Convergence Theories of the Mean Curvature Flow

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Mean curvature flow, first formulated by Mullins [33], describes the evolution of surfaces moving along its mean curvature vector. Among various boundary conditions for mean curvature flow with boundary, the free boundary condition is one of the most extensively studied formulations. Mean curvature flow with free boundary is of great significance and relates closely to free boundary minimal surfaces and constrained motion of liquid interfaces.

This thesis provides a theoretical foundation for the convergence theory of mean curvature flow with free boundary in Riemannian ambient manifolds. By reviewing the classical method by Huisken[25], we highlight the importance of an iteration scheme for proving the pinching estimate of the traceless second fundamental form. Next, we calculate the boundary derivative of the second fundamental form under the covariant formulation of mean curvature flow introduced by Andrews and Baker [6]. We also generalize [15, Theorem 3.1], an iteration scheme for proving uniform bound of functions satisfying two special inequalities, to Riemannian manifolds as the ambient space. The results of this thesis will help generalize the convergence result by Hirsch and Li [20] to mean curvature flow with free boundary in Riemannian manifolds.

摘要

平均曲率流首先由Mullins[33]公式化，描述了沿其平均曲率矢量移動的曲面的演變。在具有邊界的曲面的平均曲率流中，具有自由邊界的平均曲率流受到了最廣泛的研究。這類平均曲率流非常重要，並且與自由邊界最小曲面和液體界面的受約束運動密切相關。

本文旨在為黎曼流形中具有自由邊界的平均曲率流的收斂理論提供理論基礎。通過回顧Huisken[25]的經典方法，我們強調了迭代方案對於證明無跡第二基本形式的收縮估計的重要性。接下來，我們利用Andrews和Baker[6]引入的平均曲率流的協變量公式，計算第二基本形式的邊界導數。我們還將[15, Theorem 3.1]推廣到黎曼流形作為環境空間的迭代方案，該迭代方案用於證明滿足兩個特殊不等式的函數的統一界。本文的結果有助於將Hirsch與Li[20]的收斂結果推廣到黎曼流形中的自由邊界平均曲率流。

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Contents

Chapter 1

Introduction

1.1 Background

The last few decades have witnessed significant development in the field of geometric flow, which leads to many remarkable accomplishments in geometry, topology, physics, and computer vision. Among various geometric flows, mean curvature flow (MCF) is one of the most important geometric flows for submanifolds of Riemannian manifolds. One way of understanding MCF is to regard it as the negative gradient flow for the area functional. In other words, a surface is deforming along MCF to decrease its area as fast as possible.

The study of MCF and other related flows is a crucial area of mathematics. Not only does it lead to a series of significant results in physics and mathematics, but it is also expected to solve some long-standing conjectures in geometry and topology. In 1994, Andrews [5] applied the harmonic MCF to provide a new proof for the topological sphere theorem. Moreover, regarded as possible evidence to the cosmic censorship conjecture in General Relativity, the Riemannian Penrose inequality was proved by Huisken and Ilmanen [27] using the method of inverse MCF. For further applications, inspired by similarities between Ricci flow and MCF and the resolution of Thurston's Geometrization Conjecture by Perelman using the Ricci flow, mathematicians believe that MCF could be a possible approach to the Schoenflies Conjecture in geometric topology [13, 8].

The nonlinear nature of geometric flows leads to the possible appearance of singularities. Mathematicians have developed various methods to continue the flow through singularities. In the framework of differential geometry, by analogy with Hamilton [18] and Perelman's [36, 35] construction of Ricci flow with surgery, mathematicians attempted to perform surgeries before the formation of singularities to continue the flow. Huisken and Sinestrari [22] applied this idea successfully to 2-convex hypersurfaces under MCF in $\mathbb{R}^n (n \geq 4)$. A few years later, Brendle and Huisken [11] managed to extend the result to mean convex surfaces in $\mathbb{R}^3$. Although MCF with surgery provides better control on the topology and stays inside the smooth category, it requires technical virtuosity and deep understandings of how singularities are formed.

Another way of continuing beyond the singular time is to consider a class of generalized solutions or weak solutions allowing singularities. Using tools from geometric measure theory, Almgren [3] and Allard [1] introduced and developed theories on a generalized class of surfaces called varifolds. Brakke [9] later defined the MCF equation in the space of varifolds by certain transport inequalities and proved a general existence theorem for the flow. The flow is referred to as the Brakke flow.

In material science, MCF arises naturally in describing the evolution of interfaces that bound phases of materials. Mullins [33] first formulated the MCF equation to model grain boundaries during metal annealing. Before the 1990s, most results on MCF were established for hypersurfaces without boundary. However, although being considerably more challenging than the no boundary case, the study of MCF for surfaces with boundary is of great significance. It is a more natural way to describe physical phenomena. For instance, the deformation of grain boundaries or the evolution of interfaces usually happens in some containers under certain physical boundary conditions. MCF with boundary can best describe such scenarios. Applications of MCF with boundary also include describing the motion of soap film whose boundary moves freely in a fixed surface.

To define MCF for surfaces with boundary properly, mathematicians mainly focus on two geometric boundary conditions, the Dirichlet boundary condition, and the Neumann boundary condition. For the Dirichlet boundary condition where the boundary moves in a prescribed way, Huisken [26] proved a theorem for graphs under the non-parametric mean curvature flow using the classical theory of Lieberman [30] regarding general quasilinear parabolic equations with Dirichlet boundary conditions. Generalizations to the Riemannian settings have been introduced by Priwitzer [37], and weak formulations for fixed boundary conditions have appeared in several studies [42, 44].

For the Neumann boundary condition, the angle between the evolving surface and the barrier is prescribed. When the contact angle is fixed to be $\pi/2$, the flow is called MCF with free boundary.Huisken [26] studied MCF with free boundary for graphs using the non-parametric method and proved the long-time existence of the solutions and the convergence of the solutions to a plane domain. After a few years, Stahl [40] established the fundamental existence and uniqueness theorem for MCF with free boundary in the parametric setting. As for boundary singularities, Buckland [12] proved a boundary monotonicity formula to classify certain boundary singularities of MCF with free boundary. Volkmann [43] also proved a monotonicity formula for compact free boundary surfaces with square-integrable mean curvature in the unit ball, which leads to a Li–Yau type inequality.

More generally, for MCF with arbitrary fixed contact angle, Altschuler and Wang [4] proved the long-time existence of the flow for graphs in $\mathbb{R}^2$ and showed that the graphs would converge to translating solutions. The existence and convergence theorems were extended by Guan [17] for higher-dimensional graphs. Furthermore, Bellettini and Kholmatov [7] are concerned about the case of possibly nonconstant prescribed contact angle to describe the motion of droplets flowing on hyperplanes.

In 1984, Huisken [25] published his seminal paper and proved that closed convex hypersurfaces in Euclidean spaces of dimension at least three would converge under MCF to a round point in finite time. Subsequently, Stahl [39] generalized Huisken's result for MCF with free boundary where the barrier is umbilic. In 2020, Hirsch and Li [20] proved a convergence theorem of MCF with free boundary on non-umbilic barriers in $\mathbb{R}^3$. It is natural to ask whether a similar convergence result holds when the barrier is defined in a general Riemannian manifold and higher dimensions.

1.2 Structure of the thesis

The purpose of this thesis is to establish necessary theoretical tools for proving convergence theorems of MCF with free boundary in Riemannian manifolds. In chapter 2, we first review some classical results for MCF in the Euclidean space to introduce the essential ingredients for the convergence theory of MCF. Then we briefly introduce the generalization of the convergence results in the free boundary setting and discuss the similarities and differences between classical MCF and MCF with free boundary. In chapter 3, we compute the boundary derivative of the second fundamental form and prove an iteration scheme for

MCF with free boundary in a general Riemannian manifold.

□ **End of chapter.**

Chapter 2

Classical Results of MCF

This chapter outlines a general methodology proposed by Huisken [25] for proving the convergence of surfaces to a round point. Following the similar idea, mathematicians [6, 31, 20] have generalized the convergence theorem to different conditions. Therefore, it is essential to review Huisken's [25] classical arguments and capture the idea behind them.

Throughout this chapter, we let M be a compact uniformly convex hypersurface smoothly embedded in $\mathbb{R}^{n+1}$. Any such M can be represented locally by the following diffeomorphism:

$$F\colon U \subset \mathbb{R}^n \to M \subset \mathbb{R}^{n+1}.$$

The metric $g = \{g_{ij}\}$ and the second fundamental form $A = \{h_{ij}\}$ at $F(\vec{x}) \in M$ can be written as

$$g_{ij}(\vec{x}) = \left(\frac{\partial F(\vec{x})}{\partial x_i}, \frac{\partial F(\vec{x})}{\partial x_j}\right), \quad h_{ij}(\vec{x}) = \left(-\nu(\vec{x}), \frac{\partial^2 F(\vec{x})}{\partial x_i \partial x_j}\right)$$

where $\nu(\vec{x}) \in \mathbb{R}^{n+1}$ is the outward normal to M at $F(\vec{x})$ and $(\cdot, \cdot)$ is the standard inner product in $\mathbb{R}^{n+1}$. The Levi-Civita connection on M induced from the standard connection on $\mathbb{R}^{n+1}$ is given by

$$\Gamma^k{}_{ij} = \frac{1}{2} g^{kl} \left(g_{il,j} + g_{jl,i} - g_{ij,l}\right)$$

where $g_{ij,k} = \frac{\partial}{\partial x_k} g_{ij}$. For a vector field $X = X^i \frac{\partial}{\partial x_i}$ on M, the covariant derivative of X is

$$(\nabla_i X)^j = \frac{\partial}{\partial x_i} X^j + \Gamma^j{}_{ik} X^k.$$

The Riemann curvature tensor on M is defined as

$$R_{ijkl} = \left\langle (\nabla_i \nabla_j - \nabla_j \nabla_i) \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l} \right\rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner product for tensors on M induced from g. By Gauss equation, we have that

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}. \tag{2.1}$$

The Ricci tensor and scalar curvature are thus given by

$$R_{ik} = H h_{ik} - h_i{}^j h_{jk}, \quad R = H^2 - |A|^2$$

where $H = g^{ij} h_{ij}$, $|A|^2 = h^{ij} h_{ij}$ and the metric tensor g is used to raise or lower indices.

Now we denote M by M_0 and F by F_0. We say a family of maps $F(\cdot, t)$ satisfies the MCF equation with initial condition F_0 if

$$\begin{aligned} &\frac{\partial}{\partial t} F(\vec{x}, t) = -H(\vec{x}, t) \cdot \nu(\vec{x}, t), \quad \vec{x} \in U, \\ &F(\cdot, 0) = F_0, \end{aligned} \tag{2.2}$$

where $H(\vec{x}, t)$ is the mean curvature on M_t.

To prove that M_t converges to a round point as $t \to T$, we first show that the hypersurface M_t converges to a point. Then we normalize the flow by keeping the total area of M_t fixed and prove that M_t converges to a round sphere under the normalized flow.

For the second part, we rescale the solution F at each time $t \in [0, T)$ by a positive constant $\psi(t)$ such that

$$\int_{\tilde{M}_t} d\tilde{\mu}_t = |M_0| \text{ for all } 0 \le t < T \tag{2.3}$$

where the surface $\tilde{M}_t$ is given by local diffeomorphisms

$$\tilde{F}(\cdot, t) = \psi(t) \cdot F(\cdot, t).$$

By introducing a new time variable $\tilde{t}(t) = \int_0^t \psi^2(\tau) d\tau$, we derive in section 2.5 that $\tilde{F}$ satisfies the following normalized equation on a different maximal time interval $\tilde{t} \in [0, \tilde{T})$:

$$\frac{\partial \tilde{F}}{\partial \tilde{t}} = -\tilde{H}\tilde{\nu} + \frac{1}{n}\tilde{h}\tilde{F} \tag{2.4}$$

where $\tilde{h} = \frac{\int \tilde{H}^2 d\tilde{\mu}}{\int d\tilde{\mu}}$ is the average of the squared mean curvature on $\tilde{M}_t$. Now we can state the convergence theorem by Huisken.

Theorem 2.0.1 ([25, Theorem 1.1])**.** *Under the assumption that $n \geq 2$ and M_0 is uniformly convex, Equation 2.2 has a smooth solution on $[0, T)$ for $T < \infty$, and M_t converges to a point as $t \to T$. The solution to the normalized Equation 2.4 exists for all time $0 \leq \tilde{t} < \infty$ and converges to a sphere in the C^∞-topology as $\tilde{t} \to \infty$.*

2.1 Maximum principles and Simons' identity

Parabolic maximum principles are essential PDE tools in the analysis of MCF. We will briefly introduce two frequently used versions in this section. One is the standard parabolic maximum principle for scalar functions.

Theorem 2.1.1 ([16, p.397])**.** *Let M be a closed smooth manifold and $f \colon M \times [0, T) \to \mathbb{R}$ be a scalar function on M varying along time t. Suppose $f(\cdot, 0) \geq 0$ and*

$$\frac{\partial f}{\partial t} \geq \Delta f + b^i \nabla_i f + cf$$

for some smooth function b^i, c, where $c \geq 0$. Then

$$\min_M f(\cdot, t) \geq \min_M f(\cdot, 0).$$

Furthermore, if there exist some $p \in M$ *and* $t_0 \geq 0$ *such that* $f(p, t_0) = \min_M f(\cdot, 0)$*, then* $f \equiv \min_M f(\cdot, 0)$ *for* $0 \leq t \leq t_0$*.*

Later we will see that the mean curvature of the evolving hypersurface satisfies such a parabolic inequality. By the strong maximum principle, we can show that the positivity of H is preserved throughout the flow.

