrm{d}(t)$$

imply

$$\frac{x_2(t) - x_1(t)}{|x_2(t) - x_1(t)|} = \nabla d_2(t, x_2(t)) = \nabla d_1(t, x_1(t)).$$

Namely, $\dfrac{x_2(t) - x_1(t)}{|\hat{x}_2(t) - x_1(t)|}$ coincides with the unit normal vector to $\partial f_2(t)$ at $x_2(t)$ pointing toward $\mathbb{R}^n \setminus f_2(t)$, which in turn coincides with the unit normal vector to $\partial f_1(t)$ at $x_1(t)$ pointing toward $\mathbb{R}^n \setminus f_1(t)$. Denote for simplicity[(6)] such a unit vector by $-\nu$.

From (5.17) we compute, for $t \in [a, \theta)$,

$$\begin{aligned}
&\frac{\partial \mathfrak{u}}{\partial t}(t, s_1(t), s_2(t)) \\
=&\langle \frac{x_2(t) - x_1(t)}{|x_2(t) - x_1(t)|}, \frac{\partial \varphi_2}{\partial t}(t, s_2(t)) - \frac{\partial \varphi_1}{\partial t}(t, s_1(t)) \rangle \\
=&\langle -\nu, \frac{\partial \varphi_2}{\partial t}(t, s_2(t)) - \frac{\partial \varphi_1}{\partial t}(t, s_1(t)) \rangle.
\end{aligned} \tag{5.21}$$

From formula (3.11) we have

$$\langle \nu, \frac{\partial \varphi_1}{\partial t}(t, s_1(t)) \rangle = \mathrm{V}_1(t, s_1(t)), \qquad \langle \nu, \frac{\partial \varphi_2}{\partial t}(t, s_2(t)) \rangle = \mathrm{V}_2(t, s_2(t)),$$

(6) Recall our convention (3.6).

where V_i is the outer normal velocity of φ_i for $i = 1, 2$. On the other hand f_1 and f_2 are smooth mean curvature flows, so that from (3.11) and (3.16) we have

$$\mathrm{V}_1(t, s_1(t)) = \Delta d_1(t, x_1(t)), \quad \mathrm{V}_2(t, s_2(t)) = \Delta d_2(t, x_2(t)). \tag{5.22}$$

From (5.18), (5.20), (5.21), and (5.22) we get

$$\begin{aligned}\lim_{\tau\downarrow 0} \frac{1}{\tau}(\mathrm{d}(t+\tau) - \mathrm{d}(t)) &= \mathrm{V}_1(t, s_1(t)) - \mathrm{V}_2(t, s_2(t)) \\ &= \Delta d_1(t, x_1(t)) - \Delta d_2(t, x_2(t)).\end{aligned} \tag{5.23}$$

Now, let us consider the translated set

$$f_1^{\mathrm{tr}}(t) := f_1(t) - \mathrm{d}(t)\nu.$$

Then

$$f_1^{\mathrm{tr}}(t) \subseteq f_2(t) \quad \text{and} \quad x_2(t) \in \partial(f_1^{\mathrm{tr}}(t)) \cap \partial f_2(t). \tag{5.24}$$

Moreover if $d_1^{\mathrm{tr}}(\cdot, t) = d(\cdot, f_1^{\mathrm{tr}}(t))$ denotes the signed distance from $\partial f_1^{\mathrm{tr}}(t)$ negative in the interior of $f_1^{\mathrm{tr}}(t)$, we have

$$\Delta d_1^{\mathrm{tr}}(t, x_2(t)) = \Delta d_1(t, x_1(t)). \tag{5.25}$$

By (5.24), using Lemma 5.1 we deduce

$$\Delta d_1^{\mathrm{tr}}(t, x_2(t)) \geq \Delta d_2(t, x_2(t)).$$

From (5.25) we then get

$$\Delta d_1(t, x_1(t)) \geq \Delta d_2(t, x_2(t)). \tag{5.26}$$

Claim (5.19) then follows from (5.23) and (5.26).

Let us now show that from (5.19) it follows that d is nondecreasing in $[a, \theta]$. Assume by contradiction that there exist t_1 and t_2 with $a \leq t_1 < t_2 \leq \theta$ and such that $\mathrm{d}(t_2) < \mathrm{d}(t_1)$. Let $P : \mathbb{R} \to \mathbb{R}$ be a linear decreasing function such that $P(t_1) = \mathrm{d}(t_1)$ and $P(t_2) > \mathrm{d}(t_2)$. Let

$$t^* := \inf\{t \in [t_1, \theta] : \mathrm{d}(\tau) \leq P(\tau) \quad \text{for any} \qquad \tau \in [t, \theta]\}.$$

Then $t^* < t_2 \leq \theta$, $P(t^*) = \mathrm{d}(t^*)$, and $\frac{\mathrm{d}(t^*+\tau)-\mathrm{d}(t^*)}{\tau} < \frac{P(t^*+\tau)-P(t^*)}{\tau}$ for $\tau \in (0, \theta - t^*)$. Therefore

$$\lim_{\tau\downarrow 0} \frac{1}{\tau}\big(\mathrm{d}(t^*+\tau) - \mathrm{d}(t^*)\big) \leq P'(t^*) < 0,$$

a contradiction with (5.19).

Hence d is nondecreasing in $[a, \theta]$, and therefore $\mathrm{d}(\theta) \geq \mathrm{d}(a) > 0$. To conclude the proof, it is enough to observe that $\theta = b$. Indeed, the above arguments and the continuity of d imply that $\mathrm{d}(t) \geq \mathrm{d}(a)$ for any $t \in [a, b]$. $\square$

Remark 5.5. If we drop the assumptions that $\partial f_1(a)$ and $\partial f_2(a)$ are connected, we still deduce that d is nondecreasing, since Theorem 5.4 ensures that the distance of a connected component of $\partial f_1(t)$ from a connected component of $\partial f_2(t)$ is nondecreasing. More generally, it is possible to prove (see [203]) that d is nondecreasing when one of the two flows is a smooth compact mean curvature flow, and the other one is just a smooth mean curvature flow, provided (5.16) holds(7).

Remark 5.6. With the same proof, one shows that the theses of Theorem 5.4 still hold if $f_1 : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ is a smooth compact subsolution to mean curvature flow, $f_2 : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ is a smooth compact supersolution to mean curvature flow, and the function d (defined as in (5.15)) satisfies (5.16).

Taking into account Remark 5.5 we have the following result.

Corollary 5.7 (Inclusion principle, I). *Let $f_1 : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth compact mean curvature flow, and let $f_2 : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth mean curvature flow. The following assertions hold:*

- *if $f_1(a) \subseteq f_2(a)$ and $\partial f_1(a) \cap \partial f_2(a) = \emptyset$ then $f_1(t) \subseteq f_2(t)$ for any $t \in [a, b]$;*
- *if $f_1(a) \cap f_2(a) = \emptyset$ then $f_1(t) \cap f_2(t) = \emptyset$ for any $t \in [a, b]$.*

A more general version of the inclusion principle for smooth mean curvature flows(8) reads as follows.

Theorem 5.8 (Inclusion principle, II). *Let $f_1 : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth compact mean curvature flow, and let $f_2 : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth mean curvature flow. Assume that*

$$f_1(a) \subseteq f_2(a).$$

Then

$$f_1(t) \subseteq f_2(t), \qquad t \in [a, b]. \tag{5.27}$$

Proof. If $\partial f_1(a) \cap \partial f_2(a) = \emptyset$ the result follows from Corollary 5.7. Therefore, we need to consider the case when $\partial f_1(a) \cap \partial f_2(a) \neq \emptyset$.

(7) In this respect, notice that conclusion (5.2) is still valid if we drop the assumption that $\mathfrak{M}$ is compact, provided we assume that $\inf_{p \in \mathfrak{M}} \mathfrak{u}(t, p) = \min_{p \in \mathfrak{M}} \mathfrak{u}(t, p)$, that $P^{\mathfrak{u}}_{\min}(t) = \cap_{\epsilon > 0} P_\epsilon(t)$ is compact for any $t \in [a, b]$, and that $L < +\infty$. A similar conclusion holds for (5.3).

(8) An alternative proof of Theorem 5.8, when f_1 and f_2 are two smooth compact mean curvature flows on a sufficiently short time interval, will be given in Theorem 17.21, as a consequence of the convergence of solutions to the singularly perturbed parabolic equations (15.2) to mean curvature flow.

In the proof we will use the continuity of mean curvature flow with respect to initial data, discussed in Chapter 7. From Theorem 7.11 it follows that we can find $\overline{\rho} > 0$ such that the smooth compact mean curvature flow f_ρ^- starting from $\overline{f_1(a)_\rho^-}$ at time a exists for all times in $[a, b]$, for any $\rho \in [0, \overline{\rho}]$. Moreover, for any $\rho \in (0, \overline{\rho}]$, from Theorem 5.4 it follows that the function

$$t \in [a, b] \to \operatorname{dist}\Big(f_\rho^-(t), \mathbb{R}^n \setminus f_2(t)\Big)$$

is nondecreasing. Hence

$$\bigcup_{\rho \in (0,\overline{\rho}]} f_\rho^-(t) \subseteq f_2(t), \qquad t \in [a, b].$$

On the other hand, using Remark 7.12 we have

$$\operatorname{int}(f_1(t)) = \bigcup_{\rho \in (0,\overline{\rho}]} f_\rho^-(t), \qquad t \in [a, b],$$

so that

$$\operatorname{int}(f_1(t)) \subseteq f_2(t), \qquad t \in [a, b]. \tag{5.28}$$

Then (5.28) and the smoothness of $f_1(t)$ imply

$$\overline{\operatorname{int}(f_1(t))} = f_1(t) \subseteq \overline{f_2(t)} = f_2(t), \qquad t \in [a, b]. \qquad \square$$

Remark 5.9. The inclusion principle will be improved further in Theorem 6.3, making use of a version of the strong maximum principle.

We point out an interesting property, which shows that embeddedness is preserved by parametric smooth compact mean curvature flow. The proof of this property needs a refinement of the arguments used in the proof of Theorem 5.4, and can be found in [203].

Theorem 5.10 (Embeddedness preserving property). *Let $\mathcal{S}$ be a smooth orientable $(n-1)$-dimensional connected compact manifold without boundary, and let*

$$\varphi \in \mathcal{X}([a, b]; \operatorname{Imm}(\mathcal{S}; \mathbb{R}^n)).$$

Assume that

- $\varphi(a, \cdot) \in \operatorname{Emb}(\mathcal{S}; \mathbb{R}^n)$*;*
- φ *is a smooth parametric compact mean curvature flow.*

Then

$$\varphi \in \mathcal{X}([a, b]; \operatorname{Emb}(\mathcal{S}; \mathbb{R}^n)).$$

Remark 5.11 (Estimate on the lifespan). As we shall see in the sequel, in general a given compact $\partial E \in \mathcal{C}^\infty$ that flows by mean curvature may develop a singularity in finite time and before the extinction. Therefore a notion of weak solution is necessary in order to describe the evolution at later times. Dealing with weak solutions requires refined tools (see [67, 139]). However, before singularities, we can show an elementary estimate, which is a direct consequence of the inclusion principle[(9)].

Let $E \subset \mathbb{R}^n$ be compact with $\partial E \in \mathcal{C}^\infty$ and let $t_{\max}(\partial E)$ be the maximal time of existence of the smooth compact mean curvature flow f

---

(9) Another estimate of $t_{\max}(\partial E)$ (see [133, 154, 139] for all details) is the following. Let $n > 1$, $E \subset \mathbb{R}^n$ be compact with $\partial E \in \mathcal{C}^\infty$, and let $f : [0, t_{\max}(\partial E)) \to \mathcal{P}(\mathbb{R}^n)$ be the smooth compact mean curvature flow starting from ∂E. Assume that the perimeter $P(f(t))$ of $f(t)$ satisfies $\lim_{t\uparrow t_{\max}(\partial E)} P(f(t)) = 0$ (note that, in view of the monotonicity of $t \in [0, t_{\max}(\partial E)) \to P(f(t))$ proved in (3.19), the limit exists and, once it vanishes, then $\lim\limits_{t\uparrow t_{\max}(\partial E)} |f(t)| = 0$, by the isoperimetric inequality [16]). Then

$$\frac{2|E|^2}{P(E)^2} \le t_{\max}(\partial E) \le C P(E)^{2/(n-1)},$$

where $C > 0$ is a constant depending only on the dimension n. Indeed, let $d(\cdot) := d(\cdot, E)$ and $d_f(t, \cdot) := d(\cdot, f(t))$. We recall the following inequality (see, *e.g.*, [208, 1.4.1]):

$$\left(\int_{\partial E} |\phi|^{\frac{n-1}{n-2}}\, d\mathcal{H}^{n-1}\right)^{\frac{n-2}{n-1}} \le c \int_{\partial E} \Big[|\delta\phi| + |\Delta d||\phi|\Big]\, d\mathcal{H}^{n-1}, \qquad \phi \in C_c^\infty(\mathbb{R}^n), \tag{5.29}$$

where $c = \frac{(n-1)4^{(n-1)}}{(n-2)\omega_{n-1}^{1/(n-1)}}$. Applying (5.29) with $\phi \equiv 1$ in a neighbourhood of ∂E yields

$$P(E)^{\frac{n-2}{n-1}} \le c \int_{\partial E} |\Delta d|\, d\mathcal{H}^{n-1} \le c \left(\int_{\partial E} (\Delta d)^2\, d\mathcal{H}^{n-1}\right)^{1/2} P(E)^{1/2}.$$

Since $\frac{n-2}{n-1} - \frac{1}{2} = \frac{n-3}{2(n-1)}$, we get $P(E)^{\frac{n-3}{n-1}} \le c^2 \int_{\partial E} (\Delta d)^2\, d\mathcal{H}^{n-1}$. Recalling formula (3.19), we apply the previous inequality with $f(t)$ in place of E, and we deduce

$$\frac{d}{dt} P(f(t)) = -\int_{\partial f(t)} (\Delta d_f)^2\, d\mathcal{H}^{n-1} \le -\frac{1}{c^2} P(f(t))^{(n-3)/(n-1)}, \qquad t \in [0, t_{\max}(\partial E)).$$

Integrating we have $P(f(t))^{2/(n-1)} - P(E)^{2/(n-1)} \le -\frac{2t}{(n-1)c^2}$. Letting $t \uparrow t_{\max}(\partial E)$ and since by assumption $\lim_{t\uparrow t_{\max}(\partial E)} P(f(t)) = 0$, we deduce $t_{\max}(\partial E) \le C P(E)^{2/(n-1)}$. Recalling also (3.20) we find

$$\begin{aligned}
-\frac{d}{dt}|f(t)| &= \int_{\partial f(t)} \Delta d_f\, d\mathcal{H}^{n-1} \le \left(\int_{\partial f(t)} (\Delta d_f)^2\, d\mathcal{H}^{n-1}\right)^{1/2} P(f(t))^{1/2} \\
&= \left(-P(f(t)) \frac{d}{dt} P(f(t))\right)^{1/2} = \left(-\frac{\frac{d}{dt} P(f(t))^2)}{2}\right)^{1/2}, \qquad t \in [0, t_{\max}(\partial E)).
\end{aligned}$$

Since $\lim\limits_{t\uparrow t_{\max}(\partial E)} P(f(t)) = \lim\limits_{t\uparrow t_{\max}(\partial E)} |f(t)| = 0$, it follows that

$$|f(0)| \le \int_0^{t_{\max}(\partial E)} \left(-\frac{\frac{d}{dt} P(f(t))^2}{2}\right)^{1/2} dt \le (t_{\max}(\partial E))^{1/2} \left(\frac{P(f(0))^2}{2}\right)^{1/2}.$$

Hence $|E| \le \left(t_{\max}(\partial E) \frac{P(E)^2}{2}\right)^{1/2}$.

starting from ∂E (see Definition 7.10). Then

$$t_{\max}(\partial E) \le \frac{(\mathrm{diam}(E))^2}{8(n-1)}. \tag{5.30}$$

Indeed, if $D_0 := \mathrm{diam}(E)$, we have $E \subseteq \overline{B_{\frac{D_0}{2}}(p)}$ for some point $p \in \mathbb{R}^n$. Therefore, by Theorem 5.8 and by the explicit expression of the mean curvature evolution of the ball $\overline{B_{\frac{D_0}{2}}(p)}$ (see Example 3.21), we have

$$f(t) \subseteq \overline{B_{R(t)}(p)}, \qquad t \in [0, t_{\max}(\partial E)),$$

where $R(t) = \sqrt{\left(\frac{D_0}{2}\right)^2 - 2(n-1)t}$. Hence $R(t) > 0$ for any $t \in [0, t_{\max}(\partial E))$, which gives (5.30).

5.1. Appendix: on the maximum and minimum principles

In this appendix we prove a version of the maximum principle[10] in a form which is useful for geometric evolution problems. We will use a parametric description of the flow.

Let $\varphi \in \mathcal{X}([a,b]; \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n))$ be a parametric smooth compact embedded mean curvature flow. The time-dependent metric tensor is denoted by $g(t) = (g_{\alpha\beta}(t))$, and its inverse by $(g^{\alpha\beta}) = (g^{\alpha\beta}(t))$.

If $u \in \mathcal{C}^\infty([a,b] \times \mathcal{S})$, we set, in a local coordinate system c taking some open subset $(\tau, T) \times O$ of an n-dimensional space into a neighbourhood of a given point $(t,s) \in [a,b] \times \mathcal{S}$,

$$(\mathrm{grad}\, u)^\alpha := g^{\alpha\beta} \frac{\partial \mathcal{U}}{\partial s_\beta}, \qquad \alpha \in \{1, \dots, n-1\},$$

$$\Delta_{g(t)} u := \frac{1}{\sqrt{\det(g_{\alpha\beta}(t))}} \frac{\partial}{\partial s_\alpha} \left(\sqrt{\det(g_{\alpha\beta}(t))} (\mathrm{grad}\, u)^\alpha \right),$$

where $\mathcal{U} := u \circ \mathrm{c}$, and $s_1, \dots, s_{n-1}$ are local coordinates around s.

Theorem 5.12. *For $\alpha \in \{1, \dots, n-1\}$ let b_α be a function of class $\mathcal{C}^\infty$ in its arguments, and let $c \in \mathcal{C}^\infty(\mathbb{R})$. Suppose that $u^+, u^- \in \mathcal{C}^\infty([a,b] \times \mathcal{S})$*

[10] See, *e.g.*, [126] and [203]. A more refined version of Theorem 5.12, valid for noncompact flowing hypersurfaces, can be found in [128, Theorem 4.3] (see also [127]).

satisfy

$$\begin{aligned}\frac{\partial u^+}{\partial t}(t,s) \geq\ & \Delta_{g(t)} u^+(t,s) \\ & + b_\alpha\Big(t,s,u^+(t,s), \text{grad } u^+(t,s)\Big)(\text{grad } u^+(t,s))^\alpha + c(u^+(t,s)), \\ \frac{\partial u^-}{\partial t}(t,s) \leq\ & \Delta_{g(t)} u^-(t,s) \\ & + b_\alpha\Big(t,s,u^-(t,s), \text{grad } u^-(t,s)\Big)(\text{grad } u^-(t,s))^\alpha + c(u^-(t,s)),\end{aligned} \tag{5.31}$$

for any $(t,s) \in [a,b] \times \mathcal{S}$. *Define*

$$\begin{aligned} u^+_{\min}(t) &:= \min\{u^+(t,s) : s \in \mathcal{S}\}, \qquad t \in [a,b], \\ u^-_{\max}(t) &:= \max\{u^-(t,s) : s \in \mathcal{S}\}, \qquad t \in [a,b]. \end{aligned}$$

Then $u^+_{\min}, u^-_{\max} \in \mathrm{Lip}([a,b])$, *and*

$$\begin{aligned} \lim_{\tau \downarrow 0} \frac{u^+_{\min}(t+\tau) - u^+_{\min}(t)}{\tau} &\geq c(u^+_{\min}(t)), \qquad t \in [a,b). \\ \lim_{\tau \downarrow 0} \frac{u^-_{\max}(t+\tau) - u^-_{\max}(t)}{\tau} &\leq c(u^-_{\max}(t)), \qquad t \in [a,b). \end{aligned} \tag{5.32}$$

Proof. The Lipschitz continuity of $u^+_{\min}$ and $u^-_{\max}$ can be proved as in the proof of Lemma 5.2. Let us show the inequality in (5.32) concerning $u^+_{\min}$, the inequality concerning $u^-_{\max}$ being similar. Define

$$P^{u^+}_{\min}(t) := \{s \in \mathcal{S} : u^+(t,s) = u^+_{\min}(t)\}, \qquad t \in [a,b].$$

Then

$$\text{grad } u^+(t,s(t)) = 0, \quad \Delta_{g(t)} u^+(t,s(t)) \geq 0, \quad t \in [a,b],\ s(t) \in P^{u^+}_{\min}(t).$$

Hence, from (5.31), it follows

$$\frac{\partial u^+}{\partial t}(t,s(t)) \geq c(u^+(t,s(t))) = c(u^+_{\min}(t)).$$

Therefore, using Lemma 5.2,

$$\begin{aligned} \lim_{\tau \downarrow 0} \frac{u^+_{\min}(t+\tau) - u^+_{\min}(t)}{\tau} &= \min\left\{\frac{\partial u^+}{\partial t}(t,s) : s \in P^{u^+}_{\min}(t)\right\} \\ &\geq c(u^+_{\min}(t)). \end{aligned}$$

□

From (5.32)[11] and the comparison principle for ordinary differential equations we can derive the following properties.

Let $y_- : \mathrm{dom}(y_-) \subseteq \mathbb{R} \to \mathbb{R}$ be the maximal smooth solution of

$$\begin{cases} y_-' = c(y_-), \\ y_-(a) \leq u^+_{\min}(a), \end{cases}$$

and let $y_+ : \mathrm{dom}(y_+) \subseteq \mathbb{R} \to \mathbb{R}$ be the maximal smooth solution of

$$\begin{cases} y_+' = c(y_+), \\ y_+(a) \geq u^-_{\max}(a). \end{cases}$$

Then

$$u^+_{\min} \geq y_- \qquad \text{in } [a, \min(b, c_-)], \quad c_- < \sup\{\mathrm{dom}(y_-)\},$$

and

$$u^-_{\max} \leq y_+ \qquad \text{in } [a, \min(b, c_+)], \quad c_+ < \sup\{\mathrm{dom}(y_+)\}.$$

A typical application of Theorem 5.12 is the following. Recall from formula (3.45) of Example 3.33 that the evolution equation of the curvature for a plane curve flowing by curvature is given by

$$\frac{\partial \mathrm{H}}{\partial t} = \partial_s^2 \mathrm{H} + \mathrm{H}^3.$$

More generally, as we shall see in Chapter 6, the evolution equation satisfied by the mean curvature for a parametric smooth mean curvature flow reads as

$$\frac{\partial \mathrm{H}}{\partial t} = \Delta_{g(t)} \mathrm{H} + \mathrm{H}|A|^2. \tag{5.33}$$

Corollary 5.13. *Let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth compact mean curvature flow. Assume that the mean curvature of $\partial f(a)$ is nonnegative everywhere, and define*

$$H_{\min}(t) := \min\{\Delta d(t, x) : x \in \partial f(t)\}, \qquad t \in [a, b].$$

Then $H_{\min}$ is nondecreasing in $[a, b]$. In particular, if $n = 2$ and $f(a)$ is uniformly convex, then $f(t)$ is uniformly convex for any $t \in [a, b]$.

(11) Since $u^+_{\min}$ and $u^-_{\max}$ are Lipschitz continuous, we have $u^{+\prime}_{\min} \geq c(u^+_{\min})$ and $u^{-\prime}_{\max} \leq c(u^-_{\max})$ almost everywhere in $[a, b]$.

Proof. Without loss of generality, we can assume that $\partial f(a)$ is connected. Then (5.33) holds in $[a, b] \times \mathcal{S}$.

Observe now[12] that $\mathrm{H}^2 \le (n-1)|A|^2$. Therefore

$$\frac{\partial \mathrm{H}}{\partial t} \ge \Delta_{g(t)} \mathrm{H} + \frac{\mathrm{H}^3}{n-1} \qquad \text{in } [a, b] \times \mathcal{S}.$$

Hence by Theorem 5.12 we have $\frac{dH_{\min}}{dt} \ge \frac{(H_{\min})^3}{n-1}$ almost everywhere in $[a, b]$. Moreover, by assumption,

$$H_{\min}(a) \ge 0.$$

Since the solution of the Cauchy problem

$$\begin{cases} y' = \dfrac{y^3}{n-1}, \\ y(a) = 0 \end{cases}$$

is

$$y \equiv 0,$$

it follows that

$$H_{\min}(t) \ge 0, \qquad t \in [a, b]. \qquad \square$$

[12] If M is a $((n-1) \times (n-1))$-matrix, then $\mathrm{tr}(M)^2 \le (n-1)\mathrm{tr}(M^2)$.

Chapter 6

Extension of the evolution equation to a neighbourhood

In this chapter we rewrite the evolution equation (3.16) expressing mean curvature flow as an equation on a time-independent neighbourhood of the flowing manifolds. We use this latter equation to compute the evolution of the normal vector, of the mean curvature, and of the square of the norm of the second fundamental form of the flowing hypersurface.

We recall that, for a smooth flow $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$, the function d and the neighbourhood Q are introduced in Definition 3.1. We also recall from Section 3.3 that:

- the projection map is defined as

$$\mathrm{pr}(t, z) := z - d(t, z)\nabla d(t, z), \qquad (t, z) \in Q,$$

  i.e., $\mathrm{pr}(t, z)$ is the unique point of $\partial f(t)$ nearest to z;
- $\overline{u}$ is the canonical extension of the restriction to $\partial f(t)$ of a function $u \in \mathcal{C}^\infty(Q)$, as defined in (3.48);
- the operator δ is defined in (3.49), and the operator $\dfrac{d}{dt}$ is defined in (3.52).

In order to rewrite the evolution equation (3.16) in the set Q, we begin with the following result proven in [138][1].

Lemma 6.1 (Constancy of $\partial d/\partial t$). *Let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth flow. For any $(t, z) \in Q$ we have*

$$\frac{\partial d}{\partial t}(t, z) = \frac{\partial d}{\partial t}(t, x), \qquad x := \mathrm{pr}(t, z). \tag{6.1}$$

[1] See also [15, Theorem 10].

Proof. Let $x := \mathrm{pr}(t, z) \in \partial f(t)$ and $\lambda \in \mathbb{R}$ be with $|\lambda|$ small enough so that

$$(t, z^\lambda(t, x)) \in Q, \qquad z^\lambda(t, x) := x + \lambda \nabla d(t, x).$$

Define

$$V(\lambda) := \frac{\partial d}{\partial t}(t, z^\lambda(t, x)).$$

In order to prove the lemma, it is enough to show that

$$V' \equiv 0. \tag{6.2}$$

Using the eikonal equation (see formula (1.7)) we have

$$|\nabla d(t, z^\lambda(t, x))|^2 = 1. \tag{6.3}$$

Notice that

$$\frac{\partial}{\partial t}\left(\nabla_i d(t, z^\lambda(t, x))\right) = \nabla_i \frac{\partial d}{\partial t}(t, z^\lambda(t, x)) + \nabla^2_{ij} d(t, z^\lambda(t, x)) \frac{\partial z^\lambda_j}{\partial t}(t, x)$$

for any $i \in \{1, \ldots, n\}$, and $\frac{\partial z^\lambda}{\partial t} = \lambda \nabla \frac{\partial d}{\partial t}$ at (t, x). Hence, differentiating (6.3) with respect to t, we obtain

$$\begin{aligned} &\langle \nabla \frac{\partial d}{\partial t}(t, z^\lambda(t, x)), \nabla d(t, z^\lambda(t, x))\rangle \\ &\quad + \lambda \langle \nabla^2 d(t, z^\lambda(t, x)) \nabla \frac{\partial d}{\partial t}(t, x), \nabla d(t, z^\lambda(t, x))\rangle = 0. \end{aligned} \tag{6.4}$$

Using (1.17) we have $\nabla^2 d \nabla d = 0$ at the point $(t, z^\lambda(t, x))$. Therefore the last addendum on the left-hand side of (6.4) vanishes, and hence

$$\langle \nabla \frac{\partial d}{\partial t}(t, z^\lambda(t, x)), \nabla d(t, z^\lambda(t, x))\rangle = 0. \tag{6.5}$$

On the other hand

$$\begin{aligned} V'(\lambda) &= \langle \nabla \frac{\partial d}{\partial t}(t, z^\lambda(t, x)), \nabla d(t, x)\rangle \\ &= \langle \nabla \frac{\partial d}{\partial t}(t, z^\lambda(t, x)), \nabla d(t, z^\lambda(t, x))\rangle, \end{aligned}$$

where we have used the equality $\nabla d(t, x) = \nabla d(t, z^\lambda(t, x))$, which follows from (1.9). Therefore, from (6.5) we deduce (6.2), hence V is constant and this concludes the proof. $\square$

Smooth compact mean curvature flows can be characterized as flows of tubular neighbourhoods. Indeed, in view of Lemma 6.1 and (1.39), we have that (3.16) is equivalent to a single equation(2) on the fixed set Q.

Theorem 6.2 (Evolution of a tubular neighbourhood). *Let* $f:[a,b]\to \mathcal{P}(\mathbb{R}^n)$ *be a smooth compact flow. Then* f *is a smooth compact mean curvature flow if and only if*

$$\frac{\partial d}{\partial t} = \mathrm{tr}\Big(\nabla^2 d\ (\mathrm{Id} - d\nabla^2 d)^{-1}\Big) \qquad \text{in } Q^{(3)}. \tag{6.6}$$

Proof. Assume that f is a smooth compact mean curvature flow. Let $(t,z) \in Q$ and $x := \mathrm{pr}(t,z) \in \partial f(t)$. Using (6.1), the equation (3.16) expressing mean curvature flow of $\partial f(t)$, and (1.40), we have

$$\begin{aligned}\frac{\partial d}{\partial t}(t,z) &= \frac{\partial d}{\partial t}(t,x) = \Delta d(t,x)\\ &= \mathrm{tr}\Big(\nabla^2 d(t,z)\big(\mathrm{Id} - d(t,z)\nabla^2 d(t,z)\big)^{-1}\Big).\end{aligned}$$

Conversely, assume that d satisfies (6.6). Then, in particular, $\dfrac{\partial d}{\partial t} = \Delta d$ on $\{d = 0\}$, so that f is a smooth compact mean curvature flow. □

Notation. We set

$$G := \Big(\mathrm{Id} - d\nabla^2 d\Big)^{-1} \qquad \text{in } Q. \tag{6.7}$$

Notice that, if $i, j \in \{1, \dots, n\}$ and G_{ij} is the ij-entry of the symmetric matrix G, we have(4)

$$G_{ij} = \mathrm{Id}_{ij} + d\nabla^2_{ij} d + d^2 \nabla^2_{il} d \nabla^2_{lj} d + \mathcal{O}(d^3) \qquad \text{in } Q. \tag{6.8}$$

(2) A proof of Theorem 6.2 can be found in [138].

(3) Thanks to (6.6) and Remark 1.22, if f is a smooth mean curvature flow we have

$$\frac{\partial d}{\partial t} - \Delta d \le 0 \quad \text{in } Q \cap \{d \le 0\}, \qquad \frac{\partial d}{\partial t} - \Delta d \ge 0 \quad \text{in } Q \cap \{d \ge 0\}.$$

These inequalities suggest a possible notion of weak solution to mean curvature flow, useful for instance when studying the geometric evolution flow with the level set method or with the reaction-diffusion equations; see [136, 248] for more.

(4) The series $\sum_{k=0}^{\infty} d^k (\nabla^2 d)^k$, which coincides with G in Q, is usually called the Neumann series [259].

Recalling (6.7) and formulas (1.40) and (1.43), equation (6.6) reads as[(5)]

$$\frac{\partial d}{\partial t} = \mathrm{tr}\left(\nabla^2 d\, G\right) = \overline{\Delta d} \qquad \text{in } Q. \tag{6.9}$$

Example 6.4 (Sphere, I). Let $\overline{B_{R(t)}}$ be a sphere flowing by mean curvature as in Example 3.21 with $z_0 = 0$, so that $\dot{R}(t) = -\dfrac{n-1}{R(t)}$ for $t \in [0, t^\dagger)$, $t^\dagger = \frac{(R(0))^2}{2(n-1)}$, and $d(t, z) = |z| - R(t)$. Then, recalling the expression (3.27) of Δd, for $(t, z) \in [0, t^\dagger) \times (\mathbb{R}^n \setminus \{0\}) =: Q$ we have

$$\overline{\Delta d}(t, z) = \Delta d\left(t, R(t)\frac{z}{|z|}\right) = \frac{n-1}{R(t)} = -\dot{R}(t) = \frac{\partial d}{\partial t}(t, z), \tag{6.10}$$

consistently with (6.9).

Evolution of ∇d and Δd

Equation (6.6) can be used for proving the short time existence theorem (see Chapter 7). It can also be used to find the evolution law of various

---

(5) Theorem 6.2 can be used to give a proof of the following version of the inclusion principle (see also [203]).

Theorem 6.3 (Strong inclusion principle). *Let $f_1, f_2 : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be two smooth compact mean curvature flows. Assume that $\partial f_1(a)$ and $\partial f_2(a)$ are connected. Suppose furthermore that $f_1(a) \subseteq f_2(a)$ and that $f_1(a) \neq f_2(a)$. Then $\partial f_1(t) \cap \partial f_2(t) = \emptyset$ for any $t \in (a, b]$.*

Proof. We follow [138]. Let $\partial E_1, \partial E_2 \in \mathcal{C}^\infty$. We say that ∂E_1 and ∂E_2 are close to each other if there exists an open set $A \subset \mathbb{R}^n$ such that $\partial E_1 \subset A$, $\partial E_2 \subset A$, and the signed distance functions from ∂E_1 and from ∂E_2 belong to $\mathcal{C}^\infty(A)$.
Now, let $d_i(t, \cdot)$ be the signed distance function from $\partial f_i(t)$ positive in $\mathbb{R}^n \setminus f_i(t)$ for $i \in \{1, 2\}$. Suppose first that $\partial f_1(a)$ and $\partial f_2(a)$ are close to each other, so that there exists an open set $A \subset \mathbb{R}^n$ such that $d_i(a, \cdot) \in \mathcal{C}^\infty(A)$ for $i \in \{1, 2\}$. Remembering (6.6), it follows that there exists $\tau > 0$ such that d_1 and d_2 are two solutions of problem (7.2) in $[a, a + \tau] \times A$. Define $w := d_1 - d_2$. Since $f_1(a) \subseteq f_2(a)$, from Theorem 5.8 it follows that $f_1(t) \subseteq f_2(t)$ for all $t \in [a, b]$, so that $w \geq 0$. Then, observing that by assumption $w(a, \cdot)$ is not identically zero, from the strong maximum principle (see for instance [134]) applied to the uniformly parabolic equation satisfied by w (see (7.38)), it follows that $w(t, z) > 0$ for any $(t, z) \in (a, a + \tau] \times A$, so that $f_1(t) \subseteq f_2(t)$ and $\partial f_1(t) \cap \partial f_2(t) = \emptyset$ for any $t \in (a, a + \tau]$. Then the same relations hold also for $t \in [a + \tau, b]$, using Theorem 5.4.
Assume now that $\partial f_1(a)$ and $\partial f_2(a)$ are not close to each other. Select a compact connected boundary $\partial F \in \mathcal{C}^\infty$ so that $f_1(a) \subseteq F \subseteq f_2(a)$, and in such a way that ∂F and $\partial f_2(a)$ are close to each other (in particular $F \neq f_1(a)$). Denote by $g : [a, a + \theta] \to \mathcal{P}(\mathbb{R}^n)$ the smooth compact mean curvature flow starting from ∂F, for some positive time $\theta < b - a$ (see Chapter 7). From the previous case we have that $g(t) \subseteq f_2(t)$ and $\partial g(t) \cap \partial f_2(t) = \emptyset$, for any $t \in (a, a + \theta]$. On the other hand, from Theorem 5.8 we have that $f_1(t) \subseteq g(t)$ for any $t \in [a, a + \theta]$. Hence $f_1(t) \subseteq f_2(t)$ and $\partial f_1(t) \cap \partial f_2(t) = \emptyset$ for any $t \in (a, a + \theta]$, and again the same relations hold also for $t \in [a + \theta, b]$. $\square$

geometric quantities, in particular of the normal vector and of the mean curvature.

Notation. Occasionally, if $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ is a smooth mean curvature flow, we will use the notation

$$\Sigma(t) := \partial f(t), \qquad t \in [a, b].$$

The next evolution equation is given in parametric form in (3.44) for $n = 2$ and, more generally, in (6.32) (see Remark 6.12).

Lemma 6.5 (Evolution of ∇d). *Let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth compact mean curvature flow and $h \in \{1, \dots, n\}$. Then*

$$\frac{d}{dt}\nabla_h d = \frac{\partial}{\partial t}\nabla_h d = \delta_h \overline{\Delta d} \qquad \text{in } Q. \tag{6.11}$$

Proof. Recalling the definition of d/dt in (3.52) and using (1.17), it follows that

$$\frac{d}{dt}\nabla_h d = \frac{\partial}{\partial t}\nabla_h d - \langle \frac{\partial d}{\partial t}\nabla d, \nabla\nabla_h d\rangle = \frac{\partial}{\partial t}\nabla_h d \qquad \text{in } Q.$$

Then, using (6.9) and (3.50), we have

$$\frac{d}{dt}\nabla_h d = \nabla_h \frac{\partial d}{\partial t} = \nabla_h \overline{\Delta d} = \delta_h \overline{\Delta d} \qquad \text{in } Q. \qquad \square$$

Example 6.6 (Sphere, II). In Example 6.4 we have

$$\frac{d}{dt}\nabla_h d = 0 \quad \text{and} \quad \delta_h \overline{\Delta d} = 0 \quad \text{in } [0, t^\dagger) \times (\mathbb{R}^n \setminus \{0\}),\ h \in \{1, \dots, n\},$$

consistently with (6.11).

We now compute the evolution equation of Δd. Its parametric counterpart is given in formula (3.45) for the curvature evolution of plane curves, and, more generally, in (6.38).

Lemma 6.7 (Evolution of Δd). *Let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be a smooth compact mean curvature flow. Then*

$$\frac{d}{dt}\Delta d = \Delta\overline{\Delta d} + \overline{\Delta d}\, |\nabla^2 d|^2 \qquad \text{in } Q. \tag{6.12}$$

Proof. From (1.35) it follows that

$$\langle \nabla \Delta d, \nabla d\rangle = -|\nabla^2 d|^2 \qquad \text{in } Q.$$

Then, using the definition of $\dfrac{d}{dt}$ and the evolution equation (6.9) we have

$$\frac{d}{dt}\Delta d = \frac{\partial}{\partial t}\Delta d - \overline{\Delta d}\ \langle \nabla d, \nabla \Delta d\rangle = \frac{\partial}{\partial t}\Delta d + \overline{\Delta d}\ |\nabla^2 d|^2 \qquad \text{in } Q.$$

Observe now that

$$\frac{\partial}{\partial t}\Delta d = \Delta \frac{\partial d}{\partial t} = \Delta \overline{\Delta d} \qquad \text{in } Q,$$

and therefore (6.12) follows □

Example 6.8 (Sphere, III). In Example 6.4, recalling formula (3.27) we have

$$\frac{\partial}{\partial t}\Delta d = 0 \qquad \text{in } Q,$$

so that

$$\begin{aligned} \frac{d}{dt}\Delta d(t,z) &= -\langle \frac{\partial d}{\partial t}(t,z)\nabla d(t,z), \nabla \Delta d(t,z)\rangle \\ &= -(n-1)\dot{R}(t)\langle \frac{z}{|z|}, \frac{z}{|z|^3}\rangle \qquad\qquad (t,z) \in Q. \\ &= -\frac{(n-1)\dot{R}(t)}{|z|^2}, \end{aligned}$$

Moreover, from (6.10),

$$\Delta \overline{\Delta d} = 0 \qquad \text{in } Q,$$

and since $|\nabla^2 d(t,z)|^2 = \frac{n-1}{|z|^2}$,

$$\overline{\Delta d}(t,z)|\nabla^2 d(t,z)|^2 = \frac{(n-1)^2}{R(t)|z|^2} = \frac{d}{dt}\Delta d(t,z), \quad (t,z) \in Q,$$

consistently with (6.12).

Evolution of $|\nabla^2 d|^2$

The evolution equations of the second fundamental form and its derivatives will be computed in the appendix of this chapter, expressing the flow in parametric form. Here, using the signed distance function, we limit ourselves to an example and to the equation for $|\nabla^2 d|^2$.

In what follows we denote by

$$\overline{\nabla^2 d}$$

the matrix having $\overline{\nabla^2_{ij} d}$ as its ij-entry, for any $i, j \in \{1, \ldots, n\}$.

Example 6.9 (Sphere, IV). Referring to Example 6.4, we have

$$\nabla^2_{ij} d(t,z) = \frac{1}{|z|}\left(\mathrm{Id}_{ij} - \frac{z_i}{|z|}\frac{z_j}{|z|}\right).$$

It follows that

$$\begin{aligned} \overline{\nabla^2_{ij} d}(t,z) &= \nabla^2_{ij} d\left(t, R(t)\frac{z}{|z|}\right) = \frac{1}{R(t)}\left(\mathrm{Id}_{ij} - \frac{z_i}{|z|}\frac{z_j}{|z|}\right), \\ |\overline{\nabla^2 d}(t,z)|^2 &= \frac{n-1}{R(t)^2}. \end{aligned}$$

Hence

$$\begin{aligned} 2|\nabla^2 d(t,z)|^4 &= 2\frac{(n-1)^2}{|z|^4}, \qquad (t,z) \in Q, \\ \frac{\partial}{\partial t}|\nabla^2 d|^2 &= 0 \qquad \text{in } Q, \\ \frac{d}{dt}|\nabla^2 d(t,z)|^2 &= -\overline{\Delta d}(t,z)\langle \nabla d(t,z), \nabla|\nabla^2 d(t,z)|^2\rangle \\ &= 2\frac{(n-1)^2}{R(t)}\langle \frac{z}{|z|}, \frac{z}{|z|^4}\rangle = 2\frac{(n-1)^2}{R(t)|z|^3}, \qquad (t,z) \in Q, \end{aligned}$$

and

$$\Delta|\overline{\nabla^2 d}|^2 = 0 \qquad \text{in } Q. \tag{6.13}$$

In particular, for $t \in [0, T]$,

$$\frac{d}{dt}|\nabla^2 d|^2 = 2|\nabla^2 d|^4 \qquad \text{on } \partial B_{R(t)}. \tag{6.14}$$

The evolution equation for the square of the norm of the second fundamental form reads as follows.

Theorem 6.10 (Evolution of $|\nabla^2 d|^2$). *Let $f:[a,b]\to\mathcal{P}(\mathbb{R}^n)$ be a smooth compact mean curvature flow. Then, for any $t \in [a,b]$, we have*

$$\begin{aligned}\frac{d}{dt}|\nabla^2 d|^2 = {} & \Delta|\overline{\nabla^2 d}|^2 - 2|\nabla\overline{\nabla^2 d}|^2 \\ & + 2|\nabla^2 d|^4 + 4\mathrm{tr}\left((\nabla^2 d)^4\right) \qquad \text{on } \partial f(t).\end{aligned} \tag{6.15}$$

Proof. For $i, j \in \{1,\dots,n\}$ we have

$$\begin{aligned}\frac{d}{dt}\nabla^2_{ij}d = {} & \frac{\partial}{\partial t}\nabla^2_{ij}d - \overline{\Delta d}\,\langle\nabla d, \nabla\nabla^2_{ij}d\rangle \\ = {} & \nabla^2_{ij}\frac{\partial}{\partial t}d - \overline{\Delta d}\,\langle\nabla d, \nabla\nabla^2_{ij}d\rangle \qquad \text{in } Q, \\ = {} & \nabla^2_{ij}\overline{\Delta d} - \overline{\Delta d}\,\langle\nabla d, \nabla\nabla^2_{ij}d\rangle \\ = {} & \nabla^2_{ij}\overline{\Delta d} + \overline{\Delta d}\,\nabla^2_{il}d\,\nabla^2_{lj}d\,,\end{aligned}$$

where the last equality follows from (1.35). Hence

$$\begin{aligned}\frac{d}{dt}|\nabla^2 d|^2 = {} & 2\nabla^2_{ij}d\,\frac{d}{dt}\nabla^2_{ij}d \\ = {} & 2\nabla^2_{ij}d\,\nabla^2_{ij}\overline{\Delta d} + 2\overline{\Delta d}\,\mathrm{tr}\left((\nabla^2 d)^3\right)\end{aligned} \qquad \text{in } Q. \tag{6.16}$$

Now, we want to exchange the operators ∇^2_{ij} and Δ in the expression $\nabla^2_{ij}\,\overline{\Delta d}$ in (6.16).

Step 1. We have

$$\begin{aligned}\nabla^2_{ij}\,\overline{\Delta d} = {} & \Delta\nabla^2_{ij}d + 2\nabla_i d\,\nabla^2_{lk}d\,\nabla^3_{jkl}d \\ & + 2\nabla_j d\,\nabla^2_{lk}d\,\nabla^3_{ikl}d + \nabla^2_{ij}d\,|\nabla^2 d|^2 \qquad \text{in } Q. \\ & + 2\nabla_i d\,\nabla_j d\,\mathrm{tr}\left((\nabla^2 d)^3\right) + \mathcal{O}(d)\end{aligned} \tag{6.17}$$

To show (6.17), let us preliminarily compute the second derivatives of the matrix G defined in (6.7). From (6.8) it follows that, in Q,

$$\begin{aligned}\nabla^2_{ij}G_{kl} = {} & \nabla_i d\,\nabla^3_{jkl}d + \nabla_j d\,\nabla^3_{ikl}d + \nabla^2_{ij}d\,\nabla^2_{kl}d \\ & + 2\nabla_i d\,\nabla_j d\,\nabla^2_{kp}d\,\nabla^2_{pl}d + \mathcal{O}(d).\end{aligned} \tag{6.18}$$

Then, using (6.8) and (6.18), we have, in Q,

$$\begin{aligned}\nabla^2_{ij}\overline{\Delta d} = {} & \nabla^2_{ij}\left(\nabla^2_{kl}d\,G_{lk}\right) \\ = {} & \nabla^4_{ijkl}d\,G_{lk} + \nabla^3_{ikl}d\,\nabla_j G_{lk} + \nabla^3_{jkl}d\,\nabla_i G_{lk} + \nabla^2_{kl}d\,\nabla^2_{ij}G_{lk} \\ = {} & \nabla^2_{ij}\Delta d + \nabla^3_{ikl}d\,\nabla_j d\,\nabla^2_{lk}d + \nabla^3_{jkl}d\,\nabla_i d\,\nabla^2_{lk}d \\ & + \nabla^2_{kl}d\Big(\nabla_i d\,\nabla^3_{jkl}d + \nabla_j d\,\nabla^3_{ikl}d + \nabla^2_{ij}d\,\nabla^2_{lk}d \\ & + 2\nabla_i d\,\nabla_j d\,\nabla^2_{kp}d\,\nabla^2_{pl}d\Big) + \mathcal{O}(d).\end{aligned}$$

This proves step 1, since $\nabla^2_{ij}\Delta d = \Delta\nabla^2_{ij}d$.

As a consequence of (6.17) and recalling that $\nabla^2_{ij}d\ \nabla_j d = 0$, we have

$$\nabla^2_{ij}d\ \nabla^2_{ij}\ \overline{\Delta d} = \nabla^2_{ij}d\ \Delta\nabla^2_{ij}d + |\nabla^2 d|^4 \qquad \text{on } \partial f(t). \tag{6.19}$$

Therefore, from (6.16) and (6.19) we deduce

$$\frac{d}{dt}|\nabla^2 d|^2 = 2\nabla^2_{ij}d\ \Delta\nabla^2_{ij}d + 2|\nabla^2 d|^4 + 2\Delta d\ \mathrm{tr}\left((\nabla^2 d)^3\right) \text{ on } \partial f(t). \tag{6.20}$$

In step 1 we have expressed $\nabla^2_{ij}\overline{\Delta d}$ in terms of $\Delta\nabla^2_{ij}d$. We now want to relate this latter expression with $\Delta\overline{\nabla^2_{ij}d}$. We first observe that from (6.18)

$$\Delta G_{kj} = 2\nabla_i d\ \nabla^3_{ikj}d + \Delta d\ \nabla^2_{kj}d + 2\nabla^2_{kp}d\ \nabla^2_{pj}d + \mathcal{O}(d) \quad \text{in } Q. \tag{6.21}$$

Step 2. We have

$$\Delta\overline{\nabla^2_{ij}d} = \Delta\nabla^2_{ij}d + \Delta d\ \nabla^2_{ik}d\ \nabla^2_{kj}d - 2\nabla^2_{il}d\ \nabla^2_{lk}d\ \nabla^2_{kj}d \quad \text{on } \partial f(t). \tag{6.22}$$

Using the fact that $\overline{\nabla^2_{ij}d} = \nabla^2_{ik}d\ G_{kj}$, and taking into account (6.21), we compute, in Q,

$$\begin{aligned}
\Delta\overline{\nabla^2_{ij}d} &= \Delta(\nabla^2_{ik}d\ G_{kj}) \\
&= \Delta\nabla^2_{ij}d + \nabla^2_{ik}d\ \Delta G_{kj} + 2\nabla^3_{lik}d\ \nabla_l G_{kj} + \mathcal{O}(d) \\
&= \Delta\nabla^2_{ij}d \\
&\quad + \nabla^2_{ik}d\Big(\Delta d\ \nabla^2_{kj}d + 2\nabla_m d\ \nabla^3_{mkj}d + 2\nabla^2_{kp}d\ \nabla^2_{pj}d\Big) \\
&\quad + 2\nabla_l d\ \nabla^2_{kj}d\nabla^3_{lik}d + \mathcal{O}(d) \\
&= \Delta\nabla^2_{ij}d + \Delta d\ \nabla^2_{ik}d\ \nabla^2_{kj}d \\
&\quad + 2\nabla_l d\ \nabla^2_{kj}d\ \nabla^3_{lik}d\ + 2\nabla_m d\ \nabla^2_{ik}d\ \nabla^3_{mkj}d \\
&\quad + 2\nabla^2_{ik}d\ \nabla^2_{kp}d\ \nabla^2_{pj}d + \mathcal{O}(d).
\end{aligned} \tag{6.23}$$

Now, in Q we have, from (1.35),

$$\nabla_l d\ \nabla^3_{ikl}d = -\nabla^2_{kl}d\ \nabla^2_{il}d.$$

Hence we can rewrite the two addenda in the penultimate line of (6.23) in terms only of second derivatives of the distance function as

$$\begin{aligned}
2\nabla_l d\ \nabla^2_{kj}d\ \nabla^3_{lik}d\ + 2\nabla_m d\ \nabla^2_{ik}d\ \nabla^3_{mkj}d = &-2\nabla^2_{li}d\ \nabla^2_{kj}d\ \nabla^2_{lk}d \\
&- 2\nabla^2_{mj}d\ \nabla^2_{ik}d\ \nabla^2_{mk}d
\end{aligned}$$

and step 2 follows.

As a consequence of (6.22), and being $\overline{\nabla^2_{ij} d} = \nabla_{ij} d$ on $\partial f(t)$, we have

$$\begin{aligned}\overline{\nabla^2_{ij} d}\ \Delta \overline{\nabla^2_{ij} d} = {} & \nabla^2_{ij} d\ \Delta \nabla^2_{ij} d \\ & + \Delta d \ \mathrm{tr}\left((\nabla^2 d)^3\right) - 2\mathrm{tr}\left((\nabla^2 d)^4\right) \qquad \text{on } \partial f(t).\end{aligned} \tag{6.24}$$

Observe now that

$$\Delta |\overline{\nabla^2 d}|^2 = 2\overline{\nabla^2_{ij} d}\ \Delta \overline{\nabla^2_{ij} d} + 2|\nabla \overline{\nabla^2 d}|^2 \qquad \text{in } Q, \tag{6.25}$$

where

$$|\nabla \overline{\nabla^2 d}|^2 := \nabla_i \overline{\nabla^2_{jk} d}\ \nabla_i \overline{\nabla^2_{jk} d}.$$

Hence, from (6.24),

$$\begin{aligned}\Delta |\overline{\nabla^2 d}|^2 = {} & 2\nabla^2_{ij} d\ \Delta \nabla^2_{ij} d + 2|\nabla \overline{\nabla^2 d}|^2 \\ & + 2\Delta d \ \mathrm{tr}\left((\nabla^2 d)^3\right) - 4\mathrm{tr}\left((\nabla^2 d)^4\right) \qquad \text{on } \partial f(t).\end{aligned} \tag{6.26}$$

Step 3. Conclusion of the proof of (6.15).

From (6.24) it follows that

$$\begin{aligned}2\nabla^2_{ij} d\ \Delta \nabla^2_{ij} d = {} & 2\nabla^2_{ij} d\ \Delta \overline{\nabla^2_{ij} d} - 2\Delta d \ \mathrm{tr}\left((\nabla^2 d)^3\right) \\ & + 4\mathrm{tr}\left((\nabla^2 d)^4\right) \qquad \text{on } \partial f(t).\end{aligned} \tag{6.27}$$

Substituting (6.27) and (6.25) in (6.20) gives, on $\partial f(t)$,

$$\begin{aligned}\frac{d}{dt}|\nabla^2 d|^2 = {} & 2\nabla^2_{ij} d\ \Delta \overline{\nabla^2_{ij} d} + 2|\nabla^2 d|^4 + 2\Delta d \ \mathrm{tr}\left((\nabla^2 d)^3\right) \\ & - 2\Delta d \ \mathrm{tr}\left((\nabla^2 d)^3\right) + 4\mathrm{tr}\left((\nabla^2 d)^4\right) \\ = {} & 2\nabla^2_{ij} d\ \Delta \overline{\nabla^2_{ij} d} + 2|\nabla^2 d|^4 + 4\mathrm{tr}\left((\nabla^2 d)^4\right) \\ = {} & \Delta |\overline{\nabla^2 d}|^2 - 2|\nabla \overline{\nabla^2 d}|^2 + 2|\nabla^2 d|^4 + 4\mathrm{tr}\left((\nabla^2 d)^4\right). \qquad \square\end{aligned}$$

Using Lemma 1.31, equation (6.15) reads as

$$\begin{aligned}\frac{d}{dt}|\nabla^2 d|^2 = {} & \Delta^{\Sigma(t)} |\nabla^2 d|^2 - 2|\nabla \overline{\nabla^2 d}|^2 \\ & + 2|\nabla^2 d|^4 + 4\mathrm{tr}\left((\nabla^2 d)^4\right) \qquad \text{on } \partial f(t).\end{aligned}$$

Example 6.11 (Sphere, V). In Example 6.4 we have

$$2|\nabla\overline{\nabla^2 d}|^2 = 4\mathrm{tr}\left((\nabla^2 d)^4\right) \qquad \text{on } \partial B_{R(t)}. \tag{6.28}$$

Indeed, for $(t, z) \in Q$,

$$\begin{aligned}\nabla_k\overline{\nabla^2_{ij} d}(t, z) &= -\frac{1}{R(t)}\nabla_k\left(\frac{z_i}{|z|}\frac{z_j}{|z|}\right)\\ &= -\frac{1}{R(t)|z|^2}\left(\mathrm{Id}_{ik}z_j + \mathrm{Id}_{jk}z_i - \frac{2z_i z_j z_k}{|z|^2}\right),\end{aligned}$$

so that the left-hand side of (6.28) reads as

$$\begin{aligned}&2|\nabla\overline{\nabla^2 d}(t, z)|^2\\ &= \frac{2}{R(t)^2|z|^4}\left(2n|z|^2 + 4|z|^2 - 4|z|^2 - 4|z|^2 + 2|z|^2\right)\\ &= \frac{4(n-1)}{R(t)^2|z|^2}.\end{aligned} \tag{6.29}$$

On the other hand, concerning the right-hand side of (6.28), we have

$$\begin{aligned}&4\mathrm{tr}\left((\nabla^2 d(t, z))^4\right)\\ &= \frac{4}{|z|^4}\left(\mathrm{Id}_{ij} - \frac{z_i z_j}{|z|^2}\right)\left(\mathrm{Id}_{jl} - \frac{z_j z_l}{|z|^2}\right)\left(\mathrm{Id}_{lk} - \frac{z_l z_k}{|z|^2}\right)\left(\mathrm{Id}_{ki} - \frac{z_k z_i}{|z|^2}\right)\\ &= \frac{4}{|z|^4}\left(\mathrm{Id}_{il} - \frac{z_i z_l}{|z|^2}\right)\left(\mathrm{Id}_{li} - \frac{z_l z_i}{|z|^2}\right) = \frac{4(n-1)}{|z|^4},\end{aligned} \tag{6.30}$$

for $(t, z) \in Q$. Hence (6.28) follows by taking the restriction of (6.29) and (6.30) on $\partial B_{R(t)}$. Observe that (6.13), (6.14) and (6.28) are consistent with (6.15).

6.1. Appendix: parametric evolution equations

We have already computed the evolution equations of the normal vector, of the mean curvature and of the second fundamental form, for a smooth boundary evolving by mean curvature. In this appendix we derive these evolution equations, and the evolution equations of the derivatives of the squared norm of the second fundamental form, using the well established formalism [168] based on the parametric description of the flow (these evolution equations have been computed in the case of plane curves in Example 3.33). The advantage of this approach stands on the fact that all evolution equations are computed on a fixed parameters space. However, the notion of covariant derivative is required.

The results of these computations, together with the maximum principle, will be used to deduce some qualitative properties of the flow. Our main application will be in the proof of the avoidance principle between barriers, in Chapter 13. A much wider discussion on the consequences that one can deduce by analyzing the above mentioned evolution equations can be found, *e.g.*, in [126, 203].

6.1.1. Evolution of geometric quantities

Let us compute the evolution equations for the second fundamental form and its derivatives, for a hypersurface evolving by mean curvature. We will follow the line of reasoning of [203], to which we refer for all details and the related references(6) concerning the derivation of the equations. All computations are made expressing the evolution in parametric form as in (3.23).

Notation. The symbol D denotes covariant differentiation of scalars and tensors with respect to $g(t) = (g_{\alpha\beta}(t))$(7), (8).

Given two tensors S, T, we denote by

$$S * T$$

a tensor formed by a sum of terms each one obtained by contracting some indices of the tensors S and T using the tensor $g_{\alpha\beta}$ and/or its inverse $g^{\alpha\beta}$, and possibly multiplied by a constant.

For a smooth parametric mean curvature flow $\varphi \in \mathcal{X}(a,b]; \mathrm{Imm}(\mathcal{S};\mathbb{R}^n))$ it is possible to prove that:

- the evolution equations for the metric tensor $(g_{\alpha\beta})$ and its inverse $(g^{\alpha\beta})$ are
$$\frac{\partial g_{\alpha\beta}}{\partial t} = -2\mathrm{H}h_{\alpha\beta}, \qquad \frac{\partial g^{\alpha\beta}}{\partial t} = 2\mathrm{H}h^{\alpha\beta}, \tag{6.31}$$
where we recall that $h_{\alpha\beta}$ are the components of the second fundamental form A (see (3.21));
- the evolution equation for the unit normal vector field ν is
$$\frac{\partial \nu}{\partial t} = -D\mathrm{H}; \tag{6.32}$$

(6) See, in particular, [168]; see also [126].

(7) We recall that, if $x \in \partial E \in \mathcal{C}^\infty$ and if X, Y are two smooth vector fields defined in a relative neighbourhood of x contained in ∂E, and tangent to ∂E, and if X^e, Y^e are two smooth vector fields defined in a neighbourhood of x in $\mathbb{R}^n$ extending X and Y respectively, we have that $D_X Y(x)$ is the orthogonal projection of $\nabla_{X^e} Y^e(x)$ on $T_x(\partial E)$, namely $D_X Y(x) = P_{T_x(\partial E)}\big(\nabla Y^e(x) X^e(x)\big)$. See, *e.g.*, [123, Example 3, page 57], [150, Proposition 2.56].

(8) In particular, the components $(D_\alpha \mathrm{grad} u)^\beta$ satisfy $(D_\alpha \mathrm{grad} u)^\alpha = \Delta_{g(t)}$.

- the evolution equation for the Christoffel symbols $\Gamma^\alpha_{\beta\gamma}$ of the metric is

$$\frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial t} = -h^\alpha_\gamma D_\beta \mathrm{H} - h^\alpha_\beta D_\gamma \mathrm{H} + h_{\beta\gamma} D^\alpha \mathrm{H} \\ - \mathrm{H}(D_\beta h^\alpha_\gamma + D_\gamma h^\alpha_\beta - D^\alpha h_{\beta\gamma}),$$

which can be written as

$$\frac{\partial \Gamma^\alpha_{\beta\gamma}}{\partial t} = (D\mathrm{H}) * A + \mathrm{H} * DA = (DA) * A. \tag{6.33}$$

Remark 6.12. The evolution equation (6.32) is consistent with the evolution equation (6.11) for the gradient of the signed distance function in the embedded case. Indeed, the right-hand side of (6.11) is, on $\partial f(t)$, the tangential gradient of Δd. To compare the left hand sides of (6.11) and (6.32), let $\varphi : [a, b] \times \mathcal{S} \to \mathbb{R}^n$ be a normal parametrization of the flow as in Remark 3.9. In particular, given $(t, s) \in [a, b] \times \mathcal{S}$, and setting $x := \varphi(t, s)$, it follows that

$$\frac{\partial \varphi}{\partial t}(t, s) = -\frac{\partial d(t, x)}{\partial t} \nabla d(t, x).$$

Differentiating with respect to t the equality $-\nu_h(t, s) = \nabla_h d(t, x)$ yields

$$-\frac{\partial \nu_h}{\partial t}(t, s) = \frac{\partial \nabla_h d}{\partial t}(t, x) - \langle \nabla\nabla_h d(t, x), \frac{\partial d(t, x)}{\partial t} \nabla d(t, x)\rangle \\ = \frac{d}{dt}\nabla_h d(t, x) = \frac{\partial}{\partial t}\nabla_h d(t, x).$$

Now, we record the following useful result.

Proposition 6.13 (Evolution of $h_{\alpha\beta}$). *The components $h_{\alpha\beta}$ of the second fundamental form satisfy*[(9)]

$$\frac{\partial h_{\alpha\beta}}{\partial t} = \Delta_{g(t)} h_{\alpha\beta} - 2\mathrm{H} h_{\alpha\delta} g^{\delta s} h_{s\beta} + |A|^2_{g(t)}\, h_{\alpha\beta}, \tag{6.34}$$

where

$$|A|^2_{g(t)} := g^{\alpha p} g^{\beta q} h_{\alpha\beta} h_{pq}. \tag{6.35}$$

(9) It is possible to prove that

$$\frac{\partial h_{\alpha\beta}}{\partial t} = D_\alpha D_\beta \mathrm{H} - \mathrm{H} h_{\alpha\delta} g^{\delta s} h_{s\beta}.$$

Then, making use of the Simons' identity $\Delta_{g(t)} h_{\alpha\beta} = D_\alpha D_\beta \mathrm{H} + \mathrm{H} h_{\alpha\delta} g^{\delta s} h_{s\beta} - |A|^2_{g(t)} h_{\alpha\beta}$, one gets (6.34). See [168] for more.

Given $k, l \in \mathbb{N}$ and a (k, l) tensor $T = T^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_l}$, the squared norm $|T|^2_{g(t)}$ of T is defined as

$$|T|^2_{g(t)} := g_{\alpha_1 s_1} \dots g_{\alpha_k s_k} g^{\beta_1 \tau_1} \dots g^{\beta_l \tau_l} T^{\alpha_1 \dots \alpha_k}_{\beta_1 \dots \beta_l} T^{s_1 \dots s_k}_{\tau_1 \dots \tau_l}. \tag{6.36}$$

The squared norm of the second fundamental form satisfies the following evolution equation[10]:

Corollary 6.14 (Evolution of $|A|^2_{g(t)}$). *We have*

$$\frac{\partial |A|^2_{g(t)}}{\partial t} = \Delta_{g(t)} |A|^2_{g(t)} - 2|DA|^2_{g(t)} + 2|A|^4_{g(t)}. \tag{6.37}$$

Proof. We have, using $g^{\alpha p} g^{\beta q} h_{pq} = h^{\alpha\beta}$ and (6.34),

$$\begin{aligned} \frac{\partial |A|^2_{g(t)}}{\partial t} &= h^{\alpha\beta} \left(\Delta_{g(t)} h_{\alpha\beta} - 2\mathrm{H} h_{\alpha\delta} g^{\delta s} h_{s\beta} + |A|^2_{g(t)} h_{\alpha\beta} \right) \\ &\quad + h_{\alpha\beta} \frac{\partial}{\partial t} \left(g^{\alpha p} g^{\beta q} h_{pq} \right). \end{aligned}$$

Next, using again (6.34) and (6.31),

$$\begin{aligned} \frac{\partial |A|^2_{g(t)}}{\partial t} &= h^{\alpha\beta} \left(\Delta_{g(t)} h_{\alpha\beta} - 2\mathrm{H} h_{\alpha\delta} g^{\delta s} h_{s\beta} + |A|^2_{g(t)} h_{\alpha\beta} \right) \\ &\quad + h_{\alpha\beta} g^{\alpha p} g^{\beta q} \left(\Delta_{g(t)} h_{pq} - 2\mathrm{H} h_{p\delta} g^{\delta s} h_{sq} + |A|^2_{g(t)} h_{pq} \right) \\ &\quad + 2\mathrm{H} h_{\alpha\beta} h^{\alpha p} g^{\beta q} h_{pq} + 2\mathrm{H} h_{\alpha\beta} h^{\beta q} g^{\alpha p} h_{pq} \\ &= 2h^{\alpha\beta} \left(\Delta_{g(t)} h_{\alpha\beta} - 2\mathrm{H} h_{\alpha\delta} g^{\delta s} h_{s\beta} + |A|^2_{g(t)} h_{\alpha\beta} \right) \\ &\quad + 4\mathrm{H} h_{\alpha\beta} h^{\beta q} g^{\alpha p} h_{pq} \\ &= \Delta_{g(t)} |A|^2_{g(t)} - 2|DA|^2_{g(t)} + 2|A|^4_{g(t)}, \end{aligned}$$

taking into account that $\Delta_{g(t)} |A|^2_{g(t)} = 2|DA|^2_{g(t)} + 2h^{\alpha\beta} \Delta_{g(t)} h_{\alpha\beta}$. □

[10] From the evolution equation (6.37) for $|A|^2_{g(t)}$ we deduce, for instance, that if $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ is a smooth compact mean curvature flow, then

$$b - a \geq \frac{1}{2} \left[\left(\max_{x \in \partial f(a)} |\nabla^2 d(a, x)|^2 \right)^{-1} - \left(\max_{x \in \partial f(b)} |\nabla^2 d(b, x)|^2 \right)^{-1} \right].$$

Indeed, if we set $\mathrm{a}(t) := \max_{s \in \mathcal{S}} |A(t, s)|_{g(t)}$, from (6.37) and Theorem 5.12 we deduce $\frac{1}{2} \frac{d}{dt} (\mathrm{a}(t)^2) \leq \mathrm{a}(t)^4$ for almost every $t \in [a, b]$. Now, we can divide by $\mathrm{a}(t)$, since $\mathrm{a}(t) = 0$ at some $t \in [a, b]$ implies that $\partial f(t)$ is a hyperplane, in contradiction with the compactness of $\partial f(t)$ (indeed, $\nabla^2 d(t, \cdot) = 0$ on $\partial f(t)$ implies, from Theorem 1.18, that $\nabla^2 d(t, \cdot) = 0$ in U. Hence d is linear in U, and its level sets are hyperplanes). In conclusion $-\frac{1}{2} \frac{d}{dt} \frac{1}{\mathrm{a}(t)^2} \leq 1$ for almost every $t \in [a, b]$. Integrating the above inequality gives the assertion.

As a consequence of (6.31) and (6.34), and taking into account that

$$\Delta_{g(t)}\mathrm{H} = g^{\alpha\beta}\Delta_{g(t)}h_{\alpha\beta},$$

the evolution equation for $\mathrm{H} = g^{\alpha\beta}h_{\alpha\beta}$ reads as

$$\begin{aligned}\frac{\partial \mathrm{H}}{\partial t} =& g^{\alpha\beta}\left(\Delta_{g(t)}h_{\alpha\beta} - 2\mathrm{H}h_{\alpha\delta}g^{\delta s}h_{s\beta} + |A|^2_{g(t)}h_{\alpha\beta}\right) + 2\mathrm{H}|A|^2_{g(t)} \\ =& \Delta_{g(t)}\mathrm{H} + \mathrm{H}|A|^2_{g(t)}.\end{aligned} \tag{6.38}$$

Recalling (6.33) we have:

Lemma 6.15 (Interchange formula). *The following interchange formula of time and covariant derivative of a tensor T holds:*

$$\frac{\partial}{\partial t}DT = D\frac{\partial T}{\partial t} + T * A * DA. \tag{6.39}$$

Proposition 6.16 (Evolution of $D^k h_{\alpha\beta}$). *Let k be a nonnegative integer. The evolution equation for the k-iterated covariant derivative of the components of the second fundamental form reads as:*

$$\frac{\partial}{\partial t}D^k h_{\alpha\beta} = \Delta_{g(t)}D^k h_{\alpha\beta} + \sum_{p+q+r=k,\ p,q,r\in\mathbb{N}} D^p A * D^q A * D^r A \tag{6.40}$$

Proof. We argue by induction on k. The case $k = 0$ is given by equation (6.34). Let $k \geq 1$. Using (6.39), we have

$$\begin{aligned}\frac{\partial}{\partial t}D^k h_{\alpha\beta} =& D\frac{\partial}{\partial t}D^{k-1}h_{\alpha\beta} + (D^{k-1}A) * A * DA \\ =& D\left(\Delta_{g(t)}D^{k-1}h_{\alpha\beta} + \sum_{p+q+r=k-1,\ p,q,r\in\mathbb{N}} D^p A * D^q A * D^r A\right) \\ &+ (D^{k-1}A) * A * DA \\ =& D\Delta_{g(t)}D^{k-1}h_{\alpha\beta} + \sum_{p+q+r=k,\ p,q,r\in\mathbb{N}} D^p A * D^q A * D^r A.\end{aligned}$$

Interchanging D with $\Delta_{g(t)}$ now gives the inductive step; see [203] for the details. □

Corollary 6.17 (Evolution of $|D^k A|^2_{g(t)}$). *Let k be a nonnegative integer. The evolution equation of the squared norm of the k-th iterated covariant derivative of A reads as:*

$$\begin{aligned}\frac{\partial}{\partial t}|D^k A|^2_{g(t)} =& \Delta_{g(t)}|D^k A|^2_{g(t)} - 2|D^{k+1}A|^2_{g(t)} \\ &+ \sum_{p+q+r=k,\ p,q,r\in\mathbb{N}} D^p A * D^q A * D^r A * D^k A.\end{aligned} \tag{6.41}$$

Proof. Recalling (6.36), the fact that the metric tensor g depends on t, and (6.31), we have

$$\frac{\partial}{\partial t}|D^k A|^2_{g(t)} = 2g\left(D^k A, \frac{\partial}{\partial t} D^k A\right) + D^k A * D^k A * A * A.$$

Then, using (6.40), we have

$$\begin{aligned}
\frac{\partial}{\partial t}|D^k A|^2_{g(t)} =& 2g\left(D^k A, \Delta_{g(t)} D^k A + \sum_{p+q+r=k,\ p,q,r\in\mathbb{N}} D^p A * D^q A * D^r A\right) \\
&+ D^k A * D^k A * A * A \\
&= 2g\left(D^k A, \Delta_{g(t)} D^k A\right) \\
&+ \sum_{p+q+r=k,\ p,q,r\in\mathbb{N}} D^p A * D^q A * D^r A * D^k A \\
&= \Delta_{g(t)} |D^k A|^2_{g(t)} - 2|D^{k+1} A|^2_{g(t)} \\
&+ \sum_{p+q+r=k,\ p,q,r\in\mathbb{N}} D^p A * D^q A * D^r A * D^k A.
\end{aligned}$$

□

Remark 6.18. We notice[(11)] that in the terms $D^p A * D^q A * D^r A * D^k A$ in formula (6.41) there can be at most two occurrences of $D^k A$.

The next theorem is an interesting consequence of the evolution equations and the maximum principle. For a proof and all details we refer the reader to [203] and references therein. It is useful to recall that, for a smooth parametric mean curvature flow $\varphi \in \mathcal{X}([0, T); \mathrm{Imm}(\mathcal{S}; \mathbb{R}^n))$ with $\varphi(0) \in \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n)$, we also have $\varphi \in \mathcal{X}([0, T); \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n))$ by the embeddedness preserving property[(12)] (see Theorem 5.10).

Theorem 6.19 (Extension of the flow at $t = T$). *Let $\mathcal{S}$ be a smooth connected compact $(n-1)$-dimensional orientable manifold without boundary. Let $\varphi_0 \in \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n)$ and let $\varphi \in \mathcal{X}([0, T); \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n))$ be the smooth compact parametric mean curvature flow starting from φ_0, for some $T > 0$, assume that*

$$\sup_{s\in\mathcal{S}} \sup_{t\in[0,T)} |A(t, s)|_{g(t)} < +\infty.$$

(11) See [168] for more.

(12) The definitions of $\mathcal{X}([0, T); \mathrm{Imm}(\mathcal{S}; \mathbb{R}^n))$ and $\mathcal{X}([0, T); \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n))$ are obtained from Definition 3.18, by replacing the interval $[a, b]$ with the interval $[0, T)$.

Then

$$\sup_{s\in\mathcal{S}}\sup_{t\in[0,T)}|D^k A(t,s)|_{g(t)} < +\infty, \qquad k\in\mathbb{N}.$$

Moreover, there exists $\varphi_T \in \mathrm{Emb}(\mathcal{S};\mathbb{R}^n)$ *such that the map*

$$\widetilde{\varphi} : [0,T]\times\mathcal{S}\to\mathbb{R}^n$$

defined as

$$\widetilde{\varphi}(t,s) := \begin{cases} \varphi(t,s) & (t,s)\in[0,T)\times S,\\ \varphi_T(s) & t=T,\ s\in S,\end{cases}$$

belongs to $\mathcal{C}^\infty([0,T]\times\mathcal{S})$. *Hence* T *cannot be a singularity time for the parametric flow.*

Chapter 7
Local well-posedness: the approach of Evans and Spruck

In this chapter we discuss the local existence and uniqueness of a smooth compact mean curvature flow starting from a solid set E with compact boundary ∂E of class $\mathcal{C}^\infty$.

Notation. We denote by $\mathrm{M}(n \times n, \mathbb{R}) \simeq \mathbb{R}^{n^2}$ the set of all $(n \times n)$-real matrices, and by $\mathrm{Sym}(n, \mathbb{R}) \subset \mathrm{M}(n \times n, \mathbb{R})$ the vector subspace of all real symmetric $(n \times n)$-matrices. For $M \in \mathrm{Sym}(n, \mathbb{R})$ we let $\lambda_1(M), \ldots, \lambda_n(M)$ be the eigenvalues of M.

Let

$$F : \mathrm{dom}(F) \to \mathbb{R}$$

be defined as follows:

$$\mathrm{dom}(F) := \Big\{(\mathrm{u}, M) \in \mathbb{R} \times \mathrm{Sym}(n, \mathbb{R}) : 1 - \mathrm{u}\lambda_i(M) \neq 0,\ i = 1, \ldots, n\Big\},$$

which we consider as a subset of $\mathbb{R} \times \mathbb{R}^{n^2}$ with the norm induced by the Euclidean norm, and

$$F(\mathrm{u}, M) := \sum_{i=1}^{n} \frac{\lambda_i(M)}{1 - \mathrm{u}\lambda_i(M)}, \qquad (\mathrm{u}, M) \in \mathrm{dom}(F). \tag{7.1}$$

Our aim is to prove the following theorem, due to Evans and Spruck [138] (see also [197]).

Theorem 7.1 (Local existence and uniqueness in $\mathcal{C}^{\frac{2+\alpha}{2}, 2+\alpha}$). *Let $E \subset \mathbb{R}^n$ be a set with compact boundary $\partial E \in \mathcal{C}^\infty$, and define*

$$d(z, E) = \mathrm{dist}(z, E) - \mathrm{dist}(z, \mathbb{R}^n \setminus E), \qquad z \in \mathbb{R}^n.$$

Take $\alpha \in (0, 1)$. Then there exist $\rho_0 > 0$ and $t_0 > 0$ such that, letting

$$\mathrm{U} := (\partial E)^+_{\rho_0},$$

the problem

$$\begin{cases} u \in \mathcal{C}^{\frac{2+\alpha}{2},\, 2+\alpha}\big([0,t_0] \times \overline{\mathrm{U}}\big), & \\ u_t = F(u, \nabla^2 u) & \text{in } (0,t_0) \times \mathrm{U}, \\ |\nabla u|^2 = 1 & \text{on } (0,t_0) \times \partial\mathrm{U}, \\ u = d & \text{on } \{t=0\} \times \mathrm{U} \end{cases} \tag{7.2}$$

has a unique solution.

The definitions of parabolic Hölder spaces and corresponding norms are given in the appendix of this chapter. Observe that, from the eikonal equation (1.7), we have the compatibility condition $|\nabla u(0,\cdot)|^2 = 1$ on $\partial\mathrm{U}$, between u and the initial datum d.

In order to prove Theorem 7.1 we record some preliminary observations.

Some properties of the function F

We first show that F can be smoothly extended on an open subset of $\mathbb{R} \times \mathrm{M}(n \times n, \mathbb{R})$, where it can be differentiated.

If $(\mathrm{u}, M) \in \mathrm{dom}(F)$ one checks that

$$M(\mathrm{Id} - \mathrm{u}M)^{-1} \in \mathrm{Sym}(n, \mathbb{R}),$$

and that $M(\mathrm{Id} - \mathrm{u}M)^{-1}$ commutes with M. Hence, if

$$M = \mathrm{diag}\Big(\lambda_1(M), \ldots, \lambda_n(M)\Big)$$

is diagonal in a suitable basis of $\mathbb{R}^n$, then $M(\mathrm{Id} - \mathrm{u}M)^{-1}$ is diagonal in the same basis, and

$$M(\mathrm{Id} - \mathrm{u}M)^{-1} = \mathrm{diag}\left(\frac{\lambda_1(M)}{1 - \mathrm{u}\lambda_1(M)}, \ldots, \frac{\lambda_n(M)}{1 - \mathrm{u}\lambda_n(M)}\right).$$

Since the trace of a matrix is independent of the choice of the basis, we have that the function F in (7.1) can also be written as

$$F(\mathrm{u}, M) = \mathrm{tr}\Big(M(\mathrm{Id} - \mathrm{u}M)^{-1}\Big), \qquad (\mathrm{u}, M) \in \mathrm{dom}(F). \tag{7.3}$$

The right-hand side of (7.3) is naturally defined on a set containing $\mathrm{dom}(F)$. Let

$$\widehat{D} := \{(\mathrm{u}, M) \in \mathbb{R} \times \mathrm{M}(n \times n, \mathbb{R}) : \mathrm{Id} - \mathrm{u}M \text{ is invertible}\}.$$

Then $\widehat{D}$ is an open subset of $\mathbb{R} \times \mathbb{R}^{n^2}$, and $\widehat{D} \supset \mathrm{dom}(F)$. The function defined as

$$\mathrm{tr}\Big(M(\mathrm{Id} - \mathrm{u}M)^{-1}\Big), \qquad (\mathrm{u}, M) \in \widehat{D}, \tag{7.4}$$

coincides with F on $\mathrm{dom}(F)$, and will be still denoted by the same symbol. Moreover

$$F(\mathrm{u}, M) = F(\mathrm{u}, M^*),$$

where M^* is the transposed of $M \in \mathrm{M}(n \times n, \mathbb{R})^{(1)}$.

We will be interested in the function F restricted to the open set

$$D := \left\{(\mathrm{u}, M) \in \widehat{D} : |\mathrm{u}| < \frac{1}{\max\{|\lambda_1(M)|, \ldots, |\lambda_n(M)|\}}\right\}.$$

From (7.4) it follows that F is analytic on D; moreover, since

$$(\mathrm{Id} - \mathrm{u}M)^{-1} = \sum_{k \geq 0} \mathrm{u}^k M^k, \qquad (\mathrm{u}, M) \in D,$$

we have

$$F(\mathrm{u}, M) = \mathrm{tr}\left(\sum_{k \geq 0} \mathrm{u}^k M^{k+1}\right), \qquad (\mathrm{u}, M) \in D.$$

Notation. If $\xi, \eta \in \mathbb{R}^n$, $\xi = (\xi_1, \ldots, \xi_n)$, $\eta = (\eta_1, \ldots, \eta_n)$, we recall that we denote by

$$\xi \otimes \eta$$

the matrix whose ij-entry is given by $\xi_i \eta_j$.

