Chapter 2

Tools of Particle Physics

2.1 Mass in Relativity

Do not get confused by the difference between *invariant mass* m_0 defined as,

$$m_0 = \sqrt{E^2/c^4 - p^2/c^2} \tag{2.1}$$

and *relativistic mass* m_R

$$m_R = E/c^2 \tag{2.2}$$

The invariant mass of a particle is **independent** of its velocity v whereas relativistic mass increases with velocity and tends to infinity as the velocity appproaches the speed of light c. The relationship between the them is:

$$m_R = m_0/\sqrt{1 - v^2/c^2} \tag{2.3}$$

The two masses are therefore equivalent only when the velocity is zero, hence the name 'rest mass' which is commonly used for m_0.

We will always take the term 'mass' to refer to **invariant mass** and give it the symbol m as is in common use in the particle physics literature.

2.2 Four-Vectors

Formulating physics theories in terms of 4-vectors is convenient to make the Lorentz transformation properties of the theory transparent. Define a space-time 4-vector X^μ ($\mu = 0, 1, 2, 3$):

$$X^0 = ct, X^1 = x, X^2 = y, X^3 = z \tag{2.4}$$

In this notation the time and space components of X^μ, when viewed from an inertial frame travelling at velocity v along the x-axis, undergo Lorentz transformations to become:

$$\begin{aligned} X'^0 &= \gamma(X^0 - \beta X^1) \\ X'^1 &= \gamma(X^1 - \beta X^0) \\ X'^2 &= X^2 \\ X'^3 &= X^3 \end{aligned} \tag{2.5}$$

where $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$.

In a more compact notation this becomes:

$$X'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} X^{\nu} \tag{2.6}$$

where,

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.7}$$

To avoid writing out lots of $\sum$ indices, we will adopt the *Einstein summation convention* where repeated indices (one as subscript, the other as superscript or vice-versa) are summed over, so that eqn.2.6 becomes:

$$X'^{\mu} = \Lambda^{\mu}_{\nu} X^{\nu} \tag{2.8}$$

Although the components of X change as we move between inertial frames the following combination is an invariant:

$$S^2 = (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 = (X'^0)^2 - (X'^1)^2 - (X'^2)^2 - (X'^3)^2 \tag{2.9}$$

The form of S^2 now allows us to define a 4-vector inner-product (or 'dot-product' or 'contraction') that is Lorentz invariant (in analogy to the usual 3-vector dot-product which is an invariant in 3D cartesian space). To do this we must first define a *covariant* space-time 4-vector X_μ(index down) in contrast to our original *contravariant* vector X^μ(index up) as follows:

$$X_\mu = g_{\mu\nu} X^\nu = (X^0; -X^1, -X^2, -X^3) \tag{2.10}$$

where $g_{\mu\nu}$ is the *metric* tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.11}$$

(N.B. sometimes the metric is defined with a sign-flip compared to eqn.2.11 so that the diagonal reads $(-1, 1, 1, 1)$. This does not change the physics since if S^2 is invariant, so too is $-S^2$.) We now define the inner product of X with itself to be

$$\begin{aligned} g_{\mu\nu} X^\nu X^\mu &= X_\mu X^\mu = X_0 X^0 + X_1 X^1 + X_2 X^2 + X_3 X^3 \\ &= (X^0)^2 - (X^1)^2 - (X^2)^2 - (X^3)^2 \\ &= (X'^0)^2 - (X'^1)^2 - (X'^2)^2 - (X'^3)^2 = S \end{aligned} \tag{2.12}$$

which we know is a Lorentz invariant quantity.

Furthermore, if we demand that *any* 4-vector we construct $(A, B, ...)$, transforms in the same way as the space-time 4-vector (i.e. $A'^{\mu} = \Lambda^{\mu}_{\nu} A^{\nu}, B'^{\mu} = \Lambda^{\mu}_{\nu} B^{\nu}, ...$) and we use the metric tensor to switch between the contravariant and covariant forms, then inner products such as

$$A_{\mu} B^{\mu} = A^{\mu} B_{\mu} = A_0 B_0 - A_1 B_1 - A_2 B_2 - A_3 B_3 \tag{2.13}$$

are automatically Lorentz invariant.