In order to study the evolution of tensors such as the second fundamental form, we extend the scalar maximum principle to tensors. Let M_{ij} be a symmetric tensor on a closed manifold M. We say $M_{ij} \geq 0$ if for any vector X on M, $M_{ij}X^iX^j \geq 0$. Let $N_{ij} = P(M_{ij}, g_{ij})$ be another symmetric tensor formed by contracting M_{ij} with itself using the metric where P is a polynomial. Then we can prove a weak maximum principle for symmetric 2-tensors by following the treatment in [14, Theorem 4.6].

Theorem 2.1.2 ([19, Theorem 9.1])**.** *Suppose* M_{ij} *is a symmetric tensor on a closed manifold* M *depending on time* t *and on* $0 \leq t < T$ *satisfies that*

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + u^k \nabla_k M_{ij} + N_{ij}$$

where u^k *is a vector on* M *and* N_{ij} *is defined as above such that*

$$N_{ij}X^iX^j \geq 0 \text{ whenever } M_{ij}X^j = 0.$$

Then if $M_{ij} \geq 0$ *at* $t = 0$*, it will remain so on* $0 \leq t < T$*.*

Proof. Given any $\tau \in [0, T)$, we claim that there exists $\delta > 0$ such that for any $t_0 \in [0, \tau - \delta]$, if $M_{ij} \geq 0$ when $t = t_0$, then $M_{ij} \geq 0$ for $t \in [t_0, t_0 + \delta]$. The theorem follows naturally from the claim.

Now we fix some $t_0 \in [0, \tau - \delta]$ and let $\tilde{M}_{ij} = M_{ij} + \epsilon(\delta + t - t_0)g_{ij}$ for $0 < \epsilon \leq 1$.Note that when $t = t_0$, $\tilde{M}_{ij} = M_{ij} + \epsilon\delta g_{ij} > M_{ij}$. To prove the claim, we only need to show that $\tilde{M}_{ij} > 0$ on $(t_0, t_0 + \delta]$ for δ independent of ϵ and let $\epsilon \to 0$.

Suppose the statement were not true. Then $\tilde{M}_{ij}$ has null eigenvectors for a first time $t_1 \in (t_0, t_0 + \delta]$ at $x_1 \in M_{t_1}$. Let $X = X^i \frac{\partial}{\partial x_i} \in T_{x_1} M_{t_1}$ be a unit null eigenvector of $\tilde{M}_{ij}$. We set $\tilde{N}_{ij} = P(\tilde{M}_{ij}, g_{ij})$. Note that $N_{ij} = P(M_{ij}, g_{ij})$ have the property that $N_{ij} Y^i Y^j \geq 0$ whenever Y is a null eigenvector of M_{ij}. Since X is a null eigenvector of $\tilde{M}_{ij}$, we have that $\tilde{N}_{ij} X^i X^j \geq 0$. Then at (x_1, t_1),

$$\begin{aligned} N_{ij} X^i X^j &= \tilde{N}_{ij} X^i X^j + (N_{ij} - \tilde{N}_{ij}) X^i X^j \\ &\geq (N_{ij} - \tilde{N}_{ij}) X^i X^j \\ &\geq - \left| N_{ij} - \tilde{N}_{ij} \right| . \end{aligned}$$

Since P is a polynomial, we have that

$$\left| N_{ij} - \tilde{N}_{ij} \right| \leq C \left| M_{ij} - \tilde{M}_{ij} \right|$$

where C is a constant depending only on $\max_{M \times [0,\tau]} \left| M_{ij} \right|$. As $t_1 \in [t_0, t_0 + \delta]$, we have that at (x_1, t_1)

$$\begin{aligned} N_{ij} X^i X^j &\geq -C \left| M_{ij} - \tilde{M}_{ij} \right| \\ &= -C \left| \epsilon(\delta + t_1 - t_0) g_{ij} \right| \\ &\geq -2C\epsilon\delta. \end{aligned}$$

The point (x_1, t_1) depends on the time t_0 we choose in $[0, \tau]$. Nevertheless, we can choose $\delta_0 > 0$ sufficiently small depending only on $\max_{M \times [0,\tau]} \left| \frac{\partial}{\partial t} g_{ij} \right|$ such that at (x_1, t_1),

$$\frac{\partial}{\partial t} g_{ij} \geq -\frac{1}{\delta_0} g_{ij}.$$

The evolution of $\tilde{M}_{ij}$ is given by

$$\frac{\partial}{\partial t} \tilde{M}_{ij} = \frac{\partial}{\partial t} M_{ij} + \epsilon g + \epsilon(\delta + t - t_0) \frac{\partial}{\partial t} g_{ij}.$$

Since $\Delta \tilde{M}_{ij} = \Delta M_{ij}$ and $\nabla \tilde{M}_{ij} = \nabla M_{ij}$, then at (x_1, t_1),

$$\begin{aligned} \frac{\partial}{\partial t} \tilde{M}_{ij} &\geq \Delta M_{ij} + u^k \nabla_k M_{ij} + N_{ij} + \epsilon g + \epsilon(\delta + t_1 - t_0) \frac{\partial}{\partial t} g_{ij} \\ &\geq \Delta \tilde{M}_{ij} + u^k \nabla_k \tilde{M}_{ij} + N_{ij} + (1 - \frac{2\delta}{\delta_0}) \epsilon g_{ij}. \end{aligned}$$

By extending the vector X^i to a parallel vector field in a neighborhood of x_1 along geodesics passing x_1 on M_{t_1} and defining X^i on $[t_0, t_0+\delta]$ independent of t, we can define a function $f = \tilde{M}_{ij}X^iX^j$ on $M \times [t_0, t_0+\delta]$. By assumption, we have that $f(x_1, t) > 0$ for $t_0 \leq t < t_1$ and $f(x_1, t_1) = 0$. Hence $\frac{\partial}{\partial t}f(x_1, t_1) \leq 0$. At $t = t_1$, we see that $f = 0$ attains a minimum at x_1; thus $\nabla f(x_1, t_1) = 0$ and $\Delta f(x_1, t_1) \geq 0$. Moreover, since X is parallel, we have that

$$\begin{aligned}
\frac{\partial}{\partial t} f &= (\frac{\partial}{\partial t}\tilde{M}_{ij})X^iX^j \\
\nabla_k f &= (\nabla_k \tilde{M}_{ij})X^iX^j \\
\Delta f &= (\Delta \tilde{M}_{ij})X^iX^j.
\end{aligned}$$

Therefore, at (x_1, t_1),

$$\begin{aligned}
0 > \frac{\partial}{\partial t} f &= (\frac{\partial}{\partial t}\tilde{M}_{ij})X^iX^j \\
&\geq \left[\Delta \tilde{M}_{ij} + u^k \nabla_k \tilde{M}_{ij} + N_{ij} + (1 - \frac{2\delta}{\delta_0})\epsilon g_{ij}\right] X^iX^j \\
&= \Delta f + u^k \nabla_k f + (1 - \frac{2\delta}{\delta_0} - 2C\delta)\epsilon \\
&\geq (1 - \frac{2\delta}{\delta_0} - 2C\delta)\epsilon.
\end{aligned}$$

Then contradiction arises when $\delta < \frac{1}{4}\min\{\delta_0, \frac{1}{C}\}$. Since now δ depends only on $\max_{M\times[0,\tau]} \left|M_{ij}\right|$ and $\max_{M\times[0,\tau]} \left|\frac{\partial}{\partial t}g_{ij}\right|$, we can let $\epsilon \to 0$ and reach the conclusion.

$\square$

To apply the maximum principles, we need the following Simons' identity to rewrite the evolution equation of the second fundamental form as a parabolic PDE.

Lemma 2.1.3 (Simons' identity)**.**

$$\Delta h_{ij} = \nabla_i \nabla_j H + H h_{li} g^{lm} h_{mj} - |A|^2 h_{ij}$$

Proof. Note that $\Delta h_{ij} = g^{mn}\nabla_m\nabla_n h_{ij}$ and $\nabla_i\nabla_j H = g^{mn}\nabla_i\nabla_j h_{mn}$. It suffices to examine the difference $\nabla_m\nabla_n h_{ij} - \nabla_i\nabla_j h_{mn}$. Since the ambient space is Euclidean, from the Codazzi equation we have that $\nabla_i h^k_j = \nabla_j h^k_i$. Hence

$$\nabla_m\nabla_n h_{ij} - \nabla_i\nabla_j h_{mn} = \nabla_m\nabla_i h_{nj} - \nabla_i\nabla_m h_{jn} = (\nabla_m\nabla_i - \nabla_i\nabla_m)h_{nj}.$$

By the product rule of connections acting on tensor product, we have that

$$(\nabla_m\nabla_i - \nabla_i\nabla_m)h_{nj} = R_{min}{}^l h_{lj} + R_{mij}{}^l h_{nl}.$$

Therefore, by Equation 2.1

$$\begin{aligned}
\Delta h_{ij} - \nabla_i\nabla_j H &= g^{mn}(R_{min}{}^l h_{lj} + R_{mij}{}^l h_{nl}) \\
&= g^{mn}g^{kl}\left\{(h_{mn}h_{ik} - h_{mk}h_{in})\, h_{lj} + (h_{mj}h_{ik} - h_{mk}h_{ij})h_{ln}\right\} \\
&= Hg^{kl}h_{ik}h_{lj} - g^{mn}g^{kl}h_{mk}h_{ln}h_{ij} \\
&= Hg^{kl}h_{ik}h_{lj} - |A|^2\, h_{ij}.
\end{aligned}$$

□

2.2 Evolution equations for geometric quantities

Since the embedding map F is evolving under time t, if we fix a point $\vec{x} \in U$, geometric quantities at $F(\vec{x}, t) \in M_t$ are also evolving under time t. By the flow equation $\frac{\partial}{\partial t}F(\vec{x}, t) = -H(\vec{x}, t)\cdot\nu(\vec{x}, t)$, we can derive evolution equations for other geometric quantities.

Lemma 2.2.1 ([25, Section 3])**.** *The following evolution equations hold.*

(1) $\frac{\partial}{\partial t}g_{ij} = -2Hh_{ij}$

(2) $\frac{\partial}{\partial t}g^{ij} = 2Hh^{ij}$

(3) $\frac{\partial \nu}{\partial t} = \nabla H$

(4) $\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2H h_{ik} g^{kl} h_{lj} + |A|^2 h_{ij}$

(5) $\frac{\partial}{\partial t} H = \Delta H + |A|^2 H$

(6) $\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2 |\nabla A|^2 + 2 |A|^4$

Proof.

(1) Since $\left(\nu, \frac{\partial F}{\partial x_i}\right) = 0$, by the product rule, we have that

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= \frac{\partial}{\partial t}\left(\frac{\partial F(\vec{x},t)}{\partial x_i}, \frac{\partial F(\vec{x},t)}{\partial x_j}\right) \\
&= \left(\frac{\partial}{\partial x_i}(-H(\vec{x},t) \cdot \nu(\vec{x},t)), \frac{\partial F}{\partial x_j})\right) \\
&\quad + \left(\frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j}(-H(\vec{x},t) \cdot \nu(\vec{x},t))\right) \\
&= -H\left[\left(\frac{\partial \nu}{\partial x_i}, \frac{\partial F}{\partial x_j}\right) + \left(\frac{\partial F}{\partial x_i}, \frac{\partial \nu}{\partial x_j}\right)\right] \\
&= -2H h_{ij}.
\end{aligned}$$

(2) Since $g_{km} g^{mj} = \delta_k^j$, we have that

$$\begin{aligned}
\frac{\partial}{\partial t}(g_{km} g^{mj}) &= 0 \\
\frac{\partial g_{km}}{\partial t} g^{mj} + g_{km} \frac{\partial g^{mj}}{\partial t} &= 0 \\
-2H h_{km} g^{mj} + g_{km} \frac{\partial g^{mj}}{\partial t} &= 0 \\
g^{ik} g_{km} \frac{\partial g^{mj}}{\partial t} &= g^{ik} 2H h_{km} g^{mj} \\
\frac{\partial}{\partial t} g^{ij} &= 2H h^{ij}.
\end{aligned}$$

(3) Since $|\nu| = 1$ is fixed, we have that $\frac{\partial \nu}{\partial t}$ lies in the tangent space of the surface. Hence we can assume that $\frac{\partial \nu}{\partial t} = V^i \frac{\partial F}{\partial x_i} \in \mathbb{R}^{n+1}$ where V^i can be determined by the following identity

$$\left(\frac{\partial \nu}{\partial t}, \frac{\partial F}{\partial x_j}\right) = g_{ij} V^i.$$

Thus, we have that

$$\begin{aligned}\frac{\partial \nu}{\partial t} &= g^{ij}\left(\frac{\partial \nu}{\partial t}, \frac{\partial F}{\partial x_j}\right)\cdot\frac{\partial F}{\partial x_i}\\ &= -g^{ij}\left(\nu, \frac{\partial}{\partial t}\frac{\partial F}{\partial x_j}\right)\cdot\frac{\partial F}{\partial x_i}\\ &= g^{ij}\left(\nu, \frac{\partial}{\partial x_j}(H(\vec{x},t)\cdot\nu(\vec{x},t))\right)\cdot\frac{\partial F}{\partial x_i}\\ &= g^{ij}\frac{\partial H}{\partial x_j}\frac{\partial F}{\partial x_i}\\ &= \nabla H.\end{aligned}$$