We denote by $F_{M_{ij}}$ the derivative of F with respect to the ij-th component of M, namely

$$\begin{aligned} F_{M_{ij}}(\mathrm{u}, M) &= \frac{dF}{dM_{ij}}(\mathrm{u}, M) \\ &:= \lim_{h \to 0} \frac{1}{h}\big(F(\mathrm{u}, M + h e_i \otimes e_j) - F(\mathrm{u}, M)\big), \end{aligned} \qquad (\mathrm{u}, M) \in D, \tag{7.5}$$

and we let $F_M(\mathrm{u}, M)$ be the matrix whose ij-entry is $F_{M_{ij}}(\mathrm{u}, M)$.

Moreover, we denote by F_u the partial derivative of F with respect to u. Observe that

$$F_\mathrm{u}(\mathrm{u}, M) = \mathrm{tr}\Big(M^2(\mathrm{Id} - \mathrm{u}M)^{-2}\Big), \qquad (\mathrm{u}, M) \in D. \tag{7.6}$$

(1) Indeed, $F(\mathrm{u}, M) = \mathrm{tr}\Big(M(\mathrm{Id} - \mathrm{u}M)^{-1}\Big) = \mathrm{tr}\Big(((\mathrm{Id} - \mathrm{u}M)^{-1})^* M^*\Big) = \mathrm{tr}\Big(((\mathrm{Id} - \mathrm{u}M)^*)^{-1} M^*\Big) = \mathrm{tr}\Big((\mathrm{Id} - \mathrm{u}M^*)^{-1} M^*\Big) = F(\mathrm{u}, M^*)$.

Lemma 7.2. *Let* $N \in \mathrm{M}(n \times n, \mathbb{R})$. *For any* $(\mathrm{u}, M) \in D$ *we have*

$$\mathrm{tr}\Big(N F_M(\mathrm{u}, M)\Big) = \mathrm{tr}\Big(N(\mathrm{Id} - \mathrm{u}M)^{-2}\Big). \tag{7.7}$$

Proof. We first observe that

$$F_{M_{ij}}(\mathrm{u}, M) = \mathrm{tr}\left(\frac{d}{dM_{ij}}\left(M \sum_{k \geq 0} \mathrm{u}^k M^k\right)\right),$$

where

$$\frac{d}{dM_{ij}}\left(M \sum_{k \geq 0} \mathrm{u}^k M^k\right)$$
$$:= \lim_{h \to 0} \frac{1}{h}\left((M + he_i \otimes e_j) \sum_{k \geq 0} \mathrm{u}^k (M + he_i \otimes e_j)^k - M \sum_{k \geq 0} \mathrm{u}^k M^k\right).$$

Consequently

$$\begin{aligned} F_{M_{ij}}(\mathrm{u}, M) =& \mathrm{tr}\left(\lim_{h \to 0} e_i \otimes e_j \sum_{k \geq 0} \mathrm{u}^k \left(M + he_i \otimes e_j\right)^k\right) \\ &+ \mathrm{tr}\left(M \lim_{h \to 0} \sum_{k \geq 0} \mathrm{u}^k \, \frac{\left(M + he_i \otimes e_j\right)^k - M^k}{h}\right) \\ =& \mathrm{tr}\left(e_i \otimes e_j \left(\sum_{k \geq 0} \mathrm{u}^k M^k + \mathrm{u}M \sum_{k \geq 0} (k+1)\mathrm{u}^k M^k\right)\right). \end{aligned}$$

Combining this formula with the equality

$$\left(\sum_{k \geq 0} \mathrm{u}^k M^k\right)^2 = \sum_{k \geq 0} (k+1)\mathrm{u}^k M^k$$

we deduce

$$F_{M_{ij}}(\mathrm{u}, M) = \mathrm{tr}\left(e_i \otimes e_j \left(\sum_{k \geq 0} \mathrm{u}^k M^k + \mathrm{u}M \left(\sum_{k \geq 0} \mathrm{u}^k M^k\right)^2\right)\right). \tag{7.8}$$

Eventually, since $\left(\sum_{k\geq 0} \mathrm{u}^k M^k\right)^2 (\mathrm{Id} - \mathrm{u}M) = \sum_{k\geq 0} \mathrm{u}^k M^k$, from (7.8) we obtain

$$\begin{aligned}
F_{M_{ij}}(\mathrm{u}, M) &= \mathrm{tr}\left(e_i \otimes e_j \left(\left(\sum_{k\geq 0} \mathrm{u}^k M^k\right)^2 (\mathrm{Id} - \mathrm{u}M + \mathrm{u}M)\right)\right) \\
&= \mathrm{tr}\Big(e_i \otimes e_j (\mathrm{Id} - \mathrm{u}M)^{-2}\Big) = (e_i \otimes e_j)_{lm}\Big((\mathrm{Id} - \mathrm{u}M)^{-2}\Big)_{ml} \\
&= \Big((\mathrm{Id} - \mathrm{u}M)^{-2}\Big)_{ij}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathrm{tr}\Big(N F_M(\mathrm{u}, M)\Big) =& N_{ij} F_{M_{ij}}(\mathrm{u}, M) \\
=& N_{ij}\Big((\mathrm{Id} - \mathrm{u}M)^{-2}\Big)_{ij} = \mathrm{tr}\Big(N(\mathrm{Id} - \mathrm{u}M)^{-2}\Big). \qquad \square
\end{aligned}$$

The function F satisfies a sort of parabolicity condition, in the following sense[(2)].

Corollary 7.3 (Parabolicity of F). *Let* $(\mathrm{u}, M) \in \mathrm{dom}(F) \cap D$. *Then*

$$F_{M_{ij}}(\mathrm{u}, M)\xi_i \xi_j \geq \mu(\mathrm{u}, M)|\xi|^2, \qquad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \tag{7.9}$$

where

$$\mu(\mathrm{u}, M) := \min\Big\{ (1 - \mathrm{u}\lambda_i(M))^{-2} : i \in \{1, \ldots, n\}\Big\} > 0.$$

Proof. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of $\mathbb{R}^n$ so that v_i is an eigenvector of M with $\lambda_i(M)$ as eigenvalue, for any $i = 1, \ldots, n$. Notice that, in the same basis, we have

$$\begin{aligned}
(\mathrm{Id} - \mathrm{u}M)^{-2} =& \mathrm{diag}\left((1 - \mathrm{u}\lambda_1(M))^{-2}, \ldots, (1 - \mathrm{u}\lambda_n(M))^{-2}\right) \\
=& \sum_{i=1}^n (1 - \mathrm{u}\lambda_i(M))^{-2}\, v_i \otimes v_i.
\end{aligned}$$

Given $\xi \in \mathbb{R}^n$, let us apply (7.7) with N replaced by $\xi \otimes \xi$. We have

$$\begin{aligned}
F_{M_{ij}}(\mathrm{u}, M)\xi_i\xi_j =& \mathrm{tr}\left(\xi \otimes \xi (\mathrm{Id} - \mathrm{u}M)^{-2}\right) = \sum_{i=1}^n \frac{\langle \xi, v_i\rangle^2}{(1 - \mathrm{u}\lambda_i(M))^2} \\
&\geq \mu(\mathrm{u}, M) \sum_{i=1}^n \langle \xi, v_i\rangle^2 = \mu(\mathrm{u}, M)|\xi|^2. \qquad \square
\end{aligned}$$

(2) Inequality (7.9) can be obtained for the original function F defined on $\mathrm{dom}(F)$, without extending it on $\widehat{D}$, by taking the incremental quotient as in (7.5), provided $e_i \otimes e_j$ is replaced by $\xi \otimes \xi$ (so that $M + h\xi \otimes \xi$ is still a symmetric matrix when M is a symmetric matrix); see [138].

7.1. Local existence

We begin by proving the local existence statement in Theorem 7.1. Since the boundary of E is compact and of class $\mathcal{C}^\infty$, for $\rho_0 > 0$ small enough we have $d \in \mathcal{C}^\infty\big(\overline{\mathrm{U}}\big)$, where

$$\mathrm{U} = (\partial E)^+_{\rho_0} = \{z \in \mathbb{R}^n : |d(z)| < \rho\},$$

and

$$\left|d(z)\lambda_i(\nabla^2 d(z))\right| \leq \frac{1}{2}, \qquad z \in \overline{\mathrm{U}}, \quad i \in \{1, \dots, n\}.$$

In particular,

$$(d(z), \nabla^2 d(z)) \in D, \qquad z \in \overline{\mathrm{U}}.$$

The study of problem (7.2) will be faced using a linearization method. For notational simplicity, set

$$\ell(z) := F(d(z), \nabla^2 d(z)), \qquad z \in \overline{\mathrm{U}}.$$

Given $t_0 > 0$ and $w \in \mathcal{C}^{\frac{2+\alpha}{2}, 2+\alpha}([0, t_0] \times \overline{\mathrm{U}})$ to be selected later (see Proposition 7.4), we look for solutions u of (7.2) of the form

$$u(t, z) = d(z) + t\ell(z) + w(t, z), \qquad (t, z) \in [0, t_0] \times \overline{\mathrm{U}}. \tag{7.10}$$

Inserting (7.10) into the first equation in (7.2) and subtracting the quantity

$$F_{M_{ij}}(d, \nabla^2 d)\nabla^2_{ij} w + F_{\mathrm{u}}(d, \nabla^2 d)w$$

from both sides, we get

$$w_t - \mathcal{A}(z, w, \nabla^2 w) = \mathcal{R}(t, z, w, \nabla^2 w), \qquad (t, z) \in (0, t_0) \times \mathrm{U}. \tag{7.11}$$

Here

- the function $\mathcal{A} : \mathrm{U} \times \mathbb{R} \times \mathrm{Sym}(n, \mathbb{R}) \to \mathbb{R}$ is defined as

  $$\mathcal{A}(z, \mathrm{u}, M) := F_{M_{ij}}(d(z), \nabla^2 d(z))M_{ij} + F_{\mathrm{u}}(d(z), \nabla^2 d(z))\mathrm{u}$$

  and it is linear with respect to M and u;
- the function $\mathcal{R} : \mathbb{R} \times \mathrm{U} \times \mathbb{R} \times \mathrm{Sym}(n, \mathbb{R}) \to \mathbb{R}$ is defined as

  $$\begin{aligned}\mathcal{R}(t,z,\mathrm{u},M) :=\, &F\big(d(z)+t\ell(z) +\mathrm{u}, \nabla^2 d(z)+ t\nabla^2\ell(z)+ M\big)\\ &-\ell(z) - F_{\mathrm{u}}(d(z),\nabla^2 d(z))\mathrm{u} - F_{M_{ij}}(d(z),\nabla^2 d(z))M_{ij},\end{aligned} \tag{7.12}$$

  and depends nonlinearly on M.

Notice that $\mathcal{R}(0, z, 0, 0) = 0$, and

$$\begin{aligned} &\mathcal{R}_{\mathrm{u}}(t, z, \mathrm{u}, M)_{|t=0,\mathrm{u}=0,M=0} = 0, \\ &\mathcal{R}_{M_{ij}}(t, z, \mathrm{u}, M)_{|t=0,\mathrm{u}=0,M=0} = 0, \qquad i, j \in \{1, \dots, n\}. \end{aligned} \tag{7.13}$$

Now, we consider the boundary condition on $(0, t_0) \times \partial\mathrm{U}$. Inserting (7.10) into the second equation in (7.2) and observing that, thanks to the eikonal equation (1.7),

$$1 = |\nabla d + t\nabla\ell + \nabla w|^2 = 1 + |t\nabla\ell + \nabla w|^2 + 2\langle\nabla d, \nabla w\rangle + 2t\langle\nabla d, \nabla\ell\rangle,$$

we get

$$\langle\nabla d, \nabla w\rangle = -\frac{1}{2}|t\nabla\ell + \nabla w|^2 - t\langle\nabla d, \nabla\ell\rangle.$$

Hence

$$\frac{\partial w}{\partial \mathrm{n_U}} = \beta(t, z, \nabla w) \qquad \text{on } (0, t_0) \times \partial\mathrm{U}. \tag{7.14}$$

Here $\mathrm{n_U} = (\mathrm{n_{U1}}, \dots, \mathrm{n_{U}}_n)$ is the unit normal to $\partial\mathrm{U}$ pointing toward $\mathbb{R}^n \setminus \overline{\mathrm{U}}$, so that

$$\mathrm{n_U} = \begin{cases} \nabla d & \text{on } \{d > 0\} \cap \partial\mathrm{U}, \\ -\nabla d & \text{on } \{d < 0\} \cap \partial\mathrm{U}, \end{cases}$$

and the function $\beta : (0, t_0) \times \mathrm{U} \times \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\beta(t, z, q) := \begin{cases} -\dfrac{1}{2}|t\nabla\ell(z) + q|^2 - t\dfrac{\partial\ell}{\partial\mathrm{n_U}}(z), \\ \qquad (t, z, q) \in (0, t_0) \times \big(\{d > 0\} \cap \partial\mathrm{U}\big) \times \mathbb{R}^n, \\ \dfrac{1}{2}|t\nabla\ell(z) + q|^2 - t\dfrac{\partial\ell}{\partial\mathrm{n_U}}(z), \\ \qquad (t, z, q) \in (0, t_0) \times \big(\{d < 0\} \cap \partial\mathrm{U}\big) \times \mathbb{R}^n, \end{cases} \tag{7.15}$$

and depends nonlinearly on q.

Finally, inserting (7.10) into the last equation of (7.2), we get

$$w = 0 \qquad \text{on } \{t = 0\} \times \mathrm{U}. \tag{7.16}$$

Collecting together equations (7.11), (7.14) and (7.16) we have:

$$\begin{cases} w_t - \mathcal{A}(z, w, \nabla^2 w) = \mathcal{R}(t, z, w, \nabla^2 w) & \text{in } (0, t_0) \times \mathrm{U}, \\ \dfrac{\partial w}{\partial\mathrm{n_U}} = \beta(t, z, \nabla w) & \text{on } (0, t_0) \times \partial\mathrm{U}, \\ w = 0 & \text{on } \{t = 0\} \times \mathrm{U}. \end{cases} \tag{7.17}$$

In Proposition 7.4 it is shown that problem (7.17) has a unique solution

$$w \in \mathcal{C}^{\frac{2+\alpha}{2},\, 2+\alpha}\big([0, t_0] \times \overline{\mathrm{U}}\big),$$

provided t_0 is chosen sufficiently small. In order to prove this assertion, we will make use of Theorem 7.13, that we will apply with the choice

$$\begin{cases} a_{ij}(t, z) = a_{ij}(z) := F_{M_{ij}}(d(z), \nabla^2 d(z)), \\ \qquad\qquad\qquad\qquad\qquad i, j \in \{1, \dots, n\}, \\ b_i \equiv 0, \qquad i \in \{1, \dots, n\}, \\ c(t, z) = c(z) := F_{\mathrm{u}}(d(z), \nabla^2 d(z)), \end{cases} \tag{7.18}$$

so that

$$a_{ij} \in \mathcal{C}^{\alpha}(\overline{\mathrm{U}}) \tag{7.19}$$

and

$$c \in \mathcal{C}^{\alpha}(\overline{\mathrm{U}}), \tag{7.20}$$

and with the choice

$$\begin{cases} \beta_i(t, z) = \beta_i(z) = \mathrm{n}_{\mathrm{U}_i}(z), \qquad i \in \{1, \dots, n\}, \\ \gamma \equiv 0, \end{cases} \tag{7.21}$$

so that[(3)]

$$\beta_i \in \mathcal{C}^{1+\alpha}(\partial \mathrm{U}). \tag{7.22}$$

Recall that (7.9) yields that the parabolicity condition (7.57) below is satisfied since, once coupled with the smoothness and compactness of ∂E, it implies that there exists a constant $\mu_1 > 0$ such that

$$F(d(x), \nabla^2 d(x))\xi_i \xi_j \geq \mu_1 |\xi|^2, \qquad x \in \partial E,\ \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n.$$

We also need the following expression, obtained by expanding $\mathcal{R}$ in (7.12) up to second order with integral remainder[(4)]:

$$\begin{aligned} \mathcal{R}(t,z,\mathrm{u},M) = {} & A_0(z)t + B_0(z)t \\ & + D_1^{ijkl}(t,z,\mathrm{u},M)(t\nabla^2_{ij}\ell(z) + M_{ij})(t\nabla^2_{kl}\ell(z) + M_{kl}) \\ & + D_2^{ij}(t, z, \mathrm{u}, M)(t\nabla^2_{ij}\ell(z) + M_{ij})(t\ell(z) + \mathrm{u}) \\ & + D_3(t, z, \mathrm{u}, M)(t\ell(z) + \mathrm{u})^2, \end{aligned} \tag{7.23}$$

---

(3) Actually, under our assumptions $a_{ij} \in \mathcal{C}^{\infty}(\overline{U})$, $c \in \mathcal{C}^{\infty}(\overline{U})$, and $\beta_i \in \mathcal{C}^{\infty}(\partial \mathrm{U})$. We refer to [197, page 177] for the definitions of Hölder spaces on the boundary of U.

(4) Remember that for a function ψ of class $\mathcal{C}^2$ of several variables, we have $\psi(p) = \psi(p_0) + \langle \nabla \psi(p_0), p - p_0 \rangle + \int_0^1 (1-\sigma) \langle \nabla^2 \psi(p_0 + \sigma(p - p_0))\, d\sigma\ (p - p_0), p - p_0 \rangle$. We apply this formula with the choice $p = (t, \mathrm{u}, M)$, $\psi(t, \mathrm{u}, M) = \mathcal{R}(t, z, \mathrm{u}, M)$ and $p_0 = (0, 0, 0)$ (recall formulas (7.13)).

where

$$
\begin{aligned}
A_0(z) &:= F_{\mathrm{u}}(d(z), \nabla^2 d(z))\ell(z), \\
B_0(z) &:= F_{M_{ij}}(d(z), \nabla^2 d(z))\nabla^2_{ji}\ell(z), \\
D_1^{ijkl}(t,z,\mathrm{u},M) &:= \int_0^1 (1-\sigma) F_{M_{ij}M_{kl}}(d+\sigma t\ell+\sigma\mathrm{u}, \nabla^2 d+\sigma t\nabla^2\ell+\sigma M)d\sigma, \\
D_2^{ij}(t,z,\mathrm{u},M) &:= 2\int_0^1 (1-\sigma) F_{M_{ij}\mathrm{u}}(d+\sigma t\ell+\sigma\mathrm{u}, \nabla^2 d+\sigma t\nabla^2\ell+\sigma M)d\sigma, \\
D_3(t,z,\mathrm{u},M) &:= \int_0^1 (1-\sigma) F_{\mathrm{uu}}(d+\sigma t\ell+\sigma\mathrm{u}, \nabla^2 d+\sigma t\nabla^2\ell+\sigma M)d\sigma,
\end{aligned}
$$

and d and ℓ in the terms $D_1^{ijk\ell}$, D_2^{ij}, D_3 are evaluated at z. Notice that the smoothness assumption on ∂E allows us to differentiate ℓ.

Proposition 7.4 (Local existence and uniqueness of w). *There exists $t_0 > 0$ such that problem* (7.17) *has a unique solution*

$$w \in \mathcal{C}^{\frac{2+\alpha}{2},\, 2+\alpha}\big([0, t_0] \times \overline{\mathrm{U}}\big).$$

Proof. The proof is based on Theorem 7.13 below, and on a fixed point argument. Define

$$Y := \left\{u \in \mathcal{C}^{\frac{2+\alpha}{2},\, 2+\alpha}\big([0, t_0] \times \overline{\mathrm{U}}\big) : u(0, \cdot) = 0\right\},$$

which turns out to be a Banach space. We define the map

$$\Gamma : Y \to Y$$

as follows: given $u \in Y$, then

$$\Gamma(u) := w.$$

Here w is the solution of the linear parabolic problem (7.59) given by Theorem 7.13, with the choices

$$
\begin{cases}
\mathrm{f}(t, z) := \mathcal{R}(t, z, u(t, z), \nabla^2 u(t, z)), \\
\mathrm{g}(t, z) := \beta(t, z, \nabla u(t, z)), \\
w_0 \equiv 0,
\end{cases}
\tag{7.24}
$$

where $\mathcal{R}$ and β are given in (7.12) and (7.15), respectively, with the choices of a_{ij}, b_i and c as in (7.18), and with the choices of β_i and γ as in (7.21).

More specifically,

$$
\begin{aligned}
& w_t(t,z) - F_{M_{ij}}(d(z),\nabla^2 d(z))\nabla^2_{ij} w(t,z) - F_{\mathrm{u}}(d(z),\nabla^2 d(z)) w(t,z) \\
& = \mathcal{R}(t,z,u(t,z),\nabla^2 u(t,z)) \\
& = F_{\mathrm{u}}(d(z),\nabla^2 d(z))\ell(z)\, t + F_{M_{ij}}(d(z),\nabla^2 d(z))\nabla^2_{ij}\ell(z)\, t \\
& \quad + \int_0^1 (1-\sigma) F_{M_{ij}M_{kl}}\Big(d(z)+\sigma t\ell(z)+\sigma u(t,z),\nabla^2 d(z) \\
& \qquad\qquad\qquad\qquad + \sigma t\nabla^2\ell(z) + \sigma\nabla^2 u(t,z)\Big)\, d\sigma \\
& \qquad \times \Big(t\nabla^2_{ij}\ell(z) + \nabla^2_{ij}u(t,z)\Big)\Big(t\nabla^2_{kl}\ell(z) + \nabla^2_{kl}u(t,z)\Big) \\
& \quad + 2\int_0^1 (1-\sigma) F_{M_{ij}\mathrm{u}}\Big(d(z)+\sigma t\ell(z)+\sigma u(t,z),\nabla^2 d(z) \\
& \qquad\qquad\qquad\qquad + \sigma t\nabla^2\ell(z) + \sigma\nabla^2 u(t,z)\Big)\, d\sigma \\
& \qquad \times \Big(t\nabla^2_{ij}\ell(z) + \nabla^2_{ij}u(t,z)\Big)\Big(t\ell(z) + u(t,z)\Big) \\
& \quad + \int_0^1 (1-\sigma) F_{\mathrm{uu}}\Big(d(z)+\sigma t\ell(z)+\sigma u(t,z),\nabla^2 d(z) \\
& \qquad\qquad\qquad\qquad + \sigma t\nabla^2\ell(z) + \sigma\nabla^2 u(t,z)\Big)\, d\sigma \\
& \qquad \times \Big(t\ell(z) + u(t,z)\Big)^2.
\end{aligned}
$$

Note that, as $u \in \mathcal{C}^{\frac{2+\alpha}{2},\ 2+\alpha}\big([0,t_0]\times\overline{\mathrm{U}}\big)$, it follows[5] that

$$
\mathrm{f} \in \mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0]\times\overline{\mathrm{U}}), \qquad \mathrm{g} \in \mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}([0,t_0]\times\partial\mathrm{U}).
$$

Therefore, remembering also (7.19), (7.20) and (7.22), the assumptions of Theorem 7.13 are satisfied.

Given $R > 0$, set

$$
Y_{t_0,R} := \left\{ u \in Y : \|u\|_{\mathcal{C}^{\frac{2+\alpha}{2},\ 2+\alpha}\left([0,t_0]\times\overline{\mathrm{U}}\right)} \le R \right\},
$$

[5] We define [138]

$$
\|\mathrm{g}\|_{\mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\partial\mathrm{U})} := \inf\left\{ \|v\|_{\mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\overline{\mathrm{U}})} : v = \mathrm{g} \text{ on } [0,T]\times\partial\mathrm{U} \right\}.
$$

which is a closed subset of Y. We will prove the following two properties:

(i) there exist $t_0 > 0$ and $R > 0$ such that $\Gamma : Y_{t_0,R} \to Y_{t_0,R}$;
(ii) there exist $t_0 > 0$ and $R > 0$ such that

$$\|\Gamma(u) - \Gamma(v)\|_{\mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}\left([0,t_0]\times\overline{\mathrm{U}}\right)} \le \frac{1}{2}\|u - v\|_{\mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}\left([0,t_0]\times\overline{\mathrm{U}}\right)},$$

for any $u, v \in Y_{t_0,R}$.

Let us prove (i). Let $u \in Y_{t_0,R}$, so that

$$\|u\|_{\mathcal{C}^{\frac{2+\alpha}{2},\ 2+\alpha}\left([0,t_0]\times\overline{\mathrm{U}}\right)} \le R. \tag{7.25}$$

Observe that there exists a constant $C_1 > 0$ such that, for $t_0 > 0$ sufficiently small,

$$\|t\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0]\times\overline{\mathrm{U}})} = \|t\|_{\mathcal{C}^{\frac{\alpha}{2}}([0,t_0])} \le C_1 t_0^{1-\frac{\alpha}{2}}. \tag{7.26}$$

Moreover, for $u_1, u_2 \in \mathcal{C}^{\frac{\alpha}{2},\alpha}([0,T]\times\overline{\mathrm{U}})$[(6)],

$$\|u_1u_2\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0]\times\overline{\mathrm{U}})} \le \|u_1\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0]\times\overline{\mathrm{U}})}\|u_2\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0]\times\overline{\mathrm{U}})}. \tag{7.27}$$

Then, from (7.26), (7.23) and (7.25) it follows[(7)] that there exists a constant $C_2 > 0$ independent of t_0 and R, such that

$$\|\mathcal{R}(t,z,u(t,z),\nabla^2 u(t,z))\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0]\times\overline{\mathrm{U}})} \le C_2\left(R^2 + t_0^{1-\frac{\alpha}{2}}\right). \tag{7.28}$$

(6) We have $|u_1(t,z)u_2(t,z) - u_1(s,y)u_2(s,y)| \le |u_1(t,z) - u_1(s,y)||u_2(t,z)| + |u_2(t,z) - u_2(s,y)||u_1(s,y)|$; hence, omitting for simplicity $[0,T]\times\overline{U}$ in the notation,

$$[u_1u_2]_{\mathcal{C}^{0,\alpha}} \le \|u_1\|_\infty[u_2]_{\mathcal{C}^{0,\alpha}} + [u_1]_{\mathcal{C}^{0,\alpha}}\|u_2\|_\infty,$$
$$[u_1u_2]_{\mathcal{C}^{\frac{\alpha}{2},0}} \le \|u_1\|_\infty[u_2]_{\mathcal{C}^{\frac{\alpha}{2},0}} + [u_1]_{\mathcal{C}^{\frac{\alpha}{2},0}}\|u_2\|_\infty,$$

and

$$\begin{aligned}\|u_1u_2\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}} =& 2\|u_1u_2\|_\infty + [u_1u_2]_{\mathcal{C}^{0,\alpha}} + [u_1u_2]_{\mathcal{C}^{\frac{\alpha}{2},0}}\\ \le& 2\|u_1\|_\infty\|u_2\|_\infty + \|u_1\|_\infty([u_2]_{\mathcal{C}^{0,\alpha}} + [u_2]_{\mathcal{C}^{\frac{\alpha}{2},0}}) + \|u_2\|_\infty([u_1]_{\mathcal{C}^{0,\alpha}} + [u_1]_{\mathcal{C}^{\frac{\alpha}{2},0}})\\ \le& \|u_1\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}}\|u_2\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}}.\end{aligned}$$

(7) The term $C_2 t_0^{1-\frac{\alpha}{2}}$ is originated by estimating the $\mathcal{C}^{\frac{\alpha}{2},\alpha}$-norm of $A_0 t$ and $B_0 t$ and using (7.26). The $\mathcal{C}^{\frac{\alpha}{2},\alpha}$-norm of the terms $D_1^{ijkl}(t,z,u(t,z),\nabla^2 u(t,z))$, $D_2^{ij}\left(t,z,u(t,z),\nabla^2 u(t,z)\right)$, $D_3\left(t,z,u(t,z),\nabla^2 u(t,z)\right)$ is bounded by a positive constant, provided t_0 and R are small enough. The $\mathcal{C}^{\frac{\alpha}{2},\alpha}$-norm of terms of the form $t^2\nabla^2_{ij}\ell\nabla^2_{kl}\ell$, $t^2\ell\nabla^2_{ij}\ell$, $t^2\ell^2$ is controlled by the linear term in t, and therefore bounded by $C_2 t_0^{1-\frac{\alpha}{2}}$. The $\mathcal{C}^{\frac{\alpha}{2},\alpha}$-norm of terms of the form $\nabla^2_{ij}u\nabla^2_{kl}u$, u^2, $u\nabla^2_{ij}u$ is controlled by C_2R^2. Using the inequality $ab \le \frac{1}{2}(a^2+b^2)$, the $\mathcal{C}^{\frac{\alpha}{2},\alpha}$-norm of terms of the form $t\nabla^2_{ij}u$, tu is controlled by $C_2(R^2 + t_0^{1-\frac{\alpha}{2}})$.

Similarly, there exists a constant $C_3 > 0$ such that, for $t_0 > 0$ sufficiently small,

$$\|t\|_{\mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}([0,t_0]\times\partial \mathrm{U})} \leq C_3 t_0^{(1-\alpha)/2}. \tag{7.29}$$

Hence, using (7.15), (7.29) and (7.25), it is possible to prove that there exists a constant $C_4 > 0$ independent of R and t_0, such that

$$\|\beta(t, z, u(t, z))\|_{\mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}([0,t_0]\times\partial \mathrm{U})} \leq C_4 \left(R^2 + t_0^{(1-\alpha)/2}\right). \tag{7.30}$$

From (7.28), (7.30), (7.24), the inequality $1 - \frac{\alpha}{2} > \frac{1-\alpha}{2}$, the definition of w and Theorem 7.13, we have

$$\|w\|_{\mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}([0,t_0]\times\overline{\mathrm{U}})} \leq C_5 \left(R^2 + t_0^{(1-\alpha)/2}\right), \tag{7.31}$$

where

$$C_5 := \mathrm{C}(C_2 + C_4),$$

and C is defined in (7.60), and can be taken independent of t_0 (and of R). Choosing

$$0 < R \leq \frac{1}{2C_5},$$

we have $C_5 R^2 \leq R/2$, so that from (7.31) we have

$$\|w\|_{\mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}([0,t_0]\times\overline{\mathrm{U}})} \leq \frac{R}{2} + C_5 t_0^{(1-\alpha)/2}.$$

Choosing then $t_0 = t_0(R)$, so that

$$0 < t_0 \leq \left(\frac{R}{2C_3}\right)^{1/((1-\alpha)/2)},$$

we get $\frac{R}{2} + C_5 t_0^{(1-\alpha)/2} \leq R$, so that

$$\|w\|_{\mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}([0,t_0]\times\overline{\mathrm{U}})} \leq R,$$

and assertion (i) follows.

To prove the strict contraction property (ii), take $u, v \in Y_{t_0,R}$, so that $\|u\|_{\mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}} < R$ and $\|v\|_{\mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}} < R$, and set

$$\begin{aligned}
B^u(t, z) &:= \mathcal{R}(t, z, u(t, z), \nabla^2 u(t, z)),\\
\mathrm{g}^u(t, z) &:= \beta(t, z, \nabla u(t, z)),\\
B^v(t, z) &:= \mathcal{R}(t, z, v(t, z), \nabla^2 v(t, z)),\\
\mathrm{g}^v(t, z) &:= \beta(t, z, \nabla v(t, z)).
\end{aligned}$$

From (7.60) and the linearity of the equations in (7.59), we have

$$\begin{aligned}&\|\Gamma(u)-\Gamma(v)\|_{\mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}([0,t_0]\times\overline{\mathrm{U}})}\\&\quad\le \mathrm{C}\Big(\|B^u-B^v\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0]\times\overline{\mathrm{U}})}+\|\mathrm{g}^u-\mathrm{g}^v\|_{\mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}([0,t_0]\times\partial\mathrm{U})}\Big).\end{aligned}\tag{7.32}$$

Observe that if $\Phi:\mathbb{R}\to\mathbb{R}$ is a function of class $\mathcal{C}^\infty$ having bounded first and second derivatives, then there exists a constant $C_6>0$ such that[(8)]

$$\begin{aligned}&\|\Phi(u)-\Phi(v)\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0]\times\overline{\mathrm{U}})}\\&\le C_6\Big((\|u\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0]\times\overline{\mathrm{U}})}+\|v\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0]\times\overline{\mathrm{U}})}+1)\|u-v\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0]\times\overline{\mathrm{U}})}\Big).\end{aligned}\tag{7.33}$$

From (7.23), (7.27), (7.28) and (7.33) it follows that there exists a constant C_7, independent of t_0 and R, such that

$$\|B^u-B^v\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0]\times\overline{\mathrm{U}})}\le C_7\left(R+t_0^{1-\frac{\alpha}{2}}\right)\|u-v\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0]\times\overline{\mathrm{U}})}.\tag{7.34}$$

Similarly, there exists a constant $C_8>0$ independent of R and t_0, such that

$$\|\mathrm{g}^u-\mathrm{g}^v\|_{\mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}([0,t_0]\times\partial U)}\le C_8\left(R+t_0^{(1-\alpha)/2}\right)\|u-v\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0]\times\overline{\mathrm{U}})}.\tag{7.35}$$

From (7.35), (7.34) and (7.32) it follows that

$$\begin{aligned}&\|\Gamma(u)-\Gamma(v)\|_{\mathcal{C}^{\frac{2+\alpha}{2},\,2+\alpha}\left([0,t_0]\times\overline{\mathrm{U}}\right)}\\&\qquad\le \mathrm{C}(C_7+C_8)\left(R+t^{(1-\alpha)/2}\right)\|u-v\|_{\mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}\left([0,t_0]\times\overline{\mathrm{U}}\right)}.\end{aligned}$$

Taking then $t_0>0$ and $R>0$ sufficiently small, assertion (ii) follows.

From properties (i) and (ii) and the fixed point theorem, we obtain that Γ has a unique fixed point in $Y_{t_0,R}$, provided $t_0>0$ and $R>0$ are sufficiently small. This concludes the proof of Proposition 7.4. □

7.2. Uniqueness

We have shown existence in $\mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}([0,t_0]\times\overline{\mathrm{U}})$ of a solution to (7.2). Now, let us show uniqueness in $\mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}([0,t_0]\times\overline{\mathrm{U}})$, provided ρ_0 and t_0

(8) Write $\Phi(u)-\Phi(v)=\Phi'(u)(v-u)+\int_0^1(1-\sigma)\Phi''(u+\sigma(v-u))\,d\sigma\,(v-u)^2$, and use (7.27).

are small enough[9]. Let $u_1, u_2 \in \mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}([0,t_0]\times\overline{\mathrm{U}})$ be two solutions of (7.2), and define

$$\overline{t} := \sup\{t \in [0,t_0] : u_1(t,z) = u_2(t,z),\ z \in \mathrm{U}\}.$$

We have to show that $\overline{t} = t_0$. Assume by contradiction that

$$\overline{t} \in [0,t_0). \tag{7.36}$$

By continuity, we have that $\nabla u_1(t,z) = \nabla u_2(t,z)$ for any $t \in [0,\overline{t}]$ and $z \in \overline{\mathrm{U}}$, in particular

$$\nabla u_1(\overline{t},\cdot) = \nabla u_2(\overline{t},\cdot).$$

Notice that, provided $\rho_0 > 0$ and $t_0 > 0$ are small enough, we have that there exists a constant $\nu_0 > 0$ so that the following nontangentiality condition holds:

$$\langle \nabla u_1, \mathrm{n_U}\rangle \geq \nu_0, \qquad \langle \nabla u_2, \mathrm{n_U}\rangle \geq \nu_0, \qquad \text{on } [0,t_0]\times\partial\mathrm{U}. \tag{7.37}$$

Set now

$$\omega := u_1 - u_2.$$

Then ω satisfies

$$\begin{cases} \omega \in \mathcal{C}^{\frac{2+\alpha}{2},\,2+\alpha}\big([0,t_0]\times\overline{\mathrm{U}}\big), & \\ \omega_t = \mathrm{a}_{ij}\nabla_{ij}\omega + \mathrm{c}\omega & \text{in } (\overline{t},t_0)\times\mathrm{U}, \\ \langle \mathrm{b}(t,z), \nabla\omega(t,z)\rangle = 0 & \text{on } [\overline{t},t_0]\times\partial\mathrm{U}, \\ \omega = 0 & \text{on } \{t=\overline{t}\}\times\mathrm{U}. \end{cases} \tag{7.38}$$

Here[10] for $i, j \in \{1,\dots,n\}$,

$$\begin{aligned} \mathrm{a}_{ij}(t,z) := \int_0^1 F_{M_{ij}}\Big(&\sigma u_1(t,z) + (1-\sigma)u_2(t,z), \sigma\nabla^2 u_1(t,z) \\ &+ (1-\sigma)\nabla^2 u_2(t,z)\Big)\, d\sigma, \\ \mathrm{c}(t,z) := \int_0^1 F_{\mathrm{u}}\Big(&\sigma u_1(t,z) + (1-\sigma)u_2(t,z), \sigma\nabla^2 u_1(t,z) \\ &+ (1-\sigma)\nabla^2 u_2(t,z)\Big)\, d\sigma, \end{aligned}$$

(9) Let $u_1, u_2 \in Y_{t_0,R}$ be two solutions of (7.2). Then, recalling (7.10), the two functions

$$w_1(t,z) := u_1(t,z) - d(z) - t\ell(z), \qquad w_2(t,z) := u_2(t,z) - d(z) - t\ell(z)$$

are both solutions to problem (7.17). Since (7.17) has a unique solution in $Y_{t_0,R}$, it follows that $w_1 = w_2$, and hence $u_1 = u_2$. This argument gives uniqueness in $Y_{t_0,R}$, and not uniqueness in $\mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}([0,t_0]\times\overline{\mathrm{U}})$.

(10) Write $F(u_1,\nabla^2u_1) - F(u_2,\nabla^2u_2) = \int_0^1 \frac{d}{d\sigma}F(\sigma u_1 + (1-\sigma)u_2, \sigma\nabla^2u_1 + (1-\sigma)\nabla^2u_2)\,d\sigma$.

and, setting

$$g(p) := |p|^2 - 1, \qquad p \in \mathbb{R}^n,$$

and writing

$$\begin{aligned} 0 &= g(\nabla u_1(t,z)) - g(\nabla u_2(t,z)) \\ &= \int_0^1 \frac{d}{d\sigma}\, g\Big(\sigma \nabla u_1(t,z) + (1-\sigma)\nabla u_2(t,z)\Big)\, d\sigma \\ &= \int_0^1 \langle \nabla g\Big(\sigma \nabla u_1(t,z) + (1-\sigma)\nabla u_2(t,z)\Big), \nabla\omega(t,z)\rangle\, d\sigma, \end{aligned}$$

we let, for $i \in \{1,\ldots,n\}$,

$$\begin{aligned} \mathrm{b}_i(t,z) &:= \int_0^1 \nabla_i g\Big(\sigma \nabla u_1(t,z) + (1-\sigma)\nabla u_2(t,z)\Big)\, d\sigma \\ &= \nabla_i u_1(t,z) + \nabla_i u_2(t,z). \end{aligned}$$

Notice that $\mathrm{a}_{ij} \in \mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0] \times \overline{\mathrm{U}})$, $\mathrm{c} \in \mathcal{C}^{\frac{\alpha}{2},\alpha}([0,t_0] \times \overline{\mathrm{U}})$, and from the parabolicity of F (see (7.9)) we have that a_{ij} satisfy (7.57) below. Moreover, from (7.37) it follows that

$$\langle \mathrm{b}(t,z), \nabla\omega(t,z)\rangle \geq 2\nu_0, \qquad (t,z) \in [0,t_0] \times \partial\mathrm{U},$$

hence b satisfies the nontangentiality condition (7.58). Then ω solves a uniformly parabolic linear problem, hence (see [190]) $\omega \equiv 0$ in $[\bar{t}, t_0]$, which contradicts (7.36). We deduce that $\bar{t} = t_0$, and this concludes the proof of Theorem 7.1. □

Looking at the linear evolution equation[11] satisfied by the incremental ratios

$$\frac{u(t, z+he_k) - u(t,z)}{h}, \qquad h \neq 0,$$

and passing to the limit as $h \to 0$, it is possible to show the following result (see [197, Proposition 8.5.6] for the proof).

Proposition 7.5 (Gradient regularity). *Let u be the solution given by Theorem 7.1 and let* U' *be an open set relatively compact in* U. *Then*

$$\nabla_i u \in \mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}([0,t_0] \times \overline{\mathrm{U}'}), \qquad i \in \{1,\ldots,n\}. \tag{7.39}$$

[11] Which turns out to be uniformly parabolic.

The next proposition (see [138]) shows that the condition $|\nabla u|^2 = 1$ is valid also in the interior $(0, t_0) \times \mathrm{U}$ of the time-space domain. In the proof we will make use of Proposition 7.5, since we will need to consider third order spatial derivatives of u.

Theorem 7.6 (Eikonal equation). *Let u be the solution given by Theorem* 7.1. *Then*

$$|\nabla u|^2 = 1 \qquad \text{in } [0, t_0] \times \overline{\mathrm{U}}. \tag{7.40}$$

Proof. We set

$$v := |\nabla u|^2 - 1.$$

From (7.2) we have

$$v = 0 \qquad \text{on } [0, t_0] \times \partial \mathrm{U} \tag{7.41}$$

and, by the properties of the signed distance function, also

$$v = 0 \qquad \text{on } \{t = 0\} \times \mathrm{U}. \tag{7.42}$$

By Proposition 7.5 we have

$$v \in \mathcal{C}^{\frac{2+\alpha}{2},\, 2+\alpha}\Big([0, t_0] \times \overline{\mathrm{U}'}\Big)$$

for any open set U' relatively compact in U. In addition

$$\nabla^2_{ij} v = 2\nabla_k u \nabla^3_{ijk} u + 2\nabla^2_{ik} u \nabla^2_{kj} u, \qquad i, j \in \{1, \dots, n\}. \tag{7.43}$$

Differentiating the first equation in (7.2) with respect to z_k and using (7.43) gives

$$\begin{aligned} v_t &= 2\nabla_k u \nabla_k u_t = 2\nabla_k u \big[F_{M_{ij}}(u, \nabla^2 u)\nabla^3_{ijk} u + F_{\mathrm{u}}(u, \nabla^2 u)\nabla_k u\big] \\ &= F_{M_{ij}}(u, \nabla^2 u)\nabla^2_{ij} v - 2F_{M_{ij}}(u, \nabla^2 u)\nabla^2_{ik} u \nabla^2_{kj} u + 2F_{\mathrm{u}}(u, \nabla^2 u)|\nabla u|^2. \end{aligned}$$

Observe now that

$$F_{\mathrm{u}}(u, \nabla^2 u) = F_{M_{ij}}(u, \nabla^2 u)\nabla^2_{ik} u \nabla^2_{kj} u. \tag{7.44}$$

Indeed, applying (7.7) with $N = \nabla^2 u \nabla^2 u$ we have

$$\begin{aligned} F_{M_{ij}}(u, \nabla^2 u)\nabla_{ik} u \nabla_{kj} u &= \operatorname{tr}\left(\nabla^2 u \nabla^2 u (\mathrm{Id} - u\nabla^2 u)^{-2}\right) \\ &= \sum_{i=1}^{n} \frac{(\lambda_i(\nabla^2 u))^2}{(1 - u\lambda_i(\nabla^2 u))^2}. \end{aligned} \tag{7.45}$$

On the other hand, from (7.6) it follows that $F_{\mathrm{u}}(u, \nabla^2 u)$ coincides with the right-hand side of (7.45).

From (7.44) we then have

$$v_t = F_{M_{ij}}(u, \nabla^2 u)\nabla^2_{ij} v + 2F_{\mathrm{u}}(u, \nabla^2 u)(|\nabla u|^2 - 1),$$

so that

$$v_t = F_{M_{ij}}(u, \nabla^2 u)\nabla^2_{ij} v + 2F_{\mathrm{u}}(u, \nabla^2 u)v. \tag{7.46}$$

Equation (7.46) is a linear partial differential equation in the unknown v, and it is uniformly parabolic thanks to Corollary 7.3. Hence, from (7.41) and (7.42), it follows[12] that

$$v \equiv 0 \qquad \text{in } [0, t_0] \times \mathrm{U}'.$$

Since U' is an arbitrary open set relatively compact in U, (7.40) follows. □

Remark 7.7 (Further regularity). The solution u given by Theorem 7.1 not only satisfies (7.39), but also

$$u \in \mathcal{C}^\infty((0, t_0) \times \overline{\mathrm{U}'}) \tag{7.47}$$

for any open set U' relatively compact in U.

Indeed, from Theorem 7.1 and Proposition 7.5 we have, for $i, j, k \in \{1, \dots, n\}$,

$$\begin{aligned} &u \in \mathcal{C}^{\frac{2+\alpha}{2}, 2+\alpha}([0, t_0] \times \mathrm{U}), \\ &\nabla_k u \in \mathcal{C}^{\frac{2+\alpha}{2}, 2+\alpha}([0, t_0] \times \overline{\mathrm{U}'}), \\ &\nabla^2_{ij} u \in \mathcal{C}^{\frac{1+\alpha}{2}, 1+\alpha}([0, t_0] \times \overline{\mathrm{U}'}). \end{aligned}$$

Differentiating the first equation in (7.2) with respect to z_k gives

$$\frac{\partial}{\partial t}\nabla_k u = F_{\mathrm{u}}(u, \nabla^2 u)\nabla_k u + F_{M_{ij}}(u, \nabla^2 u)\nabla^2_{ij}\nabla_k u. \tag{7.48}$$

Since

$$F_{\mathrm{u}}(u, \nabla^2 u) \in \mathcal{C}^{\frac{1+\alpha}{2}, 1+\alpha}((0, t_0) \times \overline{\mathrm{U}'})$$

and

$$F_{M_{ij}}(u, \nabla^2 u) \in \mathcal{C}^{\frac{1+\alpha}{2}, 1+\alpha}((0, t_0) \times \overline{\mathrm{U}'}),$$

(12) See, *e.g.*, [230, Theorem 8, page 176].

from (7.48) it follows[(13)] that

$$\nabla_k u \in \mathcal{C}^{\frac{3+\alpha}{2},3+\alpha}((0,t_0)\times\overline{\mathrm{U}'}),$$

and therefore

$$\nabla^2_{ij} u \in \mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}((0,t_0)\times\overline{\mathrm{U}'}). \tag{7.49}$$

Then, equation (7.2) implies

$$u_t \in \mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}((0,t_0)\times\overline{\mathrm{U}'}). \tag{7.50}$$

Using (7.49) in equation (7.48) gives

$$\frac{\partial}{\partial t}\nabla_k u \in \mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}((0,t_0)\times\overline{\mathrm{U}'}), \tag{7.51}$$

hence

$$\frac{\partial}{\partial t}\nabla^2_{ij} u \in \mathcal{C}^{\frac{\alpha}{2},\alpha}((0,t_0)\times\overline{\mathrm{U}'}). \tag{7.52}$$

From (7.50), (7.51) and (7.52) we deduce

$$u \in \mathcal{C}^{\frac{4+\alpha}{2},4+\alpha}((0,t_0)\times\overline{\mathrm{U}'}),$$

and the smoothness (7.47) of u in $(0,t_0)\times\overline{\mathrm{U}'}$ follows by iteration.

Actually, one can prove[(14)] that $u \in \mathcal{C}^\infty([0,t_0]\times \mathrm{U})$.

Remark 7.8 (The solution u is a signed distance function). Using (7.40) it turns out[(15)] that

$$u(t,z) = \mathrm{dist}(z, E(t)) - \mathrm{dist}(z, \mathbb{R}^n \setminus E(t)), \qquad (z,t)\in[0,t_0]\times \mathrm{U},$$

where

$$E(t) = \{u(t,\cdot)\le 0\}, \qquad t\in[0,t_0].$$

Then, recalling Theorem 6.2, it follows that $t\in[0,t_0]\to\{u(t,\cdot)\le 0\}$ is the smooth compact mean curvature flow on $[0,t_0]$ starting from $\overline{E}$.

(13) See [197] for the definitions of the space $\mathcal{C}^{\frac{k+\alpha}{2},k+\alpha}([0,t_0]\times\overline{\mathrm{U}}')$ for $k\in\mathbb{N}, k\ge 3$.

(14) A proof relying on the parametric description of the flow is the following. Take a connected component C of ∂E, and choose $\varphi_0 \in \mathrm{Emb}(\mathcal{S};\mathbb{R}^n)$ such that $\varphi(\mathcal{S}) = C$. Consider the unique smooth parametric mean curvature evolution $\varphi \in \mathcal{X}([0,t_0];\mathrm{Emb}(\mathcal{S};\mathbb{R}^n))$ starting from φ_0 (see, *e.g.*, [203]). Then, the signed distance function from $\varphi(t,\mathcal{S})$ negative in the bounded connected component of $\mathbb{R}^n\setminus C$ is a solution of (7.2) of class $\mathcal{C}^\infty([0,t_0]\times\mathrm{U})$.

(15) In general, the following result holds (see, *e.g.*, [15, 206]). Let $\partial E\in\mathcal{C}^\infty$ be compact, and let $u:\mathbb{R}^n\to\mathbb{R}$ be a Lipschitz function such that $\{u<0\}=\mathrm{int}(E)$, and $\{u=0\}=\partial E$. Suppose that u is of class $\mathcal{C}^\infty$ in $U\setminus\overline{E}$, where U is a tubular neighbourhood of ∂E, and that $|\nabla u|^2 = 1$ in $U\setminus\overline{E}$. Then $u(z) = \mathrm{dist}(z,\{u=0\})$ for any $z\in U\setminus\overline{E}$.

We can summarize the above results as follows.

Theorem 7.9 (Smooth compact mean curvature flow). *Let $\partial E \in \mathcal{C}^\infty$ be compact. Then there exist $t_0 > 0$ and a unique smooth compact mean curvature flow $f : [0, t_0] \to \mathcal{P}(\mathbb{R}^n)$ starting from ∂E at time 0.*

We can now introduce the following definition[(16)].

Definition 7.10 (Maximal existence time). Let $\partial E \in \mathcal{C}^\infty$ be compact. We define the maximal time of existence for the compact mean curvature flow starting from ∂E (or equivalently from $\overline{E}$) as

$$\begin{aligned} t_{\max}(\partial E) := \sup \Big\{ & t \in (0, +\infty) : \exists f : [0, t] \to \mathcal{P}(\mathbb{R}^n), \\ & f \text{ smooth compact mean curvature flow, } f(0) = \overline{E} \Big\}. \end{aligned}$$

Note that the smooth compact mean curvature flow $f : [0, t_{\max}(\partial E)) \to \mathcal{P}(\mathbb{R}^n)$ starting from $\overline{E}$ is uniquely defined. We will see in Theorem 13.1 of Chapter 13 that

$$t_{\max}(\partial E) \geq \frac{1}{2 \max\limits_{x \in \partial E} |\nabla^2 d(x, E)|^2}. \tag{7.53}$$

We conclude this section by pointing out some results on the continuous dependence of a smooth compact mean curvature flow with respect to the initial set E. To this purpose, we recall the notation introduced in (1.3): for any $B \subset \mathbb{R}^n$ and any $\rho > 0$, the two sets $B^\pm_\rho$ are defined as

$$\begin{aligned} B^+_\rho &:= \{z \in \mathbb{R}^n : \operatorname{dist}(z, B) < \rho\}, \\ B^-_\rho &:= \{z \in \mathbb{R}^n : \operatorname{dist}(z, \mathbb{R}^n \setminus B) > \rho\}. \end{aligned}$$

In the following result we make use of estimate (7.53).

Theorem 7.11 (Uniform existence time). *Let $\partial E \in \mathcal{C}^\infty$ be compact. Let*

$$0 < t_0 < \frac{1}{2 \max\limits_{x \in \partial E} |\nabla^2 d(x, E)|^2},$$

and let $f : [0, t_0] \to \mathcal{P}(\mathbb{R}^n)$ be the smooth compact mean curvature flow starting from ∂E given by Theorem 7.9. Then there exists $\overline{\rho} > 0$ such that, for any $\rho \in [0, \overline{\rho}]$, the sets E^-_ρ and E^+_ρ have a unique smooth compact mean curvature flow on $[0, t_0]$.

(16) We point out that, using the embeddedness preserving property (Theorem 5.10), it follows that $t_{\max}(\partial E)$ coincides with the supremum of all $t > 0$ such that there exists a smooth parametric compact mean curvature flow on $[0, t]$ starting from ∂E.

Proof. Since ∂E is smooth and compact, there exists $\widetilde{\rho} > 0$ such that ∂E_ρ^- and ∂E_ρ^+ are smooth and compact for any $\rho \in [0, \widetilde{\rho}]$. Let $\rho \in [0, \widetilde{\rho}]$, and let us consider the case of E_ρ^+, the discussion for E_ρ^- being similar. Set

$$d(\cdot) := d(\cdot, E), \qquad d_\rho(\cdot) := d(\cdot, E_\rho^+).$$

We first observe that

$$d_\rho = d - \rho \qquad \text{in } \mathbb{R}^n.$$

Hence

$$\nabla^2 d_\rho(y) = \nabla^2 d(y), \qquad y \in \partial E_\rho^+.$$

From (1.34) we then have, for any $i \in \{1, \ldots, n\}$,

$$\begin{aligned}\lambda_i(\nabla^2 d_\rho(y)) &= \lambda_i(\nabla^2 d(y)) \\ &= \frac{\lambda_i(\nabla^2 d(x))}{1 + \rho\lambda_i(\nabla^2 d(x))}, \quad y \in \partial E_\rho^+, \ x := \mathrm{pr}(y, \partial E).\end{aligned} \tag{7.54}$$

Define

$$\mathrm{a}^2 := \max_{x \in \partial E} |\nabla^2 d(x)|^2, \qquad \mathrm{a}_\rho^2 := \max_{y \in \partial E_\rho} |\nabla^2 d_\rho(y)|^2,$$

set

$$\epsilon := \frac{1}{2\mathrm{a}^2} - t_0 > 0,$$

and take

$$0 < \sigma < \frac{2\epsilon \mathrm{a}^4}{1 - 2\epsilon \mathrm{a}^2}. \tag{7.55}$$

Choose $\overline{\rho} \in (0, \widetilde{\rho}]$ so that

$$\sum_{i=1}^n \frac{(\lambda_i(\nabla^2 d(x)))^2}{(1 - \overline{\rho}|\lambda_i(\nabla^2 d(x))|)^2} \leq \mathrm{a}^2 + \sigma, \qquad x \in \partial E.$$

From (7.54) it follows that

$$\mathrm{a}_\rho^2 \leq \mathrm{a}^2 + \sigma, \qquad \rho \in [0, \overline{\rho}]. \tag{7.56}$$

Then, using (7.53) with E_ρ^+ in place of E and (7.56), we have

$$t_{\max}(\partial E_\rho^+) \geq \frac{1}{2\mathrm{a}_\rho^2} \geq \frac{1}{2\mathrm{a}^2 + 2\sigma} \geq \frac{1}{2\mathrm{a}^2} - \epsilon = t_0,$$

where the last inequality follows from the choice of σ in (7.55). $\square$

Remark 7.12 (Continuity with respect to the initial set). Using the continuity results of the flow with respect to the initial data proven in [203], it is possible to show that if E, $t_0 > 0$, f and $\overline{\rho}$ are as in Theorem 7.11, and if $f_{E^\pm_\rho} : [0, t_0] \to \mathcal{P}(\mathbb{R}^n)$ are the smooth compact mean curvature flows starting from $\overline{E^\pm_\rho}$ for any $\rho \in [0, \overline{\rho}]$, then

$$\bigcup_{\rho\in(0,\overline{\rho}]} f_{E^-_\rho}(t) = \mathrm{int}(f(t)), \qquad \bigcap_{\rho\in(0,\overline{\rho}]} f_{E^+_\rho}(t) = f(t), \qquad t \in [0, t_0].$$

7.3. Appendix

Let

$$\alpha \in (0, 1).$$

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class $\mathcal{C}^{2+\alpha}$ and $T > 0$. We recall the definitions of the various parabolic Hölder spaces (see for instance [197]). We let

$$\mathcal{C}^\alpha([0,T]) := \left\{ u \in \mathcal{C}([0,T]) : [u]_{\mathcal{C}^\alpha([0,T])} := \sup_{t,s\in[0,T],t\neq s} \frac{|u(t)-u(s)|}{|t-s|^\alpha} < +\infty \right\}$$

be endowed with the norm

$$\|u\|_{\mathcal{C}^\alpha([0,T])} := \|u\|_{L^\infty([0,T])} + [u]_{\mathcal{C}^\alpha([0,T])},$$

and

$$\mathcal{C}^\alpha(\overline{\Omega}) := \left\{ u \in \mathcal{C}(\overline{\Omega}) : [u]_{\mathcal{C}^\alpha(\overline{\Omega})} := \sup_{x,s\in\overline{\Omega},x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < +\infty \right\}$$

be endowed with the norm

$$\|u\|_{\mathcal{C}^\alpha(\overline{\Omega})} := \|u\|_{L^\infty(\overline{\Omega})} + [u]_{\mathcal{C}^\alpha(\overline{\Omega})}.$$

We let

$$\mathcal{C}^{\alpha,0}([0,T]\times\overline{\Omega}) := \Big\{ u \in \mathcal{C}([0,T]\times\overline{\Omega}) : u(\cdot, z) \in \mathcal{C}^\alpha([0,T])\ \forall z \in \overline{\Omega}, \\ \|u\|_{\mathcal{C}^{\alpha,0}([0,T]\times\overline{\Omega})} := \sup_{z\in\overline{\Omega}} \|u(\cdot, z)\|_{\mathcal{C}^\alpha([0,T])} < +\infty \Big\}$$

be endowed with the norm $\|u\|_{\mathcal{C}^{\alpha,0}([0,T]\times\overline{\Omega})}$, and

$$\mathcal{C}^{0,\alpha}([0,T]\times\overline{\Omega}) := \Big\{ u \in \mathcal{C}([0,T]\times\overline{\Omega}) : u(t, \cdot) \in \mathcal{C}^\alpha(\overline{\Omega})\ \forall t \in [0,T], \\ \|u\|_{\mathcal{C}^{0,\alpha}([0,T]\times\overline{\Omega})} := \sup_{t\in[0,T]} \|u(t, \cdot)\|_{\mathcal{C}^\alpha(\overline{\Omega})} < +\infty \Big\}$$

be endowed with the norm $\|u\|_{\mathcal{C}^{0,\alpha}([0,T]\times\overline{\Omega})}$.

We let

$$\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,T]\times\overline{\Omega}) := \mathcal{C}^{\frac{\alpha}{2},0}([0,T]\times\overline{\Omega}) \cap \mathcal{C}^{0,\alpha}([0,T]\times\overline{\Omega})$$

be endowed with the norm

$$\|u\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,T]\times\overline{\Omega})} := \|u\|_{\mathcal{C}^{\frac{\alpha}{2},0}([0,T]\times\overline{\Omega})} + \|u\|_{\mathcal{C}^{0,\alpha}([0,T]\times\overline{\Omega})}.$$

We let

$$\mathcal{C}^{1+\alpha}(\overline{\Omega}) := \left\{u \in \mathcal{C}^1(\overline{\Omega}) : \nabla_k u \in \mathcal{C}^{\alpha}(\overline{\Omega}), k \in \{1,\dots,n\}\right\}$$

be endowed with the norm

$$\|u\|_{\mathcal{C}^{1+\alpha}(\overline{\Omega})} := \|u\|_{L^\infty(\overline{\Omega})} + \sum_{k=1}^{n} \|\nabla_k u\|_{L^\infty(\overline{\Omega})} + \sum_{k=1}^{n} [\nabla_k u]_{\mathcal{C}^{\alpha}(\overline{\Omega})}.$$

We let

$$\begin{aligned}
&\mathcal{C}^{0,1+\alpha}([0,T]\times\overline{\Omega}) \\
&\quad := \Big\{u \in \mathcal{C}([0,T]\times\overline{\Omega}) : u(t,\cdot) \in \mathcal{C}^{1+\alpha}(\overline{\Omega})\ \forall t \in [0,T], \\
&\qquad\qquad \|u\|_{\mathcal{C}^{0,1+\alpha}([0,T]\times\overline{\Omega})} := \sup_{t\in[0,T]} \|u(t,\cdot)\|_{\mathcal{C}^{1+\alpha}(\overline{\Omega})} < +\infty\Big\}
\end{aligned}$$

be endowed with the norm $\|u\|_{\mathcal{C}^{0,1+\alpha}([0,T]\times\overline{\Omega})}$.

We let

$$\mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\overline{\Omega}) := \mathcal{C}^{\frac{1+\alpha}{2},0}([0,T]\times\overline{\Omega}) \cap \mathcal{C}^{0,1+\alpha}([0,T]\times\overline{\Omega})$$

be endowed with the norm

$$\|u\|_{\mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\overline{\Omega})} := \|u\|_{\mathcal{C}^{\frac{1+\alpha}{2},0}([0,T]\times\overline{\Omega})} + \|u\|_{\mathcal{C}^{0,1+\alpha}([0,T]\times\overline{\Omega})}.$$

We let

$$\begin{aligned}
\mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}([0,T]\times\overline{\Omega}) := \Big\{&u \in \mathcal{C}([0,T]\times\overline{\Omega}) : \\
&u_t, \nabla_{ij} u \in \mathcal{C}^{\frac{\alpha}{2},\alpha}([0,T]\times\overline{\Omega}),\ i,j \in \{1,\dots,n\}\Big\}
\end{aligned}$$

be endowed with the norm

$$\begin{aligned}
\|u\|_{\mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}([0,T]\times\overline{\Omega})} :=& \|u\|_{L^\infty([0,T]\times\overline{\Omega})} + \sum_{k=1}^{n} \|\nabla_k u\|_{L^\infty([0,T]\times\overline{\Omega})} \\
&+ \|u_t\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,T]\times\overline{\Omega})} + \sum_{i,j=1}^{n} \|\nabla_{ij} u\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,T]\times\overline{\Omega})}.
\end{aligned}$$

Similar definitions are obtained by replacing $[0, T]$ with $(0, T)$ and $\overline{\Omega}$ with Ω.

We conclude this chapter with the following result on second order linear parabolic partial differential equations with Hölder coefficients (see [190](17)).

Theorem 7.13 (Linear case). *Let* $\alpha \in (0, 1)$, $T > 0$, *and let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set of class* $\mathcal{C}^{2+\alpha}$. *Let*

$$a_{ij}, b_i, c, \mathrm{f} \in \mathcal{C}^{\frac{\alpha}{2},\alpha}([0, T] \times \overline{\Omega}), \qquad i, j \in \{1, \dots, n\},$$

and

$$\beta_i, \gamma, \mathrm{g} \in \mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}([0, T] \times \partial\Omega), \qquad i \in \{1, \dots, n\}.$$

Take

$$w_0 \in \mathcal{C}^{2+\alpha}(\overline{\Omega}).$$

For $\varphi \in \mathcal{C}^{2+\alpha}([0, T] \times \overline{\Omega})$ *set*

$$\mathcal{A}\varphi := a_{ij}\nabla_{ij}\varphi + b_i\nabla_i\varphi + c\varphi,$$

$$\mathcal{B}\varphi := \beta_i\nabla_i\varphi + \gamma\varphi,$$

and suppose that the following compatibility condition holds(18)*:*

$$\mathcal{B}(0, z)w_0(z) = \mathrm{g}(0, z), \qquad z \in \partial\Omega.$$

Assume that there exist a constant $\mu > 0$ *such that*

$$a_{ij}(t, z)\xi_i\xi_j \geq \mu|\xi|^2, \quad t \in [0, T],\ z \in \overline{\Omega},\ \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n, \tag{7.57}$$

and a constant $\nu_0 > 0$ *such that*

$$|\langle\beta(t, z), \mathrm{n}_\Omega(z)\rangle| \geq \nu_0, \qquad (t, z) \in [0, T] \times \partial\Omega. \tag{7.58}$$

Then the problem

$$\begin{cases} w_t - \mathcal{A}w = \mathrm{f} & \text{in } (0, T) \times \Omega, \\ \mathcal{B}w = \mathrm{g} & \text{on } (0, T] \times \partial\Omega, \\ w = w_0 & \text{on } \{t = 0\} \times \overline{\Omega}, \end{cases} \tag{7.59}$$

(17) See also [197] and references therein.

(18) $\mathcal{B}(0, z)$ means to take $\beta_i(0, z)$ and $\gamma(0, z)$ in the expression of $\mathcal{B}$.

has a unique solution

$$w \in \mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}([0,T] \times \overline{\Omega}).$$

Moreover, there exists a constant $\mathrm{C} > 0$ *such that*

$$\begin{aligned} &\|w\|_{\mathcal{C}^{\frac{2+\alpha}{2},2+\alpha}([0,T]\times\overline{\Omega})} \\ &\leq \mathrm{C}\left(\|w_0\|_{\mathcal{C}^{2+\alpha}(\overline{\Omega})} + \|\mathrm{f}\|_{\mathcal{C}^{\frac{\alpha}{2},\alpha}([0,T]\times\overline{\Omega})} + \|\mathrm{g}\|_{\mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}([0,T]\times\partial\Omega)}\right). \end{aligned} \tag{7.60}$$

The constant C *depends on* Ω*, on the* $\mathcal{C}^{\frac{\alpha}{2},\alpha}$*-norm of* a_{ij}*,* b_i*,* c*, on the* $\mathcal{C}^{\frac{1+\alpha}{2},1+\alpha}$*-norm of* β_i *and* γ*, on the constants* μ*,* ν_0*, on the space dimension* n*, and on* T*, and is increasing with respect to* T*. In particular,* C *remains bounded as* T *goes to zero.*

Chapter 8
Grayson's example

We have seen in Chapter 7 that, if ∂E is smooth and compact, then for short times there exists a unique smooth compact mean curvature flow starting from ∂E. We have also seen in Example 3.21 that the sphere of radius R_0 shrinks to a point in the finite time $R_0^2/(2(n-1))$. This time can be interpreted as a singularity time of the flow, even if the evolving sphere reduces to a point. In this chapter we describe an example, due to Grayson [159], of a smooth compact mean curvature flow in $\mathbb{R}^3$ which develops a singularity, and does not reduce to a point at the same time: this example and the proof that we present here follows that of [159]. In [126] the reader can find various differences in the proof.

Let r and l be two positive real numbers such that

$$r < \frac{2l}{\pi}. \tag{8.1}$$

Let $h \in \mathcal{C}^\infty(\mathbb{R}; (0, +\infty))$ be a function whose graph over the z_1-axis has the shape depicted in Figure 8.1.

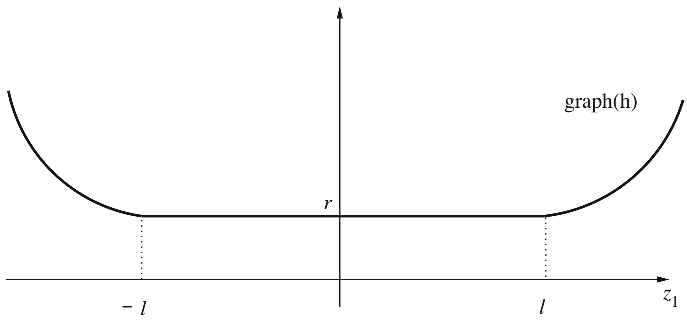

Figure 8.1. Qualitative shape of the function $h : \mathbb{R} \to (0, +\infty)$.

Let us rotate the graph of h around the horizontal z_1-axis, and denote by $F_h \subset \mathbb{R}^3$ the closed set containing the z_1-axis enclosed by the rotated graph, that is

$$F_h := \left\{ (z_1, z_2, z_3) \in \mathbb{R}^3 : z_2^2 + z_3^2 \le (h(z_1))^2 \right\}.$$

It is possible to choose the function h in such a way that

$$\text{the mean curvature of } \partial F_h \text{ is nonnegative everywhere.} \tag{8.2}$$

The role of F_h will be the following: using condition (8.2), F_h will act as a (fixed) barrier for certain smooth mean curvature flows starting inside it (see in particular inclusion (8.12)).

Now, select a real positive number R with

$$\frac{R^2}{4} > \frac{2lr^2}{2l - \pi r}, \tag{8.3}$$

and two points $p, q \in F_h$ with p (respectively q) belonging to the negative (respectively positive) part of the z_1-axis, and far enough from the origin, so that

$$\overline{B_R(p)} \cap \overline{B_R(q)} = \emptyset \qquad \text{and} \qquad \overline{B_R(p)} \cup \overline{B_R(q)} \subset \mathrm{int}(F_h).$$

Recall that if $B \subset \mathbb{R}^3$ is Borel measurable, then $|B|$ is the Lebsegue measure of B.

The result is the following.

Theorem 8.1 (Grayson's example). *Suppose that* r, l, R *satisfy* (8.1) *and* (8.3)*, and let* F_h*,* p*, and* q *be as above. Let* $E \subset \mathbb{R}^3$ *be a bounded connected closed set with* $\partial E \in \mathcal{C}^\infty$*, such that*

$$B_R(p) \cup B_R(q) \subset E \subset \mathrm{int}(F_h). \tag{8.4}$$

Let $f : [0, t_{\max}(\partial E)) \to \mathcal{P}(\mathbb{R}^3)$ *be the smooth compact mean curvature flow starting from* ∂E *in the maximal time interval* $[0, t_{\max}(\partial E))$*. Then*

$$\liminf_{t \uparrow t_{\max}(\partial E)} |f(t)| > 0.$$

Hence the smooth compact mean curvature flow starting from ∂E develops a singularity before reducing to a point. The idea is that the long thin "neck" of E[(1)] must, by comparison, shrink and singularize before the whole evolution $f(t)$ has reduced to a point, as a consequence of the fact that (still by comparison) $f(t)$ must contain the mean curvature flow of the two balls.

(1) Think of r to be small with respect to l.

8.1. The proof

Let us take a smooth periodic function $v_0 \in \mathcal{C}^\infty(\mathbb{R}; (0, +\infty))$ such that, if we consider the set

$$I_{v_0} := \left\{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_2^2 + z_3^3 \le (v_0(z_1))^2\right\}$$

enclosed by the rotation around the z_1-axis of the graph of v_0, we have, besides

$$\partial I_{v_0} \in \mathcal{C}^\infty,$$

also

$$\mathrm{int}(I_{v_0}) \supset E \qquad \text{and} \qquad I_{v_0} \subset \mathrm{int}(F_h). \tag{8.5}$$

Let $(x_1, x_2, x_3) \in \partial I_{v_0}$, so that $(v_0(x_1))^2 = x_2^2 + x_3^3$. As ∂I_{v_0} is a surface of revolution, its mean curvature[(2)] at (x_1, x_2, x_3) is

$$-\frac{v_0''(x_1)}{\left(1 + (v_0'(x_1))^2\right)^{3/2}} + \frac{1}{v_0(x_1)\left(1 + (v_0'(x_1))^2\right)^{1/2}}.$$

By uniqueness[(3)], it follows that the smooth mean curvature flow starting from ∂I_{v_0} is given by $\partial I_v(t)$, where

$$I_v(t) = \left\{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_2^2 + z_3^2 \le (v(t, z_1))^2\right\}, \qquad t \in [0, T_{v_0}).$$

Here $I_v(t)$ is obtained by rotating around the z_1-axis the graph of the smooth function $v : [0, T_{v_0}) \times \mathbb{R} \to [0, +\infty)$, with $v(t, \cdot)$ periodic with the same period as v_0, and solving[(4)]

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{v''}{1 + (v')^2} - \dfrac{1}{v} & \text{in } (0, T_{v_0}) \times \mathbb{R}, \\[2ex] v = v_0 & \text{on } \{t = 0\} \times \mathbb{R}, \end{cases} \tag{8.6}$$

where $' = \dfrac{\partial}{\partial z_1}$ and

$$T_{v_0} := t_{\max}(\partial I_{v_0}) > 0.$$

(2) Recall Example 1.17.

(3) We refer to Theorem 5.8, which can be extended to the case when $f_1(t)$ is obtained by rotating the subgraph of a periodic function, (see also [11, Theorem 4.3]).

(4) Recall Example 3.32.

Notice that, by comparison[5] with a cylinder having axis of rotation the z_1-axis and initially containing I_{v_0}, it follows that there exists a constant $K > 0$ such that

$$|v(t, z_1)| \le K, \qquad (t, z_1) \in [0, T_{v_0}) \times \mathbb{R}. \tag{8.7}$$

Remark 8.2. For any $b \in (0, T_{v_0})$ we have

$$\min_{(t,z_1)\in[0,b]\times\mathbb{R}} v(t, z_1) > 0.$$

We now show that the only way for $\partial I_v(t)$ to singularize is to intersect the z_1-axis.

Proposition 8.3 (Intersection with the z_1-axis as $t \uparrow T_{v_0}$). *We have*

$$\inf_{(t,z_1)\in[0,T_{v_0})\times\mathbb{R}} v(t, z_1) = 0.$$

Proof. Assume by contradiction that there exists $\sigma > 0$ such that

$$\inf_{(t,z_1)\in(0,T_{v_0})\times\mathbb{R}} v(t, z_1) \ge \sigma. \tag{8.8}$$

Let

$$b \in (0, T_{v_0}).$$

We divide the proof into three steps.

Step 1. We have[6]

$$\begin{aligned} -\|v'(0,\cdot)\|_{L^\infty(\mathbb{R})}\, e^{\frac{t}{K^2}} &\le v'(t, z_1) \\ &\le \|v'(0,\cdot)\|_{L^\infty(\mathbb{R})}\, e^{\frac{t}{\sigma^2}}, \qquad (t, z_1) \in (0, b] \times \mathbb{R}. \end{aligned}$$

Indeed, differentiating the partial differential equation in (8.6) with respect to z_1, we obtain

$$\frac{\partial v'}{\partial t} = \frac{v'''}{1 + (v')^2} - \frac{2\, v'\, (v'')^2}{\big(1 + (v')^2\big)^2} + \frac{v'}{v^2}.$$

(5) As already remarked, Theorem 5.8 is still valid when $\partial f_1(t)$, instead of being compact, is obtained by rotating the graph of a smooth periodic function. The mean curvature evolution of the cylinder is that given in Example 3.22; see also [85]. Observe also that $T_{v_0} < +\infty$. Indeed, assume by contradiction that $T_{v_0} = +\infty$. Then, still comparing $I_v(t)$ with the mean curvature evolution of the above mentioned cylinder, there exists $(t, z_1) \in (0, T_{v_0}) \times \mathbb{R}$ such that $v(t, z_1) = 0$. Hence $\partial I_v(t)$ is singular at z_1 and this contradicts the definition of T_{v_0}. A more precise upper bound on T_{v_0} will be given in (8.9).

(6) See also [11, Theorem 4.3].

Define

$$v'_{\max}(t) := \max\{v'(t, z_1) : z_1 \in \mathbb{R}\}, \qquad t \in (0, b].$$

Then[7], using also hypothesis (8.8),

$$\frac{d}{dt} v'_{\max} \leq \frac{v'_{\max}}{v^2} \leq \frac{v'_{\max}}{\sigma^2} \qquad \text{a.e. in } (0, b).$$

Consequently

$$v'(t, z_1) \leq \|v'(0, \cdot)\|_{L^\infty(\mathbb{R})} \, e^{\frac{t}{\sigma^2}}, \qquad (t, z_1) \in (0, b] \times \mathbb{R}.$$

Arguing in a similar way for $v'_{\min}(t) := \min\{v'(t, z_1) : z_1 \in \mathbb{R}\}$ we get, using (8.7),

$$\frac{d}{dt} v'_{\min} \geq \frac{v'_{\min}}{v^2} \geq \frac{v'_{\min}}{K^2} \qquad \text{a.e. in } (0, b).$$

Therefore

$$v'(t, z_1) \geq -\|v'(0, \cdot)\|_{L^\infty(\mathbb{R})} \, e^{\frac{t}{K^2}}, \qquad (t, z_1) \in (0, b] \times \mathbb{R},$$

and the proof of step 1 is concluded.

Step 2. We have

$$\begin{aligned} -\left\| \frac{\partial v}{\partial t}(0, \cdot) \right\|_{L^\infty(\mathbb{R})} e^{\frac{t}{K^2}} &\leq \frac{\partial v}{\partial t}(t, z_1) \\ &\leq \left\| \frac{\partial v}{\partial t}(0, \cdot) \right\|_{L^\infty(\mathbb{R})} e^{\frac{t}{\sigma^2}}, \quad (t, z_1) \in (0, b] \times \mathbb{R}. \end{aligned}$$

Indeed, from (8.6) we compute

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial t}\right) = \frac{(\frac{\partial v}{\partial t})''}{1 + (v')^2} - \frac{2\, v'\, v''\, (\frac{\partial v}{\partial t})'}{\left(1 + (v')^2\right)^2} + \frac{\frac{\partial v}{\partial t}}{v^2}.$$

Then the assertion follows by arguing as in the proof of step 1.

The next step follows from (8.8), from steps 1 and 2 and writing the equation in (8.6) in the form

$$v'' = \left(1 + (v')^2\right) \frac{\partial v}{\partial t} + \frac{1}{v} \left(1 + (v')^2\right).$$

[7] See, for instance, Theorem 5.12.

Step 3. The function $|v''|$ increases at most exponentially in $(0, b] \times \mathbb{R}$.

From steps 1 and 3 it follows that the length of the second fundamental form[8] of $\partial I_v(t)$ remains bounded in $[0, b] \times \mathbb{R}$. Hence, by Theorem 6.19, it follows that $\partial I_v(t)$ cannot singularize in $[0, b] \times \mathbb{R}$. Now, we observe that the bound for the length of the second fundamental of $I_v(t)$ is independent of b. It is then possible to prove[9] that one can define a smooth surface $\partial I_v(T_{v0})$ as the limit of the surfaces $\partial I_v(t)$ as $t \uparrow T_{v0}$. Using the short time existence result, the mean curvature flow starting from $\partial I_v(T_{v0})$ can be continued, at least for a short positive time, after T_{v0}, and this contradicts the definition of T_{v0}, and concludes the proof of the proposition. $\square$

We now show that $\partial I_v(t)$ singularizes at some time less than $\dfrac{R^2}{4}$.

Lemma 8.4 (Estimate of T_{v0}). *We have*

$$T_{v0} < \frac{R^2}{4}. \tag{8.9}$$

Proof. Let us consider

$$\|v(t,\cdot)\|_{L^1((-l,l))} = \int_{-l}^{l} v(t, z_1)\, dz_1, \qquad t \in [0, T_{v0}).$$

Using the evolution equation (8.6) for v we have, for $t \in (0, T_{v0})$,

$$\begin{aligned}\frac{d}{dt}\|v(t,\cdot)\|_{L^1((-l,l))} &= \int_{-l}^{l} \frac{\partial v}{\partial t}(t, z_1)\, dz_1 \\ &= \int_{-l}^{l} \left(\frac{v''(t, z_1)}{1 + (v'(t, z_1))^2} - \frac{1}{v(t, z_1)} \right) dz_1 \\ &= -\int_{\mathrm{graph}(v(t,\cdot))\cap((-l,l)\times\mathbb{R})} \kappa \, d\mathcal{H}^1 - \int_{-l}^{l} \frac{1}{v(t, z_1)}\, dz_1,\end{aligned} \tag{8.10}$$

[8] Remember from (1.25) that the second fundamental form of $\partial I_v(t)$ equals $\left(\mathrm{Id} - \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}\right) \frac{\nabla^2 u}{|\nabla u|} \left(\mathrm{Id} - \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}\right) \frac{\nabla u}{|\nabla u|}$, and that from (1.31) and (1.32) we have

$$\frac{\nabla^2 u}{|\nabla u|} = \frac{\mathrm{diag}(-vv'' - (v')^2, 1, 1)}{v\sqrt{1 + (v')^2}}.$$

[9] See, *e.g.*, [203].

where

$$\kappa(t, z_1, z_2) := -\frac{v''(t, z_1)}{(1 + (v'(t, z_1))^2)^{3/2}}, \qquad z_2 = v(t, z_1),$$

is the curvature of the plane curve graph$(v(t, \cdot))$. From (8.10) it follows that

$$\frac{d}{dt}\|v(t, \cdot)\|_{L^1((-l,l))} \leq \pi - \int_{-l}^{l} \frac{1}{v(t, z_1)}\, dz_1, \qquad t \in (0, T_{v_0}). \quad (8.11)$$

By (8.2), (8.5) and the inclusion principle (Theorem 5.4), it follows that

$$I_v(t) \subset F_h, \qquad t \in [0, T_{v_0}). \quad (8.12)$$

From (8.12) it follows that

$$v(t, z_1) \leq r, \qquad t \in [0, T_{v_0}),\ z_1 \in [-l, l].$$

Hence $-\dfrac{1}{v(t, z_1)} \leq -\dfrac{1}{r}$ for any $(t, z_1) \in [0, T_{v_0}) \times [-l, l]$, and therefore, from (8.11), we deduce

$$\frac{d}{dt}\|v(t, \cdot)\|_{L^1((-l,l))} \leq \pi - \frac{2l}{r}, \qquad t \in (0, T_{v_0}).$$

Since

$$\|v(0, \cdot)\|_{L^1((-l,l))} \leq 2lr,$$

we obtain

$$\begin{aligned} \|v(t, \cdot)\|_{L^1((-l,l))} &\leq \left(\pi - \frac{2l}{r}\right) t + \|v(0, \cdot)\|_{L^1((-l,l))} \\ &\leq \left(\pi - \frac{2l}{r}\right) t + 2lr, \end{aligned} \quad (8.13)$$

for any $t \in [0, T_{v_0})$. The right-hand side of (8.13) vanishes at time

$$\frac{2lr^2}{2l - \pi r}.$$

Remember now that, in view of assumption (8.3), we have

$$\frac{2lr^2}{2l - \pi r} < \frac{R^2}{4}.$$

Hence, as time t increases to some value less than $R^2/4$, the function $v(t, \cdot)$ tends to vanish somewhere in $[-l, l]$. Therefore the norm of the second fundamental form of $\partial I_v(t)$ tends to blow up at those points. This proves (8.9). $\square$

We are now in a position to conclude the proof of Theorem 8.1.