We can define 4-vector derivatives in a similar way:

$$\begin{aligned} \partial^{\mu} &= \frac{\partial}{\partial x_{\mu}} = \left(\frac{\partial}{\partial t}; -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z} \right) \\ \Rightarrow \partial^{\mu} &= \left(\frac{\partial}{\partial t}; -\underline{\nabla} \right) \\ \partial_{\mu} &= \left(\frac{\partial}{\partial t}; \underline{\nabla} \right) \end{aligned} \tag{2.14}$$

Hence for any 4-vector V,

$$\partial_{\mu} V^{\mu} = \frac{\partial V^0}{\partial t} + \underline{\nabla} \cdot \underline{V} \tag{2.15}$$

It also follows that,

$$\partial_{\mu} \partial^{\mu} = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2} \tag{2.16}$$

$$= \frac{\partial^2}{\partial t^2} - \underline{\nabla}^2 = \partial^{\mu} \partial_{\mu} \tag{2.17}$$

In relativistic kinematics, energy and momentum form a 4-vector P^{μ} where energy plays the role of the 'time' component and 3-momentum forms the spatial components

$$P^{\mu} = \left(P^0; \underline{p} \right) = \left(\frac{E}{c}, p_x, p_y, p_z \right) \tag{2.18}$$

and,

$$P^2 = P^{\mu} P_{\mu} = \left(\frac{E}{c} \right)^2 - (\underline{p})^2 = m^2 c^2 \tag{2.19}$$

which is manifestly invariant between frames since m is the rest mass. As expected, the Lorentz transformations of the energy-momentum 4-vector are of the same form as the space-time 4-vector 2.5:

$$\begin{aligned} E' &= \gamma(E - \beta p_x c) \\ p'_x c &= \gamma(p_x c - \beta E) \\ p'_y &= p_y \\ p'_z &= p_z \end{aligned} \tag{2.20}$$

Finally, note that vectors are, more generally, tensors of 'rank 1'. A tensor of rank 2 ($Y^{\mu\nu}$) carries two indices, has $4^2 = 16$ components and transforms by applying the transformation, Λ, twice i.e.

$$Y'^{\mu\nu} = \Lambda^{\mu}_{\alpha}\Lambda^{\nu}_{\beta}Y^{\alpha\beta} \tag{2.21}$$

Covariant and 'mixed' tensors (i.e. one index up and the other down) can be formed by operating with the metric tensor as before,

$$Y_{\mu\nu} = g_{\mu\alpha}g_{\nu\beta}Y^{\alpha\beta} \tag{2.22}$$
$$Y^{\mu}_{\nu} = g_{\nu\beta}Y^{\mu\beta} \tag{2.23}$$

2.3 Units

SI units:

$[E]$	J
$[M]$	kg
$[L]$	m
$[T]$	s

Not such a practical choice for particle physics ($m_{\text{proton}} \sim 10^{-27}$kg!)
Instead use 'natural units' based on the dimensional scales relevant to relativity and quantum mechanics:

- Energy measured in electron volts (1eV $= 1.6 \times 10^{-19}$J, 1GeV $\equiv 10^9$eV)
- Velocity of light, $c = 3.0 \times 10^8$m/s
- Unit of action in QM, $\hbar = h/2\pi = 6.6 \times 10^{-22}$MeVs ($1.1 \times 10^{-34}$Js)

Exercise 1.1 *An aside on electron-volt units of energy: 1eV is the energy gained by an electron when it is accelerated through a potential difference of 1V. The Equipartition Principle states that for a particle in space* $\langle KE \rangle = \frac{3kT}{2}$, k *=Boltzmann's constant* $= 1 \times 10^{-4}$ eV/K.
eg 1) For a particle at room temperature: $T \sim 290K$, $\langle KE \rangle = \frac{3}{2} \times 10^{-4} \cdot 290 = 0.04eV$
eg 2) For a particle at surface of the Sun: $T \sim 6000K$, $\langle KE \rangle \sim 20 \times 0.04 = 0.8eV$)

Energy	GeV
Momentum (N.B. $KE = pc$)	GeV/c
Time (N.B. $\Delta E \Delta t \sim \hbar$)	$\hbar$/GeV
Length (N.B. $\hbar c = 200$ MeVfm)	$\hbar c$/GeV
Area	$(\hbar c/\text{GeV})^2$

Examples

1)

$$\hbar c = 1.1 \times 10^{-34}\,[\text{Js}] \cdot 3.0 \times 10^{8}\left[\text{ms}^{-1}\right]$$
$$\hbar c = \frac{1.1 \times 10^{-34}}{1.6 \times 10^{-13}}\,[\text{MeVs}] \cdot 3.0 \times 10^{8}\left[\text{ms}^{-1}\right] \quad (\text{N.B. 1eV} = 1.6 \times 10^{-19}\text{J})$$
$$\hbar c = 2 \times 10^{-13}\,[\text{MeVm}] = 200\,MeV fm$$

N.B. This form can be very handy.

