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KY FAN INEQUALITIES

MOHAMMAD SAL MOSLEHIAN

Dedicated to the memory of Professor Ky Fan

Abstract. There are several inequalities in the literature carrying the name of Ky Fan. We survey these well-known Ky Fan inequalities and some other significant inequalities generalized by Ky Fan and review some of their recent developments.

1. Introduction

According to MathSciNet, Professor Ky Fan (1914-2010) published 126 papers and books and coauthored with 15 mathematicians. His earliest indexed item goes back to 1940 [19]. It is notable that contributions of Ky Fan to mathematics have provided a lot of influence in the development of nonlinear analysis, convex analysis, approximation theory, operator theory, linear algebra, mathematical programming and mathematical economics; see e.g. [55]. In the literature, there are several inequalities due to Ky Fan in various fields; cf. [12]. In this article we try to briefly survey most important ones by using some information in MathSciNet and Zentralblatt MATH for some unavailable old papers.

2. Preliminaries on matrix analysis

An n-tuple $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ with $x_1 \geq \cdots \geq x_n$ is said to be weakly majorized by an n-tuple $y = (y_1, \ldots, y_n)$ of real numbers with $y_1 \geq \cdots \geq y_n$, denoted by $x \prec_w y$, if $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$ for $k = 1, \ldots, n$. If, in addition,

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$\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, then x is said to be majorized by y and it is denoted by $x \prec y$. Let $\mathrm{Re}(x)$ stand for $(\mathrm{Re}(x_1), \ldots, \mathrm{Re}(x_n))$ for an $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$.

Let $\mathbb{M}_n$ be the matrix algebra of all $n \times n$ matrices with entries in the complex field identified with the algebra $\mathbb{B}(\mathbb{C}^n)$ of all linear operators on the Hilbert space $(\mathbb{C}^n, \langle\cdot,\cdot\rangle)$. We denote by $\mathbb{H}_n$ the set of all Hermitian matrices in $\mathbb{M}_n$. By I_n (or I if there is no ambiguity) we denote the identity matrix of $\mathbb{M}_n$. A matrix $A \in \mathbb{H}_n$ is called positive-semidefinite if $\langle Ax, x\rangle \geq 0$ holds for every $x \in \mathbb{C}^n$ and then we write $A \geq 0$. In particular, if A is invertible, then it is called positive-definite and we write $A > 0$. The absolute value of A is defined by $|A| = (A^*A)^{1/2}$, where A^* denotes the conjugate transpose of A. For $A, B \in \mathbb{H}_n$, we say $A \leq B$ if $B - A \geq 0$.

For a matrix $A \in \mathbb{H}_n$, we denote by $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ the eigenvalues of A arranged in the decreasing order with their multiplicities counted. The notation $\lambda(A)$ stands for the row vector $(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$. The eigenvalue inequality $\lambda(A) \leq \lambda(B)$ means $\lambda_j(A) \leq \lambda_j(B)$ $(j = 1, 2, \ldots, n)$. The trace and determinant of a matrix $A \in \mathbb{M}_n$ are defined by $\mathrm{tr}(A) = \sum_{i=1}^n \lambda_i(A)$ and $\det(A) = \prod_{i=1}^n \lambda_i(A)$, respectively. A norm $|||\cdot|||$ on $\mathbb{M}_n$ is said to be unitarily invariant if $|||UAV||| = |||A|||$ for all $A \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$. The Ky Fan norms, defined as $\|A\|_{(k)} = \sum_{j=1}^k s_j(A)$ for $k = 1, 2, \ldots, n$, provide a significant family of unitarily invariant norms; see also [11, 56]. Here $s(A) = (s_1(A), \ldots, s_n(A))$ denotes the n-tuple of the singular values of A, i.e. the eigenvalues of $|A|$, arranged in the decreasing order.

3. Ky Fan matrix inequalities

The contributions of Ky Fan to matrix theory and operator theory are substantial. In this section we study matrix inequalities due to Ky Fan. The interested reader is referred to [56, 11, 50] to see details of proofs.

