Convergence Theories of the Mean Curvature Flow

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Thesis Defense Presentation
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June 10, 2021

Mean Curvature Flow

Definition

hypersurfaces $M_t \subset \mathbb{R}^n$ evolving by

$$\frac{\partial \vec{x}}{\partial t} = -H\vec{n}$$

where H is the mean curvature, $\vec{n}$ is the unit normal of M_t at $\vec{x}$.

First studied in material science.
Applications: Topological sphere theorem, Riemann Penrose inequality.

Mean Curvature Flow with Free Boundary

We can consider MCF for manifolds with boundary. Here we focus on the free boundary condition where boundary of the evolving manifolds moves freely in a prescribed barrier and the manifolds is orthogonal to the barrier. Let $(\bar{M}, \bar{g})$ be an $(n+1)$-dimensional Riemannian manifold with the Levi-Civita connection $\bar{\nabla}$. Let Σ be a two-sided smooth n-dimensional manifold with non-empty boundary $\partial\Sigma$. A smooth immersion $F\colon \Sigma \to \bar{M}$ defines a free boundary hypersurface if $F(\partial\Sigma) \subset S$ and $F_* N = \nu_S \circ F$ where N is the outward unit normal of $\partial\Sigma \subset \Sigma$ with respect to the metric induced from $\bar{M}$ by F.

Convergence of Mean Curvature Flow

The nonlinear nature of geometric flows leads to the possible appearance of singularities. By Huisken's monotocity formula, singularities are modeled after self-similar solutions. One of the most important models is the shrinker which evolves only homothetically under the flow. For example, the round sphere is an example of shrinkers. There are many examples of shrinkers which make it hard to classify singularity models completely.
One possible direction is to find certain initial geometric assumption that forces hypersurfaces converge to simple singularities, say, the singularity of shrinking spheres.

Theorem

A compact, convex hypersurface in $\mathbb{R}^n$ converges to a round point under MCF.

General Strategies for Proving Convergence

Theorem

A compact, convex hypersurface in $\mathbb{R}^n$ with free boundary on a sphere converges to a round half-point under MCF.

1. Preservation of convexity
2. Pinching estimate
   Show that the quantity

$$|A|^2 - \frac{1}{n}H^2 = \frac{1}{n}\sum_{i<j}^{n}(\kappa_i - \kappa_j)^2$$

which measures the sum of differences between eigenvalues κ_i of the second fundamental form A is relatively small.

Main Results

In this thesis, we focus on mean curvature flow with free boundary in an ambient Riemannian manifolds $\bar{M}$ and obtain the following two results:

- compute the boundary derivative of the second fundamental form
- establish an iteration scheme for the flow.

Boundary Derivatives

Traditionally, the boundary derivatives are computed by fixing a local coordinate system. Since the boundary is involved, one needs to carefully choose the coordinate system to simplify computations. We work with the tangent and normal bundles over the space-time domain $\Sigma \times [0, T)$. In this way, we can compute the boundary derivative of the second fundamental form. The only new term brounght by the ambient geomtry is the last term, so we expect to control the boundary derivative in a similar manner as long as the ambient geometry is assumed to be preserved.

Theorem

Let $p \in \partial\Sigma$. For $u, v \in T_p\partial\Sigma$,

$$\begin{aligned}\nabla_N h(u,v) = &\left(\nabla_{F_*u} A^S(\iota\nu, F_*v) + A^S(\bar{\nabla}^S_{F_*u}\iota\nu, F_*v)\right)\nu \\ &+ A^S(F_*u, F_*v)h(N,N) - h(\nabla_u N, v) \\ &+ A^S(\iota\nu, \iota\nu)h(u,v) + \overset{\perp}{\pi}(F^* R_\nabla(u,N)(F_*v)).\end{aligned}$$

Stampacchia's Iteration

We prove that if a function on the hypersurface evolving under the MCF equation satisfies two special inequalities, then this function can be uniformly bounded in spacetime. Most of the arguments are similar to the proof of Edelen where the major differences are the use of a Michael–Simon type inequality for Riemannian submanifolds and the dependence of the uniform bound on ambient geometry.

Future Directions—Convergence Theory

The boundary derivatives are essential for applying maximum principles to prove that certain inequalities are preserved under the flow. When the barrier surface is not umbilic, cross terms will appear in the boundary derivatives. These terms are impossible to control; thus making maximum principles not applicable.

To cancel problematic cross terms, one could use a perturbation argument of the second fundamental form. We need to examine the requirement on the ambient geometry and the barrier to make a valid perturbation.

Future Directions—Higher Codimension

The covariant formulation used to compute boundary derivatives was introduced by Andrews and Baker to study MCF in higher codimensions. They proved that higher-codimension submanifolds satisfying certain non-convex initial condition converge to a round point under MCF.

We are interested in generalize the above result to the free boundary case. What makes the generalization complicated is that the dimension of the barrier may vary and affects the nature of the problem drastically.

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