Let $f : [0, t_{\max}(\partial E)) \to \mathcal{P}(\mathbb{R}^3)$ be the smooth compact mean curvature flow starting from ∂E. Observe that the mean curvature flow starting from $\overline{B_R(p)}$ (respectively $\overline{B_R(q)}$) is given by $\overline{B_{R(t)}(p)}$ (respectively $\overline{B_{R(t)}(q)}$), where $R(t) = \sqrt{R^2 - 4t}$. In particular, it is well defined for all $t \in [0, \frac{R^2}{4})$. Hence, using (8.4) and the inclusion principle, it follows that

$$B_{R(t)}(q) \cup B_{R(t)}(p) \subset f(t), \qquad t \in \left[0, \min\left(\frac{R^2}{4}, t_{\max}(\partial E)\right)\right).$$

Still by the inclusion principle, it follows

$$f(t) \subset I_v(t), \qquad t \in \Big[0, \min\big(T_{v_0}, t_{\max}(\partial E)\big)\Big).$$

From (8.9) it then follows

$$B_{R(t)}(q) \cup B_{R(t)}(p) \subset f(t) \subset I_v(t), \quad t \in \Big[0, \min\big(T_{v_0}, t_{\max}(\partial E)\big)\Big). \tag{8.14}$$

As we have seen in Proposition 8.3, $I_v(t)$ tends to touch the z_1-axis as $t \uparrow T_{v_0}$. Therefore, from (8.14) it follows $t_{\max}(\partial E) \le T_{v_0}$. The assertion of the theorem follows.

Chapter 9
De Giorgi's barriers

We start this chapter with the definitions of barrier and minimal barrier. We recall that the motivation for introducing barriers derives from the presence of singularities in mean curvature flow: the aim is then to provide a notion of weak solution to the flow which is meaningful for all positive times, and which coincides with the smooth evolution till the latter exists[(1)].

Let us begin by introducing the family $\mathfrak{F}$, the elements of which take the role of test evolutions.

Definition 9.1 (The family $\mathfrak{F}$). We denote by $\mathfrak{F}$ a family of maps of a real variable which satisfies the following property: for any $f \in \mathfrak{F}$ there exist two real numbers a, b such that $a < b$ and

$$f : [a, b] \to \mathcal{P}(\mathbb{R}^n).$$

In the examples we are interested in, the elements of $\mathfrak{F}$ can be considered as flowing smooth test hypersurfaces.

The notion of barrier reads as follows.

Definition 9.2 (Barrier). We say that a function ϕ of a real variable is a barrier for the family $\mathfrak{F}$, and we write

$$\phi \in \mathcal{B}(\mathfrak{F}), \tag{9.1}$$

if there exists an interval $J \subseteq \mathbb{R}$ such that $\phi : J \to \mathcal{P}(\mathbb{R}^n)$, and whenever a, b and f satisfy the conditions

$$[a, b] \subseteq J, \ \ f : [a, b] \to \mathcal{P}(\mathbb{R}^n), \ \ f \in \mathfrak{F}, \ \ f(a) \subseteq \phi(a),$$

then

$$f(b) \subseteq \phi(b).$$

(1) Our presentation in this chapter, as well as in Chapters 10, 11, 13, 14, follows mainly [54, 45, 46, 48, 40].

If necessary, the map ϕ in Definition 9.2 will be called a barrier in J, and sometimes we will write[(2)]

$$\phi \in \mathcal{B}(\mathfrak{F}, J).$$

It is clear that if $\phi \in \mathcal{B}(\mathfrak{F}, J)$, then the restriction of ϕ to any subinterval $\mathcal{I}$ of J is a barrier in $\mathcal{I}$.

Two trivial examples of barrier in $[0, +\infty)$ are given by the maps $\phi_1 : [0, +\infty) \to \mathcal{P}(\mathbb{R}^n)$, $\phi_2 : [0, +\infty) \to \mathcal{P}(\mathbb{R}^n)$ defined as $\phi_1(t) := \emptyset$ for any $t \in [0, +\infty)$, and $\phi_2(t) = \mathbb{R}^n$ for any $t \in [0, +\infty)$.

Remark 9.3 (Intersection of barriers is a barrier). Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval, let Λ be a set of indices, and for any $i \in \Lambda$ let ϕ_i be a barrier in $\mathcal{I}$. Then the map

$$\bigcap_{i \in \Lambda} \phi_i : \mathcal{I} \to \mathcal{P}(\mathbb{R}^n),$$

that associates with each $t \in \mathcal{I}$ the set $\bigcap_{i \in \Lambda} \phi_i(t)$, is a barrier for the family $\mathfrak{F}$ in $\mathcal{I}$.

Different choices of the family $\mathfrak{F}$ in general may lead to different barriers. In this book we will be mainly concerned with the family $\mathfrak{F} = \mathcal{F}$, which is defined as follows[(3)].

Definition 9.4 (The family $\mathcal{F}$). We denote by $\mathcal{F}$ the family of maps which satisfy the following properties: for any $f \in \mathcal{F}$

- there exist $a < b$ such that $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$;
- for any $t \in [a, b]$ the set $f(t)$ is closed and $\partial f(t)$ is compact;
- there exists an open set $A \subset \mathbb{R}^n$ such that $A \supset \partial f(t)$ for any $t \in [a, b]$, and
$$d := d_f \in \mathcal{C}^\infty([a, b] \times A);$$
- $\dfrac{\partial d}{\partial t}(t, x) = \Delta d(t, x)$ for any $t \in [a, b]$ and $x \in \partial f(t)$.

---

(2) To enphasize the choice of the order relation $\subseteq$ between subsets of $\mathbb{R}^n$, we could also write $\phi \in \mathcal{B}(\subseteq, \mathfrak{F}, J)$ in place of (9.1) (see Section 9.5); to simplify the presentation, we will in general not use this notation.

(3) In more general cases, like for example those where the normal velocity is expressed by a function of the time-space position, the normal vector and the second fundamental form (as for instance for a geometric [86] F as in (14.24)), it is sometimes necessary to replace the last equation in Definition 9.4 with a suitable inequality: see [45] for some details. For motion by mean curvature, it is possible to show that the minimal barrier $\mathcal{M}(E, \mathcal{F})$ in Definition 9.13 below coincides with $\mathcal{M}(E, \mathcal{F}^{\geq})$, where $\mathcal{F}^{\geq}$ is the family of all maps as in the first three items of Definition 9.4, and where the equation in the last item is replaced by $\frac{\partial d(t,x)}{\partial t} \geq \Delta d(t, x)$ for any $t \in [a, b]$ and $x \in \partial f(t)$.

The family $\mathcal{F}$ is contained in the family of smooth compact mean curvature flows considered in Definition 3.11. Indeed, requiring $d \in \mathcal{C}^\infty([a,b] \times A)$ is slightly more restrictive than the smoothness assumption (3.2) required on d in Definition 3.2. However, the class $\mathcal{B}(\mathcal{F})$ coincides with the class of all barriers for the family of smooth compact mean curvature flows(4).

In the original definition (see [108]), the map $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ is additionally required to satisfy that $f(t)$ is compact for any $t \in [a, b]$. Therefore, the definition reads as follows.

Definition 9.5 (The family $\mathcal{F}_c$). We denote by $\mathcal{F}_c$ the family of maps which satisfy the following properties: for any $f \in \mathcal{F}_c$

- there exist $a < b$ such that $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$;
- for any $t \in [a, b]$ the set $f(t)$ is compact;
- there exists an open set $\mathrm{A} \subset \mathbb{R}^n$ such that $\mathrm{A} \supset \partial f(t)$ for any $t \in [a, b]$, and
$$d := d_f \in \mathcal{C}^\infty([a, b] \times \mathrm{A});$$
- $\dfrac{\partial d}{\partial t}(t, x) = \Delta d(t, x)$ for any $t \in [a, b]$ and $x \in \partial f(t)$.

It is clear that $\mathcal{F}_c \subset \mathcal{F}$.

Example 9.6 (Tests are barriers for mean curvature flow). Let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{F}$. By the inclusion principle between smooth compact mean curvature flows (Theorem 5.8) f is a barrier for the family $\mathcal{F}$ in $[a, b]$, *i.e.*, $f \in \mathcal{B}(\mathcal{F}, [a, b])$. Hence

$$\mathcal{F} \subset \mathcal{B}(\mathcal{F}). \tag{9.2}$$

Remark 9.7 (Time translations). If $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{F}$, and $\tau \in \mathbb{R}$, then the map $g : [a-\tau, b-\tau] \to \mathcal{P}(\mathbb{R}^n)$ defined by $g(t) := f(t+\tau)$ for any $t \in [a - \tau, b - \tau]$, belongs to $\mathcal{F}$. Moreover, if $\phi \in \mathcal{B}(\mathcal{F}, J)$, then the map $\psi : J - \tau \to \mathcal{P}(\mathbb{R}^n)$ defined by $\psi(t) := \phi(t + \tau)$ for any $t \in J - \tau$, belongs to $\mathcal{B}(\mathcal{F}, J - \tau)$.

(4) Indeed, let $\phi \in \mathcal{B}(\mathcal{F}, J)$ and let $f : [a, b] \subseteq J \to \mathcal{P}(\mathbb{R}^n)$ be a smooth compact mean curvature flow such that $f(a) \subseteq \phi(a)$. We have to show that $f(b) \subseteq \phi(b)$. We observe that there exist $k \in \mathbb{N}$, $k \geq 1$, and points $\tau_0 := a < \tau_1 < \cdots < \tau_{k-1} < \tau_k := b$ such that $f_{|[\tau_i, \tau_{i+1}]}$ is an element of $\mathcal{F}$, for any $i \in \{0, \ldots, k-1\}$. The inclusion $f(a) \subseteq \phi(a)$ and the assumption $\phi \in \mathcal{B}(\mathcal{F}, J)$ then imply that $f(\tau_1) \subseteq \phi(\tau_1)$. Since the evolution starting from $f(\tau_1)$ at time $t \in [\tau_1, \tau_2]$ coincides with $f(t)$, it follows using the same reasoning that $f(\tau_2) \subseteq \phi(\tau_2)$. Repeating k-times this argument we deduce that $f(b) \subseteq \phi(b)$. Hence ϕ is a barrier for the family of all smooth compact mean curvature flows.

Notation. If $\phi : J \subseteq \mathbb{R} \to \mathcal{P}(\mathbb{R}^n)$ is a map, we denote by

$$\mathrm{int}(\phi) \tag{9.3}$$

the map defined as $t \in J \to \mathrm{int}(\phi(t))$ for any $t \in J$, and by

$$\overline{\phi} \tag{9.4}$$

the map defined as $t \in J \to \overline{\phi(t)}$ for any $t \in J$.

If $\rho > 0$, we let

$$\phi_\rho^-, \qquad \phi_\rho^+ \tag{9.5}$$

be the maps defined[(5)] as $t \in J \to \phi(t)_\rho^-$ and $t \in J \to \phi(t)_\rho^+$, respectively. Furthermore, if $y \in \mathbb{R}^n$, by

$$\phi + y$$

we mean the map defined as $t \in J \to \phi(t) + y$. Here, given $F \subseteq \mathbb{R}^n$,

$$F + y := \{z + y : z \in F\}.$$

Also, given two maps $\phi_1, \phi_2 : J \subseteq \mathbb{R} \to \mathcal{P}(\mathbb{R}^n)$, we write

$$\phi_1 \subseteq \phi_2 \quad (\text{respectively } \phi_1 = \phi_2, \phi_1 \setminus \phi_2) \tag{9.6}$$

to mean that $\phi_1(t) \subseteq \phi_2(t)$ (respectively $\phi_1(t) = \phi_2(t), \phi_1(t) \setminus \phi_2(t)$) for any $t \in J$.

The next example shows that barriers for mean curvature flow cannot shrink too fast. Define

$$\varrho(\tau) := \sqrt{2(n-1)\tau}, \qquad \tau \in [0, +\infty). \tag{9.7}$$

Example 9.8 (Comparison with balls). Let $\phi \in \mathcal{B}(\mathcal{F}, J)$ and $s \in J$. Then

$$\mathrm{int}\,(\phi(t)) \supseteq \Big\{ z \in \mathbb{R}^n : \mathrm{dist}\,\big(z, \mathbb{R}^n \setminus \phi(s)\big) > \varrho(t-s) \Big\}, \qquad t \in J,\ t > s. \tag{9.8}$$

Indeed, let $t \in J$, $t > s$, and $z \in \mathbb{R}^n$ be such that

$$\mathrm{dist}\,\big(z, \mathbb{R}^n \setminus \phi(s)\big) > \varrho(t-s),$$

and choose

$$\overline{\varrho} \in \Big(\varrho(t-s)\ ,\ \mathrm{dist}\,\big(z, \mathbb{R}^n \setminus \phi(s)\big) \Big).$$

(5) Recall the notation introduced in formula (1.3).

Let $B_{\overline{\varrho}}(z)$ be the open ball centered at z with radius $\overline{\varrho}$, so that

$$B_{\overline{\varrho}}(z) \subset \phi(s).$$

Denote by $\overline{B_{R(\tau)}(z)}$ the evolution by mean curvature starting at time s from $\overline{B_{\overline{\varrho}}(z)}$, for $\tau \in [s, s + \frac{\overline{\varrho}^2}{2(n-1)})$. Notice that

$$R(t) = \sqrt{\overline{\varrho}^2 - 2(n-1)(t-s)} = \sqrt{\overline{\varrho}^2 - (\varrho(t-s))^2} > 0.$$

Since $\phi \in \mathcal{B}(\mathcal{F}, J)$, it follows that $\overline{B_{R(t)}(z)} \subseteq \phi(t)$. Then $R(t) > 0$ implies

$$z \in \operatorname{int}(\phi(t)).$$

Example 9.9. Let $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{F}$ and $s \in [a, b)$. Then

$$f(t) \subseteq \left\{z \in \mathbb{R}^n : \operatorname{dist}(z, f(s)) \leq \varrho(t-s)\right\}, \quad t \in [a, b],\ t > s. \tag{9.9}$$

Indeed, the map[6]

$$f^{c}(t) := \overline{\mathbb{R}^n \setminus f(t)}, \qquad t \in [a, b],$$

belongs to $\mathcal{F}$, and hence it is a barrier for the family $\mathcal{F}$ in $[a, b]$ by Example 9.6 Applying (9.8) with J replaced by $[a, b]$ and with ϕ replaced by f^{c} we get inclusion (9.9).

Remark 9.10. Barriers for the family $\mathcal{F}$ can jump at a certain time and instantly enlarge. Therefore, given $\phi \in \mathcal{B}(\mathcal{F})$, the map $\mathbb{R}^n \setminus \phi$ is not necessarily an element of $\mathcal{B}(\mathcal{F})$.

Example 9.11. Let $\rho_1 : [0, +\infty) \to (0, +\infty)$ and $\rho_2 : [0, +\infty) \to (0, +\infty)$ be defined as

$$\rho_1(t) := \begin{cases} 1 & \text{if } t \in [0, 1], \\ 2 & \text{if } t \in (1, +\infty), \end{cases}, \qquad \rho_2(t) := \begin{cases} 1 & \text{if } t \in [0, 1), \\ 2 & \text{if } t \in [1, +\infty). \end{cases}$$

Then the two maps defined as

$$t \in [0, +\infty) \to \overline{B_{\rho_1(t)}(0)}, \qquad t \in [0, +\infty) \to \overline{B_{\rho_2(t)}(0)}$$

are barriers for the family $\mathcal{F}$ in $[0, +\infty)$. On the other hand, the two maps

$$t \in [0, +\infty) \to \mathbb{R}^n \setminus B_{\rho_1(t)}(0), \qquad t \in [0, +\infty) \to \mathbb{R}^n \setminus B_{\rho_2(t)(0)}$$

are not barriers for the family $\mathcal{F}$ in $[0, +\infty)$.

The comparison with balls discussed in Example 9.8 implies also an inequality between the distance of a smooth flow f and the complement of a barrier.

[6] If we take $f \in \mathcal{F}_c$, then $f^{c} \notin \mathcal{F}_c$. However, it is still true that f^{c} is barrier for the family $\mathcal{F}_c$ (and for the family $\mathcal{F}$) in its domain.

Lemma 9.12. *Let $\phi \in \mathcal{B}(\mathcal{F}, J)$ and $f : [a,b] \subseteq J \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{F}$, be such that*

$$f(a) \subseteq \phi(a).$$

Define

$$\delta(t) := \operatorname{dist}(f(t), \mathbb{R}^n \setminus \phi(t)), \qquad t \in [a,b].$$

Then

$$s, t \in [a,b],\ t > s \quad \Rightarrow \quad \delta(t) \geq \delta(s) - 2\varrho(t-s). \tag{9.10}$$

In particular,

$$\delta(t) \leq \liminf_{\tau \downarrow 0} \delta(t+\tau), \qquad t \in [a,b). \tag{9.11}$$

Proof. Let $s, t \in [a,b], t > s$; choose

$$x \in \partial f(t) \text{ and } y \in \overline{\mathbb{R}^n \setminus \phi(t)} \quad \text{such that} \quad |x-y| = \delta(t). \tag{9.12}$$

Using the comparison with balls (Example 9.8) we have

$$\mathbb{R}^n \setminus \operatorname{int}(\phi(t)) \subseteq \{z \in \mathbb{R}^n : \operatorname{dist}(z, \mathbb{R}^n \setminus \phi(s)) \leq \varrho(t-s)\},$$

hence

$$\operatorname{dist}\big(y, \mathbb{R}^n \setminus \phi(s)\big) \leq \varrho(t-s). \tag{9.13}$$

Furthermore, by Example 9.9 we have

$$\operatorname{dist}(x, f(s)) \leq \varrho(t-s). \tag{9.14}$$

Using the triangular property of the distance, and formulas (9.12), (9.13) and (9.14), we then have

$$\begin{aligned}\delta(s) &\leq \operatorname{dist}(x, f(s)) + |x-y| + \operatorname{dist}(y, \mathbb{R}^n \setminus \phi(s)) \\ &= \operatorname{dist}(x, f(s)) + \delta(t) + \operatorname{dist}\big(y, \mathbb{R}^n \setminus \phi(s)\big) \leq \delta(t) + 2\varrho(t-s),\end{aligned}$$

and (9.10) follows.

Inequality (9.11) follows from (9.10), writing $t+\tau$ in place of t, t in place of s, and letting $\tau \downarrow 0$. $\square$

9.0.1. The minimal barrier

Let $t_0 \in \mathbb{R}$, and fix

$$I := [t_0, +\infty).$$

The notion of minimal barrier reads as follows.

Definition 9.13 (Minimal barrier). Let $E \subseteq \mathbb{R}^n$ and let $\mathfrak{F}$ be as in Definition 9.1. The minimal barrier

$$\mathcal{M}\,(E, \mathfrak{F}, t_0) : I \to \mathcal{P}(\mathbb{R}^n)$$

with respect to $\mathfrak{F}$, starting from E at time t_0, is defined as

$$\begin{aligned} \mathcal{M}(E, \mathfrak{F}, t_0)(t) := \bigcap \Big\{ \phi(t) :\ & \phi : I \to \mathcal{P}(\mathbb{R}^n), \\ & \phi \in \mathcal{B}(\mathfrak{F}, I), \phi(t_0) \supseteq E \Big\}, \end{aligned} \tag{9.15}$$

for any $t \in I$.

Notation. When no confusion is possible, we drop the dependence on t_0 in the notation of the minimal barrier; therefore the left-hand side of (9.15) will be often denoted by

$$\mathcal{M}(E, \mathfrak{F})(t).$$

Remark 9.14. The notion of barrier is a sort of "supersolution" (note however that no differential inequality is involved in the definition), and the minimal barrier is a sort of "minimal supersolution". It is possible to define a sort of "subsolution", and of "maximal subsolution" (see [48] for some details). We will not need these two latter notions in the next chapters.

Note that, in view of Remark 9.3, we have

$$\mathcal{M}\,(E, \mathfrak{F}) \in \mathcal{B}(\mathfrak{F}, I). \tag{9.16}$$

Assertions (i) and (ii) of the next proposition show that the minimal barrier takes the initial datum and satisfies the inclusion principle. Assertion (iii) shows a monotonicity property of the minimal barrier with respect to the family $\mathfrak{F}$. Assertion (iv) compares an element f of $\mathfrak{F}$ with the minimal barrier starting from $f(a)$.

Proposition 9.15. *The following properties hold.*

(i) *Let $E \subseteq \mathbb{R}^n$. Then*

$$\mathcal{M}(E, \mathfrak{F}, t_0)(t_0) = E. \tag{9.17}$$

(ii) *Let* $E, F \subseteq \mathbb{R}^n$. *Then*

$$E \subseteq F \ \Rightarrow\ \mathcal{M}(E, \mathfrak{F}) \subseteq \mathcal{M}(F, \mathfrak{F}). \tag{9.18}$$

(iii) *Let* $\mathfrak{G}$ *and* $\mathfrak{H}$ *be two families of maps as in Definition* 9.1. *Then*

$$\mathfrak{G} \subseteq \mathfrak{H} \Rightarrow \mathcal{B}(\mathfrak{G}) \supseteq \mathcal{B}(\mathfrak{H}),$$

hence

$$\mathfrak{G} \subseteq \mathfrak{H} \Rightarrow \mathcal{M}(E, \mathfrak{G}) \subseteq \mathcal{M}(E, \mathfrak{H}), \qquad E \subseteq \mathbb{R}^n.$$

(iv) *Let* $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathfrak{F}$. *Then*

$$f(t) \subseteq \mathcal{M}(f(a), \mathfrak{F}, a)(t), \qquad t \in [a, b]. \tag{9.19}$$

Proof. By (9.15) it follows that $E \subseteq \mathcal{M}(E, \mathfrak{F}, t_0)(t_0)$; hence, from (9.16), we have that the map

$$t \in I \to \begin{cases} E & \text{if } t = t_0 \\ \mathcal{M}(E, \mathfrak{F}, t_0)(t) & \text{if } t > t_0, \end{cases}$$

is a barrier for the family $\mathfrak{F}$ in I, and (i) follows. Assertions (ii), (iii) and (iv) are direct consequences of Definitions 9.2 and 9.13. □

The next result concerns the choice $\mathfrak{F} = \mathcal{F}$ (Definition 9.4) and shows that, if ∂E is compact and smooth, then its smooth (local in time) mean curvature evolution coincides with the minimal barrier: this is the consistency property of the minimal barrier. Hence inclusion (9.19) is, in this case, strengthened into an equality.

Proposition 9.16 (Consistency). *Let* $\partial E \in \mathcal{C}^\infty$ *be compact and let* $f : [a, b] \subset I \to \mathcal{P}(\mathbb{R}^n)$ *be the smooth compact mean curvature flow starting from* ∂E *at time* a, *where* $b > a$ *is any time smaller than the maximal time* $t_{\max}(\partial E)$ *of smooth existence. Then*

$$f(t) = \mathcal{M}(E, \mathcal{F}, a)(t), \qquad t \in [a, b].$$

Proof. As observed in Example 9.6, the map $t \in [a, b] \to f(t) \in \mathcal{P}(\mathbb{R}^n)$ is a barrier for the family $\mathcal{F}$ in $[a, b]$. Therefore

$$\mathcal{M}(E, \mathcal{F})(t) \subseteq f(t), \qquad t \in [a, b].$$

The converse inclusion is proved in (9.19), since $f(a) = E$. □

Example 9.17. Let $\partial E \in \mathcal{C}^\infty$ be compact, and let $f : [0, t_{\max}(\partial E)) \to \mathcal{P}(\mathbb{R}^n)$ be the smooth compact mean curvature flow starting from E. Suppose that the perimeter $P(f(t))$ of $f(t)$ (see Chapter 2) satisfies

$$\lim_{t \uparrow t_{\max}(\partial E)} P(f(t)) = 0.$$

Then

$$\mathcal{M}(E, \mathcal{F})(t) = \begin{cases} f(t) & \text{if } t \in [0, t_{\max}(\partial E)), \\ \emptyset & \text{if } t \geq t_{\max}(\partial E), \end{cases} \tag{9.20}$$

since the map on the right-hand side of (9.20) is a barrier for the family $\mathcal{F}$ in $[0, +\infty)$.

Once the minimal barrier is empty or has zero Lebesgue measure, it remains empty. More precisely, the following holds.

Example 9.18 (Empty minimal barrier, I). If $\mathcal{M}(E, \mathcal{F})(\bar{t}) = \emptyset$ for some $\bar{t} \in I$, then $\mathcal{M}(E, \mathcal{F})(t) = \emptyset$ for any $t \geq \bar{t}$. Also, if $\bar{t} > t_0$ and $\mathcal{M}(E, \mathcal{F})(\bar{t})$ has zero Lebesgue measure, then $\mathcal{M}(E, \mathcal{F})(t) = \emptyset$ for any $t \geq \bar{t}$. Indeed, the map $\phi : I \to \mathcal{P}(\mathbb{R}^n)$ defined as

$$\phi(t) := \begin{cases} \mathcal{M}(E, \mathcal{F})(t) & \text{if } t \in [t_0, \bar{t}), \\ \emptyset & \text{if } t \geq \bar{t}, \end{cases}$$

belongs to $\mathcal{B}(\mathcal{F}, I)$.

It may happen that the minimal barrier must be compared with a smooth evolution which is not an element of the family of test evolutions[7], as we show now in a situation related to an example proposed by Ilmanen in [182, 183].

Example 9.19 (Noncompact evolution with noncompact complement). Let $t_0 = 0$, $n = 2$ and let $v_0 : \mathbb{R} \to (0, 1)$ be an even positive function of class $\mathcal{C}^\infty(\mathbb{R})$, decreasing in $(0, +\infty)$, converging to zero as $z_1 \to +\infty$, and such that the Lebesgue measure of the set

$$E := \left\{(z_1, z_2) \in \mathbb{R}^2 : |z_2| \leq v_0(z_1)\right\}$$

is finite. Then $\partial E \in \mathcal{C}^\infty$ and

$$\partial E = \operatorname{graph}(-v_0) \cup \operatorname{graph}(v_0).$$

---

[7] See inclusion (11.3). Compare also the Dirichlet evolution in Figure 11.9 of Chapter 11.

Note that there is not a tubular neighbourhood of ∂E where $d(\cdot, E)$ is smooth[(8)].

It is possible to prove[(9)] that there exists a smooth curvature evolution starting from graph(v_0), and given by the graph of a function $v \in \mathcal{C}^\infty((0,+\infty) \times \mathbb{R}) \cap \mathcal{C}^0([0,+\infty) \times \mathbb{R})$ taking values in $(0, 1)$.

From Theorem 5.8 it follows that the map

$$t \in (0,+\infty) \to \mathrm{f}(t) := \{z = (z_1, z_2) \in \mathbb{R}^2 : |z_2| \leq v(t, z_1)\}$$

is a barrier for the family $\mathcal{F}$ in $(0, +\infty)$, hence

$$\mathcal{M}(E, \mathcal{F})(t) \subseteq \mathrm{f}(t), \qquad t \in (0, +\infty). \tag{9.21}$$

Observe also that, given $a, b > 0$ with $a < b$, if we consider the restriction $f_{|[a,b]}$ of f to $[a, b]$, then

$$\mathrm{f}_{|[a,b]} \notin \mathcal{F}, \tag{9.22}$$

because $\partial\mathrm{f}(t)$ is not compact.

In general, as we shall see in Section 10.1, it may happen that, because of (9.22), the analog of inclusion (9.19) fails. Indeed, in this example equality does not hold in (9.21). Therefore, the consistency property of minimal barriers with respect to $\mathcal{F}$ does not extend, in general, to smooth mean curvature flows which are not compact.

9.1. Translation invariance

Sometimes, imposing further properties on the elements of the family $\mathfrak{F}$ implies corresponding properties on the minimal barriers.

Let us start by showing that, under a suitable assumption on the family of test evolutions, the minimal barrier satisfies the semigroup property.

Definition 9.20 (Splitting property). We say that the family $\mathfrak{F}$ has the splitting property if

$$f : [a, b] \subset I \to \mathcal{P}(\mathbb{R}^n),\ f \in \mathfrak{F},\ t \in (a,b) \Rightarrow f_{|[a,t]} \in \mathfrak{F} \text{ and } f_{|[t,b]} \in \mathfrak{F}. \tag{9.23}$$

Example 9.21. The family $\mathfrak{F} = \mathcal{F}$ has the splitting property.

(8) Observe also that the set E is the closed 1-sublevel set of the function $(z_1, z_2) \to \frac{z_2^2}{(v_0(z_1))^2}$, which is continuous but not uniformly continuous in $\mathbb{R}^2$.

(9) See [127], and also Section 4.1.

Lemma 9.22 (Semigroup property). *Assume that $\mathfrak{F}$ has the splitting property. Then, for any $E \subseteq \mathbb{R}^n$ and $t_1, t_2 \in I$ with $t_0 \le t_1 \le t_2$, we have*

$$\mathcal{M}(E, \mathfrak{F}, t_0)(t_2) = \mathcal{M}\Big(\mathcal{M}(E, \mathfrak{F}, t_0)(t_1), \mathfrak{F}, t_1\Big)(t_2).$$

Proof. Define the map $\phi : I \to \mathcal{P}(\mathbb{R}^n)$ as follows:

$$t \in I \to \phi(t) := \begin{cases} \mathcal{M}(E, \mathfrak{F}, t_0)(t) & \text{if } t \in [t_0, t_1], \\ \mathcal{M}\Big(\mathcal{M}(E, \mathfrak{F}, t_0)(t_1), \mathfrak{F}, t_1\Big)(t) & \text{if } t > t_1. \end{cases}$$

Then $\phi(t_0) = E$ by (9.17), and using (9.16), (9.19) and (9.23) we have

$$\phi \in \mathcal{B}(\mathfrak{F}, I).$$

Hence

$$\mathcal{M}(E, \mathfrak{F}, t_0)(t_2) \subseteq \phi(t_2) = \mathcal{M}\Big(\mathcal{M}(E, \mathfrak{F}, t_0)(t_1), \mathfrak{F}, t_1\Big)(t_2).$$

Conversely, since $\mathcal{M}(E, \mathfrak{F}, t_0)$ is a barrier for the family $\mathfrak{F}$ in $[t_1, +\infty)$ which coincides with $\phi(t_1)$ at $t = t_1$, we have

$$\mathcal{M}(E, \mathfrak{F}, t_0)(t_2) \supseteq \mathcal{M}(\phi(t_1), \mathfrak{F}, t_1)(t_2) = \phi(t_2). \qquad \square$$

Definition 9.23 (Translation invariance). We say that the family $\mathfrak{F}$ is translation invariant if the following property holds:

$$y \in \mathbb{R}^n, \quad f : [a, b] \to \mathcal{P}(\mathbb{R}^n), \ f \in \mathfrak{F} \ \Rightarrow \ f + y \in \mathfrak{F}.$$

Example 9.24. The family $\mathfrak{F} = \mathcal{F}$ is translation invariant.

The last item of the next proposition will be useful, for instance, in Theorem 10.4.

Proposition 9.25. *Assume that $\mathfrak{F}$ is translation invariant. Then*

(i) *for any $y \in \mathbb{R}^n$ we have*

$$\phi \in \mathcal{B}(\mathfrak{F}, J) \iff \phi + y \in \mathcal{B}(\mathfrak{F}, J);$$

(ii) *for any $y \in \mathbb{R}^n$ and any $E \subseteq \mathbb{R}^n$ we have*

$$\mathcal{M}(E + y, \mathfrak{F}) = \mathcal{M}(E, \mathfrak{F}) + y;$$

(iii) *for any $E \subseteq \mathbb{R}^n$ and any $\rho > 0$ we have*

$$\mathcal{M}(E, \mathfrak{F})^+_\rho \subseteq \mathcal{M}(E^+_\rho, \mathfrak{F}). \tag{9.24}$$

Proof. Assertion (i) is immediate. Now, given $\phi \in \mathcal{B}(\mathfrak{F}, I)$, and setting

$$\psi(t) := \phi(t) + y, \qquad t \in I,$$

we have, using also (i),

$$\begin{aligned}\mathcal{M}(E, \mathfrak{F})(t) =& \bigcap \{\psi(t) - y : \ \ \psi \in \mathcal{B}(\mathfrak{F}, I)\,,\ \psi(t_0) \supseteq E + y\}\\ =& \bigcap \{\psi(t) : \ \ \psi \in \mathcal{B}(\mathfrak{F}, I)\,,\ \psi(t_0) \supseteq E + y\} - y\\ =& \mathcal{M}(E + y, \mathfrak{F})(t) - y,\end{aligned}$$

which is property (ii).

If $\rho > 0$, $y \in \mathbb{R}^n$ and $t \in I$, we have, using (ii),

$$\begin{aligned}\bigcup_{y \in B_\rho(0)} \mathcal{M}(E + y, \mathfrak{F})(t) = & \bigcup_{y \in B_\rho(0)} \Big(\mathcal{M}(E, \mathfrak{F})(t) + y\Big)\\ & = \mathcal{M}(E, \mathfrak{F})(t) + B_\rho(0)\\ & = \mathcal{M}(E, \mathfrak{F})(t)^+_\rho.\end{aligned} \tag{9.25}$$

Observe now that $E + y \subset E + B_\rho(0) = E^+_\rho$ for any $y \in B_\rho(0)$, hence

$$\mathcal{M}(E + y, \mathfrak{F}) \subseteq \mathcal{M}(E^+_\rho, \mathfrak{F}), \qquad y \in B_\rho(0).$$

As a consequence,

$$\bigcup_{y \in B_\rho(0)} \mathcal{M}(E + y, \mathfrak{F}) \subseteq \mathcal{M}(E^+_\rho, \mathfrak{F}),$$

and (iii) follows from (9.25). □

An example of strict inclusion in (9.24) can be constructed by taking $n = 1$, $\mathfrak{F} = \mathcal{F}$ and E a singleton.

Another consequence of the translation invariance is given by the following monotonicity property of the distance.

Lemma 9.26 (Monotonicity of the distance). *Suppose that $\mathfrak{F}$ has the splitting property and is translation invariant. Let $\phi \in \mathcal{B}(\mathfrak{F}, J)$, and let $f : [a, b] \subseteq J \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathfrak{F}$, be such that*

$$f(a) \subseteq \phi(a).$$

Define

$$\delta(t) := \operatorname{dist}\big(f(t), \mathbb{R}^n \setminus \phi(t)\big)\,, \qquad t \in [a, b].$$

Then

$$t_1, t_2 \in [a, b],\ t_1 < t_2,\ \ \delta(t_1) > 0 \ \ \Rightarrow \ \ \delta(t_1) \leq \delta(t_2). \tag{9.26}$$

Proof. Assume by contradiction that there are $t_1, t_2 \in [a, b]$ such that $t_1 < t_2, \delta(t_1) > 0$, and

$$\delta(t_1) > \delta(t_2). \tag{9.27}$$

The definition of $\delta(t_1)$ implies

$$f(t_1)^+_{\delta(t_1)} \subseteq \phi(t_1). \tag{9.28}$$

Write

$$f(t_1)^+_{\delta(t_1)} = \bigcup_{\nu \in \mathbb{S}^{n-1}, \lambda \in [0, \delta(t_1))} (f(t_1) + \lambda\nu). \tag{9.29}$$

Since $\mathfrak{F}$ is translation invariant, the map

$$t \in [a, b] \to f(t) + \lambda\nu$$

belongs to $\mathfrak{F}$ for any $\nu \in \mathbb{S}^{n-1}$ and $\lambda \in [0, \delta(t_1))$. Hence, by (9.28), (9.29) and by the splitting property of $\mathfrak{F}$ and the barrier property,

$$f(t) + \lambda\nu \subseteq \phi(t), \qquad t \in [t_1, b], \ \nu \in \mathbb{S}^{n-1}, \ \lambda \in [0, \delta(t_1)).$$

Therefore

$$\bigcup_{\nu \in \mathbb{S}^{n-1}, \lambda \in [0, \delta(t_1))} (f(t) + \lambda\nu) \subseteq \phi(t), \qquad t \in [t_1, b]. \tag{9.30}$$

Considering inclusion (9.30) at time t_2 implies $\delta(t_2) \geq \delta(t_1)$, a contradiction with (9.27). $\square$

Note that, in addition to (9.26), the barrier property implies $f(t) \subseteq \phi(t)$ for any $t \in [a, b]$.

Definition 9.27 (Elements with compact boundary). We say that the family $\mathfrak{F}$ has elements with compact boundary if the following property holds:

$$\begin{aligned} &f : [a, b] \to \mathcal{P}(\mathbb{R}^n),\ f \in \mathfrak{F} \Rightarrow f(t) \ \text{ is closed} \\ &\text{and } \partial f(t) \text{ is compact for any } t \in [a, b]. \end{aligned}$$

Definition 9.28 (Compact elements). We say that the family $\mathfrak{F}$ has compact elements if the following property holds:

$$f : [a, b] \to \mathcal{P}(\mathbb{R}^n),\ f \in \mathfrak{F} \Rightarrow f(t) \text{ is compact for any } t \in [a, b].$$

Example 9.29. The family $\mathcal{F}$ has elements with compact boundary, and the family $\mathcal{F}_c$ has compact elements.

Lemma 9.30 (Interior of a barrier is a barrier). *Assume that $\mathfrak{F}$ has the splitting property, is translation invariant and has elements with compact boundary. Then*

(i) $\phi \in \mathcal{B}(\mathfrak{F}, J) \Rightarrow \mathrm{int}(\phi) \in \mathcal{B}(\mathfrak{F}, J)$;
(ii) *if $A \subseteq \mathbb{R}^n$ is an open set then $\mathcal{M}(A, \mathfrak{F})(t)$ is an open set for any $t \in I$.*

Proof. Let $\phi \in \mathcal{B}(\mathfrak{F}, J)$. To prove (i) we have to show that, if

$$f : [a, b] \subseteq J \to \mathcal{P}(\mathbb{R}^n), \quad f \in \mathfrak{F} \quad \text{and } f(a) \subseteq \mathrm{int}\,(\phi(a))\,,$$

then

$$f(b) \subseteq \mathrm{int}\,(\phi(b))\,.$$

Since $\partial f(a)$ is compact, we have $\mathrm{dist}\,(f(a), \mathbb{R}^n \setminus \phi(a)) > 0$. By Lemma 9.26 it follows that $\mathrm{dist}\,(f(b), \mathbb{R}^n \setminus \phi(b)) > 0$, hence $f(b) \subseteq \mathrm{int}\,(\phi(b))$.

As $\mathcal{M}(A, \mathfrak{F}) \in \mathcal{B}(\mathfrak{F}, I)$, by property (i) it follows that

$$\phi := \mathrm{int}\,(\mathcal{M}(A, \mathfrak{F})) \in \mathcal{B}(\mathfrak{F}, I).$$

Since A is open, from (9.17) we deduce

$$\phi(t_0) = \mathrm{int}(A) = A,$$

and therefore $\phi = \mathcal{M}(A, \mathfrak{F})$. In particular, $\mathcal{M}(A, \mathfrak{F})(t)$ is open for any $t \in I$. □

Remark 9.31. Let $A \subseteq \mathbb{R}^n$ be an open set. Then

$$\mathcal{M}(A, \mathcal{F}) = \bigcup_{K \subset A,\ K \text{ compact}} \mathcal{M}(K, \mathcal{F}). \tag{9.31}$$

Indeed, denote by ϕ the right-hand side of (9.31). From (9.18) it follows that $\mathcal{M}(A, \mathcal{F}) \supseteq \phi$. On the other hand, the converse inclusion follows by observing that $\phi \in \mathcal{B}(\mathcal{F})$. To show this, let us first suppose that A is bounded. We observe that, as a consequence of (9.18), we have $\phi = \bigcup_{\rho>0} \mathcal{M}(A^-_\rho, \mathcal{F})$. Therefore, given $f : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ with $f(a) \subseteq \bigcup_{\rho>0} \mathcal{M}(A^-_\rho, \mathcal{F})(a)$, by the compactness of $\partial f(a)$ and (ii) of Lemma 9.30, there exists $\rho > 0$ such that $f(a) \subset \mathcal{M}(A^-_\rho, \mathcal{F})(a)$. Hence $f(b) \subset \mathcal{M}(A^-_\rho, \mathcal{F})(b)$, and therefore $f(b) \subseteq \phi(b)$.

Assume now that A is an arbitrary open set. Then

$$\phi = \bigcup_{A' \subset A,\ A' \text{ open bounded}} \mathcal{M}(A', \mathcal{F}),$$

and the assertion follows by arguing as in the previous case.

The next proposition asserts the left upper semicontinuity[(10)] of the set-valued map

$$t \in (t_0, +\infty) \to \mathbb{R}^n \setminus \mathcal{M}(A, \mathfrak{F})(t) \in \mathcal{P}(\mathbb{R}^n),$$

when $A \subseteq \mathbb{R}^n$ is an open set.

Proposition 9.32 (Upper semicontinuity). *Suppose that $\mathfrak{F}$ satisfies all assumptions of Lemma* 9.30. *Let $A \subseteq \mathbb{R}^n$ be an open set, $(t, z) \in (t_0, +\infty) \times \mathbb{R}^n$, and $(t_h) \subset I$, $(z_h) \subset \mathbb{R}^n$ be two sequences with*

$$t_h < t, \qquad z_h \in \mathbb{R}^n \setminus \mathcal{M}(A, \mathfrak{F})(t_h), \qquad h \in \mathbb{N},$$

and such that

$$\lim_{h \to +\infty} (t_h, z_h) = (t, z).$$

Then

$$z \in \mathbb{R}^n \setminus \mathcal{M}(A, \mathfrak{F})(t).$$

Proof. Define the two maps $\psi, \widetilde{\psi} : I \to \mathcal{P}(\mathbb{R}^n)$ as follows:

$$\psi := \mathbb{R}^n \setminus \mathcal{M}(A, \mathfrak{F}),$$

$$\widetilde{\psi}(t) := \begin{cases} \mathbb{R}^n \setminus A & \text{if } t = t_0, \\ \displaystyle\bigcap_{0<\tau<t-t_0} \overline{\bigcup_{s \in (t-\tau, t]} \psi(s)} & \text{if } t > t_0. \end{cases}$$

Observe that $\widetilde{\psi}(t_0) = \psi(t_0) = \overline{\psi(t_0)}$ and $\widetilde{\psi} \supseteq \psi$, *i.e.*,

$$\mathbb{R}^n \setminus \widetilde{\psi} \subseteq \mathcal{M}(A, \mathfrak{F}). \tag{9.32}$$

We claim that

$$\widetilde{\psi} = \psi. \tag{9.33}$$

To show the claim, let us prove that

$$\mathbb{R}^n \setminus \widetilde{\psi} \in \mathcal{B}(\mathfrak{F}, I). \tag{9.34}$$

(10) We say that a map $\phi : I \to \mathcal{P}(\mathbb{R}^n)$ is left upper semicontinuous in I if $\phi(t_0) = \overline{\phi(t_0)}$ and $\phi(t) \supseteq \displaystyle\bigcap_{0<\tau<t-t_0} \overline{\bigcup_{s \in (t-\tau, t]} \phi(s)}$ for any $t \in I, t > t_0$ (see also [19, 15]).

Let $f : [a, b] \subset I \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathfrak{F}$, and assume that

$$f(a) \subseteq \mathbb{R}^n \setminus \widetilde{\psi}(a). \tag{9.35}$$

Suppose first that $a > t_0$. Then the right-hand side of (9.35) equals

$$\bigcup_{0<\tau<a-t_0} \mathrm{int}\left(\bigcap_{s\in(a-\tau,a]} \big(\mathbb{R}^n \setminus \psi(s)\big)\right).$$

It follows that there exists $\tau \in (0, a - t_0)$ such that

$$f(a) \subseteq \mathrm{int}\left(\bigcap_{s\in(a-\tau,a]} \big(\mathbb{R}^n \setminus \psi(s)\big)\right).$$

Since $\partial f(a) \subset f(a)$ and $\partial f(a)$ is compact, there exists $\rho > 0$ such that

$$f(a)^+_\rho \subseteq \bigcap_{s\in(a-\tau,a]} (\mathbb{R}^n \setminus \psi(s)).$$

Hence

$$f(a)^+_\rho \subseteq \mathbb{R}^n \setminus \psi(s) = \mathcal{M}(A, \mathfrak{F})(s), \qquad s \in (a-\tau, a].$$

Since $\mathcal{M}(A, \mathfrak{F}) \in \mathcal{B}(\mathfrak{F}, I)$, and $\mathfrak{F}$ has the splitting property and is translation invariant, applying Lemma 9.26 it follows that

$$f(b)^+_\rho \subseteq \mathcal{M}(A, \mathfrak{F})(s), \qquad s \in (b-\tau, b].$$

This implies that

$$f(b)^+_\rho \subseteq \bigcap_{s\in(b-\tau,b]} \mathcal{M}(A, \mathfrak{F})(s) = \bigcap_{s\in(b-\tau,b]} \big(\mathbb{R}^n \setminus \psi(s)\big), \quad s \in (b-\tau, b],$$

and hence $f(b) \subseteq \mathbb{R}^n \setminus \widetilde{\psi}(b)$.

Suppose now $a = t_0$. Since A is open and $\partial f(a)$ is compact, there exists $\rho > 0$ such that $f(a)^+_\rho \subseteq A = \mathcal{M}(A, \mathfrak{F})(a)$, and therefore $f(s)^+_\rho \subseteq \mathcal{M}(A, \mathfrak{F})(s)$ for any $s \in [a, b]$. The smoothness of f implies that there exists $\tau > 0$ such that $f(b)^+_{\rho/2} \subseteq f(s)^+_\rho$ for any $s \in (b-\tau, b]$. We deduce that $f(b)^+_{\rho/2} \subseteq \mathcal{M}(A, \mathfrak{F})(s)$ for any $s \in (b-\tau, b]$, hence $f(b) \subseteq \mathbb{R}^n \setminus \widetilde{\psi}(b)$. The proof of (9.34) is therefore concluded.

From (9.34) we deduce

$$\mathbb{R}^n \setminus \widetilde{\psi} \supseteq \mathcal{M}(A, \mathfrak{F}).$$

Hence, recalling (9.32), claim (9.33) is proven. The assertion of the proposition then follows, since the set-valued map $\widetilde{\psi}$ is left upper semicontinuous by definition. $\square$

Example 9.33 (Empty minimal barrier, II). Let $t_0 = 0$, $R_0 > 0$, $z_0 \in \mathbb{R}^n$, $A = B_R(z_0)$, and $R(t) = \sqrt{R_0^2 - 2(n-1)t}$ for any $t \in [0, t^\dagger)$, where $t^\dagger := \frac{R_0^2}{2(n-1)}$. We can use Proposition 9.32 to show that

$$\mathcal{M}(A, \mathcal{F})(t^\dagger) = \emptyset. \tag{9.36}$$

Indeed, take a sequence $(t_h) \subset (0, t^\dagger)$ converging to $t^\dagger$ as $h \to +\infty$, and a sequence (z_h) of points with $z_h \in \partial B_{R(t_h)}(z_0)$, so that in particular

$$z_h \in \mathbb{R}^n \setminus B_{R(t_h)}(z_0) = \mathbb{R}^n \setminus \mathcal{M}(A, \mathcal{F})(t_h),$$

and (z_h) converges to z_0 as $h \to +\infty$. Then Proposition 9.32 implies

$$z_0 \notin \mathcal{M}(A, \mathcal{F})(t^\dagger),$$

and (9.36) follows.

9.2. Examples

In order to clarify the definition of minimal barrier in this section we give some examples. Our first example should be compared with Example 9.38.

Example 9.34 (The closed ball without an interior point). Let $n = 2$, $t_0 = 0$, $p := (0, 0)$, $R_0 > 0$ and

$$E := \overline{B_{R_0}(p)} = \left\{(z_1, z_2) \in \mathbb{R}^2 : z_1^2 + z_2^2 \le R_0^2\right\}.$$

Recall that[11]

$$\mathcal{M}(E, \mathcal{F})(t) = \begin{cases} \overline{B_{R(t)}(p)}, & t \in [0, t^\dagger), \\ \emptyset, & t \ge t^\dagger, \end{cases} \tag{9.37}$$

where $t^\dagger = \frac{R_0^2}{2}$, and $R(t) = \sqrt{R_0^2 - 2t}$ for $t \in [0, t^\dagger)$.

We claim that

$$\mathcal{M}\big(E \setminus \{p\}, \mathcal{F}\big)(t) = \mathcal{M}(E, \mathcal{F})(t), \qquad t > 0, \tag{9.38}$$

$$\mathcal{M}\big(\mathrm{int}(E) \setminus \{p\}, \mathcal{F}\big)(t) = \mathcal{M}\left(\mathrm{int}(E), \mathcal{F}\right)(t), \qquad t > 0. \tag{9.39}$$

[11] One checks that the right-hand side of (9.37) is a barrier.

Let us show (9.38). From property (9.18) it follows that

$$\mathcal{M}\big(E \setminus \{p\}, \mathcal{F}\big) \subseteq \mathcal{M}(E, \mathcal{F}),$$

in particular $\mathcal{M}\big(E \setminus \{p\}, \mathcal{F}\big)(t)$ is empty for $t \geq t^\dagger$.
Let us prove that

$$\mathcal{M}\big(E \setminus \{p\}, \mathcal{F}\big)(t) \supseteq \overline{B_{R(t)}(p)} \setminus \{p\}, \qquad t \in \big(0, t^\dagger\big). \tag{9.40}$$

Let $t \in (0, t^\dagger)$ and $q \in \overline{B_{R(t)}(p)} \setminus \{p\}$. We have to show that $q \in \mathcal{M}\big(E \setminus \{p\}, \mathcal{F}\big)(t)$. We construct a circular annulus centered at the origin, initially contained in $E \setminus \{p\}$, the curvature evolution of which contains q at time t. We have $0 < |q| \leq \sqrt{R_0^2 - 2t}$. Choose $\overline{R}$ with

$$2t < \overline{R}^2 < |q|^2 + 2t.$$

Let

$$f(0) := \{z \in E : |z| \geq \overline{R}\},$$

and let $f : [0, \overline{R}^2/2) \to \mathcal{P}(\mathbb{R}^2)$ be the smooth curvature evolution starting from $f(0)$, so that $f_{|[0, \frac{\overline{R}^2}{2} - \epsilon]} \in \mathcal{F}$ for any $\epsilon \in (0, \frac{\overline{R}^2}{2})$. Then $q \in f(t)$ by construction, since $\overline{R}^2 - 2t < |q|^2$. As $f(0) \subset E \setminus \{p\}$, by the barrier property it follows that $f(t) \subseteq \mathcal{M}(E \setminus \{p\}, \mathcal{F})(t)$. Therefore

$$q \in \mathcal{M}(E \setminus \{p\}, \mathcal{F})(t),$$

and (9.40) follows.

To conclude the proof of (9.38), it is sufficient to show that

$$p \in \mathcal{M}\big(E \setminus \{p\}, \mathcal{F}\big)(t), \qquad t \in (0, t^\dagger). \tag{9.41}$$

This time we will let evolve a suitable circular annulus centered at a point slightly different from the origin.

Take $a \in [0, t^\dagger)$, $\epsilon \in (0, R(a)/3)$, and define

$$f_\epsilon(a) := \Big\{z = (z_1, z_2) \in \overline{B_{R(a)}(p)} : (z_1 - \epsilon)^2 + z_2^2 \geq 4\epsilon^2\Big\}.$$

Then $\partial f_\epsilon(a) \in \mathcal{C}^\infty$, and $p \notin f_\epsilon(a)$. Denote by

$$f_\epsilon : [0, 2\epsilon^2) \to \mathcal{P}(\mathbb{R}^2)$$

the smooth curvature evolution starting from $f_\epsilon(a)$, so that $f_{\epsilon|[0,2\epsilon^2-\delta]} \in \mathcal{F}$ for any $\delta \in (0, 2\epsilon^2)$.

Now, we observe that given $\tau \in (0, R(a)^2/2)$, we can find $\epsilon(\tau) \in (0, R(a)/3)$ so that $f_{\epsilon(\tau)}(a+\tau)$ contains the origin. Hence

$$p \in f_{\epsilon(\tau)}(a+\tau) \subset \mathcal{M}\big(E \setminus \{p\}, \mathcal{F}\big)(a+\tau).$$

Since a is arbitrary, inclusion (9.41) follows.

Formula (9.39) can be proved in a similar way as (9.38), with minor modifications.

Example 9.35 (The closed ball without an interior segment). Let $n = 2$, $t_0 = 0$, $p = (0,0)$, $R_0 > 0$ and $E := \overline{B_{R_0}(p)}$. Denote by S a closed segment contained in the interior of E. Then

$$\mathcal{M}\big(E \setminus S, \mathcal{F}\big)(t) = \mathcal{M}\big(E, \mathcal{F}\big)(t), \qquad t > 0, \tag{9.42}$$

$$\mathcal{M}\left(\mathrm{int}(E) \setminus S, \mathcal{F}\right)(t) = \mathcal{M}\left(\mathrm{int}(E), \mathcal{F}\right)(t), \qquad t > 0. \tag{9.43}$$

Equality (9.42) can be proved with arguments similar to those used in the proof of (9.38). In particular, we have

$$S \subset \mathcal{M}(E \setminus S, \mathcal{F})(t), \qquad t \in (0, t^\dagger). \tag{9.44}$$

To show (9.44), let us take $a \in [0, t^\dagger)$ and a smooth set $f(a) \subseteq E \setminus S$, such that

$$\partial f(a) = \{|z| = R(a)\} \cup \partial A_\epsilon,$$

where $\epsilon > 0$ is small enough, and $A_\epsilon \subset \mathrm{int}(E)$ is an ellipse, with $S_\epsilon^+ \subset A_\epsilon$ and such that:

- the area $|A_\epsilon|$ of the ellipse is of order $\mathcal{O}(\epsilon)$,
- the two focuses of A_ϵ lie on a line l parallel to the line containing S,
- $0 < \mathrm{dist}(l, S) = \mathcal{O}(\epsilon)$.

Recall from (3.20) that[(12)] $t_{\max}(\partial A_\epsilon) = \frac{|A_\epsilon|}{2\pi}$.

Denote by $f_\epsilon : [0, t_{\max}(\partial A_\epsilon)) \to \mathcal{P}(\mathbb{R}^2)$ the smooth curvature evolution starting from $f_\epsilon(a)$, so that $f_{|[0, t_{\max}(\partial A_\epsilon)]-\delta} \in \mathcal{F}$ for any $\delta \epsilon (0, t_{\max}(\partial A_\epsilon))$. Then, given $\tau \in (0, \frac{R(a)^2}{2})$, we can find $\epsilon(\tau) > 0$ such that $f_\epsilon(a+\tau)$ contains S. Hence

$$S \subset \mathcal{M}\left(E \setminus S, \mathcal{F}\right)(a+\tau).$$

Since a is arbitrary, equality (9.42) follows. The proof of (9.43) is similar.

(12) The smooth mean curvature evolution of a bounded convex set $K \subset \mathbb{R}^n$ with $\partial K \in \mathcal{C}^\infty$, exists up to the time when it reduces to a point [149, 168].

The following examples are motivations for introducing the regularizations in Chapter 10, since $\mathcal{M}(E, \mathcal{F})$ turns out to be very sensible with respect to modifications of the set E on subsets with zero Lebesgue measure, while the regularizations are not.

Example 9.36 (The closed square instantly looses the boundary). Let $n = 2$, $t_0 = 0$, and

$$E := [-1, 1]^2.$$

Then

$$\mathcal{M}(E, \mathcal{F})(t) = \mathcal{M}\big(\mathrm{int}(E), \mathcal{F}\big)(t), \qquad t > 0. \tag{9.45}$$

In particular, recalling (ii) of Proposition 9.30, we have that $\mathcal{M}(E, \mathcal{F})(t)$ is open for any $t > 0$.

To prove (9.45) it is enough to show that the map

$$t \in I \to \begin{cases} E & \text{if } t = 0, \\ \mathcal{M}\big(\mathrm{int}(E), \mathcal{F}\big)(t) & \text{if } t > 0 \end{cases} \tag{9.46}$$

belongs to $\mathcal{B}(\mathcal{F})$. Denote by $\{C^\rho\}_{\rho \in (0,1)}$ a family of open convex subsets of $\mathbb{R}^2$ of class $\mathcal{C}^\infty$, symmetric with respect to both the z_1 and z_2-axes, with the following four properties:

- if $0 < \rho' < \rho < 1$ then $C^\rho \subset C^{\rho'} \subset E$;
- $\displaystyle\bigcup_{\rho \in (0,1)} C^\rho = \mathrm{int}(E)$;
- $\partial C^\rho \cap \partial E = \big([-1+\rho, 1-\rho] \times \{-1, 1\}\big) \cup \big(\{-1, 1\} \times [-1+\rho, 1-\rho]\big)$;
- if $0 < \rho' < \rho < 1$ then $\partial C^\rho \cap \partial C^{\rho'} \cap \mathrm{int}(E) = \emptyset$.

Essentially, the boundary of C^ρ is obtained from ∂E by smoothing the corners. Denote by

$$t \in [0, t^\dagger_\rho) \subset I \to f_\rho(t)$$

the smooth curvature evolution starting from $\overline{C^\rho}$ at time zero, where $t^\dagger_\rho := t_{\max}(\partial C^\rho)$[13].

If $F \subseteq E$ and $\partial F \in \mathcal{C}^\infty$, then ∂F cannot contain any corner of E. It follows that $F \subseteq f_\rho(0)$ for some $\rho \in (0, 1)$. Consequently, since the smooth curvature evolution starting from F is contained in the smooth evolution starting from $f_\rho(0)$, in order to prove that the map in (9.46)

(13) Remember that the curvature evolution of C^ρ exists up to the time when it is reduced to a point.

belongs to $\mathcal{B}(\mathcal{F})$, it is enough to show the following assertion: for any $\rho \in (0, 1)$,

$$f_\rho(b) \subseteq \mathcal{M}\big(\mathrm{int}(E), \mathcal{F}\big)(b), \qquad b \in (0, t_\rho^\dagger). \tag{9.47}$$

Let us fix $\rho \in (0, 1)$ and $b \in (0, t_\rho^\dagger)$. As a consequence of Theorem 6.3 we have $t_\rho^\dagger < t_{\rho/2}^\dagger$ and

$$\mathrm{dist}\,\big(\partial f_\rho(t), \partial f_{\rho/2}(t)\big) > 0, \qquad t \in (0, b]. \tag{9.48}$$

For any $\lambda \in (0, 1]$ let

$$F_\lambda(0) := \lambda f_{\rho/2}(0).$$

Clearly $F_\lambda(0) \subseteq F_1(0)$ and, if $\lambda \in (0, 1)$, then

$$F_\lambda(0) \subseteq \mathrm{int}(E). \tag{9.49}$$

Remembering Theorem 7.11, if λ is sufficiently close to 1, the curvature evolution $F_\lambda(t)$ starting from $F_\lambda(0)$ is (nonempty and) smooth for any $t \in [0, b]$. Note also that (9.49) and Lemma 9.30 imply

$$F_\lambda(b) \subseteq \mathcal{M}(\mathrm{int}(E), \mathcal{F})(b). \tag{9.50}$$

Now $F_\lambda(\lambda^2 t)$ coincides with $\lambda F_1(t)$[14], and $F_1(t) = f_{\rho/2}(t)$ is smooth for $t \in [0, b]$. It follows that there exists $\lambda \in (0, 1)$ such that the quantity $\sup_{t\in[0,b]} d_{\mathcal{H}}\big(\partial F_\lambda(t), \partial f_{\rho/2}(t)\big)$ is arbitrarily small, where

$$\begin{aligned} & d_{\mathcal{H}}\big(\partial F_\lambda(t), \partial f_{\rho/2}(t)\big) \\ & := \max\left\{ \max_{x\in\partial F_\lambda(t)} \mathrm{dist}(x, \partial f_{\rho/2}(t)),\ \max_{x\in\partial f_{\rho/2}(t)} \mathrm{dist}(x, \partial F_\lambda(t)) \right\} \end{aligned}$$

is the Hausdorff distance[15] between $\partial F_\lambda(t)$ and $\partial f_{\rho/2}(t)$. Hence, by (9.48), there exists $\lambda \in (0, 1)$ such that

$$f_\rho(t) \subseteq F_\lambda(t), \qquad t \in [0, b].$$

From (9.50) it follows that

$$f_\rho(b) \subseteq F_\lambda(b) \subseteq \mathcal{M}\big(\mathrm{int}(E), \mathcal{F}\big)(b).$$

---

[14] Invariance property under parabolic scaling was already used in Proposition 3.24.

[15] See [16].

Remark 9.37. If $C \subseteq \mathbb{R}^n$ is a closed set and $t > t_0$, in general $\mathcal{M}(C, \mathcal{F})(t)$ is neither closed nor open: compare Proposition 9.16 and Example 9.36.

Example 9.38 (The closed ball without one boundary point). Let $n = 2$, $t_0 = 0$, $p = (0, 0)$ and $E := \overline{B_1(p)}$. Let $q \in \partial E$. Then

$$\mathcal{M}(E \setminus q, \mathcal{F})(t) = \mathcal{M}\big(\mathrm{int}(E), \mathcal{F}\big)(t), \qquad t > 0.$$

The proof is similar to that given in Example 9.36. Indeed, we have to prove that the map in (9.46) belongs to $\mathcal{B}(\mathcal{F})$. Denote by $\{C^\rho\}_{\rho \in (0,1)}$ a family of closed subsets of E, with boundary of class $\mathcal{C}^\infty$, satisfying the following four properties:

- if $0 < \rho' < \rho < 1$ then $C^\rho \subset C^{\rho'} \subset E$;
- $\bigcup_{\rho \in (0,1)} C^\rho = \mathrm{int}(E)$;
- $\partial C^\rho \cap \partial E = \partial E \setminus B_{\epsilon_\rho}(q)$, where $\epsilon_\rho > 0$ and $\epsilon_\rho \downarrow 0$ as $\rho \downarrow 0$;
- if $0 < \rho' < \rho < 1$ then $\partial C^\rho \cap \partial C^{\rho'} \cap \mathrm{int}(E) = \emptyset$.

If $F \subset E$ and $\partial F \in \mathcal{C}^\infty$, then $F \subseteq C^\rho$ for some $\rho \in (0, 1)$. Hence it is enough to show the following assertion: for any $\rho \in (0, 1)$, if we denote by $f_\rho(t)$, $t \in [0, t^\dagger_\rho) \subset I$, the smooth curvature evolution[(16)] starting from C^ρ at time zero, then for any $b \in (0, t^\dagger_\rho)$ inclusion (9.47) holds. The proof is similar to that given in Example 9.36.

9.3. On the topological regularity of the minimal barrier

Given $E \subseteq \mathbb{R}^n$, we set

$$E^r := E \cap \overline{\mathrm{int}(E)}. \tag{9.51}$$

Moreover, given a map $\phi : J \to \mathcal{P}(\mathbb{R}^n)$, we let $\phi^r : J \to \mathcal{P}(\mathbb{R}^n)$ be the map defined as

$$\phi(t) := \phi(t)^r, \qquad t \in J. \tag{9.52}$$

Definition 9.39 (Topological regularity). We say that the family $\mathfrak{F}$ is topologically regular if the following property holds:

$$f : [a, b] \to \mathcal{P}(\mathbb{R}^n), \;\; f \in \mathfrak{F}, \;\; t \in [a, b] \Rightarrow f(t) = \overline{\mathrm{int}(f(t))}.$$

Example 9.40. The family $\mathcal{F}$ is topologically regular.

(16) Recall [158] that the smooth curvature flow starting from C^ρ first evolves the set in a convex set, and then shrinks it to a point.

Proposition 9.41. *Suppose that $\mathfrak{F}$ is topologically regular. Then*

$$\phi \in \mathcal{B}(\mathfrak{F}, J) \Rightarrow \phi^r \in \mathcal{B}(\mathfrak{F}, J). \tag{9.53}$$

Moreover, for any $E \subseteq \mathbb{R}^n$, we have

$$\mathcal{M}(E, \mathfrak{F})(t) = \mathcal{M}(E, \mathfrak{F})(t)^r = \mathcal{M}(E^r, \mathfrak{F})(t), \qquad t > t_0. \tag{9.54}$$

In particular,

$$\mathcal{M}(E, \mathfrak{F}) \subseteq \mathcal{M}\left(\overline{\mathrm{int}(E)}, \mathfrak{F}\right),$$

and

$$C \text{ closed} \Rightarrow \mathcal{M}(C, \mathfrak{F}) = \mathcal{M}\left(\overline{\mathrm{int}(C)}, \mathfrak{F}\right). \tag{9.55}$$

Proof. Let $f : [a, b] \subseteq J \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathfrak{F}$ and assume that

$$f(a) \subseteq \phi(a)^r.$$

Since $\phi(a)^r \subseteq \phi(a)$, we have $f(a) \subseteq \phi(a)$, and therefore $f(b) \subseteq \phi(b)$, because $\phi \in \mathcal{B}(\mathfrak{F}, J)$. In particular, $\mathrm{int}(f(b)) \subseteq \mathrm{int}(\phi(b))$. As $\mathfrak{F}$ is topologically regular, it follows that

$$f(b) = \overline{\mathrm{int}(f(b))} \subseteq \overline{\mathrm{int}(\phi(b))}.$$

Hence

$$f(b) \subseteq \phi(b)^r,$$

so that $\phi^r \in \mathcal{B}(\mathfrak{F}, J)$.

Now, let us show (9.54). Since $\mathcal{M}(E, \mathfrak{F}) \supseteq \mathcal{M}(E, \mathfrak{F})^r$, to show the first equality in (9.54) it is sufficient to prove that the map

$$\phi : t \in I \to \phi(t) := \begin{cases} E & \text{if } t = t_0, \\ \mathcal{M}(E, \mathfrak{F})(t)^r & \text{if } t > t_0 \end{cases}$$

belongs to $\mathcal{B}(\mathfrak{F})$. Let $f : [a, b] \subset I \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathfrak{F}$ and assume that $f(a) \subseteq \phi(a)$. If $a > t_0$ then, from (9.53), it follows that $f(b) \subseteq \phi(b)$. If $a = t_0$ then, using the topological regularity of $\mathfrak{F}$, we have

$$f(a) = \overline{\mathrm{int}(f(a))} \subseteq \overline{\mathrm{int}(E)},$$

so that

$$f(a) \subseteq E^r = \phi(a)^r = \mathcal{M}(E, \mathfrak{F})(a)^r.$$

Recalling that $\mathcal{M}(E, \mathfrak{F}) \in \mathcal{B}(\mathfrak{F})$ and using (9.53), it then follows

$$f(b) \subseteq \mathcal{M}(E, \mathfrak{F})(b)^r = \phi(b).$$

Since $\mathcal{M}(E, \mathfrak{F}) \supseteq \mathcal{M}(E^r, \mathfrak{F})$, to show the second equality in (9.54) it is sufficient to prove that the map

$$\psi : t \in I \to \psi(t) := \begin{cases} E & \text{if } t = t_0, \\ \mathcal{M}(E^r, \mathfrak{F})(t) & \text{if } t > t_0 \end{cases}$$

belongs to $\mathcal{B}(\mathfrak{F})$. Let $f : [a, b] \subset I \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathfrak{F}$ and assume that $f(a) \subseteq \psi(a)$. If $a > t_0$ then $f(b) \subseteq \psi(b)$, because $\mathcal{M}(E^r, \mathfrak{F})$ is a barrier for the family $\mathfrak{F}$. If $a = t_0$ then $f(a) \subseteq E^r = \mathcal{M}(E^r, \mathfrak{F})(a)$, and again it follows that $f(b) \subseteq \mathcal{M}(E^r, \mathfrak{F})(b) = \psi(b)$.

The last two assertions of the proposition follow respectively from (9.54), oberving that $E^r \subseteq \overline{\mathrm{int}(E)}$, and that we have the inclusion $C \supseteq \overline{\mathrm{int}(C)}$. □

Note that (9.55) gives another proof of (9.38) and (9.42).

Remark 9.42. As a corollary of Proposition 9.41 and Example 9.40, it follows that if $E, F \subseteq \mathbb{R}^n$ are such that

$$E^r = F^r,$$

then

$$\mathcal{M}(E, \mathcal{F})(t) = \mathcal{M}(F, \mathcal{F})(t), \qquad t > t_0.$$

9.4. On the complement of the minimal barrier

We have seen in Remark 9.10 that, in general, the complement of a barrier is not a barrier. However, the following result holds.

Proposition 9.43 (Complement of the minimal barrier). *Let $E \subseteq \mathbb{R}^n$. Then*

$$\mathbb{R}^n \setminus \mathcal{M}(E, \mathcal{F}) \in \mathcal{B}(\mathcal{F}, I).$$

Therefore

$$\mathcal{M}\left(\mathbb{R}^n \setminus E, \mathcal{F}\right) \subseteq \mathbb{R}^n \setminus \mathcal{M}(E, \mathcal{F}). \tag{9.56}$$

Moreover, this inclusion can be strict.

Proof. Assume by contradiction that $\mathbb{R}^n \setminus \mathcal{M}(E, \mathcal{F}) \notin \mathcal{B}(\mathcal{F})$. Then there exists $f : [a, b] \subset I \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{F}$, satisfying

$$f(a) \subseteq \mathbb{R}^n \setminus \mathcal{M}(E, \mathcal{F})(a),$$

and

$$f(b) \cap \mathcal{M}(E, \mathcal{F})(b) \neq \emptyset.$$

Define

$$f^c := \overline{\mathbb{R}^n \setminus f} : [a, b] \to \mathcal{P}(\mathbb{R}^n).$$

We have:

- $f^c \in \mathcal{F}$;
- $\mathcal{M}(E, \mathcal{F})(a) \subseteq \operatorname{int}(f^c(a))$;
- there exists a point of $\mathcal{M}(E, \mathcal{F})(b)$ which does not belong to $\operatorname{int}(f^c(b))$.

Now, let us define $\phi : I \to \mathcal{P}(\mathbb{R}^n)$ as follows:

$$\phi(t) := \begin{cases} \mathcal{M}(E, \mathcal{F})(t) \cap \operatorname{int}(f^c(t)) & \text{if } t \in [a, b], \\ \mathcal{M}(E, \mathcal{F})(t) & \text{if } t \in I \setminus [a, b]. \end{cases}$$

Since $\phi(b)$ is strictly contained in $\mathcal{M}(E, \mathcal{F})(b)$, to reach a contradiction it is enough to show that

$$\phi \in \mathcal{B}(\mathcal{F}, I).$$

Hence, let $h : [c, d] \subset I \to \mathcal{P}(\mathbb{R}^n)$, $h \in \mathcal{F}$, and assume that $h(c) \subseteq \phi(c)$. We have to show that

$$h(d) \subseteq \phi(d). \tag{9.57}$$

If $[c, d] \cap [a, b] = \emptyset$ then $h(d) \subseteq \phi(d)$, since $\mathcal{M}(E, \mathcal{F}) \in \mathcal{B}(\mathcal{F}, I)$. Suppose now $[c, d] \cap [a, b] \neq \emptyset$. We can assume that $c \geq a$ since, if $c < a < d$ we have $h(a) \subseteq \mathcal{M}(E, \mathcal{F})(a) = \phi(a)$. Hence let

$$a \leq c < d \leq b.$$

Since $h(c) \subseteq \mathcal{M}(E, \mathcal{F})(c)$ we have $h(d) \subseteq \mathcal{M}(E, \mathcal{F})(d)$, and since $h(c) \subseteq \operatorname{int}(f^c(c))$, by the comparison principle between smooth compact mean curvature evolutions (see Theorem 5.4) we have also $h(d) \subseteq \operatorname{int}(f^c(d))$. Hence (9.57) holds, and this leads to a contradiction.

To prove that inclusion (9.56) can be strict, let $n = 2$, $t_0 = 0$, and

$$E := [-1, 1]^2.$$

Then $\mathcal{M}(E, \mathcal{F})(t)$ is open for any $t > 0$ (see Example 9.36), hence

$$\mathbb{R}^2 \setminus \mathcal{M}(E, \mathcal{F})(t) \text{ is closed for any } t > 0.$$

On the other hand $\mathbb{R}^2 \setminus E$ is open, hence $\mathcal{M}(\mathbb{R}^2 \setminus E, \mathcal{F})(t)$ is open for any $t > 0$ (see Proposition 9.30). $\square$

9.5. Appendix: abstract definition of barrier

In this appendix we recall the abstract definitions of barrier and minimal barrier, as originally proposed by De Giorgi in [108].

Let $\mathbb{S}$ be a set, $r \subseteq \mathbb{S}^2$ be the graph of a binary relation r on $\mathbb{S}$. Suppose that any point of $\mathbb{S}$ is either first or second element of a pair belonging to r (hence each point of $\mathbb{S}$ is in relation with some element of $\mathbb{S}$)[(17)].

Definition 9.44. Let $\mathfrak{F}$ be a family of functions of a real variable which satisfies the following property: for any $f \in \mathfrak{F}$ there exist two real numbers a, b such that $a < b$ and

$$f : [a, b] \to \mathbb{S}.$$

We say that a function ϕ is a barrier associated with the pair $(r, \mathfrak{F})$, and we write

$$\phi \in \mathcal{B}(r, \mathfrak{F}),$$

if there exists a convex set $J \subseteq I$ such that $\phi : J \to \mathbb{S}$ and, whenever a, b, f satisfy the conditions

$$[a, b] \subseteq J, \quad f : [a, b] \to \mathbb{S}, \quad f \in \mathfrak{F}, \qquad (f(a), \phi(a)) \in r,$$

then

$$(f(b), \phi(b)) \in r.$$

The case considered in Definition 9.2 corresponds to the choice

$$\mathbb{S} := \mathcal{P}(\mathbb{R}^n), \tag{9.58}$$

and

$$r := \{(E, F) \in \mathbb{S}^2 : E \subseteq F\}. \tag{9.59}$$

Example 9.45 (Ordinary differential equations). Let

$$\psi \in (\mathcal{C}^\infty(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n))^n.$$

Take

$$\mathbb{S} := \mathcal{P}(\mathbb{R}^n)$$

and

$$r := \{(E, F) \in \mathbb{S}^2 : E \subseteq F\}.$$

(17) Namely, $\mathbb{S}$ is the ambient of r, that is $\mathbb{S} = \bigcap \left\{D : D^2 \supseteq r\right\}$.

We choose also $\mathfrak{F}$ as follows: $f \in \mathfrak{F}$ if there exist $a, b \in \mathbb{R}$ with $a < b$ and $x \in \mathcal{C}^\infty([a,b]; \mathbb{R}^n)$, such that

$$\dot{x}(t) = \psi(x(t)), \qquad t \in [a,b],$$

and

$$f(t) = \{x(t)\}, \qquad t \in [a,b].$$

Let A be an open subset of $\mathbb{R}^n$. Then the function $\phi : I \to \mathcal{P}(\mathbb{R}^n)$ defined as

$$\phi(t) := \{z \in \mathbb{R}^n : \operatorname{dist}(z, A) < t\|\psi\|_{L^\infty(\mathbb{R}^n)}\}, \qquad t \in I,$$

is a barrier associated with the pair $(r, \mathfrak{F})$.

The definition of barrier enforces a global inclusion property, from which it is possible, under suitable assumptions, to derive a partial differential inequality. This procedure is in some sense the converse of the usual one, where one derives a comparison property starting from subsolutions and supersolutions.

Example 9.46 (Heat equation). Choose $\mathbb{S} := \mathbb{R}^{\mathbb{R}^n}$ as the set of all functions from $\mathbb{R}^n$ to $\mathbb{R}$, and let

$$r := \{(f, g) \in \mathbb{S}^2 : f \leq g\}$$

be the graph of the usual pointwise order relation between two real-valued functions. We choose $\mathfrak{F}$ as follows: $f \in \mathfrak{F}$ if there exist $a, b \in \mathbb{R}$ with $a < b$ and $f \in \mathcal{C}^\infty([a,b] \times \mathbb{R}^n) \cap L^\infty([a,b] \times \mathbb{R}^n)$ such that

$$\frac{\partial f}{\partial t} - \Delta f \leq 0 \qquad \text{in } [a,b] \times \mathbb{R}^n.$$

If $\phi \in \mathcal{B}(r, \mathfrak{F})$ and

$$\phi \in \mathcal{C}^\infty([a,b] \times \mathbb{R}^n) \cap L^\infty([a,b] \times \mathbb{R}^n),$$

then

$$\frac{\partial \phi}{\partial t} - \Delta \phi \geq 0 \qquad \text{in } [a,b] \times \mathbb{R}^n.$$

Now, we proceed to the definition of the minimal barrier in the general setting of Definition 9.44. For any $T \in \mathbb{S}$ define the set of minorants and majorants of T respectively, as

$$\operatorname{Minor}(r, T) := \{\eta \in \mathbb{S} : (\eta, \chi) \in r \quad \forall \chi \in T\},$$
$$\operatorname{Maior}(r, T) := \{\eta \in \mathbb{S} : (\chi, \eta) \in r \quad \forall \chi \in T\}.$$

Set also

$$\begin{aligned}\mathrm{Mini}(r,T) &:= T \cap \mathrm{Minor}(r,T),\\ \mathrm{Maxi}(r,T) &:= T \cap \mathrm{Maior}(r,T),\end{aligned}$$

and finally

$$\begin{aligned}M^-(r,T) &:= \mathrm{Maxi}(r, \mathrm{Minor}(r,T)),\\ M^+(r,T) &:= \mathrm{Mini}(r, \mathrm{Maior}(r,T)).\end{aligned}$$

Definition 9.47. Let $T \in \mathbb{S}$. If the set $M^-(r,T)$ (respectively $M^+(r,T)$) consists of only one element, such an element will be denoted by $r - \min(T)$ (respectively $r - \max(T)$):

$$\begin{aligned} r - \min(T) = m &\iff \{m\} = M^-(r,T),\\ r - \max(T) = \mu &\iff \{\mu\} = M^+(r,T).\end{aligned} \tag{9.60}$$

We denote by $\mathcal{B}(r,\mathcal{F},I)$ the set of all barriers for which $J = I$ in Definition 9.44.

The definition of minimal barrier reads as follows.

Definition 9.48. Let $\eta \in \mathbb{S}$. If there exists a function $\sigma : I \to \mathbb{S}$ defined, for any $t \in I$, by the formula

$$\sigma(t) = r - \min\{\phi(t) : \ \phi : I \to \mathbb{S},\ \phi \in \mathcal{B}(r,\mathfrak{F},I),\ (\eta,\phi(t_0)) \in r\},$$

we say that σ is the minimal barrier associated with $\eta, r, \mathfrak{F}, I$, and we write

$$\sigma = \mathcal{M}(\eta, r, \mathfrak{F}, I).$$

Example 9.49. Let us consider the case of (9.58) and (9.59), and let $t \in I$. For a given family of maps $\phi_i : J \to \mathcal{P}(\mathbb{R}^n)$, where $i \in \Lambda$ and Λ is a set of indices indicizing the barriers, take

$$T := \{\phi_i(t) : i \in \Lambda\} \subseteq \mathcal{P}(\mathbb{R}^n).$$

Therefore

$$\begin{aligned}\mathrm{Minor}(r,T) &= \{A \subseteq \mathbb{R}^n : A \subseteq \phi_i(t), i \in \Lambda\} \subseteq \mathcal{P}(\mathbb{R}^n),\\ \mathrm{Maior}(r,T) &= \{A \subseteq \mathbb{R}^n : A \supseteq \phi_i(t), i \in \Lambda\} \subseteq \mathcal{P}(\mathbb{R}^n)\end{aligned}$$

and

$$\begin{aligned} M^-(r,T) &= \mathrm{Minor}(r,T) \cap \mathrm{Maior}(r,\mathrm{Minor}(r,T)) = \bigcap_{i\in\Lambda} \phi_i(t),\\ M^+(r,T) &= \mathrm{Maior}(r,T) \cap \mathrm{Minor}(r,\mathrm{Maior}(r,T)) = \bigcup_{i\in\Lambda} \phi_i(t).\end{aligned}$$

Beside motion by mean curvature, applications of the minimal barrier theory have been found in motion by mean curvature in arbitrary codimension [108, 19, 47], and in some special systems of ordinary differential equations [42][18].

[18] In contrast with what happens in Example 9.18, in the example of [42] the minimal barrier is empty at some $\bar{t} > 0$, and nonempty for t belonging to an interval of the form $(\bar{t}, \bar{t} + \delta)$, for some $\delta > 0$. In this case the semigroup property of the minimal barrier fails.

