Convergence Theories of the Mean Curvature Flow

WANG Yizi

A Thesis Submitted in Partial Fulfilment

of the Requirements for the Degree of

Master of Philosophy

in

Mathematics

Supervised by

Prof. Li Man-chun, Martin

Abstract of thesis entitled:

Convergence Theories of the Mean Curvature Flow

Submitted by WANG Yizi

for the degree of Master of Philosophy

at The Chinese University of Hong Kong in May 2020

This is the abstract in no more than 350 words.

Acknowledgement

I would like to thank my supervisor...

This work is dedicated to...

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Chapter 1

Introduction

1.1 Background

The last few decades have witnessed a significant development in the field of geometric flow, which leads to many remarkable accomplishments in geometry, topology, physics and computer vision. Among various geometric flows, the Mean Curvature Flow is one of the most important geometric flows for submanifolds of Riemannian manifolds. One way of understanding the Mean Curvature Flow is to regard it as the negative gradient flow for area. In other words, a surface is deforming along the

Mean Curvature Flow to decrease its area as fast as possible.

The study of mean curvature flow and the related field is a critical area of mathematics. Not only does it lead to a series of significant results in physics and mathematics, but it is also expected to solve some long-standing conjectures in geometry and topology. In 1994, Andrews [And94] applied the harmonic mean curvature flow to provide a new proof for the topological sphere theorem and improve the result of homeomorphism to a weaker version of diffeomorphism. Moreover, regarded as possible evidence to the cosmic censorship conjecture, the Riemannian Penrose inequality in general relativity was solved by Huisken and Ilmanen [HI01] using the method of inverse mean curvature flow.

Mullins [Mul56] first formulated the mean curvature flow equation to model grain boundaries during metal annealing. Before the 1990s, most results on mean curvature flow were established for hypersurfaces without boundary. However, although being considerably more challenging than the no boundary case, the study of mean curvature flow for hypersurfaces with boundaries is of great significance. It is a more natural way of describing physical phenomena. For instance, the deformation of grain boundaries usually happens in some containers which provide constraints for the evolution. Such a scenario can be best described by mean curvature flow with boundaries. Applications of mean curvature flow with boundaries also include describing the motion of soap film whose boundary moves freely in a fixed surface.

To define the mean curvature flow for surfaces with boundaries properly, one needs to prescribe certain geometric boundary conditions. One of the most extensively studied boundary conditions is the Neumann boundary condition where the surface's boundary could move freely in a prescribed barrier surface. Moreover, the angle between the surface and the barrier is fixed. When the fixed contact angle is 90 degrees, the flow is then called Mean Curvature Flow with free boundary. For simplicity, it will be referred to as MCF with free boundary in the rest of the thesis.

In 1984, Huisken [Hui84] published his seminal paper and proved that closed convex hypersurfaces in Euclidean spaces of dimension at least three would converge under mean curvature flow to a round point in finite time. Subsequently, Stahl [Sta96a] generalized Huisken's result for free boundary MCF where the barrier is umbilic. In 2020, Hirsch and Li [HL20] proved the convergence thoerem of free boundary MCF with non-umbilic barriers in $\mathbb{R}^3$. It is natural to ask whether a similar convergence result holds when the barrier is defined in a general Riemannian manifold.

1.2 Structure of the Thesis

This thesis aims at building a theoretical foundation for the convergence theory of the MCF with free boundary in a general

Riemannian manifold.

We first review some classical results for the MCF of convex hypersurfaces in the Euclidean space to introduce the essential ingredients for the convergence theory of the MCF. Then we briefly introduce the generalization of the convergence results in the free boundary setting and discuss the similarities and differences between the classical MCF and the MCF with free boundary. Finally. we compute the boundary derivative for the second fundamental form and prove a Stampacchia's iteration scheme for the MCF with free boundary in a general Riemannian manifold.

□ **End of chapter.**

Chapter 2

Classical Results of the MCF

Throughout this section, we let M be a compact uniformly convex n-dimensional surface smoothly embedded in $\mathbb{R}^{n+1}$. Then M can be represented by the following local diffeomorphism:

$$F : U \subset \mathbb{R}^n \to M \subset \mathbb{R}^{n+1}.$$

Then the metric $g = \{g_{ij}\}$ and the second fundamental form $A = \{h_{ij}\}$ at $F(\vec{x}) \in M$ can be written as

$$g_{ij}(\vec{x}) = \left(\frac{\partial F(\vec{x})}{\partial x_i}, \frac{\partial F(\vec{x})}{\partial x_j} \right), \quad h_{ij}(\vec{x}) = \left(-\nu(\vec{x}), \frac{\partial^2 F(\vec{x})}{\partial x_i \partial x_j} \right)$$

where $\nu(\vec{x}) \in \mathbb{R}^{n+1}$ is the outward normal to M at $F(\vec{x})$ and $(\cdot, \cdot)$ is the standard inner product in $\mathbb{R}^{n+1}$. The Levi-Civita connection on M induced from the standard connection on $\mathbb{R}^{n+1}$ is given by

$$\Gamma^k{}_{ij} = \frac{1}{2} g^{kl} \left(g_{il,j} + g_{jl,i} - g_{ij,l} \right)$$

where $g_{ij,k} = \frac{\partial}{\partial x_k} g_{ij}$. For a vector field $X = X^i \frac{\partial}{\partial x_i}$ on M, the covariant derivative of X is

$$(\nabla_i X)^j = \frac{\partial}{\partial x_i} X^j + \Gamma^j{}_{ik} X^k.$$

The Riemann curvature tensor on M is defined as

$$R_{ijkl} = \left\langle (\nabla_i \nabla_j - \nabla_j \nabla_i) \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_k} \right\rangle$$

where $\langle \cdot, \cdot \rangle$ is the inner product for tensors on M induced from g. By the Gauss' equation, we have that

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}.$$

The Ricci tensor and scalar curvature are thus given by

$$R_{ik} = H h_{ik} - h_i{}^j h_{jk}, \quad R = H^2 - |A|^2$$

where $H = g^{ij} h_{ij}$, $|A|^2 = h^{ij} h_{ij}$ and the metric tensor g is used to raise or lower indices.

Now we denote M by M_0 and F by F_0. We say a family of maps $F(\cdot, t)$ satisfies the mean curvature flow equation with initial condition F_0 if

$$\frac{\partial}{\partial t}F(\vec{x}, t) = -H(\vec{x}, t) \cdot \nu(\vec{x}, t), \quad \vec{x} \in U,$$
$$F(\cdot, 0) = F_0,$$

where $H(\vec{x}, t)$ is the mean curvature on M_t.

2.1 Maximum Principles and Preliminary Geometric Identities

Parabolic maximum principles are essential PDE tools in the analysis of mean curvature flow. We will briefly introduce two frequently used versions in this section. One is the standard parabolic maximum principle for scalar functions. The other is the parabolic maximum principle for symmetric two-tensors.

Theorem 1. *[Strong maximum principle for parabolic equations] Let M be a closed smooth manifold and $f : M \times [0, T) \to$*

$\mathbb{R}$ *be a scalar function on* M *varying along time* t. *Suppose* $f(\cdot, 0) \geq 0$ *and*

$$\frac{\partial f}{\partial t} \geq \Delta f + b^i \nabla_i f + cf$$

for some smooth function b^i, c, *where* $c \geq 0$. *Then*

$$\min_M f(\cdot, t) \geq \min_M f(\cdot, 0).$$

Furthermore, if there exist some $p \in M$ *and* $t_0 \geq 0$ *such that* $f(p, t_0) = \min_M f(\cdot, 0)$, *then* $f \equiv \min_M f(\cdot, 0)$ *for* $0 \leq t \leq t_0$.

Now we extend the maximum principle to tensors. Let M_{ij} be a symmetric tensor on a closed manifold M. We say $M_{ij} \geq 0$ if for any vector X on M, $M_{ij}X^iX^j \geq 0$. Let $N_{ij} = P(M_{ij}, g_{ij})$ be another symmetric tensor formed by contracting M_{ij} with itself using the metric where p is a polynomial. Then we have the following verison of the maximum principle:

Theorem 2 (Strong maximum principle for symmetric two-tensors)**.** *Suppose* M_{ij} *is a symmetric tensor on a closed manifold*

M depending on time t and on $0 \leq t < T$ satisfies that

$$\frac{\partial}{\partial t} M_{ij} = \Delta M_{ij} + u^k \nabla_k M_{ij} + N_{ij}$$

where u^k is a vector on M and N_{ij} is defined as above such that

$$N_{ij} X^i X^j \geq 0 \text{ whenever } M_{ij} X^j = 0.$$

Then if $M_{ij} \geq 0$ at $t = 0$, it will remain so on $0 \leq t \leq T$.

Proof. Let $\delta > 0$ be a constant depending only on $\max \left| M_{ij} \right|$. Set

$$\tilde{M}_{ij} = M_{ij} + \epsilon(\delta + t) g_{ij}$$

for some $\epsilon > 0$. Now it suffices to show that $\tilde{M}_{ij} > 0$ on $0 \leq t \leq \delta$ for all $\epsilon > 0$. Suppose for contradiction that the above assertion is not true. Then there exists $t_0 \in (0, \delta]$ and a unit vector X^i at $x_0 \in M$ such that $\tilde{M}_{ij} X^j = 0$ for all i at x_0. Note that $N_{ij} = P(M_{ij}, g_{ij})$, we set $\tilde{N}_{ij} = P(\tilde{M}_{ij}, g_{ij})$. By the assumption, since $\tilde{M}_{ij} X^j = 0$, we have that $\tilde{N}_{ij} X^i X^j \geq 0$. Then at (x_0, t_0),

$$
\begin{aligned}
N_{ij}X^iX^j &= \tilde{N}_{ij}X^iX^j + (N_{ij} - \tilde{N}_{ij})X^iX^j \\
&\geq (N_{ij} - \tilde{N}_{ij})X^iX^j \\
&\geq -\left|N_{ij} - \tilde{N}_{ij}\right|.
\end{aligned}
$$

Since P is a polynomial, we have that

$$\left|N_{ij} - \tilde{N}_{ij}\right| \leq C\left|M_{ij} - \tilde{M}_{ij}\right|$$

where C is a constant depending only on $\max\left|M_{ij}\right|$ if we keep $\epsilon, \delta \leq 1$. Hence as $t_0 \leq \delta$,

$$
\begin{aligned}
N_{ij}X^iX^j &\geq -C\left|M_{ij} - \tilde{M}_{ij}\right| \\
&= -C\left|\epsilon(\delta + t_0)g_{ij}\right| \qquad (2.1) \\
&\geq -2C\epsilon\delta.
\end{aligned}
$$

Let $f = \tilde{M}_{ij}X^iX^j$. Observe that $f(x_0, t) > 0$ for $t < t_0$ and $f(x_0, t_0) = 0$ which imply that $\frac{\partial}{\partial t}f \leq 0$ for $t < t_0$. At $t = t_0$, we see that $f = 0$ attains a minimum at x_0. Hence $\nabla f = 0$ and $\Delta f \geq 0$ at (x_0, t_0).