2) Rest energy of a proton:
Proton mass, $m_p = 1.67 \times 10^{-27}$kg
Proton rest energy $= m_p c^2 = 1.67 \times 10^{-27} \cdot (3.0 \times 10^8)^2 = 1.50 \times 10^{-10}$ [kg(ms^{-1})^2]
but 1J = 1kg(ms^{-1})^2 (e.g. since $KE = \frac{1}{2}mv^2$)

$$\begin{aligned} \Rightarrow m_p c^2 &= \frac{1.50 \times 10^{-10}}{1.60 \times 10^{-19}} \mathrm{eV} = 937.5\mathrm{MeV} \\ (\text{or, } m_p &= 937.5\mathrm{MeV/c^2}) \end{aligned}$$

2.3.1 Natural Units: $\hbar = c = 1$

Often convenient to simplify equations by setting $\hbar = c = 1$ i.e. using *natural units*.In this scheme the relativistic energy becomes $E^2 = p^2 + m^2$ and all quanitities are expressed in units of energy.

Energy	GeV
Momentum	GeV
Mass	GeV
Time	GeV^{-1}
Length	GeV^{-1}
Area	GeV^{-2}

$\Rightarrow$ Convert back into SI units at the end of a calculation by putting back the missing factors of $\hbar$ and c:

Example

Illustration of convenience of natural units: If a proton has kinetic energy of 1GeV, what is its momentum?

(a) **in SI units**: Must first find the total energy E, and then use $E^2 = p^2c^2 + m^2c^4$

$$\begin{aligned} E &= KE + E_{\mathrm{REST}} = KE + m_p c^2 \\ &= 1 \times 10^9 \cdot 1.60 \times 10^{-19} [\mathrm{J}] + 1.67 \times 10^{-27} (3.0 \times 10^8)^2 [\mathrm{kgm^2s^{-2}}] \ (1\mathrm{eV} = 1.60 \times 10^{-19}\mathrm{J}) \\ &= 1.6 \times 10^{-10} + 15.03 \times 10^{-11} = 3.1 \times 10^{-10}\mathrm{J} \end{aligned}$$

Now we can evaluate p,

$$\begin{aligned} p^2 &= \frac{E^2}{c^2} - m^2c^2 \\ p^2 &= \frac{(3.1 \times 10^{-10})^2}{(3.0 \times 10^8)^2} [\mathrm{kg^2m^2s^{-2}}] - (1.67 \times 10^{-27})^2 (3.0 \times 10^8)^2 [\mathrm{kg^2m^2s^{-2}}] \\ p^2 &= 1.07 \times 10^{-36} - 0.25 \times 10^{-36} = 0.82 \times 10^{-36} [\mathrm{kg^2m^2s^{-2}}] (\text{or } \mathrm{J^2m^{-2}s^2}) \\ \Rightarrow p &= 0.91 \times 10^{-18} \mathrm{Jm^{-1}s} \end{aligned}$$

(b) **in natural units**: Set $\hbar = c = 1$, so that $E^2 = p^2 + m^2$

$$E = KE + m_p = 1.0 + 0.94 = 1.94\mathrm{GeV}$$

evaluate p,

$$\begin{aligned} p^2 &= E^2 - m_p^2 = (1.94)^2 - (0.94)^2 = 2.88\text{GeV}^2 \\ \Rightarrow p &= 1.70\text{GeV} \end{aligned}$$

Put back factor of c, $p = 1.70$ GeV/c. Much simpler/quicker!
Cross check the answer agrees with (a):

$$\begin{aligned} p &= 1.70\,[\text{GeV}/c] \\ &= \frac{1.70 \times 10^9 \cdot 1.6 \times 10^{-19}}{3.0 \times 10^8}\text{Jm}^{-1}\text{s} = 0.91 \times 10^{-18}\text{Jm}^{-1}s \end{aligned}$$

Setting $\hbar = c = 1$, note these very useful relationships:

$$\gamma = \tfrac{E}{m}$$
$$\beta = \tfrac{p}{E}$$

2.4 Relativistic Kinematics

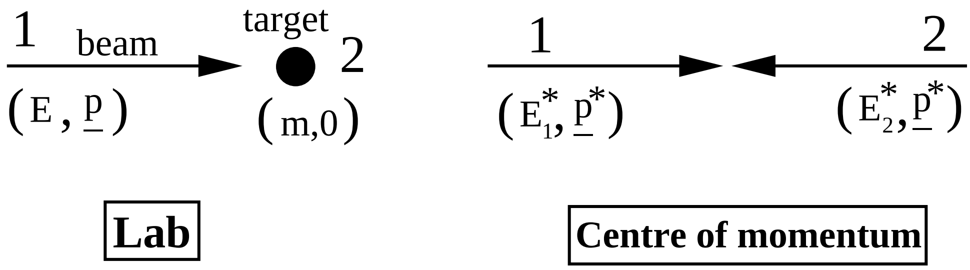

Figure 2.1: Initial states for a collison process in both the lab and COM frame.