(4) By the Gauss-Weingarten relations, we have that

$$\begin{cases}\frac{\partial^2 F}{\partial x_i \partial x_j} = {\Gamma^k}_{ij}\frac{\partial F}{\partial x_k} - h_{ij}\nu\\ \frac{\partial \nu}{\partial x_j} = h_{jl}g^{lm}\frac{\partial F}{\partial x_m}.\end{cases}$$

Hence

$$\begin{aligned}\frac{\partial}{\partial t}h_{ij} &= -\frac{\partial}{\partial t}\left(\nu, \frac{\partial^2 F}{\partial x_i \partial x_j}\right)\\ &= -\left(g^{pq}\frac{\partial H}{\partial x_p}\frac{\partial F}{\partial x_q}, \frac{\partial^2 F}{\partial x_i\partial x_j}\right) + \left(\nu, \frac{\partial^2}{\partial x_i \partial x_j}(H\cdot\nu)\right)\\ &= -\left(g^{pq}\frac{\partial H}{\partial x_p}\frac{\partial F}{\partial x_q}, {\Gamma^k}_{ij}\frac{\partial F}{\partial x_k} - h_{ij}\nu\right)\\ &\quad + \frac{\partial}{\partial x_j}\left(\nu, \frac{\partial}{\partial x_i}(H\cdot\nu)\right) - \left(h_{jl}g^{lm}\frac{\partial F}{\partial x_m}, \frac{\partial}{\partial x_i}(H\cdot\nu)\right)\\ &= -g^{pq}\frac{\partial H}{\partial x_q}{\Gamma^k}_{ij}g_{pk} + \frac{\partial^2 H}{\partial x_i \partial x_j}\\ &\quad - H\cdot\left(h_{jl}g^{lm}\frac{\partial F}{\partial x_m}, h_{il'}g^{l'm'}\frac{\partial F}{\partial x_{m'}}\right)\\ &= \frac{\partial^2 H}{\partial x_i\partial x_j} - {\Gamma^q}_{ij}\frac{\partial H}{\partial x_q} - H h_j^m h_i^n g_{mn}.\end{aligned}$$

Since H is a scalar function, we have that

$$\nabla_i\nabla_j H = \frac{\partial^2 H}{\partial x_i \partial x_j} - {\Gamma^q}_{ij}\frac{\partial H}{\partial x_q}$$

where ∇ is the Levi-Civita connection on M_t.

Hence, by Lemma 2.1.3,

$$\begin{aligned}\frac{\partial}{\partial t}h_{ij} &= \frac{\partial^2 H}{\partial x_i \partial x_j} - \Gamma^q{}_{ij}\frac{\partial H}{\partial x_q} - Hh_j^m h_i^n g_{mn} \\ &= \Delta h_{ij} - (Hh_{li}g^{lm}h_{mj} - |A|^2 h_{ij}) - Hh_j^m h_i^n g_{mn} \\ &= \Delta h_{ij} - 2Hh_{li}g^{lm}h_{mj} + |A|^2 h_{ij}.\end{aligned}$$

(5) Since $H = g^{ij}h_{ij}$, we have that

$$\begin{aligned}\frac{\partial}{\partial t}H = \frac{\partial}{\partial t}(g^{ij}h_{ij}) &= \frac{\partial g^{ij}}{\partial t}h_{ij} + g^{ij}\frac{\partial h_{ij}}{\partial t} \\ &= 2Hh^{ij}h_{ij} + g^{ij}(\Delta h_{ij} - 2Hh_{li}g^{lm}h_{mj} + |A|^2 h_{ij}) \\ &= \Delta H + |A|^2 H.\end{aligned}$$

(6) Combining previous results, we can deduce the following evolution equation

$$\begin{aligned}\frac{\partial}{\partial t}h_i{}^j &= \frac{\partial}{\partial t}(h_{ik}g^{kj}) \\ &= (\Delta h_{ik} - 2Hh_{li}g^{lm}h_{mk} + |A|^2 h_{ik})g^{kj} + h_{ik}(2Hh^{kj}) \\ &= \Delta h_i{}^j - 2Hh_{ik}h^{kj} + |A|^2 h_i{}^j - 2Hh_{ik}h^{kj} \\ &= \Delta h_i{}^j + |A|^2 h_i{}^j.\end{aligned}$$

Since $|A|^2 = h^{ij}h_{ij} = h_i{}^j h^i{}_j$, we have that

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 &= \frac{\partial}{\partial t}(h_i{}^j h^i{}_j) \\ &= (\Delta h_i{}^j + |A|^2 h_i{}^j)h^i{}_j + h_i{}^j(\Delta h^i{}_j + |A|^2 h^i{}_j) \\ &= 2(h^{ij}\Delta h_{ij} + |A|^4).\end{aligned}$$

Since the connection ∇ is compatible with the metric g, we have that

$$\begin{aligned}\Delta |A|^2 &= g^{mn}\nabla_m\nabla_n(h^{ij}h_{ij}) \\ &= 2g^{mn}\nabla_m(h^{ij}\nabla_n h_{ij}) \\ &= 2(g^{mn}\nabla_m\nabla_n h_{ij})h^{ij} + 2g^{mn}(\nabla_m h^{ij})(\nabla_n h_{ij}) \\ &= 2h^{ij}\Delta h_{ij} + 2|\nabla A|^2.\end{aligned}$$

It follows that

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 &= 2(h^{ij}\Delta h_{ij} + |A|^4) \\ &= \Delta |A|^2 - 2|\nabla A|^2 + |A|^4.\end{aligned}$$

□

2.3 Preservation of convexity and the pinching condition

Combining the maximum principles and the evolution equations, we can prove the following two theorems in [25, Section 4].

Theorem 2.3.1. *If $h_{ij} \geq 0$ at $t = 0$, then it remains so for $0 \leq t < T$.*

Proof. We have that

$$\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - 2Hh_{li}g^{lm}h_{mj} + |A|^2 h_{ij}.$$

Let $M_{ij} = h_{ij}$ and $N_{ij} = |A|^2 h_{ij} - 2Hh_{li}g^{lm}h_{mj}$. If a vector X^j satisfies that $h_{ij}X^j = 0$ for all i, then

$$N_{ij}X^j = |A|^2 (h_{ij}X^j) - 2Hh_{li}g^{lm}(h_{mj}X^j) = 0.$$

Hence we can apply Theorem 2.1.2 to conclude. □

Theorem 2.3.2. *If $\epsilon H g_{ij} \leq h_{ij} \leq \beta H g_{ij}$, and $H \geq 0$ at $t = 0$, then it remains true for $t > 0$.*

Proof. First, since $\frac{\partial}{\partial t}H = \Delta H + |A|^2 H$, by Theorem 2.1.1 we have that if $H \geq 0$ at $t = 0$, $H \geq 0$ for all $t \geq 0$. Let $M_{ij} = h_{ij} - \epsilon H g_{ij}$. Then

$$\begin{aligned}\frac{\partial}{\partial t}M_{ij} &= \frac{\partial}{\partial t}h_{ij} - \epsilon(\frac{\partial}{\partial t}H)g_{ij} - \epsilon H\frac{\partial}{\partial t}g_{ij} \\ &= \Delta h_{ij} - 2Hh_{li}g^{lm}h_{mj} + |A|^2 h_{ij} - \epsilon g_{ij}(\Delta H + |A|^2 H) - \epsilon H(-2Hh_{ij}) \\ &= \Delta M_{ij} + |A|^2 h_{ij} + 2\epsilon H^2 h_{ij} - \epsilon |A|^2 H g_{ij} - 2Hh_{li}g^{lm}h_{mj}.\end{aligned}$$

Let $N_{ij} = |A|^2 h_{ij} + 2\epsilon H^2 h_{ij} - \epsilon |A|^2 H g_{ij} - 2H h_{li} g^{lm} h_{mj}$. From direct computation we have that

$$\begin{aligned} N_{ij} &= |A|^2 (h_{ij} - \epsilon H g_{ij}) - 2H(h_{li} g^{lm} h_{mj} - \epsilon H h_{ij}) \\ &= |A|^2 M_{ij} - 2H(h_{li} g^{lm} h_{mj} - \epsilon H h_{li} g^{lm} g_{mj}) \\ &= |A|^2 M_{ij} - 2H h_i^m (h_{mj} - \epsilon H g_{mj}) \\ &= |A|^2 M_{ij} - 2H h_i^m M_{mj}. \end{aligned}$$

Then for a null vector X^i of M_{ij}, we have that

$$N_{ij} X^j = |A|^2 (M_{ij} X^j) - 2H h_i^m (M_{mj} X^j) = 0.$$

Then the result follows from Theorem 2.1.2. □

2.4 Stampacchia's iteration

One essential step for proving Theorem 2.0.1 is to show that the quantity

$$|A|^2 - \frac{1}{n} H^2$$

becomes small compared to H^2.

Theorem 2.4.1. *There are constants $C_0 < \infty$ and $\delta > 0$ depending only on M_0 such that*

$$|A|^2 - \frac{1}{n} H^2 \leq C_0 H^{2-\delta}$$

for all times $t \in [0, T)$.

The rationale behind the idea is that

$$|A|^2 - \frac{1}{n} H^2 = \frac{1}{n} \sum_{i<j}^{n} (\kappa_i - \kappa_j)^2$$

measures the sum of differences between eigenvalues κ_i of the second fundamental form A.

An iteration scheme known as Stampacchia's iteration is used to reach the goal. In this section, we introduce the general idea for Stampacchia's iteration.

The principal components of Stampacchia's iteration are Lemma 2.4.2, an algebraic lemma by Stampacchia [41], and Lemma 2.4.3, a version of the Sobolev inequality by Michael and Simon [32].

Lemma 2.4.2 ([41, Lemma 4.1]). *Let $f\colon [\bar{x}, \infty) \to \mathbb{R}$ be a non-negative and non-increasing function. Suppose for $C > 0, p > 0$ and $\gamma > 1$,*

$$(y-x)^p f(y) \leq C f(x)^\gamma, \quad \forall y \geq x \geq \bar{x}.$$

Then $f(y) = 0$ for $y \geq \bar{x} + d$ where $d^p = C f(\bar{x})^{\gamma-1} 2^{\frac{p\gamma}{\gamma-1}}$

Proof. Without loss of generality, we can assume that $\bar{x} = 0$. Let $g = (\frac{f}{f(0)})^{\frac{1}{p}}$ and $A = (Cf(0)^{\gamma-1})^{\frac{1}{p}}$. For $y \geq x \geq 0$, we have that

$$\begin{aligned}
(y-x)^p f(y) &\leq C f(x)^\gamma \\
A^p (y-x)^p f(y) &\leq A^p C f(x)^\gamma \\
(y-x)^p g(y)^p f(0)^\gamma &\leq C f(0)^{\gamma-1} g(x)^{p\gamma} f(0)^\gamma \\
(y-x) g(y) &\leq A g(x)^\gamma.
\end{aligned}$$

Now fix $y > 0$, let $x_n = y(1 - \frac{1}{2^n})$. Note that $\lim_{n\to\infty} x_n = y$ and $x_0 = 0$. Hence, we have that $g(x_0) = g(0) = 1$ and

$$\begin{aligned}
(x_{n+1} - x_n) g(x_{n+1}) &\leq A g(x_n)^\gamma \\
y(\frac{1}{2^n} - \frac{1}{2^{n+1}}) g(x_{n+1}) &\leq A g(x_n)^\gamma \\
g(x_{n+1}) &\leq \frac{A}{y} 2^{n+1} g(x_n)^\gamma.
\end{aligned}$$

Using the above inequality inductively, we have that

$$g(x_n) \leq (\frac{A}{y})^{1+\gamma+\dots+\gamma^{n-1}} 2^{n+(n-1)\gamma+(n-2)\gamma^2+\dots+\gamma^{n-1}}.$$

Since

$$n+(n-1)\gamma+(n-2)\gamma^2+\cdots+\gamma^{n-1}=\frac{\gamma^n+n-(n+1)\gamma}{(\gamma-1)^2},$$

if we choose y such that $\frac{A}{y}=2^{-\frac{\gamma}{\gamma-1}}$, then we have that

$$\begin{aligned} g(x_n) &\leq (\frac{A}{y})^{\frac{\gamma^n-1}{\gamma-1}}2^{\frac{\gamma^n+n-(n+1)\gamma}{(\gamma-1)^2}} \\ &\leq 2^{\frac{1}{(\gamma-1)^2}(-\gamma(\gamma^n-1)+\gamma^{n+1}+n-(n+1)\gamma)} \\ &= 2^{-\frac{n}{\gamma-1}}. \end{aligned}$$

It follows that $\lim_{n\to\infty} g(x_n)=0$. By continuity of g, we have that $g(y)=0$. Therefore, $f(y)=0$. □

Lemma 2.4.3 ([32, Theorem 2.1]). *Let v be a Lipschitz function on M. Then*

$$\left(\int_M |v|^{\frac{n}{n-1}}\,d\mu\right)^{\frac{n-1}{n}} \leq c(n)\int_M |\nabla v|+H\,|v|\,d\mu.$$

The geometric quantity we aim to bound is

$$f_\sigma=\left(|A|^2-\frac{1}{n}H^2\right)H^{\sigma-2}=\left(\frac{|A|^2}{H^2}-\frac{1}{n}\right)H^\sigma$$

for sufficient small $\sigma>0$.

Since M is uniformly convex, by Theorem 2.3.2, we have that $\epsilon H g_{ij}\leq h_{ij}\leq \beta H g_{ij}$, and $H\geq 0$ for any $t>0$. Combining previous evolution equations, we can deduce as in [25, Corollary 5.3] that

$$\frac{\partial}{\partial t}f_\sigma \leq \Delta f_\sigma+\frac{2(1-\sigma)}{H}\left\langle \nabla_i H,\nabla_i f_\sigma\right\rangle-\epsilon^2\frac{1}{H^{2-\sigma}}|\nabla H|^2+\sigma|A|^2 f_\sigma \tag{2.5}$$

for all $0\leq t<T$ and $\sigma>0$.