We can extend the vector X^i to a parallel vector field in a neighborhood of x_0 along geodesics passing x_0 and define X^i on $[0, t_0]$ independent of t. Then we have that

$$\frac{\partial}{\partial t} f = (\frac{\partial}{\partial t} \tilde{M}_{ij}) X^i X^j$$

$$\nabla_k f = (\nabla_k \tilde{M}_{ij}) X^i X^j = (\nabla_k M_{ij}) X^i X^j$$

$$\Delta f = (\Delta \tilde{M}_{ij}) X^i X^j = (\Delta M_{ij}) X^i X^j$$

Therefore,

$$\begin{aligned}
\frac{\partial}{\partial t} f &= (\frac{\partial}{\partial t} \tilde{M}_{ij}) X^i X^j \\
&= (\frac{\partial}{\partial t} (M_{ij} + \epsilon(\delta + t) g_{ij})) X^i X^j \\
&= (\frac{\partial}{\partial t} M_{ij}) X^i X^j + \epsilon g_{ij} X^i X^j + \epsilon(\delta + t)(\frac{\partial}{\partial t} g_{ij}) X^i X^j \\
&= (\frac{\partial}{\partial t} M_{ij}) X^i X^j + \epsilon \\
&= \Delta f + u^k \nabla_k f + N_{ij} X^i X^j + \epsilon \\
&= (1 - 2c\delta)\epsilon.
\end{aligned}$$

Then contradiction arises when $2c\delta < 1$. □

To apply the maximum principles, we need the following Simon's identity to rewrite the evolution equation of the second fundamental form to a parabolic PDE.

Lemma 1 (Simon's identity)**.**

$$\Delta h_{ij} = \nabla_i \nabla_j H + H h_{li} g^{lm} h_{mj} - |A|^2 h_{ij}$$

Proof. Note that $\Delta h_{ij} = g^{mn} \nabla_m \nabla_n h_{ij}$ and $\nabla_i \nabla_j H = g^{mn} \nabla_i \nabla_j h_{mn}$. It suffices to examine the difference $\nabla_m \nabla_n h_{ij} - \nabla_i \nabla_j h_{mn}$. Since the ambient space is Euclidean, from the Codazzi equation we have that $\nabla_i h_j^k = \nabla_j h_i^k$. Hence

$$\nabla_m \nabla_n h_{ij} - \nabla_i \nabla_j h_{mn} = \nabla_m \nabla_i h_{nj} - \nabla_i \nabla_m h_{jn} = (\nabla_m \nabla_i - \nabla_i \nabla_m) h_{nj}.$$

By the product rule of connections acting on tensor product, we have that

$$(\nabla_m \nabla_i - \nabla_i \nabla_m) h_{nj} = R_{min}{}^l h_{lj} + R_{mij}{}^l h_{nl}.$$

Therefore, by Gauss equation

$$
\begin{aligned}
\Delta h_{ij} - \nabla_i \nabla_j H &= g^{mn}(R_{min}{}^{l} h_{lj} + R_{mij}{}^{l} h_{nl}) \\
&= g^{mn} g^{kl} \{(h_{mn} h_{ik} - h_{mk} h_{in}) h_{lj} + (h_{mj} h_{ik} - h_{mk} h_{ij}) h_{ln}\} \\
&= H g^{kl} h_{ik} h_{lj} - g^{mn} g^{kl} h_{mk} h_{ln} h_{ij} \\
&= H g^{kl} h_{ik} h_{lj} - |A|^2 h_{ij}.
\end{aligned}
$$

□

2.2 Evolution Equations for Geometric Quantities

Since the embedding map F is evolving under time t, if we fix a point $\vec{x} \in U$, we have that geometric quantities on $F(\vec{x}, t) \in M_t$ are also evolving under time t. By the flow equation $\frac{\partial}{\partial t} F(\vec{x}, t) = -H(\vec{x}, t) \cdot \nu(\vec{x}, t)$ for F, we can derive evolution equations for other geometric quantities.

Lemma 2. *The following evolution equations hold.*

1. $\frac{\partial}{\partial t} g_{ij} = -2H h_{ij}$

2. $\frac{\partial}{\partial t} g^{ij} = 2Hh^{ij}$

3. $\frac{\partial \nu}{\partial t} = \nabla H$

4. $\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2Hh_{ik}g^{kl}h_{lj} + |A|^2 h_{ij}$

5. $\frac{\partial}{\partial t} H = \Delta H + |A|^2 H$

6. $\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2 |\nabla A|^2 + 2 |A|^4$

Proof. 1. Since $\left(\nu, \frac{\partial F}{\partial x_i}\right) = 0$, by the product rule, we have that

$$\begin{aligned}
\frac{\partial}{\partial t} g_{ij} &= \frac{\partial}{\partial t}\left(\frac{\partial F(\vec{x},t)}{\partial x_i}, \frac{\partial F(\vec{x},t)}{\partial x_j}\right) \\
&= \left(\frac{\partial}{\partial x_i}(-H(\vec{x},t)\cdot\nu(\vec{x},t)), \frac{\partial F}{\partial x_j})\right) + \left(\frac{\partial F}{\partial x_i}, \frac{\partial}{\partial x_j}(-H(\vec{x},t)\cdot\nu(\vec{x},t))\right) \\
&= -H(\left(\frac{\partial \nu}{\partial x_i}, \frac{\partial F}{\partial x_j}\right) + \left(\frac{\partial F}{\partial x_i}, \frac{\partial \nu}{\partial x_j}\right)) \\
&= -2Hh_{ij}
\end{aligned}$$

2. Since $g_{km}g^{mj} = \delta_k^j$, we have that

$$\begin{aligned}
\frac{\partial}{\partial t}(g_{km}g^{mj}) &= 0 \\
\frac{\partial g_{km}}{\partial t}g^{mj} + g_{km}\frac{\partial g^{mj}}{\partial t} &= 0 \\
-2Hh_{km}g^{mj} + g_{km}\frac{\partial g^{mj}}{\partial t} &= 0 \\
g^{ik}g_{km}\frac{\partial g^{mj}}{\partial t} &= g^{ik}2Hh_{km}g^{mj} \\
\frac{\partial}{\partial t}g^{ij} &= 2Hh^{ij}.
\end{aligned}$$

3. Since $|\nu| = 1$ is fixed, we have that $\frac{\partial \nu}{\partial t}$ lies in the tangent space of the surface. Hence we can assume that $\frac{\partial \nu}{\partial t} = V^i \frac{\partial F}{\partial x_i} \in \mathbb{R}^{n+1}$ where V^i can be determined by the following identity

$$\left(\frac{\partial \nu}{\partial t}, \frac{\partial F}{\partial x_j}\right) = g_{ij}V^i$$

. Thus, we have that

$$
\begin{aligned}
\frac{\partial \nu}{\partial t} &= g^{ij}\left(\frac{\partial \nu}{\partial t}, \frac{\partial F}{\partial x_j}\right) \cdot \frac{\partial F}{\partial x_i} \\
&= -g^{ij}\left(\nu, \frac{\partial}{\partial t}\frac{\partial F}{\partial x_j}\right) \cdot \frac{\partial F}{\partial x_i} \\
&= g^{ij}\left(\nu, \frac{\partial}{\partial x_j}(H(\vec{x},t)\cdot \nu(\vec{x},t))\right) \cdot \frac{\partial F}{\partial x_i} \\
&= g^{ij}\frac{\partial}{\partial x_j} H \frac{\partial F}{\partial x_i} \\
&= \nabla H
\end{aligned}
$$

4. By the Gauss-Weingarten relations, we have that

$$
\begin{cases}
\frac{\partial^2 F}{\partial x_i \partial x_j} = \Gamma^k{}_{ij}\frac{\partial F}{\partial x_k} - h_{ij}\nu \\
\\
\frac{\partial \nu}{\partial x_j} = h_{jl} g^{lm}\frac{\partial F}{\partial x_m}.
\end{cases}
$$

Hence

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} &= -\frac{\partial}{\partial t}\left(\nu, \frac{\partial^2 F}{\partial x_i \partial x_j}\right) \\
&= -\left(g^{pq}\frac{\partial}{\partial x_p}H\frac{\partial F}{\partial x_q}, \frac{\partial^2 F}{\partial x_i \partial x_j}\right) + \left(\nu, \frac{\partial^2}{\partial x_i \partial x_j}(H \cdot \nu)\right) \\
&= -\left(g^{pq}\frac{\partial}{\partial x_p}H\frac{\partial F}{\partial x_q}, \Gamma^k{}_{ij}\frac{\partial F}{\partial x_k} - h_{ij}\nu\right) \\
&\quad + \frac{\partial}{\partial x_j}\left(\nu, \frac{\partial}{\partial x_i}(H \cdot \nu)\right) - \left(h_{jl}g^{lm}\frac{\partial F}{\partial x_m}, \frac{\partial}{\partial x_i}(H \cdot \nu)\right) \\
&= -g^{pq}\frac{\partial H}{\partial x_q}\Gamma^k{}_{ij}g_{pk} + \frac{\partial^2 H}{\partial x_i \partial x_j} \\
&\quad - H \cdot \left(h_{jl}g^{lm}\frac{\partial F}{\partial x_m}, h_{il'}g^{l'm'}\frac{\partial F}{\partial x_{m'}}\right) \\
&= \frac{\partial^2 H}{\partial x_i \partial x_j} - \Gamma^q{}_{ij}\frac{\partial H}{\partial x_q} - H h_j^m h_i^n g_{mn}
\end{aligned}$$

Since H is a scalar function, we have that

$$\nabla_i \nabla_j H = \frac{\partial^2 H}{\partial x_i \partial x_j} - \Gamma^q{}_{ij}\frac{\partial H}{\partial x_q}$$

where ∇ is the Levi-Civita connection on M_t.

Hence, by Lemma 1,

$$\begin{aligned}
\frac{\partial}{\partial t}h_{ij} &= \frac{\partial^2 H}{\partial x_i \partial x_j} - \Gamma^q{}_{ij}\frac{\partial H}{\partial x_q} - H h_j^m h_i^n g_{mn} \\
&= \Delta h_{ij} - (H h_{li} g^{lm} h_{mj} - |A|^2 h_{ij}) - \Gamma^q{}_{ij}\frac{\partial H}{\partial x_q} - H h_j^m h_i^n g_{mn} \\
&= \Delta h_{ij} - 2H h_{li} g^{lm} h_{mj} + |A|^2 h_{ij}.
\end{aligned}$$

5. Since $H = g^{ij}h_{ij}$, we have that

$$\begin{aligned}\frac{\partial}{\partial t}H = \frac{\partial}{\partial t}(g^{ij}h_{ij}) &= \frac{\partial g^{ij}}{\partial t}h_{ij} + g^{ij}\frac{\partial h_{ij}}{\partial t} \\ &= 2Hh^{ij}h_{ij} + g^{ij}(\Delta h_{ij} - 2Hh_{li}g^{lm}h_{mj} + |A|^2 h_{ij}) \\ &= \Delta H + |A|^2 H.\end{aligned}$$

6. Combining previous results, we can deduce the following evolution equation

$$\begin{aligned}\frac{\partial}{\partial t}{h_i}^j &= \frac{\partial}{\partial t}(h_{ik}g^{kj}) \\ &= (\Delta h_{ik} - 2Hh_{li}g^{lm}h_{mk} + |A|^2 h_{ik})g^{kj} + h_{ik}(2Hh^{kj}) \\ &= \Delta {h_i}^j - 2Hh_{ik}h^{kj} + |A|^2 {h_i}^j - 2Hh_{ik}h^{kj} \\ &= \Delta {h_i}^j + |A|^2 {h_i}^j.\end{aligned}$$

Since $|A|^2 = h^{ij}h_{ij} = {h_i}^j {h^i}_j$, we have that

$$\begin{aligned}\frac{\partial}{\partial t}|A|^2 &= \frac{\partial}{\partial t}({h_i}^j {h^i}_j) \\ &= (\Delta {h_i}^j + |A|^2 {h_i}^j){h^i}_j + {h_i}^j(\Delta {h^i}_j + |A|^2 {h^i}_j) \\ &= 2(h^{ij}\Delta h_{ij} + |A|^4)\end{aligned}$$