Energy and momentum are separately conserved in the lab. and center of momentum frame. For a system of N particles:

$$\left(\sum_{i=1,N} E_i, \sum_{i=1,N} \underline{p_i} \right)_{\text{initial}} = \left(\sum_{i=1,N} E_i, \sum_{i=1,N} \underline{p_i} \right)_{\text{final}} \tag{2.24}$$

In addition, $(4 - \text{momentum})^2$ is invariant between frames since it corresponds to the $(\text{invariant mass})^2$ of the system. For N particles:

$$P^2 = \left(\sum_{i=1,N} E_i \right)^2 - \left(\sum_{i=1,N} \underline{p}_i \right)^2 \tag{2.25}$$

Example: What is the energy threshold for pion production in $p-p$ collisions?

$$p+p \quad \rightarrow \quad (p+p)+\pi, \; (m_p = 0.94\text{GeV}, \; m_\pi = 0.14\text{GeV})$$

(i) Consider the initial state in the **lab frame**(see Fig. 2.1):

$$\begin{aligned} P^2 &= \left(\sum E\right)^2 - \left(\sum \underline{\mathbf{p}}\right)^2 = (E+m_p)^2 - |\underline{p}|^2 \\ &= E^2 + 2m_pE + m_p^2 - |\underline{p}|^2 \\ &= 2m_pE + m_p^2 + m_p^2 = 2m_p(E+m_p) \end{aligned}$$

(ii) Consider the final state in the **COM frame**:
threshold production $\Rightarrow$ final state particles are produced at rest (and so $E = m$)

$$P^2 = \left(\sum E\right)^2 - \left(\sum \underline{p}\right)^2 = (E_p + E_p + E_\pi)^2 - 0 = (2m_p + m_\pi)^2$$

Apply conservation of P^2 between frames:

$$\begin{aligned} 2m_p(E+m_p) &= (m_\pi + 2m_p)^2 \\ E &= \frac{(m_\pi + 2m_p)^2}{2m_p} - m_p = \frac{(0.14+1.88)^2}{1.88} - 0.94 = 1.23\text{GeV} \end{aligned}$$

2.5 Energy in the Centre-of-Momentum Frame

Consider the collision of two particles, $(E_1, \underline{p}_1, m_1), (E_2, \underline{p}_2, m_2)$ at an incident angle θ.

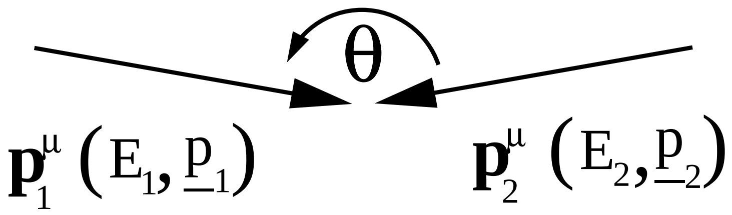

Figure 2.2:

The following quantity is a Lorentz invariant:

$$
\begin{aligned}
s &= (P_1^\mu + P_2^\mu)(P_{1\mu} + P_{2\mu}) \\
s &= (P_1^\mu P_{1\mu} + P_2^\mu P_{2\mu} + 2P_1^\mu P_{2\mu}) \\
s &= (E_1^2 - \underline{p}_1^2) + (E_2^2 - \underline{p}_2^2) + 2(E_1 E_2 - \underline{p}_1 \cdot \underline{p}_2) \\
s &= m_1^2 + m_2^2 + 2(E_1 E_2 - |\underline{p}_1||\underline{p}_2| \cos\theta)
\end{aligned}
$$

s is a so-called **Mandelstam variable** and $\sqrt{s}$ corresponds to the energy, in the COM frame, that is available for new particle production as a result of the collision.

2.6 Fixed target vs Collider

(1) Fixed Target
Let m_1 be the projectile and m_2 the target:

$$
\begin{aligned}
s &= m_1^2 + m_2^2 + 2E_1 m_2 \\
s &= 2E_1 m_2, \text{ if } E_1 >> m_1, m_2 \\
\sqrt{s} &\simeq \sqrt{2E_1 m_2}
\end{aligned}
$$

(2) Collider Experiment
Mass m_1 and m_2 in a head-on collision i.e. $\cos\theta = \cos\pi = -1$:

$$
s = m_1^2 + m_2^2 + 2(E_1 E_2 + |\underline{p}_1||\underline{p}_2|)
$$

If $E >> m_1, m_2$, then $|\underline{p}| \simeq E$ and

$$
s \simeq 2(E_1 E_2 + E_1 E_2) = 4E_1 E_2
$$

For the special case of identicle particles of equal momentum, colliding head-on, the COM frame is at rest in the lab and,

$$
\begin{aligned}
s &= 2m^2 + 2E^2 + 2p^2 = 4E^2 \\
\Rightarrow \sqrt{s} &= 2E
\end{aligned}
$$

Therefore, all (or almost all) the beam energy is available for new particle creation in collider mode and rises linearly with beam energy in contrast to the fixed target situation where most of the beam energy is wasted giving momentum to the target and $\sqrt{s}$ rises only as $\sqrt{E}$.