Chapter 10
Inner and outer regularizations

The examples in Chapter 9 show that, for a general initial set E, describing the minimal barrier $\mathcal{M}(E, \mathcal{F})$ can be difficult. Therefore it can be convenient to "regularize" the concept of minimal barrier, by considering inner and outer approximations of E. This is the aim of this chapter. Recall our notation: if $E \subseteq \mathbb{R}^n$ and $\rho > 0$,

$$E_\rho^+ := \{z \in \mathbb{R}^n : \mathrm{dist}(z, E) < \rho\}, \quad E_\rho^- := \{z \in \mathbb{R}^n : \mathrm{dist}(z, \mathbb{R}^n \setminus E) > \rho\}.$$

Morever, as usual we let

$$I = [t_0, +\infty),$$

for some $t_0 \in \mathbb{R}$.

Definition 10.1 (Regularizations). Let $E \subseteq \mathbb{R}^n$. For any $t \in I$ we set

$$\begin{aligned}
\mathcal{M}_* (E, \mathfrak{F}, t_0)(t) &:= \bigcup_{\rho>0} \mathcal{M}\left(E_\rho^-, \mathfrak{F}, t_0\right)(t), \\
\mathcal{M}^* (E, \mathfrak{F}, t_0)(t) &:= \bigcap_{\rho>0} \mathcal{M}\left(E_\rho^+, \mathfrak{F}, t_0\right)(t).
\end{aligned} \tag{10.1}$$

Notation. When no confusion is possible, we drop the dependence on t_0 in the notation; in this case the left-hand sides of (10.1) will be denoted by $\mathcal{M}_*(E, \mathfrak{F})(t)$ and $\mathcal{M}^*(E, \mathfrak{F})(t)$ respectively. Moreover, in this chapter $\mathcal{B}(\mathfrak{F})$ denotes the set of all barriers in the interval I, namely

$$\mathcal{B}(\mathfrak{F}) \text{ stands for } \mathcal{B}(\mathfrak{F}, I).$$

Note that

$$\mathcal{M}^* (E, \mathfrak{F}) \in \mathcal{B}(\mathfrak{F}),$$

and from (9.17) we have

$$\begin{aligned}
\mathcal{M}_*(E, \mathfrak{F})(t_0) &= \bigcup_{\rho>0} E_\rho^- = \mathrm{int}(E), \\
\mathcal{M}^*(E, \mathfrak{F})(t_0) &= \bigcap_{\rho>0} E_\rho^+ = \overline{E}.
\end{aligned}$$

From the definitions it also follows that

$$\begin{aligned} \mathcal{M}_*(E, \mathfrak{F}) &= \mathcal{M}_* (\mathrm{int}\,(E)\,, \mathfrak{F}) \subseteq \mathcal{M}(E, \mathfrak{F}) \subseteq \mathcal{M}^*(E, \mathfrak{F}) \\ &= \mathcal{M}^*(\overline{E}, \mathfrak{F}), \end{aligned} \tag{10.2}$$

$$E \subseteq F \subseteq \mathbb{R}^n \Rightarrow \begin{cases} \mathcal{M}_*(E, \mathfrak{F}) \subseteq \mathcal{M}_*(F, \mathfrak{F}), \\ \mathcal{M}^*(E, \mathfrak{F}) \subseteq \mathcal{M}^*(F, \mathfrak{F}), \end{cases} \tag{10.3}$$

and

$$\begin{aligned} &B, C \in \mathcal{P}(\mathbb{R}^n), C \subset B, \mathrm{dist}(C, \mathbb{R}^n \setminus B) > 0 \\ &\Rightarrow \mathcal{M}^*(C, \mathfrak{F}) \subseteq \mathcal{M}_*(B, \mathfrak{F}). \end{aligned} \tag{10.4}$$

For instance, (10.4) applies when C is compact, B is open and $C \subset B$.

Remark 10.2. The inclusion in (10.4) between the two regularizations is false in general, if C is just closed, B is open and $C \subset B$. This is shown by an example (see Section 10.1), using formulas (10.9) and (10.16), with the choice $B = \mathrm{int}(E)$.

Remark 10.3. Suppose that $\mathfrak{F}$ has the splitting property, is translation invariant and has elements with compact boundary, and let $E \subseteq \mathbb{R}^n$. Then, as a consequence of Lemma 9.30(ii) and the fact that E_ρ^- is open for any $\rho > 0$, we have that $\mathcal{M}_*(E, \mathfrak{F})(t)$ is open for any $t \in I$. This property is valid in more general situations, as shown in the next result.

Theorem 10.4 (Preserving open and closed sets). *Assume that $\mathfrak{F}$ is translation invariant. Then for any $E \subseteq \mathbb{R}^n$ and for any $t \in I$*

$$\begin{aligned} &\mathcal{M}_*(E, \mathfrak{F})(t) \text{ is open.} \\ &\mathcal{M}^*(E, \mathfrak{F})(t) \text{ is closed.} \end{aligned}$$

Proof. Given $\rho > 0$ and $\sigma > 0$, we observe that $E_\sigma^- \supseteq (E_{\sigma+\rho}^-)_{\rho/2}^+$. Therefore[(1)]

$$\mathcal{M}(E_\sigma^-, \mathfrak{F}) \supseteq \mathcal{M}\left((E_{\sigma+\rho}^-)_{\rho/2}^+, \mathfrak{F}\right).$$

By inclusion (9.24) it follows that

$$\mathcal{M}\left((E_{\sigma+\rho}^-)_{\rho/2}^+, \mathfrak{F}\right) \supseteq \mathcal{M}(E_{\sigma+\rho}^-, \mathfrak{F})_{\rho/2}^+.$$

Hence

$$\mathcal{M}(E_\sigma^-, \mathfrak{F}) \supseteq \mathcal{M}(E_{\sigma+\rho}^-, \mathfrak{F})_{\rho/2}^+.$$

(1) Remember the notations that we have introduced in (9.3), (9.4), (9.5) and (9.6).

Consequently

$$\mathcal{M}(E_\sigma^-, \mathfrak{F}) \supseteq \bigcup_{\rho>0} \mathcal{M}(E_{\sigma+\rho}^-, \mathfrak{F})_{\rho/2}^+. \tag{10.5}$$

Since the right-hand side of (10.5) is an open set, we obtain

$$\operatorname{int}\left(\mathcal{M}(E_\sigma^-, \mathfrak{F})\right) \supseteq \bigcup_{\rho>0} \mathcal{M}(E_{\sigma+\rho}^-, \mathfrak{F})_{\rho/2}^+ \supseteq \bigcup_{\rho>0} \mathcal{M}(E_{\sigma+\rho}^-, \mathfrak{F}). \tag{10.6}$$

Being (10.6) valid for any $\sigma > 0$, we get

$$\bigcup_{\sigma>0} \operatorname{int}\big(\mathcal{M}(E_\sigma^-, \mathfrak{F})\big) \supseteq \bigcup_{\sigma,\rho>0} \mathcal{M}(E_{\sigma+\rho}^-, \mathfrak{F}) = \mathcal{M}_*(E, \mathfrak{F}). \tag{10.7}$$

Inclusion (10.7) implies

$$\mathcal{M}_*(E, \mathfrak{F}) = \bigcup_{\sigma>0} \operatorname{int}\left(\mathcal{M}(E_\sigma^-, \mathfrak{F})\right).$$

Hence $\mathcal{M}_*(E, \mathfrak{F})(t)$ is open for any $t \in I$.

Let us show that $\mathcal{M}^*(E, \mathfrak{F})(t)$ is a closed set. If $\rho > 0$, by inclusion (9.24) and equality (1.4), we have

$$\mathcal{M}(E_{2\rho}^+, \mathfrak{F}) \supseteq (\mathcal{M}(E_\rho^+, \mathfrak{F}))_{\rho/2}^+ \supseteq \overline{\mathcal{M}(E_\rho^+, \mathfrak{F})}.$$

Therefore

$$\mathcal{M}^*(E, \mathfrak{F}) = \bigcap_{\rho>0} \mathcal{M}(E_{2\rho}^+, \mathfrak{F}) \supseteq \bigcap_{\rho>0} \overline{\mathcal{M}(E_\rho^+, \mathfrak{F})},$$

so that

$$\mathcal{M}^*(E, \mathfrak{F}) = \bigcap_{\rho>0} \overline{\mathcal{M}(E_\rho^+, \mathfrak{F})}.$$

Hence $\mathcal{M}^*(E, \mathfrak{F})(t)$ is a closed set for any $t \in I$. □

Remark 10.5 (Strict inclusions). For $\mathfrak{F} = \mathcal{F}$ all inclusions in (10.2) can be strict. Indeed, let $n = 2$, $t_0 = 0$, $p = (0, 0)$, and $E := \overline{B_1(p)}$. Then, if $t \in [0, \frac{1}{2})$, we have from Proposition 9.16 that

$$\mathcal{M}(E, \mathcal{F})(t) = \overline{B_{R(t)}(p)},$$

where $R(t) = \sqrt{1 - 2t}$. In particular, $\mathcal{M}(E, \mathcal{F})(t)$ is a closed set. On the other hand $\mathcal{M}_*(E, \mathcal{F})(t)$ is open, so that

$$\mathcal{M}_*(E, \mathcal{F})(t) \subsetneq \mathcal{M}(E, \mathcal{F})(t), \qquad t \in [0, 1/2).$$

Moreover $\mathcal{M}(\mathrm{int}(E), \mathcal{F})(t)$ is open (by Lemma 9.30) and

$$\mathcal{M}^*(\mathrm{int}(E), \mathcal{F})(t) = \mathcal{M}^*(\overline{\mathrm{int}(E)}, \mathcal{F})(t)$$

is closed (by Proposition 10.4), so that

$$\mathcal{M}(\mathrm{int}(E), \mathcal{F})(t) \subsetneq \mathcal{M}^*(\mathrm{int}(E), \mathcal{F})(t), \qquad t \in (0, 1/2).$$

The next remark shows that the inner and outer regularizations avoid some of the phenomena related to sets with zero Lebesgue measure, that we pointed out for the minimal barrier in Chapter 9.

Remark 10.6. In the case of Example 9.34 we have

$$\mathcal{M}_*(E\setminus\{p\}, \mathcal{F}) = \mathcal{M}_*(\mathrm{int}(E), \mathcal{F}), \qquad \mathcal{M}^*(E\setminus\{p\}, \mathcal{F}) = \mathcal{M}^*(E, \mathcal{F}).$$

In the case of Example 9.35 we have

$$\mathcal{M}_*(E \setminus S, \mathcal{F}) = \mathcal{M}_*(\mathrm{int}(E), \mathcal{F}), \qquad \mathcal{M}^*(E \setminus S, \mathcal{F}) = \mathcal{M}^*(E, \mathcal{F}).$$

In the case of Example 9.38 we have

$$\mathcal{M}_*(E\setminus\{p\}, \mathcal{F}) = \mathcal{M}_*(\mathrm{int}(E), \mathcal{F}), \qquad \mathcal{M}^*(E\setminus\{p\}, \mathcal{F}) = \mathcal{M}^*(E, \mathcal{F}).$$

Proposition 10.7. *Assume that $\mathfrak{F}$ is translation invariant and has elements with compact boundary. Then for any $E \subseteq \mathbb{R}^n$ we have*

$$\mathcal{M}_*(E, \mathfrak{F}) \in \mathcal{B}(\mathfrak{F}).$$

Proof. Let $f : [a, b] \subset I \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathfrak{F}$, $f(a) \subseteq \mathcal{M}_*(E, \mathfrak{F})(a)$. By Theorem 10.4 and the compactness of $\partial f(a)$, it follows that there exists $\rho > 0$ such that

$$f(a) \subseteq \mathcal{M}(E^-_\rho, \mathfrak{F})(a).$$

Then

$$f(b) \subseteq \mathcal{M}(E^-_\rho, \mathfrak{F})(b) \subseteq \mathcal{M}_*(E, \mathfrak{F})(b),$$

and the assertion follows. □

Note that, if $\mathfrak{F}$ is translation invariant and has elements with compact boundary, then

$$\mathcal{M}_*(A, \mathfrak{F}) = \mathcal{M}(A, \mathfrak{F})$$

for any open set $A \subseteq \mathbb{R}^n$. Indeed, this is a consequence of Proposition 10.7 and the fact that

$$\mathcal{M}_*(A, \mathfrak{F})(t_0) = \mathrm{int}(A) = A = \mathcal{M}(A, \mathfrak{F})(t_0).$$

Example 10.8. Let $n = 2$, $t_0 = 0$, $R > 0$, $p = (0,0)$, $A = B_R(p)$ and $R(t) = \sqrt{R^2 - 2t}$ for any $t \in [0, t^\dagger)$, where $t^\dagger := \frac{R^2}{2}$. For any $\rho > 0$ we have $p \in \mathcal{M}(A_\rho^+, \mathcal{F})(t^\dagger)$, and therefore

$$\mathcal{M}^*(A, \mathcal{F})(t^\dagger) = \{p\}.$$

On the other hand, recalling Example 9.33, we have

$$\mathcal{M}_*(A, \mathcal{F})(t^\dagger) = \emptyset.$$

We conclude this section by pointing out a first inclusion relation between the evolution of the complement and the complement of the evolution.

Proposition 10.9 (Complement of the evolution). *Let $E \subseteq \mathbb{R}^n$. Then*

$$\mathcal{M}_*(\mathbb{R}^n \setminus E, \mathcal{F}) \subseteq \mathbb{R}^n \setminus \mathcal{M}^*(E, \mathcal{F}). \tag{10.8}$$

Proof. By inclusion (9.56) we have

$$\begin{aligned} \mathcal{M}_*\left(\mathbb{R}^n \setminus E, \mathcal{F}\right) &= \bigcup_{\rho>0} \mathcal{M}\left(\left(\mathbb{R}^n \setminus E\right)_\rho^-, \mathcal{F}\right) \\ &= \bigcup_{\rho>0} \mathcal{M}\left(\mathbb{R}^n \setminus \overline{E_\rho^+}, \mathcal{F}\right) \subseteq \bigcup_{\rho>0} \left[\mathbb{R}^n \setminus \mathcal{M}\left(\overline{E_\rho^+}, \mathcal{F}\right)\right] \\ &= \mathbb{R}^n \setminus \bigcap_{\rho>0} \mathcal{M}\left(\overline{E_\rho^+}, \mathcal{F}\right) = \mathbb{R}^n \setminus \mathcal{M}^*(E, \mathcal{F}). \end{aligned}$$ □

Inspecting the converse inclusion of (10.8) will be one of the aims of Chapter 13.

10.1. On the evolution of an unbounded set

In this section we discuss some further properties of the evolution of the unbounded boundary ∂E considered in Example 9.19, where

$$E := \left\{(z_1, z_2) \in \mathbb{R}^2 : |z_2| \leq v_0(z_1)\right\}.$$

Although we are considering an evolution in the plane, we will see that the unboundedness of ∂E is source of various difficulties.

Let $t_0 = 0$ and let

$$C := \{(z_1, z_2) \in \mathbb{R}^2 : z_2 = 0\}$$

be the horizontal axis. Clearly

$$C \subset E.$$

We intend to illustrate some qualitative properties of $\mathcal{M}_*(E,\mathcal{F})$ and $\mathcal{M}^*(E,\mathcal{F})$. As a consequence we will get some informations on the difference[(2)]

$$\mathcal{M}^*(E,\mathcal{F}) \setminus \mathcal{M}_*(E,\mathcal{F}).$$

The first observation is that it is possible to prove that

$$\mathcal{M}^*(E,\mathcal{F})(t) \supset C, \qquad t \in [0,+\infty). \tag{10.9}$$

Even if inclusion (10.9) seems to be obvious, we give here some details on its proof.
For any $\rho > 0$ the set E^+_ρ contains the horizontal strip

$$C^+_\rho = \{(z_1,z_2) \in \mathbb{R}^2 : |z_2| < \rho\},$$

so that

$$\mathcal{M}(E^+_\rho,\mathcal{F}) \supseteq \mathcal{M}(C^+_\rho,\mathcal{F}). \tag{10.10}$$

We claim that

$$\mathcal{M}(C^+_\rho,\mathcal{F})(t) = C^+_\rho, \qquad t \in [0,+\infty). \tag{10.11}$$

Observe that the constant map

$$t \in [0,+\infty) \to C^+_\rho$$

belongs to $\mathcal{B}(\mathcal{F})$, and hence

$$\mathcal{M}(C^+_\rho,\mathcal{F})(t) \subseteq C^+_\rho, \qquad t \in [0,+\infty).$$

Therefore, in order to prove claim (10.11), one has to show that

$$\mathcal{M}(C^+_\rho,\mathcal{F})(t) \supseteq C^+_\rho, \qquad t \in [0,+\infty). \tag{10.12}$$

Recall that

$$\mathcal{M}(C^+_\rho,\mathcal{F})(0) = C^+_\rho$$

and that, by Lemma 9.30, the set $\mathcal{M}(C^+_\rho,\mathcal{F})(t)$ is open for any $t \in [0,+\infty)$. By symmetry arguments we will assume that $\mathcal{M}(C^+_\rho,\mathcal{F})(t)$ is a horizontal open strip (symmetric with respect to the horizontal axis)

---

(2) A more detailed analysis on the difference $\mathcal{M}^*(E,\mathcal{F}) \setminus \mathcal{M}_*(E,\mathcal{F})$ will be given in Chapter 11, in a specific example.

of width $\alpha(t)$, for a function $\alpha : [0, +\infty) \to [0, +\infty)$ with $\alpha(0) = \rho$, namely

$$\mathcal{M}(C^+_\rho, \mathcal{F})(t) = \{(z_1, z_2) \in \mathbb{R}^2 : |z_2| < \alpha(t)\}, \qquad t \in [0, +\infty).$$

Since by Proposition 9.43 we have

$$\mathbb{R}^2 \setminus \mathcal{M}(C^+_\rho, \mathcal{F}) \in \mathcal{B}(\mathcal{F}),$$

taking sufficiently large balls inside $\mathbb{R}^2 \setminus C^+_\rho$ and tangent to ∂C^+_ρ, one can show, by comparison, that α is nonincreasing. Since $\mathcal{M}(C^+_\rho, \mathcal{F})$ is a barrier for the family $\mathcal{F}$, by taking sufficiently elongated ellipses inside C^+_ρ enclosing an arbitrarily large area, from the monotonicity of α and by comparison, it follows that the function α must be continuous; in particular it remains positive for some positive time.

Assume now by contradiction that (10.12) is false; therefore, we can find $\overline{t} \in (0, +\infty)$ such that

$$0 < \alpha < \rho \qquad \text{in a neighbourhood of } \overline{t}.$$

Set

$$\overline{\rho} := \alpha(\overline{t}),$$

and

$$p := (0, \overline{\rho}) \in \partial C^+_{\overline{\rho}}.$$

Given $R > 0$, define

$$z := (0, \overline{\rho} - R),$$

where $B_R(z) \subset \{(z_1, z_2) \in \mathbb{R}^2 : z_2 \le \overline{\rho}\}$ is so that $p \in \partial B_R(z)$; see Figure 10.1. We can further choose $\overline{t}$ and a large enough R such that the curvature evolution $\overline{B_{R(t)}(z)}$ starting from $\overline{B_R(z)}$ at time $\overline{t}$, has the following property:

$$B_{R(t)}(z) \text{ lies locally outside } \mathcal{M}(C^+_{\overline{\rho}}, \mathcal{F})(t) \text{ around } p \tag{10.13}$$

for t in a right neighbourhood of $\overline{t}$.

Choose now a sufficiently elongated open ellipse $\mathcal{E} = \mathcal{E}(\overline{t})$ symmetric with respect to the two coordinate axes, satisfying the following properties:

- $\mathcal{E} \subset C^+_{\overline{\rho}}$;
- $p \in \partial\mathcal{E}$,

and so that there exists $r > 0$ such that

$$\mathcal{E} \cap B_r(p) \supset B_R(z) \cap B_r(p).$$

Let $\mathcal{E}(t)$, for $t \in [\bar{t}, \bar{t} + t^\dagger)$, be the curvature evolution of $\mathcal{E}$ starting at $\bar{t}$, where $t^\dagger$ is the extinction time of $\mathcal{E}$. We need to estimate the normal velocity of the boundary of the ellipse at the point p. Since $\mathcal{E} \subset C^+_{\bar{\rho}}$, by the barrier property we have

$$\mathcal{E}(t) \subseteq \mathcal{M}(C^+_{\bar{\rho}}, \mathcal{F})(t), \qquad t \in [\bar{t}, \bar{t} + t^\dagger). \tag{10.14}$$

Let q^- (respectively q^+) be the (transverse) intersection of $\partial\mathcal{E}$ with $\partial B_R(z)$ on the left (respectively on the right) of the z_2-axis, so that

$$\partial\mathcal{E} \cap \partial B_R(z) = \{q^-, p, q^+\}.$$

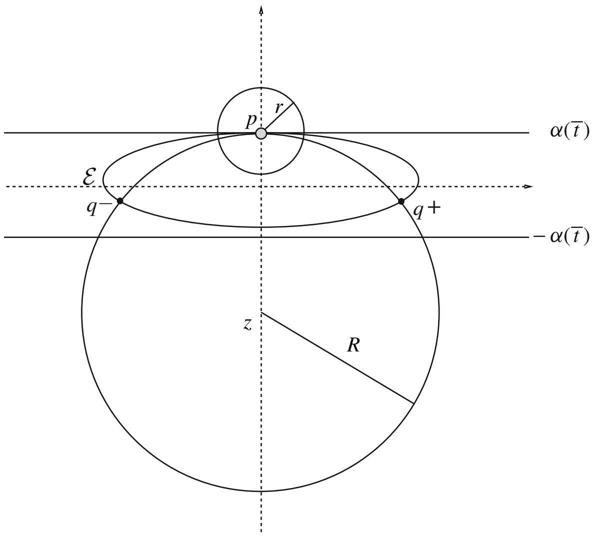

Figure 10.1. The example discussed in Section 10.1.

For $\sigma > 0$ sufficiently small, let $t \in [\bar{t}, \bar{t} + \sigma] \to q^-(t) \in \mathbb{R}^2$ (respectively $t \in [\bar{t}, \bar{t} + \sigma] \to q^+(t) \in \mathbb{R}^2$) be the continuous map defined as follows: $q^-(t)$ (respectively $q^+(t)$) is the intersection of $\partial\mathcal{E}(t)$ with $\partial B_{R(t)}(z)$ such that $q^-(\bar{t}) = q^-$ (respectively $q^+(\bar{t}) = q^+$).

Let us follow the trajectory of the point p, once for the evolution of $\partial B_{R(t)}(z)$, and denote it by

$$p^{\text{circle}}(t) = (0, p_2^{\text{circle}}(t)),$$

and once for the evolution of $\partial\mathcal{E}(t)$, and denote it by

$$p^{\text{ellipse}}(t) = (0, p_2^{\text{ellipse}}(t)),$$

so that $p^{\text{circle}}(\bar{t}) = p^{\text{ellipse}}(\bar{t}) = p$. It is possible to prove[(3)] that there exists $\delta \in (0, \sigma)$ such that, for $t \in (\bar{t}, \bar{t} + \delta)$, we have

$$p_2^{\text{ellipse}}(t) > p_2^{\text{circle}}(t), \tag{10.15}$$

and

$$\partial\mathcal{E}(t) \cap \partial B_{R(t)}(z) = \{q^-(t), q^+(t)\}.$$

Inequalities (10.15) and (10.13) force $\mathcal{E}(t)$ to lye locally outside $\mathcal{M}(C_\rho^+, \mathcal{F})(t)$ around p, for $t \in (\bar{t}, \bar{t} + \sigma)$, which contradicts (10.14), and concludes the proof of claim (10.11).

From (10.10) and (10.11) we deduce (10.9), since

$$\mathcal{M}^*(E, \mathcal{F}) \supseteq \bigcap_{\rho>0} \mathcal{M}(C_\rho^+, \mathcal{F}) = \bigcap_{\rho>0} C_\rho^+ = C.$$

Our next observation is that

$$\mathcal{M}_*(E, \mathcal{F})(t) = \emptyset, \qquad t \in \left(\frac{|E|}{2\pi}, +\infty\right), \tag{10.16}$$

where we recall that the area $|E|$ of E is finite.

Indeed, given $\rho > 0$ sufficiently small, and setting $t_\rho^\dagger := \frac{|E_\rho^-|}{2\pi}$, it is possible to prove[(4)] that there exists a unique curvature flow $t \in [0, t_\rho^\dagger) \to E_\rho^-(t)$ starting from $\overline{E_\rho^-}$ at time 0, which is smooth for $t \in (0, t_\rho^\dagger)$, and is such that $E_\rho^-(t)$ shrinks to a point as $t \uparrow t_\rho^\dagger$. By the consistency property (Proposition 9.16) we then have

$$\mathcal{M}\left(\overline{E_\rho^-}, \mathcal{F}\right)(t) = E_\rho^-(t), \qquad t \in (0, t_\rho^\dagger);$$

---

(3) Inequality (10.15) is a consequence of a Theorem due to Angenent (see [25, Section 5]). In [23, Theorem 1.3] the author proves a statement which, in the case of curvature flows, reads as follows: let $f_1, f_2 : [a, b] \to \mathcal{P}(\mathbb{R}^2)$ be two smooth compact curvature flows, with $f_1(a) \neq f_2(a)$. Then, given $t \in (a, b]$, the number of intersection points of $\partial f_1(t)$ with $\partial f_2(t)$ is finite. This number does not increase with time, and decreases exactly at those times t for which $\partial f_1(t)$ and $\partial f_2(t)$ have a nontransversal intersection. Moreover, the set of such times is a discrete subset of (a, b).

(4) We refer to [23, 180] for mean curvature flow starting from Lipschitz initial data.

in addition (see Remark 9.18)

$$\mathcal{M}(E_\rho^-, \mathcal{F})(t) = \emptyset, \qquad t \in (t_\rho^\dagger, +\infty).$$

Then (10.16) follows by recalling the definition of $\mathcal{M}_*(E, \mathcal{F})$.

Notice eventually that, in view of (10.9) and (10.16),

$$\mathcal{M}^*(E, \mathcal{F})(t) \setminus \mathcal{M}_*(E, \mathcal{F})(t) \text{ equals } \mathcal{M}^*(E, \mathcal{F})(t) \text{ for all } t > \frac{|E|}{2\pi}.$$

Chapter 11
An example of fattening

In this chapter we want to analyze the formation of a special type of singularity in mean curvature flow, called fattening, considering only a particularly simple situation, namely that of evolving plane curves. Our interest in fattening is due mainly to two reasons. The first one is that this kind of singularity is described in a rather natural way with the language of barriers. The second reason is that fattening can be related to a sort of instability of weak solutions to mean curvature flow(1).

Since for an embedded smooth closed plane curve fattening cannot appear(2), we need either to modify the evolution law, or to consider a compact nonsmooth initial datum. We choose this second alternative, so that our initial curve has a point where it is singular: the proof that we present here can be extracted from [52](3).

Let us define what we mean by fattening for mean curvature flow of hypersurfaces, in the language of barriers. For notational simplicity, we take $I = [0, +\infty)$ (namely $t_0 = 0$) in the definition of the regularizations $\mathcal{M}_*(E, \mathcal{F})$ and $\mathcal{M}^*(E, \mathcal{F})$ (see Definition 10.1).

Definition 11.1 (Fattening). Let $E \subset \mathbb{R}^n$ be nonempty and such that ∂E has empty interior, and let $t > 0$. We say that ∂E has developed fattening at time t if the interior of the set

$$\mathcal{M}^*(E, \mathcal{F})(t) \setminus \mathcal{M}_*(E, \mathcal{F})(t)$$

is nonempty.

(1) Remember that the minimal barrier (Chapter 9) is always unique, as well as the level set evolution considered in Section 14.3.

(2) A smooth simple closed plane curve evolves smoothly by curvature till it becomes extinct [158] (Grayson's theorem).

(3) In [52] the initial plane curve is smooth (the union of two disjoint circles), but the evolution law is curvature flow with a forcing term, that can even be taken equal to one. See also [40] for the figures displayed in this chapter and concerning Section 11.2, and Section 11.3.

The appearance of fattening can be interpreted as an instability phenomenon. In the specific example that we will consider, the initial datum ∂E can give raise to different (suitably defined) curvature evolutions, depending on how ∂E is approximated. Hence, small perturbations of the initial datum could lead to completely different mean curvature flows. The idea is that we can choose several evolutions inside the "fat" region[(4)] $\mathcal{M}^*(E,\mathcal{F})(t) \setminus \mathcal{M}_*(E,\mathcal{F})(t)$, each of which can be continued for some subsequent time.

11.1. Fattening of two tangent circles

Recall our notation: for any $z \in \mathbb{R}^2$ and $R > 0$, we let $B_R(z) := \{w \in \mathbb{R}^2 : |w - z| < R\}$. We also set $B_R := B_R((0,0))$. The example of fattening is described in the following result.

Theorem 11.2 (Fattening example). *Let $n = 2$, and*

$$E := \overline{B_1((-1,0))} \cup \overline{B_1((1,0))},$$

as in Figure 11.1. Then there exists $T > 0$ such that ∂E has developed fattening at any time $t \in (0,T)$.

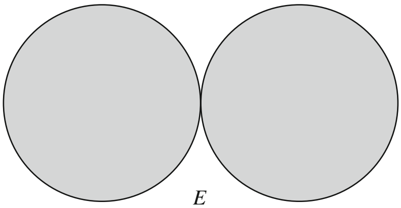

Figure 11.1. The initial set $E \subset \mathbb{R}^2$ (dark region) is the union of two closed balls tangent at the origin, which is therefore a singular point of the boundary of E.

Proof. Recalling the explicit expression of the curvature evolution for a sphere (see Example 3.21) and the definition (10.1) of $\mathcal{M}_*(E,\mathcal{F})$, it follows that

$$\mathcal{M}_*(E,\mathcal{F})(t) = \mathrm{int}\Big(B^-(t)\Big) \cup \mathrm{int}\Big(B^+(t)\Big), \qquad t \in \Big[0, \frac{1}{2}\Big),$$

(4) Roughly speaking, the fat region is expected to evolve continuously in time.

where

$$B^-(t) := \overline{B_{R(t)}((-1,0))}, \qquad B^+(t) := \overline{B_{R(t)}((0,1))}$$

and $R(t) = \sqrt{1-2t}$. In particular, there exist $\alpha, \beta \in I$, with

$$0 < \alpha < \beta,$$

and there exists $\sigma > 0$ such that $\mathcal{M}_*(E,\mathcal{F})(t) \neq \emptyset$ for any $t \in [\alpha, \beta]$, and also

$$\overline{B_\sigma} \cap \mathcal{M}_*(E,\mathcal{F})(t) = \emptyset, \qquad t \in [\alpha, \beta]; \tag{11.1}$$

see Figure 11.2.

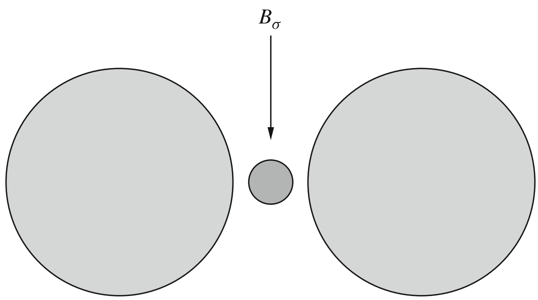

Figure 11.2. After some positive time t, the lower regularization $\mathcal{M}_*(E,\mathcal{F})(t)$ consists of the open balls (see (11.1)), which are well separated, so that we can place a ball B_σ as in the figure.

Now, let us focus the attention on the barrier $\mathcal{M}^*(E,\mathcal{F})$. Since E contains $B^-(0)$ and $B^+(0)$, it follows that

$$\mathcal{M}^*(E,\mathcal{F})(t) \supseteq B^-(t) \cup B^+(t), \qquad t \in \left[0, \frac{1}{2}\right).$$

Recalling formula (10.1), we have to consider the curvature evolution of the dumbell shaped sets E_ρ^+ (which are symmetric with respect to the vertical and horizontal axes), for $\rho > 0$ small enough; see Figure 11.3. Let us consider the auxiliary ball

$$B_1((0,1)) =: B_{\text{aux}};$$

see Figure 11.4. For any $t \in [0, 1/2)$, denote by

- $\overline{B_{\text{aux}}(t)} = \overline{B_{R(t)}((0,1))}$ the curvature evolution starting from $\overline{B_{\text{aux}}}$;
- $E_\rho^+(t)$ the curvature evolution starting from $\overline{E_\rho^+}$.

Remember that[5] E_ρ^+smoothly evolves by curvature for all positive times before the extinction time, and this time can be bounded from below (by comparison) by $1/2$.

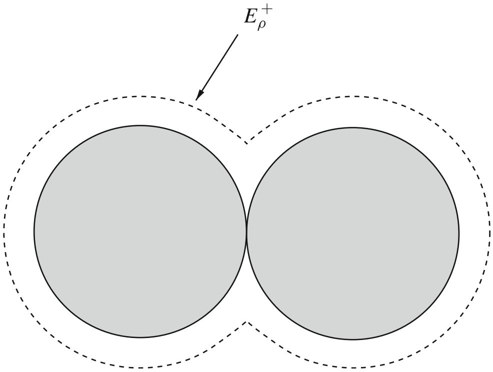

Figure 11.3. The dotted line is the boundary of the Lipschitz symmetric dumbell shaped set E_ρ^+. Recall that $\mathcal{M}^*(E, \mathcal{F})(t)$ is defined as the intersection over $\rho > 0$ of the evolutions of E_ρ^+ at time t.

The crucial point is to prove that $E_\rho^+(t)$, after some positive time, contains a ball centered at the origin, and this ball is independent of ρ.

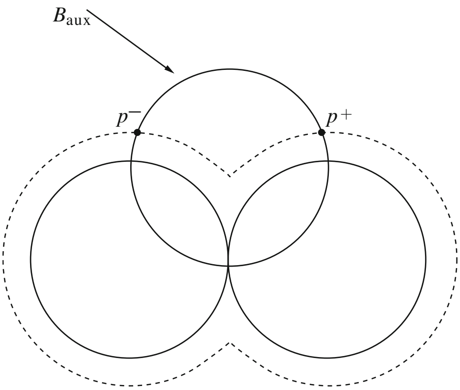

Figure 11.4. The auxiliary ball B_{aux} is needed to confine the evolution of the dotted arc of $\partial(E_\rho^+)$ between p^- and p^+ (with clockwise orientation). The aim is to show that such an arc necessarily must remain inside $B_{\mathrm{aux}}(t)$ for a suitable interval J of positive times.

[5] The set E_ρ^+ has Lipschitz boundary; concerning its curvature evolution, see footnote n. 3 in Chapter 10.

Recalling the explicit expression of the curvature evolution of a ball, we can take a nonempty compact interval $J \subset (0, +\infty)$ of times, such that

the interior of the set $\overline{B_{\rm aux}(t)} \cap B^+(t)$ is non empty for all $t \in J$.

Possibly reducing $\sigma > 0$ in (11.1), we can assume that

$$\overline{B_\sigma} \cap \left(\overline{B_{\rm aux}(t)} \cup B^+(t)\right) = \emptyset, \qquad t \in J;$$

see Figure 11.5. Set

$$\{p^-, p^+\} := \partial B_{\rm aux} \cap \partial(E^+_\rho);$$

see Figure 11.4.

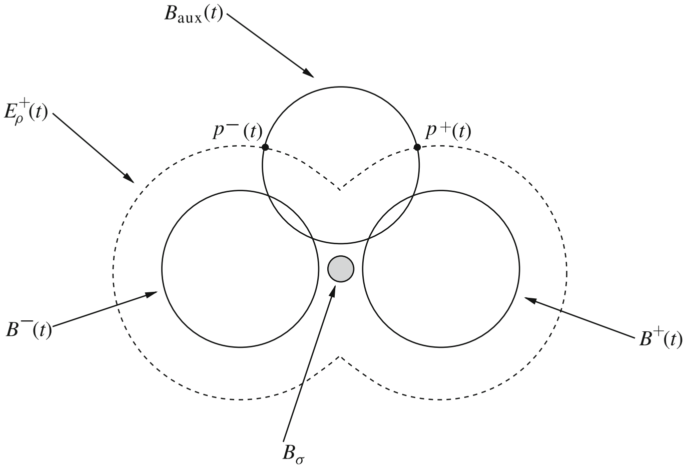

Figure 11.5. We choose an interval J of positive times such that $B_{\rm aux}(t) \cap \mathrm{int}(B^+(t)) \neq \emptyset$ for any $t \in J$. In this time interval the arc on the dotted curve $\partial(E^+_\rho(t))$ connecting $p^-(t)$ and $p^+(t)$ (with clockwise orientation) must be contained in $B_{\rm aux}(t)$. This allows us to choose a ball B_σ *independent of* ρ contained in $E^+_\rho(t)$.

For convenience, we recall once more a result (already used in Section 10.1) of Angenent, on the behaviour of the intersections between two closed curves evolving separately by curvature: in [25, Section 5] (see also [23, Theorems 1.3 and 3.2]) the author proves a statement which, in the case of curvature flows, reads as follows. Let $f_1, f_2 : [a, b] \to \mathcal{P}(\mathbb{R}^2)$ be two smooth compact curvature flows, with $f_1(a) \neq f_2(a)$. Then, given $t \in (a, b]$, the number of intersection points of $\partial f_1(t)$ with $\partial f_2(t)$ is finite. This number does not increase with time, and decreases exactly at those times t for which $\partial f_1(t)$ and $\partial f_2(t)$ have a nontransverse intersection[(6),(7)].

Therefore, we can write

$$\{p^-(t), p^+(t)\} := \partial B_{\mathrm{aux}}(t) \cap \partial(E_\rho^+(t))$$

where $p^-(t)$ is the continuous evolution of $p^- = p^-(0)$ and $p^+(t)$ is the continuous evolution of $p^+ = p^+(0)$ (see Figure 11.5).

Let us show that $p^-(t)$ and $p^+(t)$ are distinct for some time interval $\widetilde{J}$ after $t = 0$, and therefore there are no other intersection points between $\partial B_{\mathrm{aux}}(t)$ and $\partial(E_\rho^+(t))$. Indeed:

- by symmetry, the two points $p^-(t)$ and $p^+(t)$ can meet together only at the north pole or at the south pole of $B_{\mathrm{aux}}(t)$;
- by comparison with a closed ball centered at the north pole of B_{aux} and disjoint from $\overline{E_\rho^+}$, it follows that $p^-(t)$ and $p^+(t)$ cannot meet at the north pole of $B_{\mathrm{aux}}(t)$, for all t in a suitable time interval $\widetilde{J}$;
- by the comparison principle, $p^-(t)$ and $p^+(t)$ are forced to lie in the complement of $B^-(t) \cup B^+(t)$ for all $t \in \widetilde{J}$. Therefore $p^-(t)$ and $p^+(t)$ cannot meet at the south pole of $B_{\mathrm{aux}}(t)$, for all $t \in \widetilde{J}$.

Possibly reducing J, we can assume that $\widetilde{J} = J$.

It follows that the relatively open arc belonging to $\partial(E_\rho^+(t))$ and connecting $p^-(t)$ and $p^+(t)$ (with clockwise orientation) must lye in the interior of $B_{\mathrm{aux}}(t)$. Hence, possibly reducing the length of the interval J and of $\sigma > 0$, we deduce that

$$B_\sigma \subset \mathcal{M}^*(E, \mathcal{F})(t), \qquad t \in J. \tag{11.2}$$

From (11.1) and (11.2) it follows that B_σ is contained in the set $\mathcal{M}^*(E, \mathcal{F})(t) \setminus \mathcal{M}_*(E,\mathcal{F})(t)$ for any $t \in J$, and the assertion of the theorem follows. □

The following remark, which is a consequence of Theorem 13.5 in Chapter 13, shows that the regularized disjoint and joint set properties still hold (see Definition 11.5, below), in spite of the presence of fattening.

Remark 11.3. We have

$$\mathcal{M}_*(\mathbb{R}^2 \setminus E, \mathcal{F}) = \mathbb{R}^2 \setminus \mathcal{M}^*(E, \mathcal{F}).$$

(6) Moreover, the set of such times is a discrete subset of (a, b).

(7) Angenent's theorem does not exclude that $\partial B_{\mathrm{aux}}(t) \cap \partial(E_\rho^+(t))$ consists of two distinct intersections, one of them being when $p^-(t)$ and $p^+(t)$ coincide, and the other being, for instance, at the south pole of $B_{\mathrm{aux}}(t)$. We will see that this is impossible in the time interval J.

11.2. An alternative proof

In this section we briefly sketch an alternative proof[8] of Theorem 11.2. Let us consider the singular plane curve ∂E, which is the union of the boundaries of the two tangent balls of Figure 11.1 (see Figure 11.6 left), and let us smoothly parametrize it with a regular map $\gamma(0, \cdot) : \mathbb{S}^1 \to \mathbb{R}^2$. Let us flow $\gamma(0, \cdot)$ by curvature in the parametric immersed sense (see Definition 3.18, and [22, 23]). The evolved curves $\gamma(t, \cdot) : \mathbb{S}^1 \to \mathbb{R}^2$, for $t > 0$ sufficiently small, are qualitatively depicted in Figure 11.6 (right). At the origin, $\gamma(t, \cdot)$ has a transverse self-intersection. By symmetry, $\gamma(t, \cdot)$ is a figure eight curve which is union of two parts $\gamma_{\mathrm{left}}(t, \cdot)$, $\gamma_{\mathrm{right}}(t, \cdot)$, which intersect each other only at the origin, and flow separately by curvature keeping the origin fixed.

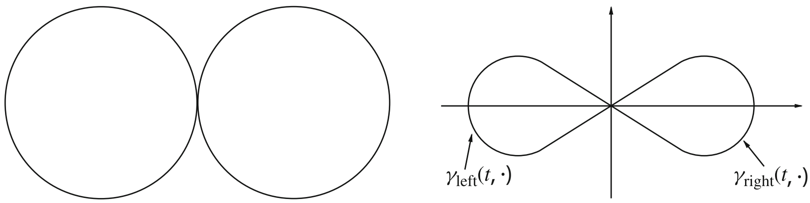

Figure 11.6. In the right picture we show the curve $\gamma(t, \cdot)$ (for small positive times) obtained by evolving ∂E in the left picture by curvature in the sense of immersed curves.

It turns out[9] that, for $t \geq 0$ sufficiently close to zero, say $t \in [0, T]$ for some $T > 0$, we have

$$\mathcal{M}^*(\partial E, \mathcal{F})(t) \supseteq \gamma(t, \mathbb{S}^1) \cup \big(\partial B^-(t) \cup \partial B^+(t)\big).$$

Observe that the inclusion $\mathcal{M}^*(\partial E, \mathcal{F})(t) \supseteq \partial B^-(t) \cup \partial B^+(t)$ is expected by the properties of barriers, since the evolutions $t \in [0, T] \to B^{\pm}(t)$ are elements of the family $\mathcal{F}$. On the other hand the inclusion

$$\mathcal{M}^*(\partial E, \mathcal{F})(t) \supseteq \gamma(t, \mathbb{S}^1) \tag{11.3}$$

is not immediate, since the $t \to \gamma(t, \cdot)$ is not an element of $\mathcal{F}$.

Once we know that $\gamma(t, \mathbb{S}^1)$ is contained in $\mathcal{M}^*(\partial E, \mathcal{F})(t)$, we can flow separately the Lipschitz curves $\gamma_{\mathrm{left}}(0, \cdot)$ and $\gamma_{\mathrm{right}}(0, \cdot)$ starting at subsequent times, and prove that these flows, at time t, are contained in $\mathcal{M}^*(\partial E, \mathcal{F})(t)$.

[8] We refer to [51] for more details, applied however in a case of space curves (see also [40]).

[9] In [48] it is proved that $\mathcal{M}^*(\partial E, \mathcal{F}) = \mathcal{M}^*(E, \mathcal{F}) \setminus \mathcal{M}_*(E, \mathcal{F})$.

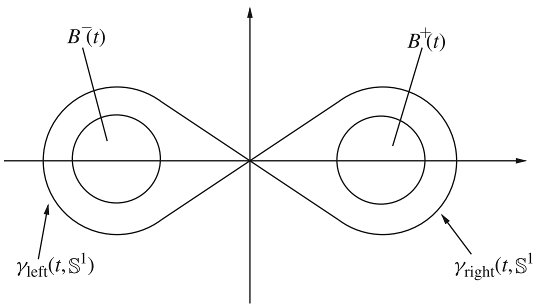

Figure 11.7. The two-dimensional annular region between $\gamma_{\mathrm{left}}(t, \cdot)$ and $\partial B^-(t)$ is contained in $\mathcal{M}^*(\partial E, \mathcal{F})(t)$. The same holds for the annular region between $\gamma_{\mathrm{right}}(t, \cdot)$ and $\partial B^+(t)$.

In this way it is possible to conclude that, for any $t \in (0, T]$, the barrier $\mathcal{M}^*(\partial E, \mathcal{F})(t)$ contains two open sets; one is the annular region bounded by the curves $\gamma_{\mathrm{left}}(t, \mathbb{S}^1)$ and $\partial B^-(t)$, and the other one is the annular region bounded by the curves $\partial B^+(t)$ and $\gamma_{\mathrm{right}}(t, \mathbb{S}^1)$; see Figure 11.7. In particular, for any $t_1, t_2 \in (0, T]$, with $0 < t_1 < t_2$, the barrier $\mathcal{M}^*(\partial E, \mathcal{F})(t)$ contains two triangles meeting at the origin O for any $t \in [t_1, t_2]$; see Figure 11.8. We remark that, once t_1, t_2 are chosen, the

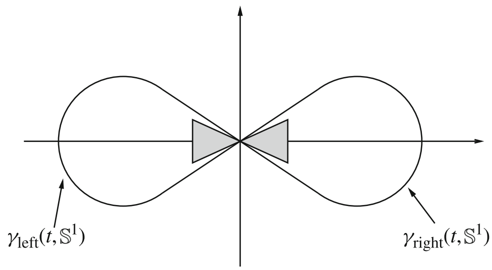

Figure 11.8. The bold triangles are contained in $\mathcal{M}^*(\partial E, \mathcal{F})(t)$, for any $t \in [t_1, t_2]$.

two triangles are independent of time. Now, let P be any point belonging to the vertical edge of the left triangle, and let Q be one vertex, different from the origin O, of the right triangle; see Figure 11.9. Let us consider the initial Lipschitz curve, union of the segments PO and OQ as in Figure 11.9 (left). Let us flow by curvature such an initial curve keeping fixed the points P and Q, starting at any time $t \in [t_1, t_2]$. This is a Dirichlet curvature flow, which is not an element of the family $\mathcal{F}$, and the resulting evolved curve is depicted in Figure 11.9 (right) for short times. Using these Dirichlet curvature flows it is possible to prove the fattening assertion (see [51] for more). This concludes the sketch of the proof.

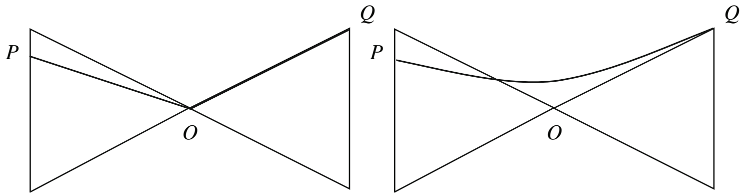

Figure 11.9. We depict (and zoom) the two triangles of Figure 11.8. In the left picture we show (in bold) the initial Lipschitz curve, and in the right picture we qualitatively show (in bold) its Dirichlet curvature evolution (the points P and Q are kept fixed).

Remark 11.4. Arguing in a similar manner to one of the two ways we have proved Theorem 11.2, one can show that an initial eight-shaped set E (such as the region inside the curve in Figure 11.6 (right)) having a transverse self-intersection, develops fattening. The same observation holds for the initial set given by the union of the first and the third quadrant in the plane: this latter example can be related to a possible nonuniqueness of (properly defined) weak solutions to curvature flow.

11.3. Notes

We list here some of the results that are known on the fattening phenomenon for mean curvature flow (see also [48]).

- If $E \subset \mathbb{R}^2$ has smooth compact boundary then ∂E does not develop fattening, as a particular consequence of the results of Grayson [158].
- If $E \subset \mathbb{R}^2$ is the union of two suitable balls (of different radii) whose closure are disjoint, then ∂E develops fattening under the evolution law
$$\frac{\partial d}{\partial t} = \Delta d - 1, \tag{11.4}$$
i.e., under motion by curvature with a forcing term constantly equal to one (see [52]). In [38] an example of fattening is given, for curvature flow for a smooth compact curve evolving with a time dependent forcing term. See also [137] and [248].
- There exist smooth sets $E \subset \mathbb{R}^2$ with non compact boundary developing fattening, as shown in [182] (for instance the set E considered in Example 9.19 and Section 10.1 develops fattening instantaneously). Any cone in $\mathbb{R}^2$ (with aperture less than π) develops fattening.
- There exist smooth sets $E \subset \mathbb{R}^2$ with compact boundary developing fattening under anisotropic curvature flow [228], for a suitable nonsymmetric anisotropy.

- All Lipschitz entire graphs evolving by mean curvature in $\mathbb{R}^n$ do not develop fattening: this follows, in particular, as a consequence of a result proved by Ecker and Huisken in [127], which states that any Lipschitz entire graph admits a unique smooth global evolution by mean curvature (compare also Section 4.1). In [128] it is proved that any locally Lipschitz continuous entire graph (actually, any continuous entire graph [27]) admits a global smooth evolution by mean curvature. However, for such initial data a uniqueness result is presently not known (unless $n = 2$, see the paper [35], and also [36]), hence it is not known whether or not fattening is possible. Sufficient conditions on the behaviour of the graph at infinity that guarantee that the solution does not develop fattening have been considered in [34, 63].
- The boundary of a bounded open convex set $E \subset \mathbb{R}^n$ flowing by mean curvature does not develop fattening, as a particular consequence of the results of Huisken in [168].
- Sufficient conditions that guarantee that the boundary of a smooth set $E \subset \mathbb{R}^n$ does not develop fattening have been investigated in [38], which in particular cover the case of surfaces having positive mean curvature everywhere.
- Conjectures relating fattening and the limit behaviour of solutions to the parabolic semilinear equations (15.2) can be found in [103] and [107] (see also [102, 104, 110]).
- Non fattening for mean curvature flow of suitable torii was proved by Soner-Souganidis in [252] (see also [229, 11]).
- Numerical simulations in [146] suggest the existence of a suitable torus in $\mathbb{R}^4$ which should develop fattening.
- A not completely rigorous example of fattening in $\mathbb{R}^3$, however supported by numerical experiments, starting from a smooth unbounded initial boundary, has been proposed in [28]. In the same paper the evolution and the possible development of fattening of the cones in $\mathbb{R}^n$ are considered (see also [186] and the references therein).
- An example of a set $E \subset \mathbb{R}^3$ with smooth compact boundary developing fattening has been announced in [268].
- The nonfattening for the evolution of suitable unbounded triply periodic surfaces in $\mathbb{R}^3$ has been proved in [266].
- Fattening for the boundaries of suitable evolving balls in $\mathbb{R}^n$ can be found in [163].
- Connections between fattening and stochastic motion by mean curvature have been investigated in [122] (see also [254]).
- Another proof of the formation of fattening for the problem addressed with equation (11.4) has been recently given in [195].

- Fattening for space curves evolving by curvature has been considered in [51].

We conclude this chapter by recalling the following notions, introduced and developed further in [45].

Definition 11.5 (Regularized disjoint and joint sets property). Let $\mathfrak{F}$ and $\mathfrak{G}$ be two family of maps as in Definition 9.1. Let $E_1, E_2 \in \mathcal{P}(\mathbb{R}^n)$. We say that the regularized disjoint sets property with respect to $\mathfrak{F}$ and $\mathfrak{G}$ holds if

$$E_1 \cap E_2 = \emptyset \Rightarrow \mathcal{M}_*(E_1, \mathfrak{F}, t_0)(t) \cap \mathcal{M}^*(E_2, \mathfrak{G}, t_0)(t) = \emptyset, \quad t > t_0. \quad (11.5)$$

We say that the regularized joint sets property with respect to $\mathfrak{F}$ and $\mathfrak{G}$ holds if

$$E_1 \cup E_2 = \mathbb{R}^n \Rightarrow \mathcal{M}_*(E_1, \mathfrak{F}, t_0)(t) \cup \mathcal{M}^*(E_2, \mathfrak{F}, t_0)(t) = \mathbb{R}^n, \quad t > t_0. \quad (11.6)$$

The regularized disjoint and joints set properties are related to the fattening phenomenon, and to the comparison results proven in Chapter 14 between minimal barriers and other generalized notions of solutions to mean curvature flow[(10)].

(10) Compare with Proposition 10.9 and with Theorem 13.5.

Chapter 12
Ilmanen's interposition lemma

In this chapter we want to prove Ilmanen's interposition lemma (see [183]). The proof presented here follows closely that given by Cardaliaguet in [75, Appendix]. We refer also to [77, 141] and [60] for related results. We will make use of Ilmanen's interposition lemma in the proof of Theorem 13.3, where we will show that the distance between the complement of two barriers is nondecreasing. Theorem 13.3 will be used, in turn, to compare(1) minimal barriers and their regularizations with other generalized notions of mean curvature evolution (Theorem 14.4).

Recall that the distance between two sets $B_1, B_2 \subseteq \mathbb{R}^n$ is defined as

$$\operatorname{dist}(B_1, B_2) := \inf\{|z - w| : z \in B_1, w \in B_2\}.$$

Given two open sets A_1, A_2, we write $A_1 \subset\subset A_2$ if there exists a compact set F such that $A_1 \subset F \subset A_2$.

Remember also our notation in formula (1.3): if $B \subseteq \mathbb{R}^n$ and $\rho > 0$, we set $B^+_\rho := \{z \in \mathbb{R}^n : \operatorname{dist}(z, B) < \rho\}$.

Ilmanen's interposition lemma reads as follows.

Theorem 12.1 (Interposition lemma). *Let $C \subset \mathbb{R}^n$ be a nonempty closed set and let $K \subset \mathbb{R}^n \setminus C$ be a nonempty compact set. Then there exists a bounded open set $A \subset \mathbb{R}^n$ with the following properties:*

(i) *∂A is of class $\mathcal{C}^{1,1}$;*
(ii) *$K \subset A \subset\subset \mathbb{R}^n \setminus C$;*
(iii) $\operatorname{dist}(K, C) = \operatorname{dist}(K, \partial A) + \operatorname{dist}(\partial A, C)$.

In the appendix of this chapter we briefly recall some properties of boundaries of sets of class $\mathcal{C}^{1,1}$, and some useful properties of the distance function.

(1) Theorem 14.4 could be proved without using Ilmanen's interposition lemma (see [45] and [46]).

Example 12.2 (Nonuniqueness and optimal regularity of ∂A). Let $n = 2$, and take

$$K := [-1, 1]^2, \qquad C := \mathbb{R}^2 \setminus (-2, 2)^2, \tag{12.1}$$

so that $\mathrm{dist}(K, C) = 1$. In this case there are several open sets A satisfying (i), (ii) and (iii) of Theorem 12.1. For instance, for $\rho \in (0, \mathrm{dist}(K, C))$, we can take

$$A = \{z \in \mathbb{R}^2 : \mathrm{dist}(z, K) < \rho\} = K_\rho^+. \tag{12.2}$$

Notice, instead, that the square

$$A = (-1 - \rho, 1 + \rho)^2 \tag{12.3}$$

satisfies (ii) and (iii), but its boundary is just Lipschitz continuous and not of class $\mathcal{C}^{1,1}$. Observe that for K and C as in (12.1), it is possible to find open sets A with $\partial A \in \mathcal{C}^\infty$ satisfying (ii) and (iii)[(2)].

However, in general, the regularity of ∂A is optimal, as it can be seen by considering the following choices: $n = 2$,

$$K := [-1, 1]^2, \qquad C := \mathbb{R}^2 \setminus K_1^+,$$

so that $\mathrm{dist}(K, C) = 1$. In this case, for $\rho \in (0, \mathrm{dist}(K, C))$,

$$A = K_\rho^+ ,$$

which has boundary of class $\mathcal{C}^{1,1}$ and not of class $\mathcal{C}^2$.

Proof of Theorem 12.1. Since K is compact and has empty intersection with the closed set C, we have

$$0 < \Delta := \mathrm{dist}(K, C) < +\infty.$$

Set

$$\begin{aligned} d_K(z) &:= \mathrm{dist}(z, K), & z &\in \mathbb{R}^n, \\ d_C^-(z) &:= \Delta - \mathrm{dist}(z, C), & z &\in \mathbb{R}^n. \end{aligned}$$

(2) These sets can be constructed "in between" the set (12.2) and the set (12.3). Observe also that, for general C and K as in the statement of Theorem 12.1, the set $(K_\rho^+)^-_{\rho/2} = \{z \in \mathbb{R}^n : \mathrm{dist}(z, \mathbb{R}^n \setminus K_\rho^+) > \rho/2\}$, for certain $\rho \in (0, \mathrm{dist}(K, C))$, does not necessarily fulfill (i)-(iii), as it can be seen considering as $K \subset \mathbb{R}^2$ the union of two distinct points.

Then

- d_K is locally semiconcave[3] in $\mathbb{R}^n \setminus K$,
- d_C^- is locally semiconvex in $\mathbb{R}^n \setminus C$,
- the triangular inequality implies that

$$d_C^- \leq d_K. \tag{12.4}$$

Let

$$\rho \in (0, \Delta)$$

and set[4]

$$d_\rho(z) := \operatorname{dist}(z, \partial K_\rho^+), \qquad z \in \mathbb{R}^n.$$

Roughly, we would like to define ∂A as the inverse image of a certain value of d_ρ, but d_ρ is not smooth enough. We will then show that ∂A can be taken as the inverse image of a regular value of a smoothed version q of d_ρ (see (12.9)).

We divide the proof of the theorem into four steps. The main step is step 2, the proof of which is postponed in Section 12.1.

Step 1. We have

$$d_C^-(x) \leq \rho - d_\rho(x) \leq d_K(x), \qquad x \in K_\rho^+ \setminus K. \tag{12.5}$$

Indeed, let $x \in K_\rho^+ \setminus K$ and let

$$y \in \operatorname{pr}(x, \partial K_\rho^+),$$

so that

$$d_K(y) = \rho \qquad \text{and} \qquad |y - x| = d_\rho(x).$$

The triangular inequality implies that

$$d_K(y) \leq |y - x| + d_K(x),$$

hence

$$d_K(x) \geq d_K(y) - |y - x| = \rho - |y - x| = \rho - d_\rho(x),$$

which gives the right inequality in (12.5).

(3) We refer to the appendix of this chapter for the definitions of local semiconcavity and local semiconvexity.

(4) For the regularity of ∂K_ρ^+ we refer to the notes at the end of the chapter. It is possible to prove (see for instance [141]) that $\overline{K_\rho^+} = \{z \in \mathbb{R}^n : \operatorname{dist}(z, K) \leq \rho\}$, $\partial K_\rho^+ = \{x \in \mathbb{R}^n : \operatorname{dist}(x, K) = \rho\}$ and that $\operatorname{dist}(z, K) = \operatorname{dist}(z, \partial K_\rho^+) + \rho$ for any $z \notin K_\rho^+$. In [73] it is also proved that K_ρ^+ has finite perimeter.

To show the left inequality in (12.5), take $x \in K_\rho^+ \setminus K$ and let

$$z \in \mathrm{pr}(x, C),$$

so that

$$|z - x| \geq d_\rho(x) \qquad \text{and} \qquad |z - x| = \mathrm{dist}(x, C).$$

Note that $|z - x| > 0$, since $\rho \in (0, \Delta)$. Set

$$y' := x + \frac{(z - x)}{|z - x|}\, d_\rho(x).$$

Then

$$\begin{aligned}\mathrm{dist}(y', C) \leq |y' - z| = |z - x| \left|1 - \frac{d_\rho(x)}{|z - x|}\right| &= \big||z - x| - d_\rho(x)\big| \\ &= |z - x| - d_\rho(x).\end{aligned}$$

Since K_ρ^+ contains the open ball centered at x with radius $d_\rho(x)$, it follows that $y' \in \overline{K_\rho^+}$, hence $d_K(y') \leq \rho$. Therefore, evaluating (12.4) at y', we get

$$\Delta - \rho \leq \Delta - d_K(y') \leq \mathrm{dist}(y', C).$$

We then have

$$\Delta - \rho \leq \mathrm{dist}(y', C) \leq |z - x| - d_\rho(x) = \mathrm{dist}(x, C) - d_\rho(x).$$

Hence

$$d_C^-(x) = \Delta - \mathrm{dist}(x, C) \leq \rho - d_\rho(x),$$

which is the left inequality in (12.5). The proof of step 1 is concluded.

Before passing to the next step, we introduce some notation. We define

$$S_{\frac{\rho}{3}, \frac{2\rho}{3}} := \left\{x \in K_\rho^+ : \frac{\rho}{3} < d_\rho(x) < \frac{2\rho}{3}\right\},$$

and[5]

$$\Theta := \left\{x \in S_{\frac{\rho}{3}, \frac{2\rho}{3}} : \mathrm{dist}(x, K) + \mathrm{dist}(x, C) = \Delta\right\}. \tag{12.6}$$

[5] Observe that it may happen that $\Theta = S_{\frac{\rho}{3}, \frac{2\rho}{3}}$, for instance when $K = \overline{B_1(0)}$ and $C = \mathbb{R}^n \setminus B_2(0)$.

Notice that

$$\Theta = \left\{ x \in S_{\frac{\rho}{3},\frac{2\rho}{3}} : d_C^-(x) = d_K(x) \right\}.$$

The next step is a consequence of Theorem 12.4, below.

Step 2. There exists an open set $O \subset K_\rho^+ \setminus K$ with

$$\Theta \subset O$$

such that

$$d_\rho \in \mathcal{C}^{1,1}(\overline{O}).$$

Collecting the informations[(6)] from steps 1 and 2, we have

$$\rho - d_\rho \in \mathcal{C}^{1,1}(\overline{O})$$

and

$$d_C^- \le \rho - d_\rho \le d_K \qquad \text{in } \overline{O}.$$

Moreover, being d_ρ the distance from ∂K_ρ^+, we have

$$|\nabla(\rho - d_\rho)| = 1 \qquad \text{in } \overline{O}. \tag{12.7}$$

Step 3. Construction of the function q.

Let O be as in step 2. Let $O' \subset \mathbb{R}^n$ be an open set such that

$$\Theta \subset O' \subset\subset O.$$

By mollification[(7)], following [75] and [183], and recalling (12.5) and (12.6), we can construct a function

$$q : K_\rho^+ \setminus K \to \mathbb{R}$$

having the following properties[(8)]:

(1) $q = d_\rho$ in O';
(2) $q \in \mathcal{C}^{1,1}(S_{\frac{\rho}{3},\frac{2\rho}{3}}) \cap \mathcal{C}^\infty(S_{\frac{\rho}{3},\frac{2\rho}{3}} \setminus \overline{O})$;

(6) Concerning the insertion of a locally $\mathcal{C}^{1,1}$ function in between a locally semiconvex function and a locally semiconcave function, we refer the reader to [183, 141] and [60].

(7) In [141], using a suitable density argument concerning smooth maps with a prescribed regular value, it is shown the existence of a function q and a regular value $\overline{\sigma}$ as in steps 3, 4, and moreover such that $d_C^- < \rho - q < d_K$ in $S_{\frac{\rho}{3},\frac{2\rho}{3}} \setminus \Theta$.

(8) The case $S_{\frac{\rho}{3},\frac{2\rho}{3}} \setminus \overline{O} = \emptyset$ is not excluded.

(3) $\|q - d_\rho\|_{L^\infty(K_\rho^+ \setminus K)} \le \frac{\rho}{12}$;

(4) $\|\nabla q - \nabla d_\rho\|_{L^\infty(O)} \le \frac{1}{2}$;

(5) $d_C^- \le \rho - q \le d_K$ in $S_{\frac{\rho}{3},\frac{2\rho}{3}}$.

Observe that, given $\sigma \in \left(\frac{5\rho}{12}, \frac{7\rho}{12}\right)$, we have

$$\{x \in K_\rho^+ \setminus K : q(x) = \sigma\} \subset S_{\frac{\rho}{3},\frac{2\rho}{3}}. \tag{12.8}$$

Indeed, if $q(x) = \sigma \in \left(\frac{5\rho}{12}, \frac{7\rho}{12}\right)$ then, using also (3),

$$\frac{\rho}{3} = -\frac{\rho}{12} + \frac{5\rho}{12} < -\frac{\rho}{12} + \sigma \le d_\rho(x) \le \sigma + \frac{\rho}{12} < \frac{7\rho}{12} + \frac{\rho}{12} = \frac{2\rho}{3},$$

and (12.8) follows.

Step 4. There exists $\overline{\sigma} \in \left(\frac{5\rho}{12}, \frac{7\rho}{12}\right)$ such that

$$A := K \cup \{x \in K_\rho^+ \setminus K : q(x) < \overline{\sigma}\} \tag{12.9}$$

satisfies the thesis of the theorem.

From (12.7) and property (4) it follows that

$$\nabla q(x) \ne 0, \qquad x \in \overline{O}. \tag{12.10}$$

Let us consider the open set $S_{\frac{\rho}{3},\frac{2\rho}{3}} \setminus \overline{O}$, where the function q is of class $\mathcal{C}^\infty$: from Sard's lemma there exists $\overline{\sigma} \in \left(\frac{5\rho}{12}, \frac{7\rho}{12}\right)$ such that

$$\nabla q(z) \ne 0 \quad \text{for any } z \in \{x \in \mathbb{R}^n : q(x) = \overline{\sigma}\} \cap (S_{\frac{\rho}{3},\frac{2\rho}{3}} \setminus \overline{O}). \tag{12.11}$$

Combining (12.11) with (12.10), it then follows that $\{x \in \mathbb{R}^n : q(x) = \overline{\sigma}\}$ is a compact hypersurface of class $\mathcal{C}^{1,1}$ without boundary, contained in $S_{\frac{\rho}{3},\frac{2\rho}{3}}$. Define now the open set A as in (12.9). Then A satisfies (ii) by construction, and $\partial A = \{q = \overline{\sigma}\}$ is of class $\mathcal{C}^{1,1}$.

It remains to prove assertion (iii). If $x \in K$ and $y \in C$ are such that

$$\Delta = \operatorname{dist}(K, C) = |x - y|,$$

and if $z \in \partial A$ is a point belonging to the intersection between ∂A and the segment joining x and y, we have

$$|x-y| = |x-z| + |z-y| \geq \text{dist}(x, \partial A) + \text{dist}(y, \partial A) \\ \geq \text{dist}(K, \partial A) + \text{dist}(C, \partial A).$$

Therefore

$$\Delta \geq \text{dist}(K, \partial A) + \text{dist}(C, \partial A).$$

Hence, to conclude the proof we need to show that

$$\Delta \leq \text{dist}(K, \partial A) + \text{dist}(C, \partial A). \tag{12.12}$$

For any $x \in \partial A$ we have $q(x) = \overline{\sigma}$; since $x \in S_{\frac{\rho}{3},\frac{2\rho}{3}}$ by (12.8), applying the right inequality in property (5) of step 3, it follows that

$$d_K(x) \geq \rho - \overline{\sigma}.$$

Therefore

$$\text{dist}(\partial A, K) \geq \rho - \overline{\sigma}. \tag{12.13}$$

Similarly, applying the left inequality in property (5), we obtain

$$\Delta - \text{dist}(x, C) \leq \rho - \overline{\sigma}.$$

Hence

$$\text{dist}(\partial A, C) \geq \Delta - \rho + \overline{\sigma}. \tag{12.14}$$

Adding together (12.13) and (12.14) gives (12.12). This concludes the proof of Theorem 12.1. $\square$

As a consequence of the interposition lemma, in [76, Proposition 3.7] the following result is proven.

Corollary 12.3. *Let $C \subset \mathbb{R}^n$ be a nonempty closed set and let $K \subset \mathbb{R}^n \setminus C$ be a nonempty compact set. Let $x \in K$ and $y \in C$ be such that*

$$\text{dist}(K, C) = |x-y|.$$

Then there exists a bounded open set $B \subset \mathbb{R}^n$ with the following properties:

- *∂B is of class $\mathcal{C}^{1,1}$;*
- *$K \subset B$ and $x \in \partial B$;*
- *$B + y - x \subset \mathbb{R}^n \setminus C$ and $y \in \partial(B + y - x)$.*

12.1. $\mathcal{C}^{1,1}$-regularity of d_ρ

We now show step 2 of the proof of Theorem 12.1.

Theorem 12.4. *Let $x_0 \in K_\rho^+ \setminus K$ be such that*

$$\mathrm{dist}(x_0, K) + \mathrm{dist}(x_0, C) = \Delta. \tag{12.15}$$

Then there exists $\epsilon_0 > 0$ such that $d_\rho \in \mathcal{C}^{1,1}(B_{\epsilon_0}(x_0))$.

The proof of Theorem 12.4 is based on two preliminary lemmas and a proposition. Given a set $F \subseteq \mathbb{R}^n$, we denote by $\mathrm{co}(F)$ the closed convex hull of F, and by

$$\mathrm{diam}(F) := \sup\{|x - y| : x, y \in F\}$$

the diameter of F.

Lemma 12.5. *Let $x \in K_\rho^+$, $y \in \mathrm{pr}(x, \partial K_\rho^+)$ and assume that*[9]

$$y \notin \mathrm{co}\Big(\mathrm{pr}(y, K)\Big). \tag{12.16}$$

Then there exist nonnegative real numbers $\lambda_1, \ldots, \lambda_{n+1}$ and points $z_1, \ldots, z_{n+1}$ in $\mathrm{pr}(y, K)$, such that

$$y - x = \sum_{i=1}^{n+1} \lambda_i (y - z_i). \tag{12.17}$$

In Figure 12.1 we show an example where the assumptions of Lemma 12.5 are satisfied, while in Figure 12.2 the assumptions are not satisfied.

Note that, if $\mathrm{co}\Big(\mathrm{pr}(y, K)\Big) = \{z\}$, then (12.17) is satisfied. Indeed in this case[10] the distance function from K is differentiable at y. Hence ∂K_ρ^+ admits tangent hyperplane $T_y K$ at y and both $y - x$ and $y - z$ are orthogonal to $T_y K$, so that $y - z$ and $y - x$ are parallel. Thus x belongs to the segment joining y and z, and therefore (12.17) is satisfied with $\lambda_2 = \cdots = \lambda_{n+1} = 0$ and for a suitable $\lambda_1 \in [0, 1)$.

[9] A point $y \notin K$ satisfying $y \in \mathrm{co}(\mathrm{pr}(y, K))$ is called a critical point of d_K (see [144]). Indeed, the condition $y \in \mathrm{co}(\mathrm{pr}(y, K))$ is equivalent to say that 0 belongs to the superdifferential of d_K at y [74, Corollary 3.4.5].

[10] See Remark 1.1 in Chapter 1.

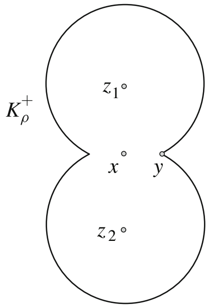

Figure 12.1. K consists of the two points z_1 and z_2. We have $y \in \mathrm{pr}(x, \partial K_\rho^+)$, $\{z_1, z_2\} = \mathrm{pr}(y, K)$, and $y \notin \mathrm{co}\big(\mathrm{pr}(y, K)\big)$.

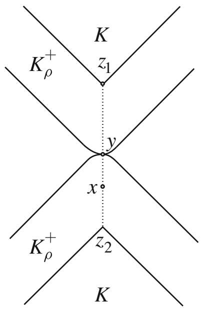

Figure 12.2. We have $y \in \mathrm{pr}(x, \partial K_\rho^+)$, $z_1, z_2 \in \mathrm{pr}(y, K)$, but $y \in \mathrm{co}\big(\mathrm{pr}(y, K)\big)$, so that (12.16) is not satisfied.

Proof. We claim that

$$\text{if } v \in \mathbb{S}^{n-1} \text{ is such that } \min_{z \in \mathrm{pr}(y,K)} \langle v, y - z \rangle \geq 0 \text{ then } \langle v, y - x \rangle \geq 0. \quad (12.18)$$

Let us define

$$\eta_K(z) := \frac{1}{2}\mathrm{dist}(z, K)^2, \qquad z \in \mathbb{R}^n,$$

which is a semiconcave function in $\mathbb{R}^n$. Given $v \in \mathbb{S}^{n-1}$, there exists the directional semiderivative $\nabla_v^+ \eta_K(y)$ of η_K at y, defined as

$$\nabla_v^+ \eta_K(y) := \lim_{h \downarrow 0} \frac{\eta_K(y + hv) - \eta_K(y)}{h},$$

and has the following representation[(11)]:

$$\nabla_v^+ \eta_K(y) = \min_{z \in \mathrm{pr}(y,K)} \langle v, y - z \rangle.$$

Therefore claim (12.18) can be rewritten as follows:

$$\text{if } v \in \mathbb{S}^{n-1} \text{ is such that } \nabla_v^+ \eta_K(y) \geq 0 \text{ then } \langle v, y - x \rangle \geq 0. \qquad (12.19)$$

The two closed compact convex sets $\{y\}$ and $\mathrm{co}\big(\mathrm{pr}(y, K)\big)$ are disjoint by assumption (12.16); hence there exists $w \in \mathbb{R}^n \setminus \{0\}$ such that

$$\langle w, y - z \rangle > 0, \qquad z \in \mathrm{pr}(y, K). \qquad (12.20)$$

Let $v \in \mathbb{S}^{n-1}$ be as in (12.19), $\epsilon \in (0, 1)$ and set

$$v_\epsilon := \frac{v + \epsilon w}{|v + \epsilon w|}.$$

Since

$$\nabla_{v_\epsilon}^+ \eta_K(y) = \min_{z \in \mathrm{pr}(y,K)} \langle v_\epsilon, y - z \rangle,$$

from (12.19) and (12.20) we obtain

$$\nabla_{v_\epsilon}^+ \eta_K(y) > 0.$$

This implies that, for $h > 0$ small enough,

$$\eta_K(y + h v_\epsilon) > \eta_K(y).$$

Since $y \in \partial K_\rho^+$, it follows that

$$y + h v_\epsilon \notin K_\rho^+.$$

Since $y \in \mathrm{pr}(x, \partial K_\rho^+)$, we have

$$|y - x|^2 \leq |y' - x|^2 \qquad \text{for any } y' \notin K_\rho^+.$$

Then, taking $y' = y + h v_\epsilon$, we have

$$|y - x|^2 \leq |y + h v_\epsilon - x|^2 = |y - x|^2 + 2h \langle v_\epsilon, y - x \rangle + h^2.$$

(11) See, *e.g.*, [74, Theorem 3.3.6 and Corollary 3.4.5].

This inequality implies that

$$\langle v_\epsilon, y - x\rangle \geq 0.$$

Since $\epsilon \in (0, 1)$ is arbitrary, it follows that $\langle y - x, v\rangle \geq 0$, and the claim is proved.

Observe now that $y - x$ can be written as a finite sum of vectors of the form $\lambda(y - z)$, with $z \in \mathrm{co}(\mathrm{pr}(y, K))$ and $\lambda \geq 0$. Indeed, if not, by the separation theorem there would exist a vector $v \in \mathbb{S}^{n-1}$ such that $\langle v, y - z\rangle \geq 0$ for any $z \in \mathrm{pr}(y, K)$, and $\langle v, y - x\rangle < 0$, in contradiction with (12.19).

Since $y - x$ can be written as a finite sum of vectors of the form $\lambda(y - z)$, with $z \in \mathrm{co}\Big(\mathrm{pr}(y, K)\Big)$ and $\lambda \geq 0$, the thesis of the lemma follows from Caratheodory's theorem (see [232, Theorem 17.1]). $\square$

Lemma 12.6. *Let x, y, $\lambda_1, \ldots, \lambda_{n+1}$ and $z_1, \ldots, z_{n+1}$ be as in Lemma* 12.5. *Suppose that*

$$\exists\, \epsilon \in \Big(0, \rho - d_\rho(x)\Big) : \mathrm{diam}\Big(\mathrm{pr}(y, K)\Big) \leq \Big\{2\Big[\rho^2 - \big(d_\rho(x) + \epsilon\big)^2\Big]\Big\}^{1/2}. \tag{12.21}$$

Then

$$\sum_{i=1}^{n+1} \lambda_i \leq \frac{d_\rho(x)}{d_\rho(x) + \epsilon}. \tag{12.22}$$

Proof. Observe that

$$|y - z_i|^2 = |y - z_j|^2 = \rho^2, \qquad i, j \in \{1, \ldots, n + 1\}. \tag{12.23}$$

Since $y \in \mathrm{pr}(x, \partial K_\rho^+)$, from (12.17) we have

$$\begin{aligned}
(d_\rho(x))^2 = |y - x|^2 &= \sum_{i,j=1}^{n+1} \lambda_i \lambda_j \langle y - z_i, y - z_j\rangle \\
&= \sum_{i,j=1}^{n+1} \lambda_i \lambda_j \langle y - z_i, y - z_i + z_i - z_j\rangle \\
&= \sum_{i,j=1}^{n+1} \lambda_i \lambda_j \Big(|y - z_i|^2 + \langle y - z_i, z_i - z_j\rangle\Big) \\
&= \rho^2 \left(\sum_{i=1}^{n+1} \lambda_i\right)^2 + \sum_{i,j=1}^{n+1} \lambda_i \lambda_j \langle y - z_i, z_i - z_j\rangle,
\end{aligned} \tag{12.24}$$

where in the last equality we make use of (12.23).

We now rewrite the last addendum on the right-hand side of (12.24). Using again (12.23), we have

$$\begin{aligned}-\langle y - z_i, z_i - z_j\rangle &= -\langle y - z_i, z_i - y + y - z_j\rangle \\ &= \rho^2 - \langle y - z_i, y - z_j\rangle = \frac{1}{2}|z_i - y + y - z_j|^2 \\ &= \frac{1}{2}|z_i - z_j|^2.\end{aligned}$$

Substituting these expressions in (12.24) and using the inequality

$$|z_i - z_j| \leq \mathrm{diam}\Big(\mathrm{pr}(y, K)\Big),$$

we have

$$\begin{aligned}(d_\rho(x))^2 =& \rho^2 \left(\sum_{i=1}^{n+1} \lambda_i\right)^2 - \frac{1}{2}\sum_{i,j=1}^{n+1} \lambda_i\lambda_j |z_i - z_j|^2 \\ &\geq \left\{\rho^2 - \frac{1}{2}\Big[\mathrm{diam}\big(\mathrm{pr}(y, K)\big)\Big]^2\right\} \left(\sum_{i=1}^{n+1} \lambda_i\right)^2.\end{aligned}$$

By hypothesis (12.21), we have that $\epsilon \in (0, \rho - d_\rho(x))$ satisfies

$$\rho^2 - \frac{1}{2}\Big[\mathrm{diam}\big(\mathrm{pr}(y, K)\big)\Big]^2 \geq (d_\rho(x) + \epsilon)^2.$$

Hence we obtain

$$(d_\rho(x))^2 \geq \big(d_\rho(x) + \epsilon\big)^2 \left(\sum_{i=1}^{n+1} \lambda_i\right)^2.$$

Therefore

$$d_\rho(x) \geq \big(d_\rho(x) + \epsilon\big) \sum_{i=1}^{n+1} \lambda_i,$$

and (12.22) follows. □

Proposition 12.7. *Suppose that*

- $\overline{x}, \underline{x} \in K_\rho^+ \setminus K$ *and* $d_\rho(\overline{x}) = d_\rho(\underline{x})$*;*
- $\overline{y} \in \mathrm{pr}(\overline{x}, \partial K_\rho^+)$ *and* $\underline{y} \in \mathrm{pr}(\underline{x}, \partial K_\rho^+)$*;*
- $\overline{y} \notin \mathrm{co}(\mathrm{pr}(\overline{y}, K))$*;*
- $\exists \epsilon \in (0, \rho - d_\rho(\overline{x}))$ *such that*

$$\mathrm{diam}\Big(\mathrm{pr}(\overline{y}, K)\Big) \leq \Big\{2\Big[\rho^2 - (d_\rho(\overline{x}) + \epsilon)^2\Big]\Big\}^{1/2}.$$

Then

$$|\overline{y} - \underline{y}| \leq \frac{2(d_\rho(\overline{x}) + \epsilon)}{\epsilon} |\overline{x} - \underline{x}|. \tag{12.25}$$

Proof. Applying Lemma 12.5 with the choices, $x = \overline{x}$ and $y = \overline{y}$, it follows that there exist nonnegative real numbers $\lambda_1, \ldots, \lambda_{n+1}$ and points $z_1, \ldots, z_{n+1}$ in $\mathrm{pr}(\overline{y}, K)$ such that

$$\overline{y} - \overline{x} = \sum_{i=1}^{n+1} \lambda_i (\overline{y} - z_i). \tag{12.26}$$

Moreover, from Lemma 12.6 we have

$$1 - \sum_{i=1}^{n+1} \lambda_i \geq \frac{\epsilon}{d_\rho(\overline{x}) + \epsilon}. \tag{12.27}$$

Since $\overline{y}, \underline{y} \in \partial K_\rho^+$ and $z_i \in \mathrm{pr}(\overline{y}, K)$ for any $i \in \{1, \ldots, n+1\}$, we have

$$\rho^2 = |\overline{y} - z_i|^2 \leq |\underline{y} - z_i|^2.$$

Consequently

$$|\overline{y} - z_i|^2 \leq |\underline{y} - \overline{y} + \overline{y} - z_i|^2 = |\underline{y} - \overline{y}|^2 + |\overline{y} - z_i|^2 + 2\langle \overline{y} - z_i, \underline{y} - \overline{y} \rangle.$$

Therefore

$$2\langle \overline{y} - z_i, \underline{y} - \overline{y} \rangle \geq -|\underline{y} - \overline{y}|^2. \tag{12.28}$$

Taking the scalar product of (12.26) with $2(\underline{y} - \overline{y})$ and using (12.28), we have

$$2\langle \overline{y} - \overline{x}, \underline{y} - \overline{y} \rangle = 2 \sum_{i=1}^{n+1} \lambda_i \langle \overline{y} - z_i, \underline{y} - \overline{y} \rangle \geq -|\underline{y} - \overline{y}|^2 \sum_{i=1}^{n+1} \lambda_i. \tag{12.29}$$

The inclusions $\underline{y} \in \mathrm{pr}(\underline{x}, \partial K_\rho^+)$ and $\overline{y} \in \partial K_\rho^+$ imply

$$|\underline{x} - \underline{y}|^2 \leq |\underline{x} - \overline{y}|^2,$$

and therefore

$$|\underline{x} - \overline{y}|^2 \geq |\underline{x} - \overline{y} + \overline{y} - \underline{y}|^2 = |\underline{x} - \overline{y}|^2 + |\underline{y} - \overline{y}|^2 + 2\langle \underline{x} - \overline{y}, \overline{y} - \underline{y} \rangle,$$

which gives

$$|\underline{y} - \overline{y}|^2 \leq 2\langle \overline{y} - \underline{x}, \overline{y} - \underline{y} \rangle. \tag{12.30}$$

By adding (12.29) and (12.30) we deduce

$$\left(1-\sum_{i=1}^{n+1}\lambda_i\right)|\underline{y}-\overline{y}|^2 \le 2\langle \overline{y}-\underline{x}, \overline{y}-\underline{y}\rangle + 2\langle \overline{y}-\overline{x}, \underline{y}-\overline{y}\rangle$$
$$= 2\langle \underline{x}-\overline{x}, \underline{y}-\overline{y}\rangle \le 2|\underline{x}-\overline{x}||\underline{y}-\overline{y}|.$$

From this inequality and (12.27) we then get

$$|\overline{y}-\underline{y}| \le 2\left(1-\sum_{i=1}^{n+1}\lambda_i\right)^{-1}|\underline{x}-\overline{x}| \le \frac{2(d_\rho(\overline{x})+\epsilon)}{\epsilon}|\overline{x}-\underline{x}|,$$

and this concludes the proof of (12.25). □

We are now in a position to prove Theorem 12.4.