Applying integration by parts and Peter-Paul inequality, we have the following Poincare-like inequality for f_σ.

Lemma 2.4.4 ([25, Lemma 5.4])**.** *Let $p \geq 2$. For any $0 < \sigma \leq \frac{1}{2}$ and any $\eta > 0$, we have that*

$$n\epsilon^2 \int f_\sigma^p H^2 d\mu \leq (2\eta p + 5) \int \frac{1}{H^{2-\sigma}} |\nabla H|^2 d\mu + \eta^{-1}(p-1) \int f_\sigma^{p-2} |f_\sigma|^2 d\mu.$$

For a positive constant k, we let $f_{\sigma,k} = (f_\sigma - k)_+$, $A(k) = \{f_\sigma \geq k\}$ and $A(k,t) = A(k) \cap M_t$. Following Huisken's idea in [25, Lemma 5.5],we can further derive a evolution-like inequality for $f_{\sigma,k}$.

Lemma 2.4.5. *Let $p \geq 2$. For any $0 < \sigma < 1$, we have that*

$$\begin{aligned}\frac{\partial}{\partial t} \int f_{\sigma,k}^p d\mu \leq &- \frac{1}{2} p(p-1) \int f_{\sigma,k}^{p-2} |\nabla f_\sigma|^2 d\mu \\ &- p\left(\epsilon^2 - \frac{2}{p-1}\right) \int f_{\sigma,k}^{p-1} \frac{|\nabla H|^2}{H^{2-\sigma}} d\mu \\ &- \int H^2 f_{\sigma,k}^p d\mu + \sigma p \int_{A(k,t)} H^2 f_\sigma^p d\mu.\end{aligned}$$

Proof. The idea is to multiply both sides of Equation 2.5 by $p f_{\sigma,k}^{p-1}$ and integrate by parts over M_t. For the left-hand side, we have that

$$\begin{aligned}\int p f_{\sigma,k}^{p-1} \frac{\partial}{\partial t} f_\sigma d\mu &= \int \frac{\partial}{\partial t} f_{\sigma,k}^p d\mu \\ &= \frac{\partial}{\partial t} \int f_{\sigma,k}^p d\mu - \int f_{\sigma,k}^p \frac{\partial}{\partial t}(d\mu) \\ &= \frac{\partial}{\partial t} \int f_{\sigma,k}^p d\mu + \int H^2 f_{\sigma,k}^p d\mu.\end{aligned}$$

For the right-hand side,

$$\int p f_{\sigma,k}^{p-1} \Delta f_\sigma d\mu = -p(p-1) \int f_{\sigma,k}^{p-2} |\nabla f_\sigma|^2$$

and $|A|^2 \leq H^2, \langle \nabla_i H, \nabla_i f_\sigma \rangle \leq |\nabla H| |\nabla f_\sigma|$. It follows that

$$f_{\sigma,k} \leq f_\sigma = \left(|A|^2 - \frac{1}{n} H^2\right) H^{\sigma-2} \leq H^\sigma$$

and for $0 < \sigma < 1, p \geq 2$

$$\begin{aligned}\frac{2(1-\sigma)}{H} f_{\sigma,k} |\nabla H| |\nabla f_\sigma| &\leq \frac{p-1}{2} |\nabla f_\sigma|^2 + \frac{2}{p-1} \frac{|\nabla H|^2 f_{\sigma,k}^2}{H^2} \\ &\leq \frac{p-1}{2} |\nabla f_\sigma|^2 + \frac{2}{p-1} \frac{|\nabla H|^2}{H^{2-\sigma}} f_{\sigma,k}\end{aligned}$$

Hence

$$\begin{aligned}&\frac{\partial}{\partial t} \int f_{\sigma,k}^p d\mu + p(p-1) \int f_{\sigma,k}^{p-2} |\nabla f_\sigma|^2 d\mu \\ &\qquad + \epsilon^2 p \int \frac{1}{H^{2-\sigma}} f_{\sigma,k}^{p-1} |\nabla H|^2 d\mu + \int H^2 f_{\sigma,k}^p d\mu \\ &\leq 2(1-\sigma)p \int \frac{1}{H} f_{\sigma,k}^{p-1} |\nabla H| |\nabla f_\sigma| d\mu + \sigma p \int |A|^2 f_{\sigma,k}^{p-1} f_\sigma d\mu. \\ &\leq \frac{1}{2} p(p-1) \int f_{\sigma,k}^{p-2} |\nabla f_\sigma|^2 d\mu + 2\frac{p}{p-1} \int f_{\sigma,k}^{p-1} \frac{|\nabla H|^2}{H^{2-\sigma}} \\ &\qquad + \sigma p \int_{A(k,t)} H^2 f_\sigma^p d\mu.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial t} \int f_{\sigma,k}^p d\mu \leq &-\frac{1}{2} p(p-1) \int f_{\sigma,k}^{p-2} |\nabla f_\sigma|^2 d\mu \\ &- p\left(\epsilon^2 - \frac{2}{p-1}\right) \int f_{\sigma,k}^{p-1} \frac{|\nabla H|^2}{H^{2-\sigma}} d\mu \\ &- \int H^2 f_{\sigma,k}^p d\mu + \sigma p \int_{A(k,t)} H^2 f_\sigma^p d\mu.\end{aligned}$$

□

Now we have established two inequalities for the function f_σ. Notice that any compact hypersurface M in $\mathbb{R}^{n+1}$ can be enclosed by a sphere which shrinks to a point under MCF in finite time. From the avoidance principle, we have that the maximal time $T < \infty$. Then by Theorem 3.4.5, a general iteration scheme we are going to derive in the later chapter, we can bound f_σ uniformly for all times $t \in [0, T)$, which proves Theorem 2.4.1.

2.5 Convergence to a round point

A general argument from Huisken [25, Theorem 8.1] states that M_t exists on a maximal time interval $t \in [0, T)$ where $T < \infty$ and $\max_{M_t} |A|^2$ becomes unbounded as $t \to T$.

First, we show that M_t converges to a single point. To control the diameter of M_t, we need to examine the minimum value of the mean curvature $H_{\min}$. Since $|A|^2 \leq H^2$, the maximum value of the mean curvature $H_{\max} \to \infty$ as t approaches T. To compare the mean curvature at different points on M_t, we need the following gradient estimate for H.

Theorem 2.5.1 ([25, Theorem 6.1])**.** *For any $\eta > 0$, there exists a constant $C = C(\eta, M_0, n)$ such that*

$$|\nabla H|^2 \leq \eta H^4 + C.$$

Following Huisken's treatment in the proof of [25, Theorem 8.4], combining *Theorem* 2.5.1 and the preservation of curvature pinching $h_{ij} \leq \epsilon H g_{ij}$ and Myer's theorem one can prove that

$$\frac{H_{\max}}{H_{\min}} \to 1$$

as $t \to T$. Hence the diameter of M_t decreases to zero as $t \to T$.

As for the normalized hypersurface $\tilde{M}_t$ parametrized by $\tilde{F}(\cdot, t) = \psi(t) \cdot F(\cdot, t)$, geometric quantities of $\tilde{M}_t$ are rescaled as follows:

$$\tilde{g}_{ij} = \psi^2 g_{ij}, \quad \tilde{h}_{ij} = \psi h_{ij}, \quad \tilde{H} = \psi^{-1} H.$$

Hence

$$\begin{aligned}\frac{\partial}{\partial t}\sqrt{\det \tilde{g}_{ij}} &= \frac{\det \tilde{g}_{ij}}{2\sqrt{\det \tilde{g}_{ij}}}\tilde{g}^{pq}\frac{\partial \tilde{g}_{pq}}{\partial t} \\ &= \frac{\sqrt{\det \tilde{g}_{ij}}}{2}\psi^{-2}g^{pq}\left(\frac{\partial \psi^2}{\partial t}g_{pq} + \psi^2\frac{\partial g_{pq}}{\partial t}\right) \\ &= \sqrt{\det \tilde{g}_{ij}}\left(\psi^{-1}n\frac{\partial \psi}{\partial t} - \psi^2\tilde{H}^2\right)\end{aligned}$$

Therefore, differentiating Equation 2.3 yields that

$$\psi^{-1}\frac{\partial \psi}{\partial t} = \frac{1}{n}\psi^2\tilde{h}.$$

Then for $\tilde{t}(t) = \int_0^t \psi^2(\tau)d\tau$, we have the following normalized flow equation on a different maximal time interval $\tilde{t} \in [0, \tilde{T})$:

$$\frac{\partial \tilde{F}}{\partial \tilde{t}} = -\tilde{H}\tilde{\nu} + \frac{1}{n}\tilde{h}\tilde{F}$$

The evolution equations of geometric quantities under the normalized flow differ from the original evolution equations by a lower order term. Most of the computations in previous sections still hold. We can further prove that the maximal time $\tilde{T} = \infty$ and derive the exponential decay of the following geometric quantities on $\tilde{M}_t$.

Lemma 2.5.2 ([25, Lemma 10.6])**.** *There are constants $\delta > 0$, $C < \infty$ depending only on geometric quantities of M_0 and the dimension n such that*

(1) $|\tilde{A}|^2 - \frac{1}{n}\tilde{H}^2 \le Ce^{-\delta\tilde{t}}$

(2) $\left|\tilde{h}_{ij}\tilde{H} - \frac{1}{n}\tilde{h}\tilde{g}_{ij}\right| \le Ce^{-\delta\tilde{t}}$

(3) $\max_{\tilde{M}}\left|\nabla^m\tilde{A}\right| \le C_m e^{-\delta_m\tilde{t}}$

where $\delta_m > 0, C_m < \infty$ also depend on the order m.

Since the metric $\tilde{g}_{ij}$ evolves under the equation

$$\frac{\partial}{\partial \tilde{t}}\tilde{g}_{ij} = -2\tilde{h}_{ij}\tilde{H} + \frac{2}{n}\tilde{h}\tilde{g}_{ij},$$

by *Lemma* 2.5.2(2), $\tilde{g}_{ij}$ converges uniformly to a positive definite metric $\tilde{g}_{ij}(\infty)$ as $\tilde{t} \to \infty$. Then from *Lemma* 2.5.2(3) and Arzela-Ascoli thoerem, we have that $\tilde{g}_{ij}(\infty)$ is smooth. Finally *Lemma* 2.5.2(1) implies that $\tilde{g}_{ij}(\infty)$ is the metric of a sphere.

2.6 Generalizations

Huisken's convergence theorem was generalized in various settings.

In 1986, Huisken [23] managed to extend Theorem 2.0.1 for hypersurfaces in a general Riemannian manifold where hypersurfaces need to be convex enough to overcome the obstruction caused by the curvature of the ambient manifold. WHen the ambient manifold is a spherical space-form, Huisken [24] observed that the hypersurface could converge to a round point without the initial convexity condition. Similarly, the result obtained by Andrews and Baker [6] for the convergence of higher-codimension submanifold also allows some non-convex condition to be preserved.

For MCF with free boundary, Stahl [39, 40] showed that if the barrier surface in the Euclidean space is a flat hyperplane or a round hypersphere, any convex hypersurface with free boundary on the barrier will converge to a round half-point. Later in 2020, Hirsch and Li [20] managed to generalize the above result to non-umbilic barriers in $\mathbb{R}^3$. They proved that if the barrier surface satisfies a uniform bound on the exterior and interior ball curvature and certain bounds on the first and second derivative of the second fundamental form, then sufficiently convex hypersurfaces with free boundary will converge to a round half-point.

In summary, most convergence results above were obtained by following Huisken's line of argument. The key steps are Theorem 2.4.1, the pinching estimate of the traceless second fundamental form, and Theorem 2.5.1, the gradient estimate of the mean curvature. The former describes the "roundness" of the hypersurface pointwisely, while the latter enables us to compare mean curvatures of the hypersurface at different points. In particular, the gradient estimate is built upon the pinching estimate. To prove a general convergence theorem for MCF with free boundary in the Riemannian ambient space, we need to establish a proper iteration scheme to prove the pinching estimate.

□ **End of chapter.**

Chapter 3

MCF with Free Boundary in Riemannian Manifolds

In this chapter, we study the convergence theory for MCF with free boundary where the barrier surface lies in Riemannian manifolds.

To prove that a free boundary hypersurface converges to a round half-point under MCF, the standard argument from Huisken [25] also works. Hence, it suffices to prove the pinching estimate and the gradient estimate.

However, for a general barrier in a Riemannian manifold, two difficulties need to be overcome.

First, as the prerequisite of proving pinching and gradient estimates, the initial convexity condition is expected to be preserved along the flow. To apply maximum principles for surfaces with boundary, we need compute and estimate the boundary derivatives of geometric quantities. In section 3.3, we calculate the boundary derivatives for the mean curvature and the second fundamental form.

The second difficulty is the reformulation of Stampacchia's iteration in a more general setting. Edelen [15] has introduced a free boundary version of

Stampacchia's iteration, but the iteration argument only works when the barrier lies in the Euclidean space. In section 3.4, we extend the iteration argument to the Riemannian manifold. We first generalize the Michael–Simon inequality by Hoffman and Spruck [21] to the free boundary case. Then following Edelen's [15] arguments, we prove the generalized Stampacchia's iteration.