Since the connection ∇ is compatible with the metric g, we have that

$$\begin{aligned}
\Delta |A|^2 &= g^{mn}\nabla_m\nabla_n(h^{ij}h_{ij}) \\
&= 2g^{mn}\nabla_m(h^{ij}\nabla_n h_{ij}) \\
&= 2(g^{mn}\nabla_m\nabla_n h_{ij})h^{ij} + 2g^{mn}(\nabla_m h^{ij})(\nabla_n h_{ij}) \\
&= 2h^{ij}\Delta h_{ij} + 2|\nabla A|^2 .
\end{aligned}$$

It follows that

$$\begin{aligned}
\frac{\partial}{\partial t}|A|^2 &= 2(h^{ij}\Delta h_{ij} + |A|^4) \\
&= \Delta |A|^2 - 2|\nabla A|^2 + |A|^4 .
\end{aligned}$$

□

2.3 Preservation of the convexity and the pinching condition

Theorem 3. *If $h_{ij} \geq 0$ at $t = 0$, then it remains so for $0 \leq t < T$.*

Proof. We have that

$$\frac{\partial}{\partial t}h_{ij} = \Delta h_{ij} - 2Hh_{li}g^{lm}h_{mj} + |A|^2 h_{ij}.$$

Let $M_{ij} = h_{ij}$ and $N_{ij} = |A|^2 h_{ij} - 2Hh_{li}g^{lm}h_{mj}$. If vector X^j satisfies that $h_{ij}X^j = 0$ for all i, then

$$N_{ij}X^j = |A|^2 (h_{ij}X^j) - 2Hh_{li}g^{lm}(h_{mj}X^j) = 0.$$

Hence we can apply Theorem 2 to conclude. □

We can in fact prove a stronger version of the theorem above.

Theorem 4. *If $\epsilon H g_{ij} \leq h_{ij} \leq \beta H g_{ij}$, and $H \geq 0$ at $t = 0$, then it remains true for $t > 0$.*

Proof. First, since $\frac{\partial}{\partial t}H = \Delta H + |A|^2 H$, by Theorem 1 we have that if $H \geq 0$ at $t = 0$, $H \geq 0$ for all $t \geq 0$. Let $M_{ij} = h_{ij} - \epsilon H g_{ij}$. Then

$$\begin{aligned}\frac{\partial}{\partial t}M_{ij} &= \frac{\partial}{\partial t}h_{ij} - \epsilon(\frac{\partial}{\partial t}H)g_{ij} - \epsilon H\frac{\partial}{\partial t}g_{ij} \\ &= \Delta h_{ij} - 2Hh_{li}g^{lm}h_{mj} + |A|^2 h_{ij} - \epsilon g_{ij}(\delta H + |A|^2 H) - \epsilon H(-2Hh_{ij}) \\ &= \Delta M_{ij} + |A|^2 h_{ij} + 2\epsilon H^2 h_{ij} - \epsilon |A|^2 H g_{ij} - 2Hh_{li}g^{lm}h_{mj}.\end{aligned}$$

Let $N_{ij} = |A|^2 h_{ij} + 2\epsilon H^2 h_{ij} - \epsilon |A|^2 H g_{ij} - 2Hh_{li}g^{lm}h_{mj}$. From direct computation we have that

$$\begin{aligned} N_{ij} &= |A|^2 (h_{ij} - \epsilon H g_{ij}) - 2H(h_{li} g^{lm} h_{mj} - \epsilon H h_{ij}) \\ &= |A|^2 M_{ij} - 2H(h_{li} g^{lm} h_{mj} - \epsilon H h_{li} g^{lm} g_{mj}) \\ &= |A|^2 M_{ij} - 2H h_i^m (h_{mj} - \epsilon H g_{mj}) \\ &= |A|^2 M_{ij} - 2H h_i^m M_{mj}. \end{aligned}$$

Then for the null vector X^i of M_{ij}, we have that

$$N_{ij} X^j = |A|^2 (M_{ij} X^j) - 2H h_i^m (M_{mj} X^j) = 0.$$

Then the result follows from Theorem 2. □

2.4 Stampacchia's iteration

One key step for proving that M converges to a round point is to show that the geometric quantity $|A|^2 - \frac{1}{n} H^2$ becomes small compared to H^2.

Theorem 5. *There are constants $C_0 < \infty$ and $\delta > 0$ depending only on M_0 such that*

$$|A|^2 - \frac{1}{n} H^2 \leq C_0 H^{2-\delta}$$

for all times $t \in [0, T)$.

The rationale behind is that

$$|A|^2 - \frac{1}{n}H^2 = \frac{1}{n}\sum_{i<j}^{n}(\kappa_i - \kappa_j)^2$$

measures the sum of distances between eigenvalues κ_i of the second fundamental form A.

An iteration scheme named Stampacchia's iteration is used to reach the goal. In this section, we introduce the general idea for Stampacchia's iteration.

The principal components of Stampacchia's iteration are the following algebraic lemma and a version of the Sobolev inequality from [MS73]:

Lemma 3. *Let* $f : [\bar{x}, \infty) \to \mathbb{R}$ *be a non-negative and non-increasing function. Suppose for* $C > 0, p > 0$ *and* $\gamma > 1$,

$$(y - x)^p f(y) \leq C f(x)^\gamma, \quad y \geq x \geq \bar{x}.$$

Then $f(y) = 0$ *for* $y \geq \bar{x} + d$ *where* $d^p = C f(\bar{x})^{\gamma - 1} 2^{\frac{p\gamma}{\gamma - 1}}$

Proof. Without loss of generality, we can assume that $\bar{x} = 0$. Let $g = (\frac{f}{f(0)})^{\frac{1}{p}}$ and $A = (Cf(0)^{\gamma-1})^{\frac{1}{p}}$. For $y \geq x \geq 0$, we have that

$$(y-x)^p f(y) \leq Cf(x)^\gamma$$

$$A^p(y-x)^p f(y) \leq A^p Cf(x)^\gamma$$

$$(y-x)^p g(y)^p f(0)^\gamma \leq Cf(0)^{\gamma-1} g(x)^{p\gamma} f(0)^\gamma$$

$$(y-x)g(y) \leq Ag(x)^\gamma.$$

Now fix $y > 0$, let $x_n = y(1-\frac{1}{2^n})$. Note that $\lim_{n\to\infty} x_n = y$ and $x_0 = 0$. Hence, we have that $g(x_0) = g(0) = 1$ and

$$\begin{aligned} (x_{n+1}-x_n)g(x_{n+1}) &\leq Ag(x_n)^\gamma \\ y(\frac{1}{2^n} - \frac{1}{2^{n+1}})g(x_{n+1}) &\leq Ag(x_n)^\gamma \\ g(x_{n+1}) &\leq \frac{A}{y}2^{n+1}g(x_n)^\gamma. \end{aligned}$$

Using the above inequality inductively, we have that

$$g(x_n) \leq (\frac{A}{y})^{1+\gamma+\cdots+\gamma^{n-1}} 2^{n+(n-1)\gamma+(n-2)\gamma^2+\cdots+\gamma^{n-1}}.$$

Since

$$n + (n-1)\gamma + (n-2)\gamma^2 + \cdots + \gamma^{n-1} = \frac{\gamma^n + n - (n+1)\gamma}{(\gamma-1)^2},$$

if we choose y such that $\frac{A}{y} = 2^{-\frac{\gamma}{\gamma-1}}$, then we have that

$$\begin{aligned} g(x_n) &\leq (\frac{A}{y})^{\frac{\gamma^n-1}{\gamma-1}} 2^{\frac{\gamma^n+n-(n+1)\gamma}{(\gamma-1)^2}} \\ &\leq 2^{\frac{1}{(\gamma-1)^2}(-\gamma(\gamma^n-1)+\gamma^{n+1}+n-(n+1)\gamma)} \\ &= 2^{-\frac{n}{\gamma-1}}. \end{aligned}$$

It follows that $\lim_{n\to\infty} g(x_n) = 0$. By continuity of g, we have that $g(y) = 0$. Therefore, $f(y) = 0$. □

Lemma 4. *Let v be a Lipschitz function on M. Then*

$$\left(\int_M |v|^{\frac{n}{n-1}}\, d\mu\right)^{n-\frac{1}{n}} \leq c(n) \int_M |\nabla v| + H\,|v|\, d\mu.$$

The geometric quantity we aim to bound is

$$f_\sigma = \left(|A|^2 - \frac{1}{n}H^2\right) H^{2-\sigma} = \left(\frac{|A|^2}{H^2} - \frac{1}{n}\right) H^\sigma$$

for sufficient small $\sigma > 0$.

Since M is uniformly convex, by Theorem 4, we have that $\epsilon H g_{ij} \leq h_{ij} \leq \beta H g_{ij}$, and $H \geq 0$ for any $t > 0$. Combining previous evolution equations, we can deduce that

$$\frac{\partial}{\partial t} f_\sigma \leq \Delta f_\sigma + \frac{2(1-\sigma)}{H} \langle \nabla_i H, \nabla_i f_\sigma \rangle - \epsilon^2 \frac{1}{H^{2-\sigma}} |\nabla H|^2 + \sigma |A|^2 f_\sigma \tag{2.2}$$

for all $0 \leq t < T$ and $\sigma > 0$.

Applying integration by parts and Peter-Paul inequality, we have the following Poincare-like inequality for f_σ.

Lemma 5. *Let $p \geq 2$. For any $0 < \sigma \leq \frac{1}{2}$ and any $\eta > 0$, we have that*

$$\begin{aligned} n\epsilon^2 \int f_\sigma^p H^2 d\mu \leq (2\eta p + 5) \int \frac{1}{H^{2-\sigma}} |\nabla H|^2 d\mu \\ + \eta^{-1} (p-1) \int f_\sigma^{p-2} |f_\sigma|^2 d\mu. \end{aligned} \tag{2.3}$$

For a positive constant k, we let $f_{\sigma,k} = (f_\sigma - k)_+$, $A(k) = \{f_\sigma \geq k\}$ and $A(k,t) = A(k) \cap M_t$. Integration by parts also yields the following evolution-like inequality.