Example: 100GeV proton hitting a proton at rest:

$$\sqrt{s} \simeq \sqrt{2E_p m_p} = \sqrt{2 \cdot 100 \cdot 1} = 14\text{GeV}$$

Example: 100GeV proton colliding head-on with another 100GeV proton:

$$\sqrt{s} = 2 \cdot 100 = 200\text{GeV}$$

2.7 Boson and fermion QM descriptions

2.7.1 Schrödinger Equation

Classically (i.e. non relativistically) the energy of a free particle is given by,

$$E = K.E. + P.E. = \frac{\mathbf{p}^2}{2m} + V(\underline{\mathbf{r}}) \tag{2.26}$$

Lets assume $V(\underline{\mathbf{r}}) = 0$ and replace momentum and energy by their quantum mechanical operators, to obtain the Schrödinger equation for a free particle associated with wave function $\phi(\underline{r}, t)$:

$$\underline{p} \to \frac{\hbar}{i}\underline{\nabla} \qquad E \to \frac{\hbar}{i}\frac{\partial}{\partial t} \tag{2.27}$$

$$\Rightarrow \frac{\hbar}{i}\frac{\partial\phi(\underline{\mathbf{r}}, t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\phi(\underline{\mathbf{r}}, t) \tag{2.28}$$

which has plane wave solutions: $\phi(\underline{\mathbf{r}}, t) = Ne^{-i(Et - \underline{\mathbf{p}}\cdot\underline{\mathbf{r}})}$.

Note that the Schrödinger equation is 1st order in the time derivative ($\frac{\partial}{\partial t}$) but 2nd order in space derivatives (∇^2) and is therefore not Lorentz invariant! The Schrödinger equation cannot be used to describe relativisitic particles which is a problem since in particle physics we deal almost exclusive with highly relativistic particles.

2.7.2 Klein-Gordan Equation

We need to build an equation that is *covariant* i.e. one whose predictions are invariant under Lorentz transformations. This is assured by basing our equation on the relativistic energy-momentum relationship,

$$E^2 = \underline{\mathbf{p}}^2 c^2 + m^2 c^4 \tag{2.29}$$

Replace momentum and energy by their quantum mechanical operators acting on a particle state $\psi(\underline{\mathbf{r}})$,

$$\begin{aligned} -\hbar^2 \frac{\partial^2\psi(\underline{\mathbf{r}})}{\partial t^2} &= -\hbar^2 c^2 \nabla^2 \psi(\underline{\mathbf{r}}) + m^2 c^4 \psi(\underline{\mathbf{r}}) \\ \Rightarrow \left(\nabla^2 - \frac{\partial^2}{c^2 \partial t^2} - \frac{m^2 c^2}{\hbar^2}\right)\psi(\underline{\mathbf{r}}) &= 0 \\ \text{or } \left(\Box^2 + m^2\right)\psi(\underline{\mathbf{r}}) &= 0 \end{aligned} \tag{2.30}$$

where we have set $\hbar = c = 1$ in the last step. This is the Klein-Gordan equation, which is second order in both time and space derivatives, and is by construction, Lorentz invariant. It also has plane wave solutions of the form, $e^{-i(Et - \underline{\mathbf{p}}\cdot\underline{\mathbf{r}})}$, with energies

$$E = \pm\sqrt{\underline{\mathbf{p}}^2 c^2 + m^2 c^4} \tag{2.31}$$

It turns out that the Klein-Gordan equation, can describe (spinless) boson fields $\psi(\underline{\mathbf{r}})$. If we want to describe particles with spin, things get a bit more complicated!

Aside on Anti-Particles:

- Any theory based on

$$E^2 = \underline{\mathbf{p}}^{\mathbf{2}}\mathbf{c^2} + \mathbf{m^2c^4}$$

  will naturally contain negative energy solutions, (that travel backward in time!)

- They can be interpretted as negative energy *anti-particles* travelling forward in time.

- Each particle has an anti-particle which has the same mass, lifetime and spin but opposite values of all quantum numbers such as charge, colour charge and flavour etc e.g. e^+/e^-, π^+/π^-, W^+/W^- and p^+/p^-.