3.1 Definitions and notations

Let $(\bar{M}, \bar{g})$ be an $(n+1)$-dimensional Riemannian manifold with the Levi-Civita connection $\bar{\nabla}$. We denote by $\sigma_x(P)$ the sectional curvature of a 2-plane P at $x \in \bar{M}$ and by $i_x(\bar{M})$ the injectivity radius of $\bar{M}$ at x.

Consider a properly embedded, orientable, smooth hypersurface $S \subset \bar{M}$ without boundary. We refer to S as the *barrier surface* or the *barrier*. We write $f = O(g)$ to indicate that $|f| \le c(n, S, \bar{M})\,|g|$. By fixing a smooth global unit normal ν_S on S, we can define the second fundamental form $A^S\colon TS \times TS \to \mathbb{R}$ by

$$A^S(u, v) = -\bar{g}(\bar{\nabla}_u v, \nu_S).$$

Let Σ be a two-sided smooth n-dimensional manifold with non-empty boundary $\partial\Sigma$. A smooth immersion $F\colon \Sigma \to \bar{M}$ defines a free boundary hypersurface if $F(\partial\Sigma) \subset S$ and $F_* N = \nu_S \circ F$ where N is the outward unit normal of $\partial\Sigma \subset \Sigma$ with respect to the metric induced from $\bar{M}$ by F.

3.2 Covariant formulation of Mean Curvature Flow

From the previous chapter, we can see that Huisken considered a family of maps F_t from an open set $U \subset \mathbb{R}^n$ to $\mathbb{R}^{n+1}$ which evolve along the mean curvature vector of their images. In this way, we can fix a local coordinate system and analyze geometric quantities of the images along the flow using this coordinate system. On the one hand, the structure of the general evolution equation is more evident in local coordinates. The short-time existence of the flow follows naturally by the theory of quasilinear parabolic differential equations. On the other hand, one needs to carefully choose the local coordinate system to simplify the computation without losing information on derivatives. A more modern treatment is to consider an invariant form of evolution equations independent of local coordinates. Following [6, Section 2], we extend the metric and connection on vector bundles over the space-time domain and derive structure equations and evolution equations for geometric quantities in the new vector bundle terminology.

3.2.1 Subbundles.

Definition 1. *Let K, E be two vector bundles over a manifold $\bar{M}$. We say K is a subbundle of E if there exists an injective vector bundle homomorphism $\iota_K\colon K \to E$ covering the identity map on $\bar{M}$.*

Now let E be a vector bundle over a manifold $\bar{M}$. We can consider two complementary subbundles K and L of E , in the sense that for each $x \in \bar{M}$, the fiber $E_x = \iota_K(K_x) \oplus \iota_L(L_x)$. Let $\pi_K\colon E \to K$ and $\pi_L\colon E \to L$ be the correseponding projections from E onto K and L where we have the following relations

$$\pi_K \circ \iota_K = \mathrm{Id}_K \quad \pi_L \circ \iota_L = \mathrm{Id}_L$$

$$\pi_K \circ \iota_L = 0 \quad \pi_L \circ \iota_K = 0$$

$$\iota_K \circ \pi_K + \iota_L \circ \pi_L = \mathrm{Id}_E.$$

Similar to the way of defining the second fundamental form for submanifolds, we can extend a connection ∇ on E to a connection $\overset{K}{\nabla}$ on its subbundle K and define the second fundamental form $h^K \in \Gamma(T^*(\bar{M}) \otimes K^* \otimes L)$ of K where

$$\overset{K}{\nabla}_u \xi = \pi_K(\nabla_u(\iota_K \xi)) \qquad h^K(u, \xi) = \pi_L(\nabla_u(\iota_K \xi)),$$

for any $\xi \in \Gamma(K)$ and $u \in T\bar{M}$.

Then we can derive the following Gauss equation relating the curvature R^K of $\overset{K}{\nabla}$ to the curvature R_∇ of ∇ and the second fundamental forms h^L and h^K:

$$R^K(u, v)\xi = \pi_k(R_\nabla(u, v)\iota_K \xi) + h^L(u, h^K(v, \xi)) - h^L(v, h^K(u, \xi))$$

for any $u, v \in T_x\bar{M}$ and $\xi \in \Gamma(K)$. If we also have a connection defined on $T\bar{M}$, then we can define the covariant derivative of the second fundamental form h_K by

$$\nabla_u h^K(v, \xi) = \overset{L}{\nabla}_u(h^K(v, \xi)) - h^K(\nabla_u v, \xi) - h^K(v, \overset{K}{\nabla}_u \xi)$$

for any $u, v \in T_x\bar{M}$ and $\xi \in \Gamma(K)$. Assume in addition that the connection on $T\bar{M}$ is symmetric, we have the following Codazzi identity:

$$\nabla_u h^K(v, \xi) - \nabla_v h^K(u, \xi) = \pi_L(R_\nabla(u, v)(\iota_K \xi)). \tag{3.1}$$

Furthermore, if E admits a metric g compatible with ∇ and K, L are orthogonal to the metric in the sense that

$$g(\iota_K \xi, \iota_L \eta) = 0$$

for any $\xi \in \Gamma(K)$ and $\eta \in \Gamma(L)$. Then the metric g induces naturally metrics g_K, g_L on subbundles K, L respectively and gives us the Weingarten relation associating the second fundamental forms h^K and h^L by

$$g^L(h^K(u, \xi), \eta) + g^K(\xi, h^L(u, \eta)) = 0.$$

3.2.2 Time-dependent immersion.

Let I be an interval in $\mathbb{R}$. Then the tangent bundle $T(\Sigma \times I)$ splits into $\mathcal{H} \oplus \mathbb{R}\partial_t$ where $\mathcal{H} := \{u \in T(\Sigma \times I) : dt(u) = 0\}$ is the 'spatial' tangent bundle.

Let $F\colon \Sigma\times I \to \bar{M}$ be a smooth map such that $F(\cdot,t)\colon \Sigma \to \bar{M}$ defines a free boundary hypersurface with respect to the barrier S. Note that the pullback bundle $F^*T\bar{M}$ is equipped with a metric $\bar{g}_F$ and a connection ${}^F\bar{\nabla}$ induced from the ambient manifold $\bar{M}$.

The pushforward map of the spatial tangent vector $F_*\colon \mathcal{H} \to F^*T\bar{M}$ defines a subbundle of $F^*T\bar{M}$ of rank n . We denote by $\mathcal{N}$ the orthogonal complement of $F_*(\mathcal{H})$ in $F^*T\bar{M}$. Then $\mathcal{N}$ is a subbundle of $F^*T\bar{M}$ of rank 1, which is referred to as the (spacetime) normal bundle.

Now $\mathcal{H}$ and $\mathcal{N}$ are subbundles of $F^*T\bar{M}$ with inclusion maps

$$F_*\colon \mathcal{H} \to F^*T\bar{M} \qquad \iota\colon \mathcal{N} \to F^*T\bar{M}$$

and projection maps

$$\pi\colon F^*T\bar{M} \to \mathcal{H} \qquad \overset{\perp}{\pi}\colon F^*T\bar{M} \to \mathcal{N}.$$

Then by subsection 3.2.1, we can define the metric $g(u,v) := \bar{g}_F(F_*u, F_*v)$, the connection $\nabla := \pi \circ {}^F\bar{\nabla} \circ F_*$ on the bundle $\mathcal{H}$. Similarly, on the bundle $\mathcal{N}$, we can define the metric $\overset{\perp}{g}(\xi,\eta) := \bar{g}_F(\iota\xi, \iota\eta)$, the connection $\overset{\perp}{\nabla} := \overset{\perp}{\pi} \circ {}^F\bar{\nabla} \circ \iota$.

By restricting the first argument of the second fundamental form

$$h^{\mathcal{H}} = \overset{\perp}{\pi} \circ {}^F\bar{\nabla} \circ F_* \in \Gamma(T(\Sigma \times I)^* \otimes \mathcal{H}^* \otimes \mathcal{N})$$

to $\mathcal{H}$, we can define the symmetric bilinear form $h \in \Gamma(\mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{N})$ on $\mathcal{H}$ with values in $\mathcal{N}$. The mean curvature vector $\vec{H} \in \Gamma(\mathcal{N})$ on Σ is thus defined as $\vec{H} := \mathrm{Tr}_g(h)$.

Let $I = [0,T)$. We say a time-dependent immersion $F\colon \Sigma \times I \to \bar{M}$ is a solution to MCF with free boundary if

$$F_*\partial_t = \iota\vec{H}.$$

Note that in the case of MCF with free boundary, the remaining components of $h^{\mathcal{H}}$ are given by

$$\begin{aligned} h^{\mathcal{H}}(\partial_t, v) &= \overset{\perp}{\pi}({}^F\bar{\nabla}_{\partial_t} F_* v) \\ &= \overset{\perp}{\pi}({}^F\bar{\nabla}_v F_* \partial_t + F_*([\partial_t, v])) \\ &= \overset{\perp}{\nabla}_v \vec{H} \end{aligned}$$

where $\overset{\perp}{\pi} \circ F_*([\partial_t, v]) = 0$ for $[\partial_t, v] = (\partial_t v^i)\partial_i \in \mathcal{H}$.

3.3 Boundary derivatives

Since $\mathcal{N}$ is a trivial subbundle of $F^*T\bar{M}$ of rank 1, we can fix a global unit section $\nu \in \Gamma(\mathcal{N})$. Let H be a function over $\Sigma \times I$ defined by $H := -\overset{\perp}{g}(\vec{H}, \nu)$. Then $\vec{H} = -H\nu$.

Theorem 3.3.1. $N(H) = HA^S(\iota\nu, \iota\nu)$

Proof.

$$\begin{aligned} N(H) &= -\overset{\perp}{g}(\overset{\perp}{\nabla}_N \vec{H}, \nu) \\ &= -\overset{\perp}{g}(h^{\mathcal{H}}(\partial_t, N), \nu) \\ &= -\bar{g}_F({}^F\bar{\nabla}_{\partial_t} F_* N, \iota\nu) \\ &= -\bar{g}_F({}^F\bar{\nabla}_{\partial_t} \nu_S \circ F, \iota\nu) \\ &= -\bar{g}(\bar{\nabla}_{F_*\partial_t} \nu_S, \iota\nu) \\ &= -\bar{g}(\bar{\nabla}_{-H\iota\nu} \nu_S, \iota\nu) \\ &= HA^S(\iota\nu, \iota\nu) \end{aligned}$$

□

In the rest of the section, we fix a time $t_0 \in I$. Then the restrictions of $\mathcal{H}$ and $\mathcal{N}$ to $\Sigma \times \{t_0\}$ are the usual tangent and the normal bundle of F_{t_0}. Moreover, ∇ agrees with the Levi-Civita connection of $g(t_0)$ and h agrees with the usual second fundamental form of the immersion F_{t_0}.

Let $p \in \partial\Sigma$. Then for any $u \in T_p\Sigma$, we can extend u to a section of $\mathcal{H}$ in an open neighborhood of $(p, t_0) \in \Sigma \times I$. Since the quantities we are going to work with in the rest of the section are all tensorial, we can further assume that $\nabla u = \pi \circ {}^F\bar{\nabla} \circ F_* u = 0$ at p without affecting the values of the quantities. But for vectors in the tangent space of $\partial\Sigma$, such extension would make the vector leave the tangent space of the boundary. Hence we can only extend the vector to the interior of Σ along the normal direction N and have that $\nabla_N u = 0$.

Before computing the boundary derivative of the second fundamental form h on $\mathcal{H}$. We first derive a relationship between h and A^S on $F(\partial\Sigma \times \{t_0\}) \subset S$.

Lemma 3.3.2. *For $u \in T_p\partial\Sigma$, we have that*

$$h(u, N) = A^S(F_*u, \iota\nu)\nu.$$

Proof. Since $u \in T_p\partial\Sigma$, then $\bar{g}(F_*u, \nu_S) = \bar{g}_F(F_*u, F_*N) = 0$. By construction, we also have that $\bar{g}(\iota\nu, \nu_S) = \bar{g}_F(\iota\nu, F_*N) = 0$. Hence $\iota\nu$ and F_*u is tangent to the barrier S and

$$A^S(F_*u, \iota\nu) = \bar{g}(\iota\nu, \bar{\nabla}_{F_*u}\nu_S) = \bar{g}_F(\iota\nu, {}^F\bar{\nabla}_u F_*N).$$

Therefore,

$$h(u, N) = \overset{\perp}{\pi}({}^F\bar{\nabla}_u F_*N) = \bar{g}_F(\iota\nu, {}^F\bar{\nabla}_u F_*N)\nu = A^S(F_*u, \iota\nu)\nu.$$

□

Theorem 3.3.3. *For $u, v \in T_p\partial\Sigma$,*

$$\begin{aligned}\nabla_N h(u, v) = &\left(\nabla_{F_*u} A^S(\iota\nu, F_*v) + A^S(\bar{\nabla}^S_{F_*u}\iota\nu, F_*v)\right)\nu \\ &+ A^S(F_*u, F_*v)h(N, N) - h(\nabla_u N, v) \\ &+ A^S(\iota\nu, \iota\nu)h(u, v) + \overset{\perp}{\pi}(F^*R_{\nabla}(u, N)(F_*v)).\end{aligned}$$

Proof. By the Codazzi identity Equation 3.1, we have that

$$\nabla_N h(u, v) - \nabla_u h(N, v) = \overset{\perp}{\pi}(F^*R_{\nabla}(u, N)(F_*v))$$

where

$$\nabla_u h(N,v) = \overset{\perp}{\nabla}_u(h(N,v)) - h(\nabla_u N, v) - h(N, \nabla_u v).$$

Since $v \in T_p\partial\Sigma$, by Lemma 3.3.2, we have that

$$\begin{aligned}\overset{\perp}{\nabla}_u(h(N,v)) &= \overset{\perp}{\nabla}_u(A^S(F_*v, \iota\nu)\nu) \\ &= F_*u(A^S(F_*v, \iota\nu))\nu.\end{aligned}$$

Since the equation we need to derive is tensorial, we can extend the vectors u, v by parallel transport on $\partial\Sigma$ and along the direction N to the interior of Σ where $\nabla_u v = g(\nabla_u v, N)N$. Hence,

$$\begin{aligned}h(N, \nabla_u v) &= g(\nabla_u v, N)h(N,N) \\ &= \bar{g}_F({}^F\bar{\nabla}_u F_*v, F_*N)h(N,N) \\ &= \bar{g}(\bar{\nabla}_{F_*u}F_*v, \nu_S)h(N,N) \\ &= -A^S(F_*u, F_*v)h(N,N).\end{aligned}$$

Moreover, the pushforward $F_*u, F_*v \in T_pS$ can be extended to vector fields on the barrier S where $\bar{\nabla}^S_{F_*u}F_*v = \iota h(u,v)$ and

$$\begin{aligned}&F_*u(A^S(\iota\nu, F_*v)) \\ =&\nabla_{F_*u}A^S(\iota\nu, F_*v) + A^S(\bar{\nabla}^S_{F_*u}\iota\nu, F_*v) + A^S(\iota\nu, \bar{\nabla}^S_{F_*u}F_*v) \\ =&\nabla_{F_*u}A^S(\iota\nu, F_*v) + A^S(\bar{\nabla}^S_{F_*u}\iota\nu, F_*v) + A^S(\iota h(u,v), \iota\nu)\end{aligned}$$

where $\bar{\nabla}^S$ is the connection on S induced from $\bar{\nabla}$.