Lemma 6. *Let $p \geq 2$. For any $0 < \sigma < 1$, we have that*

$$\begin{aligned}\frac{\partial}{\partial t}\int f_{\sigma,k}^{p} d\mu \leq & -\frac{1}{2}p(p-1)\int f_{\sigma,k}^{p-2}\left|\nabla f_{\sigma}\right|^{2} d\mu \\ & - p\left(\epsilon^{2}-\frac{2}{p-1}\right)\int f_{\sigma,k}^{p-1}\frac{\left|\nabla H\right|^{2}}{H^{2-\sigma}}d\mu \\ & -\int H^{2} f_{\sigma,k}^{p} d\mu + \sigma p\int_{A(k,t)} H^{2} f_{\sigma}^{p} d\mu.\end{aligned} \tag{2.4}$$

Proof. The idea is to multiply both sides of Equation 2.2 by $pf_{\sigma,k}^{p-1}$ and integrate by parts over M_t. For the left-hand side, we have that

$$\begin{aligned}\int pf_{\sigma,k}^{p-1}\frac{\partial}{\partial t}f_{\sigma} d\mu &= \int \frac{\partial}{\partial t} f_{\sigma,k}^{p} d\mu \\ &= \frac{\partial}{\partial t}\int f_{\sigma,k}^{p} d\mu - \int f_{\sigma,k}^{p}\frac{\partial}{\partial t}(d\mu) \\ &= \frac{\partial}{\partial t}\int f_{\sigma,k}^{p} d\mu + \int H^{2} f_{\sigma,k}^{p} d\mu.\end{aligned} \tag{2.5}$$

For the right-hand side,

$$\int pf_{\sigma,k}^{p-1}\Delta f_{\sigma} d\mu = -p(p-1)\int f_{\sigma,k}^{p-2}\left|\nabla f_{\sigma}\right|^{2} \tag{2.6}$$

and $|A|^2 \leq H^2, \langle \nabla_i H, \nabla_i f_\sigma \rangle \leq |\nabla H|\,|\nabla f_\sigma|$. It follows that

$$f_{\sigma,k} \leq f_{\sigma} = \left(|A|^{2} - \frac{1}{n}H^{2}\right) H^{\sigma-2} \leq H^{\sigma}$$

and for $0 < \sigma < 1, p \geq 2$

$$\begin{aligned}\frac{2(1-\sigma)}{H} f_{\sigma,k} |\nabla H| |\nabla f_\sigma| &\leq \frac{p-1}{2} |\nabla f_\sigma|^2 + \frac{2}{p-1} \frac{|\nabla H|^2 f_{\sigma,k}^2}{H^2} \\ &\leq \frac{p-1}{2} |\nabla f_\sigma|^2 + \frac{2}{p-1} \frac{|\nabla H|^2}{H^{2-\sigma}} f_{\sigma,k}\end{aligned} \tag{2.7}$$

Hence

$$\begin{aligned}&\frac{\partial}{\partial t} \int f_{\sigma,k}^p d\mu + p(p-1) \int f_{\sigma,k}^{p-2} |\nabla f_\sigma|^2 d\mu \\ &\qquad + \epsilon^2 p \int \frac{1}{H^{2-\sigma}} f_{\sigma,k}^{p-1} |\nabla H|^2 d\mu + \int H^2 f_{\sigma,k}^p d\mu \\ &\quad\leq 2(1-\sigma)p \int \frac{1}{H} f_{\sigma,k}^{p-1} |\nabla H| |\nabla f_\sigma| d\mu + \sigma p \int |A|^2 f_{\sigma,k}^{p-1} f_\sigma d\mu. \\ &\quad\leq \frac{1}{2} p(p-1) \int f_{\sigma,k}^{p-2} |\nabla f_\sigma|^2 d\mu + 2\frac{p}{p-1} \int f_{\sigma,k}^{p-1} \frac{|\nabla H|^2}{H^{2-\sigma}} \\ &\qquad + \sigma p \int_{A(k,t)} H^2 f_\sigma^p d\mu.\end{aligned} \tag{2.8}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial t} \int f_{\sigma,k}^p d\mu \leq &- \frac{1}{2} p(p-1) \int f_{\sigma,k}^{p-2} |\nabla f_\sigma|^2 d\mu \\ &- p\left(\epsilon^2 - \frac{2}{p-1}\right) \int f_{\sigma,k}^{p-1} \frac{|\nabla H|^2}{H^{2-\sigma}} d\mu \\ &- \int H^2 f_{\sigma,k}^p d\mu + \sigma p \int_{A(k,t)} H^2 f_\sigma^p d\mu.\end{aligned} \tag{2.9}$$

□

Now we have established two inequalities for the function f_σ. Notice that any compact hypersurface M in $\mathbb{R}^{n+1}$ can be enclosed by a sphere which shrinks to a point under the MCF in finite time. From the avoidance principle, we have that the maximal time $T < \infty$. Then by Theorem 11, the general iteration scheme we are going to derive in the later chapter, we can bound f_σ uniformly for all times $t \in [0, T)$, which proves *Theorem* 5.

2.5 Convergence to a round point

A general argument from Huisken implies that M_t exists on a maximal time interval $t \in [0, T)$ where $T < \infty$ and $\max_{M_t} |A|^2$ becomes unbounded as $t \to T$.

To prove that M_t converges to a round point as $t \to T$, we first show that the hypersurface M_t converges to a point under the original flow. Then we normalize the flow by keeping the total area of M_t fixed and prove that M_t converges to a sphere under the normalized flow.

For the first part, to control the diameter of M_t, we need to examine the minimum value of the mean curvature $H_{\min}$. Since $|A|^2 \leq H^2$, the maximum value of the mean curvature $H_{\max} \to \infty$ as t approaches T. To compare the mean curvature at different points on M_t, we need the following gradient estimate for H.

Theorem 6. *For any $\eta > 0$, there exists a constant $C = C(\eta, M_0, n)$ such that*

$$|\nabla H|^2 \leq \eta H^4 + C.$$

Following Hamilton's argument, the preservation of curvature pinching $h_{ij} \leq \epsilon H g_{ij}$ together with *Theorem* 6 and Myer's theorem implies that $H_{\max}/H_{\min} \to 1$ as $t \to T$. Hence the diameter of M_t decreases to zero as $t \to T$.

For the second part, we rescale the solution F to the unnormalized equation $\frac{\partial}{\partial t} F(\vec{x}, t) = -H(\vec{x}, t) \cdot \nu(\vec{x}, t)$ at each time

$t \in [0, T)$ by a positive constant $\psi(t)$ such that

$$\int_{\tilde{M}_t} d\tilde{\mu}_t = |M_0| \text{ for all } 0 \le t < T \tag{2.10}$$

where the surface $\tilde{M}_t$ is given by local diffeomorphisms

$$\tilde{F}(\cdot, t) = \psi(t) \cdot F(\cdot, t).$$

Then geometric quantities of $\tilde{M}_t$ are dilated by some constants

$$\tilde{g}_{ij} = \psi^2 g_{ij}, \quad \tilde{h}_{ij} = \psi h_{ij}, \quad \tilde{H} = \psi^{-1} H.$$

Hence

$$\begin{aligned}
\frac{\partial}{\partial t}\sqrt{\det \tilde{g}_{ij}} &= \frac{\det \tilde{g}_{ij}}{2\sqrt{\det \tilde{g}_{ij}}} \tilde{g}^{pq} \frac{\partial \tilde{g}_{pq}}{\partial t} \\
&= \frac{\sqrt{\det \tilde{g}_{ij}}}{2} \psi^{-2} g^{pq} \left(\frac{\partial \psi^2}{\partial t} g_{pq} + \psi^2 \frac{\partial g_{pq}}{\partial t} \right) \\
&= \sqrt{\det \tilde{g}_{ij}} \left(\psi^{-1} n \frac{\partial \psi}{\partial t} - \psi^2 \tilde{H}^2 \right)
\end{aligned}$$

Differentiating *Equation* 2.10 yields that

$$\psi^{-1} \frac{\partial \psi}{\partial t} = \frac{1}{n} \psi^2 \tilde{h}$$

where $\tilde{h} = \frac{\int \tilde{H}^2 d\tilde{\mu}}{\int d\tilde{\mu}}$ is the average of the squared mean curvature on $\tilde{M}_t$. By introducing a new time variable $\tilde{t}(t) = \int_0^t \psi^2(\tau) d\tau$, we have the following normalized equation on a different maximal time interval $\tilde{t} \in [0, \tilde{T})$:

$$\frac{\partial \tilde{F}}{\partial \tilde{t}} = -\tilde{H}\tilde{\nu} + \frac{1}{n}\tilde{h}\tilde{F}.$$

The evolution equations of geometric quantities under the normalized flow differ from the original evolution equations by a lower order term. Most of the computations in previous sections still hold. We can further prove that the maximal time $\tilde{T} = \infty$ and derive the exponential decay of the following geometric quantities on $\tilde{M}_t$:

Lemma 7. *There are constants $\delta > 0$ and $C < \infty$ such that*

1. $|\tilde{A}|^2 - \frac{1}{n}\tilde{H}^2 \leq Ce^{-\delta\tilde{t}}$
2. $\left|\tilde{h}_{ij}\tilde{H} - \frac{1}{n}\tilde{h}\tilde{g}_{ij}\right| \leq Ce^{-\delta\tilde{t}}$
3. $\max_{\tilde{M}} \left|\nabla^m \tilde{A}\right| \leq C_m e^{-\delta_m \tilde{t}}$

Since the metric $\tilde{g}_{ij}$ evolves under the equation

$$\frac{\partial}{\partial \tilde{t}}\tilde{g}_{ij} = -2\tilde{h}_{ij}\tilde{H} + \frac{2}{n}\tilde{h}\tilde{g}_{ij},$$

by *Lemma* 7(2), $\tilde{g}_{ij}$ converges uniformly to a positive definite metric $\tilde{g}_{ij}(\infty)$ as $\tilde{t} \to \infty$. Then from *Lemma* 7(3) and Arzela-Ascoli thoerem, we have that $\tilde{g}_{ij}(\infty)$ is smooth. Finally *Lemma* 7(1) implies that $\tilde{g}_{ij}(\infty)$ is the metric of a sphere.

2.6 Generalizations

The previous convergence theorem was generalized in various settings by mathematicians.

In 1986, Huisken [Hui86] managed to extend the result for hypersurfaces in a general Riemannian manifold where the hypersurface need to be convex enough to overcome the obstruction caused by the curvature of the ambient manifold to converge to a round point. In the special case of spherical spaceform, Huisken [Hui87] observed that the hypersurface can converge to a round point without the initial convexity condition. Similarly, the result obtained by Andrews and Baker [AB10] for the converge of higher-codimension submanifold also allows some non-convex condition to be preserved.

For MCF with free boundary, Stahl [Sta96a, Sta96b] showed that if the barrier surface in the Euclidean space is a flat hyperplane for a round hypersphere, any convex hypersurface with free boundary on the barrier will converge to a round half point. Later in 2020, Hirsch and Li [HL20] managed to generalize the above result to non-umbilic barriers in $\mathbb{R}^3$. They proved that if the barrier surface satisfies a uniform bound on the exterior and interior ball curvature and certain bounds on the first and second derivative of the second fundamental form, then sufficiently convex free boundary hypersurfaces will converge to a round half point.

In sum, most convergence results above were obtained by following Huisken's line of argument where the key steps are the pinching estimate of the traceless second fundamental form and the estimate for the gradient of the mean curvature. The former describes the "roundness" of the hypersurface pointwisely while the latter enables us to compare mean curvatures of the hypersurface at different points. In particular, the gradient estimate for the mean curvature is built upon the pinching estimate. To prove a general convergence theorem for the free boundary MCF in the Riemannian ambient space, it is essential to establish a proper iteration scheme for showing the pinching estimate.

□ **End of chapter.**

Chapter 3

Free Boundary MCF in Riemannian Manifolds

To prove that a free boundary hypersurface converges to a round half-point under the MCF, the standard argument from Huisken [Hui84] also works. Hence, it suffices to prove the pinching estimate by the Stampacchia's iteration and the gradient estimate.

However, for a general barrier in a 3-manifold, two difficulties need to be overcome. First, as the prerequisite of proving pinching and gradient estimates, the initial convexity condition is expected to be preserved along the flow. To apply maximum principles for surfaces with boundary, it is essential to compute and estimate the boundary derivatives of geometric quantities.

The second difficulty is the reformulation of Stampacchia's iteration in a more general setting. Edelen [Ede16] has introduced a free boundary version of Stampacchia's iteration, but the iteration argument only works when the barrier is in the Euclidean space. To further extend the iteration argument to the Riemannian manifold, we first need to extend the Michael-Simon inequality for Riemannian manifold by Hoffman and Spruck [HS74] to the free boundary case. Then by the arguments in [Ede16], the Stampacchia's iteration could be applied once the Poincare-like inequality and the evolution-like inequality are established.