  N.B.(1) Some particles are their own antiparticle e.g. the photon and the π^0.
  N.B.(2) A common notation for these anti-particle solutions is $\psi/\bar{\psi}$ for particle/anti-particle.

2.7.3 The Dirac Equation

The Dirac equation provides a covariant description of free fermions of mass m and spin $\frac{\hbar}{2}$,

$$\left(i\gamma_\mu \frac{\partial}{\partial x_\mu} - \frac{mc}{\hbar}\right)\psi(x) \equiv (i\gamma_\mu \partial^\mu - m)\,\psi(x) = 0$$

- The Gamma-matrices satisfy:

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}, \text{ where } g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Useful quantity: $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$

- The fermion fields $\psi(x)$ are 4-component 'spinors' and, in the Dirac-Pauli representation,

$$\gamma_\mu = (\gamma_0, \gamma_i)\,,\ \gamma_i = \gamma_0\alpha_i$$
$$\gamma_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \alpha_i = \begin{pmatrix} 0 & \sigma_\mathbf{i} \\ \sigma_\mathbf{i} & 0 \end{pmatrix} \ [\sigma_i\,(i = 1, 2, 3) \equiv \text{Pauli spin matrices}]$$

- $\bar{\psi}(x) = \psi(x)^\dagger\gamma_0 = (\psi^*(x))^T\gamma_0$, are states that also satisfy the Dirac equation (**Exercise: prove this!**) and describe anti-fermions.

2.8 Helicity

- For spin $\hbar/2$ state the spin vector $\underline{\sigma}$ is either along or opposite to the direction of the momentum $\underline{\mathbf{p}}$. Define the *helicity* to be,

$$H = \frac{\underline{\sigma} \cdot \underline{p}}{|\underline{\sigma}| \, E} = \pm\beta$$

- Relativistic fermions ($\beta \simeq 1$) have either $H = 1$ i.e. spin vector aligned with $\underline{\mathbf{p}}$ (right-handed) or $H = -1$ i.e. spin vector anti-aligned (left-handed).

- The electromagnetic and strong interaction couple to both left and right-handed fermions but the weak interaction couples only to left-handed particles (or right-handed anti-particles)!
$\Rightarrow$ only left-handed(right-handed) neutrinos(anti-neutrinos) exist.

- The operators, $(1 \pm \gamma_5)$ project out helicity states i.e.

$$\begin{aligned} \frac{1}{2}(1+\gamma_5)\,\psi &= \psi_L \\ \frac{1}{2}(1-\gamma_5)\,\psi &= \psi_R \end{aligned}$$

2.9 Parity ($\hat{P}$)

- $\hat{P}$ is the operation of spatial inversion; $(x, y, z) \to (-x, -y, -z)$ or $\hat{P}\psi(\underline{\mathbf{r}}) \to \psi(-\underline{\mathbf{r}})$

- Apply parity operator twice and you end up where you started $\hat{P}\left(\hat{P}\psi(\underline{\mathbf{r}})\right) = \psi(\underline{\mathbf{r}}) \Rightarrow$ if the operator has an eigenvalue it must be $P = +1$ (EVEN) or $P = -1$ (ODD)
Example:

$$\begin{aligned} \psi &= \cos x, \hat{P}\psi = \cos(-x) = \cos x = +\psi :\Rightarrow P = +1(\text{EVEN}) \\ \psi &= \sin x, \hat{P}\psi = \sin(-x) = -\sin x = -\psi :\Rightarrow P = -1(\text{ODD}) \end{aligned}$$

- BUT, some wavefunctions may not have a well defined parity at all:

$$\psi = \frac{1}{\sqrt{2}}(\cos x + \sin x)\,, \hat{P}\psi \to \cos x - \sin x \neq \pm\psi$$

- As always, the parity of a system will be a conserved quantum number if $\left[\hat{P}, \hat{H}\right] = 0$. Any spherically symmetric potential has the property $\hat{H}(\underline{\mathbf{r}}) = \hat{H}(-\underline{\mathbf{r}})$ so that $\left[\hat{P}, \hat{H}\right] = 0 \Rightarrow$ bound states of H-atom must have a definite parity.