Since $A^S(\iota h(u,v), \iota\nu)\nu = A^S(\iota\nu, \iota\nu)h(u,v)$, combining all equations above, we can conclude that

$$\begin{aligned}\nabla_N h(u,v) = &\left(\nabla_{F_*u}A^S(\iota\nu, F_*v) + A^S(\bar{\nabla}^S_{F_*u}\iota\nu, F_*v)\right)\nu \\ &+ A^S(F_*u, F_*v)h(N,N) - h(\nabla_u N, v) \\ &+ A^S(\iota\nu, \iota\nu)h(u,v) + \overset{\perp}{\pi}(F^*R_\nabla(u,N)(F_*v)).\end{aligned}$$

□

3.4 Stampacchia's iteration

In this section, we assume that the ambient manifold $\bar{M}$ satisfies uniform bounds

$$\sigma_x(P) \leq K, \quad i_x(\bar{M}) \geq i(\bar{M})$$

for constants $K \geq 0$ and $i(\bar{M}) > 0$.

3.4.1 Michael–Simon with free boundary.

In Lemma 2.4.3, we have seen a Sobolev inequality for surfaces in $\mathbb{R}^{n+1}$. Hoffman and Spruck [21] extended Lemma 2.4.3 for Riemannian submanifolds.

Lemma 3.4.1 ([21, Theorem 2.2])**.** *Let f be a Lipschitz function on Σ vanishing on $\partial\Sigma$. Then*

$$\left(\int_\Sigma |f|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq c(n)\left(\int_\Sigma |\nabla f| + \int_\Sigma H\,|f|\right)$$

provided

$$K^2(1-\alpha)^{-\frac{2}{n}}(\omega^{-1}\,|\mathrm{supp}\ f|^{\frac{2}{n}}) \leq 1$$

and

$$2\rho_0 \leq i(N)$$

where ω_n is the volume of the unit ball and

$$\rho_0 = K^{-1}\arcsin\left\{K(1-\alpha)^{-\frac{1}{n}}\left(\omega_n^{-1}\,|\mathrm{supp}\ f|\right)^{\frac{1}{n}}\right\}.$$

Here $0 < \alpha < 1$ is a free parameter and

$$c(n) = \pi 2^{n-1}\alpha^{-1}\,(1-\alpha)^{-\frac{1}{n}}\,\frac{n}{n-1}\omega_n^{-\frac{1}{n}}.$$

We further generalize Lemma 3.4.1 to the free boundary setting.

Theorem 3.4.2. *For any Σ meeting S orthogonally, any $f \in C^1(\bar{\Sigma})$ satisfying the conditions in Lemma 3.4.1, and any positive integer $p < n$, there exists a constant $c = c(n, p, S, \bar{M})$ such that*

$$\|f\|_{\frac{np}{n-p};\Sigma} \leq c(\|\nabla f\|_{p;\Sigma} + \|Hf\|_{p;\Sigma} + \|f\|_{p;\Sigma}).$$

Lemma 3.4.3. *There exists a constant $c = c(n, S, \bar{M})$ such that for any Σ meeting S orthogonally, and any $f \in C^1(\bar{\Sigma})$*

$$\frac{1}{c}\int_{\partial\Sigma} |f| \leq \int_\Sigma |\nabla f| + \int_\Sigma |Hf| + \int_\Sigma |f| \, .$$

Proof. Fix $X \in \mathfrak{X}(\bar{M})$ which is 0 outside a neighborhood of S and $X|_S = \nu_S$. Let ν be the outward normal of $\partial\Sigma$. By the divergence theorem and product rule, we have that

$$\begin{aligned}
\int_{\partial\Sigma} |f| &= \int_{\partial\Sigma} (|f|\, X) \cdot \nu \\
&= \int_\Sigma \operatorname{div}_\Sigma \left(|f|\, X^T\right) \\
&= \int_\Sigma \nabla |f| \cdot X^T + |f| \operatorname{div}_\Sigma(X^T).
\end{aligned}$$

Since $X = X^T + X^\perp$ and $\operatorname{div}_\Sigma(X^\perp) = (X \cdot N)H$, we can conclude that

$$\begin{aligned}
\int_{\partial\Sigma} |f| &= \int_\Sigma \nabla |f| \cdot X^T + |f| \operatorname{div}_\Sigma(X^T) \\
&= \int_\Sigma \nabla |f| \cdot X^T + |f| \operatorname{div}_\Sigma(X) - |f|\,(X \cdot N)\, H \\
&\leq \max |X| \int_\Sigma |\nabla f| + n \max |\nabla X| \int_\Sigma |f| + \max |X| \int_\Sigma |Hf| \, .
\end{aligned}$$

□

Theorem 3.4.4. *There exists a constant $c = c(n)$ such that for any Σ meeting S orthogonally and any $f \in C^1_c(\bar{\Sigma})$ satisfying the conditions in Lemma 3.4.1,*

$$\frac{1}{c}\Big(\int_\Sigma |f|^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \leq \int_\Sigma |\nabla f| + \int_\Sigma |Hf| + \int_{\partial\Sigma} |f| \, .$$

Proof. Without loss of generality, we assume that $f > 0$.

Let $\Omega = \{x \in \Sigma : d(x, \partial\Sigma) \leq \epsilon\}$ where $d\colon \Sigma \times \Sigma \to \mathbb{R}$ is the distance function on Σ. Then for sufficiently small $\epsilon > 0$, we can find the diffeomorphism

$$\phi\colon [0, \epsilon] \times \partial\Sigma \to \Omega$$

with bounded Jacobian $|J\phi| \in [\frac{1}{2}, 2]$.

Hence

$$\begin{aligned} \int_\Omega f &= \int_0^\epsilon \int_{\partial\Sigma} f\,|J\phi| \\ &\leq 2 \int_0^\epsilon \int_{\partial\Sigma} f(t, x) \\ &\leq \epsilon^2\,|\partial\Sigma| \sup_\Sigma |\nabla f| + 2\epsilon \int_{\partial\Sigma} f \end{aligned} \tag{3.2}$$

where the last inequality follows from the Taylor expansion $f(t, x) = f(0, x) + t\frac{\partial}{\partial t} f(t^*(x), x)$ for some $t^*(x) \in (0, \epsilon)$ depending on $x \in \partial\Sigma$.

Let $\eta\colon \Sigma \to \mathbb{R}$ be a smooth function such that $\eta|_{\partial\Sigma} \equiv 0$, $\eta|_{\Sigma-\Omega} \equiv 1$ and $|\nabla\eta| \leq \frac{2}{\epsilon}$. By Equation 3.2, we have that

$$\begin{aligned} \int_\Sigma ((1-\eta)\, f)^{\frac{n}{n-1}} &\leq \int_\Omega f^{\frac{n}{n-1}} \\ &\leq \epsilon^2\,|\partial\Sigma| \sup_\Sigma \left|\nabla\left(f^{\frac{n}{n-1}}\right)\right| + 2\epsilon \int_{\partial\Sigma} f^{\frac{n}{n-1}} \\ &\leq \epsilon C \end{aligned}$$

for C independent of ϵ. For the function ηf which vanishes on $\partial\Sigma$, we can apply Lemma 3.4.1 and conclude that

$$\|\eta f\|_{\frac{n}{n-1}} \leq c(n) \left(\int_\Sigma |\nabla(\eta f)| + \int_\Sigma |H|\,\eta f \right).$$

Therefore, for $c = c(n)$ and all sufficiently small $\epsilon > 0$,

$$\begin{aligned}
\|f\|_{\frac{n}{n-1}} &\leq \|\eta f\|_{\frac{n}{n-1}} + \|(1-\eta)f\|_{\frac{n}{n-1}} \\
&\leq c\int_\Sigma \eta\,|\nabla f| + c\int_\Sigma |H|\,\eta f + c\int_\Sigma |\nabla\eta|\,f + \epsilon^{\frac{n-1}{n}}C \\
&\leq c\int_\Sigma |\nabla f| + c\int_\Sigma |H|\,f + \frac{2c}{\epsilon}\int_\Omega f + \epsilon^{\frac{n-1}{n}}C \\
&\leq c\int_\Sigma |\nabla f| + c\int_\Sigma |H|\,f + 4c\int_{\partial\Sigma} f \\
&\quad + 2c\epsilon\,|\partial\Sigma|\sup_\Sigma |\nabla f| + \epsilon^{\frac{n-1}{n}}C.
\end{aligned}$$

The conclusion follows by taking $\epsilon \to 0$. □

By combining Lemma 3.4.3 and Theorem 3.4.4, we can prove Theorem 3.4.2 using the argument identical to the proof of [15, Theorem 2.3].

3.4.2 Main theorem and the idea of proof.

Let $(\Sigma_t)_{t\in[0,T)}$ be a class of hypersurfaces following the MCF with free boundary on the barrier S. Assume $T < \infty$. Let f_α be a non-negative function on Σ_t where $\alpha = \alpha(S, \Sigma_0, T, n)$. Then we consider another two functions $\tilde{H} > 0, \tilde{G} \geq 0$ on Σ_t such that

$$H = O(\tilde{H}) \qquad \nabla\tilde{H} = O(\tilde{G}).$$

Finally, for another two positve constant σ and k, we let $f = f_\alpha \tilde{H}^\sigma$, $f_k = (f-k)_+$ and $A(k) = \{f \geq k\}$, $A(k,t) = A(k) \cap \Sigma_t$.

We say the function f satisfies the condition $(\star)$ if there exist constants $c = c(S, \Sigma_0, \bar{M}, T, n, \alpha)$ and $C = C(S, \Sigma_0, \bar{M}, T, n, \alpha, p, \sigma)$ such that the following two inequalities hold:

(Poincare-like)

$$\begin{aligned}\frac{1}{c}\int_{\Sigma_t} f^p\tilde{H}^2 \leq & p\left(1+\frac{1}{\beta}\right)\int_{\Sigma_t} f^{p-2}\left|\nabla f\right|^2 \\ & + (1+\beta p)\int_{\Sigma_t}\frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}}f^{p-1} \\ & + \int_{\Sigma_t} f^p + \int_{\partial\Sigma_t} f^{p-1}\tilde{H}^\sigma\end{aligned} \tag{3.3}$$

(Evolution-like)

$$\begin{aligned}\partial_t\int_{\Sigma_t} f_k^p \leq & -\frac{1}{3}p^2\int_{\Sigma_t} f_k^{p-2}\left|\nabla f\right|^2 \\ & -\frac{p}{c}\int_{\Sigma_t}\frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}}f_k^{p-1} \\ & + C\int_{A(k,t)} f^p + cp\int_{\partial\Sigma_t} f_k^{p-1}\tilde{H}^\sigma \\ & + cp\sigma\int_{A(k,t)}\tilde{H}^2 f^p - \frac{1}{5}\int_{\Sigma_t}\tilde{H}^2 f_k^p + C\left|A(k)\right|\end{aligned} \tag{3.4}$$

for any $p > p_0(n,\alpha,c), \sigma < \frac{1}{2}, k > 0, \beta > 0$.

Now we state the main theorem.

Theorem 3.4.5. *If f satisfies $(\star)$, then for sufficiently small σ depending on sufficiently large p, $f = f_\alpha\tilde{H}^\sigma$ is uniformly bounded in spacetime by a constant depending on $(S, \Sigma_0, \bar{M}, T, n, \alpha, p, \sigma)$.*

The proof of the main theorem splits into three parts. First, we find a way to handle the boundary term. Then we obtain a higher L^p bound for f by rearranging and combining the inequalities. Finally, using the higher L^p bound and the inequality in Theorem 3.4.2, we establish the iteration scheme.

3.4.3 Boundary integral estimate.

The following two lemmas are needed to handle the boundary integral.