3.1 Definitions and Notations

Let $(M, \bar{g})$ be an $(n+1)$-dimensional Riemannian manifold with the Levi-Civita connection $\bar{\nabla}$. We denote by $\sigma_x(P)$ the sectional curvature of a 2-plane P at $x \in M$ and by $i_x(M)$ the injectivity radius of M at x.

Consider a properly embedded, orientable, smooth hypersurface $S \subset M$ without boundary. We refer to S as the *barrier surface* or the *barrier*. We write $f = O(g)$ to indicate that $|f| \leq c(n, S, M)\,|g|$. By fixing a smooth global unit normal ν_S on S, we can define the second fundamental form $A^S : TS \times TS \to \mathbb{R}$ by

$$A^S(u, v) = -\bar{g}(\bar{\nabla}_u v, \nu_S).$$

Let Σ be a two-sided smooth n-dimensional manifold with non-empty boundary $\partial\Sigma$. A smooth immersion $F : \Sigma \to M$ defines a free boundary hypersurface if $F(\partial\Sigma) \subset S$ and $F_* N = \nu_S \circ F$ where N is the outward unit normal of $\partial\Sigma \subset \Sigma$ with respect to the metric induced from M by F.

3.2 Covariant Formulation of the Mean Curvature Flow

From the previous chapter, we can see that Huisken considered a family of maps F_t from an open set $U \subset \mathbb{R}^n$ to $\mathbb{R}^{n+1}$ which evolve along the mean curvature vector of their images. In this way, we can fix a local coordinate system and analyze geometric quantities of the images along the flow using this invariant coordinate system. The advantages include that the structure of the general evolution equation is clearer which enables us to prove the short-time existence of the flow using the theory of quasi-linear parabolic differential equations. On the other hand, one needs to carefully choose the local coordinate system to simplify the computation without losing the important information. A more modern treatment is to consider a rather invariant form of evolution equations independent of the local coordinates. In particular, we consider the metrics and connections on vector bundles over the space-time domain and derive structure equations and evolution equations for geometric quantities in such new vector bundle machinery.

3.2.1 Subbundles

Definition 1. *Let K, E be two vector bundles over a manifold M. We say K is a subbundle of E if there exists an injective vector bundle homomorphism $\iota_K : K \to E$ covering the identity map on M.*

Now let E be a vector bundle over a manifold M. We can consider two complementary subbundles K and L of E , in the sense that for each $x \in M$, the fiber $E_x = \iota_K(K_x) \oplus \iota_L(L_x)$. Let $\pi_K : E \to K$ and $\pi_L : E \to L$ be the correseponding projections from E onto K and L where we have the following relations

$$\pi_K \circ \iota_K = \mathrm{Id}_K \quad \pi_L \circ \iota_L = \mathrm{Id}_L$$

$$\pi_K \circ \iota_L = 0 \quad \pi_L \circ \iota_K = 0$$

$$\iota_K \circ \pi_K + \iota_L \circ \pi_L = \mathrm{Id}_E.$$

Similar to the way of defining the second fundamental form for submanifolds, we can extend a connection ∇ on E to a connection $\overset{K}{\nabla}$ on its subbundle K and define the second fundamental form $h^K \in \Gamma(T^*(M) \otimes K^* \otimes L)$ of K where

$$\overset{K}{\nabla}_u \xi = \pi_K(\nabla_u(\iota_K \xi)) \qquad h^K(u, \xi) = \pi_L(\nabla_u(\iota_K \xi)), \tag{3.1}$$

for any $\xi \in \Gamma(K)$ and $u \in TM$.

Then we can derive the following Gauss equation relating the curvature R^K of $\overset{K}{\nabla}$ to the curvature R_∇ of ∇ and the second fundamental forms h^L and h^K:

$$R^K(u, v)\xi = \pi_k(R_\nabla(u, v)\iota_K \xi) + h^L(u, h^K(v, \xi)) - h^L(v, h^K(u, \xi)) \tag{3.2}$$

for any $u, v \in T_xM$ and $\xi \in \Gamma(K)$. If we also have a connection defined on TM, then we can define the covariant derivative of the second fundamental form h_K by

$$\nabla_u h^K(v,\xi) = \overset{L}{\nabla}_u(h^K(v,\xi)) - h^K(\nabla_u v, \xi) - h^K(v, \overset{K}{\nabla}_u \xi) \quad (3.3)$$

for any $u, v \in T_xM$ and $\xi \in \Gamma(K)$. Assume in addition that the connection on TM is symmetric, we have the following Codazzi identity:

$$\nabla_u h^K(v,\xi) - \nabla_v h^K(u,\xi) = \pi_L(R_\nabla(u,v)(\iota_K \xi)). \quad (3.4)$$

Furthermore, if E admits a metric g compatible with ∇ and K, L are orthogonal with respect to the metric in the sense that

$$g(\iota_K \xi, \iota_L \eta) = 0 \quad (3.5)$$

for any $\xi \in \Gamma(K)$ and $\eta \in \Gamma(L)$. Then the metric g induces naturally metrics g_K, g_L on subbundles K, L respectively and gives us the Weingarten relation associating the second fundamental forms h^K and h^L by

$$g^L(h^K(u,\xi),\eta) + g^K(\xi, h^L(u,\eta)) = 0. \quad (3.6)$$

3.2.2 Time-dependent Immersion

Let I be a real interval. Then the tangent bundle $T(\Sigma \times I)$ splits into $\mathcal{H} \oplus \mathbb{R}\partial t$ where $\mathcal{H} := \{u \in T(\Sigma \times I) : dt(u) = 0\}$ is the 'spatial' tangent bundle.

Let $F : \Sigma \times I \to M$ be a smooth map such that $F(\cdot, t) : \Sigma \to M$ defines a free boundary hypersurface with respect to the barrier S. Note that the pullback bundle F^*TM is equipped with a metric $\bar{g}_F$ and a connection ${}^F\bar{\nabla}$ induced from the ambient manifold M.

The pushforward map of the spatial tangent vector $F_* : \mathcal{H} \to F^*TM$ defines a subbundle of F^*TM of rank n . We denote by $\mathcal{N}$ the orthogonal complement of $F_*(\mathcal{H})$ in F^*TM. Then $\mathcal{N}$ is a subbundle of F^*TM of rank 1, which is referred to as the (spacetime) normal bundle.

Now $\mathcal{H}$ and $\mathcal{N}$ are subbundles of F^*TM with inclusion maps

$$F_* : \mathcal{H} \to F^*TM \qquad \iota : \mathcal{N} \to F^*TM$$

and projection maps

$$\pi : F^*TM \to \mathcal{H} \qquad \overset{\perp}{\pi} : F^*TM \to \mathcal{N}.$$

Then from the previous section we can define the metric $g(u,v) := \bar{g}_F(F_*u, F_*v)$, the connection $\nabla := \pi \circ {}^F\bar{\nabla} \circ F_*$ on the bundle $\mathcal{H}$ and the metric $\overset{\perp}{g}(\xi,\eta) := \bar{g}_F(\iota\xi, \iota\eta)$, the connection $\overset{\perp}{\nabla} := \overset{\perp}{\pi} \circ {}^F\bar{\nabla} \circ \iota$ on the bundle $\mathcal{N}$.

By restricting the first argument of the second fundamental form $h^{\mathcal{H}} = \overset{\perp}{\pi} \circ {}^F\bar{\nabla} \circ F_* \in \Gamma(T(\Sigma \times I)^* \otimes \mathcal{H}^* \otimes \mathcal{N})$ to $\mathcal{H}$, we can define the symmetric bilinear form $h \in \Gamma(\mathcal{H}^* \otimes \mathcal{H}^* \otimes \mathcal{N})$ on $\mathcal{H}$ with values in $\mathcal{N}$. The mean curvature vector $\vec{H} \in \Gamma(\mathcal{N})$ on Σ is thus defined as $\vec{H} := \mathrm{Tr}_g(h)$.

Let $I = [0,T)$. We say a time-dependent immersion $F : \Sigma \times I \to M$ is a solution to the free boundary mean curvature flow if

$$F_*\partial t = \iota\vec{H}.$$

Note that in the case of free boundary mean curvature flow, the remaining components of $h^{\mathcal{H}}$ are given by

$$\begin{aligned} h^{\mathcal{H}}(\partial_t, v) &= \overset{\perp}{\pi}({}^F\bar{\nabla}_{\partial_t} F_* v) \\ &= \overset{\perp}{\pi}({}^F\bar{\nabla}_v F_* \partial_t + F_*([\partial_t, v])) \\ &= \overset{\perp}{\nabla}_v \vec{H} \end{aligned} \tag{3.7}$$

where $\overset{\perp}{\pi} \circ F_*([\partial_t, v]) = 0$ for $[\partial_t, v] = (\partial_t v^i)\partial_i \in \mathcal{H}$.

3.3 Boundary Derivatives

Since $\mathcal{N}$ is a subbundle of F^*TM of rank 1, we can fix a global unit section $\nu \in \Gamma(\mathcal{N})$. Let H be a function over $\Sigma \times I$ defined by $H := -\overset{\perp}{g}(\vec{H}, \nu)$. Then $\vec{H} = -H\nu$.

Theorem 7. $N(H) = HA^S(\iota\nu, \iota\nu)$

Proof.

$$
\begin{aligned}
N(H) &= -\overset{\perp}{g}(\overset{\perp}{\nabla}_N \vec{H}, \nu) \\
&= -\overset{\perp}{g}(h^{\mathcal{H}}(\partial_t, N), \nu) \\
&= -\bar{g}_F({}^F\bar{\nabla}_{\partial_t} F_* N, \iota\nu) \\
&= -\bar{g}_F({}^F\bar{\nabla}_{\partial_t} \nu_S \circ F, \iota\nu) \qquad (3.8) \\
&= -\bar{g}(\bar{\nabla}_{F_*\partial_t} \nu_S, \iota\nu) \\
&= -\bar{g}(\bar{\nabla}_{-H\iota\nu} \nu_S, \iota\nu) \\
&= H A^S(\iota\nu, \iota\nu)
\end{aligned}
$$

□

In the rest of the section, we fix a time $t_0 \in I$. Then the restrictions of $\mathcal{H}$ and $\mathcal{N}$ to $\Sigma \times \{t_0\}$ are the usual tangent and the normal bundle of F_{t_0}. Moreover, ∇ agrees with the Levi-Civita connection of $g(t_0)$ and h agrees with the usual second fundamental form of the immersion F_{t_0}.

Let $p \in \partial\Sigma$. Then for any $u \in T_p\Sigma$, we can extend u to a section of $\mathcal{H}$ in an open neighborhood of $(p, t_0) \in \Sigma \times I$. **Since**

the quantities we are going to work with in the rest of the section are all tensorial, we can further assume that $\nabla u = \pi \circ {}^F\bar{\nabla} \circ F_* u = 0$ without affecting the values of the quantities. *But for vectors in the tangent space of the boundary of Sigma, such extension would make the vector leave the tangent space of the boundary.* What we could do is to extend the vector to the interior of Σ along the normal direction N. Then we have that $\nabla_N u = 0$.

Before computing the boundary derivative of the second fundamental form h on $\mathcal{H}$. We first derive a relationship between h and A^S on $F(\partial\Sigma \times \{t_0\}) \subset S$.

Lemma 8. *For $u \in T_p\partial\Sigma$, we have that*

$$h(u, N) = A^S(F_* u, \iota\nu)\nu.$$

Proof. Since $u \in T_p\partial\Sigma$, then $\bar{g}(F_* u, \nu_S) = \bar{g}_F(F_* u, F_* N) = 0$. By construction, we also have that $\bar{g}(\iota\nu, \nu_S) = \bar{g}_F(\iota\nu, F_* N) = 0$.