Ky Fan [20, 1949] proved that if $H, K \in \mathbb{H}_n$, then

$$\lambda(H + K) \prec \lambda(H) + \lambda(K) \qquad \text{(Ky Fan eigenvalue inequality)},$$

see also [69, Theorme 7.14]. An extension of this result was given by Aujla and Silva [8] as $\lambda(f(\alpha A+(1-\alpha)B)) \prec_w \lambda(\alpha f(A)+(1-\alpha)f(B))$, where f is a convex function on an interval J and $A, B \in \mathbb{H}_n$ with spectra in J. In addition, Ky Fan [20, 1949] showed that if $A \in \mathbb{M}_n$, t is a positive integer and the eigenvalues $\kappa_i^{(t)}$ of $(A^t)^*A^t$ are arranged in the decreasing order $\kappa_1^{(t)} \geq \cdots \geq \kappa_n^{(t)}$, then

$$\kappa_1^{(t)} + \cdots + \kappa_q^{(t)} \leq \kappa_1^t + \cdots + \kappa_q^t \quad (1 \leq q \leq n)\,,$$

where $k_1, \ldots, k_n$ are eigenvalues of A^*A. Ky Fan [21, 1950] established two results which are powerful tools in obtaining matrix inequalities. They assert that if $H \in \mathbb{H}_n$, then

$$\max_{UU^*=I_k} \operatorname{tr}(UHU^*) = \sum_{i=1}^{k} \lambda_i(H), \quad \min_{UU^*=I_k} \operatorname{tr}(UHU^*) = \sum_{i=1}^{k} \lambda_{n-i+1}(H) \quad (1 \leq k \leq n)$$

where maximum and minimum are taken over all $k \times n$ complex matrices U satisfying $UU^* = I_k$.

The Ky Fan dominance theorem [22, 1951] states that for $A, B \in \mathbb{M}_n$, the inequalities $\|A\|_{(k)} \leq \|B\|_{(k)}$ $(1 \leq k \leq n)$ hold if and only if $|||A||| \leq |||B|||$ for all unitarily invariant norms $|||\cdot|||$; see [57] for an application of the Ky Fan dominance theorem.

Ky Fan [22, 1951] extended an inequality of Weyl [67] (the case $m = 1$) and von Neumann [65] (the case $m = 2$): Let $A_1, \ldots, A_m \in \mathbb{M}_n$ and let $s_{j1} \geq s_{j2} \geq \cdots \geq s_{jn}$ be the singular values of A_j. Then

$$\max \left| \sum_{i=1}^{n} \langle U_1 A_1 \cdots U_m A_m x_i, x_i \rangle \right| \leq \sum_{i=1}^{n} (s_{1i} \cdots s_{mi})$$

and

$$\max \left| \det_{1 \leq i,k \leq n} \left[\langle U_1 A_1 \cdots U_m A_m x_i, x_k \rangle \right] \right| \leq \prod_{i=1}^{n} (s_{1i} \cdots s_{mi})\,,$$

where $U_1, \ldots, U_m$ run over all unitary matrices and $\{x_1, \ldots, x_n\}$ runs over all orthonormal sets in $\mathbb{C}^n$.

Furthermore, Ky Fan showed that for any matrices $A, B \in \mathbb{M}_n$,

$$s(A+B) \prec_w s(A) + s(B) \qquad \text{(Ky Fan singular value inequality)},$$

as well as

$$s_{r+t+1}(A+B) \leq s_{r+1}(A) + s_{t+1}(B),$$

where $t \geq 0, r \geq 0, r+t+1 \leq n$.

In the same paper [22, 1951], it was compared by Ky Fan that the real parts and modules of the diagonal elements $a_1, \ldots, a_n$ of an arbitrary matrix $A \in \mathbb{M}_n$ by establishing that

$$\mathrm{Re}(a_1, \ldots, a_n) \prec_w (|a_1|, \ldots, |a_n|) \prec_w s(A)\,.$$

Due to the fact that any matrix is unitarily equivalent to an upper triangular matrix, one gets another Ky Fan inequality [21, 1950] for eigenvalues as

$$\mathrm{Re}(\lambda(A)) \prec \lambda(\mathrm{Re}(A)),$$

where $\mathrm{Re}(A) = (A+A^*)/2$; cf. [68, Lemma 4.20]. A similar result for the eigenvalues of $\mathrm{Im}(A) = (A-A^*)/(2\mathrm{i})$ is presented by Amir-Moez and Horn [6]. The authors of [42, 1955] proved that

$$|\lambda_i(\mathrm{Re}(A))| \leq s_i(A) \qquad (i = 1, \ldots, n)$$

for any matrix $A \in \mathbb{M}_n$.