Lemma 3.4.6. *Let g be any non-negative funciton on Σ_t. If $r \in (0,2)$, $0 < q < p$ with $\frac{rp}{q} < 2$, then for any $\mu > 0$,*

$$\int_{\Sigma_t} g^q \tilde{H}^r \leq \frac{1}{\mu} \int_{\Sigma_t} g^p \tilde{H}^2 + C(\mu, r, q, p) \int_{\Sigma_t} g^p + |\mathrm{spt} g| .$$

Proof. By Young's inequality, since $0 < q < p$, we have that

$$\begin{aligned}\int_{\Sigma_t} g^q \tilde{H}^r &\leq \int_{\Sigma_t} (g^q \tilde{H}^r)^{\frac{p}{q}} + 1 \\ &= \int_{\Sigma_t} g^p \tilde{H}^{\frac{rp}{q}} + |\mathrm{spt} g| .\end{aligned}$$

Since $\eta := \frac{rp}{2q} < 1$, again by Young's inequality, we can deduce that

$$\begin{aligned} g^p \tilde{H}^{2\eta} &= g^{p\eta} \tilde{H}^{2\eta} g^{p(1-\eta)} \\ &= \left(\frac{1}{\mu\eta} g^p \tilde{H}^2 \right)^{\eta} \left((\mu\eta)^{\frac{\eta}{1-\eta}} g^p \right)^{1-\eta} \\ &\leq \frac{1}{\mu} g^p \tilde{H}^2 + C(\mu, r, q, p) g^p \end{aligned}$$

where $C(\mu, r, q, p) = \frac{(\mu\eta)^{\frac{\eta}{1-\eta}}}{1-\eta}$. The conclusion follows by combining the two inequalities above. □

Lemma 3.4.3, which associates integrals on the boundary and the interior for free boundary surfaces, is also needed.

Now we can prove the following lemma which estimates the boundary integral.

Lemma 3.4.7. *For any $\sigma < \frac{1}{2}, p > 4$ and $\mu > 0$, there exists constants $c = c(n, S, \bar{M})$ and $C = C(n, S, \bar{M}, \mu, p)$ such that*

$$\begin{aligned}\int_{\partial\Sigma_t} f_k^{p-1} \tilde{H}^\sigma \leq & c \int_{\Sigma_t} |\nabla f|^2 f_k^{p-2} + c\sigma \int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f_k^{p-1} \\ &+ \frac{cp^2}{\mu} \int_{A(k,t)} f^p \tilde{H}^2 + C \int_{A(k,t)} f^p + C\, |A(k,t)| .\end{aligned}$$

Proof. By Lemma 3.4.3, we have that

$$\frac{1}{c(n,S,\bar{M})}\int_{\partial\Sigma_t} f_k^{p-1}\tilde{H}^{\sigma} \leq \int_{\Sigma_t}\left|\nabla\left(f_k^{p-1}\tilde{H}^{\sigma}\right)\right| + \int_{\Sigma_t}\left|Hf_k^{p-1}\tilde{H}^{\sigma}\right| + \int_{\Sigma_t}\left|f_k^{p-1}\tilde{H}^{\sigma}\right|.$$

Since f_k and $\tilde{H}$ are non-negative, by product rule and triangle inequality, we have that

$$\left|\nabla\left(f_k^{p-1}\tilde{H}^{\sigma}\right)\right| \leq pf_k^{p-2}\tilde{H}^{\sigma}\left|\nabla f\right| + c(n,S,\bar{M})\sigma f_k^{p-1}\tilde{H}^{\sigma-1}\tilde{G}.$$

Combining the inequalities above, we have that, for some constant $c = c(n,S,\bar{M})$ and $\sigma < \frac{1}{2}$,

$$\begin{aligned}\int_{\partial\Sigma_t} f_k^{p-1}\tilde{H}^{\sigma} \leq & c\int_{\Sigma_t} f_k^{p-2}\left|\nabla f\right|^2 + cp^2\int_{\Sigma_t} f_k^{p-2}\tilde{H}^{2\sigma} \\ & + c\sigma\int_{\Sigma_t} f_k^{p-1}\frac{\tilde{G}^2}{\tilde{H}^{\sigma-2}} + c\int_{\Sigma_t} f_k^{p-1}\left(\tilde{H}^{\sigma}+\tilde{H}^{\sigma+1}\right)\end{aligned}$$

Finally, since $\sigma < \frac{1}{2}$ and $p > 4$, for any $\mu > 0$, we can apply Lemma 3.4.6 for $\int_{\Sigma_t} f_k^{p-2}\tilde{H}^{2\sigma}$, $\int_{\Sigma_t} f_k^{p-1}\tilde{H}^{\sigma}$ and $\int_{\Sigma_t} f_k^{p-1}\tilde{H}^{1+\sigma}$; thus concluding that

$$\begin{aligned}\int_{\partial\Sigma_t} f_k^{p-1}\tilde{H}^{\sigma} \leq & c\int_{\Sigma_t}\left|\nabla f\right|^2 f_k^{p-2} + c\sigma\int_{\Sigma_t}\frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}f_k^{p-1}} \\ & + \frac{cp^2}{\mu}\int_{A(k,t)} f^p\tilde{H}^2 + C\int_{A(k,t)} f^p + C\left|A(k,t)\right|\end{aligned}$$

for constants $c = c(n,S,\bar{M})$ and $C = C(n,S,\bar{M},\mu,p)$. □

3.4.4 Higher L^p bound.

Next, we establish the higher L^p bound for f.

Lemma 3.4.8. *Suppose f satisfies $(\star)$. Then there exist constants $p_0(c)$ and $c_\sigma(c)$ depending on some $c = c(S,\Sigma_0,\bar{M},T,n,\alpha)$ such that for $p > p_0(c)$ and $\sigma < \frac{c_\sigma(c)}{\sqrt{p}}$,*

$$\int_0^T\int_{\Sigma_t} f^p \leq C_1(C,T,\Sigma_0) < \infty.$$

Proof. By Equation 3.4, for $k = 0$, we have that

$$\begin{aligned}
\partial_t \int_{\Sigma_t} f^p \leq & -\frac{1}{3}p^2 \int_{\Sigma_t} f^{p-2} |\nabla f|^2 - \frac{p}{c} \int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f^{p-1} \\
& + C \int_{\Sigma_t} f^p + cp \int_{\partial\Sigma_t} f^{p-1}\tilde{H}^\sigma \\
& + cp\sigma \int_{\Sigma_t} \tilde{H}^2 f^p - \frac{1}{5} \int_{\Sigma_t} \tilde{H}^2 f^p + C |\Sigma_t| \\
\leq & -\frac{1}{3}p^2 \int_{\Sigma_t} f^{p-2} |\nabla f|^2 - \frac{p}{c} \int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f^{p-1} \\
& + C \int_{\Sigma_t} f^p + cp \int_{\partial\Sigma_t} f^{p-1}\tilde{H}^\sigma - \frac{1}{5} \int_{\Sigma_t} \tilde{H}^2 f^p + C |\Sigma_t| \\
& + cp\sigma [p \left(1 + \frac{1}{\beta}\right) \int_{\Sigma_t} f^{p-2} |\nabla f|^2 \\
& + (1 + \beta p) \int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f^{p-1} + \int_{\Sigma_t} f^p + \int_{\partial\Sigma_t} f^{p-1}\tilde{H}^\sigma]
\end{aligned}$$

where we use Equation 3.3 to estimate the term $cp\sigma \int_{\Sigma_t} \tilde{H}^2 f^p$. For the boundary integral $\int_{\partial\Sigma_t} f^{p-1}\tilde{H}^\sigma$, we apply the previous estimate Lemma 3.4.7 and conclude that

$$\begin{aligned}
\partial_t \int_{\Sigma_t} f^p \leq & \left[-\frac{1}{3}p^2 + cp^2\sigma(1 + \frac{1}{\beta}) + cp\right] \int_{\Sigma_t} f^{p-2} |\nabla f| \\
& + \left[-\frac{p}{c} + cp\sigma(1 + \beta p) + cp\sigma\right] \int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f^{p-1} \\
& + \left(\frac{cp^3}{\mu} - \frac{1}{5}\right) \int_{\Sigma_t} \tilde{H}^2 f^p \\
& + C |\Sigma_t| + C \int_{\Sigma_t} f^p
\end{aligned}$$

For $p > 12c$, we can choose constants $\mu = 10cp^3, \beta = \frac{1}{\sqrt{cp}}, \sigma = \frac{1}{6\sqrt{c^3 p}}$ such that

$$\begin{cases}
-\frac{1}{3}p^2 + cp^2\sigma(1 + \frac{1}{\beta}) + cp \leq 0 \\
-\frac{p}{c} + cp\sigma(1 + \beta p) + cp\sigma \leq 0 \\
\frac{cp^3}{\mu} - \frac{1}{5} \leq 0.
\end{cases}$$

Therefore $\int_0^T \int_{\Sigma_t} f^p \leq C_1(C, T, \Sigma_0) < \infty$ as T is finite. □

We can also simplify the evolution-like equation for f_k and obtain the following lemma.

Lemma 3.4.9. *Suppose f satisfies $(\star)$. Then for σ, p satisfying the same bounds as Lemma 3.4.8 and C independent of k,*

$$\begin{aligned}\partial_t \int_{\Sigma_t} f_k^p \leq -\frac{p^2}{12}\int_{\Sigma_t} f_k^{p-2}\left|\nabla f\right|^2 + C\int_{A(k,t)} f^p + C\left|A(k)\right| \\ + C\int_{A(k,t)} \tilde{H}^2 f^p\end{aligned}$$

Proof. By rewriting the boundary integral in Equation 3.4 using Lemma 3.4.7, we have that

$$\begin{aligned}\partial_t \int_{\Sigma_t} f_k^p \leq & -\frac{1}{3}p^2\int_{\Sigma_t} f_k^{p-2}\left|\nabla f\right|^2 + C\int_{A(k,t)} f^p \\ & -\frac{p}{c}\int_{\Sigma_t}\frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f_k^{p-1} + C\left|A(k)\right| \\ & + cp\sigma\int_{A(k,t)}\tilde{H}^2 f^p - \frac{1}{5}\int_{\Sigma_t}\tilde{H}^2 f_k^p \\ & + cp\bigg[\int_{\Sigma_t}\left|\nabla f\right|^2 f_k^{p-2} + \sigma\int_{\Sigma_t}\frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f_k^{p-1} \\ & + \frac{p^2}{\mu}\int_{A(k,t)} f^p\tilde{H}^2 + C\int_{A(k,t)} f^p + C\left|A(k,t)\right|\bigg] \\ \leq & \left(cp - \frac{1}{3}p^2\right)\int_{\Sigma_t} f_k^{p-2}\left|\nabla f\right|^2 + C\int_{A(k,t)} f^p \\ & + p\left(c\sigma - \frac{1}{c}\right)\int_{\Sigma_t}\frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f_k^{p-1} + C\left|A(k)\right| \\ & + cp\left(\sigma + \frac{p^2}{\mu}\right)\int_{A(k,t)}\tilde{H}^2 f^p - \frac{1}{5}\int_{\Sigma_t}\tilde{H}^2 f_k^p\end{aligned}$$

The conclusion follows by choosing the value of p, σ, μ as in the proof of Lemma 3.4.8.

□

3.4.5 Iteration scheme and the uniform bound.

By Theorem 3.4.2, for $n \geq 2$, there are some $q > 1$ and $c = c(n, q, |\Sigma_0|, S, \bar{M})$ such that

$$\left(\int_\Sigma v^{2q}\right)^{\frac{1}{q}} \leq c\int_\Sigma |\nabla v|^2 + c\int_\Sigma v^2 H^2 + c\int_\Sigma v^2$$

provided that v satisfies the assumptions in Lemma 3.4.1. For $n > 2$, we let $q = \frac{n}{n-2}$. For $n = 2$, we apply Corollary 2.4 and Remark 2.5 in [15].

Take $v = f_k^{\frac{p}{2}}$, then by Lemma 3.4.8, we have that

$$|\text{supp } v| = |A(k,t)| \leq \frac{1}{k}\int_{\Sigma_t} f \leq \frac{1}{k}C'$$

where C' depend on C_1 and $|\Sigma_0|$. Since $C_1 = C_1(C, T, \Sigma_0)$ and the constant C in $(\star)$ depends on $(S, \Sigma_0, \bar{M}, T, n, \alpha, p, \sigma)$, for $k \geq k_0(S, \Sigma_0, \bar{M}, T, n, \alpha, p, \sigma)$

$$\left(\int_{\Sigma_t} f_k^{pq}\right)^{\frac{1}{q}} \leq c\int_{\Sigma_t} \left|\nabla f_k^{\frac{p}{2}}\right|^2 + c\int_{\Sigma_t} f_k^p H^2 + c\int_{\Sigma_t} f_k^p. \tag{3.5}$$

Theorem 3.4.10. *Suppose there are constants p_0 and σ_0 independent of p, σ, k such that for $p > p_0$ and $\sigma < \frac{\sigma_0}{\sqrt{p}}$, we have that*

$$\int_0^T \int_{\Sigma_t} f^p < \infty$$

and

$$\partial_t \int_{\Sigma_t} f^p + \frac{1}{c}\int_{\Sigma_t} \left|\nabla f_k^{\frac{p}{2}}\right|^2 \leq C\int_{A(k,t)} \tilde{H}^2 f^p + C\int_{A(k,t)} f^p + C|A(k,t)| \tag{3.6}$$

for any $k > 0$ where C, c are constants independent of k. Then for sufficient small σ, f is uniformly bounded in spacetime and the bound will depend on $(S, \Sigma_0, \bar{M}, T, n, \alpha, p, \sigma)$.