Proof of Theorem 12.4. Let

$$\kappa_0 \in \mathrm{pr}(x_0, K) \qquad \text{and} \qquad c_0 \in \mathrm{pr}(x_0, C).$$

Using (12.15) it follows that, for any $\lambda \in (0, 1)$, we have

$$\mathrm{pr}\big(\lambda\kappa_0 + (1-\lambda)c_0, K\big) = \{\kappa_0\}, \qquad \mathrm{pr}\big(\lambda\kappa_0 + (1-\lambda)c_0, C\big) = \{c_0\}.$$

Take $\overline{\lambda} \in (0, 1)$ such that

$$\mathrm{dist}(\overline{\lambda}\kappa_0 + (1-\overline{\lambda})c_0, K) = \rho,$$

and set

$$y_1 := \overline{\lambda}\kappa_0 + (1-\overline{\lambda})c_0.$$

Then $y_1 \in \partial K_\rho^+$, and

$$\mathrm{pr}(x_0, \partial K_\rho^+) = \{y_1\}. \tag{12.31}$$

Define

$$\epsilon := \frac{\rho - d_\rho(x_0)}{3} > 0, \qquad \delta := \left[\rho^2 - (d_\rho(x_0) + 2\epsilon)^2\right]^{1/2}. \tag{12.32}$$

Since

$$\mathrm{pr}(y_1, K) = \{\kappa_0\},$$

it follows[12] that $\mathrm{pr}(\cdot, K)$ is continuous at y_1, so that

$$\exists\, \epsilon_1 > 0 \;:\; \mathrm{pr}\Big(B_{\epsilon_1}(y_1), K\Big) \subset B_{\delta/2}(\kappa_0).$$

In particular, $\mathrm{co}\Big(\mathrm{pr}(B_{\epsilon_1}(y_1), K)\Big) \subseteq B_{\delta/2}(\kappa_0)$ and

$$\mathrm{diam}\Big(\mathrm{pr}\left(B_{\epsilon_1}(y_1), K\right)\Big) \leq \mathrm{diam}(B_{\delta/2}(\kappa_0)) = \delta. \tag{12.33}$$

From (12.31) it follows that $\mathrm{pr}(\cdot, \partial K_\rho^+)$ is continuous at x_0, so that

$$\exists\, \epsilon_0 \in (0, \epsilon) \;:\; \mathrm{pr}\Big(B_{\epsilon_0}(x_0), \partial K_\rho^+\Big) \subset B_{\epsilon_1}(y_1); \tag{12.34}$$

see Figure 12.3.

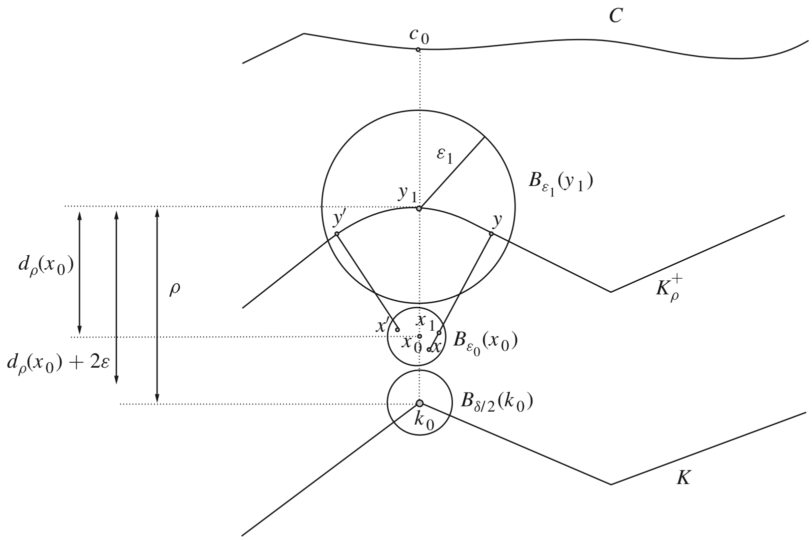

Figure 12.3. The construction in the proof of Theorem 12.4.

[12] Let $B \subset \mathbb{R}^n$ be a nonempty closed set and $z \notin B$. If $\mathrm{pr}(z, B) = \{x\}$ is a singleton, then $\mathrm{pr}(\cdot, B)$ is continuous at the point z, *i.e.*, for any $\epsilon > 0$ there exists $\delta > 0$ such that $\mathrm{pr}(B_\delta(z), B) \subset B_\epsilon(x)$. The proof of this assertion is similar to that given in [142, Theorem 4.8 (4)]. Indeed, assume by contradiction that there exist $\epsilon > 0$ and a sequence (z_n) of points with $z_n \in B_{1/n}(z)$ and $\mathrm{dist}(\mathrm{pr}(z_n, B), x) \geq \epsilon$ for any $n \in \mathbb{N}$. Then (z_n) converges to z, and there are points $x_n \in \mathrm{pr}(z_n, B)$ such that $|x_n - x| \geq \epsilon$. The sequence (x_n) is bounded, since $|x_n - z| \leq |x_n - z_n| + |z_n - z| \leq \mathrm{dist}(z_n, B) + \frac{1}{n}$. Hence, possibly passing to a (not relabelled) subsequence, we have that (x_n) converges to a point $y \in B$. But then $|y - x| \geq \epsilon$, and $y \in \mathrm{pr}(z, B)$ (see [74, Theorem 3.3.6]), a contradiction.

Now, let us prove that $\mathrm{pr}(\cdot, \partial K_\rho^+)$ is Lipschitz continuous in $B_{\epsilon_0}(x_0)$ (and in particular single valued on $B_{\epsilon_0}(x_0)$), and more precisely that

$$\begin{gathered} x, x' \in B_{\epsilon_0}(x_0),\ \ y \in \mathrm{pr}\ (x, \partial K_\rho^+),\ y' \in \mathrm{pr}(x', \partial K_\rho^+) \\ \Downarrow \\ |y - y'| \ \le \frac{4(d_\rho(x_0) + 2\epsilon)}{\epsilon}|x - x'|. \end{gathered} \tag{12.35}$$

We cannot directly apply Proposition 12.7 with the choice $\overline{x} = x$ and $\underline{x} = x'$, since in general

$$d_\rho(x) \neq d_\rho(x').$$

However, we can assume without loss of generality that

$$d_\rho(x) \ge d_\rho(x').$$

Define

$$x_1 := x + \frac{(y - x)}{|y - x|}(d_\rho(x) - d_\rho(x')). \tag{12.36}$$

Then

$$d_\rho(x_1) = d_\rho(x'),$$

and

$$y \in \mathrm{pr}(x_1, \partial K_\rho^+);$$

see Figure 12.3.

Moreover, since $x \in B_{\epsilon_0}(x_0)$, from (12.34) we have $y \in B_{\epsilon_1}(y_1)$; therefore, from (12.33) and (12.32) we have

$$\begin{aligned} \mathrm{diam}\Big(\mathrm{pr}(y, K)\Big) &\le \mathrm{diam}\Big(\mathrm{pr}\left(B_{\epsilon_1}(y_1), K\right)\Big) \le \delta \\ &= \left[\rho^2 - (d_\rho(x_0) + 2\epsilon)^2\right]^{1/2}. \end{aligned} \tag{12.37}$$

Now, note that the triangular property of the distance and the inclusion $x' \in B_{\epsilon_0}(x_0)$ imply

$$d_\rho(x') \le d_\rho(x_0) + \epsilon_0 \le d_\rho(x_0) + \epsilon.$$

Hence, from (12.37) we deduce

$$\mathrm{diam}\Big(\mathrm{pr}(y, K)\Big) \le \left[\rho^2 - (d_\rho(x') + \epsilon)^2\right]^{1/2} = \left[\rho^2 - (d_\rho(x_1) + \epsilon)^2\right]^{1/2}.$$

Note also that $y \notin \mathrm{co}(\mathrm{pr}(y, K))$, since $y \in \partial K_\rho^+$ and $\mathrm{diam}(\mathrm{pr}(y, K)) < \rho$. Therefore, observing that

$$\epsilon = \frac{\rho - d_\rho(x_0)}{3} < \rho - d_\rho(x_1),$$

we can apply Proposition 12.7 with the choices

$$\overline{x} = x_1, \quad \underline{x} = x', \quad \overline{y} = y, \quad \underline{y} = y',$$

and we obtain

$$|y - y'| \leq \frac{2(d_\rho(x_1) + \epsilon)}{\epsilon} |x_1 - x'|. \tag{12.38}$$

Since d_ρ is one-Lipschitz and $x_1 \in B_{\epsilon_0}(x_0)$, we have

$$d_\rho(x_1) - d_\rho(x_0) \leq |x_1 - x_0| \leq \epsilon_0 \leq \epsilon. \tag{12.39}$$

Our substituting (12.39) into (12.38) gives

$$|y - y'| \leq \frac{2(d_\rho(x_0) + 2\epsilon)}{\epsilon} |x_1 - x'|. \tag{12.40}$$

Recalling the definition (12.36) of x_1, it follows that

$$|x_1 - x'|^2 \leq 2|x - x'|^2 + 2(d_\rho(x) - d_\rho(x'))^2 \leq 4|x - x'|^2. \tag{12.41}$$

Inserting (12.41) into (12.40) we obtain (12.35).

Inequality (12.35) shows that $\mathrm{pr}(\cdot, \partial K_\rho^+)$ is Lipschitz continuous in $B_{\epsilon_0}(x_0)$. From the expression of $\mathrm{pr}(\cdot, \partial K_\rho^+)$ (see (1.2) in Chapter 1) it follows that $d_\rho \in \mathcal{C}^{1,1}(B_{\epsilon_0}(x_0))$. □

12.2. Appendix

In this appendix we briefly recall some properties of the distance function from a closed set (see [74] and [32]). Let $\Omega \subseteq \mathbb{R}^n$ be an open set.

Definition 12.8 (Semiconcavity). A function $u : \Omega \to \mathbb{R}$ is said to be semiconcave in Ω if there exists a constant $C \geq 0$ such that $x \to u(x) - \frac{C}{2}|x|^2$ is concave in any ball contained in Ω. If u is semiconcave on any open relatively compact subset of Ω, then u is said locally semiconcave in Ω.

If u is such that $-u$ is semiconcave in Ω, then u is said to be semiconvex in Ω. Similarly, if u is such that $-u$ is locally semiconcave in Ω, then u is said to be locally semiconvex in Ω.

Remark 12.9. It is possible to prove (see, *e.g.*, [74, Corollary 3.3.8], [60], and the references in [141]) that if u is both semiconcave and semiconvex in the convex open set Ω, then $u \in \mathcal{C}^{1,1}(\Omega)$, that is $u \in \mathrm{Lip}(\Omega)$ and $\nabla u \in \mathrm{Lip}(\Omega; \mathbb{R}^n)$.

The distance function from a closed set is not necessarily of class $\mathcal{C}^{1,1}$; however its regularity is expressed by the following optimal result (see [15, 74]).

Theorem 12.10 (Semiconcavity of the distance). *Let C be a nonempty closed subset of $\mathbb{R}^n$. Then the function* $\mathrm{dist}(\cdot, C)$ *is locally semiconcave in $\mathbb{R}^n \setminus C$, and the function* $\mathrm{dist}(\cdot, C)^2$ *is semiconcave in $\mathbb{R}^n$.*

Definition 12.11 (Sets of class $\mathcal{C}^{1,1}$). Let $E \subset \mathbb{R}^n$ be a closed set. We write $\partial E \in \mathcal{C}^{1,1}$ if there exists an open set U containing ∂E such that $d(\cdot, E) \in \mathcal{C}^{1,1}(\mathrm{U})$.

The following theorem can be found in [114, Section 5.4], [115, Theorems 5.1, 5.2], [273, Section 11, Proposition 13.8] (see also [271] for several relations of the distance function versus the regularity of the boundary).

Theorem 12.12 ($\mathcal{C}^{1,1}$-smoothness). *Let ∂E be compact. Then $\partial E \in \mathcal{C}^{1,1}$ if and only if given $x \in \partial E$, there exists a neighbourhood B of x such that $B \cap \partial E$ can be written as the graph of a $\mathcal{C}^{1,1}$ function v of $n-1$ variables with respect to a suitable orthogonal system of coordinates, and $B \cap E$ is the subgraph of v in B.*

We conclude this chapter with a brief list of some results on the distance function, which are related to the $\mathcal{C}^{1,1}$-smoothness of the boundary of a set.

Let $C \subset \mathbb{R}^n$ be a closed set. Denote by $\mathrm{Unp}(C)$ be the set of all points $z \in \mathbb{R}^n$ such that $\mathrm{pr}(z, C)$ is a singleton. Given $x \in C$, we set $\mathrm{reach}(C, x) := \sup\{r > 0 : B_x(r) \subset \mathrm{Unp}(C)\}$, and $\mathrm{reach}(C) := \inf\{\mathrm{reach}(C, x) : x \in C\}$. The set C is said to be of positive reach if $\mathrm{reach}(C) > 0$. These sets were introduced in [142].

Theorem 12.13. *Let $C \subset \mathbb{R}^n$ be a nonempty closed set. Then* $\nabla \mathrm{dist}(\cdot, C)$ *is continuous on* $\mathrm{int}(\mathrm{Unp}(C) \setminus C)$. *Moreover, if* $\mathrm{reach}(C) > 0$ *and if* $0 < s < r < \mathrm{reach}(C)$, *then* $\nabla \mathrm{dist}(\cdot, C)$ *is Lipschitz continuous on the set* $\{z \in \mathbb{R}^n : s \le \mathrm{dist}(z, C) \le r\}$.

Proof. See [142, Theorem 4.8, items (5) and (9)]. □

Theorem 12.14. *∂E is of class $\mathcal{C}^{1,1}$ if and only if ∂E has positive reach.*

Proof. See [142, 4.20], [148, Section 2.1]. □

Theorem 12.15. *Let* $u \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^n)$. *Then* subgraph$(u)$ *has positive reach if and only if* u *is semiconcave*.

Proof. See [148, Theorems 2.3, 2.6]. □

Chapter 13
The avoidance principle

In this chapter we prove a monotonicity property of the distance between the complement of two barriers (Theorem 13.3). For this latter result we will use Ilmanen's interposition lemma, proved in Chapter 12. A by-product of this theorem is a remarkable formula in the theory of barriers, that gives the relation between the outer regularization starting from a set E and the inner regularization starting from the complement $\mathbb{R}^n \setminus E$ of E (see formula (13.20)).

We begin with the following result, which is a consequence of the evolution equations derived in Section 6.1, combined with the weak maximum principle.

We recall our notation: if $\partial E \in \mathcal{C}^\infty$ we let $d(\cdot) := d(\cdot, E)$ be the signed distance function from ∂E negative in the interior of E. We also write, as usual, $\mathcal{B}(\mathcal{F})$ in place of $\mathcal{B}(\mathcal{F}, I)$, to denote the set of all barriers for the family $\mathcal{F}$ in the interval I, where $I = [t_0, +\infty)$ (see Definitions 9.2 and 9.4).

Theorem 13.1 (Lower bound for the maximal time). *Let $\partial E \in \mathcal{C}^\infty$ be compact. Let $t_{\max}(\partial E) \in (0, +\infty)$ be the maximal existence time of the smooth mean curvature flow f starting from ∂E. Then*

$$t_{\max}(\partial E) \geq \frac{1}{2 \max\limits_{x \in \partial E} |\nabla^2 d(x)|^2}.$$

Proof. We will use the evolution equations derived in Chapter 6, which are obtained by looking at the parametric version of the flow. It is enough to reason separately for each connected component of ∂E, since we know that the smooth mean curvature flow of two connected components of ∂E cannot intersect each other. Therefore we prove the theorem using the map $\varphi \in \mathcal{X}([0, t_{\max}(\partial E)); \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n))$ parametrizing f, supposing ∂E connected.

Define the function $\mathrm{a} : [0, t_{\max}(\partial E)) \to [0, +\infty)$ as

$$\mathrm{a}(t) := \max_{s \in \mathcal{S}} |A(t, s)|_{g(t)}, \qquad t \in [0, t_{\max}(\partial E)),$$

where $A(t,\cdot)$ is the second fundamental form of $\partial f(t)$ and $|A(t,\cdot)|^2_{g(t)}$ is its squared length[(1)]. Note that

$$\mathrm{a}(0) = \max_{x\in\partial E} |\nabla^2 d(x)|.$$

From Lemma 5.2 it follows that $\mathrm{a} \in \mathrm{Lip}([0,t])$ for any $t \in (0, t_{\max}(\partial E))$. From equation (6.37), which gives the evolution equation of $|A(t,\cdot)|^2_{g(t)}$, and the weak maximum principle (Theorem 5.12), it follows that

$$\frac{d}{dt}(\mathrm{a})^2 \le 2\mathrm{a}^4 \qquad \text{a.e. in } [0, t_{\max}(\partial E)). \tag{13.1}$$

Notice that $\mathrm{a}(t) > 0$ for any $t \in [0, t_{\max}(\partial E))$. Indeed, $\mathrm{a}(t) = 0$ for some $t \in (0, t_{\max}(\partial E))$ implies that $\partial f(t)$ is a hyperplane, which is not compact. Therefore, from inequality (13.1) it follows that

$$\frac{d}{dt}\mathrm{a} \le \mathrm{a}^3 \qquad \text{a.e. in } [0, t_{\max}(\partial E)).$$

The solution of the Cauchy problem

$$\begin{cases} \dot y = y^3, \\ y(0) = \mathrm{a}(0) > 0, \end{cases}$$

is given by

$$y(t) = \frac{1}{\sqrt{2}\left(\frac{1}{2(\mathrm{a}(0))^2} - t\right)^{1/2}}, \qquad t \in \left[0, \frac{1}{2(\mathrm{a}(0))^2}\right). \tag{13.2}$$

Therefore by comparison it follows that

$$\mathrm{a} \le y \tag{13.3}$$

in the interval $\left[0, \min\left(t_{\max}(\partial E), \frac{1}{2(\mathrm{a}(0))^2}\right)\right)$. Now, we can conclude the proof of the theorem, by showing that

$$t_{\max}(\partial E) \ge \frac{1}{2(\mathrm{a}(0))^2}.$$

(1) See (1.51) and (6.35).

Indeed, if by contradiction $t_{\max}(\partial E) < \dfrac{1}{2(\mathrm{a}(0))^2}$, then formulas (13.3) and (13.2) imply

$$\sup_{t\in[0,t_{\max}(\partial E))} \mathrm{a}(t) < +\infty.$$

In view Theorem 6.19, all covariant derivatives of A remain uniformly bounded in $[0, t_{\max}(\partial E))$, and the parametric mean curvature flow can be restarted at time $t_{\max}(\partial E)$. This contradicts the definition of $t_{\max}(\partial E)$, since $t_{\max}(\partial E)$ coincides with the maximal time of smooth existence of the parametric flow. □

We also record the following observation (see Lemma 9.26), being the family $\mathcal{F}$ translation invariant.

Remark 13.2. Let $\phi \in \mathcal{B}(\mathcal{F})$, $f : [a, b] \subset I \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{F}$ and assume that $f(a) \subseteq \phi(a)$. Define

$$\delta(t) := \mathrm{dist}\left(f(t), \mathbb{R}^n \setminus \phi(t)\right), \qquad t \in [a, b],$$

and suppose

$$\delta(a) > 0.$$

Then δ is nondecreasing in $[a, b]$.

Remark 13.2, in general, does not hold if f is replaced by a barrier, since in this case the barrier could grow instantly. The aim of the last part of this chapter is to generalize Remark 13.2 when f is replaced by the complement $\mathbb{R}^n \setminus \psi$ of a barrier ψ.

13.1. Distance between the complement of two barriers

In this section we want to show the following theorem[(2)], the proof of which is taken from [183, 54, 57][(3)].

Theorem 13.3 (Avoidance principle). *Let $\phi, \psi \in \mathcal{B}(\mathcal{F})$. Assume that*

$$\text{either } \mathbb{R}^n \setminus \mathrm{int}\,(\phi(t_0)) \text{ or } \mathbb{R}^n \setminus \mathrm{int}\,(\psi(t_0)) \text{ is compact.}$$

Define the function

$$\eta(t) := \mathrm{dist}\Big(\mathbb{R}^n \setminus \psi(t), \mathbb{R}^n \setminus \phi(t)\Big), \qquad t \in I,$$

(2) With a slightly different notation, this result was called avoidance principle by Ilmanen [183, Section 4E].

(3) In [54] and [57] the validity of the avoidance principle is extended to mean curvature flow with a forcing term.

and suppose that

$$\eta(t_0) > 0. \tag{13.4}$$

Then η is nondecreasing in I.

In order to prove Theorem 13.3 we need some preparation.

Lemma 13.4 (Approximation of a compact boundary of class $\mathcal{C}^{1,1}$). *Let $E \subset \mathbb{R}^n$ be an open set with $\partial E \in \mathcal{C}^{1,1}$ and compact. Then there exists a sequence (E_h) of open sets with $\partial E_h \in \mathcal{C}^\infty$ such that, denoting by*

$$d_h(z) := \mathrm{dist}(z, E_h) - \mathrm{dist}(z, \mathbb{R}^n \setminus E_h), \qquad z \in \mathbb{R}^n, \tag{13.5}$$

the signed distance function from ∂E_h negative in the interior of E_h, the following holds: for any $\delta > 0$ there is $\overline{h} \in \mathbb{N}$ so that

(i) $\{d_h = 0\} \subset (\partial E)^+_\delta$ *for any $h \in \mathbb{N}$, $h \geq \overline{h}$,*

(ii) $\sup\limits_{h \geq \overline{h}} \sup\limits_{x \in \partial E_h} |\nabla^2 d_h(x)| < +\infty.$

Proof. Recalling Definition 12.11, we can choose $\delta > 0$ such that $d \in \mathcal{C}^{1,1}((\partial E)^+_{3\delta})$, where $d(\cdot) := d(\cdot, E)$ is the signed distance from ∂E negative in the interior of E. For $h \in \mathbb{N}$, let J_h be a standard smooth nonnegative radially symmetric convolution kernel having support in the ball $B_{\frac{1}{h}}(0)$ of radius $\frac{1}{h}$ and centered at the origin, and define

$$g_h(z) := d * J_h(z) = \int_{B_{\frac{1}{h}}(0)} d(z-y) J_h(y)\, dy, \qquad z \in \mathbb{R}^n.$$

Then $g_h \in \mathcal{C}^\infty(\mathbb{R}^n) \cap \mathrm{Lip}(\mathbb{R}^n)$. Set now

$$V := (\partial E)^+_\delta.$$

Let $\tilde{h} \in \mathbb{N}$ be such that $\tilde{h} > \dfrac{1}{\delta}$. We claim that

$$\sup_{h \geq \tilde{h}} \sup_{z \in V} |\nabla^2 g_h(z)|^2 < +\infty. \tag{13.6}$$

Indeed, for $i, j \in \{1, \dots, n\}$, $h \geq \tilde{h}$, and for every $z \in V$, we have, using Jensen's inequality,

$$\begin{aligned}
\left(\nabla^2_{ij} g_h(z)\right)^2 &= \left(\int_{B_{\frac{1}{h}}(0)} \nabla^2_{ij} d(z-y) g_h(y)\, dy\right)^2 \\
&\leq \int_{B_{\frac{1}{h}}(0)} \left(\nabla^2_{ij} d(z-y)\right)^2 g_h(y)\, dy \leq \sup_{z \in (\partial E)^+_{2\delta}} |\nabla^2 d(z)|^2.
\end{aligned}$$

Since the square norm of the Hessian of d in $(\partial E)^+_{2\delta}$ is essentially bounded, (13.6) follows.

Moreover, for $h \in \mathbb{N}$ large enough, we have

$$V \subset \{1/2 \leq |\nabla g_h| \leq 3/2\}, \tag{13.7}$$

because the sequence (g_h) converges to d in $\mathcal{C}^1(K)$ for any compact set $K \subset \mathbb{R}^n$, and d satisfies the eikonal equation

$$|\nabla d|^2 = 1 \qquad \text{in } V.$$

We now observe that the uniform convergence of (g_h) to d on compact subsets of $\mathbb{R}^n$ implies that, for $h \in \mathbb{N}$ large enough,

$$\{g_h = 0\} \subset V, \tag{13.8}$$

since g_h must be negative on $\{d = -\delta\}$ (respectively, positive on $\{d = \delta\}$).

Define

$$E_h := \{g_h \leq 0\}.$$

Then by (13.8), (13.7) and the implicit function theorem, it follows that ∂E_h is a compact $(n-1)$-dimensional manifold of class $\mathcal{C}^\infty$ and, for d_h defined as in (13.5),

$$\partial E_h = \{d_h = 0\} \subset V. \tag{13.9}$$

By (1.21) and (1.25) we have, for $i, j \in \{1, \dots, n\}$,

$$\nabla^2_{ij} d_h = \left(\mathrm{Id}_{il} - \frac{\nabla_i g_h}{|\nabla g_h|}\frac{\nabla_l g_h}{|\nabla g_h|}\right)\frac{\nabla^2_{lm} g_h}{|\nabla g_h|}\left(\mathrm{Id}_{mj} - \frac{\nabla_m g_h}{|\nabla g_h|}\frac{\nabla_j g_h}{|\nabla g_h|}\right) \quad \text{on } \partial E_h.$$

Whence, from (13.7), (13.6) and (13.9), it follows that there exists $\overline{h} \in \mathbb{N}$ such that

$$\sup_{h \geq \overline{h}} \|\nabla^2 d_h\|_{L^\infty(\partial E_h)} < +\infty. \qquad \square$$

We are now in a position to prove Theorem 13.3.

13.2. Proof of Theorem 13.3

Without loss of generality, we suppose that

$$\mathbb{R}^n \setminus \operatorname{int}(\psi(t_0)) \text{ is compact.}$$

Let $s \in I$; recalling Example 9.8, for any $t \in I$ with $t > s$ we have

$$\mathbb{R}^n \setminus \mathrm{int}\,(\phi(t)) \subseteq \{z \in \mathbb{R}^n : \mathrm{dist}\,(z, \mathbb{R}^n \setminus \phi(s)) \leq \varrho(t-s)\},$$
$$\mathbb{R}^n \setminus \mathrm{int}\,(\psi(t)) \subseteq \{z \in \mathbb{R}^n : \mathrm{dist}\,(z, \mathbb{R}^n \setminus \psi(s)) \leq \varrho(t-s)\},$$

where the function ϱ is defined in (9.7), namely

$$\varrho(\tau) := \sqrt{2(n-1)\tau}, \qquad \tau \in [0, +\infty).$$

In particular,

$$\mathbb{R}^n \setminus \mathrm{int}\,(\psi(t)) \qquad \text{is compact for any } t \in I. \tag{13.10}$$

We now intend to show that

$$\eta(t) \geq \eta(t_0), \qquad t \in I. \tag{13.11}$$

Let us argue by contradiction. Assume that (13.11) is false, and let

$$t^* := \inf\{t \in I : \eta(t) < \eta(t_0)\} < +\infty.$$

Suppose first that $t^* > t_0$. Using the triangular property of η and reasoning in a similar way to the proof of inequality (9.10), it follows that

$$s, t \in I,\ t > s \quad \Rightarrow \quad \eta(t) \geq \eta(s) - 2\varrho(t-s). \tag{13.12}$$

From (13.12) it follows

$$\eta(t^*) \geq \eta(s) - 2\varrho(t^* - s), \qquad s \in [t_0, t^*).$$

Using the definition of t^* we have $\eta(s) \geq \eta(t_0)$ for $s \in [t_0, t^*)$, hence

$$\eta(t^*) \geq \eta(t_0) - 2\varrho(t^* - s), \qquad s \in [t_0, t^*).$$

Letting $s \uparrow t^*$ it follows that

$$\eta(t^*) \geq \eta(t_0).$$

In particular, $\eta(t^*) > 0$. On the other hand, if $t^* = t_0$, we have $\eta(t^*) > 0$ by assumption (13.4).

Recalling (13.10), we can now apply Ilmanen's interposition lemma (Theorem 12.1) with the following choice:

$$C = \mathbb{R}^n \setminus \mathrm{int}(\phi(t^*)), \qquad K = \mathbb{R}^n \setminus \mathrm{int}(\psi(t^*)).$$

It follows that there exists a bounded open set $A \subset \mathbb{R}^n$ with the following properties:

$$\begin{cases} \partial A \in \mathcal{C}^{1,1}, \\ \mathbb{R}^n \setminus \mathrm{int}(\psi(t^*)) \subset A \subset\subset \mathrm{int}(\phi(t^*)), \\ \eta(t^*) = \mathrm{dist}\Big(\mathbb{R}^n \setminus \mathrm{int}\,(\phi(t^*))\,,\, \partial A\Big) + \mathrm{dist}\Big(\partial A, \mathbb{R}^n \setminus \mathrm{int}\,(\psi(t^*))\Big). \end{cases}$$

Now, we would like to flow ∂A by mean curvature; however ∂A is not smooth enough to apply the local existence result proven in Chapter 7, and therefore it must be regularized. Using Lemma 13.4, for any $\epsilon \in (0, 1)$ we can find a bounded open set A_ϵ satisfying the following properties[(4)]:

$$\partial A_\epsilon \in \mathcal{C}^\infty,$$

$$\mathbb{R}^n \setminus \mathrm{int}(\psi(t^*)) \subset A_\epsilon \subset\subset \mathrm{int}(\phi(t^*)),$$

$$\begin{aligned} &\mathrm{dist}\left(\mathbb{R}^n \setminus \mathrm{int}\left(\phi(t^*)\right), \partial A_\epsilon\right) + \mathrm{dist}\left(\partial A_\epsilon, \mathbb{R}^n \setminus \mathrm{int}\left(\psi(t^*)\right)\right) \\ &\geq \eta(t^*) - \epsilon. \end{aligned} \tag{13.13}$$

Moreover the L^∞ norm of the second fundamental form of ∂A_ϵ is uniformly bounded with respect to $\epsilon \in (0, 1)$. Taking into account this latter uniform bound, we now apply Theorem 13.1: it follows that there exists

$$\tau > 0,$$

that may depend on t^* but can be chosen independently of ϵ, such that the smooth compact mean curvature flow f_ϵ starting from $\overline{A_\epsilon}$ is well defined for $t \in [t^*, t^* + \tau]$, for any $\epsilon \in (0, 1)$. Therefore

$$\text{the map } f_\epsilon : t \in [t^*, t^* + \tau] \to f_\epsilon(t) \text{ belongs to } \mathcal{F}. \tag{13.14}$$

Using the triangular property of the distance we have

$$\begin{aligned} \eta(t) \geq\ &\mathrm{dist}\left(\mathbb{R}^n \setminus \phi(t), f_\epsilon(t)\right) \\ &+ \mathrm{dist}\left(\mathbb{R}^n \setminus f_\epsilon(t), \mathbb{R}^n \setminus \psi(t)\right), \quad t \in [t^*, t^* + \tau]. \end{aligned} \tag{13.15}$$

[(4)] Observe that, for E, E_h as in Lemma 13.4, given $\epsilon > 0$, there exists $\underline{h} \in \mathbb{N}$ such that, for any $h \geq \underline{h}$,

$$\begin{aligned} &E \subset \{z \in \mathbb{R}^n : \mathrm{dist}(z, E_h) < \epsilon\}, \qquad E_h \subset \{z \in \mathbb{R}^n : \mathrm{dist}(z, E) < \epsilon\}, \\ &\partial E \subset \{z \in \mathbb{R}^n : \mathrm{dist}(z, \partial E_h) < \epsilon\}, \qquad \partial E_h \subset \{z \in \mathbb{R}^n : \mathrm{dist}(z, \partial E) < \epsilon\}. \end{aligned}$$

From (13.14) and Remark 13.2[5] we have

$$\mathrm{dist}\left(\mathbb{R}^n \setminus \phi(t), f_\epsilon(t)\right) \geq \mathrm{dist}\left(\mathbb{R}^n \setminus \phi(t^*),\ f_\epsilon(t^*)\right), \quad t \in [t^*, t^*+\tau]. \quad (13.16)$$

Similarly

$$\begin{aligned} &\mathrm{dist}(\mathbb{R}^n \setminus f_\epsilon(t), \mathbb{R}^n \setminus \psi(t)) \\ \geq &\,\mathrm{dist}(\mathbb{R}^n \setminus f_\epsilon(t^*), \mathbb{R}^n \setminus \psi(t^*)),\ \ t \in [t^*, t^*+\tau]. \end{aligned} \quad (13.17)$$

From (13.15), (13.16), (13.17) and (13.13) we deduce

$$\begin{aligned} \eta(t) &\geq \mathrm{dist}\left(\mathbb{R}^n \setminus \phi(t^*), f_\epsilon(t^*)\right) + \mathrm{dist}\left(\mathbb{R}^n \setminus f_\epsilon(t^*), \mathbb{R}^n \setminus \psi(t^*)\right) \\ &\geq \eta(t^*) - \epsilon, \qquad t \in [t^*, t^* + \tau]. \end{aligned}$$

Letting $\epsilon \downarrow 0$ we get

$$\eta(t) \geq \eta(t^*) \geq \eta(t_0), \qquad t \in [t^*, t^* + \tau],$$

which contradicts the definition of t^*. The proof of inequality (13.11) is therefore concluded.

Let us now observe that, from (13.11), it follows that η is nondecreasing. Indeed, let $s, t \in I$ be with $s < t$. From (13.11) we have $\eta(s) \geq \eta(t_0)$, hence $\eta(s) > 0$. From (13.10) we have that $\mathbb{R}^n \setminus \mathrm{int}(\psi(s))$ is compact. Moreover, the restrictions $\phi_{|[s,+\infty)}$, $\psi_{|[s,+\infty)}$ are barriers for the family $\mathcal{F}$ in the interval $[s, +\infty)$. Applying what we have just proved, it follows that $\eta(\sigma) \geq \eta(s)$ for any $\sigma \geq s$, in particular $\eta(t) \geq \eta(s)$. $\square$

We can use Theorem 13.3 to improve our knowledge on the evolution of the complement of a set E. Remember that in Proposition 10.9 we have proven that

$$\mathcal{M}_\star(\mathbb{R}^n \setminus E, \mathcal{F}) \subseteq \mathbb{R}^n \setminus \mathcal{M}^\star(E, \mathcal{F}). \quad (13.18)$$

A consequence of the avoidance principle is the following.

Theorem 13.5 (Complement of the inner and outer regularizations). *Let E be a subset of $\mathbb{R}^n$ such that*

$$\text{either } \overline{E} \text{ or } \mathbb{R}^n \setminus \mathrm{int}(E) \text{ is compact.} \quad (13.19)$$

Then

$$\mathcal{M}_\star(\mathbb{R}^n \setminus E, \mathcal{F}) = \mathbb{R}^n \setminus \mathcal{M}^\star(E, \mathcal{F}). \quad (13.20)$$

[5] Applied with $[a, b]$ replaced by $[t^*, t^* + \tau]$.

Proof. In view of (13.18), we need to prove that

$$\mathcal{M}_\star(\mathbb{R}^n \setminus E, \mathcal{F}) \supseteq \mathbb{R}^n \setminus \mathcal{M}^\star(E, \mathcal{F}). \tag{13.21}$$

We claim that, for any $\rho, \epsilon > 0$, we have

$$\mathcal{M}\left(E^+_{\rho+\epsilon}, \mathcal{F}\right) \supseteq \mathbb{R}^n \setminus \mathcal{M}\left(\mathbb{R}^n \setminus E^+_\rho, \mathcal{F}\right). \tag{13.22}$$

To prove the claim, set

$$\phi := \mathcal{M}\left(E^+_{\rho+\epsilon}, \mathcal{F}\right), \qquad \psi := \mathcal{M}\left(\mathbb{R}^n \setminus E^+_\rho, \mathcal{F}\right).$$

Then $\phi \in \mathcal{B}(\mathcal{F})$ and $\psi \in \mathcal{B}(\mathcal{F})$, and by our assumption (13.19) on E,

$$\text{either } \mathbb{R}^n \setminus \mathrm{int}(\phi(t_0)) \text{ or } \mathbb{R}^n \setminus \mathrm{int}(\psi(t_0)) \text{ is compact.}$$

Set

$$\eta(t) := \mathrm{dist}(\mathbb{R}^n \setminus \phi(t), \mathbb{R}^n \setminus \psi(t)), \qquad t \in I,$$

so that in particular

$$\eta(t_0) = \mathrm{dist}\left(\mathbb{R}^n \setminus E^+_{\rho+\epsilon}, E^+_\rho\right) = \epsilon > 0.$$

We can therefore apply the avoidance principle, and we obtain

$$\eta(t) > 0, \qquad t \in I.$$

Hence

$$\mathcal{M}\left(E^+_{\rho+\epsilon}, \mathcal{F}\right) \supseteq \mathbb{R}^n \setminus \mathrm{int}\,(\psi) \supseteq \mathbb{R}^n \setminus \psi = \mathbb{R}^n \setminus \mathcal{M}\left(\mathbb{R}^n \setminus E^+_\rho, \mathcal{F}\right),$$

which proves claim (13.22).

Recalling the definitions of $\mathcal{M}_\star(\mathbb{R}^n \setminus E, \mathcal{F})$ and $\mathcal{M}^\star(E, \mathcal{F})$ and using inclusion (13.22), we obtain

$$\begin{aligned}\mathcal{M}_\star(\mathbb{R}^n \setminus E, \mathcal{F}) &= \bigcup_{\rho>0} \mathcal{M}\left(\mathbb{R}^n \setminus E^+_\rho, \mathcal{F}\right) \\ &\supseteq \bigcup_{\rho,\epsilon>0} \left[\mathbb{R}^n \setminus \mathcal{M}\left(E^+_{\rho+\epsilon}, \mathcal{F}\right)\right] \\ &= \mathbb{R}^n \setminus \bigcap_{\rho,\epsilon>0} \mathcal{M}\left(E^+_{\rho+\epsilon}, \mathcal{F}\right) = \mathbb{R}^n \setminus \mathcal{M}^\star(E, \mathcal{F}).\end{aligned}$$

This proves (13.21), and concludes the proof of (13.20). $\square$

The avoidance principle will be used in the next chapter, for comparing the barriers with a generalized evolution of sets.

Chapter 14
Comparison between barriers and a generalized evolution

Recalling that the family $\mathcal{F}$ consists of smooth compact local in time mean curvature flows (see Definition 9.4), in this chapter we prove a result[1] which relates the evolutions $\mathcal{M}(E,\mathcal{F})$, $\mathcal{M}_\star(E,\mathcal{F})$ and $\mathcal{M}^\star(E,\mathcal{F})$ defined in Chapters 9 and 10 to a suitable abstract evolution law R, that we call comparison flow. As a consequence, we will obtain the relations between $\mathcal{M}(E,\mathcal{F})$, $\mathcal{M}_\star(E,\mathcal{F})$, $\mathcal{M}^\star(E,\mathcal{F})$ and the level set evolution for mean curvature flow.

14.1. Comparison flows

Let us give the definition of comparison flow. We take as usual $I = [t_0, +\infty)$ for some $t_0 \in \mathbb{R}$. We also denote by $\mathcal{B}(\mathcal{F}) = \mathcal{B}(\mathcal{F}, I)$ the set of all barriers with respect to $\mathcal{F}$ in the interval I.

Notation. We denote by

$$\mathscr{L} \subseteq \mathcal{P}(\mathbb{R}^n)$$

a family of subsets of $\mathbb{R}^n$ which contains the open sets A such that either A or $\mathbb{R}^n \setminus A$ is bounded, and the closed sets C such that either C or $\mathbb{R}^n \setminus C$ is bounded.

Definition 14.1 (Comparison flow). We say that R is a comparison flow if, given

$$(E, \tau) \in \mathscr{L} \times I,$$

the following properties hold:

(1) $R(E,\tau) : [\tau, +\infty) \to \mathscr{L}$ and

$$R(E,\tau)(\tau) = E;$$

(1) See [54], and also [53, 57].

(2) for any $A \in \mathscr{L}$, any $t_1, t_2 \in I$ with $t_1 \leq t_2$, if we set

$$B := R(A, t_1)(t_2),$$

then

$$R(A, t_1)(t) = R(B, t_2)(t), \qquad t \in [t_2, +\infty);$$

(3) if $f : [a, b] \subset I \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{F}$, then

$$f(t) = R(f(a), a)(t), \qquad t \in [a, b];$$

(4) if $A_1, A_2 \in \mathscr{L}$ are such that

$$A_1 \subseteq A_2,$$

and if $t \in I$, then

$$R(A_1, t)(\tau) \subseteq R(A_2, t)(\tau), \qquad \tau \in [t, +\infty).$$

For notational simplicity, we have dropped the dependence on $\mathscr{L}$ of the comparison flow R.

Items (2), (3) and (4) in Definition 14.1 express the semigroup property, the consistency property and the inclusion property respectively, of a comparison flow (for A_1 and A_2 with compact boundary).

In order to specify the relations between a comparison flow and the barriers theory (Theorem 14.4), we need some preliminary observations.

Lemma 14.2 (A comparison flow and its interior are barriers). *Let R be a comparison flow and let $E \in \mathscr{L}$. Then*

$$R(E, t_0) \in \mathcal{B}(\mathcal{F}) \tag{14.1}$$

and

$$\mathrm{int}\,(R(E, t_0)) \in \mathcal{B}(\mathcal{F})\,. \tag{14.2}$$

Proof. Let $f : [a, b] \subset I \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{F}$, and assume that

$$f(a) \subseteq R(E, t_0)(a).$$

In order to prove (14.1) we have to show that

$$f(b) \subseteq R(E, t_0)(b).$$

As $f \in \mathcal{F}$, by the consistency property (3) of Definition 14.1, we have

$$f(t) = R(f(a), a)(t), \qquad t \in [a, b].$$

Therefore, using the inclusion property (4) and the semigroup property (2) of a comparison flow, we get

$$f(b) = R(f(a), a)(b) \subseteq R(R(E, t_0)(a), a)(b) = R(E, t_0)(b).$$

Let us prove (14.2). Let $f : [a, b] \subset I \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{F}$, and assume that

$$f(a) \subseteq \operatorname{int}(R(E, t_0)(a)). \tag{14.3}$$

We have to show that

$$f(b) \subseteq \operatorname{int}(R(E, t_0)(b)).$$

Define

$$\phi(t) := R(E, t_0)(t), \qquad t \in I.$$

Then $\phi \in \mathcal{B}(\mathcal{F})$ by inclusion (14.1). Let

$$\delta(t) := \operatorname{dist}\big(f(t), \mathbb{R}^n \setminus \phi(t)\big), \qquad t \in [a, b].$$

Recall that $\partial f(t)$ is compact for any $t \in [a, b]$ so that, in particular, $\partial f(a)$ is compact. Hence

$$\delta(a) > 0.$$

From Lemma 9.26 it follows that δ is nondecreasing in $[a, b]$. Therefore we obtain $\delta(b) > 0$, which implies that $f(b) \subseteq \operatorname{int}(R(E, t_0)(b))$. □

In general the complement of a barrier is not a barrier (see Remark 9.10). A comparison flow, however, has the following property.

Lemma 14.3 (Interior of the complement of a comparison flow). *Let R be a comparison flow and $E \in \mathscr{L}$. Then*

$$\mathbb{R}^n \setminus \overline{R(E, t_0)} \in \mathcal{B}(\mathcal{F}). \tag{14.4}$$

Proof. Let $f : [a, b] \subset I \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{F}$, and assume that

$$f(a) \subseteq \mathbb{R}^n \setminus \overline{R(E, t_0)(a)} = \operatorname{int}\big(\mathbb{R}^n \setminus R(E, t_0)(a)\big). \tag{14.5}$$

We have to show that

$$f(b) \subseteq \mathbb{R}^n \setminus \overline{R(E, t_0)(b)}. \tag{14.6}$$

Since $\operatorname{dist}\Big(f(a), \overline{R(E, t_0)(a)}\Big) > 0$ by (14.5), we can find $\rho > 0$ such that $f(a)^+_\rho \in \mathcal{C}^\infty$ has compact boundary, and

$$f(a) \subseteq f(a)^+_\rho \subseteq \overline{f(a)^+_\rho} \subseteq \operatorname{int}\Big(\mathbb{R}^n \setminus R(E, t_0)(a)\Big). \tag{14.7}$$

Using the estimate from below on the existence time of a smooth compact mean curvature flow given by Theorem 13.1, we can also choose $\rho > 0$ so that, if f_ρ denotes the mean curvature evolution starting from $\overline{f_\rho(a)^+}$ at time a, then

$$f_\rho : [a, b] \to \mathcal{P}(\mathbb{R}^n), \qquad f_\rho \in \mathcal{F}.$$

Define now

$$f_\rho^{\mathrm{c}} := \overline{\mathbb{R}^n \setminus f_\rho}.$$

We have $f_\rho^{\mathrm{c}} : [a, b] \to \mathcal{P}(\mathbb{R}^n)$, $f_\rho^{\mathrm{c}} \in \mathcal{F}$ and

$$f_\rho^{\mathrm{c}}(a) \supseteq \overline{R(E, t_0)(a)}$$

by (14.7). Therefore, by properties (2), (3) and (4) of Definition 14.1, it follows that

$$f_\rho^{\mathrm{c}}(b) = R\left(f_\rho^{\mathrm{c}}(a), a\right)(b) \supseteq R\left(\overline{R(E, t_0)(a)}, a\right)(b) \supseteq R(E, t_0)(b).$$

Hence, being $f_\rho^{\mathrm{c}}(b)$ a closed set, also

$$f_\rho^{\mathrm{c}}(b) \supseteq \overline{R(E, t_0)(b)}. \tag{14.8}$$

From Theorem 5.4 it follows that

$$\operatorname{dist}\left(f(b), \mathbb{R}^n \setminus f_\rho(b)\right) \geq \operatorname{dist}\left(f(a), \mathbb{R}^n \setminus f(a)_\rho^+\right) = \rho > 0,$$

and therefore, from (14.8), we obtain (14.6). $\square$

14.2. Barriers and comparison flows

In this section we prove the result concerning the relations between barriers and comparison flows[(2)]. Note that, as a consequence of the fact that $R(E, t_0)$ is a barrier for the family $\mathcal{F}$ in I (Lemma 14.2), we have

$$\mathcal{M}\,(E, \mathcal{F}) \subseteq R(E, t_0). \tag{14.9}$$

Moreover, applying (14.9) with E replaced by E_ρ^-, taking the union over $\rho > 0$, and recalling the definition of $\mathcal{M}_\star\,(E, \mathcal{F})$, we obtain

$$\mathcal{M}_\star\,(E, \mathcal{F}) \subseteq \bigcup_{\varrho > 0} R(E_\varrho^-, t_0). \tag{14.10}$$

(2) Recall our conventions: if $\phi, \psi : I \to \mathcal{P}(\mathbb{R}^n)$ are two maps, by $\phi \subseteq \psi$ we mean $\phi(t) \subseteq \psi(t)$ for any $t \in I$. See also the notation in (9.3), (9.4) and (9.6).

Similarly, applying (14.9) with E replaced by E^+_ρ, taking the intersection over $\rho > 0$, and recalling the definition of $\mathcal{M}^\star(E,\mathcal{F})$, we obtain

$$\mathcal{M}^\star(E,\mathcal{F}) \subseteq \bigcap_{\varrho>0} R\big(E^+_\varrho, t_0\big). \tag{14.11}$$

The next theorem shows, in particular, that $\mathcal{M}_\star(E,\mathcal{F})$ and $\mathcal{M}^\star(E,\mathcal{F})$ provide a lower and an upper bound respectively, to any comparison flow. In the proof we will need the avoidance principle, that we have proved in Theorem 13.3.

Theorem 14.4 (Barriers and comparison flows). *Let R be a comparison flow. Let $E \in \mathscr{L}$ be a nonempty set such that*

$$\text{either } \ E \ \text{ or } \ \mathbb{R}^n \setminus E \ \text{ is bounded.} \tag{14.12}$$

Then

$$\begin{aligned}
\mathcal{M}_\star(E,\mathcal{F}) &= \bigcup_{\varrho>0} R(E^-_\varrho, t_0) = \bigcup_{\varrho>0} \overline{R(E^-_\varrho, t_0)} \\
&\subseteq \mathcal{M}(E,\mathcal{F}) \subseteq R(E,t_0) \subseteq \overline{R(E,t_0)} \subseteq \bigcap_{\varrho>0} R(E^+_\varrho, t_0) \\
&= \bigcap_{\varrho>0} \overline{R(E^+_\varrho, t_0)} = \mathcal{M}^\star(E,\mathcal{F}).
\end{aligned}$$

Proof. Let us prove that

$$\mathcal{M}(E,\mathcal{F}) \supseteq \overline{R(E^-_\rho, t_0)}, \qquad \rho > 0. \tag{14.13}$$

For any $t \in I$ define

$$\phi(t) := \mathcal{M}(E,\mathcal{F})(t), \qquad \psi(t) := \mathbb{R}^n \setminus \overline{R(E^-_\rho, t_0)(t)}.$$

Then $\phi \in \mathcal{B}(\mathcal{F})$ and $\phi(t_0) = E$ (see Section 9.0.1). Moreover $\psi \in \mathcal{B}(\mathcal{F})$ by (14.4), and by (1) of Definition 14.1 also $\psi(t_0) = \mathbb{R}^n \setminus \overline{E^-_\rho}$. By our assumption (14.12) on E, it follows that

$$\text{either} \quad \mathbb{R}^n \setminus \mathrm{int}\,(\phi(t_0)) \quad \text{or} \quad \mathbb{R}^n \setminus \mathrm{int}\,(\psi(t_0)) \quad \text{is compact.}$$

Set

$$\eta(t) := \mathrm{dist}\big(\mathbb{R}^n \setminus \phi(t), \mathbb{R}^n \setminus \psi(t)\big), \qquad t \in I.$$

Then

$$\eta(t_0) = \mathrm{dist}\left(\mathbb{R}^n \setminus E, E^-_\rho\right) = \rho > 0.$$

From the avoidance principle (Theorem 13.3) it follows that

$$\eta(t) > 0, \qquad t \in I,$$

so that

$$\mathbb{R}^n \setminus \psi(t) \subseteq \phi(t), \qquad t \in I.$$

Hence (14.13) is proved, and therefore

$$\bigcup_{\varrho>0} \overline{R(E_\varrho^-, t_0)(t)} \subseteq \mathcal{M}\,(E, \mathcal{F})\,(t), \qquad t \in I. \tag{14.14}$$

Let us show that

$$\mathcal{M}_\star\,(E, \mathcal{F})\,(t) = \bigcup_{\varrho>0} R(E_\varrho^-, t_0)(t) = \bigcup_{\varrho>0} \overline{R(E_\varrho^-, t_0)(t)}, \quad t \in I. \tag{14.15}$$

If $\varrho > 0$, by (14.14) and property (4) in Definition 14.1, we have

$$\mathcal{M}\left(E_\varrho^-, \mathcal{F}\right)(t) \supseteq \bigcup_{\delta>0} \overline{R((E_\varrho^-)_\delta^-, t_0)(t)} \supseteq \overline{R(E_{2\varrho}^-, t_0)(t)}, \qquad t \in I.$$

Hence

$$\mathcal{M}_\star\big(E,\mathcal{F}\big)(t) \supseteq \bigcup_{\varrho>0} \overline{R\big(E_{2\varrho}^-, t_0\big)(t)} = \bigcup_{\varrho>0} \overline{R\big(E_\varrho^-, t_0\big)(t)}, \quad t \in I. \tag{14.16}$$

Then (14.15) follows from (14.10) and (14.16).

Let us show that

$$\overline{R(E, t_0)(t)} \subseteq \bigcap_{\rho>0} R\left(E_\rho^+, t_0\right)(t), \qquad t \in I. \tag{14.17}$$

Let $\rho > 0$ and, for any $t \in I$, set

$$\phi(t) := R\left(E_\rho^+, t_0\right)(t), \qquad \psi(t) := \mathbb{R}^n \setminus \overline{R(E, t_0)(t)}.$$

Then by (14.1) and (14.4) we have $\phi \in \mathcal{B}(\mathcal{F})$ and $\psi \in \mathcal{B}(\mathcal{F})$. If we define once more $\eta(t) := \mathrm{dist}\big(\mathbb{R}^n \setminus \phi(t), \mathbb{R}^n \setminus \phi(t)\big)$ for any $t \in I$, we have

$$\eta(t_0) = \mathrm{dist}\left(\mathbb{R}^n \setminus E_\rho^+, E\right) = \rho > 0.$$

By Theorem 13.3 it follows that $\eta(t) > 0$ for any $t \in I$. Hence

$$\mathbb{R}^n \setminus \psi(t) = \overline{R(E, t_0)(t)} \subseteq \phi(t) = R(E_\rho^+, t_0)(t), \qquad t \in I,$$

and (14.17) follows.

Since inclusion (14.9) has been already proven at the beginning of this section, it remains to show that

$$\bigcap_{\varrho>0} R(E_\varrho^+, t_0)(t) = \bigcap_{\varrho>0} \overline{R(E_\varrho^+, t_0)(t)} = \mathcal{M}^\star(E, \mathcal{F})(t), \qquad t \in I.$$

Recalling (14.11), it is enough to show that

$$\mathcal{M}^\star(E, \mathcal{F})(t) \supseteq \bigcap_{\varrho>0} \overline{R(E_\varrho^+, t_0)(t)}, \qquad t \in I. \tag{14.18}$$

If $\varrho > 0$ we have, by inclusion (14.14),

$$\mathcal{M}\left(E_\varrho^+, \mathcal{F}\right)(t) \supseteq \bigcup_{\delta>0} \overline{R((E_\varrho^+)_\delta^-, t_0)(t)} \supseteq \overline{R(E_{\varrho/2}^+, t_0)(t)}, \qquad t \in I,$$

and (14.18) follows. □

Theorem 14.4 is motivated by the existence of different notions of generalized mean curvature flow[(3)], that could be therefore compared with the barriers.

We conclude this section with an observation. We have seen in Lemma 14.2 that a comparison flow and its interior are barriers. Concerning the closure of a comparison flow, the following holds.

Remark 14.5 (Closure of a comparison flow). Let R be a comparison flow and let $E \in \mathscr{L}$ be a set such that either E or $\mathbb{R}^n \setminus E$ is bounded. If

$$\mathrm{int}\left(\overline{R(E, t_0)(t)}\right) \subseteq R(E, t_0)(t), \qquad t \in I, \tag{14.19}$$

then

$$\overline{R(E, t_0)} \in \mathcal{B}(\mathcal{F}).$$

Indeed, let $f : [a, b] \subset I \to \mathcal{P}(\mathbb{R}^n)$, $f \in \mathcal{F}$, and assume that $f(a) \subseteq \overline{R(E, t_0)(a)}$. We have to show that

$$f(b) \subseteq \overline{R(E, t_0)(b)}.$$

If $\rho > 0$ is sufficiently small, we have that $\partial(f(a))_\rho^- \in \mathcal{C}^\infty$ is compact, and, using (14.19),

$$(f(a))_\rho^- \subseteq \mathrm{int}(f(a)) \subseteq \mathrm{int}\left(\overline{R(E, t_0)(a)}\right) \subseteq R(E, t_0)(a).$$

(3) See, *e.g.*, [67, 185, 9, 196, 105, 14, 17, 209, 132, 109, 43, 79, 201] and [216].

Moreover, from Theorem 13.1, it follows that we can choose $\rho > 0$ so that, if f_ρ^- denotes the mean curvature evolution starting from $\overline{(f_\rho^-(a))^-}$ at time a, then $f_\rho^- : [a,b] \to \mathcal{P}(\mathbb{R}^n)$ and $f_\rho^- \in \mathcal{F}$.

Using properties (3) and (4) of Definition 14.1, and property (1), we have

$$\begin{aligned} f_\rho^-(b) &= R\left(f_\rho^-(a), a\right)(b) \\ &\subseteq R\left(\mathrm{int}\left(\overline{R(E,t_0)(a)}\right), a\right)(b) \subseteq R\left(R(E,t_0)(a), a\right)(b) \\ &= R(E,t_0)(b) \subseteq \overline{R(E,t_0)(b)}. \end{aligned}$$

Therefore, using Remark 7.12, we deduce

$$f(b) = \overline{\bigcup_{\rho>0} f_\rho(b)} \subseteq \overline{R(E,t_0)(b)}.$$

14.3. Barriers and level set evolution

As an application of the results of the previous section, we can compare the barriers evolution with the so-called level set flow. We choose in this section $\mathscr{L}$ as the family consisting of all open sets $A \subseteq \mathbb{R}^n$ such that either A or $\mathbb{R}^n \setminus A$ is bounded, and of all closed subsets $C \subseteq \mathbb{R}^n$ such that either C or $\mathbb{R}^n \setminus C$ is bounded.

A remarkable example of comparison flow is the following(4).

Example 14.6 (Level set flow). Let $E \in \mathscr{L}$ be nonempty, and define $u_0 : \mathbb{R}^n \to \mathbb{R}$ as follows:

$$u_0(z) := \max\Big(-1, \min\big(d(z,E), 1\big)\Big), \qquad z \in \mathbb{R}^n.$$

The function u_0 is bounded and Lipschitz continuous in $\mathbb{R}^n$.

Let us denote by $u : I \times \mathbb{R}^n \to \mathbb{R}$ the unique bounded Lipschitz continuous solution of the Cauchy problem

$$\begin{cases} u_t - |\nabla u| \mathrm{div}\left(\dfrac{\nabla u}{|\nabla u|}\right) = 0 & \text{in } (t_0, +\infty) \times \mathbb{R}^n, \\ u = u_0 & \text{on } \{t = t_0\} \times \mathbb{R}^n, \end{cases}$$

in the viscosity sense(5).

(4) See [224, 225].

(5) The definition of viscosity solution, its main properties, and a list of related references can be found in [96] and [33]. The viscosity solution for mean curvature flow has been studied in [137, 86, 139, 140, 153] (see also the book [152]). The global existence and uniqueness of u is a consequence of the results in [137] and [86]. The function u is usually called the uniformly continuous viscosity solution (or also the level set solution) to mean curvature flow. It is unique also in presence of fattening (see Chapter 11); in this case, there are times $t \in I$ such that $\overline{\{z \in \mathbb{R}^n : u(t,z) < 0\}}$ is strictly contained in $\{z \in \mathbb{R}^n : u(t,z) \le 0\}$.

It is then possible to prove that, given $(E, \tau) \in \mathscr{L} \times I$, the map associating with any $t \in [\tau, +\infty)$ the set $R(E, \tau)(t) := \{z \in \mathbb{R}^n : u(t, z) < 0\}$ if E is open, and the set $R(E, \tau)(t) := \{z \in \mathbb{R}^n : u(t, z) \leq 0\}$ if E is closed, is a comparison flow. In addition, if $E \in \mathscr{L}$ is open, we have

$$R(E, \tau)(t) = \bigcup_{\rho>0} R(E_\rho^-, \tau)(t), \qquad t \in [\tau, +\infty),$$

and if $E \in \mathscr{L}$ is closed, we have

$$R(E, \tau)(t) = \bigcap_{\rho>0} R(E_\rho^+, \tau)(t), \qquad t \in [\tau, +\infty).$$

We are now in a position to compare the barriers with the level set evolution[(6)].

Theorem 14.7 (Barriers and level set evolution). *Let $\mathscr{L}$, E and u be as Example* 14.6. *Then, for any $t \in I$, we have*

$$\begin{aligned} \mathcal{M}_\star(E, \mathcal{F})(t) = \{z \in \mathbb{R}^n : u(t, z) < 0\} &\subseteq \mathcal{M}(E, \mathcal{F})(t) \\ \subseteq \{z \in \mathbb{R}^n : u(t, z) \leq 0\} &= \mathcal{M}^\star(E, \mathcal{F})(t). \end{aligned} \tag{14.20}$$

Proof. This follows from Theorem 14.4. □

Remark 14.8. We recall that in Theorem 13.5 we have shown that if either $\overline{E}$ or $\mathbb{R}^n \setminus \mathrm{int}(E)$ is compact, then

$$\mathcal{M}_*(\mathbb{R}^n \setminus E, \mathcal{F}) = \mathbb{R}^n \setminus \mathcal{M}^*(E, \mathcal{F}). \tag{14.21}$$

It is worth noticing that (14.21) is also a consequence of (14.20).

Remark 14.9. From (14.20) and the coincidence of $\mathcal{M}(\cdot, \mathcal{F})$ with $\mathcal{M}_\star(\cdot, \mathcal{F})$ on the open sets (see Proposition 10.7 (ii)), it follows that, if $E \in \mathcal{L}$ is open, we have

$$\mathcal{M}(E, \mathcal{F})(t) = \{z \in \mathbb{R}^n : u(t, z) < 0\}, \qquad t \in I. \tag{14.22}$$

We conclude this section with a further remark on the relations between barriers and the level set flow.

(6) Theorem 14.7 is proved in [54, 57] in the more general case of mean curvature flow with a forcing term having suitable regularity properties. Without forcing term, this result is similar to the result of Ilmanen [183] concerning the comparison between his notion of set-theoretic subsolution and the level set evolution.

Proposition 14.10. *Let E and u be as in Example* 14.6. *If in addition*

$$\operatorname{int}\left(\overline{\{z \in \mathbb{R}^n : u(t,z) < 0\}}\right) \subseteq \{z \in \mathbb{R}^n : u(t,z) < 0\}, \qquad t \in I, \quad (14.23)$$

then the map

$$t \in I \to \overline{\{z \in \mathbb{R}^n : u(t,z) < 0\}}$$

belongs to $\mathcal{B}(\mathcal{F})$. In particular,

$$\mathcal{M}(E, \mathcal{F})(t) \subseteq \overline{\{z \in \mathbb{R}^n : u(t,z) < 0\}}, \qquad t \in I.$$

Proof. It is a consequence of Remark 14.5. □

Assumption (14.23) is necessary, as shown in the next example.

Example 14.11 (Tangent squares). Let $n = 2$, $t_0 = 0$, and let

$$E := C_- \cup C_+, \quad C_- := (-1, 0) \times (0, 1), \quad C_+ := (0, 1) \times (0, 1).$$

Let $u_0 : \mathbb{R}^2 \to \mathbb{R}$ be a bounded uniformly continuous function such that

$$\{z \in \mathbb{R}^2 : u_0(z) < 0\} = E.$$

Note that

$$\operatorname{int}\left(\overline{\{z \in \mathbb{R}^2 : u_0(z) < 0\}}\right) = (-1, 1) \times (0, 1),$$

which is not contained in E, so that (14.23) is not satisfied at $t = 0 \in I$.

Let us denote by

- $C_-(t)$ the curvature evolution starting from C_-,
- $C_+(t)$ the curvature evolution starting from C_+,
- $Q(t)$ the curvature evolution starting from $(-1, 1) \times (0, 1)$,

in some common time interval $[0, T]$[(7)]. Then it is possible to prove that

$$\overline{\{z \in \mathbb{R}^2 : u(t,z) < 0\}} = \overline{C_-(t)} \cup \overline{C_+(t)}, \qquad t \in [0, T].$$

It follows that the map

$$t \in I \to \overline{\{z \in \mathbb{R}^2 : u(t,z) < 0\}}$$

cannot belong to $\mathcal{B}(\mathcal{F})$, since

$$\overline{\{z \in \mathbb{R}^2 : u_0(z) < 0\}} = [-1, 1] \times [0, 1].$$

(7) A square in the plane uniquely evolves by curvature, and instantly regularizes and becomes strictly convex.

14.4. Notes

Without referring to the notion of level set evolution, starting from the barrier evolution of sets, one can construct a corresponding function $\mathcal{M}(u_0, \mathcal{F})$ as follows. Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a bounded uniformly continuous function. Define the function

$$\mathcal{M}(u_0, \mathcal{F}) : I \times \mathbb{R}^n \to \mathbb{R}$$

as follows: for any $(t, z) \in I \times \mathbb{R}^n$

$$\mathcal{M}(u_0, \mathcal{F})(t, z) := \inf \left\{ \lambda \in \mathbb{R} : \mathcal{M}\Big(\{z \in \mathbb{R}^n : u_0(z) < \lambda\}, \mathcal{F}\Big)(t) \ni x \right\}.$$

Then one can consider the problem of studying the function $\mathcal{M}(u_0, \mathcal{F})$, its relations with other notions of weak solutions and its regularity: see [108, 46, 45], also for comparison results between barriers and level set solutions for more general geometric equations.

As a consequence of Theorem 14.7, we have that

$$\mathcal{M}(u_0, \mathcal{F}) = u,$$

provided $\{z \in \mathbb{R}^n : u_0(z) < \lambda\}$ has compact boundary.

Barriers for geometric evolution equations of the form

$$u_t + F(\nabla u, \nabla^2 u) = 0, \tag{14.24}$$

for F not degenerate elliptic, have been considered in [45] (see also [48] for the same problem in the context for viscosity solutions) where, under suitable assumptions, it is proved that the minimal barrier coincides with the minimal barrier for the smallest degenerate elliptic function above F. A useful tool introduced in [45] is the notion of local (in space) barrier, which reads as follows.

Definition 14.12 (Local barrier). A function ϕ of a real variable is a local barrier for the family $\mathfrak{F}$, if there exists an interval $J \subseteq \mathbb{R}$ such that $\phi : J \to \mathcal{P}(\mathbb{R}^n)$, and the following property holds: for any $z \in \mathbb{R}^n$ there exists $R > 0$ such that if $f : [a, b] \subset J \to \mathcal{P}(\mathbb{R}^n)$ belongs to $\mathfrak{F}$, and

$$f(a) \subseteq \phi(a) \cap B_R(z),$$

then $f(b) \subseteq \phi(b)$.

The properties of the distance function from the level set flow (hence, in view of Theorem 14.7, from the regularizations $\mathcal{M}_\star(E, \mathcal{F})$ and $\mathcal{M}^\star(E, \mathcal{F})$) have been investigated in [248, 136, 19].

Other weak definitions for geometric evolution problems that can be related to barriers can be found in [248, 75] and [39]. The papers [19, 15, 47, 245] contain further results on barriers in the case of mean curvature flow of manifolds having arbitrary codimension.

Chapter 15
One-dimensional analysis related to a reaction-diffusion equation

In the previous chapters we have considered the geometric evolution of a hypersurface by its mean curvature. What we want to show now is that the geometric motion can be approximated by zero level sets of solutions to a suitable parabolic partial differential equation, thus reducing, in some sense, the geometric problem to a more standard one.

Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded connected open set, $T > 0$ and $\epsilon \in (0, 1]$. Let

$$u_\epsilon^0 \in \mathcal{C}^{1,1}(\Omega) \cap L^\infty(\Omega) \tag{15.1}$$

be an initial datum, possibly depending on ϵ. In the sequel of the book, we often be interested in the asymptotic behaviour[(1)] as $\epsilon \downarrow 0$ of solutions u_ϵ to the parabolic problem

$$\begin{cases} \epsilon \dfrac{\partial u}{\partial t} - \epsilon \Delta u + \epsilon^{-1} W'(u) = 0 & \text{in } (0, T) \times \Omega, \\ \dfrac{\partial u}{\partial \mathrm{n}_\Omega} = 0 & \text{on } (0, T) \times \partial\Omega, \\ u = u_\epsilon^0 & \text{on } \{t = 0\} \times \Omega, \end{cases} \tag{15.2}$$

where n_Ω is the outward unit normal to $\partial\Omega$ and W' is a suitable cubic-like nonlinearity[(2), (3)]. The scaling parameter ϵ has a geometrical meaning and, as we shall see, it measures the thickness of a diffuse interface approximating an evolving boundary. The way equation (15.2) is rescaled

---

(1) Instead of sequences of functions (or functionals, or points) indicized by an integer parameter tending to $+\infty$, we often will consider families of functions (or functionals, or points) labelled by a continuous parameter (usually denoted by ϵ) tending to zero. With a small abuse of language, these families will still be called sequences, and denoted for instance by (u_ϵ).

(2) The partial differential equation in (15.2) is called reaction-diffusion equation, and sometimes Allen–Cahn's equation, or also parabolic Ginzburg-Landau's equation.

(3) We will limit ourselves to consider zero Neumann boundary conditions; for Dirichlet boundary conditions see [226].

with ϵ can be understood looking for instance at a formal asymptotic expansion; this is the proper rescaling for obtaining mean curvature flow in the limit $\epsilon \downarrow 0$.

Remark 15.1. Existence, uniqueness, smoothness and qualitative properties of a solution to parabolic semilinear equations such as that in (15.2) can be found for instance in [165, 120] and [15] (see also [246, 197, 193]).

The scope of this chapter is to collect some preliminary material that will be used in the asymptotic analysis of (15.2) in Chapters 16 and 17.

Cubic-like nonlinearity and double-well potential

Let γ_-, γ_+ be two real numbers with

$$\gamma_- < \gamma_+,$$

and let $w \in \mathcal{C}^\infty(\mathbb{R})$ be a strictly convex function negative inside the interval (γ_-, γ_+) and positive outside $[\gamma_-, \gamma_+]$, so that in particular

$$w^{-1}(0) = \{\gamma_-, \gamma_+\}.$$

Associated with w we define the function $W' : \mathbb{R} \to \mathbb{R}$ as

$$W'(\mathrm{y}) := w(\mathrm{y})w'(\mathrm{y}), \qquad \mathrm{y} \in \mathbb{R}, \tag{15.3}$$

which has exactly three zeroes

$$\gamma_- < \gamma_c < \gamma_+. \tag{15.4}$$

The function W' is sometimes called cubic-like nonlinearity. We denote by W the primitive of W' vanishing at $\gamma_\pm$, so that

$$W := \frac{1}{2}w^2.$$

In Figure 15.1 we draw an example of W and W', with $\gamma_\pm = \pm 1$ and $\gamma_c = 0$.

The function W is usually called a double-well potential.

We set

$$\alpha_\pm := W''(\gamma_\pm), \tag{15.5}$$

so that

$$\alpha_\pm = (w'(\gamma_\pm))^2 > 0. \tag{15.6}$$

Example 15.2 (Typical double-well potential). The prototypical example is given by

$$\gamma_- = -1, \qquad \gamma_+ = 1,$$

$$w(\mathrm{y}) = \frac{1}{2}(\mathrm{y}^2 - 1), \qquad \mathrm{y} \in \mathbb{R}. \tag{15.7}$$

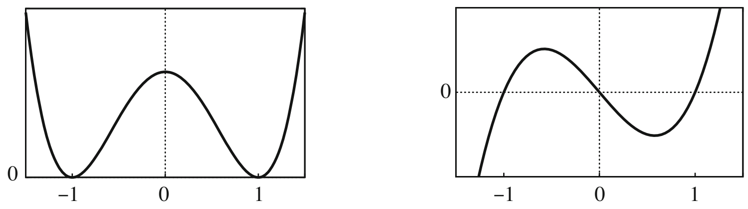

Figure 15.1. Plot of the graph of a double well potential W (left) and of its derivative W' (right).

Therefore

$$W'(y) = \frac{1}{2}(y^2 - 1)y, \qquad y \in \mathbb{R}, \tag{15.8}$$

so that

$$\gamma_c = 0$$

and

$$\alpha_\pm = 1.$$

In this case, the double-well potential takes the form

$$W(y) = \frac{1}{8}(y^2 - 1)^2, \qquad y \in \mathbb{R}. \tag{15.9}$$

Associated functionals

Given $\epsilon \in (0, 1]$, and denoting by $H^1(\Omega)$ the Sobolev space of functions $v \in L^2(\Omega)$ having distributional gradient $\nabla v \in L^2(\Omega; \mathbb{R}^n)$, we define(4) the functionals $\mathcal{F}_\epsilon(\cdot, \Omega) : L^2(\Omega) \to [0, +\infty]$ as

$$\mathcal{F}_\epsilon(v,\Omega) := \begin{cases} \displaystyle\int_\Omega \left(\epsilon \, \frac{|\nabla v|^2}{2} + \epsilon^{-1} \, W(v) \right) dz \\ \qquad\qquad \text{if} \quad v \in H^1(\Omega) \text{ and } W(v) \in L^1(\Omega), \\ +\infty \qquad \text{elsewhere.} \end{cases} \tag{15.10}$$

The limit(5) as $\epsilon \downarrow 0$ of the sequence $(\mathcal{F}_\epsilon)$ has been analyzed by various auhors; see [112, 215] (and also [65, 66] and references therein). Here

(4) The functionals $\mathcal{F}_\epsilon$ are sometimes called the Allen–Cahn's functionals, or also the Modica-Mortola's functionals, or even the scalar Ginzburg-Landau's functionals.

(5) If we replace $L^2(\Omega)$ with $L^1(\Omega)$ in (15.10), it turns out that the family $(\mathcal{F}_\epsilon(\cdot, \Omega))$, for $\epsilon = 1/h$, $h \in \mathbb{N}$, $\Gamma - L^1(\Omega)$-converges as $h \to +\infty$, to the perimeter functional multiplied by the constant σ_0 given in (15.12) (the pointwise limit is not interesting and gives a limit functional which takes only the values 0 and $+\infty$): concerning Γ-convergence we refer to [97] and references therein. The asymptotic behaviour of $(\mathcal{F}_\epsilon(\cdot, \Omega))$ has been investigated also when $\Omega = \mathbb{R}^n$, replacing $L^1(\Omega)$ with $L^1_{\rm loc}(\Omega)$, or for W of the form $W(y) = 1 + (\cos(y/\epsilon^{1/2}))^2$. For much more general results

we just mention that it is possible to prove that such a limit is the perimeter functional, multiplied by a positive constant σ_0 (defined in formula (15.12), and sometimes called surface tension)[(6)].

Concerning the initial datum u_ϵ^0 in (15.1), we will assume the bound[(7), (8), (9)]

$$\sup_{\epsilon\in(0,1]} \mathcal{F}_\epsilon(u_\epsilon^0, \Omega) < +\infty, \tag{15.11}$$

which is a natural requirement, in view of the variational structure of problem (15.2). Indeed, the parabolic partial differential equation in (15.2) is the so-called $L^2(\Omega)$-gradient flow[(10)] of the functional $\mathcal{F}_\epsilon(\cdot, \Omega)$,

---

in this direction we refer for instance to [13]. We have considered the "ambient" space $L^2(\Omega)$ in (15.10), instead of $L^1(\Omega)$, in order to set the functionals $\mathcal{F}_\epsilon(\Omega, \cdot)$ in a Hilbert space, which can be more convenient when we look at the associated gradient flows. Beside the above mentioned Γ-convergence property, it is worthwhile to recall that $(\mathcal{F}_\epsilon(\cdot, \Omega))$ enjoys another property, called equicoercivity. More precisely, let there be given a sequence (v_ϵ) of functions satisfying a uniform bound of the form $\sup_{\epsilon\in(0,1]} \mathcal{F}_\epsilon(v_\epsilon, \Omega) < +\infty$. Then (v_ϵ) admits a subsequence converging in $L^1(\Omega)$ to a function equals to γ_- in a set E of finite perimeter in Ω and to γ_+ in $\Omega \setminus E$.

(6) If $n = 1$, the following scaling identity holds: $\mathcal{F}_\epsilon(v, \Omega) = \mathcal{F}_1(v_\epsilon, \frac{1}{\epsilon}\Omega)$, where $v_\epsilon(x) := v(\epsilon x)$ for $x \in \frac{1}{\epsilon}\Omega$. This property is of importance for various reasons, and in particular for understanding the way the functionals $\mathcal{F}_\epsilon(\cdot, \Omega)$ are rescaled.

(7) Observe that (15.11) does not imply that $\sup_{\epsilon\in(0,1]} \int_\Omega |\nabla u_\epsilon^0| \, dz < +\infty$, as it happens for instance with the choice $n = 1, \Omega = (-1, 1)$, W as in (15.9), and $u_\epsilon^0(z) = 1 + \epsilon^{1/2}\sin(z/\epsilon)$ for any $z \in (-1, 1)$. Indeed, in this case we have $\int_{-1}^1 |u_{\epsilon z}^0| \, dz = \epsilon^{-1/2} \int_{-1}^1 |\cos(z/\epsilon)| \, dz = \epsilon^{1/2} \int_{-1/\epsilon}^{1/\epsilon} |\cos y| \, dy = \mathcal{O}(\epsilon^{-1/2})$, while $\int_{-1}^1 \frac{\epsilon}{2}(u_{\epsilon z}^0)^2 \, dz = \frac{1}{2}\int_{-1}^1 (\cos(z/\epsilon))^2 \, dz = \frac{\epsilon}{2}\int_{-1/\epsilon}^{1/\epsilon} (\cos y)^2 \, dy = \mathcal{O}(1)$, and $\int_{-1}^1 \epsilon^{-1} W(u_\epsilon^0) \, dz = \frac{1}{8\epsilon}\int_{-1}^1 [\epsilon(\sin(z/\epsilon))^2 + 2\epsilon^{1/2}\sin(z/\epsilon)]^2 \, dz = \mathcal{O}(1)$.

(8) A natural choice of a sequence of initial data satisfying (15.11) is the following. Let for simplicity $\Omega = \mathbb{R}^n$, let $\partial E \in \mathcal{C}^\infty$ be compact and d be the signed distance from ∂E negative in the interior of E, and define

$$v_\epsilon^0(z) := \gamma(d(z)/\epsilon), \qquad z \in \mathbb{R}^n,$$

where γ is as in Section 15.1. Then $v_\epsilon^0 \in \mathrm{Lip}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n; (-1, 1))$, and using the coarea formula (Theorem 2.3),

$$\begin{aligned}
\mathcal{F}_\epsilon(v_\epsilon^0, \mathbb{R}^n) &= \frac{1}{2\epsilon}\int_{\mathbb{R}^n} \left[(\gamma'(d/\epsilon))^2 + (w(\gamma(d/\epsilon)))^2\right] dz \\
&= \frac{1}{2\epsilon}\int_{\mathbb{R}} \left[(\gamma'(\lambda/\epsilon))^2 + (w(\gamma(\lambda/\epsilon)))^2\right] \mathcal{H}^{n-1}(\{d = \lambda\}) \, d\lambda \\
&= \frac{1}{2}\int_{\mathbb{R}} \left[(\gamma'(\lambda))^2 + (w(\gamma(\lambda)))^2\right] \mathcal{H}^{n-1}(\{d = \epsilon\lambda\}) \, d\lambda,
\end{aligned}$$

which converges to $\sigma_0 P(E)$ as $\epsilon \to 0^+$ (see (15.20)). We can additionally modify the expression of v_ϵ^0 in order to fulfill the regularity requirement (15.1); see Definition 17.2 or also the initial condition in (17.54).

(9) Much more general initial conditions have been considered in [81, 120, 249, 250].