Several inequalities for eigenvalues and singular values of two Hermitian matrices H, K and $H + \mathrm{i}K$ is surveyed by Cheng, Horn and Li [14]. There are still other inequalities due to Ky Fan.

Ky Fan [24, 1954] compared the characteristic roots of diagonal blocks of a Hermitian matrix and those of the matrix itself. Ky Fan proved that if $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$ is an $n \times n$ Hermitian block matrix and $C(H) = \begin{pmatrix} H_{11} & 0 \\ 0 & H_{22} \end{pmatrix}$ is its pinching, then $\lambda(C(H)) \prec \lambda(H)$. This is true for any pinching of H; see [11, page 50]. In the next year, Ky Fan [25, 1955] proved that for H positive-definite

$$\prod_{i=1}^{m} \lambda_{n-i+1}(H) \leq \prod_{i=1}^{m} \lambda_{k-i+1}(H_{11}) \ (1 \leq m \leq k),$$

where the matrix H_{11} and H_{22} are $k \times k$ and $m \times m$ matrices, respectively.

Ky Fan collaborated with Hoffman [42, 1955] and proved the following: Let $|||\cdot|||$ be a unitarily invariant norm on $\mathbb{M}_n$. Then

(i) If $A = U|A|$ is the polar decomposition of A, where U is unitary, and V is any unitary matrix, then

$$|||A - U||| \leq |||A - V||| \leq |||A + U|||\,.$$

(ii) Let $H \in \mathbb{H}_n$. It follows from the identity $A - \mathrm{Re}(A) = \frac{A-H}{2\mathrm{i}} - \frac{(A-H)^*}{2\mathrm{i}}$ that

$$|||A - \mathrm{Re}(A)||| \leq |||A - H|||\,.$$

(iii) If $H, K \in \mathbb{H}_n$ and $U = (H - \mathrm{i}I)(H + \mathrm{i}I)^{-1}$, $V = (K - \mathrm{i}I)(K + \mathrm{i}I)^{-1}$ are their Cayley transforms, then

$$|||U - V||| \leq 2|||H - K|||\,.$$

Ky Fan also worked on systems of inequalities. For instance, Ky Fan [10, 1963] gave some necessary and sufficient conditions for the existence of Hermitian matrices X_j satisfying the system of linear inequalities

$$\sum_{j=1}^{n}(A_{ij}X_j + X_jA_{ij}{}^*) \geq B_i \ (1 \leq i \leq m) \quad \text{and} \quad \mathrm{tr}\left(\sum_{j=1}^{n} C_jX_j\right) \geq c\,,$$

where $A_{ij} \in \mathbb{M}_n$ and $B_i, C_j \in \mathbb{H}_n$ and c is a real number.

4. Results of Ky Fan involving integrals

For real functions f and g on $[0,1]$, we say that $f \prec g$ whenever $\int_0^x f(t)dt \leq \int_0^x g(t)dt$ for all $0 \leq x \leq 1$ and $\int_0^1 f(t)dt = \int_0^1 g(t)dt$. In [43, 1954], Ky Fan and Lorentz gave sufficient and necessary conditions in terms of mixed second differences of a suitable differentiable function Φ in order that

$$\int_0^1 \Phi(t, f_1(t), \ldots, f_n(t))dt \leq \int_0^1 \Phi(t, g_1(t), \ldots, g_n(t))dt$$

for any bounded functions f_i, g_i with $f_i \prec g_i$.

5. Ky Fan–Taussky–Todd inequality

The Wirtinger inequality reads as follows: If f is a periodic function of the period 2π with $f, f' \in L^2([0,2\pi])$ and $\int_0^{2\pi} f = 0$, then

$$\int_0^{2\pi} f^2 \leq \int_0^{2\pi} f'^2$$

with equality if and only if $f(x) = a\cos x + b\sin x$ for some constants a, b. Ky Fan, Taussky and Todd [44, 1955] proved discrete analogs of Wirtinger type. They showed that if $a_1, \ldots, a_n$ are real numbers and $a_0 = a_{n+1} = 0$, then

$$2\left(1 - \cos\frac{\pi}{n+1}\right)\sum_{k=1}^{n} a_k^2 \leq \sum_{k=1}^{n+1}(a_k - a_{k-1})^2$$

with equality if and only if $a_k = c\sin(k\pi/(n+1))$ $(k = 1, \ldots, n)$ for some real constant c.