Proof. Integrating Equation 3.6 and Equation 3.5 over $[0, T)$ yields that

$$\sup_{t\in[0,T)} \int_{\Sigma_t} f^p + \frac{1}{c}\int_0^T \int_{\Sigma_t} \left|\nabla f_k^{\frac{p}{2}}\right|^2 \leq C\iint_{A(k)} \tilde{H}^2 f^p + C\iint_{A(k)} f^p + C|A(k)|$$

and

$$\int_0^T \left(\int_{\Sigma_t} f_k^{pq}\right)^{\frac{1}{q}} \leq c\int_0^T \int_{\Sigma_t} \left|\nabla f_k^{\frac{p}{2}}\right|^2 + c\iint_{A(k)} f_k^p H^2 + c\iint_{A(k)} f_k^p.$$

provided that $k \geq k_0(S, \Sigma_0, \bar{M}, T, n, \alpha, p, \sigma)$. Then by adjust the constants to absorb the term $\int_0^T \int_{\Sigma_t} \left|\nabla f_k^{\frac{p}{2}}\right|^2$, we have that

$$\begin{aligned}&\max\left\{\sup_{t\in[0,T)} \int_{\Sigma_t} f_k^p, \int_0^T \left(\int_{\Sigma_t} f_k^{pq}\right)^{\frac{1}{q}}\right\}\\ \leq& C\iint_{A(k)} \tilde{H}^2 f^p + C\iint_{A(k)} f^p + C\left|A(k)\right|.\end{aligned}$$

Hence by Holder's inequality,

$$\begin{aligned}\int_0^T \int_{\Sigma_t} f_k^{p\frac{2q-1}{q}} &\leq \int_0^T \int_{\Sigma_t} f_k^p f_k^{p\frac{q-1}{q}}\\ &\leq \int_0^T \left(\int_{\Sigma_t} f_k^{pq}\right)^{\frac{1}{q}} \left(\int_{\Sigma_t} f_k^p\right)^{\frac{q-1}{q}}\\ &\leq \left(\sup_{t\in[0,T)} \int_{\Sigma_t} f_k^p\right)^{\frac{q-1}{q}} \int_0^T \left(\int_{\Sigma_t} f_k^{pq}\right)^{\frac{1}{q}}\\ &\leq \left(C\iint_{A(k)} \tilde{H}^2 f^p + C\iint_{A(k)} f^p + C\left|A(k)\right|\right)^{\frac{2q-1}{q}}.\end{aligned} \tag{3.7}$$

For any function g defined on $A(k)$, for any $r > 1$, we can apply the Holder's inequality to have that

$$\iint_{A(k)} g \leq \left|A(k)\right|^{1-\frac{1}{r}} \left(\iint_{A(k)} g^r\right)^{\frac{1}{r}}.$$

Hence

$$\begin{aligned}\int_0^T \int_{\Sigma_t} f_k^{p\frac{2q-1}{q}} \leq& C\left|A(k)\right|^{\frac{2q-1}{q}\left(1-\frac{1}{r}\right)} [\left(\iint_{A(k)} f^{pr}\right)^{\frac{1}{r}}\\ &+ \left(\iint_{A(k)} \tilde{H}^{2r} f^{pr}\right)^{\frac{1}{r}} + \left|A(k)\right|^{\frac{1}{r}}]^{\frac{2q-1}{q}}.\end{aligned}$$

For p sufficiently large relative to r, we have that

$$\iint_{A(k)} f^{pr} < +\infty$$

and

$$\iint_{A(k)} \left(\tilde{H}^2 f^p\right)^r = \iint_{A(k)} \left(f_\alpha \tilde{H}^{\sigma+\frac{2}{p}}\right)^{pr} < +\infty.$$

By fixing r sufficiently large, we let $\gamma = \frac{2q-1}{q}\left(1-\frac{1}{r}\right) > 1$ and $\beta = p\frac{2q-1}{q} > 0$. Thus, for any $l > k$, Equation 3.7 implies that

$$|l-k|^\beta |A(k)| \leq \iint_{A(l)} f_k^\beta \leq C\,|A(k)|^\gamma$$

where the constant C is independent of l and k.

Therefore, by Lemma 2.4.2, $A(k) = 0$ for $k > k_1(\alpha, \beta, C)$. □

3.5 Conclusions and directions for future research

In this thesis, we first provide a background discussion, including PDE analysis and geometric properties of MCF. Then we sketch a proof for Huisken's seminal result in 1984 concerning the convergence of convex hypersurfaces in Euclidean spaces to a round point. In chapter 3, we compute the boundary derivative of the second fundamental form and establish the iteration scheme in Riemannian manifolds following the argument of Edelen.

As discussed at the beginning of this chapter, the boundary derivatives are essential for applying maximum principles to prove that certain inequalities are preserved under the flow. When the barrier surface is not umbilic, cross terms will appear in the boundary derivatives. These terms are impossible to control; thus making maximum principles not applicable.

To cancel problematic cross terms, one could use a perturbation argument of the second fundamental form. The method of perturbation was first introduced by Huisken and Sinestrari [28]. When the barrier is in the Euclidean space of dimension three, Hirsch and Li [20] defined a perturbation tensor which kills off cross terms on the boundary and enables them to apply the maximum principle.

To obtain information on the original second fundamental form, one needs to do the pertubation in a controlled way. In Hirsch and Li's work, one major factor influencing the estimates of the perturbed form is the ball curvature of the barrier. Such property can be well defined in the Euclidean space, but in a 3-manifold, we only have locally defined balls. Brendle [10] introduced the method of using local balls to define ball curvatures in Riemannian manifolds. By combining ideas from Hirsch and Li, and Brendle, it is believed that the difficulty of estimating boundary derivatives could be overcome.

Another future research direction involves certain non-convex initial conditions for convergence of free boundary hypersurfaces in the unit ball. For MCF with free boundary on the standard hypersphere, it is known that any convex free boundary hypersurface will converge to a round half-point [39]. Considering Huisken's study on MCF in spherical spaceforms [24], it is natural to ask whether the convexity condition can be replaced by some non-convex curvature pinching condition.

The analogy between free boundary minimal surfaces in the unit ball and closed minimal surfaces in the standard sphere are reflected in various research results [2, 34, 38, 29]. Such similarities would be helpful towards this research topic. Moreover, the study of MCF in spheres by Huisken [24] would inspire the proposed research topic greatly by its setting of initial condition which implies the positivity of intrinsic curvature of the surfaces.

□ **End of chapter.**

Bibliography

[1] W. K. Allard, *On the first variation of a varifold*, Ann. of Math. (2) **95** (1972), no. 3, 417.

[2] F. J. Almgren, *Some interior regularity theorems for minimal surfaces and an extension of bernstein's theorem*, Ann. of Math. (2) **84** (1966), no. 2, 277–292.

[3] F. Almgren, K. Brakke, and J. Sullivan, *Plateau's problem: an invitation to varifold geometry*, Student mathematical library, W.A. Benjamin, 1966.

[4] S. J. Altschuler and L. F. Wu, *Translating surfaces of the non-parametric mean curvature flow with prescribed contact angle*, Calc. Var. Partial Differential Equations **2** (1994), no. 1, 101–111.

[5] B. Andrews, *Contraction of convex hypersurfaces in Riemannian spaces*, J. Differential Geom. **39** (1994), no. 2, 407–431, 25.

[6] B. Andrews and C. Baker, *Mean curvature flow of pinched submanifolds to spheres*, J. Differential Geom. **85** (2010), no. 3, 357–396, 40.

[7] G. Bellettini and S. Kholmatov, *Minimizing movements for mean curvature flow of droplets with prescribed contact angle*, Journal de Mathématiques Pures et Appliquées **117** (2018), 1–58.

[8] J. Bernstein and L. Wang, *Closed hypersurfaces of low entropy in $\mathbb{R}^4$ are isotopically trivial*, 2020, arXiv: 2003.13858 [math.DG].

[9] K. A. Brakke, *The motion of a surface by its mean curvature. (MN-20)*, Princeton University Press, 1978.

[10] S. Brendle, *An inscribed radius estimate for mean curvature flow in Riemannian manifolds*, 2013, arXiv: 1310.3439 [math.DG].

[11] S. Brendle and G. Huisken, *Mean curvature flow with surgery of mean convex surfaces in* $\mathbb{R}^3$, Invent. Math. **203** (2016), no. 2, 615–654.

[12] J. A. Buckland, *Mean curvature flow with free boundary on smooth hypersurfaces*, J. Reine Angew. Math. **586** (2005), 71–90.

[13] O. Chodosh et al., *Mean curvature flow with generic initial data*, 2020, arXiv: 2003.14344 [math.DG].

[14] B. Chow and D. Knopf, *The Ricci flow: an introduction*, Mathematical surveys and monographs v. 110, Providence, R.I: American Mathematical Society, 2004.

[15] N. Edelen, *Convexity estimates for mean curvature flow with free boundary*, Adv. Math. **294** (2016), 1–36.

[16] L. C. Evans, *Partial differential equations*, 2nd ed, Graduate studies in mathematics v. 19, Providence, R.I: American Mathematical Society, 2010.

[17] B. Guan, *Mean curvature motion of nonparametric hypersurfaces with contact angle condition*, Elliptic and parabolic methods in geometry, 1996, pp. 47–56.

[18] R. Hamilton, *Four-manifolds with positive isotropic curvature*, Comm. Anal. Geom. **5** (1997), 1–92.

[19] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. **17** (1982), no. 2, 255–306.

[20] S. Hirsch and M. Li, *Contracting convex surfaces by mean curvature flow with free boundary on convex barriers*, 2020, arXiv: 2001.01111 [math.AP].

[21] D. Hoffman and J. Spruck, *Sobolev and isoperimetric inequalities for riemannian submanifolds*, Comm. Pure Appl. Math. **27** (1974), no. 6, 715–727.

[22] G. Huisken and C. Sinestrari, *Mean curvature flow with surgeries of two–convex hypersurfaces*, Invent. Math. **175** (2009), 137–221.

[23] G. Huisken, *Contracting convex hypersurfaces in Riemannian manifolds by their mean curvature*, Invent. Math. **84** (1986), no. 3, 463–480.

[24] G. Huisken, *Deforming hypersurfaces of the sphere by their mean curvature*, Math. Z. **195** (1987), no. 2, 205–219.

[25] G. Huisken, *Flow by mean curvature of convex surfaces into spheres*, J. Differential Geom. **20** (1984), no. 1, 237–266.

[26] G. Huisken, *Non-parametric mean curvature evolution with boundary conditions*, J. Differential Equations **77** (1989), no. 2, 369–378.

[27] G. Huisken and T. Ilmanen, *The Inverse Mean Curvature Flow and the Riemannian Penrose Inequality*, J. Differential Geom. **59** (2001), no. 3, 353–437, 85.

[28] G. Huisken and C. Sinestrari, *Convexity estimates for mean curvature flow and singularities of mean convex surfaces*, Acta Math. **183** (1999), no. 1, 45–70.

[29] M. Li, *Free boundary minimal surfaces in the unit ball: recent advances and open questions*, Proceedings of the International Consortium of Chinese Mathematicians, 2017, International Press of Boston, Inc, 2020, pp. 401–436.

[30] G. M. Lieberman, *The first initial-boundary value problem for quasilinear second order parabolic equations*, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze **Ser. 4, 13** (1986), no. 3, 347–387.

[31] K. Liu, H. Xu, and E. Zhao, *Mean curvature flow of higher codimension in Riemannian manifolds*, 2012, arXiv: `1204.0107 [math.DG]`.

[32] J. H. Michael and L. M. Simon, *Sobolev and mean-value inequalities on generalized submanifolds of Rn*, Comm. Pure Appl. Math. **26** (1973), no. 3, 361–379.

[33] W. W. Mullins, *Two-dimensional motion of idealized grain boundaries*, J. Appl. Phys. **27** (1956), no. 8, 900–904.

[34] J. C. C. Nitsche, *Stationary partitioning of convex bodies*, Arch. Ration. Mech. Anal. **89** (1985), no. 1, 1–19.

[35] G. Perelman, *Ricci flow with surgery on three-manifolds*, 2003, arXiv: `math/0303109 [math.DG]`.

[36] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, 2002, arXiv: `math/0211159 [math.DG]`.

[37] B. Priwitzer, *Mean curvature flow with Dirichlet boundary conditions in Riemannian manifolds with symmetries*, Ann. Global Anal. Geom. **23** (2003), no. 2, 157–171.

[38] A. Ros, *Stability of minimal and constant mean curvature surfaces with free boundary*, Mat. Contemp (2008), 221–240.

[39] A. Stahl, *Convergence of solutions to the mean curvature flow with a Neumann boundary condition*, Calc. Var. Partial Differential Equations **4** (1996), no. 5, 421–441.

[40] A. Stahl, *Regularity estimates for solutions to the mean curvature flow with a Neumann boundary condition*, Calc. Var. Partial Differential Equations **4** (1996), no. 4, 385–407.

[41] G. Stampacchia, *Équations elliptiques du second ordre à coefficients discontinus*, Séminaire Jean Leray (1963-1964), no. 3, 1–77.

[42] S. Stuvard and Y. Tonegawa, *An existence theorem for Brakke flow with fixed boundary conditions*, Calc. Var. Partial Differential Equations **60** (2021), no. 1, 43.

[43] A. Volkmann, *A monotonicity formula for free boundary surfaces with respect to the unit ball*, Comm. Anal. Geom. **24** (2016), no. 1, 195–221.

[44] B. White, *Mean Curvature Flow with Boundary*, 2021, arXiv: 1901.03008 [math.DG].