(10) See, *e.g.*, [69, 105, 14, 17].

after replacing t by ϵt. Note that

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_\epsilon(u_\epsilon,\Omega) &= \int_\Omega \left(\epsilon\langle\nabla u_\epsilon, \nabla\frac{\partial u_\epsilon}{\partial t}\rangle + \epsilon^{-1}W'(u_\epsilon)\frac{\partial u_\epsilon}{\partial t}\right)\,dz \\ &= \int_\Omega \left(-\epsilon\Delta u_\epsilon + \epsilon^{-1}W'(u_\epsilon)\right)\frac{\partial u_\epsilon}{\partial t}\,dz \\ &= -\int_\Omega \left(\epsilon\Delta u_\epsilon - \epsilon^{-1}W'(u_\epsilon)\right)^2\,dz \le 0,\end{aligned}$$

so that the function $t \in (0,+\infty) \to \mathcal{F}_\epsilon(u_\epsilon(t,\cdot),\Omega)$ is nonincreasing.

Remark 15.3. The fact that the Γ-limit of $(\mathcal{F}_\epsilon)$ is σ_0 times the perimeter is an important indication that the gradient flow of $\mathcal{F}_\epsilon$ could be related, in the limit $\epsilon \downarrow 0$, to the gradient flow of the perimeter, namely with mean curvature flow(11). In Chapters 16 and 17 we will make this convergence assertion more precise.

Remark 15.4. The main theorem we are interested in is the convergence to a smooth compact mean curvature flow of the zero level set of u_ϵ. This convergence result will be proved in Chapter 17 for short times, and under further restrictions on u^0_ϵ (see formulas (16.3) and (17.6)), which are suggested by the formal asymptotic expansion performed in Chapter 16.

All material discussed in this chapter can be considered as preparatory for the asymptotic analysis of the sequence (u_ϵ).

15.1. The minimizer γ

We start with the study of some properties of the functional $\mathcal{F}_\epsilon$ in dimension $n = 1$ and for $\epsilon = 1$, and when Ω is replaced by the whole of $\mathbb{R}$. We recall that $L^2_{\mathrm{loc}}(\mathbb{R})$ (respectively $H^1_{\mathrm{loc}}(\mathbb{R})$) is the space of all functions ζ such that $\zeta \in L^2(K)$ (respectively $\zeta \in H^1(K)$) for any bounded open interval $K \subset \mathbb{R}$.

We define the functional

$$F : L^2_{\mathrm{loc}}(\mathbb{R}) \to [0,+\infty]$$

as follows:

$$\mathrm{dom}(F) := \left\{\zeta \in L^2_{\mathrm{loc}}(\mathbb{R}) : \zeta' \in L^2(\mathbb{R}),\, W(\zeta) \in L^1(\mathbb{R})\right\},$$

(11) See also [240] and references therein.

and

$$F(\zeta) := \begin{cases} \displaystyle\int_{\mathbb{R}} \left(\frac{(\zeta')^2}{2} + W(\zeta) \right) dy = \frac{1}{2} \int_{\mathbb{R}} \left((\zeta')^2 + (w(\zeta))^2 \right) dy \\ \qquad\qquad \text{if } \zeta \in \mathrm{dom}(F), \\ \\ +\infty \qquad \text{elsewhere in } L^2_{\mathrm{loc}}(\mathbb{R}). \end{cases}$$

We also define(12)

$$\sigma_0 := \inf \left\{ F(\zeta) : \ \zeta \in \mathrm{dom}(F), \lim_{y \to \pm\infty} \zeta(y) = \gamma_\pm \right\}. \tag{15.12}$$

It is clear that $\sigma_0 \geq 0$. Moreover

$$\sigma_0 < +\infty,$$

since for instance $F(\xi) < +\infty$, where

$$\xi(y) = \min \left[\gamma_+, \max \left(\gamma_-, y + \frac{\gamma_- + \gamma_+}{2} \right) \right], \qquad y \in \mathbb{R}.$$

Remark 15.5 (Truncation). If $\zeta \in \mathrm{dom}(F)$ and we truncate ζ at levels $\gamma_\pm$, namely if we consider the function ζ_{tr} defined as

$$\zeta_{\mathrm{tr}}(y) := \min \left[\gamma_+, \max \left(\gamma_-, \zeta(y) \right) \right], \qquad y \in \mathbb{R},$$

then $\zeta_{\mathrm{tr}} \in \mathrm{dom}(F)$ and

$$F(\zeta_{\mathrm{tr}}) \leq F(\zeta).$$

Therefore, there is no loss of generality in restricting the minimum problem in (15.12) to functions ζ satisfying $\zeta(\mathbb{R}) \subseteq [\gamma_-, \gamma_+]$; namely

$$\sigma_0 = \inf\{F(\zeta) : \zeta \in \mathcal{X}\}, \tag{15.13}$$

where

$$\mathcal{X} := \left\{ \zeta \in \mathrm{dom}(F) : \ \zeta : \mathbb{R} \to [\gamma_-, \gamma_+], \ \lim_{y \to \pm\infty} \zeta(y) = \gamma_\pm \right\}.$$

(12) The condition $\lim_{y \to \pm\infty} \zeta(y) = \gamma_\pm$ in the minimum problem (15.12) can be weakened; see, *e.g.*, [3], and also the definition of $\mathcal{X}'$ in the proof of Theorem 15.10.

Remark 15.6 (Translation invariance). Let $\zeta \in \mathcal{X}$. Let $\eta \in \mathbb{R}$ and set

$$\tau_\eta \zeta(y) := \zeta(y - \eta), \qquad y \in \mathbb{R}.$$

Then $\tau_\eta \zeta \in \mathcal{X}$ and

$$F(\tau_\eta \zeta) = F(\zeta).$$

As shown in Theorem 15.10, the minimum problem in (15.13) has a solution (and hence infinitely many, as a consequence of Remark 15.6). Note that the existence of a solution implies that

$$\sigma_0 > 0.$$

Since our argument is based on a rearrangement technique (see [4]), we start with a definition. We denote as usual by $|B|$ the Lebesgue measure of a Borel set $B \subseteq \mathbb{R}$.

Definition 15.7 (Right rearrangement of a set). Let $a, b \in \mathbb{R}$ be such that $a < b$. For every Borel set $B \subset \mathbb{R}$ such that

$$[a, +\infty) \supseteq B \supseteq [b, +\infty)$$

we define the right rearrangement B^* of B as the half-line

$$B^* := [c, +\infty),$$

where

$$c := b - \big|B \cap [a, b]\big|.$$

Notice that

$$[a, +\infty) \supseteq B^* \supseteq [b, +\infty),$$

and $|B^* \cap I| = |B \cap I|$ for every interval I which includes $[a, b]$.

Definition 15.8 (Increasing rearrangement of a function). Let $\zeta \in \mathcal{X}$. We define the increasing rearrangement of ζ as the function $\zeta^* : \mathbb{R} \to [\gamma_-, \gamma_+]$ such that

$$\big\{y \in \mathbb{R} : \ \ell \le \zeta^*(y)\big\} := \big\{y \in \mathbb{R} : \ \ell \le \zeta(y)\big\}^*, \qquad \ell \in (\gamma_-, \gamma_+).$$

One can verify that the function ζ^* is nondecreasing and belongs to $\mathcal{X}$.

Lemma 15.9 (Monotonicity of F under rearrangements). *Let $\zeta \in \mathcal{X}$. Then*

$$F(\zeta^*) \le F(\zeta). \tag{15.14}$$

Hence

$$\sigma_0 = \inf\{F(\zeta) : \zeta \in \mathcal{X}_{\mathrm{nd}}\}, \tag{15.15}$$

where

$$\mathcal{X}_{\mathrm{nd}} := \Big\{\zeta \in \mathcal{X} : \zeta \text{ nondecreasing}\Big\}.$$

Proof. In [4] it is proven that

$$\int_{\mathbb{R}} W(\zeta^*)\, dy = \int_{\mathbb{R}} W(\zeta)\, dy.$$

Moreover, in [3] it is shown that

$$\int_{\mathbb{R}} ((\zeta^*)')^2\, dy \le \int_{\mathbb{R}} (\zeta')^2\, dy.$$

Then inequality (15.14) follows.

Formula (15.15) is a consequence of (15.14). Indeed, for every $\zeta \in \mathcal{X}$, set

$$P\zeta := \zeta^*.$$

Then P is the identity on $\mathcal{X}_{\mathrm{nd}}$, and is a projection of $\mathcal{X}$ onto $\mathcal{X}_{\mathrm{nd}}$ which does not increase the functional F. $\square$

Theorem 15.10 (Existence of minimizers). *The infimum in* (15.13) *is attained.*

Proof. Set

$$\mathcal{X}' := \Big\{\zeta \in \mathcal{X}_{\mathrm{nd}} : \zeta(y) \le (\gamma_- + \gamma_+)/2 \text{ for } y<0,\ \zeta(y) \ge (\gamma_- + \gamma_+)/2 \text{ for } y>0\Big\}.$$

For any $\zeta \in \mathcal{X}_{\mathrm{nd}}$ there exists at least one point $\eta \in \mathbb{R}$ (depending on ζ) such that

$$\zeta(\eta) \le (\gamma_- + \gamma_+)/2 \quad \text{for } y<\eta \quad \text{and} \quad \zeta(\eta) \ge (\gamma_- + \gamma_+)/2 \quad \text{for } y>\eta.$$

Remembering the definition of $\tau_\eta\zeta$ in Remark 15.6, we have $\tau_\eta\zeta \in \mathcal{X}'$, and also $F(\zeta) = F(\tau_\eta\zeta)$. Therefore, recalling that σ_0 can be expressed as in (15.15), we have

$$\sigma_0 = \inf\Big\{F(\zeta) : \zeta \in \mathcal{X}'\Big\}.$$

Hence, to conclude the proof it is enough to show that

$$\inf\Big\{F(\zeta) : \zeta \in \mathcal{X}'\Big\} = \min\Big\{F(\zeta) : \zeta \in \mathcal{X}'\Big\}.$$

Since one can prove that the functional F is $L^1_{\mathrm{loc}}(\mathbb{R})$-lower semicontinuous, we need only to prove[13] that from every minimizing sequence $(\zeta_m) \subset \mathcal{X}'$ we can extract a subsequence which converges in $L^1_{\mathrm{loc}}(\mathbb{R})$ to some $\zeta \in \mathcal{X}'$. Consider therefore a sequence $(\zeta_m) \subset \mathcal{X}'$ such that

$$\lim_{m\to+\infty} F(\zeta_m) = \inf\left\{F(\zeta) : \zeta \in \mathcal{X}'\right\} < +\infty.$$

For any $m \in \mathbb{N}$, the distributional derivative of ζ_m is a positive Borel measure on $\mathbb{R}$ with total variation given by $\gamma_+ - \gamma_-$. Hence the sequence (ζ_m) is bounded in $BV_{\mathrm{loc}}(\mathbb{R})$ and relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R})$ (see [16]), and we may extract a subsequence (ζ_{m_k}) which converges in $L^2_{\mathrm{loc}}(\mathbb{R})$ and almost everywhere, to some ζ in $BV_{\mathrm{loc}}(\mathbb{R})$ as $k \to +\infty$.

Then ζ is (almost everywhere equal to) a nondecreasing function belonging to $\mathcal{X}'$. Hence it remains to show that $\zeta(y)$ converges to γ_+ (respectively to γ_-) as y tends to $+\infty$ (respectively to $-\infty$). Since ζ is nondecreasing, there exist

$$\zeta_- := \lim_{y\to-\infty} \zeta(y) \quad \text{and} \quad \zeta_+ := \lim_{y\to+\infty} \zeta(y)\,,$$

and

$$\gamma_- \le \zeta_- \le \frac{1}{2}(\gamma_- + \gamma_+) \le \zeta_+ \le \gamma_+.$$

If we assume by contradiction that either $\zeta_- \ne \gamma_-$ or $\zeta_+ \ne \gamma_+$, and we recall that W is strictly positive in (γ_-, γ_+), we obtain that

$$\int_{\mathbb{R}} W\big(\zeta(y)\big)\, dy = +\infty.$$

Hence $F(\zeta) = +\infty$, and this is a contradiction because

$$F(\zeta) \le \liminf_{k\to+\infty} F(\zeta_{m_k}) = \lim_{m\to+\infty} F(\zeta_m) < +\infty. \qquad \square$$

Remark 15.11 (Alternative proof of existence of minimizers). Let us briefly illustrate another proof of Theorem 15.10, independent of the notion of rearrangement[14]. Let $\zeta \in \mathcal{X}$. Using the inequality $a^2+b^2 \ge 2ab$, that we apply with the choice

$$a = \frac{\zeta'(y)}{\sqrt{2}}, \qquad b = \frac{|w(\zeta(y))|}{\sqrt{2}},$$

(13) See, *e.g.*, [97].

(14) See, *e.g.*, [2].

we have

$$F(\zeta) \geq \int_{\mathbb{R}} \zeta' \, |w(\zeta)| \, dy = \int_{\gamma_-}^{\gamma_+} |w(s)| \, ds.$$

The equality in $a^2 + b^2 \geq 2ab$ holds only when $a = b$; this suggests to consider the Cauchy problem

$$\begin{cases} \zeta' = -w(\zeta) & \text{in } \mathbb{R}, \\ \zeta(0) = \dfrac{1}{2}(\gamma_- + \gamma_+). \end{cases} \tag{15.16}$$

The constant functions $\gamma_\pm$ are maximal solutions to the equation $\zeta' = -w(\zeta)$. By the assumptions on w it follows that (15.16) has a unique smooth strictly increasing solution[(15)]

$$\gamma \in \mathcal{X} \cap \mathcal{C}^\infty(\mathbb{R}; (\gamma_-, \gamma_+)).$$

Note that, differentiating the equation in (15.16), gives $-\gamma'' + W'(\gamma) = 0$, so that, in particular, γ' satisfies $-(\gamma')'' + W''(\gamma)\gamma' = 0$.

By the above arguments, γ is a solution of the minimum problem (15.13), and hence

$$\sigma_0 = F(\gamma).$$

---

(15) The function $\gamma_\pm - \gamma$ decays with exponential rate to zero at $\pm\infty$ (see, *e.g.*, [78]). Indeed, we have

$$\int_0^y \frac{\gamma'}{w(\gamma)} dy = L(\gamma(y)) = -y,$$

where $L(\mathrm{y}) := \int_{\frac{1}{2}(\gamma_- + \gamma_+)}^{\mathrm{y}} \frac{1}{w(s)} ds$ for any $\mathrm{y} \in \left(\frac{1}{2}(\gamma_- + \gamma_+), \gamma_+\right)$.

Using a Taylor expansion of $\frac{1}{w(s)}$, one can show that there exists a costant $c \in (0, +\infty)$ such that

$$L(\mathrm{y}) = -\frac{1}{\sqrt{\alpha_+}} \log\left(\frac{\gamma_+ - \mathrm{y}}{\frac{1}{2}(\gamma_+ - \gamma_-)}\right) + c + o(1) \quad \text{as } \mathrm{y} \uparrow \gamma_+,$$

which implies that there exists $\lim_{y \to +\infty} e^{\sqrt{\alpha_+} y}(\gamma_+ - \gamma(y)) \in (0, +\infty)$.
Similarly, one shows that there exists $\lim_{y \to -\infty} e^{-\sqrt{\alpha_-} y}(\gamma(y) - \gamma_-) \in (0, +\infty)$.
The exponential convergence of γ to its asymptotic values guarantees, in particular, that $F(\gamma) < +\infty$. Also, all derivatives of γ decay exponentially fast to zero as $y \to \pm\infty$, as it can be inferred from the behaviour of γ at $\pm\infty$, and from (15.16). In particular,

$$\begin{aligned} A_1 e^{\sqrt{\alpha_-} y} < \gamma'(y) < A_2 e^{\sqrt{\alpha_-} y} \qquad & \text{as } y \to -\infty, \\ B_1 e^{-\sqrt{\alpha_+} y} < \gamma'(y) < B_2 e^{-\sqrt{\alpha_+} y} \qquad & \text{as } y \to +\infty, \end{aligned}$$

for suitable constants A_1, A_2, B_1, B_2 with $0 < A_1 < A_2$ and $0 < B_1 < B_2$. Note that the exponential decay of γ' at $\pm\infty$ is governed by the solutions to the linear ordinary differential equations

$$-\xi'' + \alpha_\pm \xi = 0$$

which, in general, are given by real linear combinations of $e^{-\sqrt{\alpha_\pm} y}$ and $e^{\sqrt{\alpha_\pm} y}$.

Remark 15.12. If ζ_1 and ζ_2 are two minimizers of F in $\mathcal{X}$, then there exists $\vartheta \in \mathbb{R}$ such that

$$\zeta_1(y) = \zeta_2(y - \vartheta), \qquad y \in \mathbb{R}.$$

Indeed, necessarily

- $\zeta_1, \zeta_2 \in \mathcal{C}^\infty(\mathbb{R})$,
- ζ_1, ζ_2 satisfy the ordinary differential equation in (15.16) (in particular they are strictly increasing),
- $\zeta_1(\mathbb{R}) = \zeta_2(\mathbb{R}) = (\gamma_-, \gamma_+)$.

By uniqueness, we have that either $\zeta_1 < \zeta_2$ or $\zeta_2 < \zeta_1$. Then, setting $\vartheta := \zeta_2^{-1}(\zeta_1(0))$, we have $\zeta_1(y) = \zeta_2(y + \vartheta)$.

Summarizing, there exists a unique absolute minimizer γ of the problem

$$\inf \Big\{ F(\zeta) : \zeta \in \mathcal{X}_0 \Big\}, \tag{15.17}$$

where

$$\mathcal{X}_0 := \left\{ \zeta \in \mathcal{X} : \zeta(0) = \frac{1}{2}(\gamma_- + \gamma_+), \lim_{y \to \pm\infty} \zeta(y) = \gamma_\pm \right\}.$$

The function γ belongs to $\mathcal{C}^\infty(\mathbb{R}; (\gamma_-, \gamma_+))$, it is strictly increasing, and satisfies the Euler-Lagrange equation of F,

$$-\gamma''(y) + W'(\gamma(y)) = 0, \qquad y \in \mathbb{R}. \tag{15.18}$$

Moreover

$$\gamma'(y) = -w(\gamma(y)), \qquad y \in \mathbb{R}, \tag{15.19}$$

and the minimal value σ_0 of F is given by

$$\begin{aligned} \sigma_0 &= \int_{\mathbb{R}} \left(\frac{1}{2}(\gamma')^2 + W(\gamma) \right) dy = \frac{1}{2} \int_{\mathbb{R}} \left((\gamma')^2 + (w(\gamma))^2 \right) dy \\ &= \int_{\mathbb{R}} (\gamma')^2 \, dy = \int_{\mathbb{R}} 2W(\gamma) \, dy = \int_{\gamma_-}^{\gamma_+} |w(s)| \, ds. \end{aligned} \tag{15.20}$$

Example 15.13 (Hyperbolic tangent). If W is as in Example 15.2, then

$$\gamma(y) = \mathrm{tgh}(y/2), \qquad y \in \mathbb{R}.$$

15.2. Linearization around γ: the operator $\mathcal{L}$

In our analysis of the parabolic partial differential equation in (15.2), it is useful to study the properties of a one-dimensional linear operator, that will be denoted by $\mathcal{L}$, obtained as the linearization around γ of a nonlinear operator. In this section we record some of the properties of $\mathcal{L}$. The reader can look through the references [191, Example 5, page 73], [120, 82, 237, 231, 113] and [166] for related discussions.

Assumptions. In this section, and up to the end of the chapter, we take for simplicity $w(y) = \frac{1}{2}(y^2 - 1)$ (as in Example 15.2), hence $\gamma(y) = \mathrm{tgh}(y/2)$ for any $y \in \mathbb{R}$. In particular γ' is even, and

$$\alpha_- = \alpha_+ = 1 =: \alpha. \tag{15.21}$$

We have already observed that, differentiating (15.18),

$$-\gamma''' + W''(\gamma)\gamma' = 0 \qquad \text{in } \mathbb{R}. \tag{15.22}$$

For future reference(16), we notice that $W''(\gamma)$ is even, smooth, bounded and its asymptotic value at $\pm\infty$ is positive. Hence there exists a constant $c > 0$ such that the function $c\gamma' + W''(\gamma)$ is uniformly positive on $\mathbb{R}$, that is,

$$\inf\left\{c\gamma'(y) + W''(\gamma(y)) : y \in \mathbb{R}\right\} > 0.$$

15.2.1. The operator $\mathcal{L}$

Denote by $\mathcal{L}$ the unbounded linear operator

$$\mathcal{L} : \mathrm{dom}(\mathcal{L}) = H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad \mathcal{L}\zeta := -\zeta'' + W''(\gamma)\zeta,$$

which is the linearization(17) of the operator $\zeta \in H^2(\mathbb{R}) \subset L^2(\mathbb{R}) \to -\zeta'' + W'(\zeta)$ around $\zeta = \gamma$. Notice that the relevant norm here is the $L^2(\mathbb{R})$-norm.

Remark 15.14. The operator $\mathcal{L}$ is densely defined, closed and symmetric (see, *e.g.*, [70, Sections 2.6 and 7.4]). Moreover, if $\mathcal{L}^\star$ denotes the adjoint of $\mathcal{L}$, we have

$$\ker(\mathcal{L}^\star)^\perp = \overline{\mathrm{range}(\mathcal{L})}, \tag{15.23}$$

where the symbol $^\perp$ and the symbol of closure mean orthogonality and closure with respect to the scalar product of $L^2(\mathbb{R})$, respectively (see, *e.g.*, [70, Corollary 2.18]).

(16) See inequality (17.44) in Chapter 17.

(17) Observe that the equation $\mathcal{L}\zeta = 0$ is the Euler-Lagrange equation of the functional $\zeta \in H^1(\mathbb{R}) \to \frac{1}{2}\int_{\mathbb{R}} \left[(\zeta')^2 + W''(\gamma)\zeta^2\right] dy$.

It is also possible to prove(18) that the range of $\mathcal{L}$ is closed, and therefore

$$\ker(\mathcal{L}^\star)^\perp = \mathrm{range}(\mathcal{L}). \tag{15.24}$$

Another useful property holds(19).

Lemma 15.15 (Self-adjointness of $\mathcal{L}$). *The operator $\mathcal{L}$ is self-adjoint, that is*

$$\mathcal{L} = \mathcal{L}^\star. \tag{15.25}$$

Proof. We begin to show that $\mathrm{dom}(\mathcal{L}^\star) = \mathrm{dom}(\mathcal{L})$, namely that

$$\mathrm{dom}(\mathcal{L}^\star) = H^2(\mathbb{R}). \tag{15.26}$$

If $\xi \in H^2(\mathbb{R})$, integrating by parts(20), we have

$$\int_{\mathbb{R}} \xi \; \mathcal{L}\varphi \; dy = \int_{\mathbb{R}} \left(-\xi'' + W''(\gamma)\xi\right) \varphi \; dy, \qquad \varphi \in H^2(\mathbb{R}). \tag{15.27}$$

---

(18) To show (15.24), one way is to introduce the following linear operator L, defined by

$$L : H^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad L\zeta := -\zeta'' + W''(\gamma)\zeta, \qquad \zeta \in H^2(\mathbb{R}) .$$

Observe that L is bounded and closed. Notice also that $\mathrm{range}(L) = \mathrm{range}(\mathcal{L})$, so that proving that the range of $\mathcal{L}$ is closed is equivalent to show that the range of L is closed.
Let M be the multiplication operator defined by

$$M : H^2(\mathbb{R}) \to L^2(\mathbb{R}), \qquad M\zeta := (W''(\gamma) - \alpha)\zeta,$$

where we recall that α is defined in (15.21). Notice that M is a compact operator, as a consequence of the exponential convergence of $W''(\gamma) - \alpha$ to zero at $\pm\infty$, and of the dominated convergence theorem.
Write

$$L\zeta = L_\alpha\zeta + M\zeta, \qquad \zeta \in H^2(\mathbb{R}),$$

where L_α is the linear operator defined as follows:

$$L_\alpha : H^2\mathbb{R}) \to L^2(\mathbb{R}), \qquad L_\alpha\zeta := -\zeta'' + \alpha\zeta, \qquad \zeta \in H^2(\mathbb{R}).$$

Note that L_α is an injective and surjective operator. Indeed, denote by $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the Fourier transform, and remember that $\alpha > 0$. Given $r \in L^2(\mathbb{R})$, a solution $\zeta \in H^2(\mathbb{R})$ to $L_\alpha\zeta = r$ is $\mathcal{F}^{-1}\left(\frac{\mathcal{F}(r)}{\alpha+|\xi|^2}\right)$, and so L_α is surjective. On the other hand, if $L_\alpha\zeta = 0$, then ζ is a linear combination of the hyperbolic sine and the hyperbolic cosine, which is not an element of $L^2(\mathbb{R})$, so that $\zeta = 0$, and hence $\ker(L_\alpha) = 0$.
Then the fact that the range of L is closed is a consequence of the properties of L_α and [167, Corollary 19.1.8].

(19) See also [1, Theorem 1.1], [166], [120, pages 1535-1536], [58, Chapter 2] and [178] for related arguments.

(20) Recall that, if $v \in H^1(\mathbb{R})$, then $\lim_{y\to\pm\infty} v(y) = 0$. Indeed, if $u := v^2$, then $u \in L^1(\mathbb{R})$, and $u' = 2vv' \in L^1(\mathbb{R})$. Consequently, $u(y) = u(0) + \int_0^y u'(\sigma)\, d\sigma$ has a limit as $y \to +\infty$, and if such a limit is not zero, then $u \notin L^1(\mathbb{R})$. The same argument holds when $y \to -\infty$.

Hence there exists a constant $c \geq 0$ such that

$$\int_{\mathbb{R}} \xi \mathcal{L}\varphi \, dy \leq c\|\varphi\|_{L^2(\mathbb{R})}, \qquad \varphi \in H^2(\mathbb{R}),$$

and therefore[21]

$$\mathrm{dom}(\mathcal{L}^\star) \supseteq H^2(\mathbb{R}). \tag{15.28}$$

Let us now show the opposite inclusion, namely that $\mathrm{dom}(\mathcal{L}^\star) \subseteq H^2(\mathbb{R})$. Let $\xi \in \mathrm{dom}(\mathcal{L}^\star) \subseteq L^2(\mathbb{R})$, so that

$$\int_{\mathbb{R}} \mathcal{L}^\star \xi \; \varphi \, dy = \int_{\mathbb{R}} \xi \; \mathcal{L}\varphi \, dy, \qquad \varphi \in H^2(\mathbb{R}).$$

Then

$$\int_{\mathbb{R}} \xi \varphi'' \, dy = \int_{\mathbb{R}} \big(W''(\gamma)\xi - \mathcal{L}^\star \xi\big) \varphi \, dy, \qquad \varphi \in H^2(\mathbb{R}). \tag{15.29}$$

Define

$$\Phi(y) := \int_0^y \int_0^\sigma \Big(W''(\gamma(\tau))\xi(\tau) - \mathcal{L}^\star \xi(\tau)\Big) \, d\tau \, d\sigma, \qquad y \in \mathbb{R}.$$

Then, recalling also that $W''(\gamma)$ is bounded, we have

$$\Phi'' = W''(\gamma)\xi - \mathcal{L}^\star \xi \in L^2(\mathbb{R}).$$

From (15.29) it then follows that

$$\int_{\mathbb{R}} \xi \varphi'' \, dy = \int_{\mathbb{R}} \Phi'' \varphi \, dy, \qquad \varphi \in H^2(\mathbb{R}).$$

Consequently

$$\Phi'' = \xi''$$

in the distributional sense[22] and so $\xi'' \in L^2(\mathbb{R})$. Since $\xi \in L^2(\mathbb{R})$, we deduce that $\xi \in H^2(\mathbb{R})$. This, together with (15.28), concludes the proof of (15.26). Eventually, (15.25) follows from equality (15.27). $\square$

Lemma 15.16 (Kernel of $\mathcal{L}^\star$). *We have*

$$\ker(\mathcal{L}^\star) = \mathrm{span}\{\gamma'\}. \tag{15.30}$$

(21) Recall that, by definition, $\mathrm{dom}(\mathcal{L}^\star) = \{\xi \in L^2(\mathbb{R}) : \exists c \geq 0 : |\langle \xi, \mathcal{L}\varphi \rangle| \leq c\|\varphi\|_{L^2(\mathbb{R})} \ \forall \varphi \in \mathrm{dom}(\mathcal{L})\}$.

(22) Therefore $\Phi - \xi$ is a linear function.

Proof. From (15.22) and (15.25) it follows that $\gamma' \in \ker(\mathcal{L}^\star)$, hence

$$\ker(\mathcal{L}^\star) \supseteq \mathrm{span}\{\gamma'\}.$$

Let us show the opposite inclusion. Take

$$\zeta_0, \zeta_1 \in \mathbb{R}, \qquad \zeta_0 > 0, \qquad \zeta_1 > 0. \tag{15.31}$$

Recalling (15.5), (15.6) and $\lim_{y \to +\infty} \gamma(y) = \gamma_+$, we select y_0 with the following properties:

$$y_0 > 0 \ \text{ and } \ W''(\gamma(y_0)) > 0.$$

Let $\zeta \in \mathcal{C}^\infty(\mathbb{R})$ be the unique solution of the Cauchy problem

$$\begin{cases} -\zeta'' + W''(\gamma)\zeta = 0, \\ \zeta(y_0) = \zeta_0, \\ \zeta'(y_0) = \zeta_1. \end{cases} \tag{15.32}$$

Using (15.31) it follows that $\lim_{y \to +\infty} \zeta(y) = +\infty$, so that

$$\zeta \notin \ker(\mathcal{L}^\star). \tag{15.33}$$

Since the linear space of solutions to the ordinary differential equation in (15.32) is two-dimensional, and (15.33) holds, we deduce that $\ker(\mathcal{L}^\star)$ must be one-dimensional, and (15.30) follows. □

We are now in a position to prove the following result[(23)].

Theorem 15.17 (Alternative). *Suppose that $r \in L^2(\mathbb{R})$.*

(i) *We have*

$$\ker(\mathcal{L}^\star)^\perp = \mathrm{range}(\mathcal{L}),$$

(23) Another proof of Theorem 15.17(i) is the following (see [55]). Consider the operator $\mathfrak{L} : H^1(\mathbb{R}) \to H^{-1}(\mathbb{R})$ defined by $\mathfrak{L}\xi := -\xi'' + W''(\gamma)\xi$. One can verify that $\mathfrak{L}$ is self-adjoint and that $\mathrm{Ker}(\mathfrak{L}) = \mathrm{span}\{\gamma'\}$. Denote by $H^1_\alpha(\mathbb{R})$ the space $H^1(\mathbb{R})$ endowed with the scalar product $(\xi, \eta)_\alpha := \int_{\mathbb{R}} \xi'\eta' \, dy + \alpha \int_{\mathbb{R}} \xi\eta \, dy$, and by $H^{-1}_\alpha(\mathbb{R})$ the dual of $H^1_\alpha(\mathbb{R})$, endowed with the dual norm. Let $\mathcal{R} : H^{-1}_\alpha(\mathbb{R}) \to H^1_\alpha(\mathbb{R})$ be the Riesz (linear isometric) operator, and let $\mathcal{B} : H^1_\alpha(\mathbb{R}) \to H^{-1}_\alpha(\mathbb{R})$ be the linear operator defined by $\mathcal{B}(\xi) := -\xi'' + \alpha\xi$, so that $\mathcal{R}\mathcal{B} = \mathrm{Id}$ on $H^1_\alpha(\mathbb{R})$. We write the operator $\mathcal{R}\mathfrak{L} : H^1_\alpha(\mathbb{R}) \to H^1_\alpha(\mathbb{R})$ as follows: $\mathcal{R}\mathfrak{L} = \mathrm{Id} + \mathcal{R}(\mathfrak{L} - \mathcal{B})$, and we observe that the operator $\mathfrak{L} - \mathcal{B} : H^1_\alpha(\mathbb{R}) \to H^{-1}_\alpha(\mathbb{R})$ is compact, so that also the operator $\mathcal{R}(\mathfrak{L} - \mathcal{B})$ is compact. Then assertion (i) of Theorem 15.17 follows.

hence the problem

$$\mathcal{L}\zeta = r$$

has a solution $\zeta \in H^2(\mathbb{R})$ *if and only if*

$$\int_{\mathbb{R}} r\gamma' \, dy = 0. \tag{15.34}$$

(ii) *The solution* ζ *in* (i) *is unique up to the addition of a real multiple of* γ'.

(iii) *If* r *is odd then the solution* $\zeta \in H^2(\mathbb{R})$ *of*

$$\begin{cases} \mathcal{L}\zeta = r, \\ \zeta(0) = 0 \end{cases} \tag{15.35}$$

is unique and odd.

Proof. From (15.24) it follows that assertions (i) and (ii) are a consequence of Lemma 15.16. Assume now that r is odd (so that (15.34) is satisfied) and let $\{\zeta + \lambda\gamma' : \lambda \in \mathbb{R}\}$ be the set of all solutions of $\mathcal{L}\zeta = r$. From $\zeta(0) = 0$ and $\gamma'(0) = 1$, it follows that λ is uniquely determined, so that (15.35) admits a unique solution. Moreover, such a solution is odd, since the function $\frac{1}{2}(\zeta(y) - \zeta(-y)))$, which is still a solution of (15.35) (recall that $W''(\gamma)$ is even), is odd. $\square$

15.3. The shape function

We now introduce a special function, needed in the asymptotic expansion described in Chapter 16 and in the sub/supersolutions construction in Chapter 17. Observe that the exponential convergence at $\pm\infty$ to zero of γ' implies that the function

$$y \to y\gamma'(y) \tag{15.36}$$

belongs to $L^2(\mathbb{R})$. Therefore, we can invoke Theorem 15.17, to ensure that the following definition is well-posed.

Definition 15.18 (The function ξ). We denote by $\xi \in H^2(\mathbb{R})$ the solution of the problem

$$\begin{cases} \mathcal{L}\xi(y) = -y\gamma'(y), \\ \xi(0) = 0. \end{cases} \tag{15.37}$$

Indeed, the function ξ exists by Theorem 15.17, since the function in (15.36) is odd, and so

$$\int_{\mathbb{R}} y(\gamma'(y))^2 \, dy = 0.$$

Furthermore, ξ is odd and of class $\mathcal{C}^\infty(\mathbb{R})$. We call the function ξ the shape function.

In the proof of the convergence Theorem 17.5 we will need the knowledge of the asymptotic behaviour of ξ and of its derivative, at points of order $|\log \epsilon|$. Therefore, we record the following result.

Lemma 15.19 (Asymptotic behaviour of ξ). *There exists a constant $C_\xi > 0$ such that*

$$\begin{aligned} |\xi(y)| &\le C_\xi(1+y^2)\gamma'(y), \qquad && y \in \mathbb{R}, \\ |\xi'(y)| &\le C_\xi(1+y^2)\gamma'(y), \qquad && y \in \mathbb{R}. \end{aligned} \tag{15.38}$$

Proof. It is a consequence of the next lemma, applied with $m = 1$, observing that[24] $\xi \in H^3(\mathbb{R})$. ☐

Lemma 15.20. *Let $\zeta \in H^2(\mathbb{R})$, and suppose that there exist a constant $c > 0$ and $m \in \mathbb{N}$ such that*

$$|\mathcal{L}\zeta(y)| \le c(1+|y|^m)\gamma'(y), \qquad y \in \mathbb{R}. \tag{15.39}$$

Then there exists a constant $C > 0$ such that

$$|\zeta(y)| \le C(1+|y|^{m+1})\gamma'(y), \qquad y \in \mathbb{R}. \tag{15.40}$$

If in addition $\zeta \in H^3(\mathbb{R})$ and

$$|(\mathcal{L}\zeta)'(y)| \le c(1+|y|^m)\gamma'(y), \qquad y \in \mathbb{R}, \tag{15.41}$$

then

$$|\zeta'(y)| \le C(1+|y|^{m+1})\gamma'(y), \qquad y \in \mathbb{R}. \tag{15.42}$$

Proof. From (15.18) and (15.19) it follows that

$$\begin{aligned} \lim_{y\to+\infty} \frac{\gamma''(y)}{\gamma'(y)} &= -\lim_{y\to+\infty} \frac{W'(\gamma(y))}{w(\gamma(y))} \\ &= -\lim_{y\to+\infty} \frac{w(\gamma(y))w'(\gamma(y))}{w(\gamma(y))} = -1 =: -a. \end{aligned}$$

Therefore we can select $y_0 \ge m+1$ so large that

$$\gamma''(y) < -\frac{a}{2}\gamma'(y), \qquad y \ge y_0, \tag{15.43}$$

(24) We have $\xi'' = W''(\gamma)\xi + y\gamma' \in H^1(\mathbb{R})$.

and at the same time we can fix $\delta > 0$ so that

$$W''(\gamma(y)) \geq \delta > 0, \qquad y \geq y_0. \tag{15.44}$$

Take $k \geq 1$ in such a way that

$$|\zeta(y_0)| \leq k y_0^{m+1} \gamma'(y_0), \tag{15.45}$$

and define

$$z(y) := k y^{m+1} \gamma'(y), \qquad y \in \mathbb{R}. \tag{15.46}$$

We claim that

$$\zeta(y) - z(y) \leq 0, \qquad y \geq y_0. \tag{15.47}$$

First of all, by definition and by (15.45) we have

$$\zeta(y_0) - z(y_0) \leq 0. \tag{15.48}$$

Next, recalling from (15.22) that $\mathcal{L}\gamma' = 0$ and using (15.43), we have, for $y \geq 0$,

$$\begin{aligned} -\mathcal{L}z(y) &= z''(y) - W''(\gamma(y))z(y) \\ &= -ky^{m+1}\mathcal{L}\gamma'(y) + 2(m+1)ky^m\gamma''(y) + (m+1)mky^{m-1}\gamma'(y) \\ &= 2(m+1)ky^m\gamma''(y) + (m+1)mky^{m-1}\gamma'(y) \\ &\leq -k(m+1)\left(ay^m - my^{m-1}\right)\gamma'(y). \end{aligned}$$

Hence, using assumption (15.39), we have

$$\begin{aligned} \mathcal{L}(\zeta(y) - z(y)) &= \mathcal{L}\zeta(y) - \mathcal{L}z(y) \\ &\leq |\mathcal{L}\zeta(y)| - k(m+1)\left(ay^m - my^{m-1}\right)\gamma'(y) \\ &\leq 0, \end{aligned} \tag{15.49}$$

for any $y \geq y_0$, provided k is large enough (depending on c). Then, by (15.48), (15.49), and the comparison principle[(25)] (recall (15.44) and the fact that $\lim_{y \to +\infty} \zeta(y) = \lim_{y \to +\infty} z(y) = 0$), claim (15.47) follows[(26)].

(25) See for instance [134, Section 6.4].

(26) Indeed, assume by contradiction that there exists $\overline{y} \geq y_0$ such that $\zeta(\overline{y}) - z(\overline{y}) > 0$. Note that from (15.46) it follows that $\lim_{y \to \pm\infty} z(y) = 0$, and from $\zeta \in H^2(\mathbb{R})$ it also follows that $\lim_{y \to \pm\infty} \zeta(y) = 0$. Consequently the function $\zeta - z$, considered in $[y_0, +\infty)$, admits a maximum $y_{\max} \in [y_0, +\infty)$. We have $-(\zeta''(y_{\max}) - z''(y_{\max})) \geq 0$, and $\zeta(y_{\max}) - z(y_{\max}) > 0$. Taking into account that $W''(\gamma(y_0)) > 0$ (recall (15.44)), it follows that $W''(\gamma) > 0$ in $[y_0, +\infty)$. Therefore, we are led to a contradiction with (15.49) at the point $y_{\max}$.

Inequality (15.47) reads as

$$\zeta(y) \le k y^{m+1} \gamma'(y), \qquad y \ge y_0. \tag{15.50}$$

Similarly, since from (15.45) we have

$$-\zeta(y_0) - z(y_0) \le 0,$$

and

$$\mathcal{L}(-\zeta(y) - z(y)) \le 0$$

for any $y \ge y_0$, provided k is large enough, we obtain

$$\zeta(y) \ge -k y^{m+1} \gamma'(y), \qquad y \ge y_0. \tag{15.51}$$

Since analogous results are valid on $(-\infty, -y_0]$, assertion (15.40) follows from (15.50), (15.51), and the compactness of $[-y_0, y_0]$, provided C is large enough.

It remains to prove (15.42). Remembering the definition of $\mathcal{L}$, it follows that

$$|\mathcal{L}(\zeta')| \le |(\mathcal{L}\zeta)'| + |W'''(\gamma)\gamma'\zeta|.$$

Since $W'''(\gamma) = 3\gamma \in L^\infty(\mathbb{R})$ and $\zeta \in L^\infty(\mathbb{R})$, using assumption (15.41) we get that there exists a constant $\bar{c} > 0$ such that

$$|\mathcal{L}(\zeta'(y))| \le \bar{c}(1 + |y|^m)\gamma'(y), \qquad y \in \mathbb{R}.$$

Then inequality (15.42) follows from the first part of the lemma. $\square$

Chapter 16
Parabolic singular perturbations: formal matched asymptotics

For each $\epsilon \in (0, 1]$ we denote by u_ϵ the solution to the singularly perturbed parabolic problem (15.2), which for convenience of the reader we rewrite in the following form:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \epsilon^{-2}\, W'(u) = 0 & \text{in } (0, T) \times \Omega, \\ \dfrac{\partial u}{\partial \mathrm{n}_\Omega} = 0 & \text{on } (0, T) \times \partial\Omega, \\ u = u_\epsilon^0 & \text{on } \{t = 0\} \times \Omega. \end{cases} \tag{16.1}$$

In this chapter we perform two asymptotic expansions of u_ϵ, which will be suitably matched one each other. In spite of the fact that the argument is formal, it eventually leads to a rigorous proof of convergence of $\{u_\epsilon(t, \cdot) = 0\}$ to a mean curvature flow as $\epsilon \downarrow 0$, valid for short times (see Chapter 17). Asymptotic expansions for reaction-diffusion equations of the type in (16.1) have been performed, among other places, in [145, 118, 120] (see also [59, 226] and [81, 6]). Here we will closely follow the arguments of [229][(1)].

Because of the strong reaction term $\epsilon^{-2} W'(u)$, we expect that the solution u_ϵ takes values near the two minima $\gamma_\pm$ of the potential W in most of the domain, with a thin, smooth, transition region between the two phases[(2)], where it transversally crosses the unstable zero $\mathrm{y} = \gamma_c$ of W'. This motivates the use of matched asymptotics in the exterior and interior phases (outer expansion) and in the transition layer (inner expansion). As a consequence, at least formally, the front generated by (16.1) will propagate by mean curvature.

(1) In [229] the operator Δu is replaced by an elliptic operator of the form $\mathrm{div}(a(x)\nabla u)$.

(2) The two phases are the set of points in time-space where the function u_ϵ is approximately equal to 1 (exterior phase) or -1 (interior phase); see formula (16.27).

Assumptions. We suppose that there exists a compact set E with

$$E \subset \Omega, \qquad \partial E \in \mathcal{C}^\infty, \ \partial E \text{ connected}, \qquad \partial E \cap \partial\Omega = \emptyset, \tag{16.2}$$

(see Definition 1.2) such that

$$\{x \in \Omega : u_\epsilon^0(x) = 0\} = \partial E, \qquad \epsilon \in (0, 1]. \tag{16.3}$$

For any $t \in [0, T]$ we also suppose that $\{x \in \Omega : u_\epsilon(t, x) = 0\}$ is relatively compact in Ω and

$$\nabla u_\epsilon(t, y) \neq 0 \quad \text{for any } y \in \{x \in \Omega : u_\epsilon(t, x) = 0\}.$$

The set $\{x \in \Omega : u_\epsilon(t, x) = 0\}$ is therefore a smooth compact $(n-1)$-dimensional oriented embedded connected submanifold without boundary, which does not intersect the boundary of Ω(3).
Furthermore, we assume for simplicity

$$\gamma_- = -1, \qquad \gamma_+ = 1,$$

and

$$w(\mathrm{y}) = \frac{1}{2}(\mathrm{y}^2 - 1), \qquad \mathrm{y} \in \mathbb{R},$$

as in Example 15.2 (even if the results of this chapter are valid in more generality). In particular,

$$W'(\mathrm{y}) = \frac{1}{2}(\mathrm{y}^2 - 1)\mathrm{y}, \qquad \mathrm{y} \in \mathbb{R},$$

and $\gamma_c = 0$.

Notation. As usual

$$d(z) := \mathrm{dist}(z, E) - \mathrm{dist}(z, \mathbb{R}^n \setminus E), \quad z \in \mathbb{R}^n, \tag{16.4}$$

is the signed distance function from ∂E negative in the interior of E.
For $\epsilon \in (0, 1]$ we write

$$E_\epsilon(t) := \{z \in \Omega : u_\epsilon(t, z) \leq 0\}, \qquad t \in [0, T], \tag{16.5}$$

so that

$$\partial E_\epsilon(t) = \{x \in \Omega : u_\epsilon(t, x) = 0\}, \qquad t \in [0, T].$$

(3) Since $\{x \in \Omega : u_\epsilon(t, x) = 0\}$ remains "far" from $\partial\Omega$, in this chapter we shall not take into account the Neumann boundary condition in (16.1).

We denote by

$$d_\epsilon(t, z) := \mathrm{dist}(z, E_\epsilon(t)) - \mathrm{dist}(z, \mathbb{R}^n \setminus E_\epsilon(t)), \quad t \in [0, T], z \in \mathbb{R}^n, \tag{16.6}$$

the signed distance function from $\partial E_\epsilon(t)$ negative in the interior of $E_\epsilon(t)$. We also suppose that $T > 0$ is sufficiently small so that there exists $\rho > 0$ such that

$$d_\epsilon \in \mathcal{C}^\infty([0, T] \times \mathrm{U}), \qquad \epsilon \in (0, 1],$$

where

$$\mathrm{U} = \{z \in \mathbb{R}^n : |d(z)| < \rho\} = (\partial E)^+_\rho. \tag{16.7}$$

Parametrization of $\partial E_\epsilon(t)$

For any $\epsilon \in (0, 1]$ and $t \in [0, T]$ we parametrize $\partial E_\epsilon(t)$ on a fixed reference manifold as follows. Let $\mathcal{S} \subset \mathbb{R}^n$ be a smooth embedded oriented compact connected $(n-1)$-dimensional submanifold without boundary. We suppose that there exists a map[(4)]

$$\varphi_\epsilon \in \mathcal{X}([0, T]; \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n))$$

such that

$$\varphi_\epsilon(t, \mathcal{S}) = \partial E_\epsilon(t), \qquad t \in [0, T].$$

We also assume that φ_ϵ depends smoothly on $\epsilon \in (0, 1]$.

Notice that

$$u_\epsilon(t, \varphi_\epsilon(t, s)) = 0, \qquad t \in [0, T], \ s \in \mathcal{S}. \tag{16.8}$$

Given $(t, s) \in [0, T] \times \mathcal{S}$, and setting

$$x := \varphi_\epsilon(t, s) \in \partial E_\epsilon(t),$$

we denote[(5)] by

- $\nu_\epsilon(t, s)$ the unit normal to $\partial E_\epsilon(t)$ at x, pointing toward the interior of $E_\epsilon(t)$,
- $\mathrm{H}_\epsilon(t, s)$ the mean curvature of $\partial E_\epsilon(t)$ at x,
- $A_\epsilon(t, s)$ the second fundamental form of $\partial E_\epsilon(t)$ at x.

We recall that

(4) See Definition 3.5. We do not necessarily assume here the parametrization φ_ϵ to be normal, in the sense of Remark 3.9.

(5) See Section 1.5.

- $\nabla d_\epsilon(t,x) = -\nu_\epsilon(t,s)$,
- the normal velocity vector $\mathbf{V}_\epsilon(t,s)$ and the outer normal velocity $-\mathrm{V}_\epsilon(t,s)$ of $\partial E_\epsilon(t)$ at x are given respectively by[6]

$$\begin{aligned}\mathbf{V}_\epsilon(t,s) =& \langle \nu_\epsilon(t,s), \frac{\partial \varphi_\epsilon}{\partial t}(t,s)\rangle\, \nu_\epsilon(t,s),\\ \mathrm{V}_\epsilon(t,s) =& \langle \nu_\epsilon(t,s), \frac{\partial \varphi_\epsilon}{\partial t}(t,s)\rangle,\end{aligned}$$

  and we have

$$-\frac{\partial d_\epsilon}{\partial t}(t,x)\nabla d_\epsilon(t,x) = \mathbf{V}_\epsilon(t,s), \qquad \frac{\partial d_\epsilon}{\partial t}(t,x) = \mathrm{V}_\epsilon(t,s).$$

Remember also that[7]

- $\Delta d_\epsilon(t,x) = \mathrm{H}_\epsilon(t,s)$,
- $|\nabla^2 d_\epsilon(t,x)| = |A_\epsilon(t,s)|_{g_\epsilon(t)}$ is the length of the second fundamental form, where $|A_\epsilon(t,s)|_{g_\epsilon(t)}$ is defined in (6.35), and $g_\epsilon(t)$ is the metric tensor on $\partial E_\epsilon(t)$ (see (1.49)).

We conclude this section with the following definition, that allows us to define a reparametrization of a neighbourhood of the flowing manifolds, and will be useful in the subsequent formal analysis. Observe that such a reparametrization depends on ϵ.

Definition 16.1 (The map s_ϵ). We denote by

$$s_\epsilon : [0,T] \times \mathrm{U} \to \mathcal{S}$$

the map defined as follows: given $(t,z) \in [0,T] \times \mathrm{U}$, $s_\epsilon(t,z) \in \mathcal{S}$ is the parameter such that $\varphi_\epsilon(t, s_\epsilon(t,z))$ is the point on $\partial E_\epsilon(t)$ nearest to z, namely

$$z - d_\epsilon(t,z)\nabla d_\epsilon(t,z) = \varphi_\epsilon(t, s_\epsilon(t,z)), \qquad (t,z) \in [0,T] \times \mathrm{U}. \tag{16.9}$$

Note that if $\lambda \in \mathbb{R}$ and $|\lambda|$ is sufficiently small, and if $x \in \partial E_\epsilon(t)$, then

$$s_\epsilon(t,x) = s_\epsilon(t, x + \lambda \nabla d_\epsilon(t,x)),$$

and this implies

$$0 = \frac{d}{d\lambda} s_\epsilon(t, x + \lambda\nabla d_\epsilon(t,x))_{|\lambda=0} = \mathrm{d}_z s_\epsilon(t,x)\nabla d_\epsilon(t,x), \tag{16.10}$$

where $\mathrm{d}_z s_\epsilon$ denotes the differential of s_ϵ with respect to z.

[6] See formulas (3.7), (3.8) and (3.11).

[7] See Chapter 1.

16.1. Formal series expansions

We begin by formally expanding various quantities with respect to ϵ.

16.1.1. Expansion of φ_ϵ

We suppose that φ_ϵ admits the expansion

$$\varphi_\epsilon(t,s)=\varphi_0(t,s)+\epsilon\varphi_1(t,s)+\mathcal{O}(\epsilon^2), \quad (t,s)\in[0,T]\times\mathcal{S}, \tag{16.11}$$

where

$$\varphi_i \in \mathcal{X}([0,T];\mathrm{Emb}(\mathcal{S};\mathbb{R}^n)), \qquad i\in\{0,1\}.$$

We define

$$\Sigma_0(t) := \varphi_0(t,\mathcal{S}), \qquad t\in[0,T]. \tag{16.12}$$

16.1.2. Expansions of V_ϵ, ν_ϵ, H_ϵ and $|A_\epsilon|^2_{g_\epsilon}$

We suppose that:

- V_ϵ admits the expansion

  $$\mathrm{V}_\epsilon(t,s) = V_0(t,s)+\epsilon V_1(t,s)+\mathcal{O}(\epsilon^2), \quad (t,s)\in[0,T]\times\mathcal{S}, \tag{16.13}$$

  where we assume that

  - $V_i\in\mathcal{C}^\infty([0,T]\times\mathcal{S})$ for $i\in\{0,1\}$,
  - $-V_0(t,s)$ is the outer normal velocity of $\Sigma_0(t)$ at $\varphi_0(t,s)$.
- ν_ϵ admits the expansion

  $$\nu_\epsilon(t,s) = \nu_0(t,s)+\epsilon\nu_1(t,s)+\mathcal{O}(\epsilon^2), \quad (t,s)\in[0,T]\times\mathcal{S}, \tag{16.14}$$

  where we assume that

  - $\nu_i\in\mathcal{C}^\infty([0,T]\times\mathcal{S};\mathbb{R}^n)$ for $i\in\{0,1\}$,
  - $\nu_0(t,s)$ is the inward unit normal[8] to $\Sigma_0(t)$ at $\varphi_0(t,s)$.
- H_ϵ admits the expansion

  $$\mathrm{H}_\epsilon(t,s)=\mathrm{H}_0(t,s)+\epsilon\mathrm{H}_1(t,s)+\mathcal{O}(\epsilon^2), \quad (t,s)\in[0,T]\times\mathcal{S}, \tag{16.15}$$

  where we assume that

  - $\mathrm{H}_i\in\mathcal{C}^\infty([0,T]\times\mathcal{S})$ for $i\in\{0,1\}$,
  - $\mathrm{H}_0(t,s)$ is the mean curvature of $\Sigma_0(t)$ at $\varphi_0(t,s)$.

---

[8] See also (16.27) below.

- $|A_\epsilon|^2_{g_\epsilon}$ admits the expansion

$$|A_\epsilon(t,s)|^2_{g_\epsilon(t)} = \mathcal{A}_0(t,s) + \mathcal{O}(\epsilon), \qquad (t,s) \in [0,T] \times \mathcal{S},$$

  where we assume that
  - $\mathcal{A}_0 \in \mathcal{C}^\infty([0,T] \times \mathcal{S})$,
  - $\mathcal{A}_0(t,s)$ is the squared length of the second fundamental form of $\Sigma_0(t)$ at $\varphi_0(t,s)$.

16.1.3. Expansion of s_ϵ

We suppose that s_ϵ admits the expansion

$$s_\epsilon(t,z) = s_0(t,z) + \epsilon s_1(t,z) + \mathcal{O}(\epsilon^2), \qquad (t,z) \in [0,T] \times \mathrm{U}, \tag{16.16}$$

where s_0 and s_1 are of class $\mathcal{C}^\infty$ in a time-space neighbourhood of $\Sigma_0(\cdot)$, and $s_0(t,z)$ is the parameter on $\mathcal{S}$ such that $\varphi_0(t, s_0(t,z))$ is the point on $\Sigma_0(t)$ nearest to z.

16.1.4. Initial interface

The independence of ϵ of the initial boundary $\{x \in \Omega : u_\epsilon(0,x) = 0\}$ in (16.3)[9] implies that there exists a map

$$\varphi_E \in \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n)$$

such that

$$\varphi_\epsilon(0,\cdot) = \varphi_E(\cdot), \qquad \epsilon \in (0,1]. \tag{16.17}$$

From assumption (16.3) and the definition (16.6) of d_ϵ it follows that

$$d_\epsilon(0,\cdot) = d(\cdot), \qquad \epsilon \in (0,1], \tag{16.18}$$

where d is defined in (16.4).

Since φ_E is independent of ϵ, from (16.11) and (16.17) it follows that

$$\varphi_0(0,s) = \varphi_E(s), \qquad s \in \mathcal{S} \tag{16.19}$$

and

$$\varphi_1(0,s) = 0, \qquad s \in \mathcal{S}. \tag{16.20}$$

We denote by

$$\nu_E(s) = -\nabla d(x) \tag{16.21}$$

[9] This is consistent with the choice of u^0_ϵ that we will make in Chapter 17.

and by

$$\mathrm{H}_E(s) = \Delta d(x)$$

the unit normal to ∂E pointing toward the interior of E, and the mean curvature of ∂E at $x = \varphi_E(s)$, respectively.

From $\nabla d_\epsilon(0, x) = -\nu_\epsilon(0, s)$, (16.14), (16.18), (16.17) and (16.21), it follows that

$$\nu_0(0, s) = \nu_E(s), \qquad s \in \mathcal{S},$$

and

$$\nu_1(0, s) = 0, \qquad s \in \mathcal{S}.$$

Similarly

$$\mathrm{H}_0(0, s) = \mathrm{H}_E(s), \qquad s \in \mathcal{S},$$

and

$$\mathrm{H}_1(0, s) = 0, \qquad s \in \mathcal{S}. \tag{16.22}$$

We denote by

$$s_E : \mathrm{U} \to \mathcal{S}$$

the map defined as follows: given $z \in \mathrm{U}$, the point $\varphi_E(s_E)$ is the point on ∂E nearest to z, namely

$$z - d(z)\nabla d(z) = \varphi_E(s_E(z)).$$

Note that

$$s(0, \cdot) = s_E(\cdot).$$

16.1.5. Extensions in $[0, T] \times \mathrm{U}$

Recall from Lemma 6.1 that

$$\frac{\partial d_\epsilon}{\partial t}(t, z) = \frac{\partial d_\epsilon}{\partial t}(t, x) = \mathrm{V}_\epsilon(t, s_\epsilon(t, z)), \qquad (t, z) \in [0, T] \times \mathrm{U},$$

where

$$x := z - d_\epsilon(t, z)\nabla d_\epsilon(t, z).$$

Then, using the expansion (16.13) of V_ϵ and the expansion (16.16) of the map s_ϵ, we have

$$\begin{aligned}\frac{\partial d_\epsilon}{\partial t}(t, z) = {} & V_0(t, s_0(t, z)) \\ & + \epsilon\Big[V_1(t, s_0(t, z)) + \mathrm{d}_s V_0\big(t, s_0(t, z)\big)s_1(t, z)\Big] \\ & + \mathcal{O}(\epsilon^2),\end{aligned} \tag{16.23}$$

where d_s denotes the differential with respect to s in a local chart.

From (1.9) we have

$$\nabla d_\epsilon(t,z) = \nabla d_\epsilon(t,x) = -\nu_\epsilon(t, s_\epsilon(t,z)), \qquad (t,z) \in [0,T] \times \mathrm{U}.$$

Then, using the expansion (16.14) of ν_ϵ and the expansion (16.16) of the map s_ϵ, we have

$$\begin{aligned} -\nabla d_\epsilon(t,z) = {} & \nu_0(t, s_0(t,z)) \\ & + \epsilon\Big[\nu_1(t, s_0(t,z)) + \mathrm{d}_s\nu_0\big(t, s_0(t,z)\big)s_1(t,z)\Big] + \mathcal{O}(\epsilon^2). \end{aligned}$$

From formula (1.41), we have the expansion

$$\Delta d_\epsilon(t,z) = \Delta d_\epsilon(t,x) - d_\epsilon(t,z)|\nabla^2 d_\epsilon(t,x)|^2 + \mathcal{O}\left((d_\epsilon(t,z))^2\right). \tag{16.24}$$

Owing to (16.15) and (16.24), we deduce

$$\begin{aligned} \Delta d_\epsilon(t,z) = {} & \mathrm{H}_0(t, s_0(t,z)) \\ & + \epsilon\Bigg[\mathrm{H}_1(t, s_0(t,z)) + \mathrm{d}_s\mathrm{H}_0(t, s_0(t,z))s_1(t,z) \\ & \qquad - \frac{d_\epsilon(t,z)}{\epsilon}\mathcal{A}_0(t, s_0(t,z))\Bigg] + \mathcal{O}(\epsilon^2). \end{aligned} \tag{16.25}$$

We are now in a position to begin the procedure of matched asymptotic expansion[(10)].

16.2. Expansion far from the interface

Remembering the definition of $\Sigma_0(t)$ in (16.12), we suppose that, for each $t \in [0,T]$ and $z \in \Omega \setminus \Sigma_0(t)$, the function u_ϵ can be expanded[(11)] as

$$u_\epsilon(t,z) = \mathrm{u}_0(t,z) + \epsilon\mathrm{u}_1(t,z) + \epsilon^2\mathrm{u}_2(t,z) + \mathcal{O}(\epsilon^3), \tag{16.26}$$

where u_i are sufficiently smooth in $\Omega \setminus \Sigma_0(t)$, for $i \in \{0,1,2\}$. We suppose also that

$$\begin{cases} I_0(t) := \{z \in \Omega \setminus \Sigma_0(t) : \mathrm{u}_0(t,z) < 0\} \neq \emptyset, \\ O_0(t) := \{z \in \Omega \setminus \Sigma_0(t) : \mathrm{u}_0(t,z) > 0\} \neq \emptyset, \\ \Sigma_0(t) = \Omega \setminus \Big(I_0(t) \cup O_0(t)\Big). \end{cases} \tag{16.27}$$

(10) We have not expanded d_ϵ in powers of ϵ; see, *e.g.*, [6] for a case where such an expansion is performed.

(11) This expansion is usually called outer expansion.

For each $t \in [0, T]$ and $z \in \Omega \setminus \Sigma_0(t)$ we have

$$W'(u_\epsilon) = W'(\mathrm{u}_0) + \epsilon W''(\mathrm{u}_0)\mathrm{u}_1 + \epsilon^2 \left(W''(\mathrm{u}_0)\mathrm{u}_2 + \frac{1}{2}\mathrm{u}_1^2 W'''(\mathrm{u}_0) \right) + \mathcal{O}(\epsilon^3). \tag{16.28}$$

Our substituting (16.26) and (16.28) into the parabolic equation in (16.1) yields

$$\frac{1}{\epsilon^2} W'(\mathrm{u}_0) + \frac{1}{\epsilon} W''(\mathrm{u}_0)\mathrm{u}_1 + \partial_t \mathrm{u}_0 - \Delta \mathrm{u}_0 + W''(\mathrm{u}_0)\mathrm{u}_2 + \frac{1}{2}\mathrm{u}_1^2 W'''(\mathrm{u}_0) + \mathcal{O}(\epsilon) = 0. \tag{16.29}$$

We now equate in (16.29) the same powers of ϵ.

• Order ϵ^{-2}. We have

$$W'(\mathrm{u}_0) = 0.$$

Hence, from (16.27) and the smoothness of u_0 in $\Omega \setminus \Sigma_0(t)$, we have[(12)]

$$\mathrm{u}_0(t, z) = \begin{cases} \gamma_- & \text{if } z \in I_0(t), \\ \gamma_+ & \text{if } z \in O_0(t), \end{cases} \qquad t \in [0, T]. \tag{16.30}$$

• Order ϵ^{-1}. Using (16.30) and (15.6) we obtain

$$\mathrm{u}_1(t, z) = 0, \qquad t \in [0, T],\ z \in \Omega \setminus \Sigma_0(t). \tag{16.31}$$

• Order ϵ^0. Equation (16.30) implies that $\partial_t \mathrm{u}_0 = 0$ and $\nabla \mathrm{u}_0 = 0$ at each point (t, z) with $z \notin \Sigma_0(t)$. Consequently

$$\mathrm{u}_2(t, z) = 0, \qquad t \in [0, T],\ z \in \Omega \setminus \Sigma_0(t). \tag{16.32}$$

Remark 16.2. Formulas (16.30), (16.31) and (16.32) will be used to find the asymptotic values for the corresponding expressions in the inner asymptotic expansion (matching conditions).

(12) We exclude that u_0 takes the value γ_c (see (15.4)) in the open sets $I_0(t)$ and $O(t)$, since γ_c is an unstable zero of W'.

16.3. Expansion in a tubular neighbourhood of the interface

Recalling the definition of s_ϵ in (16.9), we consider the smooth map that associates with the point

$$(t, z) \in [0, T] \times \mathrm{U}$$

the point

$$(t, s_\epsilon(t, z), d_\epsilon(t, z)) \in [0, T] \times \mathcal{S} \times \mathbb{R}.$$

Note that such a map depends smoothly also on ϵ; moreover there exists $\overline{\epsilon} > 0$ such that the image of this map contains $[0, T] \times \mathcal{S} \times (-\rho/2, \rho/2)$ for any $\epsilon \in (0, \overline{\epsilon})$, where ρ is given in (16.7). On $[0,T]\times\mathcal{S}\times(-\rho/2,\rho/2)$ we can consider the smooth inverse map, namely

$$(t, s, d) \in [0, T] \times \mathcal{S} \times (-\rho/2, \rho/2) \to (t, z),$$

where

$$z = z(t, s, d) = \varphi_\epsilon(t, s) + d\ \nu_\epsilon(t, s).$$

Define now the map $\Psi_\epsilon \in \mathcal{C}^\infty\left([0, T] \times \mathcal{S} \times (-\frac{\rho}{2\epsilon}, \frac{\rho}{2\epsilon}); \mathrm{U}\right)$ as follows:

$$\Psi_\epsilon(t, s, y) = (t, z), \qquad z = z(t, s, d) = \varphi_\epsilon(t, s) + \epsilon y \nu_\epsilon(t, s),$$

where the stretched variable y is defined as

$$y := \frac{d}{\epsilon} \in \left(-\frac{\rho}{2\epsilon}, \frac{\rho}{2\epsilon}\right).$$

The idea is to write the partial differential equation in (16.1) in terms of the new coordinates (t, s, y). Therefore we define

$$U_\epsilon : [0, T] \times \mathcal{S} \times \left(-\frac{\rho}{2\epsilon}, \frac{\rho}{2\epsilon}\right) \to \mathbb{R}$$

as

$$U_\epsilon(t, s, y) := u_\epsilon(t, z) = u_\epsilon(\Psi_\epsilon(t, s, y)), \tag{16.33}$$

where

$$z = z(t, s, \epsilon y) := \varphi_\epsilon(t, s) + \epsilon y \nu_\epsilon(t, s).$$

Suppose that U_ϵ can be expanded[(13)] as

$$U_\epsilon(t, s, y) = U_0(t, s, y) + \epsilon U_1(t, s, y) + \epsilon^2 U_2(t, s, y) + \mathcal{O}(\epsilon^3), \tag{16.34}$$

where we assume $U_i \in \mathcal{C}^\infty([0, T] \times \mathcal{S} \times \mathbb{R})$ for $i \in \{0, 1, 2\}$.

[(13)] This expansion is usually called inner expansion.

Note that from (16.8) it follows that

$$U_\epsilon(t,s,0) = u_\epsilon(t,\varphi_\epsilon(t,s)) = 0, \qquad (t,s) \in [0,T] \times \mathcal{S},$$

hence

$$U_0(t,s,0) = 0, \qquad (t,s) \in [0,T] \times \mathcal{S}, \tag{16.35}$$

$$U_1(t,s,0) = 0, \qquad (t,s) \in [0,T] \times \mathcal{S}, \tag{16.36}$$

$$U_2(t,s,0) = 0, \qquad (t,s) \in [0,T] \times \mathcal{S}. \tag{16.37}$$

The conditions at infinity for U_i are obtained by imposing the matching with the outer expansion. Therefore, from (16.30), (16.31) and (16.32) we impose respectively

$$\lim_{y\to\pm\infty} U_0(t,s,y) = \gamma_\pm = \pm 1, \qquad (t,s) \in [0,T] \times \mathcal{S}, \tag{16.38}$$

and

$$\lim_{y\to\pm\infty} U_1(t,s,y) = 0, \qquad (t,s) \in [0,T] \times \mathcal{S}, \tag{16.39}$$

$$\lim_{y\to\pm\infty} U_2(t,s,y) = 0, \qquad (t,s) \in [0,T] \times \mathcal{S}. \tag{16.40}$$

Employing the notation

$$\begin{aligned} U'_\epsilon &:= \partial_y U_\epsilon = U'_0 + \epsilon U'_1 + \epsilon^2 U'_2 + \mathcal{O}(\epsilon^3), \\ U''_\epsilon &:= \partial_{yy} U_\epsilon = U''_0 + \epsilon U''_1 + \epsilon^2 U''_2 + \mathcal{O}(\epsilon^3), \end{aligned}$$

and denoting by

$$U_{\epsilon s} = U_{0s} + \epsilon U_{1s} + \mathcal{O}(\epsilon^2)$$

the differential of U_ϵ with respect to s, we have, from (16.33), (16.34) and (16.23),

$$\begin{aligned} \frac{\partial u_\epsilon}{\partial t} &= \frac{1}{\epsilon} U'_\epsilon \frac{\partial d_\epsilon}{\partial t} + U_{\epsilon s} \frac{\partial s_\epsilon}{\partial t} + \frac{\partial U_\epsilon}{\partial t} \\ &= \frac{1}{\epsilon} U'_0 V_0 + U'_0 V_1 + U'_1 V_0 + U'_0 \, \mathrm{d}_s V_0 \, s_1 + U_{0s} \frac{\partial s_0}{\partial t} \\ &\quad + \frac{\partial U_0}{\partial t} + \mathcal{O}(\epsilon). \end{aligned} \tag{16.41}$$

Indicating by $\mathrm{d}_z s_\epsilon$ the $(n-1)\times n$ Jacobian matrix of s_ϵ with respect to z and by $(\mathrm{d}_z s_\epsilon)^*$ its transposed, from (16.33) we have

$$\nabla u_\epsilon = \frac{1}{\epsilon} U'_\epsilon \nabla d_\epsilon + (\mathrm{d}_z s_\epsilon)^* U_{\epsilon s}.$$

Recalling that d_ϵ satisfies the eikonal equation

$$|\nabla d_\epsilon|^2 = 1 \qquad \text{in } [0,T] \times \mathrm{U},$$

we then have

$$\Delta u_\epsilon = \frac{1}{\epsilon^2} U_\epsilon'' + \frac{1}{\epsilon} U_\epsilon' \Delta d_\epsilon + U_{\epsilon s} \Delta s_\epsilon + \mathrm{tr}\left((\mathrm{d}_z s_\epsilon)^* U_{\epsilon ss} \mathrm{d}_z s_\epsilon\right), \tag{16.42}$$

where Δs_ϵ is the laplacian of s_ϵ with respect to z.

Hence, from (16.25),

$$\begin{aligned} \Delta u_\epsilon =& \frac{1}{\epsilon^2} U_0'' + \frac{1}{\epsilon}\left(U_1'' + U_0' \mathrm{H}_0\right) \\ &+ U_2'' + U_0'\Big(\mathrm{H}_1 + \mathrm{d}_s \mathrm{H}_0\, s_1 - y \mathcal{A}_0\Big) + U_1' \mathrm{H}_0 + U_{0s} \Delta s_0 \\ &+ \mathrm{tr}\left((\mathrm{d}_z s_0)^*\ U_{0ss}\ \mathrm{d}_z s_0\right) + \mathcal{O}(\epsilon). \end{aligned} \tag{16.43}$$

Finally

$$\begin{aligned} \frac{1}{\epsilon^2} W'(U_\epsilon) = & \frac{1}{\epsilon^2} W'(U_0) + \frac{1}{\epsilon} W''(U_0) U_1 \\ &+ \left(W''(U_0) U_2 + \frac{1}{2} U_1^2 W'''(U_0) \right) + \mathcal{O}(\epsilon). \end{aligned} \tag{16.44}$$

Now, we insert (16.41), (16.43) and (16.44) into the parabolic equation in (16.1), and we equate the terms in front of the same powers of ϵ.

• Order ϵ^{-2}. Owing to (16.43) and (16.44) we deduce, for each $t \in [0,T]$ and $s \in \mathcal{S}$, the equation

$$-U_0''(t,s,y) + W'(U_0(t,s,y)) = 0, \qquad y \in \mathbb{R}.$$

This equation is coupled with condition (16.35) at $y = 0$, and with the limit conditions (16.38). It follows[14] that U_0 is explicit and independent of (t,s). Specifically, for any $t \in [0,T]$ and $s \in \mathcal{S}$,

$$U_0(t,s,y) = \gamma(y), \qquad y \in \mathbb{R}. \tag{16.45}$$

where γ is the minimizer studied in Section 15.1

[14] If $\zeta \in \mathcal{C}^\infty(\mathbb{R})$ is a bounded function satisfying

$$-\zeta''(y) + W'(\zeta(y)) = 0, \qquad y \in \mathbb{R},$$

and $\lim_{y \to \pm\infty} \zeta(y) = \pm 1$, then there exists $\tau \in \mathbb{R}$ such that $\zeta(y) = \gamma(y+\tau)$ for any $y \in \mathbb{R}$.

• Order ϵ^{-1}. From (16.41), (16.43) and (16.44) we obtain

$$U_0' V_0 - (U_1'' + U_0' \mathrm{H}_0) + W''(U_0) U_1 = 0,$$

that is

$$-U_1'' + W''(U_0) U_1 = -U_0'(V_0 - \mathrm{H}_0).$$

Employing (16.45), for any $t \in [0, T]$ and $s \in \mathcal{S}$ we get

$$-U_1''(t,s,y) + W''(\gamma(y)) U_1(t,s,y) = -\gamma'(y)\,(V_0(t,s) - \mathrm{H}_0(t,s)), \tag{16.46}$$

which has to be coupled with (16.36) and (16.39). For (16.46), (16.36) and (16.39) to be solvable, a compatibility condition on the right-hand side must be enforced. Invoking Theorem 15.17, we obtain

$$0 = \int_{\mathbb{R}} \gamma' \left[-\gamma'(V_0 - \mathrm{H}_0)\right] dy = -(V_0 - \mathrm{H}_0) \int_{\mathbb{R}} (\gamma')^2 \, dy.$$

We deduce the remarkable formal result

$$V_0(t,s) = \mathrm{H}_0(t,s), \qquad t \in [0,T], \quad s \in \mathcal{S}, \tag{16.47}$$

which suggests that problem (16.1) approximates a mean curvature flow.

Now, we insert (16.47) into (16.46) and we obtain, for any $t \in [0, T]$ and $s \in \mathcal{S}$,

$$-U_1''(t,s,y) + W''(\gamma(y)) U_1(t,s,y) = 0, \qquad y \in \mathbb{R}.$$

Remembering (16.36) and (16.39), we deduce[(15)]

$$U_1 = 0. \tag{16.48}$$

Remark 16.3. Equation (16.47), together with the initial condition (16.19), the regularity assumption (16.2) on ∂E, and the short-time existence and uniqueness result (Theorem 7.9), determine uniquely $\Sigma_0(t)$. Consequently φ_0, V_0, ν_0, H_0 and $\mathcal{A}_0$ are determined.

• Order ϵ^0. From (16.41), (16.43), (16.44), (16.45), and the independence of γ of t and s, it follows that

$$\begin{aligned} &\gamma' V_1 + U_1' V_0 + \gamma' \mathrm{d}_s V_0\, s_1 \\ &- U_2'' - \gamma'(\mathrm{H}_1 + \mathrm{d}_s \mathrm{H}_0\, s_1 - y\mathcal{A}_0) - U_1' \mathrm{H}_0 + W''(\gamma) U_2 \\ &+ \frac{1}{2} U_1^2 W'''(\gamma) = 0. \end{aligned} \tag{16.49}$$

(15) The vanishing of U_1 was observed in [120, page 1537].

Upon substituting (16.47) into (16.49) and using (16.48), we obtain

$$-U_2'' + W''(\gamma)U_2 = -\gamma'\big(V_1 - \mathrm{H}_1 + (\mathrm{d}_s V_0 - \mathrm{d}_s \mathrm{H}_0)s_1\big) - y\mathcal{A}_0\gamma'.$$

Using (16.47) we get

$$\mathrm{d}_s V_0 - \mathrm{d}_s \mathrm{H}_0 = 0, \qquad t \in [0, T], \quad s \in \mathcal{S},$$

hence

$$-\,U_2'' + W''(\gamma)U_2 = -\gamma'(V_1 - \mathrm{H}_1) - y\mathcal{A}_0\gamma'. \tag{16.50}$$

This equation must be coupled with (16.37) and (16.40). Also in this case, for (16.50) to be solvable, a compatibility condition for the right-hand side must be enforced (Theorem 15.17). This condition reads as

$$\begin{aligned} 0 &= \int_{\mathbb{R}} \gamma' \left[-\gamma'(V_1 - \mathrm{H}_1) - y\mathcal{A}_0\gamma'\right] \, dy \\ &= -\,(V_1 - \mathrm{H}_1) \int_{\mathbb{R}} (\gamma')^2 \, dy - \mathcal{A}_0 \int_{\mathbb{R}} (\gamma')^2 y \, dy \\ &= -\,\sigma_0 \,(V_1 - \mathrm{H}_1)\,, \end{aligned} \tag{16.51}$$

where in the last equality we employed (15.20). A solution φ_1 to (16.20) and (16.51) is given by

$$\varphi_1(t, s) = 0, \qquad t \in [0, T], \quad s \in \mathcal{S},$$

and

$$\mathrm{H}_1(t, s) = 0, \qquad t \in [0, T], \quad s \in \mathcal{S}$$

(recall also (16.22)). Consequently[16]

$$V_1(t, s) = 0, \qquad (t, s) \in [0, T] \times \mathcal{S}. \tag{16.52}$$

This relation, together with (16.47), suggests that problem (16.1) approximates a mean curvature flow with an error of order ϵ^2.

Inserting (16.51) into (16.50) gives, for any $t \in [0, T]$ and $s \in \mathcal{S}$,

$$-U_2''(t, s, y) + W''(\gamma(y))U_2(t, s, y) = -y\mathcal{A}_0(t, s)\gamma'(y), \qquad y \in \mathbb{R}.$$

Enforcing the orthogonality condition for the right-hand side of this equation, and remembering Definition 15.18, where the shape function ξ is introduced, we deduce

$$U_2(t, s, y) = \mathcal{A}_0(t, s)\xi(y). \tag{16.53}$$

The knowledge of U_2 in the inner asymptotic expansion will allow us to gain an error estimate of order ϵ^2 (up to logarithmic terms) in the convergence Theorem 17.5.

[16] Equality (16.52) shows that the normal component of $\frac{\partial \varphi_1}{\partial t}$ to $\Sigma_0(t)$ vanishes.

Chapter 17
Parabolic singular perturbations: convergence and error estimate

In this chapter we show the convergence of solutions of the parabolic problem (15.2) to mean curvature flow as $\epsilon \downarrow 0$[(1)]. The convergence proof is valid for short times, in particular before the onset of singularities in mean curvature flow. We also provide an error estimate between the approximate solutions and the original geometric evolution[(2)]. As a consequence of the convergence result, in Theorem 17.21 we prove the inclusion principle for smooth compact mean curvature flows. In the notes at the end of the chapter we illustrate some more informations and qualitative properties on solutions to the Allen-Cahn's equation.

17.1. Statement of the main theorem

Let us assume hypotheses (16.2) on E, so that in particular ∂E is compact and contained in Ω. According to the results discussed in Chapter 7, we know that there exists a unique smooth compact mean curvature flow starting from ∂E, that we denote by

$$f : [0, T] \to \mathcal{P}(\mathbb{R}^n),$$

where

$$T \in \big(0, t_{\max}(\partial E)\big),$$

and

$$f(0) = E.$$

We will assume that $T > 0$ is small enough so that

$$f(t) \subset \Omega, \qquad t \in [0, T].$$

(1) We mainly follow the proof given in [55].

(2) The error estimate proved in Theorem 17.5 is of the order ϵ^2 up to logarithmic corrections. This estimate turns out to be optimal; see the arguments in [221] concerning the approximation of mean curvature flow with the so-called double obstacle problem.

We let as usual

$$d(t, z) := \mathrm{dist}(z, f(t)) - \mathrm{dist}(z, \mathbb{R}^n \setminus f(t)), \quad (t, z) \in [0, T] \times \mathbb{R}^n,$$

be the signed distance from $\partial f(t)$ negative inside the interior of $f(t)$, so that

$$\frac{\partial d}{\partial t} = \Delta d \quad \text{on} \quad \partial f(t), \quad t \in [0, T]. \tag{17.1}$$

Assumptions. We fix a positive number ϱ in such a way that the closure of the ϱ-tubular neighbourhood

$$\mathrm{A}(t) := (\partial f(t))^+_\varrho \tag{17.2}$$

of $\partial f(t)$ is contained in Ω, for each $t \in [0, T]$. Setting

$$Q := \bigcup_{t \in [0,T]} \Big(\{t\} \times \mathrm{A}(t)\Big),$$

we can select[3] $\varrho > 0$ small enough in such a way that

$$d \in \mathcal{C}^\infty(Q). \tag{17.3}$$

We also suppose that

$$\gamma_- = -1, \qquad \gamma_+ = 1,$$

and

$$w(\mathrm{y}) = \frac{1}{2}(\mathrm{y}^2 - 1), \qquad \mathrm{y} \in \mathbb{R}, \tag{17.4}$$

as in Example 15.2 (even if Theorem 17.5 is still valid taking a function W with the same qualitative properties as those of the function in (15.9)). In particular, we recall that

$$\gamma' = \frac{1}{2}(1 - \gamma^2) \qquad \text{in } \mathbb{R}, \tag{17.5}$$

and

$$\gamma(y) = \mathrm{tgh}(y/2), \qquad y \in \mathbb{R}.$$

We now choose the initial datum u^0_ϵ that we will insert in problem (15.2)[4]. To understand the definition of u^0_ϵ it is necessary to keep in mind the formal asymptotic expansion in Chapter 16, in particular equation (16.45).

(3) Since ϱ is fixed from now on, we will not point out the dependence on ϱ of the various constants that will be introduced in the remainder.

(4) The convergence is valid under far less stringent requirements on the initial datum. More general initial data can be treated at the expence of analyzing the consequent initial transient [81, 120, 250].