$\Rightarrow$ Recall solutions of H-atom (ignoring spin-effects):

$$\psi(r,\theta,\phi) \;=\; R(r)\cdot Y_l^m(\theta,\phi) \propto R(r)\cdot P_l^m(\cos\theta)e^{im\phi}$$

Spatial inversion $\underline{\mathbf{r}} \rightarrow -\underline{\mathbf{r}}$ is equivalent to $\begin{cases} \theta \rightarrow \pi - \theta \\ \phi \rightarrow \pi + \phi \end{cases}$

$\Rightarrow$

$$\begin{aligned} R(\underline{\mathbf{r}}) \rightarrow R(-\underline{\mathbf{r}}) &\equiv R(\underline{\mathbf{r}}) \\ e^{im\phi} \rightarrow e^{im(\pi+\phi)} &= (-1)^m e^{im(\phi)} \\ P_l^m(\cos\theta) \rightarrow P_l^m(\cos(\pi-\theta)) &= (-1)^{l+m} P_l^m(\cos\theta) \\ \Rightarrow \hat{P} Y_l^{m=0}(\theta,\phi) &= (-1)^l Y_l^{m=0}(\theta,\phi) \end{aligned}$$

s,d,g,..... atomic states have $P = +1$ EVEN parity
p,f,h,..... atomic states have $P = -1$ ODD parity

2.9.1 Parity Conservation

- Parity is a conserved quantity in strong and electromagnetic interactions (not in weak)
- The conserved quantity is: Total parity= intrinsic x orbital
- Dirac equation predicts fermions/anti-fermions have opposite intrinsic parity. Bosons/anti-bosons have the same intrinsic parity.
- It is a *multiplicative* quantum number i.e. for a composite system, $\psi = \phi_a\phi_b,\ P_\psi = P_{\phi_a}\cdot P_{\phi_b}\cdot(-1)^l$

  Example: Mesons ($q\bar{q}$)
  Intrinsic parity, $P(q) = -P(\bar{q})$ and the relative orbital angular momentum of the ($q\bar{q}$) system contributes $(-1)^l$ (spherical harmonic functions) $\Rightarrow P(\text{meson}) = (+1)(-1)(-1)^l = (-1)^{l+1}$

2.9.2 The Weak Interaction and Parity

- Under $\hat{P}$ transformation, $\underline{\mathbf{p}} \rightarrow -\underline{\mathbf{p}}$ (vector) and particle spin vector $\underline{\sigma} \rightarrow \underline{\sigma}$ (axial vector).
- Therefore helicity, $\underline{H} = \frac{\sigma\cdot p}{|\underline{\sigma}|E}$ flips sign i.e. $L.H. \rightarrow R.H.$
- Since only L.H. particles or R.H. anti-particles take part in the weak interaction $\Rightarrow$ parity is maximally violated

2.9.3 Summary: Parity

- $P(\mathrm{fermion}) = -1P(\mathrm{anti-fermion}),\ P(\mathrm{boson}) = P(\mathrm{anti-boson})$
- $P(\mathrm{spherical\,harmonic\,function}) = (-1)^l$
- $P(\mathrm{meson}) = (-1)^{l+1}$
- Conserved by strong and electromagnetic interactions but is maximally violated by the weak interaction

2.10 Charge Conjugation ($\hat{C}$)

- $\hat{C}$ transforms **Particles** $\rightleftharpoons$ **Anti-Particles**
- Signs of all associated (internal)quantum numbers flip: electric charge, baryon number, lepton number, strangeness, magnetic moment etc (but does not affect mass, energy, momentum or spin)

2.10.1 Charge-Conjugation of the π^0

For charged pions (and up to an arbitrary constant phase factor): $\hat{C}\pi^+ \to \pi^- \neq \pi^+$. Charged pions π^+, π^- are therefore not eigenstates $\hat{C}$. This is not the case however for the π^0, which transforms into itself (is its own anti-particle) under $\hat{C}$ i.e.

$\hat{C}\pi^0 \to \eta\pi^0$ where η is some constant phase factor which can always appear in quantum mechanics but does not change the physics.

Apply the $\hat{C}$ operator twice and you end up with the state you started with,

$$\Rightarrow \eta^2 = 1 \quad \Rightarrow \quad \hat{C}\pi^0 \to \pm\pi^0 \tag{2.32}$$

This tells us the magnitude of the eigenvalue, but what sign is it?

- Emag. fields (photons) are produced by currents which flip sign Under $\hat{C}$ $\Rightarrow \gamma$ must have $C = -1$
- C quantum number is multiplicative (as for parity) $\Rightarrow$ system of n photons has $C = (-1)^n$
- Since $\pi^0 \to 2\gamma,\ \ C(\pi^0) = +1(\mathrm{EVEN})$
- An immediate consequence of this is that the decay $\pi^0 \to 3\gamma$ should be forbidden $(C = (-1)^3 = -1)$. Indeed it is found experimentally that ,

$$\frac{\pi^0 \to 3\gamma}{\pi^0 \to 2\gamma} \ < \ 4 \times 10^{-7}$$

  which provides good evidence for the fact that the electromagnetic interaction conserves $\hat{C}$.