Hence $\iota\nu$ and F_*u is tangent to the barrier S and

$$A^S(F_*u, \iota\nu) = \bar{g}(\iota\nu, \bar{\nabla}_{F_*u}\nu_S) = \bar{g}_F(\iota\nu, {}^F\bar{\nabla}_u F_*N).$$

Therefore,

$$h(u, N) = \overset{\perp}{\pi}({}^F\bar{\nabla}_u F_*N) = \bar{g}_F(\iota\nu, {}^F\bar{\nabla}_u F_*N)\nu = A^S(F_*u, \iota\nu)\nu.$$

$\square$

Theorem 8. *For $u, v \in T_p\partial\Sigma$,*

$$\begin{aligned}\nabla_N h(u, v) = &\left(\nabla_{F_*u}A^S(\iota\nu, F_*v) + A^S(\bar{\nabla}^S_{F_*u}\iota\nu, F_*v)\right)\nu \\ &+ A^S(F_*u, F_*v)h(N, N) - h(\nabla_u N, v) \\ &+ A^S(\iota\nu, \iota\nu)h(u, v) + \overset{\perp}{\pi}(F^*R_\nabla(u, N)(F_*v)).\end{aligned}$$

Proof. By the Codazzi identity Equation 3.4, we have that

$$\nabla_N h(u, v) - \nabla_u h(N, v) = \overset{\perp}{\pi}(F^*R_\nabla(u, N)(F_*v))$$

where

$$\nabla_u h(N, v) = \overset{\perp}{\nabla}_u(h(N, v)) - h(\nabla_u N, v) - h(N, \nabla_u v).$$

Since $v \in T_p\partial\Sigma$, by Lemma 8, we have that

$$\begin{aligned} \overset{\perp}{\nabla}_u(h(N,v)) &= \overset{\perp}{\nabla}_u(A^S(F_*v, \iota\nu)\nu) \\ &= F_*u(A^S(F_*v, \iota\nu))\nu. \end{aligned} \tag{3.9}$$

Since the equation we need to derive is tensorial, we can extend the vectors u, v parallel on $\partial\Sigma$ and along the direction N to the interior of Σ where

$$\nabla_u v = g(\nabla_u v, N)N.$$

Hence,

$$\begin{aligned} h(N, \nabla_u v) &= g(\nabla_u v, N)h(N,N) \\ &= \bar{g}_F({}^F\bar{\nabla}_u F_*v, F_*N)h(N,N) \\ &= \bar{g}(\bar{\nabla}_{F_*u}F_*v, \nu_S)h(N,N) \\ &= -A^S(F_*u, F_*v)h(N,N). \end{aligned} \tag{3.10}$$

Moreover, the pushforward $F_*u, F_*v \in T_pS$ can be extended to vector fields on the barrier S where $\bar{\nabla}^S_{F_*u} F_* v = \iota h(u,v)$ and

$$\begin{aligned}
&F_*u(A^S(\iota\nu, F_*v))\\
=&\nabla_{F_*u} A^S(\iota\nu, F_*v) + A^S(\bar{\nabla}^S_{F_*u}\iota\nu, F_*v) + A^S(\iota\nu, \bar{\nabla}^S_{F_*u}F_*v)\\
=&\nabla_{F_*u} A^S(\iota\nu, F_*v) + A^S(\bar{\nabla}^S_{F_*u}\iota\nu, F_*v) + A^S(\iota h(u,v), \iota\nu)
\end{aligned} \tag{3.11}$$

where $\bar{\nabla}^S$ is the connection on S induced from $\bar{\nabla}$.

Since $A^S(\iota h(u,v), \iota\nu)\nu = A^S(\iota\nu, \iota\nu)h(u,v)$, combining all equations above, we can conclude that

$$\begin{aligned}
\nabla_N h(u,v) = &\left(\nabla_{F_*u} A^S(\iota\nu, F_*v) + A^S(\bar{\nabla}^S_{F_*u}\iota\nu, F_*v)\right)\nu\\
&+ A^S(F_*u, F_*v)h(N,N) - h(\nabla_u N, v)\\
&+ A^S(\iota\nu, \iota\nu)h(u,v) + \overset{\perp}{\pi}(F^* R_{\nabla}(u,N)(F_*v)).
\end{aligned}$$

□

3.4 Stampacchia's Iteration

In this section, we assume that the ambient manifold M satisfies uniform bounds

$$\sigma_x(P) \le K, \quad i_x(M) \ge i(M)$$

for constants $K \geq 0$ and $i(M) > 0$.

3.4.1 Michael-Simon with free boundary

Lemma 9. *There exists a constant $c = c(n, S, M)$ such that for any Σ meeting S orthogonally, and any $f \in C^1(\bar{\Sigma})$*

$$\frac{1}{c} \int_{\partial\Sigma} |f| \leq \int_{\Sigma} |\nabla f| + \int_{\Sigma} |Hf| + \int_{\Sigma} |f| .$$

Proof. Fix $X \in \mathfrak{X}(\mathbb{R}^{n+1})$ which is 0 outside a neighborhood of S and $X|_S = \nu_S$. Let ν be the outward normal of $\partial\Sigma$. By the divergence theorem and product rule, we have that

$$\begin{aligned}
\int_{\partial\Sigma} |f| &= \int_{\partial\Sigma} (|f|\, X) \cdot \nu \\
&= \int_{\Sigma} \operatorname{div}_{\Sigma} \left(|f|\, X^T\right) \\
&= \int_{\Sigma} \nabla |f| \cdot X^T + |f| \operatorname{div}_{\Sigma}(X^T).
\end{aligned}$$

Since $X = X^T + X^{\perp}$ and $\operatorname{div}_{\Sigma}(X^{\perp}) = (X \cdot N)H$, we can conclude that

$$\begin{aligned}
\int_{\partial\Sigma} |f| &= \int_{\Sigma} \nabla |f| \cdot X^T + |f| \operatorname{div}_{\Sigma}(X^T) \\
&= \int_{\Sigma} \nabla |f| \cdot X^T + |f| \operatorname{div}_{\Sigma}(X) - |f| (X \cdot N) H \\
&\leq \max |X| \int_{\Sigma} |\nabla f| + n \max |\nabla X| \int_{\Sigma} |f| + \max |X| \int_{\Sigma} |Hf| .
\end{aligned}$$

$\square$

Lemma 10. *Let f be a Lipschitz function on Σ vanishing on $\partial\Sigma$. Then*

$$\left(\int_{\Sigma} |f|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \leq c(n) \left(\int_{\Sigma} |\nabla f| + \int_{\Sigma} H |f| \right)$$

provided

$$K^2 (1-\alpha)^{-\frac{2}{n}} (\omega^{-1} |\text{supp } f|^{\frac{2}{n}}) \leq 1$$

and

$$2\rho_0 \leq i(N)$$

where ω_n is the volume of the unit ball and

$$\rho_0 = K^{-1} \arcsin \left\{ K (1-\alpha)^{-\frac{1}{n}} \left(\omega_n^{-1} |\text{supp } f| \right)^{\frac{1}{n}} \right\} .$$

Here $0 < \alpha < 1$ is a free parameter and

$$c(n) = \pi 2^{n-1} \alpha^{-1} (1-\alpha)^{-\frac{1}{n}} \frac{n}{n-1} \omega_n^{-\frac{1}{n}} .$$

Theorem 9. *There exists a constant $c = c(n)$ such that for any Σ meeting S orthogonally and any $f \in C^1_C(\bar{\Sigma})$ satisfying the conditions in Lemma 10,*

$$\frac{1}{c}\left(\int_\Sigma |f|^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}} \leq \int_\Sigma |\nabla f| + \int_\Sigma |Hf| + \int_{\partial\Sigma} |f| \,.$$

Proof. Without loss of generality, we assume that $f > 0$.

Let $d : \Sigma \times \Sigma \to \mathbb{R}$ be the distance function on Σ. Let $\Omega = \{x \in \Sigma : d(x, \partial\Sigma) \leq \epsilon\}$. Then for sufficiently small $\epsilon > 0$, we can find the diffeomorphism $\phi : [0, \epsilon] \times \partial\Sigma \to \Omega$ with bounded Jacobian $|J\phi| \in [\frac{1}{2}, 2]$.

Hence

$$\begin{aligned}\int_\Omega f &= \int_0^\epsilon \int_{\partial\Sigma} f \,|J\phi| \\ &\leq 2 \int_0^\epsilon \int_{\partial\Sigma} f(t, x) \\ &\leq \epsilon^2 \,|\partial\Sigma| \sup_\Sigma |\nabla f| + 2\epsilon \int_{\partial\Sigma} f\end{aligned} \tag{3.12}$$

where the last inequality follows from the Taylor expansion $f(t, x) = f(0, x) + t\frac{\partial}{\partial t} f(t^*(x), x)$ for some $t^*(x) \in (0, \epsilon)$ depending on $x \in \partial\Sigma$.

Let $\eta : \Sigma \to \mathbb{R}$ be a smooth function such that $\eta|_{\partial\Sigma} \equiv 0$, $\eta|_{\Sigma-\Omega} \equiv 1$ and $|\nabla\eta| \leq \frac{2}{\epsilon}$. By Equation 3.12, we have that

$$\begin{aligned}
\int_\Sigma \left((1-\eta)\, f\right)^{\frac{n}{n-1}} &\leq \int_\Omega f^{\frac{n}{n-1}} \\
&\leq \epsilon^2 \left|\partial\Sigma\right| \sup_\Sigma \left|\nabla\left(f^{\frac{n}{n-1}}\right)\right| + 2\epsilon \int_{\partial\Sigma} f^{\frac{n}{n-1}} \\
&\leq \epsilon C
\end{aligned}$$

for C independent of ϵ. For the function ηf which vanishes on $\partial\Sigma$, we can apply Lemma 10 and conclude that

$$\|\eta f\|_{\frac{n}{n-1}} \leq c(n)\left(\int_\Sigma |\nabla(\eta f)| + \int_\Sigma |H|\,\eta f\right).$$

Therefore, for $c = c(n)$ and all sufficiently small $\epsilon > 0$,

$$\begin{aligned}
\|f\|_{\frac{n}{n-1}} &\leq \|\eta f\|_{\frac{n}{n-1}} + \|(1-\eta) f\|_{\frac{n}{n-1}} \\
&\leq c\int_\Sigma \eta\,|\nabla f| + c\int_\Sigma |H|\,\eta f + c\int_\Sigma |\nabla\eta|\, f + \epsilon^{\frac{n-1}{n}} C \\
&\leq c\int_\Sigma |\nabla f| + c\int_\Sigma |H|\, f + \frac{2c}{\epsilon}\int_\Omega f + \epsilon^{\frac{n-1}{n}} C \\
&\leq c\int_\Sigma |\nabla f| + c\int_\Sigma |H|\, f + 4c\int_{\partial\Sigma} f \\
&\quad + 2c\epsilon\,|\partial\Sigma| \sup_\Sigma |\nabla f| + \epsilon^{\frac{n-1}{n}} C.
\end{aligned}$$

The conclusion follows by taking $\epsilon \to 0$. □

Finally, by combining Lemma 9 and Theorem 9, we can derive the following Michael-Simon inequality for free boundary hypersurfaces in Riemannian manifold using the argument identical to the proof of Theorem 2.3 in [ref:Edelen].