They also proved that if $a_1, \ldots, a_n$ are real numbers and $a_0 = 0$, then

$$2\left(1 - \cos\frac{\pi}{2n+1}\right)\sum_{k=1}^{n} a_k^2 \leq \sum_{k=1}^{n}(a_k - a_{k-1})^2$$

with equality if and only if $a_k = c\sin(k\pi/(2n+1))$ $(k = 1, \ldots, n)$, where c is a real constant. Both constants $2\left(1 - \cos\frac{\pi}{n+1}\right)$ and $2\left(1 - \cos\frac{\pi}{2n+1}\right)$ are best possible. The converses of these two inequalities are given by Alzer [1]. Another related inequality is that of Ozeki; see [57] for its operator versions.

6. Ky Fan–Todd determinantal inequality

Ky Fan and Todd [45, 1955] showed that if $\begin{pmatrix} a_1 & \ldots & a_n \\ b_1 & \ldots & b_n \end{pmatrix}$ is a matrix none of whose two-column minor is singular and (p_{ij}) is a symmetric matrix such that $p = \sum_{1\leq i<j\leq n} p_{ij} \neq 0$, then

$$\left[\sum_{i=1}^{n} a_i^2\right]\left[\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - (\sum_{i=1}^{n} a_i b_i)^2\right]^{-1} \leq p^{-2}\sum_{i=1}^{n}\left[\sum_{j=1, j\neq i}^{n} p_{ij}a_j(a_j b_i - a_i b_j)^{-1}\right]^2.$$

The special case where $p_{ij} = 1$, $a_i = \cos\theta_i$ and $b_i = \sin\theta_i$ for some θ_i and all i, j was obtained by Chassan [13] by some statistical arguments. Also a refinement of it was given by Chong [15].

It is worthy noting that there are a variety of other determinantal inequalities due to Ky Fan. Ky Fan [23, 1953] (see [58, p. 214]) proved that if $|C|_k$ denotes the product of the first k-th smallest eigenvalues of a real positive-definite matrix C and if $A, B \in \mathbb{M}_n$ are real positive-definite matrices and $0 \leq \lambda \leq 1$, then

$$|\lambda A + (1-\lambda)B|_k \geq |A|_k^{\lambda}|B|_k^{1-\lambda}.$$

Also Ky Fan [26, 1955] (see [56, p. 687]) proved that if $H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$ and $K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$ are positive-definite $n \times n$ matrices and H_{11} and K_{11} are $k \times k$ matrices, then

$$\left(\frac{\det(H+K)}{\det(H_{11}+K_{11})}\right)^{\frac{1}{n-k}} \geq \left(\frac{\det(H)}{\det(H_{11})}\right)^{\frac{1}{n-k}} + \left(\frac{\det(K)}{\det(K_{11})}\right)^{\frac{1}{n-k}}.$$

In fact, Ky Fan studied many convex/increasing functions on certain subsets of Hermitian matrices, e.g. $f_k(H) = \sum_{i=1}^{k} \lambda_i(H)$ $(H \in \mathbb{H}_n)$, from which one can get some inequalities such as the latter one.

In the same paper [26, 1955], Ky Fan proved that if $(i_1, i_2, \ldots, i_k)$ denotes the principal submatrix of a positive-definite matrix $H \in \mathbb{H}_n$ with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$, then

$$\lambda_1\lambda_2 \ldots \lambda_{h+k} \leq \det(p+1, p+2, \ldots, p+k) \prod_{i=1}^{h} \frac{\det(i, p+1, p+2, \ldots, n)}{\det(p+1, p+2, \ldots, n)}$$

in which $1 \leq h \leq p < n$ and $0 \leq k \leq n-p$ and for $k = 0$ we assume that $\det(p+1, \ldots, p+0) = 1$.

If $A \in \mathbb{M}_n$ is a real matrix such that $A + A^* > 0$, then Ky Fan [33, 1973] showed that

$$\det(\mathrm{Re}(A)) \leq \det(A)\frac{1+\mathrm{Re}(\lambda_i(A^{-1}A^*))}{2}$$

for all i. If $n \geq 4$ and $A - A^*$ is invertible, then

$$0 < \det\left(\frac{A-A^*}{2}\right) < \det(A)\frac{1-\mathrm{Re}(\lambda_i(A^{-1}A^*))}{2}$$

for all i.