Definition 17.1 (Well-prepared initial datum). For any $\epsilon \in (0, 1]$ we set

$$u_\epsilon^0(z) := \gamma_\epsilon\Big(\epsilon^{-1} d(0, z)\Big), \qquad z \in \Omega. \tag{17.6}$$

The function

$$\gamma_\epsilon : \mathbb{R} \to \mathbb{R}$$

appearing in (17.6) coincides with γ in a neighbourhood of the origin of size of order $|\log \epsilon|$ and attains the asymptotic values $\gamma_\pm$ outside a neighbourhood of double size, and is defined as follows. Take

$$N \in \mathbb{N}, \quad N \geq 4.$$

For any $\epsilon \in (0, 1]$ let

$$\ell_\epsilon := N|\log \epsilon|. \tag{17.7}$$

Then[5]

$$\gamma(\ell_\epsilon) = (1 - \epsilon^N)(1 + \epsilon^N)^{-1} = 1 + \mathcal{O}\left(\epsilon^N\right), \tag{17.8}$$

and, from (17.5),

$$\gamma'(\ell_\epsilon) = \frac{1}{2}\left(1 - (\gamma(\ell_\epsilon))^2\right) = \mathcal{O}\left(\epsilon^N\right).$$

Definition 17.2 (The function γ_ϵ). We denote by

$$\gamma_\epsilon \in \mathcal{C}^{1,1}(\mathbb{R}) \cap \mathcal{C}^\infty\Big(\mathbb{R} \setminus \{\pm\ell_\epsilon, \pm 2\ell_\epsilon\}\Big)$$

the odd function which coincides with γ in the interval $[0, \ell_\epsilon)$, is equal to $1 = \gamma_+$ in the interval $(2\ell_\epsilon, +\infty)$, and in $[\ell_\epsilon, 2\ell_\epsilon]$ is the cubic polynomial p_γ so that

- $\mathrm{p}_\gamma(\ell_\epsilon) = \gamma(\ell_\epsilon), \mathrm{p}_\gamma'(\ell_\epsilon) = \gamma'(\ell_\epsilon)$,
- $\mathrm{p}_\gamma(2\ell_\epsilon) = 1, \mathrm{p}_\gamma'(2\ell_\epsilon) = 0$.

Namely[6]

$$\mathrm{p}_\gamma(y) = \kappa_\epsilon \left(\frac{2\ell_\epsilon - y}{\ell_\epsilon}\right)^3 + h_\epsilon \left(\frac{2\ell_\epsilon - y}{\ell_\epsilon}\right)^2 + 1, \qquad y \in [\ell_\epsilon, 2\ell_\epsilon],$$

---

(5) In this section and in Section 17.2, for $\mathrm{a} \geq 0$ and $\mathrm{b} \geq 0$, the notation $g = \mathcal{O}\left(\epsilon^{\mathrm{a}}|\log \epsilon|^{\mathrm{b}}\right)$ stands for $|g| \leq C\epsilon^{\mathrm{a}}|\log \epsilon|^{\mathrm{b}}$ for $\epsilon > 0$ sufficiently small, where $C > 0$ is a constant independent of $\epsilon \in (0, 1]$.

(6) Note that p_γ is not nondecreasing, since $\mathrm{p}_\gamma''(2\ell_\epsilon) = \frac{2h_\epsilon}{(\ell_\epsilon)^2}$ which, remembering (17.5), equals $\frac{1-\gamma(\ell_\epsilon)}{\ell_\epsilon^2}\big(\ell_\epsilon\big(1 + \gamma(\ell_\epsilon)\big) - 6\big)$, and is positive for $\epsilon > 0$ small enough.

where

$$\kappa_\epsilon = 2\Big(1-\gamma(\ell_\epsilon)\Big) - \ell_\epsilon\gamma'(\ell_\epsilon), \quad h_\epsilon = -3\Big(1-\gamma(\ell_\epsilon)\Big) + \ell_\epsilon\gamma'(\ell_\epsilon).$$

In what follows we need to estimate the quantities

$$\|\mathrm{p}_\gamma - 1\|_{L^\infty((\ell_\epsilon,2\ell_\epsilon))}, \quad \|\mathrm{p}'_\gamma\|_{L^\infty((\ell_\epsilon,2\ell_\epsilon))}, \quad \|\mathrm{p}''_\gamma\|_{L^\infty((\ell_\epsilon,2\ell_\epsilon))}$$

in terms of ϵ. Clearly $\frac{2\ell_\epsilon - y}{\ell_\epsilon}$ is uniformly bounded with respect to ϵ for $y \in [\ell_\epsilon, 2\ell_\epsilon]$. Moreover, using (17.5), we have

$$\kappa_\epsilon = \Big(1-\gamma(\ell_\epsilon)\Big)\Big(2 - \frac{\ell_\epsilon}{2}\,(1+\gamma(\ell_\epsilon))\Big) \tag{17.9}$$

and

$$h_\epsilon = \Big(1-\gamma(\ell_\epsilon)\Big)\Big(-3 + \frac{\ell_\epsilon}{2}\,(1+\gamma(\ell_\epsilon))\Big). \tag{17.10}$$

Hence, using (17.7) and (17.8),

$$\kappa_\epsilon = \mathcal{O}\left(\epsilon^N|\log\epsilon|\right), \qquad h_\epsilon = \mathcal{O}\left(\epsilon^N|\log\epsilon|\right).$$

Therefore

$$\|\mathrm{p}_\gamma - 1\|_{L^\infty((\ell_\epsilon,2\ell_\epsilon))} = \mathcal{O}\left(\epsilon^N|\log\epsilon|\right).$$

Differentiating, we also obtain

$$\|\mathrm{p}'_\gamma\|_{L^\infty((\ell_\epsilon,2\ell_\epsilon))} = \mathcal{O}\left(\epsilon^N\right), \quad \|\mathrm{p}''_\gamma\|_{L^\infty((\ell_\epsilon,2\ell_\epsilon))} = \mathcal{O}\left(\epsilon^N|\log\epsilon|^{-1}\right). \tag{17.11}$$

In conclusion

$$\begin{aligned}\|\gamma_\epsilon - \gamma\|_{L^\infty(\mathbb{R})} &= \|\gamma'_\epsilon\|_{L^\infty\big((-2\ell_\epsilon,-\ell_\epsilon)\cup(\ell_\epsilon,2\ell_\epsilon)\big)} \\ &= \|\gamma''_\epsilon\|_{L^\infty\big((-2\ell_\epsilon,-\ell_\epsilon)\cup(\ell_\epsilon,2\ell_\epsilon)\big)} = \mathcal{O}\left(\epsilon^{N-1}\right).\end{aligned} \tag{17.12}$$

Remark 17.3 (Initial zero level set). For all $\epsilon \in (0,1]$ the zero level set of u^0_ϵ coincides with $\partial E = \partial f(0)$; in particular, it is independent of ϵ.

Remark 17.4 (Compatibility). The initial and boundary data of problem (15.2) are compatible, namely

$$\frac{\partial u_0^\epsilon}{\partial \mathrm{n}_\Omega} = 0 \qquad \text{on } \partial\Omega,$$

since $\gamma_\epsilon(\epsilon^{-1}d(0,z))$ takes the constant value γ_+ out of a suitable neighbourhood[7] of $f(0)$, and $f(0)$ is compact and contained in Ω by assumption.

[7] The set of all points having distance $2N\epsilon|\log\epsilon|$ from $f(0)$.

Notation and assumptions. We let u_ϵ be the solutions(8) to (15.2) with u_ϵ^0 as in (17.6), and (consistently with (16.5)) we set

$$E_\epsilon(t) := \{z \in \Omega : u_\epsilon(t, z) \le 0\}. \tag{17.13}$$

We will further restrict the choice of $T \in \big(0, t_{\max}(\partial E)\big)$, by supposing that

$$\nabla u_\epsilon(t, x) \neq 0, \qquad t \in [0, T],\ x \in \{y \in \Omega : u_\epsilon(t, y) = 0\}. \tag{17.14}$$

In particular, for any $t \in [0, T]$,

$$\partial E_\epsilon(t) := \{x \in \Omega : u_\epsilon(t, x) = 0\}$$

is a smooth compact hypersurface embedded in Ω.

The convergence result reads as follows.

Theorem 17.5 (Convergence of the zero level set of u_ϵ). *Under the above assumptions, there exist $\overline{\epsilon} \in (0, 1]$ and a constant $\overline{C} > 0$ depending only on ∂E and T, such that, for each $\epsilon \in (0, \overline{\epsilon})$,*

$$\partial E_\epsilon(t) \subseteq \Big\{z \in \Omega : \mathrm{dist}\big(z, \partial f(t)\big) \le \overline{C}\epsilon^2 |\log \epsilon|^2\Big\}, \quad t \in [0, T], \tag{17.15}$$

and

$$\partial f(t) \subseteq \Big\{z \in \Omega : \mathrm{dist}\big(z, \partial E_\epsilon(t)\big) \le \overline{C}\epsilon^2 |\log \epsilon|^2\Big\}, \quad t \in [0, T]. \tag{17.16}$$

In other words, for sufficiently short times, the Hausdorff distance between the mean curvature flow $\partial f(t)$ and the zero level set $\{u_\epsilon(t, \cdot) = 0\}$ is bounded by $\overline{C}\epsilon^2 |\log \epsilon|^2$, uniformly with respect to $t \in [0, T]$, and the constant $\overline{C}$ is independent of ϵ.

The sequel of this chapter is devoted to demonstrate Theorem 17.5: we will follow the proof given in [55].

17.2. Definition of $v_\epsilon^\pm$

The proof of Theorem 17.5 relies on the construction of suitable functions v_ϵ^-, v_ϵ^+ which we will prove to be a subsolution and a supersolution respectively, to problem (15.2). Their definition is suggested by the asymptotic expansion made in Chapter 16.

(8) The function u_ϵ belongs to $L^\infty((0, +\infty); H^1(\Omega)) \cap H^1_{\mathrm{loc}}([0, +\infty); L^2(\Omega)) \cap \mathcal{C}^\infty((0, +\infty) \times \Omega)$. Concerning the definitions of parabolic Sobolev spaces and their properties, we refer, *e.g.*, to [194, 134].

We set[9]

$$K := \max_{t\in[0,T]} \|\nabla^2 d(t,\cdot)\|^2_{L^\infty(\partial f(t))}. \tag{17.17}$$

Definition 17.6 (The function ϑ and the constant c). We define the function $\vartheta : [0,T] \to (0,+\infty)$ as

$$\vartheta(t) := c\, e^{(1+K)t}, \quad t \in [0,T], \tag{17.18}$$

where c is a positive constant to be selected later[10] independently of ϵ.

Now, we can introduce the following perturbations[11] of the signed distance function d.

Definition 17.7 (Modified signed distance function $d^\pm_\epsilon$). For any $\epsilon \in (0,1]$ we set

$$d^\pm_\epsilon(t,z) := d(t,z) \pm \vartheta(t)\epsilon^2|\log\epsilon|^2, \quad (t,z) \in [0,T]\times\mathbb{R}^n. \tag{17.19}$$

Accordingly, we define:

$$\mathrm{A}^\pm_\epsilon(t) := \Big\{z \in \mathbb{R}^n : |d^\pm_\epsilon(t,z)| < 2\epsilon\,\ell_\epsilon\Big\},$$

$$\mathfrak{O}^\pm_{\epsilon-}(t) := \Big\{z \in \mathbb{R}^n : d^\pm_\epsilon(t,z) < -2\epsilon\,\ell_\epsilon\Big\}, \qquad t \in [0,T],$$

$$\mathfrak{O}^\pm_{\epsilon+}(t) := \Big\{z \in \mathbb{R}^n : d^\pm_\epsilon(t,z) > 2\epsilon\,\ell_\epsilon\Big\},$$

and

$$Q^\pm_\epsilon := \bigcup_{t\in[0,T]} \Big(\{t\}\times \mathrm{A}^\pm_\epsilon(t)\Big),$$

$$\mathfrak{O}^\pm_{\epsilon-} := \bigcup_{t\in[0,T]} \Big(\{t\}\times \mathfrak{O}^\pm_{\epsilon-}(t)\Big),$$

$$\mathfrak{O}^\pm_{\epsilon+} := \bigcup_{t\in[0,T]} \Big(\{t\}\times \mathfrak{O}^\pm_{\epsilon+}(t)\Big).$$

[9] Since K is fixed from now on, we will not point out the dependence on K of the various constants that will be introduced in the remaining part of this chapter.

[10] See the arguments concerning inequality (17.44).

[11] One should not confuse the functions defined in (17.19) with the function defined in (16.6).

Remark 17.8. Recalling the definition of the neighbourhood $\mathrm{A}(t)$ given in (17.2), it follows that there exists $\epsilon_1 \in (0, 1]$, possibly depending[12] on c, such that

$$\mathrm{A}_\epsilon^\pm(t) \subset \mathrm{A}(t), \qquad t \in [0, T], \qquad \epsilon \in (0, \epsilon_1). \tag{17.20}$$

Hence

$$Q_\epsilon^\pm \subseteq Q, \qquad \epsilon \in (0, \epsilon_1),$$

that is

$$|d_\epsilon^\pm(t, z)| < 2N\epsilon |\log \epsilon| \ \Rightarrow\ |d(t, z)| < \varrho.$$

In particular, from (17.3) it follows that

$$d \in \mathcal{C}^\infty(Q_\epsilon^\pm), \qquad \epsilon \in (0, \epsilon_1).$$

Since $\nabla d_\epsilon^\pm = \nabla d$ in $Q^\pm$, we deduce

$$|\nabla d_\epsilon^\pm|^2 = 1 \qquad \text{in } Q_\epsilon^\pm, \qquad \epsilon \in (0, \epsilon_1). \tag{17.21}$$

Furthermore, there exists a constant $\mathrm{C}_1 > 0$ such that

$$|d(t, z)| \leq \mathrm{C}_1 \epsilon |\log \epsilon|, \qquad (t, z) \in Q_\epsilon^\pm,\ \epsilon \in (0, \epsilon_1). \tag{17.22}$$

It is convenient to introduce the following quantities.

Definition 17.9 (The stretched variable $y_\epsilon^\pm$). We set

$$y_\epsilon^\pm(t, z) := \epsilon^{-1} d_\epsilon^\pm(t, z) = \frac{d(t, z)}{\epsilon} \pm \vartheta(t)\epsilon |\log \epsilon|^2,\ \ (t,z) \in [0, T] \times \mathbb{R}^n.$$

Remember that the shape function ξ has been defined in Section 15.3.

Definition 17.10 (The modified shape function ξ_ϵ). We denote by

$$\xi_\epsilon \in \mathcal{C}^{1,1}(\mathbb{R}) \cap \mathcal{C}^\infty\Big(\mathbb{R} \setminus \{\pm\ell_\epsilon, \pm 2\ell_\epsilon\}\Big)$$

the odd function which coincides with ξ in $[0, \ell_\epsilon)$, it is equal to the asymptotic value 0 in the interval $(2\ell_\epsilon, +\infty)$, and in $[\ell_\epsilon, 2\ell_\epsilon]$ it is the cubic polynomial p_ξ so that

- $p_\xi(\ell_\epsilon) = \xi(\ell_\epsilon)$, $p_\xi'(\ell_\epsilon) = \xi_\epsilon'(\ell_\epsilon)$,
- $p_\xi(2\ell_\epsilon) = 0$, $p_\xi'(2\ell_\epsilon) = 0$.

[12] The number ϵ_1 depends on ϑ, which in turn is determined by c. At the end of the argument, the number c (as well as the constant Θ appearing in (17.24)) will be selected before the choice of $\overline{\epsilon}$.

Namely

$$\mathrm{p}_\xi(y) = C_\epsilon \left(\frac{2\ell_\epsilon - y}{\ell_\epsilon}\right)^3 + D_\epsilon \left(\frac{2\ell_\epsilon - y}{\ell_\epsilon}\right)^2, \qquad y \in [\ell_\epsilon, 2\ell_\epsilon],$$

where

$$C_\epsilon = -2\xi(\ell_\epsilon) - \ell_\epsilon \xi'(\ell_\epsilon), \qquad D_\epsilon = 3\xi(\ell_\epsilon) + \ell_\epsilon \xi'(\ell_\epsilon).$$

Then, using the estimates on the asymptotic behaviour of ξ and ξ' given by Lemma 15.19, one gets

$$C_\epsilon = \mathcal{O}(\epsilon^{N-1}), \qquad D_\epsilon = \mathcal{O}(\epsilon^{N-1}),$$

and

$$\begin{aligned} \|\xi_\epsilon - \xi\|_{L^\infty(\mathbb{R})} &= \|\xi_\epsilon'\|_{L^\infty\left((-2\ell_\epsilon,-\ell_\epsilon)\cup(\ell_\epsilon,2\ell_\epsilon)\right)} \\ &= \|\xi_\epsilon''\|_{L^\infty\left((-2\ell_\epsilon,-\ell_\epsilon)\cup(\ell_\epsilon,2\ell_\epsilon)\right)} = \mathcal{O}\left(\epsilon^{N-1}\right). \end{aligned} \tag{17.23}$$

We are now in a position to define what we will prove to be a subsolution and a supersolution to problem (15.2).

Definition 17.11 (The functions $v_\epsilon^\pm$). We define[13]

$$v_\epsilon^\pm : [0,T] \times \mathbb{R}^n \to \mathbb{R}$$

as follows:

$$v_\epsilon^\pm(t,z) := \begin{cases} \gamma_\epsilon\big(y_\epsilon^\pm(t,z)\big) + \epsilon^2 \big|\overline{\nabla^2 d}(t,z)\big|^2 \xi_\epsilon\big(y_\epsilon^\pm(t,z)\big) \pm \Theta\epsilon^3 |\log\epsilon|^2 \\ \qquad\qquad\qquad\qquad \text{if } (t,z) \in Q_\epsilon^\pm, \\ \gamma_+ \pm \Theta\epsilon^3|\log\epsilon|^2 \quad \text{if } (t,z) \in \mathfrak{O}_{\epsilon+}^\pm, \\ \gamma_- \pm \Theta\epsilon^3|\log\epsilon|^2 \quad \text{if } (t,z) \in \mathfrak{O}_{\epsilon-}^\pm, \end{cases} \tag{17.24}$$

where Θ is a positive constant to be selected later independently of ϵ.

We recall[14] that by definition

$$\overline{\nabla^2 d}(t,z) = \nabla^2 d\big(t, \mathrm{pr}(t,z)\big),$$

where

$$\mathrm{pr}(t,z) := \mathrm{pr}(z, \partial f(t)) = z - d(t,z)\nabla d(t,z).$$

[13] The restrictions of $v_\epsilon^\pm$ to $[0,T] \times \Omega$ will be still denoted by $v_\epsilon^\pm$.

[14] See Chapter 6.

Remark 17.12. Notice that $v_\epsilon^\pm$ are bounded, and owing to the regularity of $d_\epsilon^\pm$ and the properties of the functions γ_ϵ and ξ_ϵ, it follows that $v_\epsilon^\pm \in L^2((0,T); H^1(\Omega)) \cap H^1([0,T]; L^2(\Omega))$. Furthermore, $v_\epsilon^\pm \in \mathcal{C}^\infty(([0,+\infty) \times \Omega) \setminus \partial Q_\epsilon^\pm) \cap \mathcal{C}^{1,1}([0,+\infty) \times \Omega)$.

Remark 17.13. The right-hand side of (17.24) does not contain terms linear in ϵ.

From now on we focus the attention on v_ϵ^-, since the arguments for v_ϵ^+ are similar.

Remark 17.14 (Meaning of the definition of v_ϵ^-). It is worthwhile to observe the following properties of v_ϵ^-.

- The term $\gamma_\epsilon(y_\epsilon^-)$ corresponds to a right shift of the function γ_ϵ of order ϵ (up to logarithmic corrections), because:
  - $d(t,\cdot)$ is negative in the interior of $f(t)$,
  - the right-hand side of (17.19) has the minus sign, namely it reads as $d(t,z) - \vartheta(t)\epsilon^2|\log\epsilon|^2$,
  - γ is increasing.
- The term $\epsilon^2\left|\overline{\nabla^2 d}\right|^2\xi_\epsilon(y_\epsilon^-)$ is a shape correction, suggested by the formal asymptotic expansion given in Section 16.3 (see in particular formula (16.53)). It turns out that this correction is of higher order with respect to the previous right translation.
- The term $-\Theta\epsilon^3|\log\epsilon|^2$ provides a further downward translation, and is necessary both for the comparison with the initial datum and to provide control of "bad" terms far from the interface.

17.3. Comparison with boundary and initial data

We first notice that

$$\frac{\partial v_\epsilon^-}{\partial \mathrm{n}_\Omega} = 0 = \frac{\partial u_\epsilon}{\partial \mathrm{n}_\Omega} \qquad \text{on } [0,T] \times \partial\Omega, \tag{17.25}$$

where the first equality is based upon the observation that $\partial f(t)$ is relatively compact in Ω for any $t \in [0,T]$, and γ_ϵ and ξ_ϵ attain the asymptotic values γ_+ and 0 respectively, at $y \geq 2N|\log\epsilon|$.

Before comparing $v_\epsilon^-(0,\cdot)$ with $u_\epsilon^0(\cdot)$, we record a preliminary observation. Recall that ϵ_1 is defined in Remark 17.8 and that $c = \vartheta(0)$.

Lemma 17.15 (Estimate of γ'). *There exists $\epsilon_2 \in (0,\epsilon_1)$, possibly depending on c, such that*

$$\frac{1}{2}\gamma'\left(y_\epsilon^-(0,z)\right) \leq \gamma'(y) \leq 2\gamma'\left(y_\epsilon^-(0,z)\right) \tag{17.26}$$

for each $z \in \Omega$, $y \in [y_\epsilon^-(0,z), \epsilon^{-1}d(0,z)]$ and $\epsilon \in (0,\epsilon_2)$.

Proof. Recalling (15.18), (15.19), (15.7) and (15.8), we have

$$(\log(\gamma'))' = \frac{\gamma''}{\gamma'} = -\gamma \in L^\infty(\mathbb{R}).$$

Therefore $\log(\gamma') \in \mathrm{Lip}(\mathbb{R})$ with Lipschitz constant $\gamma_+ = 1$. Hence

$$\left|\log\left(\frac{\gamma'(y_2)}{\gamma'(y_1)}\right)\right| = |\log(\gamma'(y_2)) - \log(\gamma'(y_1))| \le |y_2 - y_1|, \qquad y_1, y_2 \in \mathbb{R}.$$

If $|y_2 - y_1| \le \vartheta(0)\epsilon|\log\epsilon|^2$, from the previous inequality it follows that

$$e^{-\vartheta(0)\epsilon|\log\epsilon|^2} \le e^{-|y_2-y_1|} \le \frac{\gamma'(y_1)}{\gamma'(y_2)} \le e^{|y_2-y_1|} \le e^{\vartheta(0)\epsilon|\log\epsilon|^2}.$$

As

$$\lim_{\epsilon\downarrow 0} e^{\pm\vartheta(0)\epsilon|\log\epsilon|^2} = 1$$

and

$$|\epsilon^{-1}d(0,z) - \epsilon^{-1}d_\epsilon^-(0,z)| = \vartheta(0)\epsilon|\log\epsilon|^2,$$

the assertion follows. □

Lemma 17.16 (Comparison with the initial datum). *There exists* $\epsilon_3 \in (0, \epsilon_2)$*, possibly depending on* c *and* Θ*, such that*

$$v_\epsilon^-(0,z) \le u_\epsilon^0(z), \qquad z \in \Omega, \ \epsilon \in (0, \epsilon_3). \tag{17.27}$$

Proof. Define, for $(t,z) \in [0,T] \times \Omega$,

$$w_\epsilon(t,z) := \gamma(y_\epsilon^-(t,z)) + \epsilon^2\left|\overline{\nabla^2 d}(t,z)\right|^2 \xi(y_\epsilon^-(t,z)) - \frac{\Theta}{2}\epsilon^3|\log\epsilon|^2$$

(notice the presence of the unperturbed γ and ξ in the definition of w_ϵ).

By (17.12) and (17.23) we have

$$\begin{aligned}
&\left|v_\epsilon^- - w_\epsilon + \frac{\Theta}{2}\epsilon^3|\log\epsilon|^2\right| \\
&\le |\gamma_\epsilon(y_\epsilon^-(t,z)) - \gamma(y_\epsilon^-(t,z))| \\
&\quad + \epsilon^2\left|\overline{\nabla^2 d}(t,z)\right|^2 |\xi_\epsilon(y_\epsilon^-(t,z)) - \xi(y_\epsilon^-(t,z))| = \mathcal{O}\left(\epsilon^{N-1}\right).
\end{aligned}$$

Now, we observe that, in order to prove (17.27), it is enough to show

$$w_\epsilon(0,z) \le u_\epsilon^0(z), \qquad z \in \Omega. \tag{17.28}$$

Indeed, in these circumstances,

$$\begin{aligned} v_\epsilon^-(0,z) =& w_\epsilon(0,z) + \left(v_\epsilon^-(0,z) - w_\epsilon(0,z)\right) \\ \le& w_\epsilon(0,z) - \frac{\Theta}{2}\epsilon^3|\log\epsilon|^2 + \mathcal{O}\left(\epsilon^{N-1}\right) \\ \le& u_\epsilon^0(z) - \frac{\Theta}{2}\epsilon^3|\log\epsilon|^2 + \mathcal{O}\left(\epsilon^{N-1}\right) \le u_\epsilon^0(z), \end{aligned}$$

for $\epsilon \in (0, \epsilon_2)$ small enough (recall that $N \ge 4$).

Let us show (17.28). We have

$$\begin{aligned} w_\epsilon(0,z) - u_\epsilon^0(z) = & \underbrace{\left(\gamma\left(y_\epsilon^-(0,z)\right) - \gamma_\epsilon\left(\frac{d(0,z)}{\epsilon}\right)\right)}_{:=\mathrm{I}_\epsilon} \\ & + \underbrace{\epsilon^2\left|\overline{\nabla^2 d}(0,z)\right|^2\xi(y_\epsilon^-(0,z))}_{:=\mathrm{II}_\epsilon} - \frac{\Theta}{2}\epsilon^3|\log\epsilon|^2. \end{aligned}$$

Using the fact that $\dfrac{d(0,z)}{\epsilon} - y_\epsilon^-(0,z) = \vartheta(0)\epsilon|\log\epsilon|^2$, from Lemma 17.15[(15)] we have, recalling also (17.12),

$$\begin{aligned} \mathrm{I}_\epsilon \le & -\frac{1}{2}\gamma'(y_\epsilon^-(0,z))\vartheta(0)\epsilon|\log\epsilon|^2 \\ & + \gamma\left(\frac{d(0,z)}{\epsilon}\right) - \gamma_\epsilon\left(\frac{d(0,z)}{\epsilon}\right) \qquad \epsilon \in (0,\epsilon_2), \\ = & -\frac{1}{2}\gamma'(y_\epsilon^-(0,z))\vartheta(0)\epsilon|\log\epsilon|^2 + \mathcal{O}(\epsilon^{N-1}). \end{aligned}$$

In addition, by (15.38) and the boundedness of $|\nabla^2 d(t,z)|$, it follows that there exists a constant $C > 0$ independent of $\epsilon \in (0,1]$ such that

$$|\mathrm{II}_\epsilon| \le C\epsilon^2\Big(1 + (y_\epsilon^-(0,z))^2\Big)\gamma'(y_\epsilon^-(0,z)).$$

Let us distinguish two cases. If $|y_\epsilon^-(0,z)| > 2|\log\epsilon|$, by direct computation we have $\gamma'(y_\epsilon^-(0,z)) < \delta\epsilon^4$ for some positive constant δ, hence

(15) By Lagrange's theorem we have

$$\gamma\left(\frac{d(0,z)}{\epsilon}\right) - \gamma\left(y_\epsilon^-(0,z)\right) = \gamma'(\eta)\vartheta(0)\epsilon|\log\epsilon|^2$$

for some $\eta \in \left(y_\epsilon^-(0,z), \frac{d(0,z)}{\epsilon}\right)$, and $\gamma'(\eta)\vartheta(0)\epsilon|\log\epsilon|^2 \ge \frac{1}{2}\gamma'\left(y_\epsilon^-(0,z)\right)\vartheta(0)\epsilon|\log\epsilon|^2$ from inequality (17.26).

the term II_ϵ can be controlled by the negative term $-\frac{\Theta}{2}\epsilon^3|\log\epsilon|^2$. If $|y_\epsilon^-(0,z)| \le 2|\log\epsilon|$, the term II_ϵ is controlled by the term I_ϵ. This proves (17.28), and so the proof of (17.27) is concluded. $\square$

17.4. Comparison

We start with the following definition.

Definition 17.17 (Sub/supersolution). Let

$$u, v \in L^2((0,T); H^1(\Omega)) \cap H^1([0,T]; L^2(\Omega)).$$

We say that u is a subsolution to (15.2) if, for any nonnegative $\varphi \in H^1(\Omega)$, we have

$$\begin{cases} \displaystyle\int_\Omega \Big(\epsilon\varphi(z)\frac{\partial u}{\partial t}(t,z) + \epsilon\langle\nabla u(t,z), \nabla\varphi(z)\rangle + \epsilon^{-1}\varphi(z)W'(u(t,z))\Big)\,dz \le 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } t \in (0,T), \\ u(0,\cdot) \le u_\epsilon^0(\cdot) \qquad\qquad\qquad\qquad \text{a.e. in } \Omega. \end{cases}$$

We say that v is a supersolution to (15.2) if, for any nonnegative $\varphi \in H^1(\Omega)$, we have

$$\begin{cases} \displaystyle\int_\Omega \Big(\epsilon\varphi(z)\frac{\partial v}{\partial t}(t,z) + \epsilon\langle\nabla v(t,z), \nabla\varphi(z)\rangle + \epsilon^{-1}\varphi(z)W'(v(t,z))\Big)\,dz \ge 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for a.e. } t \in (0,T), \\ v(0,\cdot) \ge u_\epsilon^0(\cdot) \qquad\qquad\qquad\qquad \text{a.e. in } \Omega. \end{cases}$$

The main tool in this chapter is the following comparison theorem betweeen a subsolution and a supersolution[(16)].

Theorem 17.18 (Comparison). *Let*

$$u, v \in L^2((0,T); H^1(\Omega)) \cap H^1([0,T]; L^2(\Omega))$$

be bounded. Suppose that, for any nonnegative $\varphi \in H^1(\Omega)$,

$$\int_\Omega \Big\{\epsilon\varphi(z)\frac{\partial}{\partial t}\big(u(t,z) - v(t,z)\big) + \epsilon\langle\nabla\big(u(t,z) - v(t,z)\big), \nabla\varphi(z)\rangle$$
$$+ \epsilon^{-1}\varphi(z)\big[W'(u(t,z)) - W'(v(t,z))\big]\Big\}\,dz \le 0$$

(16) See for instance [221] and [15] for similar results (in reference [15] the strong maximum principle for equation (15.2) is also addressed).

for almost every $t \in (0, T)$*, and*

$$u(0, \cdot) \le v(0, \cdot) \qquad \text{a.e. in } \Omega. \tag{17.29}$$

Then for any $t \in [0, T]$

$$u(t, \cdot) \le v(t, \cdot) \qquad \text{a.e. in } \Omega. \tag{17.30}$$

Proof. Define $\alpha(\sigma) := \max(\sigma, 0)$ for any $\sigma \in \mathbb{R}$, and

$$\mathrm{e} := \alpha(u - v).$$

Notice that

$$\mathrm{e} = \max(u - v, 0) \in L^2((0, T); H^1(\Omega)) \cap H^1([0, T]; L^2(\Omega)),$$

and by (17.29)

$$\mathrm{e}(0, \cdot) = 0 \qquad \text{a.e. in } \Omega.$$

Inequality (17.30) that we have to prove is equivalent to show that $\mathrm{e}(t, \cdot) = 0$ almost everywhere in Ω.

For any $t \in [0, T]$, define

$$I(t) := \frac{1}{2} \int_\Omega (\mathrm{e}(t, z))^2 \, dz,$$

and observe that $I(0) = 0$. We need to prove that $I = 0$ in $(0, T)$, and to do this we will find a differential inequality satisfied by I.

We start by recalling that it is possible to show (see [15, 17]) that I is absolutely continuous in $(0, T)$, and

$$I' = \int_\Omega \mathrm{e} \, \frac{\partial}{\partial t}(u - v) \, dz$$

almost everywhere in $(0, T)$. Hence, being u and v a subsolution and a supersolution to (15.2) respectively, we obtain

$$I' + \int_\Omega \langle \nabla(u - v), \nabla \mathrm{e} \rangle \, dz \le \epsilon^{-2} \int_\Omega \mathrm{e}\big(W'(v) - W'(u)\big) \, dz \tag{17.31}$$

almost everywhere in $(0, T)$. Now, we observe that

$$\int_\Omega \langle \nabla(u - v), \nabla \mathrm{e} \rangle \, dz = \int_{\mathrm{E}_t} |\nabla \mathrm{e}|^2 \, dz \ge 0, \tag{17.32}$$

where

$$\mathrm{E}_t := \{z \in \Omega : u(t, z) - v(t, z) > 0\}.$$

Combining (17.32) with (17.31) gives

$$I' \leq \epsilon^{-2} \int_{\Omega} \mathrm{e}\big(W'(v) - W'(u)\big)\, dz \tag{17.33}$$

almost everywhere in $(0, T)$. Split now W' as[17]

$$W' = \psi_l + \psi_i,$$

where $\psi_l \in \mathrm{Lip}(\mathbb{R})$ and ψ_i is nondecreasing on $\mathbb{R}$. Then

$$\begin{aligned} &\int_{\Omega} \mathrm{e}\big(W'(v) - W'(u)\big)\, dz \\ &= \int_{\Omega} \mathrm{e}\big(\psi_l(v) - \psi_l(u)\big)\, dz + \int_{\Omega} \mathrm{e}\big(\psi_i(v) - \psi_i(u)\big)\, dz \\ &= \int_{\Omega} \mathrm{e}\big(\psi_l(v) - \psi_l(u)\big)\, dz + \int_{\mathrm{E}_t} \mathrm{e}\big(\psi_i(v) - \psi_i(u)\big)\, dz \\ &\leq \int_{\Omega} \mathrm{e}\big(\psi_l(v) - \psi_l(u)\big)\, dz. \end{aligned}$$

Hence, from (17.33) and $I(0) = 0$, for any $t \in (0, T)$ we find

$$\begin{aligned} I(t) &\leq \epsilon^{-2} \int_0^t \int_{\Omega} \mathrm{e}(\tau, z)\, |\psi_l(v(\tau, z)) - \psi_l(u(\tau, z))|\; dz d\tau \\ &\leq \mathrm{lip}(\psi_l)\, \epsilon^{-2} \int_0^t \int_{\Omega} \mathrm{e}(\tau, z)\, |v(\tau, z) - u(\tau, z)|\, dz d\tau \\ &= \mathrm{lip}(\psi_l)\, \epsilon^{-2} \int_0^t \|\mathrm{e}(\tau, \cdot)\|^2_{L^2(\Omega)}\, d\tau = 2\mathrm{lip}(\psi_l)\, \epsilon^{-2} \int_0^t I\, d\tau, \end{aligned}$$

where $\mathrm{lip}(\psi_l)$ denotes the Lipschitz constant of ψ_l. Thus the thesis follows from Gronwall's lemma[18]. □

17.5. v_ϵ^- is a subsolution

In this section we show that v_ϵ^- is a subsolution of problem (15.2). Hereafter, for $\mathrm{a} \geq 0$ and $\mathrm{b} \geq 0$, the notation

$$g = \mathcal{O}\left(\epsilon^{\mathrm{a}} |\log \epsilon|^{\mathrm{b}}\right)$$

[17] Since $W'(\mathrm{y}) = \frac{1}{2}(\mathrm{y}^3 - \mathrm{y})$, we have $\psi_l(\mathrm{y}) = -\frac{1}{2}\mathrm{y}$ and $\psi_i(\mathrm{y}) = \frac{1}{2}\mathrm{y}^3$ for any $\mathrm{y} \in \mathbb{R}$.

[18] Let $\alpha, \beta, \omega : [0, T] \to \mathbb{R}$ be continuous. Assume that α is non-decreasing, β is nonnegative, and $\omega(t) \leq \alpha(t) + \int_0^t \beta(s)\omega(s)\, ds$ for any $t \in [0, T]$. Then $\omega(t) \leq \alpha(t) e^{\int_0^t \beta(s)\, ds}$ for any $t \in [0, T]$.

stands for $|g| \leq C\epsilon^{\mathrm{a}}|\log\epsilon|^{\mathrm{b}}$ for ϵ sufficiently small, say $\epsilon \in (0, \epsilon_0)$, where $C > 0$ is a constant independent of ϵ, of the constant c appearing in the expression (17.18) of ϑ, and of the constant Θ. The value ϵ_0, instead, might depend on ϑ and Θ. For a constant $\beta \in \mathbb{R}$, the notation

$$g = \mathcal{O}_\beta(\epsilon^{\mathrm{a}}|\log\epsilon|^{\mathrm{b}})$$

stands for $|g| \leq C\epsilon^{\mathrm{a}}|\log\epsilon|^{\mathrm{b}}$ for ϵ sufficiently small, say $\epsilon \in (0, \epsilon_0)$, where $C > 0$ is a constant possibly depending on β, but independent of ϵ.

Remembering (17.25), Lemma 17.16 and Remark 17.12, in order to demonstrate that v_ϵ^- is a subsolution to problem (15.2), it is sufficient to show the following lemma.

Lemma 17.19. *There exist constants $c > 0$, $\Theta > 0$, and there exists $\epsilon_4 \in (0, \epsilon_3)$ possibly depending on c and Θ, such that the function v_ϵ^- defined in* (17.24) *satisfies*

$$\frac{\partial v_\epsilon^-}{\partial t} - \Delta v_\epsilon^- + \epsilon^{-2}W'(v_\epsilon^-) \leq 0 \quad \text{in } \big((0,T)\times\Omega\big) \setminus \partial Q_\epsilon^-, \tag{17.34}$$

for each $\epsilon \in (0, \epsilon_4)$.

Proof. Let us first check inequality (17.34) in Q_ϵ^-. At each point $(t, z) \in Q_\epsilon^-$ we have

$$\begin{aligned}\frac{\partial v_\epsilon^-}{\partial t} &= \epsilon^{-1}\gamma_\epsilon'(y_\epsilon^-)\frac{\partial d_\epsilon^-}{\partial t} + \epsilon\xi_\epsilon'(y_\epsilon^-)\left|\overline{\nabla^2 d}\right|^2\frac{\partial d_\epsilon^-}{\partial t} + \epsilon^2\xi_\epsilon(y_\epsilon^-)\frac{\partial}{\partial t}\left|\overline{\nabla^2 d}\right|^2 \\ &= \epsilon^{-1}\gamma_\epsilon'(y_\epsilon^-)\frac{\partial d_\epsilon^-}{\partial t} + \mathcal{O}(\epsilon) + \mathcal{O}_c(\epsilon^3|\log\epsilon|^2),\end{aligned}$$

and (being $\nabla d_\epsilon(t,z) = \nabla d(t,z)$)

$$\nabla v_\epsilon^- = \epsilon^{-1}\gamma_\epsilon'(y_\epsilon^-)\nabla d + \epsilon\xi_\epsilon'(y_\epsilon^-)\left|\overline{\nabla^2 d}\right|^2\nabla d + \epsilon^2\xi_\epsilon(y_\epsilon^-)\nabla\left|\overline{\nabla^2 d}\right|^2. \tag{17.35}$$

Enforcing (17.21), from (17.35) we get, in Q_ϵ^-,

$$\Delta v_\epsilon^- = \epsilon^{-2}\gamma_\epsilon''(y_\epsilon^-) + \epsilon^{-1}\gamma_\epsilon'(y_\epsilon^-)\Delta d + \xi_\epsilon''(y_\epsilon^-)\left|\overline{\nabla^2 d}\right|^2 + \mathcal{O}(\epsilon).$$

Hence

$$\begin{aligned}\frac{\partial v_\epsilon^-}{\partial t} - \Delta v_\epsilon^- &= -\epsilon^{-2}\gamma_\epsilon''(y_\epsilon^-) + \epsilon^{-1}\gamma_\epsilon'(y_\epsilon^-)\left(\frac{\partial d_\epsilon^-}{\partial t} - \Delta d\right) \\ &\quad - \xi_\epsilon''(y_\epsilon^-)\left|\overline{\nabla^2 d}\right|^2 + \mathcal{O}(\epsilon) + \mathcal{O}_c\left(\epsilon^3|\log\epsilon|^2\right).\end{aligned} \tag{17.36}$$

From (17.19) it follows that

$$\frac{\partial d_\epsilon^-}{\partial t} - \Delta d = \frac{\partial d}{\partial t} - \Delta d - \vartheta' \epsilon^2 |\log \epsilon|^2. \tag{17.37}$$

On the other hand, if $(t, z) \in Q_\epsilon^-$ and if

$$x := \mathrm{pr}(z, \partial f(t)) = z - d(t, z) \nabla d(t, z),$$

recalling (6.1), (17.1) and (1.41) we have

$$\begin{aligned} &\frac{\partial d}{\partial t}(t, z) = \frac{\partial d}{\partial t}(t, x) = \Delta d(t, x) \\ &= \Delta d(t, z) + d(t, z) |\nabla^2 d(t, x)|^2 + \mathcal{O}\left((d(t, z))^2\right) \\ &= \Delta d(t, z) + d(t, z) \left|\overline{\nabla^2 d}(t, z)\right|^2 + \mathcal{O}\left((d(t, z))^2\right). \end{aligned} \tag{17.38}$$

Substituting (17.38) into (17.37), and using once more (17.19), yields

$$\begin{aligned} \frac{\partial d_\epsilon^-}{\partial t} - \Delta d = &\ \frac{\partial d}{\partial t} - \Delta d - \vartheta' \epsilon^2 |\log \epsilon|^2 \\ = &\ d \left|\overline{\nabla^2 d}\right|^2 - \vartheta' \epsilon^2 |\log \epsilon|^2 + \mathcal{O}\left(d^2\right) \\ = &\ d_\epsilon^- \left|\overline{\nabla^2 d}\right|^2 + \epsilon^2 |\log \epsilon|^2 \left(\vartheta \left|\overline{\nabla^2 d}\right|^2 - \vartheta'\right) \\ &+ \mathcal{O}\left(d^2\right) \\ = &\ d_\epsilon^- \left|\overline{\nabla^2 d}\right|^2 + \epsilon^2 |\log \epsilon|^2 \left(\vartheta \left|\overline{\nabla^2 d}\right|^2 - \vartheta'\right) \\ &+ \mathcal{O}\left(\epsilon^2 |\log \epsilon|^2\right), \end{aligned} \tag{17.39}$$

where in the last equality we employed formula (17.22).

Inserting (17.39) into (17.36) and recalling that $d_\epsilon^- = \epsilon\, y_\epsilon^-$ we obtain, in Q_ϵ^-,

$$\begin{aligned} \frac{\partial v_\epsilon^-}{\partial t} - \Delta v_\epsilon^- = &-\epsilon^{-2} \gamma_\epsilon''(y_\epsilon^-) + \left|\overline{\nabla^2 d}\right|^2 \Big(-\xi_\epsilon''(y_\epsilon^-) + y_\epsilon^- \gamma_\epsilon'(y_\epsilon^-)\Big) \\ &+ \epsilon |\log \epsilon|^2 \gamma_\epsilon'(y_\epsilon^-) \left(\vartheta \left|\overline{\nabla^2 d}\right|^2 - \vartheta'\right) \\ &+ \mathcal{O}\left(\epsilon |\log \epsilon|^2\right) + \mathcal{O}_c(\epsilon^3 |\log \epsilon|^2). \end{aligned}$$

As[19]

$$\epsilon^{-2} W'(v_\epsilon^-) = \epsilon^{-2} W'(\gamma_\epsilon(y_\epsilon^-)) + \left|\overline{\nabla^2 d}\right|^2 W''(\gamma_\epsilon(y_\epsilon^-))\xi_\epsilon(y_\epsilon^-)$$
$$- \epsilon|\log\epsilon|^2 \Theta W''(\gamma_\epsilon(y_\epsilon^-)) + \mathcal{O}_\Theta(\epsilon),$$

we finally get

$$\frac{\partial v_\epsilon^-}{\partial t} - \Delta v_\epsilon^- + \epsilon^{-2} W'(v_\epsilon^-) = \mathrm{I}_\epsilon + \mathrm{II}_\epsilon + \mathrm{III}_\epsilon + \mathcal{O}\left(\epsilon|\log\epsilon|^2\right) + \mathcal{O}_c(\epsilon^3|\log\epsilon|^2) + \mathcal{O}_\Theta(\epsilon) \quad \text{in } Q_\epsilon^-, \tag{17.40}$$

where

$$\mathrm{I}_\epsilon := \epsilon^{-2}\Big(- \gamma_\epsilon''(y_\epsilon^-) + W'(\gamma_\epsilon(y_\epsilon^-))\Big),$$

$$\mathrm{II}_\epsilon := \left|\overline{\nabla^2 d}\right|^2 \Big(- \xi_\epsilon''(y_\epsilon^-) + W''(\gamma_\epsilon(y_\epsilon^-))\xi_\epsilon(y_\epsilon^-) + y_\epsilon^- \gamma_\epsilon'(y_\epsilon^-)\Big),$$

$$\mathrm{III}_\epsilon := \epsilon|\log\epsilon|^2 \Bigg(\underbrace{\gamma_\epsilon'(y_\epsilon^-)\,\vartheta\,\left|\overline{\nabla^2 d}\right|^2}_{:=\mathrm{III}_\epsilon^{(1)}} \underbrace{-\gamma_\epsilon'(y_\epsilon^-)\,\vartheta'}_{:=\mathrm{III}_\epsilon^{(2)}} \underbrace{-\Theta\, W''(\gamma_\epsilon(y_\epsilon^-))}_{:=\mathrm{III}_\epsilon^{(3)}} \Bigg),$$

and we notice that the terms $\mathrm{III}_\epsilon^{(i)}$, for $i \in \{1, 2, 3\}$, are multiplied by $\epsilon|\log\epsilon|^2$, which is of the same order of the reminder $\mathcal{O}\left(\epsilon|\log\epsilon|^2\right)$ appearing as the last addendum of the first line on the right-hand side of (17.40).

Let us estimate the terms I_ϵ, II_ϵ and III_ϵ. Concerning I_ϵ, since $\gamma_\epsilon = \gamma$ in $(-\ell_\epsilon, \ell_\epsilon)$, we have (see (15.18))

$$-\gamma_\epsilon''(y) + W'(\gamma_\epsilon(y)) = 0, \qquad y \in (-\ell_\epsilon, \ell_\epsilon). \tag{17.41}$$

The last estimate in (17.12) implies

$$\gamma_\epsilon''(y) = \mathcal{O}(\epsilon^{N-1}), \qquad y \in (-2\ell_\epsilon, -\ell_\epsilon) \cup (\ell_\epsilon, 2\ell_\epsilon),$$

while the first one implies

$$W'(\gamma_\epsilon(y)) = \mathcal{O}(\epsilon^{N-1}), \qquad y \in (-2\ell_\epsilon, -\ell_\epsilon) \cup (\ell_\epsilon, 2\ell_\epsilon).$$

---

(19) We expand the function $W'(v_\epsilon^-)$ around $\gamma_\epsilon(y_\epsilon^-)$, taking into account that

$$\epsilon^2 \xi_\epsilon(y_\epsilon^-)\left|\overline{\nabla^2 d}\right|^2 - \Theta\epsilon^3|\log\epsilon|^2 = \mathcal{O}_\Theta(\epsilon^2).$$

Therefore

$$\mathrm{I}_\epsilon = \mathcal{O}(\epsilon^{N-3}) \qquad \text{in } Q_\epsilon^-. \tag{17.42}$$

Concerning the term II_ϵ, since $\xi_\epsilon = \xi$ in $(-\ell_\epsilon, \ell_\epsilon)$, we have (see (15.37))

$$-\xi_\epsilon''(y) + W''(\gamma_\epsilon(y))\xi_\epsilon(y) + y\gamma_\epsilon'(y) = 0, \qquad y \in (-\ell_\epsilon, \ell_\epsilon).$$

On the other hand, from (17.23) and (17.11) it follows that

$$\xi_\epsilon''(y) = W''(\gamma_\epsilon(y))\xi_\epsilon(y) = y\gamma_\epsilon'(y) = \mathcal{O}(\epsilon^{N-1})$$

for any $y \in (-2\ell_\epsilon, -\ell_\epsilon) \cup (\ell_\epsilon, 2\ell_\epsilon)$.

Therefore

$$\mathrm{II}_\epsilon = \mathcal{O}(\epsilon^{N-1}) \qquad \text{in } Q_\epsilon^-. \tag{17.43}$$

Now, let us estimate the term III_ϵ. Notice that the term $\mathrm{III}_\epsilon^{(1)}$ is positive, and therefore has the wrong sign. The idea is to choose the function ϑ and the constant Θ so that the negative term $\mathrm{III}_\epsilon^{(2)}$ absorbs $\mathrm{III}_\epsilon^{(1)}$ and then, together with $\mathrm{III}_\epsilon^{(3)}$, also the remainder in (17.40) (and the remainders in (17.42) and (17.43)).

Recalling the expression (17.18) of ϑ, we have

$$\vartheta' = \vartheta\,(1 + K)\,,$$

hence, remembering the definition (17.17) of K, we have, in Q_ϵ^-,

$$\mathrm{III}_\epsilon^{(1)} + \mathrm{III}_\epsilon^{(2)} = -\vartheta\gamma_\epsilon'(y_\epsilon^-)\left[1 + K - \left|\overline{\nabla^2 d}\right|^2\right] \le -\vartheta\gamma_\epsilon'(y_\epsilon^-).$$

Therefore

$$\mathrm{III}_\epsilon \le -\epsilon|\log\epsilon|^2\left[\vartheta\ \gamma_\epsilon'(y_\epsilon^-) + \Theta\ W''(\gamma_\epsilon(y_\epsilon^-))\right] \qquad \text{in } Q_\epsilon^-.$$

Since $\vartheta(t) \ge c$ for any $t \in [0, +\infty)$, we obtain

$$\begin{aligned}\mathrm{III}_\epsilon &\le -\epsilon|\log\epsilon|^2\left[c\ \gamma_\epsilon'(y_\epsilon^-) + \Theta\ W''(\gamma_\epsilon(y_\epsilon^-))\right]\\ &= -\Theta\epsilon|\log\epsilon|^2\left[\frac{c}{\Theta}\gamma_\epsilon'(y_\epsilon^-) + W''(\gamma_\epsilon(y_\epsilon^-))\right].\end{aligned}$$

In view of the properties of the functions γ_ϵ' and $W''(\gamma_\epsilon)$, we can choose the constants $c > 0$ and $\Theta > 0$ large enough and independent of ϵ, and a number $\overline{\epsilon}_3 \in (0, \epsilon_3)$, possibly depending on c and Θ, so that the following two conditions are satisfied:

- the function $\dfrac{c}{\Theta}\gamma_\epsilon'(y_\epsilon^-) + W''(\gamma_\epsilon(y_\epsilon^-))$ is uniformly positive with respect to $\epsilon \in (0, \overline{\epsilon}_3)$, namely there exists $\sigma > 0$ such that

$$\inf_{(t,z)\in Q_\epsilon^-}\left(\frac{c}{\Theta}\gamma_\epsilon'(y_\epsilon^-(t,z)) + W''(\gamma_\epsilon(y_\epsilon^-(t,z)))\right) \ge \sigma, \qquad \epsilon \in (0, \overline{\epsilon}_3); \tag{17.44}$$

- the positive bound σ in (17.44), multiplied by $\Theta\epsilon|\log\epsilon|^2$, absorbs[20] twice the remainder $\mathcal{O}(\epsilon|\log\epsilon|^2)$ in (17.40) (and those in (17.42) and (17.43)), for $\epsilon \in (0, \overline{\epsilon}_3)$. Possibly reducing the value of $\overline{\epsilon}_3 > 0$, we can finally suppose that

$$\mathrm{III}_\epsilon + \mathcal{O}(\epsilon|\log\epsilon|^2) + \mathcal{O}_c(\epsilon^3|\log\epsilon|^2) + \mathcal{O}_\Theta(\epsilon) \leq 0, \qquad \epsilon \in (0, \overline{\epsilon}_3).$$

We conclude that there exist two constants $c > 0$, $\Theta > 0$ independent of ϵ, and there exists $\widetilde{\epsilon}_3 > 0$ possibly depending on c and Θ, such that

$$\frac{\partial v_\epsilon^-}{\partial t} - \Delta v_\epsilon^- + \epsilon^{-2} W'(v_\epsilon^-) \leq 0 \qquad \text{in } Q_\epsilon^-, \qquad \epsilon \in (0, \widetilde{\epsilon}_3).$$

In order to complete the proof of the lemma, it remains to check inequality (17.34) in $([0,T] \times \Omega) \setminus \overline{Q_\epsilon^-}$. At points of $([0,T] \times \Omega) \setminus \overline{Q_\epsilon^-}$ we have

$$v_\epsilon^- = \gamma_- - \Theta\epsilon^3|\log\epsilon|^2 \quad \text{in} \quad \mathfrak{O}_{\epsilon-}^-,$$

and

$$v_\epsilon^- = \gamma_+ - \Theta\epsilon^3|\log\epsilon|^2 \quad \text{in} \quad \mathfrak{O}_{\epsilon+}^-,$$

so that $\dfrac{\partial v_\epsilon^-}{\partial t} = \Delta v_\epsilon^- = 0$. Moreover, recalling that $\alpha = W''(\gamma_\pm)$,

$$\epsilon^{-2} W'(v_\epsilon^-) = -\alpha\Theta\epsilon|\log\epsilon|^2 + \mathcal{O}(\epsilon).$$

Therefore

$$\frac{\partial v_\epsilon^-}{\partial t} - \Delta v_\epsilon^- + \epsilon^{-2} W'(v_\epsilon^-) = -\Theta\alpha\epsilon|\log\epsilon|^2 + \mathcal{O}(\epsilon),$$

so that (remembering that $\alpha > 0$) there exists $\epsilon_4 \in (0, \widetilde{\epsilon}_3)$, such that $\dfrac{\partial v_\epsilon^-}{\partial t} - \Delta v_\epsilon^- + \epsilon^{-2} W'(v_\epsilon^-) \leq 0$ for $\epsilon \in (0, \epsilon_4)$ in $\mathfrak{O}_{\epsilon-}^- \cup \mathfrak{O}_{\epsilon+}^-$, for any choice of the positive constant Θ. □

Corollary 17.20. *There exist constants $c > 0$ and $\Theta > 0$, and there exists $\epsilon_0 > 0$ possibly depending on c and Θ, such that, if $v_\epsilon^\pm$ denote the functions defined in* (17.24), *and if u_ϵ denotes the solution of the parabolic problem* (15.2) *with well-prepared initial datum* (17.6), *then*

$$v_\epsilon^- \leq u_\epsilon \leq v_\epsilon^+ \qquad \text{in } [0,T] \times \Omega, \tag{17.45}$$

for any $\epsilon \in (0, \epsilon_0)$.

Proof. The left inequality follows by applying Theorem 17.18 with u replaced by u_ϵ and v replaced by v_ϵ^-. The right inequality is similar. □

[20] Recall that $N \geq 4$.

17.5.1. Conclusion of the proof of Theorem 17.5

Let us prove (17.15). Fix $t \in [0,T]$ and $z \in \partial E_\epsilon(t)$. We have to show that

$$|d(t,z)| \leq \mathcal{O}(\epsilon^2 |\log \epsilon|^2). \tag{17.46}$$

Let c, Θ, ϵ_0 and ϵ be as in Corollary 17.20. We first observe that z belongs to the neighbourhood $\mathrm{A}(t)$ defined in (17.2). Indeed, $z \in \partial E_\epsilon(t)$ and Corollary 17.20 imply

$$v_\epsilon^-(t,z) \leq u_\epsilon(t,z) = 0 \leq v_\epsilon^+(t,z). \tag{17.47}$$

If by contradiction $z \notin \mathrm{A}(t)$, by inclusions (17.20) we have $z \notin \mathrm{A}_\epsilon^-(t)$ and $z \notin \mathrm{A}_\epsilon^+(t)$. Recalling the definitions of $v_\epsilon^\pm$, we deduce that $v_\epsilon^-(t,z)$ and $v_\epsilon^+(t,z)$ are both either positive or negative, a contradiction with (17.47).

From the expression of v_ϵ^- and the left inequality in (17.47), we have

$$v_\epsilon^-(t,z) = \gamma_\epsilon\left(\frac{d_\epsilon^-(t,z)}{\epsilon}\right) + \mathcal{O}(\epsilon) \leq 0,$$

so that

$$\gamma_\epsilon\left(\frac{d_\epsilon^-(t,z)}{\epsilon}\right) \leq \mathcal{O}(\epsilon).$$

As $\gamma'(0) = 1$, we deduce

$$\frac{d_\epsilon^-(t,z)}{\epsilon} \leq \mathcal{O}(\epsilon).$$

Hence, recalling the definition of d_ϵ^- in (17.19), it follows that

$$d(t,z) \leq \mathcal{O}(\epsilon^2 |\log \epsilon|^2).$$

A similar argument applied with v_ϵ^- replaced by v_ϵ^+ implies

$$d(t,z) \geq \mathcal{O}(\epsilon^2 |\log \epsilon|^2),$$

and concludes the proof of inequality (17.46), and hence of inclusion (17.15).

Let us prove (17.16). Fix $t \in [0,T]$ and $x \in \partial f(t)$. We have to show that

$$\mathrm{dist}(x, \partial E_\epsilon(t)) \leq \mathcal{O}(\epsilon^2 |\log \epsilon|^2).$$

We denote by I the connected component of the intersection between $\mathrm{A}(t)$ and the normal line to $\partial f(t)$ at x which contains x. Write

$$\{z^-, z^+\} = I \cap \partial \mathrm{A}(t),$$

where $v_\epsilon^-(t, z^+) > 0$.

We claim that there exist points

$$z_\epsilon^\pm \in I \cap \{z \in \Omega : v_\epsilon^\pm(t, z) = 0\}$$

such that

$$|z_\epsilon^\pm - x| \le C\epsilon^2 |\log \epsilon|^2, \tag{17.48}$$

for a suitable positive constant C independent of ϵ. Indeed, using also the fact that $d(t, x) = 0$, we have

$$v_\epsilon^-(t, x) = \gamma_\epsilon\big(-\vartheta(t)\epsilon|\log \epsilon|^2\big) + \mathcal{O}(\epsilon) < 0,$$

and similarly

$$v_\epsilon^-(t, z^+) > 0,$$

so that there exists a point $z_\epsilon^- \in I$ lying between x and z^+ (and different from x and z^+) such that

$$v_\epsilon(t, z_\epsilon^-) = 0. \tag{17.49}$$

Moreover, since $\gamma'(0) = 1$,

$$|v_\epsilon^-(t, z_\epsilon^-) - v_\epsilon^-(t, x)| = |v_\epsilon^-(t, x)| \le C_1\epsilon|\log \epsilon|^2, \tag{17.50}$$

for some positive constant C_1 independent of ϵ. On the other hand, $\nabla d(t, x) = \dfrac{x - z_\epsilon^-}{|x - z_\epsilon^-|}$, and

$$|v_\epsilon^-(t, z_\epsilon^-) - v_\epsilon^-(t, x)| = |z_\epsilon^- - x|\,|\langle \nabla v_\epsilon^-(t, \zeta), \nabla d(t, x)\rangle|, \tag{17.51}$$

for a suitable point ζ belonging to the segment joining z_ϵ^- and x. Moreover, recalling that $\nabla d(t, x) = \nabla d(t, \zeta)$, and using (17.49), (17.50) and the explicit expression of v_ϵ^-,

$$\begin{aligned} |\langle \nabla v_\epsilon^-(t,\zeta), \nabla d(t, x)\rangle| &= \epsilon^{-1}\gamma_\epsilon'(y_\epsilon^-(t, \zeta))|\langle \nabla d(t, \zeta), \nabla d(t, x)\rangle| + \mathcal{O}(\epsilon) \\ &= \epsilon^{-1}\gamma_\epsilon'(y_\epsilon^-(t, \zeta)) + \mathcal{O}(\epsilon) \ge C_2\epsilon^{-1}, \end{aligned} \tag{17.52}$$

for a positive constant C_2 independent of ϵ. From (17.51) and (17.52) we infer

$$|v_\epsilon^-(t, z_\epsilon^-) - v_\epsilon^-(t, x)| \ge C_2\epsilon^{-1}|z_\epsilon^- - x|. \tag{17.53}$$

Then (17.50) and (17.53) yield

$$|z_\epsilon^- - x| \le C\epsilon^2|\log \epsilon|^2,$$

where $C = \frac{C_1}{C_2}$. A similar argument applied with v_ϵ^+ replacing v_ϵ^- concludes the proof of the claim.

As $v_\epsilon^\pm(t, z_\epsilon^\pm) = 0$ we have, using (17.45),

$$u_\epsilon(t, z_\epsilon^+) \leq v_\epsilon^+(t, z_\epsilon^+) = 0, \qquad 0 = v_\epsilon^-(t, z_\epsilon^-) \leq u_\epsilon(t, z_\epsilon^-).$$

Hence there exists a point $z \in I$ belonging to the segment joining z_ϵ^- and z_ϵ^+, such that $u_\epsilon(t, z) = 0$, that is, $z \in \partial E_\epsilon(t)$. Therefore, using (17.48),

$$\mathrm{dist}(x, \partial E_\epsilon(t)) \leq |x - z| \leq |z_\epsilon^+ - z_\epsilon^-| \leq |z_\epsilon^+ - x| + |x - z_\epsilon^-| \leq 2C\epsilon^2 |\log \epsilon|^2,$$

and this gives (17.15), and concludes the proof of Theorem 17.5. □

17.6. Inclusion principle for smooth compact mean curvature flows

We demonstrate the usefulness of Theorem 17.5 by proving the comparison principle between two smooth compact mean curvature flows[(21)]. This gives an alternative proof with respect to that given in Theorem 5.8. It is valid for sufficiently short times, but it covers the case when the two initial boundaries intersect.

Let $f_1, f_2 : [a, b] \to \mathcal{P}(\mathbb{R}^n)$ be two smooth compact mean curvature flows, with

$$f_1(t) \subset\subset \Omega, \qquad f_2(t) \subset\subset \Omega, \qquad t \in [a, b].$$

For each $\epsilon \in (0, 1]$ and $i \in \{1, 2\}$ let $u_{i\epsilon}$ be the solution to

$$\begin{cases} \epsilon \dfrac{\partial u}{\partial t} - \epsilon \Delta u + \epsilon^{-1} W'(u) = 0 & \text{in } (a, +\infty) \times \Omega, \\ \dfrac{\partial u}{\partial \mathrm{n}_\Omega} = 0 & \text{on } (a, +\infty) \times \partial\Omega, \\ u(a, \cdot) = \overline{\gamma}_\epsilon \left(\dfrac{d_{f_i}}{\epsilon} \right) & \text{on } \{t = a\} \times \Omega. \end{cases} \tag{17.54}$$

Here

$$d_{f_i}(t,z) := \mathrm{dist}(z, f_i(t)) - \mathrm{dist}(z, \mathbb{R}^n \setminus f_i(t)), \qquad (t,z) \in [a, b] \times \mathbb{R}^n,\ i \in \{1,2\}.$$

(21) This observation has been made for instance in [41] and [49], in the more general context of anisotropic and crystalline mean curvature flow.

Moreover, $\overline{\gamma}_\epsilon : \mathbb{R} \to \mathbb{R}$ is the odd nondecreasing function[22] of class $\mathcal{C}^{1,1}(\mathbb{R}) \cap \mathcal{C}^\infty(\mathbb{R} \setminus \{\pm\ell_\epsilon, \pm r_\epsilon\})$, which coincides with γ in the interval $[0, \ell_\epsilon]$, it is equal to 1 in the interval $[r_\epsilon, +\infty)$ for some $r_\epsilon > \ell_\epsilon$, and in $(\ell_\epsilon, r_\epsilon)$ it is the quadratic polynomial q_γ so that[23]

- $\mathrm{q}_\gamma(\ell_\epsilon) = \gamma(\ell_\epsilon), \mathrm{q}'_\gamma(\ell_\epsilon) = \gamma'(\ell_\epsilon)$,
- $\mathrm{q}_\gamma(r_\epsilon) = 1, \mathrm{q}'_\gamma(r_\epsilon) = 0$.

Suppose that $b - a$ is small enough so that all assumptions in Section 17.1 are satisfied for f_1, f_2, and also the analog of assumption (17.14) is satisfied for $u_{1\epsilon}$ and $u_{2\epsilon}$, provided $[0, T]$ is replaced by $[a, b]$.

Theorem 17.21 (Inclusion principle). *Suppose that*

$$f_1(a) \subseteq f_2(a). \tag{17.55}$$

Then

$$f_1(t) \subseteq f_2(t), \qquad t \in [a, b]. \tag{17.56}$$

Proof. Inclusion (17.55) and the special form of the initial data in (17.54) imply

$$u_{1\epsilon}(a, \cdot) \geq u_{2\epsilon}(a, \cdot) \qquad \text{in } \Omega.$$

Consequently, from the comparison Theorem 17.18 it follows that

$$u_{1\epsilon} \geq u_{2\epsilon} \qquad \text{in } [a, +\infty) \times \Omega,$$

so that

$$\begin{aligned} E_{1\epsilon}(t) :=& \{z \in \Omega : u_{1\epsilon}(t, z) \leq 0\} \\ \subseteq& \{z \in \Omega : u_{2\epsilon}(t, z) \leq 0\} =: E_{2\epsilon}(t), \quad t \in [a, +\infty). \end{aligned}$$

By the convergence Theorem 17.5, it follows that, for any $t \in [a, b]$, the Hausdorff distance between $\partial f_i(t)$ and $\partial E_{i\epsilon}(t)$ converges to zero as $\epsilon \to 0^+$. Then inclusion (17.56) follows. □

[22] The function $\overline{\gamma}_\epsilon$ could be used in place of γ_ϵ in Theorem 17.5 as well. The qualitative difference between $\overline{\gamma}_\epsilon$ and γ_ϵ is that $\overline{\gamma}_\epsilon$ is nondecreasing.

[23] For $y \in (\ell_\epsilon, r_\epsilon)$ we have $\mathrm{q}_\gamma(y) = i_\epsilon \left(\frac{r_\epsilon - y}{\ell_\epsilon}\right)^2 + 1$, where $i_\epsilon = \frac{(\gamma'(\ell_\epsilon))^2 \ell_\epsilon^2}{4(\gamma(\ell_\epsilon) - 1)} < 0$ and $i_\epsilon = \mathcal{O}(\epsilon^N |\log \epsilon|^2)$ and $r_\epsilon = \ell_\epsilon + \frac{2(1-\gamma(\ell_\epsilon))}{\gamma'(\ell_\epsilon)} = \mathcal{O}(\ell_\epsilon)$. Note that $\|\overline{\gamma}_\epsilon - \gamma\|_{L^\infty(\mathbb{R})} = \mathcal{O}(\epsilon^N)$ and $\|\mathrm{q}'_\gamma\|_{L^\infty((-r_\epsilon, -\ell_\epsilon) \cup (\ell_\epsilon, r_\epsilon))} = \mathcal{O}(\epsilon^N)$, $\|\mathrm{q}''_\gamma\|_{L^\infty((-r_\epsilon, -\ell_\epsilon) \cup (\ell_\epsilon, r_\epsilon))} = \mathcal{O}(\epsilon^N)$.

17.7. Notes

In this section we introduce some relevant quantities and briefly list some results and references[(24)], related to the asymptotic convergence of solutions to (16.1).

Remark 17.22 (Implicit representation of the solution). In view of the strong maximum principle (see, *e.g.*, [230] and [15]) we have, for the solution u_ϵ of (16.1) with initial condition $\gamma(\epsilon^{-1}d(0,\cdot))$ (and for definitiveness $\Omega = \mathbb{R}^n$),

$$u_\epsilon(t,z) \in (-1,1), \qquad (t,z) \in [0,+\infty) \times \mathbb{R}^n.$$

Therefore we can define the function $q_\epsilon : [0,+\infty) \times \mathbb{R}^n \to \mathbb{R}$ from the relation

$$u_\epsilon(t,z) = \gamma\left(\frac{q_\epsilon(t,z)}{\epsilon}\right), \qquad (t,z) \in [0,+\infty) \times \mathbb{R}^n. \tag{17.57}$$

Inserting this expression in (16.1) gives

$$\gamma'\left(\frac{q_\epsilon}{\epsilon}\right)\left(\frac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon\right) - \epsilon^{-1}\gamma''\left(\frac{q_\epsilon}{\epsilon}\right)|\nabla q_\epsilon|^2 + \epsilon^{-1}W'\left(\gamma\left(\frac{q_\epsilon}{\epsilon}\right)\right)=0.$$

Remembering that $-\gamma'' + W'(\gamma) = 0$ we obtain

$$\gamma'\left(\frac{q_\epsilon}{\epsilon}\right)\left(\frac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon\right) + W'\left(\gamma\left(\frac{q_\epsilon}{\epsilon}\right)\right)\frac{1-|\nabla q_\epsilon|^2}{\epsilon} = 0.$$

That is,

$$\frac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon + \frac{W'\left(\gamma\left(\frac{q_\epsilon}{\epsilon}\right)\right)}{\gamma'\left(\frac{q_\epsilon}{\epsilon}\right)}\left(\frac{1-|\nabla q_\epsilon|^2}{\epsilon}\right) = 0,$$

or equivalently

$$\frac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon + \frac{\gamma''\left(\frac{q_\epsilon}{\epsilon}\right)}{\gamma'\left(\frac{q_\epsilon}{\epsilon}\right)}\left(\frac{1-|\nabla q_\epsilon|^2}{\epsilon}\right) = 0.$$

In terms of the function w, using (15.3) and (15.16) we obtain

$$\frac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon - w'\left(\gamma\left(\frac{q_\epsilon}{\epsilon}\right)\right)\frac{1-|\nabla q_\epsilon|^2}{\epsilon} = 0. \tag{17.58}$$

(24) As usual, we make no claim of completeness on the reference list.

With our choice of w in (17.4) we have

$$\frac{\partial q_\epsilon}{\partial t} - \Delta q_\epsilon - \gamma\left(\frac{q_\epsilon}{\epsilon}\right)\frac{1-|\nabla q_\epsilon|^2}{\epsilon} = 0.$$

This equation has been used for instance in [81, 184, 38] and [249].

Remark 17.23 (ϵ-Brakke's formulation). Formula (3.18) has a counterpart for solutions u_ϵ of equation (16.1) (and $\Omega = \mathbb{R}^n$), as shown by Ilmanen in [184]. Indeed, for any function

$$v \in L^2_{\mathrm{loc}}([0,+\infty); H^1_{\mathrm{loc}}(\mathbb{R}^n)) \cap H^1_{\mathrm{loc}}([0,+\infty); L^2_{\mathrm{loc}}(\mathbb{R}^n)),$$

set(25)

$$\ell_\epsilon(v) := \epsilon\frac{|\nabla v|^2}{2} + \epsilon^{-1}W(v),$$

$$\mu^t_{\epsilon,v} := \left(\epsilon\frac{|\nabla v(t,\cdot)|^2}{2} + \epsilon^{-1}W(v(t,\cdot))\right)\mathcal{L}^n,$$

and

$$\mu_{\epsilon,v} := \ell_\epsilon(v)\,\mathcal{L}^n \otimes dt,$$

where we recall that $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$, and $\otimes$ stands here for the product symbol between measures.
Then, if $\theta \in \mathcal{C}^2_c(\mathbb{R}\times\mathbb{R}^n)$ we have

$$\frac{d}{dt}\int_{\mathbb{R}^n}\theta\ d\mu^t_{\epsilon,u_\epsilon} = \int_{\mathbb{R}^n}\frac{\partial\theta}{\partial t}\,d\mu^t_{\epsilon,u_\epsilon} + \mathcal{B}_\epsilon(u_\epsilon,\theta), \tag{17.59}$$

where(26)

$$\mathcal{B}_\epsilon(u_\epsilon,\theta) := \underbrace{-\int_{\mathbb{R}^n}\Big(\epsilon\Delta u_\epsilon - \epsilon^{-1}W'(u_\epsilon)\Big)\langle\nabla u_\epsilon,\nabla\theta\rangle\,dz}_{:=\mathcal{T}_\epsilon(u_\epsilon,\theta)}$$

$$\underbrace{-\int_{\mathbb{R}^n}\theta\Big(\epsilon\Delta u_\epsilon - \epsilon^{-1}W'(u_\epsilon)\Big)^2\epsilon^{-1}\,dz}_{:=\mathcal{W}_\epsilon(u_\epsilon,\theta)}.$$

(25) Note (see [251] and references therein) that

$$\frac{\partial}{\partial t}\ell_\epsilon(u_\epsilon) = -\epsilon\left(\frac{\partial u_\epsilon}{\partial t}\right)^2 + \epsilon\,\mathrm{div}\left(\nabla u_\epsilon\frac{\partial}{\partial t}u_\epsilon\right),$$

$$\nabla_i\ell_\epsilon(u_\epsilon) = -\epsilon\nabla_i u_\epsilon\frac{\partial u_\epsilon}{\partial t} + \epsilon\nabla_j(\nabla_i u_\epsilon\nabla_j u_\epsilon), \qquad i\in\{1,\dots,n\}.$$

(26) Compare formula (4.1).

In order to prove (17.59), we compute

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} \theta \; d\mu^t_{\epsilon,u_\epsilon} - \int_{\mathbb{R}^n} \frac{\partial \theta}{\partial t} \, d\mu^t_{\epsilon,u_\epsilon} \\
&= \int_{\mathbb{R}^n} \theta \frac{\partial}{\partial t}\left(\frac{\epsilon}{2}|\nabla u_\epsilon|^2 + \epsilon^{-1} W(u_\epsilon)\right) \, dz \\
&= \int_{\mathbb{R}^n} \theta \Big(\epsilon \langle \nabla u_\epsilon, \nabla \frac{\partial u_\epsilon}{\partial t}\rangle + \epsilon^{-1} W'(u_\epsilon) \frac{\partial u_\epsilon}{\partial t}\Big) \, dz \\
&= \int_{\mathbb{R}^n} -\theta \Big(\epsilon \Delta u_\epsilon - \epsilon^{-1} W'(u_\epsilon)\Big) \frac{\partial u_\epsilon}{\partial t} \, dz \\
&\quad - \int_{\mathbb{R}^n} \langle \nabla u_\epsilon, \nabla \theta \rangle \, \epsilon \, \frac{\partial u_\epsilon}{\partial t} \, dz,
\end{aligned} \tag{17.60}$$

where the last equality is obtained with an integration by parts. Then (17.59) follows by substituting the partial differential equation in (16.1) into the right-hand side of formula (17.60).

Formula (17.59) can be called ϵ-Brakke's formulation of mean curvature flow. Passing to the (subsequential) limit as $\epsilon \to 0^+$ into (17.59), and relating this limit with (3.18), are some of the results of Ilmanen in [184], valid also after the onset of singularities.

Definition 17.24 (ϵ-stress tensor). The stress tensor $T_\epsilon(v)$ is defined[27] as

$$T_\epsilon(v) := \ell_\epsilon(v)\mathrm{Id} - \epsilon \nabla v \otimes \nabla v.$$

Namely, for for any $i, j \in \{1, \dots, n\}$, the ij-component $T_{\epsilon ij}(v)$ of the stress tensor is given by

$$T_{\epsilon ij}(v) := \ell_\epsilon(v) \, \mathrm{Id}_{ij} - \epsilon \nabla_i v \nabla_j v,$$

and

$$T_{\epsilon ij}(v) = T_{\epsilon ji}(v).$$

Notice that the trace of the ϵ-stress tensor is

$$T_{\epsilon ii}(v) = n\ell_\epsilon(v) - \epsilon |\nabla v|^2,$$

hence (assuming $n \geq 2$) and observing that $2\ell_\epsilon(v) \geq \epsilon |\nabla v|^2$,

$$T_{\epsilon ii}(v) \geq (n-2)\ell_\epsilon(v). \tag{17.61}$$

[27] A rough meaning of the stress tensor can be obtained as follows. If we assume that v takes the form $\gamma(d(t,z)/\epsilon)$, then $T_{\epsilon ij}(v)$ becomes, recalling (15.19), $\epsilon^{-1}(\gamma'(d/\epsilon))^2 \left(\mathrm{Id}_{ij} - \nabla_i d \nabla_j d\right)$, which is, up to a positive factor, close to an orthogonal projection on an $(n-1)$-dimensional subspace (see Section 1.1 and Remark 1.15).

We have (supposing that $\ell_\epsilon(v_\epsilon)$ never vanishes)

$$T_\epsilon(v)\ \mathcal{L}^n \otimes dt = \left(\mathrm{Id} - \frac{\nabla v \otimes \nabla v}{\ell_\epsilon(v)}\right) d\mu_{\epsilon,v},$$

and if we let $T^t_\epsilon(v) := T_\epsilon(v(t,\cdot))$, also

$$T^t_\epsilon(v)\ \mathcal{L}^n = \left(\mathrm{Id} - \frac{\nabla v(t,\cdot) \otimes \nabla v(t,\cdot)}{\ell_\epsilon(v(t,\cdot))}\right) d\mu^t_{\epsilon,v}.$$

The divergence $\nabla_i T_{\epsilon ij}(v)$ is strictly related to the right-hand side of equation (15.2). More specifically, supposing enough smoothness for v,

$$\begin{aligned}\nabla_i T_{\epsilon ij}(v) &= \epsilon \nabla_i v \nabla^2_{ij} v + \epsilon^{-1} W'(v) \nabla_j v \\ &\quad - \epsilon \Delta v\ \nabla_j v - \epsilon \nabla^2_{ij} v\ \nabla_i v \\ &= \nabla_j v \Big(- \epsilon \Delta v + \epsilon^{-1} W'(v)\Big).\end{aligned} \tag{17.62}$$

In particular, if $X = (X_1, \ldots, X_n) \in \mathcal{C}^2_c((0,+\infty) \times \mathbb{R}^n; \mathbb{R}^n)$, we have the following integration by parts formula:

$$\begin{aligned}&\int_{\{t\}\times\mathbb{R}^n} \Big(\ell_\epsilon(v)\mathrm{Id}_{ij} - \epsilon \nabla_i v \nabla_j v\Big) \nabla_i X_j\ dz \\ &\qquad = \int_{\{t\}\times\mathbb{R}^n} \Big(\epsilon \Delta v - \epsilon^{-1} W'(v)\Big)\ \nabla_j v\ X_j\ dz,\end{aligned}$$

which will be used to properly rewrite the term $\mathcal{T}_\epsilon(u_\epsilon, \theta)$ on the right-hand side of (17.59).

It will be convenient to introduce another notation.

Definition 17.25 (Discrepancies). We set

$$\xi_\epsilon(v) := \frac{\epsilon}{2} |\nabla v|^2 - \epsilon^{-1} W(v),$$

and

$$\xi^t_\epsilon(v) := \frac{\epsilon}{2} |\nabla v(t,\cdot)|^2 - \epsilon^{-1} W(v(t,\cdot)).$$

The functions $\xi_\epsilon(v)$ and $\xi^t_\epsilon(v)$ are sometimes called discrepancy functions, and the measures

$$d\xi_{\epsilon,v} := \xi_\epsilon(v)\ \mathcal{L}^n \otimes dt, \qquad d\xi^t_{\epsilon,v} := \xi^t_\epsilon(v)\ \mathcal{L}^n$$

are called discrepancy measures.

Notice that, using the representation (17.57), we have

$$\xi_\epsilon(u_\epsilon) = \frac{\epsilon^{-1}}{2}(\gamma'(q_\epsilon/\epsilon))^2 \ \big(|\nabla q_\epsilon|^2 - 1\big).$$

Ilmanen proved in [184] the following result, which is the counterpart of formula (4.3).