Theorem 10. *For any Σ meeting S orthogonally, any $f \in C^1(\bar{\Sigma})$ satisfying the conditions in Lemma 10, and any positive integer $p < n$, there exists a constant $c = c(n, p, S)$ such that*

$$\|f\|_{\frac{np}{n-p};\Sigma} \leq c(\|\nabla f\|_{p;\Sigma} + \|Hf\|_{p;\Sigma} + \|f\|_{p;\Sigma}).$$

3.4.2 Main theorem and the idea of proof

Let $(\Sigma_t)_{t\in[0,T)}$ be a class of hypersurfaces following the free boundary MCF with barrier S. Assume $T < \infty$. Let f_α be a non-negative function on Σ_t where $\alpha = \alpha(S, \Sigma_0, T, n)$. Then we consider another two functions $\tilde{H} > 0, \tilde{G} \geq 0$ on Σ_t such that

$$H = O(\tilde{H}) \qquad \nabla\tilde{H} = O(\tilde{G}).$$

Finally, for another two positve constant σ and k, we let $f = f_\alpha \tilde{H}^\sigma$, $f_k = (f-k)_+$ and $A(k) = \{f \geq k\}$, $A(k,t) = A(k) \cap \Sigma_t$.

We say the function f satisfies the condition $(\star)$ if there exist constants $c = c(S, \Sigma_0, M, T, n, \alpha)$ and $C = C(S, \Sigma_0, M, T, n, \alpha, p, \sigma)$ such that the following two inequalities hold:

(Poincare-like)

$$\begin{aligned} \frac{1}{c}\int_{\Sigma_t} f^p \tilde{H}^2 \leq & p\left(1+\frac{1}{\beta}\right)\int_{\Sigma_t} f^{p-2}\left|\nabla f\right|^2 \\ & + (1+\beta p)\int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f^{p-1} \\ & + \int_{\Sigma_t} f^p + \int_{\partial\Sigma_t} f^{p-1}\tilde{H}^\sigma \end{aligned} \tag{3.13}$$

(Evolution-like)

$$\begin{aligned} \partial_t \int_{\Sigma_t} f_k^p \leq & -\frac{1}{3}p^2\int_{\Sigma_t} f_k^{p-2}\left|\nabla f\right|^2 \\ & - \frac{p}{c}\int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f_k^{p-1} \\ & + C\int_{A(k,t)} f^p + cp\int_{\partial\Sigma_t} f_k^{p-1}\tilde{H}^\sigma \\ & + cp\sigma\int_{A(k,t)} \tilde{H}^2 f^p - \frac{1}{5}\int_{\Sigma_t} \tilde{H}^2 f_k^p + C\left|A(k)\right| \end{aligned} \tag{3.14}$$

for any $p > p_0(n, \alpha, c), \sigma < \frac{1}{2}, k > 0, \beta > 0$.

Now we state the main theorem.

Theorem 11. *If f satisfies $(\star)$, then for sufficiently small σ depending on sufficiently large p, $f = f_\alpha \tilde{H}^\sigma$ is uniformly bounded in spacetime by a constant depending on $(S, \Sigma_0, T, n, \alpha, p, \sigma)$.*

The proof of the main theorem splits into three parts. First, we find a way to handle the boundary term. Then we obtain a higher L^p bound for f by rearranging and combining the inequalities. Finally, using the higher L^p bound and the Michael-Simon inequality, we establish the iteration scheme which leads to the conclusion.

3.4.3 Boundary Integral Estimate

The following two lemmas are needed to handle the boundary integral.

Lemma 11. *Let g be any non-negative funciton on Σ_t. If $r \in (0,2)$, $0 < q < p$ with $\frac{rp}{q} < 2$, then for any $\mu > 0$,*

$$\int_{\Sigma_t} g^q \tilde{H}^r \leq \frac{1}{\mu} \int_{\Sigma_t} g^p \tilde{H}^2 + C(\mu, r, q, p) \int_{\Sigma_t} g^p + |\mathrm{spt} g| \,.$$

Proof. By Young's inequality, since $0 < q < p$, we have that

$$\int_{\Sigma_t} g^q \tilde{H}^r \leq \int_{\Sigma_t} (g^q \tilde{H}^r)^{\frac{p}{q}} + 1$$
$$= \int_{\Sigma_t} g^p \tilde{H}^{\frac{rp}{q}} + |\mathrm{spt} g| \,.$$

Since $\eta := \frac{rp}{2q} < 1$, again by Young's inequality, we can deduce that

$$\begin{aligned} g^p \tilde{H}^{2\eta} &= g^{p\eta} \tilde{H}^{2\eta} g^{p(1-\eta)} \\ &= \left(\frac{1}{\mu\eta} g^p \tilde{H}^2 \right)^{\eta} \left((\mu\eta)^{\frac{\eta}{1-\eta}} g^p \right)^{1-\eta} \\ &\leq \frac{1}{\mu} g^p \tilde{H}^2 + C(\mu, r, q, p) g^p \end{aligned}$$

where $C(\mu, r, q, p) = \frac{(\mu\eta)^{\frac{\eta}{1-\eta}}}{1-\eta}$. The conclusion follows by combining the two inequalities above. □

The Lemma 9 which associates integrals on the boundary and the interior for free boundary surfaces is also needed.

Now we can prove the following lemma which estimates the boundary integral.

Lemma 12. *For any $\sigma < \frac{1}{2}, p > 4$ and $\mu > 0$, there exists* *constants* $c = c(n, S, M)$ *and* $C = C(n, S, M, \mu, p)$ *such that*

$$\begin{aligned}\int_{\partial\Sigma_t} f_k^{p-1}\tilde{H}^\sigma \leq & c\int_{\Sigma_t} |\nabla f|^2 f_k^{p-2} + c\sigma\int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f_k^{p-1} \\ & + \frac{cp^2}{\mu}\int_{A(k,t)} f^p\tilde{H}^2 + C\int_{A(k,t)} f^p + C\,|A(k,t)|\,.\end{aligned} \tag{3.15}$$

Proof. By Lemma 9, we have that

$$\begin{aligned}\frac{1}{c(n,S,M)}\int_{\partial\Sigma_t} f_k^{p-1}\tilde{H}^\sigma \leq \int_{\Sigma_t}\left|\nabla\left(f_k^{p-1}\tilde{H}^\sigma\right)\right| + \int_{\Sigma_t}\left|Hf_k^{p-1}\tilde{H}^\sigma\right| \\ + \int_{\Sigma_t}\left|f_k^{p-1}\tilde{H}^\sigma\right|.\end{aligned}$$

Since f_k and $\tilde{H}$ are non-negative, by product rule and triangular inequality, we have that

$$\left|\nabla\left(f_k^{p-1}\tilde{H}^\sigma\right)\right| \leq pf_k^{p-2}\tilde{H}^\sigma\,|\nabla f| + c(n,S,M)\sigma f_k^{p-1}\tilde{H}^{\sigma-1}\tilde{G}.$$

Combining the inequalities above, we have that, for some constant $c = c(n, S, M)$ and $\sigma < \frac{1}{2}$,

$$\begin{aligned}\int_{\partial\Sigma_t} f_k^{p-1}\tilde{H}^\sigma \leq & c\int_{\Sigma_t} f_k^{p-2}|\nabla f|^2 + cp^2\int_{\Sigma_t} f_k^{p-2}\tilde{H}^{2\sigma} \\ & + c\sigma\int_{\Sigma_t} f_k^{p-1}\frac{\tilde{G}^2}{\tilde{H}^{\sigma-2}} + c\int_{\Sigma_t} f_k^{p-1}\left(\tilde{H}^\sigma + \tilde{H}^{\sigma+1}\right)\end{aligned}$$

Finally, since $\sigma < \frac{1}{2}$ and $p > 4$, for any $\mu > 0$, we can apply Lemma 11 for $\int_{\Sigma_t} f_k^{p-2}\tilde{H}^{2\sigma}$, $\int_{\Sigma_t} f_k^{p-1}\tilde{H}^\sigma$ and $\int_{\Sigma_t} f_k^{p-1}\tilde{H}^{1+\sigma}$; thus concluding that

$$\int_{\partial\Sigma_t} f_k^{p-1}\tilde{H}^\sigma \leq c\int_{\Sigma_t} |\nabla f|^2 f_k^{p-2} + c\sigma\int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma} f_k^{p-1}} + \frac{cp^2}{\mu}\int_{A(k,t)} f^p\tilde{H}^2 + C\int_{A(k,t)} f^p + C\,|A(k,t)|$$

for constants $c = c(n, S, M)$ and $C = C(n, S, M, \mu, p)$. □

3.4.4 Higher L^p bound

Next, we establish the higher L^p bound for f.

Lemma 13. *Suppose f satisfies $(\star)$. Then there exist constants $p_0(c)$ and $c_\sigma(c)$ depending on some $c = c(S, \Sigma_0, M, T, n, \alpha)$ such that for $p > p_0(c)$ and $\sigma < \frac{c_\sigma(c)}{\sqrt{p}}$,*

$$\int_0^T \int_{\Sigma_t} f^p \leq C_1(C, T, \Sigma_0) < \infty.$$

Proof. By Equation 3.14, for $k = 0$, we have that

$$
\begin{aligned}
\partial_t \int_{\Sigma_t} f^p \leq & -\frac{1}{3}p^2 \int_{\Sigma_t} f^{p-2} |\nabla f|^2 - \frac{p}{c} \int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f^{p-1} \\
& + C \int_{\Sigma_t} f^p + cp \int_{\partial \Sigma_t} f^{p-1} \tilde{H}^\sigma \\
& + cp\sigma \int_{\Sigma_t} \tilde{H}^2 f^p - \frac{1}{5} \int_{\Sigma_t} \tilde{H}^2 f^p + C |\Sigma_t| \\
\leq & -\frac{1}{3}p^2 \int_{\Sigma_t} f^{p-2} |\nabla f|^2 - \frac{p}{c} \int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f^{p-1} \\
& + C \int_{\Sigma_t} f^p + cp \int_{\partial \Sigma_t} f^{p-1} \tilde{H}^\sigma - \frac{1}{5} \int_{\Sigma_t} \tilde{H}^2 f^p + C |\Sigma_t| \\
& + cp\sigma [p \left(1 + \frac{1}{\beta}\right) \int_{\Sigma_t} f^{p-2} |\nabla f|^2 \\
& + (1 + \beta p) \int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f^{p-1} + \int_{\Sigma_t} f^p + \int_{\partial \Sigma_t} f^{p-1} \tilde{H}^\sigma]
\end{aligned}
$$

where we use Equation 3.13 to estimate the term $cp\sigma \int_{\Sigma_t} \tilde{H}^2 f^p$.

For the boundary integral $\int_{\partial \Sigma_t} f^{p-1} \tilde{H}^\sigma$, we apply the previous estimate Lemma 12 and conclude that

$$
\begin{aligned}
\partial_t \int_{\Sigma_t} f^p \leq & \left[-\frac{1}{3}p^2 + cp^2\sigma(1 + \frac{1}{\beta}) + cp \right] \int_{\Sigma_t} f^{p-2} |\nabla f| \\
& + \left[-\frac{p}{c} + cp\sigma(1 + \beta p) + cp\sigma \right] \int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f^{p-1} \\
& + \left(\frac{cp^3}{\mu} - \frac{1}{5} \right) \int_{\Sigma_t} \tilde{H}^2 f^p \\
& + C |\Sigma_t| + C \int_{\Sigma_t} f^p
\end{aligned} \tag{3.16}
$$

For $p > 12c$, we can choose constants $\mu = 10cp^3, \beta = \frac{1}{\sqrt{cp}}, \sigma = \frac{1}{6\sqrt{c^3 p}}$ such that

$$\begin{cases} -\frac{1}{3}p^2 + cp^2\sigma(1+\frac{1}{\beta}) + cp \leq 0 \\ -\frac{p}{c} + cp\sigma(1+\beta p) + cp\sigma \leq 0 \\ \frac{cp^3}{\mu} - \frac{1}{5} \leq 0. \end{cases}$$

Therefore $\int_0^T \int_{\Sigma_t} f^p \leq C_1(C, T, \Sigma_0) < \infty$ as T is finite. □

We can also simplify the evolution-like equation for f_k and obtain the following lemma.