Ky Fan [37, 1983] used the Jensen inequality to prove that if $A = H + iK$ with H and K Hermitian is normalizable in the sense that there is an invertible matrix T such that T^*AT is normal, and if $p \geq 2/n$, where n is the dimension of the Hilbert space, then $|\det(A)|^p \geq |\det(H)|^p + |\det(K)|^p$. Ky Fan showed that equality holds for $p > 2/n$ if and only if either $\det(A) = 0$ or $H = 0$ (or $k = 0$). Ky Fan [34, 1974] already proved the same result for a strictly dissipative matrix, that is, a matrix whose imaginary part $\mathrm{Im}(A) = (A - A^*)/2\mathrm{i}$ is positive-definite.

7. Ky Fan mean inequality

Suppose that $x_1, \ldots, x_n \in (0, \frac{1}{2})$, $\mathcal{A}_n := (1/n)\sum_{i=1}^n x_i, \mathcal{G}_n := (\prod_{i=1}^n x_i)^{1/n}, \mathcal{H}_n := n/\sum_{i=1}^n (1/x_i)$ and $\mathcal{A}'_n, \mathcal{G}'_n, \mathcal{H}'_n$ denote the unweighted arithmetic, geometric and harmonic means of the numbers x_i and $1-x_i$, respectively. The Ky Fan inequality for means states that

$$\mathcal{G}_n/\mathcal{G}'_n \leq \mathcal{A}_n/\mathcal{A}'_n \qquad \text{(Ky Fan mean inequality)}$$

with equality only if $x_1 = \cdots = x_n$. It first appeared in the book "Inequalities" by Beckenbach and Bellman [9]. It is proved by a forward and backward induction as used for establishing the arithmetic-geometric mean inequality. It can be also proved by applying Henrici inequality $\sum_{i=1}^n 1/(1+b_i) \geq n/(1+(\prod_{i=1}^n b_i)^{1/n})$ $(b_i \geq 1)$ to $b_i = 1/x_i - 1$ $(i = 1, \ldots, n)$. Utilizing the monotonicity of $(1 - x^n)/n$ as a function of n, Rooin [63] established some Ky Fan type inequalities. Applying the convexity of the function $f(x) = (e^x - \alpha)/(1 + e^x)$ $(\alpha > 0)$ and the Jensen inequality, the authors of [17] generalized the Ky Fan inequality. Of course, Ky Fan [22, 1951] proved that the function

$$f(x) = \frac{\left(\prod_{i=1}^n (1 - x_i)\right)^{1/n}}{\sum_{i=1}^n (1 - x_i)} \cdot \frac{\sum_{i=1}^n x_i}{\left(\prod_{i=1}^n x_i\right)^{1/n}}$$

is symmetric and Schur-convex.

Many extensions, refinements as well as some counterparts have been given by many authors. Levinson [54] extended the inequality by showing that if $\varphi(u)$ has third derivative for $0 < u < 2b$, with $\varphi''(u) \geq 0$, $0 < x_i \leq b$ and $0 < p_i$,

$i = 1, 2, \ldots, n$, then

$$\frac{\sum_{i=1}^n p_i\varphi(x_i)}{\sum_{i=1}^n p_i} - \varphi\left(\frac{\sum_{i=1}^n p_i x_i}{\sum_{i=1}^n p_i}\right) \leq \frac{\sum_{i=1}^n p_i\varphi(2b - x_i)}{\sum_{i=1}^n p_i} - \varphi\left(\frac{\sum_{i=1}^n p_i(2b - x_i)}{\sum_{i=1}^n p_i}\right).$$

The Ky Fan inequality is the special case when $\varphi(u) = \log u$, $p_i = 1$ and $b = 1/2$; see also [18]. It is noticed in [52] that Ky Fan's inequality can be deduced from an inequality proved by L. Lewent in 1908. The interested reader may be referred to the interesting survey [2] for some of generalizations and refinements of the Ky Fan inequality until 1995.