2.10.2 Charge-Conjugation of multiple pion states

A two-pion state of definite angular momentum L, picks up the same $(-1)^L$ factor as for parity under the charge conjugation operation:

$$\hat{C}|\pi^+\pi^-; L> \;=\; (-1)^L|\pi^+\pi^-; L>$$

This follows since $\hat{C}$ interchanges the $\pi^+\pi^-$ so reversing their relative position vector in the spatial wavefunction $\Rightarrow$ same effect as the parity operation.

2.11 $\hat{C}\hat{P}$ Invariance of the Weak Interactions

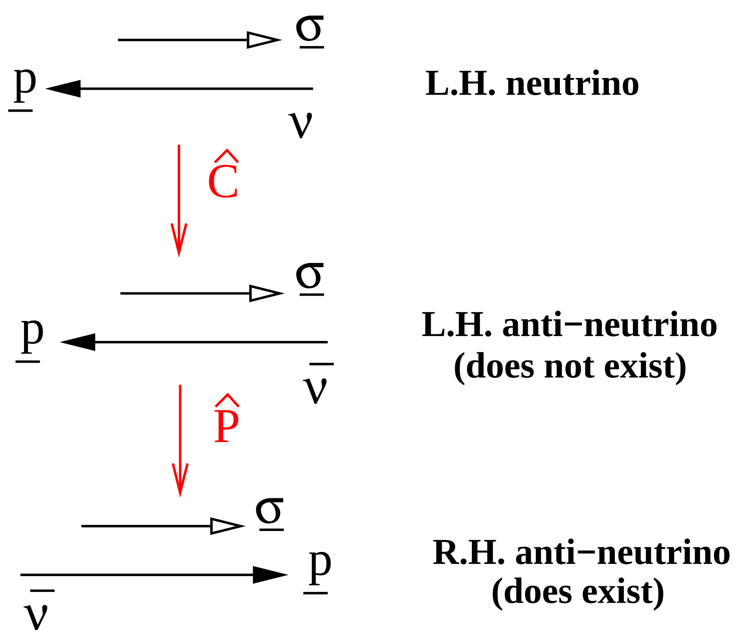

Figure 2.3:

- Parity and charge conjugation are not individually conserved in weak interations (in fact they are maximally violated)
- But the operation, $\hat{C}\hat{P}$, transforms a L.H. particle into a R.H. anti-particle (see Fig. 2.3), both of which are states that weakly interact
- We would therefore expect the weak interaction to be invariant w.r.t. $\hat{C}\hat{P}$ and this is very nearly true. Measurements of K^0 mesons and more recently, of B^0 mesons have seen that the weak interaction even violates the $\hat{C}\hat{P}$ conservation at a low level.

2.12 $\hat{C}\hat{P}\hat{T}$ Invariance

- CPT Theorem states that all interactions are invariant after the application of the three transformations $\hat{C}, \hat{P}, \hat{T}$ (in any order).

- Follows from demanding Lorentz invariance and consequences include:
  - ⇒ Fermions obey Fermi-Dirac stats. and bosons follow Bose-Einstein
  - ⇒ Particles and anti-particles have identicle masses and lifetimes (experimentally, $\frac{m(K^0)-m(\bar{K}^0)}{m(K^0)} < 10^{-14}$!!)
  - ⇒ All internal quantum numbers of particles/anti-particles are reversed

2.13 Summary of Conserved Quantities

Table 2.1:

Conserved quantity	Strong	Electromagnetic	Weak
Energy/momentum	Yes	Yes	Yes
Electric charge	Yes	Yes	Yes
Baryon number	Yes	Yes	Yes
Lepton number	Yes	Yes	Yes
Quark flavour (U,D,S,C,T,B)	Yes	Yes	No
P(Parity)	Yes	Yes	No
C(Charge conjugation)	Yes	Yes	No
CP	Yes	Yes	Yes*
CPT	Yes	Yes	Yes

$*$ small violations now seen in K^0 and B^0 decay

2.14 Tools: What You Should Know

- How to use natural units and 4-vector notation to make simple relativistic kinematical calculations.
- Energy thresholds, COM energy for new particle creation and Fixed target vs collider mode.
- Klein-Gordan equation(spinless bosons) and the Dirac equation (spin $\hbar/2$ fermions).
- Concepts of helicity, parity, charge congugation and be able to determine the overall P and C of some simple particle states.
- Know how the interaction theories transfom under $\hat{C}$,$\hat{P}$,$\hat{C}\hat{P}$ and $\hat{C}\hat{P}\hat{T}$.

2.15 Further Reading

Griffiths Chap. 3 for 4-vectors and relativistic kinematics