Theorem 17.26. *Let* $\theta \in \mathcal{C}_c^2((0,+\infty)\times\mathbb{R}^n;(0,+\infty))$. *Then*[(28)] *for any* $t \in (0,+\infty)$ *we have*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^n} \theta \, d\mu_{\mu^t_{\epsilon,u_\epsilon}} \\
&= -\int_{\mathbb{R}^n} \theta \left(\epsilon\Delta u_\epsilon - \epsilon^{-1}W'(u_\epsilon) + \frac{1}{\theta}\langle\epsilon\nabla u_\epsilon, \nabla\theta\rangle\right)^2 \epsilon^{-1}\, dz \\
&\quad + \int_{\mathbb{R}^n} \left\{\frac{\partial\theta}{\partial t} + \frac{1}{\theta}\langle\frac{\nabla u_\epsilon}{|\nabla u_\epsilon|}, \nabla\theta\rangle^2 + \left(\mathrm{Id}_{ij} - \frac{\nabla_i u_\epsilon}{|\nabla u_\epsilon|}\frac{\nabla_j u_\epsilon}{|\nabla u_\epsilon|}\right)\nabla^2_{ij}\theta\right\} d\mu^t_{\epsilon,u_\epsilon} \\
&\quad + \int_{\mathbb{R}^n} \left\{\frac{1}{\theta}\langle\frac{\nabla u_\epsilon}{|\nabla u_\epsilon|}, \nabla\theta\rangle^2 - \frac{\nabla_i u_\epsilon}{|\nabla u_\epsilon|}\frac{\nabla_j u_\epsilon}{|\nabla u_\epsilon|}\nabla^2_{ij}\theta\right\} d\xi^t_{\epsilon,u_\epsilon}.
\end{aligned} \tag{17.63}$$

Proof. Recalling formula (17.59), we properly add and subtract the quantity $\frac{1}{\theta}\langle\epsilon\nabla u_\epsilon, \nabla\theta\rangle$ in the term $\mathcal{W}_\epsilon(u_\epsilon,\theta)$: we have

$$\begin{aligned}
\mathcal{W}_\epsilon(u_\epsilon,\theta) = &-\int_{\mathbb{R}^n} \theta\left(\epsilon\Delta u_\epsilon - \epsilon^{-1}W'(u_\epsilon) + \frac{1}{\theta}\langle\epsilon\nabla u_\epsilon, \nabla\theta\rangle\right)^2 \epsilon^{-1} dz \\
&+ \int_{\mathbb{R}^n} \frac{\epsilon}{\theta}\,\langle\nabla u_\epsilon, \nabla\theta\rangle^2\, dz \\
&+ 2\int_{\mathbb{R}^n} \Big(\epsilon\Delta u_\epsilon - \epsilon^{-1}W'(u_\epsilon)\Big)\langle\nabla u_\epsilon, \nabla\theta\rangle\, dz.
\end{aligned} \tag{17.64}$$

Hence, from (17.59) we obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^n}\theta\, d\mu^t_{\epsilon,u_\epsilon} = &-\int_{\mathbb{R}^n} \theta\left(\epsilon\Delta u_\epsilon - \epsilon^{-1}W'(u_\epsilon) + \frac{1}{\theta}\langle\epsilon\nabla u_\epsilon, \nabla\theta\rangle\right)^2 \epsilon^{-1} dz \\
&+ \int_{\mathbb{R}^n}\frac{\partial\theta}{\partial t}\, d\mu^t_{\epsilon,u_\epsilon} + \int_{\mathbb{R}^n}\frac{\epsilon}{\theta}\,\langle\nabla u_\epsilon, \nabla\theta\rangle^2\, dz - \mathcal{T}_\epsilon(u_\epsilon,\theta).
\end{aligned} \tag{17.65}$$

[(28)] By analiticity, one can assume that the zero level set of $\nabla u_\epsilon(t,\cdot)$ does not contribute to the integrals in the formulas.

Since we can write

$$\int_{\mathbb{R}^n} \frac{\epsilon}{\theta} \langle \nabla u_\epsilon, \nabla \theta \rangle^2 \, dz = \int_{\mathbb{R}^n} \frac{1}{\theta} \langle \frac{\nabla u_\epsilon}{|\nabla u_\epsilon|}, \nabla \theta \rangle^2 \left(d\mu^t_{\epsilon,u_\epsilon} + d\xi^t_{\epsilon,u_\epsilon} \right),$$

from (17.65) it follows that

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^n} \theta \ d\mu^t_{\epsilon,u_\epsilon} = & - \int_{\mathbb{R}^n} \theta \left(\epsilon \Delta u_\epsilon - \epsilon^{-1} W'(u_\epsilon) + \frac{1}{\theta} \langle \epsilon \nabla u_\epsilon, \nabla \theta \rangle \right)^2 \epsilon^{-1} dz \\ & + \int_{\mathbb{R}^n} \left(\frac{\partial \theta}{\partial t} + \frac{1}{\theta} \langle \frac{\nabla u_\epsilon}{|\nabla u_\epsilon|}, \nabla \theta \rangle^2 \right) d\mu^t_{\epsilon,u_\epsilon} \\ & + \int_{\mathbb{R}^n} \frac{1}{\theta} \langle \frac{\nabla u_\epsilon}{|\nabla u_\epsilon|}, \nabla \theta \rangle^2 \, d\xi^t_{\epsilon,u_\epsilon} - \mathcal{T}_\epsilon(u_\epsilon, \theta). \end{aligned} \tag{17.66}$$

We have, using (17.62) and an integration by parts,

$$\begin{aligned} \mathcal{T}_\epsilon(u_\epsilon, \theta) &= - \int_{\mathbb{R}^n} \left(\epsilon \Delta u_\epsilon - \epsilon^{-1} W'(u_\epsilon) \right) \nabla_j \theta \ \nabla_j u_\epsilon \ dz \\ &= \int_{\mathbb{R}^n} \nabla_j \theta \ \nabla_i T_{\epsilon ij}(u_\epsilon) \ dz \\ &= - \int_{\mathbb{R}^n} \nabla^2_{ij} \theta \ T_{\epsilon ij}(u_\epsilon) \ dz \\ &= - \int_{\mathbb{R}^n} \nabla^2_{ij} \theta \ \left(\epsilon \frac{|\nabla u_\epsilon|^2}{2} + \epsilon^{-1} W(u_\epsilon) \right) \mathrm{Id}_{ij} \ dz \\ &\quad + \int_{\mathbb{R}^n} \epsilon \nabla_i u_\epsilon \ \nabla_j u_\epsilon \ \nabla^2_{ij} \theta \ dz. \end{aligned} \tag{17.67}$$

Now, we properly add and subtract to the first addendum on the right-hand side the quantity $\frac{\nabla_i u_\epsilon}{|\nabla u_\epsilon|} \frac{\nabla_j u_\epsilon}{|\nabla u_\epsilon|}$. We obtain

$$- \mathcal{T}_\epsilon(u_\epsilon, \theta) = \underbrace{\int_{\mathbb{R}^n} \nabla^2_{ij} \theta \left(\epsilon \frac{|\nabla u_\epsilon|^2}{2} + \epsilon^{-1} W(u_\epsilon) \right) \left(\mathrm{Id}_{ij} - \frac{\nabla_i u_\epsilon}{|\nabla u_\epsilon|} \frac{\nabla_j u_\epsilon}{|\nabla u_\epsilon|} \right) dz}_{:= \mathcal{T}^{(1)}_\epsilon}$$

$$+ \underbrace{\int_{\mathbb{R}^n} \nabla^2_{ij} \theta \ \frac{\nabla_i u_\epsilon}{|\nabla u_\epsilon|} \frac{\nabla_j u_\epsilon}{|\nabla u_\epsilon|} \left(\epsilon \frac{|\nabla u_\epsilon|^2}{2} + \epsilon^{-1} W(u_\epsilon) \right) dz - \int_{\mathbb{R}^n} \epsilon \nabla_i u_\epsilon \nabla_j u_\epsilon \nabla^2_{ij} \theta dz}_{:= \mathcal{T}^{(2)}_\epsilon}.$$

We have

$$\mathcal{T}^{(1)}_\epsilon = \int_{\mathbb{R}^n} \nabla^2_{ij} \theta \ \left(\mathrm{Id}_{ij} - \frac{\nabla_i u_\epsilon}{|\nabla u_\epsilon|} \frac{\nabla_j u_\epsilon}{|\nabla u_\epsilon|} \right) d\mu^t_{\epsilon,u_\epsilon} \tag{17.68}$$

and

$$\mathcal{T}_\epsilon^{(2)} = \int_{\mathbb{R}^n} \left(-\frac{\epsilon}{2}|\nabla u_\epsilon|^2 + \epsilon^{-1} W(u_\epsilon)\right) \frac{\nabla_i u_\epsilon}{|\nabla u_\epsilon|} \frac{\nabla_j u_\epsilon}{|\nabla u_\epsilon|} \nabla^2_{ij}\theta \, dz$$
$$= -\int_{\mathbb{R}^n} \frac{\nabla_i u_\epsilon}{|\nabla u_\epsilon|} \frac{\nabla_j u_\epsilon}{|\nabla u_\epsilon|} \nabla^2_{ij}\theta \, d\xi^t_{\epsilon,u_\epsilon}. \tag{17.69}$$

Collecting together (17.67), (17.68) and (17.69), we get

$$-\mathcal{T}_\epsilon(u_\epsilon, \theta) = \int_{\mathbb{R}^n} \nabla^2_{ij}\theta \left(\mathrm{Id}_{ij} - \frac{\nabla_i u_\epsilon}{|\nabla u_\epsilon|} \frac{\nabla_j u_\epsilon}{|\nabla u_\epsilon|}\right) d\mu^t_{\epsilon,u_\epsilon}$$
$$- \int_{\mathbb{R}^n} \frac{\nabla_i u_\epsilon}{|\nabla u_\epsilon|} \frac{\nabla_j u_\epsilon}{|\nabla u_\epsilon|} \nabla^2_{ij}\theta \, d\xi^t_{\epsilon,u_\epsilon}. \tag{17.70}$$

Substituting (17.70) into (17.66) gives (17.63). □

For $t_0 > 0$ and $z_0 \in \mathbb{R}^n$ let now $\rho_{(t_0,z_0)}$ be the function defined in (4.4). In [184] the following result is proven(29).

Theorem 17.27 (ϵ-monotonicity formula). *For any $t \in (0, t_0)$ we have*

$$\frac{d}{dt} \int_{\mathbb{R}^n} \rho_{(t_0,z_0)} \, d\mu_{\mu^t_{\epsilon,u_\epsilon}}$$
$$= -\int_{\mathbb{R}^n} \rho_{(t_0,z_0)} \left(\epsilon \Delta u_\epsilon - \epsilon^{-1} W'(u_\epsilon) + \frac{1}{\rho_{(t_0,z_0)}} \langle \epsilon \nabla u_\epsilon, \nabla \rho_{(t_0,z_0)} \rangle\right)^2 \epsilon^{-1} dz$$
$$+ \int_{\mathbb{R}^n} \frac{1}{2(t_0 - t)} \rho_{(t_0,z_0)} \, d\xi^t_{\epsilon,u_\epsilon}. \tag{17.71}$$

Proof. By an approximation argument (see [184]) we can substitute $\rho_{(t_0,z_0)}$ in place of θ in formula (17.63). Moreover, arguing as(30) in the proof of (4.7), it follows that the integrand in the second integral on the right-hand side of (17.63) vanishes. The assertion then follows by observing that

$$\frac{1}{\rho_{(t_0,z_0)}} \langle \frac{\nabla u_\epsilon}{|\nabla u_\epsilon|}, \nabla \rho_{(t_0,z_0)} \rangle^2 - \frac{\nabla_i u_\epsilon}{|\nabla u_\epsilon|} \frac{\nabla_j u_\epsilon}{|\nabla u_\epsilon|} \nabla^2_{ij} \rho_{(t_0,z_0)} = \frac{1}{2(t_0 - t)} \rho_{(t_0,z_0)};$$

see (4.8). □

(29) See also [251].

(30) The term $\frac{\partial \rho_{(t_0,z_0)}}{\partial t} + \frac{1}{\rho_{(t_0,z_0)}} \langle \frac{\nabla u_\epsilon}{|\nabla u_\epsilon|}, \nabla \rho_{(t_0,z_0)} \rangle^2 + \left(\mathrm{Id}_{ij} - \frac{\nabla_i u_\epsilon}{|\nabla u_\epsilon|} \frac{\nabla_j u_\epsilon}{|\nabla u_\epsilon|}\right) \nabla^2_{ij} \rho_{(t_0,z_0)}$ vanishes, as a consequence of the expression of $\rho_{(t_0,z_0)}$ and the fact that $\frac{\nabla u_\epsilon}{|\nabla u_\epsilon|}$ is a unit vector.

Remark 17.28 (Sign of the discrepancies). Choosing appropriately the initial datum of the form $u_\epsilon^0 = \gamma(\widetilde{d}/\epsilon)$, where $\widetilde{d}$ is, out of a suitable tubular neighbourhood of ∂E, a smoothing of the signed distance d, which satisfies $|\nabla \widetilde{d}| \leq 1$ everywhere in $\mathbb{R}^n$, so that (from (17.57) and (15.16))

$$\xi_\epsilon(u_\epsilon^0) \leq 0 \qquad \text{in } \mathbb{R}^n,$$

Ilmanen proved[(31)] in [184] that

$$\xi_\epsilon^t(u_\epsilon(t,z)) \leq 0, \qquad (t,z) \in (0,+\infty) \times \mathbb{R}^n. \tag{17.72}$$

This follows from the maximum principle applied to the evolution equation satisfied by $|\nabla q_\epsilon|^2$, which reads, using (17.58), as

$$\begin{aligned}
\frac{\partial}{\partial t}|\nabla q_\epsilon|^2 &= 2\langle \nabla q_\epsilon, \nabla \frac{\partial q_\epsilon}{\partial t}\rangle \\
&= 2\langle \nabla q_\epsilon, \nabla \Delta q_\epsilon + \epsilon^{-2} w''\big(\gamma(q_\epsilon/\epsilon)\big)(1 - |\nabla q_\epsilon|^2)\gamma'(q_\epsilon/\epsilon)\nabla q_\epsilon \\
&\quad - \epsilon^{-1} w'\big(\gamma(q_\epsilon/\epsilon)\big)\nabla |\nabla q_\epsilon|^2\rangle \\
&= \Delta |\nabla q_\epsilon|^2 - 2|\nabla^2 q_\epsilon|^2 \\
&\quad + 2\epsilon^{-2} w''(\gamma(q_\epsilon/\epsilon))|\nabla q_\epsilon|^2(1 - |\nabla q_\epsilon|^2)\gamma'(q_\epsilon/\epsilon) \\
&\quad - 2\epsilon^{-1} w'(\gamma(q_\epsilon/\epsilon))\langle \nabla q_\epsilon, \nabla |\nabla q_\epsilon|^2\rangle,
\end{aligned}$$

where we have used the equality $2\langle \nabla q_\epsilon, \nabla \Delta q_\epsilon\rangle = \Delta |\nabla q_\epsilon|^2 - 2|\nabla^2 q_\epsilon|^2$.

As a consequence of (17.72), from (17.71) it follows that

$$\frac{d}{dt}\int_{\mathbb{R}^n} \rho_{(t_0,z_0)}\, d\mu_{\epsilon,u_\epsilon}^t \leq 0, \qquad t \in (0,t_0),$$

which shows that the function

$$t \in (0,t_0) \to \frac{d}{dt}\int_{\mathbb{R}^n} \rho_{(t_0,z_0)}\, d\mu_{\epsilon,u_\epsilon}^t$$

is nonincreasing (compare with (4.6)).

We mention also that (17.72) allows us to improve inequality (17.61) into

$$T_{\epsilon ii}(u_\epsilon) \geq (n-1)\ell_\epsilon(u_\epsilon).$$

Indeed, we can write

$$\begin{aligned}
T_{\epsilon ii}(u_\epsilon) &= (n-1)\ell_\epsilon(u_\epsilon) + \ell_\epsilon(u_\epsilon) - \epsilon |\nabla u_\epsilon|^2 \\
&= (n-1)\ell_\epsilon(u_\epsilon) - \xi_\epsilon(u_\epsilon) \geq (n-1)\ell_\epsilon(u_\epsilon).
\end{aligned}$$

(31) See also [213, 214] in the stationary case, and [162, 5, 7] for further information. Inequality (17.72) is equivalent to prove that $|\nabla q_\epsilon|^2 \leq 1$, since we have $\frac{\frac{\epsilon}{2}|\nabla u_\epsilon|^2}{\frac{1}{\epsilon}W(u_\epsilon)} = |\nabla q_\epsilon|^2$. Here q_ϵ are as in Remark 17.22, with the above mentioned choice of the initial datum.

Remark 17.29 (Asymptotic beaviour of the discrepancies). It is possible to prove (see [184] for detailed statements) that the discrepancy measures $d\xi_{\epsilon,u_\epsilon}$ tend to vanish as $\epsilon \to 0^+$. Compare also [83], [236] and [233] for related problems.

Remark 17.30. Assume $\Omega = \mathbb{R}^n$ and that u_ϵ^0, in addition, satisfies $\Delta u_0^\epsilon \in L^2(\mathbb{R}^n)$. Multiplying the partial differential equation in (16.1) by $\frac{\partial u_\epsilon}{\partial t}$ and integrating by parts, it follows that

$$\begin{aligned}
&\frac{\epsilon}{2}\int_{\mathbb{R}^n}\left(\frac{\partial u_\epsilon}{\partial t}\right)^2 dz + \frac{1}{2}\int_{\mathbb{R}^n}\left(\epsilon\Delta u_\epsilon - \epsilon^{-1}W'(u_\epsilon)\right)^2 \epsilon^{-1}\, dz \\
&= \epsilon \int_{\mathbb{R}^n}\left(\frac{\partial u_\epsilon}{\partial t}\right)^2 dz \\
&= \int_{\mathbb{R}^n}\frac{\partial u_\epsilon}{\partial t}\left(\epsilon\Delta u_\epsilon - \epsilon^{-1}W'(u_\epsilon)\right) dz = -\frac{d}{dt}\int_{\mathbb{R}^n}\ell_\epsilon(u_\epsilon)\, dz.
\end{aligned}$$

Hence, if $0 \le \tau_1 < \tau_2$, we have

$$\begin{aligned}
&\frac{\epsilon}{2}\int_{\tau_1}^{\tau^2}\int_{\mathbb{R}^n}\left(\frac{\partial u_\epsilon}{\partial t}\right)^2 dz\, dt + \frac{1}{2}\int_{\tau_1}^{\tau^2}\int_{\mathbb{R}^n}\left(\epsilon\Delta u_\epsilon - \epsilon^{-1}W'(u_\epsilon)\right)^2 \epsilon^{-1}\, dz\, dt \\
&= \epsilon \int_{\tau_1}^{\tau_2}\int_{\mathbb{R}^n}\left(\frac{\partial u_\epsilon}{\partial t}\right)^2 dz\, dt \\
&= \int_{\{\tau_1\}\times\mathbb{R}^n}\ell_\epsilon(u_\epsilon(\tau_1, z))\, dz - \int_{\{\tau_2\}\times\mathbb{R}^n}\ell_\epsilon(u_\epsilon(\tau_2, z))\, dz.
\end{aligned}$$

Remembering the gradient flow character of the equation, and assumption (15.11), it follows in particular that

$$\sup_{\epsilon\in(0,1]} \epsilon \int_0^{+\infty}\int_{\mathbb{R}^n}\left(\frac{\partial u_\epsilon}{\partial t}\right)^2 dz\, dt < +\infty.$$

We refer to [15, 216, 241] for more information.

17.7.1. Some references

A rigorous asymptotic analysis of solutions to (16.1) to mean curvature flow before singularities was proved in [118, 119, 71, 81] and [120] (see also [234][32]).

[32] The reader can consult also [30, 235] for related subjects, and [131, 130, 68, 121], where a splitting of the reaction-diffusion operator is considered.

Various conjectures on the asymptotic behaviour of solutions to (16.1), in some cases also for unbounded hypersurfaces and after the onset of singularities, have been given in [102, 103, 104, 107] and [110].

The convergence of solutions for all times, even after the onset of singularities and provided no fattening appears, to the level set solution to mean curvature flow, has been studied, among other places, in [136, 38] and [39] (see also [187](33)). In [223] the error estimate between the zero level set of $u_\epsilon(t,\cdot)$ and the level set mean curvature flow is not, in general, of order ϵ^2 (up to logaritmic terms); subsolutions and supersolutions (without the $\mathcal{O}(\epsilon^2|\log\epsilon|^2)$ error estimate) can be constructed under various circumstances.

As already mentioned, the convergence of solutions of (16.1) to a Brakke's flow has been proved in [184] (see also [261]). The convergence of solutions to a distance solution, in the sense of Soner, has been studied in [249] and [250] (see also [251]).

When W is replaced by a non smooth potential giving raise to a double obstacle problem, various convergence results have been established in [84, 220, 221, 222].

The analog of Theorem 17.5 in the elliptic case is proven in [227] (see also [177] and [260]).

The relations between mean curvature flow and the Γ-limits of the functionals F_ϵ, and of their first variations, have been developed in [216, 217, 240] and [241].

Connections between the gradient flow of the Ginzburg-Landau's functionals and motion by mean curvature, in codimension two, have been proved in [20] and [62] (see also [61]).

Connections between the hyperbolic version of equations in (16.1) and minimal surfaces in a Lorentz space can be found in [218, 50] and [188].

(33) In [187] W has several zeroes, the number of which increases as $\epsilon \downarrow 0$.

References

[1] S. AGMON, *Bounds on exponential decay of eigenfunctions of Schrödinger operators*, In: "Scrhödinger operators" (Como, 1984), Lecture Notes in Math., Vol. 1159, Springer, Berlin, 1985, 1–38.

[2] G. ALBERTI, *Variational models for phase transitions, an approach via* Γ*-convergence*, In: "Calculus of Variations and Partial Differential Equations. Topics on Geometrical Evolution Problems and Degree Theory", Springer-Verlag, 1999, 95–114.

[3] G. ALBERTI, *Some remarks about a notion of rearrangement*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29** (2000), 457–472.

[4] G. ALBERTI and G. BELLETTINI, *A non local anisotropic model for phase transitions. Part 1: the optimal profile problem*, Math. Ann. **310** (1998), 527–560.

[5] N. ALIKAKOS, *Some basic facts on the system* $\Delta u - W_u(u) = 0$, Proc. Amer. Math. Soc. **139** (2011), 153–162.

[6] N. D. ALIKAKOS, P. W. BATES and X. CHEN, *Convergence of the Cahn-Hilliard equation to the Hele-Shaw model*, Arch. Ration. Mech. Anal. **128** (1994), 165–205.

[7] N. ALIKAKOS and A. FALIAGAS, *The stress-energy tensor and Pohozaev's identity for systems*, Acta Math. Scientia **32** (2012), 433–439.

[8] W. K. ALLARD, *On the first variation of a varifold*, Ann. of Math. **95** (1972), 417–491.

[9] F. J. ALMGREN, J. E. TAYLOR and L. WANG, *Curvature-driven flows: a variational approach*, SIAM J. Control Optim. **31** (1993), 387–437.

[10] S. ALTSCHULER, *Singularities of the curve shortening flow for space curves*, J. Differential Geom. **34** (1991), 499–514.

[11] S. ALTSCHULER, S. ANGENENT and Y. GIGA, *Mean curvature flow through singularities for surfaces of rotation*, J. Geom. Anal. **5** (1995), 293–358.

[12] S. ALTSCHULER and M. A. GRAYSON, *Shortening space curves and flow through singularities*, J. Differential Geom. **35** (1992), 283–398.
[13] L. AMBROSIO, *Metric space valued functions of bounded variation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **17** (1990), 439–478.
[14] L. AMBROSIO, *Movimenti minimizzanti*, Rend. Acc. Naz. Sc. XL, Mem. Mat. Appl. **113** (1995), 191–246.
[15] L. AMBROSIO, *Lecture notes on geometric evolution problems, distance function and viscosity solutions*, In: "Calculus of Variations and Partial Differential Equations. Topics on Geometrical Evolution Problems and Degree Theory", Springer-Verlag, 1999, 5–94.
[16] L. AMBROSIO, N. FUSCO and D. PALLARA, "Functions of Bounded Variation and Free Discontinuity Problems", Clarendon Press, Oxford, 2000.
[17] L. AMBROSIO, N. GIGLI and G. SAVARÉ, "Gradient Flows in Metric Spaces and in the Space of Probability Measures", Second edition, Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2008.
[18] L. AMBROSIO and C. MANTEGAZZA, *Curvature and distance function from a manifold*, J. Geom. Anal. **8** (1998), 719–744.
[19] L. AMBROSIO and H. M. SONER, *Level set approach to mean curvature flow in arbitrary codimension*, J. Differential Geom. **43** (1996), 693–737.
[20] L. AMBROSIO and H.-M. SONER, *A measure theoretic approach to higher codimension mean curvature flow*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997), 27–49.
[21] B. ANDREWS, *Convex solutions to the mean curvature flow*, Ann. of Math. **173** (2011), 1185-1239.
[22] S. ANGENENT, *Parabolic equations for curves on surfaces Part I. Curves with p-integrable curvature*, Ann. of Math. **132** (1990), 451–483.
[23] S. ANGENENT, *Parabolic equations for curves on surfaces Part II. Intersections, blow-up and generalized solutions*, Ann. of Math. **133** (1991), 171–215.
[24] S. ANGENENT, *On the formation of singularities in the curve shortening flow*, J. Differential Geom. **33** (1991), 601–633.
[25] S. ANGENENT, *Nodal properties of solutions of parabolic equations*, Rocky Mountain J. Math. **21** (2) (1991), 585–592.
[26] S. ANGENENT, *Shrinking doughnuts*, In: "Nonlinear Diffusion Equations and Their Equilibrium States", 3, N. G. Lloyd *et al.*

(eds.), Proc. Conf. held in Gregynog, Wales 1989, Birkhäuser, Boston, 1992.

[27] S. ANGENENT, *Some recent results on mean curvature flow*, RAM Res. Appl. Math. **30** (1994), 1–18.

[28] S. ANGENENT, D. L. CHOPP and T. ILMANEN, *A computed example of nonuniqueness of mean curvature flow in R^3*, Comm. Partial Differential Equations **20** (1995), 1937–1958.

[29] L. ANGIULI, M. MIRANDA JR., D. PALLARA and F. PARONETTO, *BV functions and parabolic initial boundary value problems on domains*, Ann. Mat. Pura Appl. **188** (2009), 297–331.

[30] D. ARONSON and H. WEINBERGER, *Multidimensional nonlinear diffusion arising in population dynamics*, Adv. Math. **30** (1978), 33–76.

[31] M. ATHANASSENAS, *Volume-preserving mean curvature flow of rotationally symmetric surfaces*, Comment. Math. Helv. **72** (1997), 52–66.

[32] M. BARDI and I. CAPUZZO DOLCETTA, "Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations", Birkhäuser, Boston, 1997.

[33] G. BARLES, "Solutions de Viscosité des Équations de Hamilton-Jacobi, Mathématiques & Applications", Paris, Springer-Verlag, 1994.

[34] G. BARLES, S. BITON and O. LEY, *A geometrical approach to the study of unbounded solutions of quasilinear parabolic equations*, Arch. Ration. Mech. Anal. **162** (2002), 287–325.

[35] G. BARLES, S. BITON and O. LEY, *Uniqueness for parabolic equations without growth condition and application to the mean curvature flow in $\mathbb{R}^2$*, J. Differential Equations **187** (2003), 456–472.

[36] G. BARLES, S. BITON and O. LEY, *Uniqueness results for quasilinear parabolic equations through viscosity solutions methods*, Calc. Var. Partial Differential Equations **18** (2003), 159–179.

[37] G. BARLES and C. GEORGELIN, *A simple proof of convergence for an approximation scheme for computing motions by mean curvature*, SIAM J. Numer. Anal. **32** (1995), 484–500.

[38] G. BARLES, H.-M. SONER and P. E. SOUGANIDIS, *Front propagation and phase field theory*, SIAM J. Control Optim. **31** (1993), 439–469.

[39] G. BARLES and P. E. SOUGANIDIS, *A new approach to front propagation problems: theory and applications*, Arch. Ration. Mech. Anal. **141** (1998), 237–296.

[40] G. Bellettini, *Barriers, fattening and mean curvature flow*, Int. J. Math. Game Theory and Algebra **14** (2004), 57–71.

[41] G. Bellettini, R. Goglione and M. Novaga, *Approximation to driven motion by crystalline curvature in two dimensions*, Adv. Math. Sci. Appl. **10** (2000), 467–493.

[42] G. Bellettini and G. F. Gronchi, *Barriers for systems of ODEs: an application to the two-body problem*, Rend. Atti Acc. Naz. Sci. XL Mem. Mat. **26** (2002), 145–160.

[43] G. Bellettini, C. Mantegazza and M. Novaga, *Singular perturbations of mean curvature flow*, J. Differential Geom. **57** (2007), 403–431.

[44] G. Bellettini, M. Masala and M. Novaga, *On a conjecture of De Giorgi on the squared distance function*, J. Convex Anal. **14** (2) (2007), 353–359.

[45] G. Bellettini and M. Novaga, *Minimal barriers for geometric evolutions*, J. Differential Equations, **139** (1997), 76–103.

[46] G. Bellettini and M. Novaga, *Comparison results between minimal barriers and viscosity solutions for geometric evolutions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), **26** (1998), 97–131.

[47] G. Bellettini and M. Novaga, *A result on motion by mean curvature in arbitrary codimension*, Diff. Geom. Appl. **11** (1999), 205–220.

[48] G. Bellettini and M. Novaga, *Some aspects of De Giorgi's barriers for geometric evolutions*, In: "Calculus of Variations and Partial Differential Equations. Topics on Geometrical Evolution Problems and Degree Theory", Springer-Verlag, 1999, 115–151.

[49] G. Bellettini and M. Novaga, *Approximation and comparison for non-smooth anisotropic motion by mean curvature in R^N*, Math. Models Methods Appl. Sci. **10** (2000), 1–10.

[50] G. Bellettini, M. Novaga and G. Orlandi, *Time-like minimal submanifolds as singular limits of nonlinear wave equations*, Phys. D (6) **239** (2010), 335–339.

[51] G. Bellettini, M. Novaga and M. Paolini, *An example of three-dimensional fattening for linked space curves evolving by curvature*, Comm. Partial Differential Equations **23** (1998), 1475–1492.

[52] G. Bellettini and M. Paolini, *Two examples of fattening for the curvature flow with a driving force*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. **5** (1994), 229–236.

[53] G. Bellettini and M. Paolini, *Teoremi di confronto tra diverse nozioni di movimento secondo la curvatura media*, Atti Ac-

cad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. (9) Mat. Appl. **6** (1995), 45–54.

[54] G. BELLETTINI and M. PAOLINI, *Some results on minimal barriers in the sense of De Giorgi applied to driven motion by mean curvature*, Rend. Atti Acc. Naz. Sci. XL Mem. Mat. **19** (1995), 43–67.

[55] G. BELLETTINI and M. PAOLINI, *Quasi-optimal error estimates for the mean curvature flow with a forcing term*, Differential Integral Equations, **8** (1995), 735–752.

[56] G. BELLETTINI and M. PAOLINI, *Mean curvature flow in the context of Finsler geometry*, Hokkaido Math. J. **25** (1996), 537–566.

[57] G. BELLETTINI and M. PAOLINI, *Errata corrige to the paper: Some results on minimal barriers in the sense of De Giorgi applied to driven motion by mean curvature*, Rend. Atti Acc. Naz. Sci. XL Mem. Mat. **26** (2002), 161–165.

[58] F.A. BEREZIN and M. A. SHUBIN, "The Schrödinger Equation", Kluwer Academic Publisher, Dordrecht, 1991.

[59] M. S. BERGER and L. E. FRAENKEL, *On the asymptotic limit of a nonlinear Dirichlet problem*, J. Math. Mech. **19** (1970), 553–585.

[60] P. BERNARD, *Lasry-Lions regularization and a lemma of Ilmanen*, Rend. Sem. Mat. Univ. Padova **124** (2010), 221–229.

[61] F. BETHUEL, G. ORLANDI and D. SMETS, *Motion of concentration sets in Ginzburg-Landau equations*, Ann. Fac. Sci. Toulouse **13**, (2004), 3–43.

[62] F. BETHUEL, G. ORLANDI and D. SMETS, *Convergence of the parabolic Ginzburg-Landau equation to motion by mean curvature*, Ann. of Math. (2) **163**, (2006), 37–163.

[63] S. BITON, P. CARDALIAGUET and O. LEY, *Nonfattening condition for the generalized evolution by mean curvature and applications*, Interfaces Free Bound. **10** (2008), 1–14.

[64] V. I. BOGACHEV, "Gaussian Measures", Mathematical Surveys and Mononographs, Vol. 62, Amer. Math. Soc., 1998.

[65] A. BRAIDES, "Approximation of Free-Discontinuity Problems", Lecture Notes in Math., Vol. 1694, Springer-Verlag, 1998.

[66] A. BRAIDES, "Γ-convergence for Beginners", Oxford University Press, New York, 2002.

[67] K. A. BRAKKE, "The Motion of a Surface by its Mean Curvature", Mathematical Notes 20, Princeton University Press, Princeton, N.J., 1978.

[68] E. BRETIN, "Mouvements par courbure moyenne et méthode de la champ de phase", Ph.D. thesis, Grenoble, 2008.

[69] H. BREZIS, “Operateurs Maximaux Monotones et Semi-Goupes de Contractions dans les Espaces de Hilbert”, North-Holland Mathematics Studies, No. 5. Notas de Matemática (50). North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[70] H. BREZIS, “Functional Analysis, Sobolev Spaces and Partial Differential Equations”, Universitext. Springer, New York, 2011.

[71] L. BRONSARD and R. V. KOHN, *Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics*, J. Differential Equations **90** (1991), 211–237.

[72] L. BRONSARD and F. REITICH, *On three-phase boundary motion and the singular limit of a vector-valued Ginzburg-Landau equation*, Arch. Ration. Mech. Anal. **124** (1993), 355–379.

[73] P. CANNARSA and P. CARDALIAGUET, *Perimeter estimates for reachable sets of control systems*, J. Convex Anal. **13** (2006), 253–367.

[74] P. CANNARSA and C. SINESTRARI, “Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control”, Progress in Nonlinear Differential Equations and Their Applications, 58, Boston, Birkhäuser, 2004.

[75] P. CARDALIAGUET, *Front propagation problems with nonlocal terms, II*. J. Math. Anal. Appl. **260** (2001), 572–601.

[76] P. CARDALIAGUET and E. ROUY, *Viscosity solutions of increasing flows of sets. Application of the Hele-Shaw problem for power-law fluids*, SIAM J. Math. Anal. **38** (2006), 143-165.

[77] P. CARDALIAGUET and O. LEY, *Some flows in shape optimization*, Arch. Ration. Mech. Anal. **183** (2007), 21–58.

[78] J. CARR and R. L. PEGO, *Metastable patterns in solutions of* $u_t = \varepsilon^2 u_{xx} - f(u)$, Comm. Pure Appl. Math. **42** (1989), 523–576.

[79] A. CHAMBOLLE, *An algorithm for mean curvature flow*, Interfaces Free Bound. **6** (2004), 195–218.

[80] A. CHAMBOLLE, *An algorithm for total variation minimization and applications*, J. Math. Imaging Vision **20** (2004), 89–97.

[81] X. CHEN, *Generation and propagation of interfaces in reaction-diffusion equations*, J. Differential Equations **96** (1992), 116–141.

[82] X. CHEN, *Spectrum for the Allen-Cahn, Cahn-Hilliard, and phase-field equations for generic interfaces*, Comm. Partial Differential Equations **19** (1994), 1371–1395.

[83] X. CHEN, *Global asymptotic limit of solutions of the Cahn-Hilliard equation*, J. Differential Geom. **44** (1996), 262–311.

[84] X. CHEN and C. M. ELLIOTT, *Asymptotics for a parabolic double obstacle problem*, Proc. Roy. Soc. London Ser. A **444** (1994), 429–445.

[85] B.-L. CHEN and L. YIN, *Uniqueness and pseudolocality theorems of the mean curvature flow*, Commun. Anal. Geom. **15** (2007), 435–490.

[86] Y. G. CHEN, Y. GIGA and S. GOTO, *Uniqueness and existence of viscosity solutions of generalized mean curvature flow equation*, J. Differential Geom. **33** (1991), 749–786.

[87] D. L. CHOPP, *Computation of self-similar solutions for mean curvature flow*, Experiment. Math. **3** (1994), 1–15.

[88] K.-S. CHOU and Y.-C. KWONG, *On quasilinear parabolic equations which admit global solutions for initial data with unrestricted growth*, Calc. Var. Partial Differential Equations **12** (2001), 281–315.

[89] B. CHOW, S.-C. CHU, D. GLICKENSTEIN, C. GUENTHER, J. ISENBERG, T. IVEY, D. KNOPF, P. LU, F. LUO and L. NI, "The Ricci Flow: Techniques and Applications", part II, Analytic Aspects, Amer. Math. Soc., 2008.

[90] B. CHOW, P. LU and L. NI, "Hamilton's Ricci Flow", Graduate Studies in Mathematics, Vol. 77, Amer. Math. Soc., 2006.

[91] J. CLUTTERBUCK, "Parabolic Equations with Continuous Initial Data", Ph.D. thesis, Australian National University, 2004.

[92] T. COLDING and W. P. MINICOZZI, *II, Sharp estimates for mean curvature flow of graphs*, J. Reine Angew. Math. **574** (2004), 187–195.

[93] T. COLDING and W.P. MINICOZZI, *Minimal surfaces and mean curvature flow*, In: "Surveys in Geometric Analysis and Relativity", Adv. Lect. Math. (ALM), Vol. 20, Int. Press, Somerville, MA, 2011, 73–143.

[94] T. COLDING and W. P. MINICOZZI, *Generic mean curvature flow I; generic singularities*, Ann. of Math. **175** (2012), 755–833.

[95] T. COLDING and W. P. MINICOZZI, *Smooth compactness of self-shrinkers*, Comm. Math. Helv. **87** (2012), 463–475.

[96] M. G. CRANDALL, H. ISHII and P.-L. LIONS, *User's guide to viscosity solutions of second order partial differential equations*, Bull. Amer. Math. Soc. (N.S.) **27** (1992), 1–67.

[97] G. DAL MASO, "An Introduction to Γ-convergence", Birkhäuser, Boston, 1993.

[98] E. DE GIORGI, *Definizione ed espressione analitica del perimetro di un insieme*, Rend. Accad. Naz. Lincei, Cl. Sci. Fis. Mat. Nat. Ser. VIII **14** (1953), 390–393.

[99] E. De Giorgi, *Su una teoria generale della misura di un insieme $(r-1)$-dimensionale in uno spazio a r dimensioni*, Ann. Mat. Pura Appl. **36** (1954), 191–212.

[100] E. De Giorgi, *Frontiere orientate di misura minima*, Seminario di Matematica della Scuola Normale Superiore di Pisa, Anno Accademico 1960-61, Pisa.

[101] E. De Giorgi, *Maggiorazioni a priori relative ad ipersuperfici minimali*, Ist. Naz. di Alta Matematica, Symposia Mathematica II (1968), 283–284.

[102] E. De Giorgi, *Congetture sui limiti delle soluzioni di alcune equazioni paraboliche quasi lineari*, In: "Nonlinear Analysis: a tribute in honour of G. Prodi", A. Ambrosetti, A. Marino (eds.), Quaderni della Scuola Normale Superiore, Pisa, 1991, 173–187.

[103] E. De Giorgi, *Conjectures on limits of some quasilinear parabolic equations and flow by mean curvature*, In: "Partial Differential Equations and Related Subjects", Proceedings of a Conference dedicated to Louis Nirenberg (Trento 1992), M. Miranda (ed.), Pitman Res. Notes in Math., Vol. 269, Longman and Wiley, 1992, 85–95.

[104] E. De Giorgi, *Some conjectures on flow by mean curvature*, In: "Methods of Real Analysis and Partial Differential Equations" (Capri 1990), M. L. Benevento, T. Bruno, C. Sbordone (eds.), Quaderni dell'Accademia Pontaniana, Vol. 14, Napoli, 1992.

[105] E. De Giorgi, *New problems on minimizing movements*, In: "Boundary Value Problems for Partial Differential Equations and Applications", C. Baiocchi and J. L. Lions (eds.), (RMA Res. Notes Appl. Math.) Masson Paris (1993), 81–98.

[106] E. De Giorgi, *Congetture riguardanti barriere, superfici minime, movimenti secondo la curvatura media*, Manuscript, Lecce November 4, 1993.

[107] E. De Giorgi, *New ideas in calculus of variations and geometric measure theory*, In: "Motion by Mean Curvature and Related Topics", Proc. Intern. Conf. Trento, july 20-24, 1992, Walter De Gruyter, Berlin, 1994, 63–69.

[108] E. De Giorgi, *Barriers, boundaries, motion of manifolds*, Conference held at the Department of Mathematics of Pavia, March 18, 1994.

[109] E. De Giorgi, *Congetture riguardanti alcuni problemi di evoluzione*, Duke Math. J. **81** (1995), 255–268.

[110] E. De Giorgi, *New conjectures on flow by mean curvature*, In: "Nonlinear Variational Problems and Partial Differential Equations" (Isola d'Elba, 1990), A. Marino, M. K. Venkatesha Murthy

(eds.), Pitman Res. Notes in Math., Vol. 320, Longman 1995, 120–186.
[111] E. DE GIORGI, *Motions of partitions*, In: *Variational methods for discontinuous structures*, Vol. 25, Birkhäuser, Basel, 1996, 1–5.
[112] E. DE GIORGI and T. FRANZONI, *Su un tipo di convergenza variazionale,* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **58** (1975), 842–850.
[113] P. DEIFT and E. TRUBOWITZ, *Inverse scattering on the line*, Comm. Pure Appl. Math. **32** (1979), 121–251.
[114] M. C. DELFOUR and J.-P. ZOLÉSIO, *Shape analysis via oriented distance function*, J. Funct. Anal. **123** (1994), 129–201.
[115] M. C. DELFOUR and J.-P. ZOLÉSIO, "Intrinsic Differential Geometry and Theory of Thin Shells", Notes of the Lectures given at the Scuola Normale Superiore di Pisa, May-June 1996, Scuola Normale Superiore Pisa, Italy.
[116] M. C. DELFOUR and J.-P. ZOLÉSIO, "Shapes and Geometries. Analysis, Differential Calculus, and Optimization", Advances in Design and Control, Vol. 4, Philadelphia, PA: SIAM, 2001.
[117] A. DE MASI, E. ORLANDI, E. PRESUTTI and L. TRIOLO, *Motion by curvature by scaling non local evolution equations*, J. Stat. Phys. **73** (1993), 543–570.
[118] P. DE MOTTONI and M. SCHATZMAN, *Évolution géométrique d'interfaces*, C.R. Acad. Sci. Paris Sér. I Math. **309** (1989), 453–458.
[119] P. DE MOTTONI and M. SCHATZMAN, *Development of interfaces in N-dimensional space*, Proc. Roy. Soc. Edinburgh **116A** (1990), 207–220.
[120] P. DE MOTTONI and M. SCHATZMAN, *Geometrical evolution of developed interfaces*, Trans. Amer. Math. Soc. **347** (1995), 1533–1589.
[121] S. DESCOMBES, *Convergence of a splitting method of high-order for reaction-diffusion systems*, Math. Comp. **70** (2000), 1481–1501.
[122] N. DIRR, S. LUCKHAUS and M. NOVAGA, *A stochastic selection principle in case of fattening for curvature flow*, Calc. Var. Partial Differential Equations **13** (2001), 405–425.
[123] M. P. DO CARMO, "Riemannian Geometry", Birkhäuser, Boston, 1992.
[124] G. DZIUK and B. KAWOHL, *On rotationally symmetric mean curvature flow*, J. Differential Equations **93** (1991), 142–149.
[125] K. ECKER, *Estimates for evolutionary surfaces of prescribed mean curvature*, Math. Z. **180** (1982), 179–192.

[126] K. ECKER, "Regularity Theory for Mean Curvature Flow", In: Progress in Nonlinear Differential Equations and their Applications, Vol. 57, Birkhäuser, Boston, Inc., MA, 2004.
[127] K. ECKER and G. HUISKEN, *Mean curvature evolution of entire graphs*, Ann. of Math. **130** (1989), 453–471.
[128] K. ECKER and G. HUISKEN, *Interior estimates for hypersurfaces moving by mean curvature*, Invent. Math. **105** (1991), 547–569.
[129] M. EMINENTI and C. MANTEGAZZA, *Some properties of the distance function and a conjecture of De Giorgi*. J. Geom. Anal. **14** (2004), 267–279.
[130] S. ESEDOGLU, S. RUUTH and R. TSAI, *Threshold dynamics for higher order geometric motions*, Interfaces Free Bound. **10** (2008), 263–282.
[131] S. ESEDOGLOU and Y.-H. R. TSAI, *Threshold dynamics for the piecewise constant Mumford-Shah functional*, J. Comput. Phys. **211** (2006), 367–384.
[132] L. C. EVANS, *Convergence of an algorithm for mean curvature motion*, Indiana Univ. Math. J. **42** (1993), 533–557.
[133] L. C. EVANS, *Regularity of fully nonlinear elliptic equations and motion by mean curvature*, In: "Viscosity Solutions and Applications", Lecture Notes in Math., Vol. 1660, Springer-Verlag, 1997, 98–133.
[134] L. C. EVANS, "Partial Differential Equations", second edition, Graduate Studies in Mathematics, Vol. 19, Amer. Math. Soc., 2010.
[135] L. C. EVANS and R. F. GARIEPY, "Measure Theory and Fine Properties of Functions", CRC Press, Studies in Advanced Mathematics, 1992.
[136] L. C. EVANS, H.-M. SONER and P. E. SOUGANIDIS, *Phase transitions and generalized motion by mean curvature*, Comm. Pure Appl. Math. **45** (1992), 1097–1123.
[137] L. C. EVANS and J. SPRUCK, *Motion of level sets by mean curvature I*, J. Differential Geom. **33** (1991), 635–681.
[138] L. C. EVANS and J. SPRUCK, *Motion of level sets by mean curvature II*, Trans. Amer. Math. Soc. **330** (1992), 321–332.
[139] L. C. EVANS and J. SPRUCK, *Motion of level sets by mean curvature III*, J. Geom. Anal. **2** (1992), 121–150.
[140] L. C. EVANS and J. SPRUCK, *Motion of level sets by mean curvature IV*, J. Geom. Anal. **5** (1995), 77–114.
[141] A. FATHI and M. ZAVIDOVIQUE, *Ilmanen's lemma and insertion of $\mathcal{C}^{1,1}$ functions*, Rend. Sem. Mat. Univ. Padova **124** (2010), 203–219.

[142] H. FEDERER, *Curvature measures*, Trans. Amer. Math. Soc. **93** (1959), 418–491.

[143] H. FEDERER, "Geometric Measure Theory", Springer-Verlag, Berlin, 1969.

[144] S. FERRY, *When ϵ-boundaries are manifolds*, Fund. Math. **90** (1976), 199–210.

[145] P. C. FIFE, "Mathematical Aspects of Reacting and Diffusing Systems", Lecture Notes in Biomathematics, Vol. 28, Springer-Verlag, Berlin, 1979.

[146] F. FIERRO and M. PAOLINI, *Numerical evidence of fattening for the mean curvature flow*, Math. Models Methods Appl. Sci. **6** (1996), 793–813.

[147] R. L. FOOTE, *Regularity of the distance function*, Proc. Amer. Math. Soc. **92** (1984), 153–155.

[148] J. H. G. FU, *Tubolar neighborhoods in euclidean spaces*, Duke Math. J. **52** (1985), 1025–1046.

[149] M. GAGE and R. S. HAMILTON, *The heat equation shrinking convex plane curves*, J. Differential Geom. **23** (1986), 69–96.

[150] S. GALLOT, D. HULINE and J. LAFONTAINE, "Riemannian Geometry", Springer-Verlag, Universitext, Berlin, 2004.

[151] M. GIAQUINTA, G. MODICA and J. SOUCEK, "Cartesian Currents in the Calculus of Variations", II, Ergebnisse der Mathematik und ihrer Grenzgebiete, Vol. 38, Springer Berlin-Heidelberg, 1998.

[152] Y. GIGA, "Surface Evolution Equations", A level set approach, Monographs in Mathematics, Vol. 99, Birkhäuser Verlag, Basel, 2006.

[153] Y. GIGA, S. GOTO, H. ISHII and M. H. SATO, *Comparison principle and convexity preserving properties for singular degenerate parabolic equations on unbounded domains*, Indiana Univ. Math. J. **40** (1991), 443–470.

[154] Y. GIGA and K. YAMA-UCHI, *On a lower bound for the extinction time of surfaces moved by mean curvature*, Calc. Var. Partial Differential Equations **1** (1993), 417–128.

[155] E. GIUSTI, *On the equation of surfaces of prescribed mean curvature. Existence and uniqueness without boundary conditions*, Inv. Math. **46** (1978), 111–137.

[156] E. GIUSTI, "Minimal Surfaces and Functions of Bounded Variation", Monographs in Mathematics, Vol. 80, Birkhäuser Verlag, Basel, 1984.

[157] M. GOLUBITSKY and V. GUILLEMIN, "Stable Mappings and Theis Singularities", Graduate Text in Mathematics, Vol. 14, New-York, Heidelberg, Berlin, 1974.

[158] M. A. GRAYSON, *The heat equation shrinks embedded plane curves to round points*, J. Differential Geom. **26** (1987), 285–314.

[159] M. A. GRAYSON, *A short note on the evolution of a surface by its mean curvature*, Duke Math. J. **58** (1989), 555–558.

[160] M. GROMOV, *Sign and geometric meaning of curvature*, Rend. Sem. Mat. Fis. Milano **61** (1991), 9–123.

[161] R. GRZHIBOVSKIS and A. HEINTZ, *A convolution-thresholding approximation of generalized curvature flow*, SIAM J. Numer. Anal. **42** (2005), 2652–2670.

[162] C. GUI, *Hamiltonian identities for elliptic partial differential equations*, J. Funct. Anal. **254** (2008), 904–933.

[163] R. GULLIVER and Y. KOO, *Sharp growth rate for generalized solutions evolving by mean curvature plus a forcing term*, J. Reine Angew. Math. **538** (2001), 1–24.

[164] R. S. HAMILTON, *Four-manifolds with positive curvature operator*, J. Differential Geom. **24** (1986), 153–179.

[165] D. HENRY, "Geometric Theory of Semilinear Parabolic Equations", Lecture Notes in Math., Vol. 840, Springer-Verlag, 1981.

[166] P. D. HISLOP and I. M. SIGAL, "Introduction to Spectral Theory with Applications to Schrödinger Operators", Springer Applied Mathematical Sciences, Vol. 113, Springer, 1996.

[167] L. HÖRMANDER, "The Analysis of Linear Partial Differential Operators. III: Pseudo-Differential Operators", Springer, Berlin, 1994.

[168] G. HUISKEN, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. **20** (1984), 237–266.

[169] G. HUISKEN, *Non-parametric mean curvature evolution with boundary conditions*, J. Differential Equations **77** (1989), 369–378.

[170] G. HUISKEN, *Asymptotic behaviour for singularities of the mean curvature flow*, J. Differential Geom. **31** (1990), 285–299.

[171] G. HUISKEN, *Local and global behaviour of hypersurfaces moving by mean curvature*, Proc. of Symp. Pure Math. **54** (1993), 175–191.

[172] G. HUISKEN, *A distance comparison principle for evolving curves*, Asian J. Math. **2** (1998), 127–133.

[173] G. HUISKEN and A. POLDEN, *Geometric evolution equations for hypersurfaces*, In: "Calculus of Variations and Geometric Evolution Problems" (Cetraro 1996), Lecture Notes in Math., 1713, Springer, Berlin, 1999, 45–84.

[174] G. HUISKEN and C. SINESTRARI, *Mean curvature flow singularities for mean convex surfaces*, Calc. Var. Partial Differential Equations **8** (1999), 1–14.

[175] G. HUISKEN and C. SINESTRARI, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Acta Math. **183** (1999), 45–70.

[176] G. HUISKEN and C. SINESTRARI, *Mean curvature flow with surgeries of two-convex hypersurfaces*, Invent. Math. **175** (2009), 137–221.

[177] J. E. HUTCHINSON and Y. TONEGAWA, *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory*, Calc. Var. Partial Differential Equations **10** (2000), 49–84.

[178] T. KATO, "Perturbation Theory for Linear Operators", Die Grundleheren der Mathematischen Wissenschaften, Band 132 Springer-Verlag, New York Inc., New York, 1966.

[179] M. KIMURA and H. NOTSU, *A level set method using the signed distance function*, Jpn. J. Indust. Appl. Math. **19** (2002), 415–446.

[180] H. KOCH and T. LAMM, *Geometric flows with rough initial data*, Asian J. Math. **16** (2012), 209–236.

[181] R. KOHN and S. SERFATY, *A deterministic-control-based approach motion by mean curvature*, Comm. Pure Appl. Math. **59** (2006), 344–407.

[182] T. ILMANEN, *Generalized flow of sets by mean curvature on a manifold*, Indiana Univ. Math. J. **41** (1992), 671–705.

[183] T. ILMANEN, *The level-set flow on a manifold*, Proc. Symposia Pure Math. **54** (1993), 193–204.

[184] T. ILMANEN, *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Differential Geom. **38** (1993), 417–461.

[185] T. ILMANEN, *Elliptic Regularization and Partial Regularity for Motion by Mean Curvature*, Mem. Amer. Math. Soc. **108** (1994).

[186] T. ILMANEN, *Lectures on mean curvature flow and related equations*, Notes for the Conference on Partial Differential Equations and Applications, 1995 ICTP, Trieste.

[187] R. L. JERRARD, *Singular limits of scalar Ginzburg-Landay equations with multiple-well potentials*, Adv. Differential Equations **1** (1997), 1–38.

[188] R. L. JERRARD, *Defects in semilinear wave equations and timelike minimal surfaces in Minkowski space*, Anal. PDE **4** (2011), 285–340.

[189] R. JERRARD and H. M. SONER, *Scaling limits and regularity for a class of Ginzburg-Landau systems*, Poincaré Anal. Non Linéaire **16** (1999), 423–466.

[190] O. A. LADYZHENSKAIA, V. A. SOLONNIKOV and N. N. URAL'CEVA, "Linear and Quasilinear Equations of Parabolic Type", Amer. Math. Soc., Transl. Math. Monographs, Providence, R.I., 1968.

[191] L. D. LANDAU and L. M. LIFSHITZ, "Quantum Mechanics. Non-Relativistic Theory", Pergamon Press, 1977.

[192] J. L. LEBOWITZ, *Microscopic origin of hydrodynamic equations: derivation and consequences*, Physica A **140** (1986), 232–239.

[193] G. M. LIEBERMAN, "Second Order Parabolic Differential Equations", World Scientific, New Jersey, 1996.

[194] J. L. LIONS and E. MAGENES, Non-Homogeneous Boundary Value Problems and Applications, Die Grundlehren der Mathematischen Wissenschaften, Berlin-Heidelberg-New-York, Springer-Verlag, 1973.

[195] Q. LIU, *Fattening and comparison principle for level-set equation of mean curvature type*, SIAM J. Control Optim. **49** (2011), 2518–2541.

[196] S. LUCKHAUS and T. STURZENHECKER, *Implicit time discretization for the mean curvature flow equation*, Calc. Var. Partial Differential Equations **3** (1995), 253–271.

[197] A. LUNARDI, "Analytic Semigroups and Optimal Regularity in Parabolic Problems", Birkhäuser, Boston, 1995.

[198] F. MAGGI, "Sets of Finite Perimeter and Geometric Variational Problems. An Introduction to Geometric Measure Theory", Cambridge Studies in Advanced Mathematics, Vol. 135, Cambridge University Press, 2012.

[199] A. MAGNI and C. MANTEGAZZA, *Some remarks on Huisken's monotonicity formula for mean curvature flow*, In: "Singularities in Nonlinear Evolution Phenomena and Applications", M. Novaga and G. Orlandi (eds.), CRM Ser. Center "Ennio De Giorgi", Pisa, Vol. 9, 2009, 157–169.

[200] A. MAGNI and C. MANTEGAZZA, *A note on Grayson's theorem*, Rend. Sem. Mat. Univ. Padova, to appear.

[201] A. MAGNI and M. RÖGER, *Variational analysis of a mean curvature flow action functional*, 2012, arxiv:1304.

[202] C. MANTEGAZZA, *Smooth geometric evolutions of hypersurfaces*, Geom. Funct. Anal. **12** (2002), 138–182.

[203] C. MANTEGAZZA, "Lecture Notes on Mean Curvature Flow", Progress in Mathematics, Vol. 290, Birkhäuser/Springer Basel AG, Basel, 2011.

[204] C. MANTEGAZZA, *Notes on the distance function from a submanifold*, available at http://cvgmt.sns.it/cgi/get.cgi/papers/man10/

[205] C. MANTEGAZZA and L. MARTINAZZI, *A note on quasilinear parabolic equations on manifolds*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **11** (2012), 1–18.

[206] C. MANTEGAZZA and A. C. G. MENNUCCI, *Hamilton-Jacobi equations and distance functions on Riemannian manifolds*, Appl. Math. Opt. **47** (2003), 1–25.

[207] C. MANTEGAZZA, M. NOVAGA and V. M. TORTORELLI, *Motion by curvature of planar networks*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **3** (2004), 235–324.

[208] U. MASSARI and M. MIRANDA, "Minimal Surfaces in Codimension One", Notas de Matemática, North-Holland, Amsterdam, 1984.

[209] B. MERRIMAN, J. K. BENCE and S. J. OSHER, *Diffusion generated motion by mean curvature*, In: "Computational Crystal Growers Workshop", J. E. Taylor (ed.), Selected Lectures in Math., Amer. Math. Soc., Providence, Rhode Island, 1992.

[210] M. MIRANDA, *Una maggiorazione integrale per le curvature delle ipersuperfici minimali*, Rend. Sem. Mat. Univ. Padova **38** (1967), 91–107.

[211] M. MIRANDA, *Movimento di superfici: approccio intrinseco*, Rend. Sem. Mat. Fis. Milano **62** (1992), 127–136.

[212] M. MIRANDA JR., D. PALLARA, F. PARONETTO and M. PREUNKERT, *Short-time heat flow and functions of bounded variation in $\mathbb{R}^N$*, Ann. Fac. Sci. Toulouse Math. **161** (2007), 125–145.

[213] L. MODICA, *A gradient bound and a Liouville theorem for nonlinear Poisson equations*, Comm. Pure Appl. Math. **38** (1985), 679–684.

[214] L. MODICA, *Monotonicity of the energy for entire solutions of semilinear elliptic equations*, In: "Partial Differential Equations and the Calculus of Variations", Progr. Nonlinear Differential Equations Appl., Vol. 2, Birkhäuser, Boston, MA, 1989, 843–850.

[215] L. MODICA and S. MORTOLA, *Un esempio di Γ-convergenza*, Boll. Un. Mat. Ital. B 5, **14** (1977), 285–299.

[216] L. MUGNAI and M. RÖGER, *The Allen-Cahn action functional in higher dimensions*, Interfaces Free Bound. **10** (2008), 45–78.

[217] L. MUGNAI and M. RÖGER, *Convergence of perturbed Allen-Cahn equations to forced mean curvature flow*, Indiana Univ. Math. J. **59** (2001), 41–76.

[218] J. C. NEU, *Kinks and the minimal surface equation in Minkowski space*, Physica D **43** (1990), 421–434.

[219] X. H. NGUYEN, *Construction of complete embedded self-similar surfaces under mean curvature flow*, Trans. Amer. Math. Soc. **361** (2009), 1683–1701.

[220] R. NOCHETTO, M. PAOLINI and C. VERDI, *Sharp error analysis for curvature dependent evolving fronts*, Math. Models Methods Appl. Sci. **3** (1993), 711–723.

[221] R. NOCHETTO, M. PAOLINI and C. VERDI, *Optimal interface error estimate for the mean curvature flow*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **21** (1994), 193–212.

[222] R. NOCHETTO, M. PAOLINI and C. VERDI, *Double obstacle formulation with variable relaxation parameter for smooth geometric front evolutions: asymptotic interface error estimates*, Asymptot. Anal. **10** (1995), 173–198.

[223] R. NOCHETTO and C. VERDI, *Convergence past singularities for a fully discrete approximation of curvature-driven interfaces*, SIAM J. Num. Anal. **34** (1997), 490–512.

[224] T. OHTA, D. JASNOW and K.KAWASAKI, *Universal scaling in the motion of random interfaces*, Phys. Rev. Lett. **49** (1982), 1223–1226.

[225] S. OSHER and J. SETHIAN, *Fronts propagating with curvature dependent speed*, J. Comput. Phys. **79** (1988), 12–49.

[226] N. OWEN, J. RUBINSTEIN and P. STERNBERG, *Minimizers and gradient flows for singularly perturbed bi-stable potentials with a Dirichlet condition*, Proc. Roy. Soc. London A **429** (1990), 505–532.

[227] M. PAOLINI, *A quasi-optimal error estimate for a discrete singularly perturbed approximation to the prescribed curvature problem*, Math. Comp. **66** (1997), 45–67.

[228] M. PAOLINI, *Fattening in two dimensions obtained with a nonsymmetric anisotropy: numerical simulations*, Acta Math. Univ. Comenianae **67** (1998), 43–55.

[229] M. PAOLINI and C. VERDI, *Asymptotic and numerical analyses of the mean curvature flow with a space-dependent relaxation parameter*, Asymptot. Anal. **5** (1992), 553–574.

[230] M. H. PROTTER and H. F. WEINBERGER, “Maximum Principles in Differential Equations”, Springer-Verlag, New-York, 1984.

[231] M. REED and B. SIMON, "Methods of Modern Mathematical Physics", Vol. I, II, III, IV. Academic Press, New-York, 1980.
[232] R. T. ROCKAFELLAR, "Convex Analysis", Princeton University Press, Princeton, 1970.
[233] M. RÖGER and R. SCHÄTZLE, *On a modified conjecture of De Giorgi*, Math. Z. **254** (2006), 675–714.
[234] J. RUBINSTEIN, P. STERNBERG and J. B. KELLER, *Fast reaction, slow diffusion and curves shortening*, SIAM J. Appl. Math. **49** (1989), 116–133.
[235] J. RUBINSTEIN, P. STERNBERG and J. B. KELLER, *Reaction diffusion processes and evolution to harmonic maps*, SIAM J. Appl. Math. **49** (1989), 1722–1733.
[236] N. SATO, *A simple proof of convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, Indiana Univ. Math. J. (2008), 1741–1749.
[237] M. SCHATZMAN, *On the stability of the saddle solution of Allen-Cahn's equation*, Proc. Roy. Soc. Edinburgh **125A** (1995), 1241–1275.
[238] O. C. SCHNÜRER, "Geometric Evolution Equations", Lecture Notes, Albach, 2007.
[239] M. SÁEZ TRUMPER and O. C. SCHNÜRER, *Mean curvature flow without singularities*, arxiv:1210.6007
[240] S. SERFATY, *Gamma-convergence of gradient flows with applications to Ginzburg-Landau*, Comm. Pure Appl. Math. **57** (2004), 1627–1672.
[241] S. SERFATY, *Γ-convergence of gradient flows on Hilbert and metric spaces and applications*, Discrete Contin. Dyn. Syst. **31** (2011), 1427–1451.
[242] L. SIMON, "Lecture Notes on Geometric Measure Theory", Australian National University, Center for Mathematical Analysis, Canberra, 1983.
[243] L. SIMON, *Survey lectures on minimal submanifolds*, In: "Seminar on Minimal Submanifolds", Bombieri (ed.), Princeton Univ. Press, 1984.
[244] C. SINESTRARI, *Formation of singularities in the mean curvature flow*, In: "Mean Curvature Flow and Isoperimetric Inequalities", V. Miquel and J. Porti (eds.), Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2010, 3–43.
[245] D. SLEPCEV, *On level set approach to motion of manifolds of arbitrary codimension*, Interfaces Free Bound. **5** (2003), 417–458.
[246] J. SMOLLER, "Shock Waves and Reaction-Diffusion Equations", Springer-Verlag, New York, 1983.

[247] J. Sokolowski and J.-P. Zolésio, "Introduction to Shape Optimization: shape sensitivity analysis", Springer-Verlag, Berlin, 1992.

[248] H.-M. Soner, *Motion of a set by the curvature of its boundary*, J. Differential Equations **101** (1993), 313–372.

[249] H.-M. Soner, *Ginzburg-Landau equation and motion by mean curvature, I. Convergence*, J. Geom. Anal. **7** (1997), 437–446.

[250] H.-M. Soner, *Ginzburg-Landau equation and motion by mean curvature, II. Development of the initial interface*, J. Geom. Anal. **7** (1997), 477–491.

[251] H.-M. Soner, *Front propagation*, In: "Boundaries, Interfaces and Transitions", CRM Proc. and Lecture Notes, M. C. Delfour (ed.), Amer. Math. Soc., **13** (1998), 185–206.

[252] H.-M. Soner and P. E. Souganidis, *Singularities and uniqueness of cylindrically symmetric surfaces moving by mean curvature*, Comm. Partial Differential Equations **18** (1993), 859–894.

[253] H.-M. Soner and N. Touzi, *A stochastic representation for mean curvature type geometric flows*, Ann. Prob. **31** (2003), 1145–1165.

[254] P. E. Souganidis and N. K. Yip, *Uniqueness of motion by mean curvature perturbed by stochastic noise*, Ann. Inst. H. Poincaré Anal. Non Linéaire **21** (2004), 1–23.

[255] M. Spivak, "A Comprehensive Introduction to Differential Geometry", Publish or Perish, Boston, 1975.

[256] H. Spohn, *Interface motion in models with stochastic dynamics*, J. Stat. Phys. **71** (1992), 1081–1132.

[257] A. Stone, *A density function and the structure of singularities of the mean curvature flow*, Calc. Var. Partial Differential Equations **2** (1994), 443–480.

[258] M. Struwe, *Geometric evolution problems*, In: "Nonlinear Partial Differential Equations in Differential Geometry", R. Hardt *et al.* (eds.), IAS/Park City Math. Ser. 2, Amer. Math. Soc., 1996, 259–339.

[259] A.E. Taylor, "Introduction to Functional Analysis", John Wiley & Sons, Inc. London, 1958.

[260] Y. Tonegawa, *Phase field model with a variable chemical potential*, Proc. Roy. Soc. Edinburgh Sec. A **132** (2001), 993–1019.

[261] Y. Tonegawa, *Integrality of varifolds in the singular limit of reaction-diffusion equations*, Hiroshima Math. J. **33** (2003), 323–341.

[262] J. J. L. Velázques, *Curvature blow-up in perturbation of minimal cones evolving by mean curvature*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **21** (1994), 595–628.

[263] M.-T. Wang, *Long-time existence and convergence of graphic mean curvature flow in arbitrary codimension*, Invent. Math. **148** (2002), 525–543.

[264] M.-T. Wang, *Gauss maps of the mean curvature flow*, Math. Res. Lett. **10** (2003), 287–299.

[265] X.-J. Wang, *Convex solutions to the mean curvature flow*, Ann. of Math. **173** (2011), 1185–1239.

[266] B. White, *Some questions of De Giorgi about mean curvature flow of triply-periodic surfaces*, In: "Motion by Mean Curvature and Related Topics", Proc. of the Intern. Conf. held in Trento, july 20-24, 1992, Walter De Gruyter, Berlin, 1994, 210–213.

[267] B. White, *The size of the singular set in mean curvature flow of mean convex surfaces*, J. Amer. Math. Soc. **13** (2000), 665–695.

[268] B. White, *Evolution of curves and surfaces by mean curvature*, Proc. Int. Congress Math., Vol I, Beijing (2002), 525–538.

[269] B. White, *The nature of singularities in mean curvature flow of mean convex surfaces*, J. Amer. Math. Soc. **16** (2003), 123–138.

[270] B. White, *A local regularity theorem for mean curvature flow*, Ann. of Math. **161** (2005), 1487–1519.

[271] J. Wloka, "Partial Differential Equations", Cambridge Univ. Press, Cambridge, 1987.

[272] X. P. Zhu, "Lectures on Mean Curvature Flow", AMS/IP Studies in Advanced Mathematics, Vol. 32, Amer. Math. Soc., Providence RI, 2002.

[273] J.-P. Zolésio, *Shape analysis and weak flow*, In: "Optimal Shape Design", Springer-Verlag, 2000, 157–341.

Index

E

F

G

H

I

L

V

W

Symbols

α: formula (15.21).
$\alpha^{\pm}$: Section 15.
A_ϵ: Section 16.
$\mathrm{A}(t)$: Section 17.1.
$A(t,s)$: formula (3.21).
$\mathrm{A}_\epsilon(t)$: Section 17.2.
$\mathcal{A}_0$: Section 16.1.

$\overline{B}$: closure of $B \subseteq \mathbb{R}^n$.
$\mathcal{B}(\mathfrak{F})$: Definition 9.2.
$\mathcal{B}(\mathfrak{F}, J)$: Definition 9.2.
$\mathcal{B}(r, \mathfrak{F})$: Definition 9.44.
$\mathcal{B}_\epsilon(u_\epsilon, \theta)$: Section 17.7.
$B_\rho(z)$: open ball of radius ρ centered at z.
B_ρ^-: ρ-enlarged of the Borel set $\mathbb{R}^n \setminus B$, formula (1.3).
B_ρ^+: ρ-enlarged of the Borel set B, formula (1.3).

$\mathcal{C}^{1,1}$: Definition 12.11.
$\mathrm{co}(F)$: closed convex hull of the set $F \subset \mathbb{R}^n$.
$\mathcal{X}([a,b]; \mathrm{Imm}(\mathcal{S}; \mathbb{R}^n))$: Definition 3.5.
$\mathcal{X}([a,b]; \mathrm{Emb}(\mathcal{S}; \mathbb{R}^n))$: Definition 3.5.

∂B: topological boundary of $B \subseteq \mathbb{R}^n$
$d(\cdot, E) = d(\cdot)$: signed distance from the boundary of B negative in the interior of B: formula (1.6). When no confusion is possible, we write $d(\cdot) = d(\cdot, z)$.
δ operator, Definition 1.6.
$d(\cdot, f(t))$: signed distance function from the flow f negative inside $f(t)$.
$d_\epsilon(t,z)$: formula (16.6).
$d_\epsilon^{\pm}(t,z)$: modified signed distance functions, formula (17.19).
$\mathrm{diam}(F)$: diameter of the set $F \subset \mathbb{R}^n$, Section 12.1.
$\mathrm{dist}(\cdot, B)$: distance from the set $B \subseteq \mathbb{R}^n$.

$\mathrm{div}^\Sigma X$: tangential divergence of X, Definition 1.28.
d/dt: material derivative, see Definition 3.34.
$\delta_h X_h$: δ-divegence, formula (1.13).
Δ^Σ: tangential Laplacian, Definition 1.30
$d\xi_{\epsilon,v}$: discrepancy measures, Section 17.7.
$d\xi^t_{\epsilon,v}$: discrepancy measures, Section 17.7.
du: differential of the function u, Chapter 1.

$E_\epsilon(t)$: formulas (16.5) and (17.13).
$\eta = d^2/2$: squared distance function.
E^r: formula (9.51).
$E^\pm_\rho$: formula (1.3).
expansion: inner and outer, Chapter 16.
$\mathrm{Emb}(\mathcal{S}; \mathbb{R}^n)$: Definition 1.33

F: one-dimensional unscaled Allen–Cahn's functional: Section 15.1.
$\mathfrak{F}$: Definition 9.1.
$\mathcal{F}$: Definition 9.4.
f^c: complement of a smooth flow, $f^c := \overline{\mathbb{R}^n \setminus f}$, Chapter 3.
$\mathcal{F}_c$: Definition 9.5.
$\mathcal{F}_\epsilon$: scaled Allen–Cahn's functionals, formula (15.10).
$\underline{F}(u, M)$: formula (7.1)
$\overline{\phi}$: formula (9.4).
φ_ϵ: Section 16.
φ_0: Section 16.1.
φ_1: Section 16.1.
ϕ^r: formula (9.52).
$\phi^\pm_\rho$: formula (9.5).
$\overline{F}(u, M)$: formula (7.1).

γ: one-dimensional minimizer, formulas (15.17), (15.18), (15.19).
$\gamma_\pm$: formula (15.4).
γ_ϵ: Section 17.1.
$\overline{\gamma}_\epsilon$: Section 17.6.
$G := \left(\mathrm{Id} - d\nabla^2 d\right)^{-1}$: formula (6.7).
$g_{\alpha\beta}$: metric tensor, formula (1.49).

$h_{\alpha\beta}$: formula (3.21).
$\mathrm{H}(t, s)$: formula (3.22).
$\mathbf{H}(t, s)$: formula (3.22).
H_ϵ: Section 16.
H_0: Section 16.1.

$\mathfrak{O}^{\pm}_{\epsilon\pm}(t)$: Section 17.2.
ω_n Lebesgue measure of the n-dimensional unit ball in $\mathbb{R}^n$.

π_k: k-component of the identity, Example 1.7.
p_γ: cubic polynomial, Definition 17.2.
p_ξ: cubic polynomial, Definition 17.10.
$\mathcal{P}(\mathbb{R}^n)$: set of all subsets of $\mathbb{R}^n$, Chapter 1.
$P_{N_x(\partial E)}$: orthogonal projection on the normal line $N_x(\partial E)$ to ∂E at $x \in \partial E$, Section 1.1.
$P_{T_x(\partial E)}$: orthogonal proyection on the tangent space $T_x(\partial E)$ to ∂E at $x \in \partial E$, Section 1.1.
$\mathrm{pr}(\cdot, B)$: projection map, Chapter 1.
$\mathrm{pr}_F(z)$: formula (1.1).

q_ϵ: Remark 17.22.
$q_\epsilon(t, z)$: Section 17.7.
$Q^{\pm}_\epsilon$: Section 17.2.
$\mathfrak{q}^r(\partial_s \mathrm{H})$: Example 3.33.

$R(E, t_0)$: comparison flow, Definition 14.1.
$\rho_{(t_0,z_0)}$: heat-type kernel of Huisken's monotonicity formula, Definiton 4.3.

$\mathcal{S}$: parameter space.
Σ; occasionally, if $\partial E \in \mathcal{C}^\infty$, we let $\Sigma = \partial E$.
s_ϵ: Definition 16.1.
s_0: Section 16.1.
σ_0: surface tension, formula (15.12).
s_1: Section 16.1. topological boundary of C.
$\mathbb{S}^{n-1} := \{z \in \mathbb{R}^n : |z| = 1\}$.
$\mathbb{S}$: Section 9.5.
$\mathrm{Sym}(n, \mathbb{R})$: $(n \times n)$-real symmetric matrices, Chapter 7.

$t_{\max}(\partial E)$: maximal existence time, Definition 7.10.
$\mathcal{T}_\epsilon(u_\epsilon, \theta)$: Section 17.7.
$T_\epsilon(v)$: ϵ-stress tensor: Definition 17.24.
$T^t_\epsilon(v)$: Section 17.7.
Θ: Definition 17.11.
$\vartheta(t)$: formula (17.18).

$\overline{u}$: canonical extension of u, formulas (1.42) and (3.48).
u_ϵ: solution to (15.2).
U_ϵ: formula (16.33).

U_0: Section 16.3.
U_1: Section 16.3.
U_2: Section 16.3.
u_ϵ^0: initial datum, see formula (15.1) (well-prepared in formula (17.6)).
u_0: Section 16.2.
u_1: Section 16.2.
u_2: Section 16.2.

V: formula (3.11).
V: formula (3.7), Definition 3.7, formula (3.8), Proposition 3.8.
$v_\epsilon^\pm$: sub/supersolutions, Definition 17.11.
V_ϵ: Chapter 16.
V_0: Section 16.1.
V_1: Section 16.1.
$\mathbf{V}_\epsilon$: Chapter 16.

W: double-well potential, Chapter 15, see formula (15.3).
w, with $W = \frac{1}{2}w^2$: Chapter 15.
$\mathcal{W}_\epsilon(u_\epsilon, \theta)$: Section 17.7.

$\overline{X}$: canonical extension of X, Definition 1.27.
ξ: shape function, Section 15.3.
ξ_ϵ: modified shape function, Definition 17.10.
$\xi_\epsilon(v)$: discrepancy function, Definition 17.25.
$\xi_\epsilon^t(v)$: discrepancy function, Definition 17.25.

$y_\epsilon^\pm(t, z)$: stretched variables, Definition 17.9.

LECTURE NOTES

This series publishes polished notes dealing with topics of current research and originating from lectures and seminars held at the Scuola Normale Superiore in Pisa.

Published volumes

1. M. TOSI, P. VIGNOLO, *Statistical Mechanics and the Physics of Fluids*, 2005 (second edition). ISBN 978-88-7642-144-0
2. M. GIAQUINTA, L. MARTINAZZI, *An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs*, 2005. ISBN 978-88-7642-168-8
3. G. DELLA SALA, A. SARACCO, A. SIMIONIUC, G. TOMASSINI, *Lectures on Complex Analysis and Analytic Geometry*, 2006. ISBN 978-88-7642-199-8
4. M. POLINI, M. TOSI, *Many-Body Physics in Condensed Matter Systems*, 2006. ISBN 978-88-7642-192-0

   P. AZZURRI, *Problemi di Meccanica*, 2007. ISBN 978-88-7642-223-2
5. R. BARBIERI, *Lectures on the ElectroWeak Interactions*, 2007. ISBN 978-88-7642-311-6
6. G. DA PRATO, *Introduction to Stochastic Analysis and Malliavin Calculus*, 2007. ISBN 978-88-7642-313-0

   P. AZZURRI, *Problemi di meccanica*, 2008 (second edition). ISBN 978-88-7642-317-8

   A. C. G. MENNUCCI, S. K. MITTER, *Probabilità e informazione*, 2008 (second edition). ISBN 978-88-7642-324-6
7. G. DA PRATO, *Introduction to Stochastic Analysis and Malliavin Calculus*, 2008 (second edition). ISBN 978-88-7642-337-6
8. U. ZANNIER, *Lecture Notes on Diophantine Analysis*, 2009. ISBN 978-88-7642-341-3
9. A. LUNARDI, *Interpolation Theory*, 2009 (second edition). ISBN 978-88-7642-342-0

10. L. AMBROSIO, G. DA PRATO, A. MENNUCCI, *Introduction to Measure Theory and Integration*, 2012.
    ISBN 978-88-7642-385-7, e-ISBN: 978-88-7642-386-4
11. M. GIAQUINTA, L. MARTINAZZI, *An Introduction to the Regularity Theory for Elliptic Systems, Harmonic Maps and Minimal Graphs*, 2012 (second edition). ISBN 978-88-7642-442-7, e-ISBN: 978-88-7642-443-4
    G. PRADISI, *Lezioni di metodi matematici della fisica*, 2012.
    ISBN: 978-88-7642-441-0
12. G. BELLETTINI, *Lecture Notes on Mean Curvature Flow, Barriers and Singular Perturbations*, 2013.
    ISBN 978-88-7642-428-1, e-ISBN: 978-88-7642-429-8

Volumes published earlier

G. DA PRATO, *Introduction to Differential Stochastic Equations*, 1995 (second edition 1998). ISBN 978-88-7642-259-1

L. AMBROSIO, *Corso introduttivo alla Teoria Geometrica della Misura ed alle Superfici Minime*, 1996 (reprint 2000).

E. VESENTINI, *Introduction to Continuous Semigroups*, 1996 (second edition 2002). ISBN 978-88-7642-258-4

C. PETRONIO, *A Theorem of Eliashberg and Thurston on Foliations and Contact Structures*, 1997. ISBN 978-88-7642-286-7

Quantum cohomology at the Mittag-Leffler Institute, a cura di Paolo Aluffi, 1998. ISBN 978-88-7642-257-7

G. BINI, C. DE CONCINI, M. POLITO, C. PROCESI, *On the Work of Givental Relative to Mirror Symmetry*, 1998. ISBN 978-88-7642-240-9

H. PHAM, *Imperfections de Marchés et Méthodes d'Evaluation et Couverture d'Options*, 1998. ISBN 978-88-7642-291-1

H. CLEMENS, *Introduction to Hodge Theory*, 1998. ISBN 978-88-7642-268-3

Seminari di Geometria Algebrica 1998-1999, 1999.

A. LUNARDI, *Interpolation Theory*, 1999. ISBN 978-88-7642-296-6

R. SCOGNAMILLO, *Rappresentazioni dei gruppi finiti e loro caratteri*, 1999.

S. RODRIGUEZ, *Symmetry in Physics*, 1999. ISBN 978-88-7642-254-6

F. STROCCHI, *Symmetry Breaking in Classical Systems*, 1999 (2000). ISBN 978-88-7642-262-1

L. AMBROSIO, P. TILLI, *Selected Topics on* "Analysis in Metric Spaces", 2000. ISBN 978-88-7642-265-2

A. C. G. MENNUCCI, S. K. MITTER, *Probabilità ed Informazione*, 2000.

S. V. BULANOV, *Lectures on Nonlinear Physics*, 2000 (2001). ISBN 978-88-7642-267-6

Lectures on Analysis in Metric Spaces, a cura di Luigi Ambrosio e Francesco Serra Cassano, 2000 (2001). ISBN 978-88-7642-255-3

L. CIOTTI, *Lectures Notes on Stellar Dynamics*, 2000 (2001).
ISBN 978-88-7642-266-9

S. RODRIGUEZ, *The Scattering of Light by Matter*, 2001.
ISBN 978-88-7642-298-0

G. DA PRATO, *An Introduction to Infinite Dimensional Analysis*, 2001.
ISBN 978-88-7642-309-3

S. SUCCI, *An Introduction to Computational Physics*: – Part I: *Grid Methods*, 2002. ISBN 978-88-7642-263-8

D. BUCUR, G. BUTTAZZO, *Variational Methods in Some Shape Optimization Problems*, 2002. ISBN 978-88-7642-297-3

A. MINGUZZI, M. TOSI, *Introduction to the Theory of Many-Body Systems*, 2002.

S. SUCCI, *An Introduction to Computational Physics*: – Part II: *Particle Methods*, 2003. ISBN 978-88-7642-264-5

A. MINGUZZI, S. SUCCI, F. TOSCHI, M. TOSI, P. VIGNOLO, *Numerical Methods for Atomic Quantum Gases*, 2004. ISBN 978-88-7642-130-0