Lemma 14. *Suppose f satisfies $(\star)$. Then for σ, p satisfying the same bounds as Lemma 13 and C independent of k,*

$$\partial_t \int_{\Sigma_t} f_k^p \leq -\frac{p^2}{12}\int_{\Sigma_t} f_k^{p-2}|\nabla f|^2 + C\int_{A(k,t)} f^p + C|A(k)| + C\int_{A(k,t)} \tilde{H}^2 f^p$$

Proof. By rewriting the boundary integral in Equation 3.14 using Lemma 12, we have that

$$
\begin{aligned}
\partial_t \int_{\Sigma_t} f_k^p \leq & -\frac{1}{3}p^2 \int_{\Sigma_t} f_k^{p-2} |\nabla f|^2 + C \int_{A(k,t)} f^p \\
& - \frac{p}{c} \int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f_k^{p-1} + C |A(k)| \\
& + cp\sigma \int_{A(k,t)} \tilde{H}^2 f^p - \frac{1}{5} \int_{\Sigma_t} \tilde{H}^2 f_k^p \\
& + cp \Bigg[\int_{\Sigma_t} |\nabla f|^2 f_k^{p-2} + \sigma \int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f_k^{p-1} \\
& + \frac{p^2}{\mu} \int_{A(k,t)} f^p \tilde{H}^2 + C \int_{A(k,t)} f^p + C |A(k,t)| \Bigg] \\
\leq & \left(cp - \frac{1}{3}p^2 \right) \int_{\Sigma_t} f_k^{p-2} |\nabla f|^2 + C \int_{A(k,t)} f^p \\
& + p \left(c\sigma - \frac{1}{c} \right) \int_{\Sigma_t} \frac{\tilde{G}^2}{\tilde{H}^{2-\sigma}} f_k^{p-1} + C |A(k)| \\
& + cp \left(\sigma + \frac{p^2}{\mu} \right) \int_{A(k,t)} \tilde{H}^2 f^p - \frac{1}{5} \int_{\Sigma_t} \tilde{H}^2 f_k^p
\end{aligned}
$$

□

The conclusion follows by choosing the value of p, σ, μ as in the proof of Lemma 13.

3.4.5 Iteration Scheme and the Uniform bound

By Theorem 10, for each $n \geq 2$, there exist some $q > 1$ and $c = c(n, q, |\Sigma_0|, S)$ such that

$$\left(\int_\Sigma v^{2q}\right)^{\frac{1}{q}} \leq c\int_\Sigma |\nabla v|^2 + c\int_\Sigma v^2 H^2 + c\int_\Sigma v^2$$

provided that v satisfies the assumptions in Lemma 10. For $n > 2$, we let $q = \frac{n}{n-2}$. For $n = 2$, we apply Corollary 2.4 and Remark 2.5 in [ref:Edelen].

Take $v = f_k^{\frac{p}{2}}$, then by Lemma 13, we have that

$$|\text{supp } v| = |A(k,t)| \leq \frac{1}{k}\int_{\Sigma_t} f \leq \frac{1}{k}C'$$

where C' depend on C_1 and $|\Sigma_0|$. Since $C_1 = C_1(C, T, \Sigma_0)$ and the constant C in $(\star)$ depends on $(S, \Sigma_0, M, T, n, \alpha, p, \sigma)$, for $k \geq k_0(S, \Sigma_0, M, T, n, \alpha, p, \sigma)$

$$\left(\int_{\Sigma_t} f_k^{pq}\right)^{\frac{1}{q}} \leq c\int_{\Sigma_t} \left|\nabla f_k^{\frac{p}{2}}\right|^2 + c\int_{\Sigma_t} f_k^p H^2 + c\int_{\Sigma_t} f_k^p. \qquad (3.17)$$

Theorem 12. *Suppose there are constants p_0 and σ_0 indepen-*

dent of p, σ, k *such that for* $p > p_0$ *and* $\sigma < \frac{\sigma_0}{\sqrt{p}}$*, we have that*

$$\int_0^T \int_{\Sigma_t} f^p < \infty$$

and

$$\partial_t \int_{\Sigma_t} f^p + \frac{1}{c} \int_{\Sigma_t} \left| \nabla f_k^{\frac{p}{2}} \right|^2 \leq C \int_{A(k,t)} \tilde{H}^2 f^p + C \int_{A(k,t)} f^p + C \left| A(k,t) \right| \tag{3.18}$$

for any $k > 0$ *where* C, c *are constants independent of* k*. Then for sufficient small* σ*,* f *is uniformly bounded in spacetime and the bound will depend on* $(S, \Sigma_0, M, T, n, \alpha, p, \sigma)$*.*

Proof. Integrating Equation 3.18 and Equation 3.17 over $[0, T)$ yields that

$$\sup_{t \in [0,T)} \int_{\Sigma_t} f^p + \frac{1}{c} \int_0^T \int_{\Sigma_t} \left| \nabla f_k^{\frac{p}{2}} \right|^2 \leq C \iint_{A(k)} \tilde{H}^2 f^p + C \iint_{A(k)} f^p + C \left| A(k) \right|$$

and

$$\int_0^T \left(\int_{\Sigma_t} f_k^{pq} \right)^{\frac{1}{q}} \leq c \int_0^T \int_{\Sigma_t} \left| \nabla f_k^{\frac{p}{2}} \right|^2 + c \iint_{A(k)} f_k^p H^2 + c \iint_{A(k)} f_k^p.$$

provided that $k \geq k_0(S, \Sigma_0, M, T, n, \alpha, p, \sigma)$. Then by adjust the constants to absorb the term $\int_0^T \int_{\Sigma_t} \left|\nabla f_k^{\frac{p}{2}}\right|^2$, we have that

$$\begin{aligned}
&\max\left\{\sup_{t\in[0,T)} \int_{\Sigma_t} f_k^p, \int_0^T \left(\int_{\Sigma_t} f_k^{pq}\right)^{\frac{1}{q}}\right\} \\
\leq& C \iint_{A(k)} \tilde{H}^2 f^p + C \iint_{A(k)} f^p + C\left|A(k)\right|.
\end{aligned}$$

Hence by Holder's inequality,

$$\begin{aligned}
\int_0^T \int_{\Sigma_t} f_k^{p\frac{2q-1}{q}} &\leq \int_0^T \int_{\Sigma_t} f_k^p f_k^{p\frac{q-1}{q}} \\
&\leq \int_0^T \left(\int_{\Sigma_t} f_k^{pq}\right)^{\frac{1}{q}} \left(\int_{\Sigma_t} f_k^p\right)^{\frac{q-1}{q}} \\
&\leq \left(\sup_{t\in[0,T)} \int_{\Sigma_t} f_k^p\right)^{\frac{q-1}{q}} \int_0^T \left(\int_{\Sigma_t} f_k^{pq}\right)^{\frac{1}{q}} \\
&\leq \left(C \iint_{A(k)} \tilde{H}^2 f^p + C \iint_{A(k)} f^p + C\left|A(k)\right|\right)^{\frac{2q-1}{q}}.
\end{aligned} \tag{3.19}$$

For any function g defined on $A(k)$, for any $r > 1$, we can apply the Holder's inequality to have that

$$\iint_{A(k)} g \leq \left|A(k)\right|^{1-\frac{1}{r}} \left(\iint_{A(k)} g^r\right)^{\frac{1}{r}}.$$

Hence

$$\begin{aligned}
\int_0^T \int_{\Sigma_t} f_k^{p\frac{2q-1}{q}} \leq& C\left|A(k)\right|^{\frac{2q-1}{q}\left(1-\frac{1}{r}\right)} [\left(\iint_{A(k)} f^{pr}\right)^{\frac{1}{r}} \\
&+ \left(\iint_{A(k)} \tilde{H}^{2r} f^{pr}\right)^{\frac{1}{r}} + \left|A(k)\right|^{\frac{1}{r}}]^{\frac{2q-1}{q}}.
\end{aligned}$$

For p sufficiently large relative to r, we have that

$$\iint_{A(k)} f^{pr} < +\infty$$

and

$$\iint_{A(k)} \left(\tilde{H}^2 f^p\right)^r = \iint_{A(k)} \left(f_\alpha \tilde{H}^{\sigma+\frac{2}{p}}\right)^{pr} < +\infty.$$

By fixing r sufficiently large, we let $\gamma = \frac{2q-1}{q}\left(1 - \frac{1}{r}\right) > 1$ and $\beta = p\frac{2q-1}{q} > 0$.

Thus, for any $l > k$, Equation 3.19 implies that

$$|l-k|^\beta |A(k)| \leq \iint_{A(l)} f_k^\beta \leq C\,|A(k)|^\gamma$$

where the constant C is independent of l and k.

Therefore, by Lemma 3, $A(k) = 0$ for $k > k_1(\alpha, \beta, C)$. □

3.5 Conclusions and Directions for Future Research

This thesis aimed at providing a theoretical foundation for the convergence theory of MCF with free boundary in the Riemannian ambient manifold. By reviewing the classical method by

Huisken, we highlight the importance of the iteration scheme for showing the pinching estimate of the traceless second fundamental form. The thesis is ended by the establishment of the iteration scheme in a Riemannian manifold following the argument of Edelen and the computation of the boundary derivative of the second fundamental form.

As discussed at the beginning of this chapter, the boundary derivatives are essential for applying maximum principle to prove the preservation of properties. Cross terms which are impossible to control will appear in the boundary derivatives when the barrier is not umbilic and make the maximum principle not applicable.

To cancel the problematic cross term, a perturbation of the second fundamental form which introduced by Huisken and Sinestrari [HS99] could be used. When the barrier is in the Euclidean space of dimension three, Hirsch and Li [HL20] defined a perturbation tensor which kills off the cross terms on the boundary and enable them to apply the maximum principle. To obtain information on the original second fundamental form, controlling the perturbed form is necessary. In Hirsch and Li's work, one major factor influencing the estimates of the perturbed form is the ball curvature of the barrier. Such property can be well defined in the Euclidean space, but in a 3-manifold, we only have locally defined balls. Brendle [Bre13] introduced the method of using local balls to define ball curvatures in Riemannian manifolds. By combining ideas from Hirsch and Li, and Brendle, it is believed that the difficulty of estimating boundary derivatives could be overcome.

Another furture research direction involves the non-convex initial conditions for convergence of free boundary hypersurfaces in the unit ball. For the free boundary MCF with barriers on the standard hypersphere, it is known that any convex free boundary hypersurface will converge to a round half point [Sta96a]. Considering Huisken's study on MCF in spherical spaceforms

[Hui87], it is natural to ask that whether the convexity condition can be replaced by some non-convex curvature pinching condition.

The similarities between free boundary minimal surfaces in the unit ball and closed minimal surfaces in the standard sphere are reflected in various research results [Alm66, Nit85, Ros08] and would be helpful to this research topic. Moreover, the study of MCF in sphere by Huisken [Hui87] would inspire the proposed research topic greatly by its setting of initial condition which implies the positivity of intrinsic curvature of the surfaces.

□ **End of chapter.**

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