In [3], the author presented the following additive version of the Ky Fan inequality:

$$\min_{1\leq i\leq n} \frac{x_i}{1 - x_i} < \frac{\mathcal{A}_n' - \mathcal{G}_n'}{\mathcal{A}_n - \mathcal{G}_n} < \max_{1\leq i\leq n} \frac{x_i}{1 - x_i}.$$

Alzer, Ruscheweyh and Salinas [5] provided the following refinement of the Ky Fan inequality:

$$\left(\frac{\mathcal{H}_n}{\mathcal{H}_n'}\right)^{n-1} \cdot \frac{\mathcal{A}_n}{\mathcal{A}_n'} \leq \left(\frac{\mathcal{G}_n}{\mathcal{G}_n'}\right)^n.$$

Some refinements of the Ky Fan inequality including means of two or more variables are obtained by Neuman and Sandor [61]. A unified approach to Sierpiński and Ky Fan inequalities by utilizing a refined Cauchy inequality and convexity of suitable functions was presented by Alzer, Ando and Nakamura [4].

8. Ky Fan inequalities involving M-matrices

A real matrix $A = (a_{ij}) \in \mathbb{M}_n$ is an M-matrix if it can be written as $A = rI - A_1$, where r is greater than the spectral radius of A_1 and A_1 has nonnegative entries. This notion was introduced by Ostrowski [62]. Ky Fan [28, 1964] gave several equivalent definitions for the notion of M-matrix. One of them is that A is an M-matrix if and only if A is nonsingular and A^{-1} has nonnegative entries. A complex matrix $B = (b_{ij})$ dominates A if $a_{ii} \leq |b_{ii}|$ for all i and $|b_{ij}| \leq |a_{ij}|$ for all $i \neq j$. If B is also an M-matrix, then we say that B proportionally dominates A whenever there exist $p_i > 0$ such that $p_i a_{ij} \leq b_{ij}$ and $a_{ij}p_j \leq b_{ij}$ for all i, j. If $\alpha \subseteq \{1, 2, \ldots, n\}$, let $A(\alpha)$ denote the determinant of the principal submatrix formed by the rows and columns with indices contained in α, and we assume that $A(\emptyset) = 1$.

An inequality of Ostrowski [62] states that $\det(A) \le |\det(B)|$ when B dominates A, in particular $\prod_{i=1}^n \lambda_i(A) \le \prod_{i=1}^n h_{ii}$, when A is an M-matrix. Ky Fan [29, 1966] extended the former inequality by showing that under conditions above on the entries of A and B if $\alpha_1, \dots, \alpha_m$ are subsets of $\{1, 2, \dots, n\}$ such that each of the indices $1, 2, \dots, n$ is contained in at most k of the sets α_i, then

$$\det(A)^k / \prod_{i=1}^m A(\alpha_i) \le |\det(B)^k / \prod_{i=1}^m B(\alpha_i)|.$$

In [30, 1967] Ky Fan proved that if A and B are M-matrices of order n and a matrix $D \in \mathbb{M}_n$ dominates $A + B$ and B proportionally dominates A, then $|\det(D)|^{1/n} \ge \det(A)^{1/n} + \det(B)^{1/n}$ and

$$\begin{aligned} &|\det(D)/D(\{k+1, \dots, n\})|^{1/k} \\ &\qquad \ge [\det(A)/A(\{k+1, \dots, n\})]^{1/k} + [\det(B)/B(\{k+1, \dots, n\})]^{1/k} \end{aligned}$$

for all $1 \le k \le n-1$.

Also, Ky Fan [27, 1960] showed that if $A, B \in \mathbb{M}_n$ are M-matrices such that $a_{ij} \le b_{ij}$ for all i, j, and

$$\Phi(A; \alpha, \beta, \gamma) := \frac{A(\alpha \cap \beta) A(\alpha \cap \gamma) A(\beta \cap \gamma) A(\alpha \cup \beta \cup \gamma)}{A(\alpha) A(\beta) A(\gamma) A(\alpha \cap \beta \cap \gamma)}\,,$$

then $\Phi(A; \alpha, \beta, \gamma) \le \Phi(A; \alpha, \beta, \gamma)$ for any subsets α, β, γ of $\{1, 2, \dots, n\}$.

Paper [28, 1964] deals with inequalities for principal subdeterminants of a special product of M-matrices. In this paper Ky Fan introduced the so-called Ky Fan product $A \odot B = \{ {}^{a_{ii} b_{ii},\ i=j}_{-a_{ij} b_{ij},\ i \ne j}$ and established that for two M-matrices A, B,

$$1 - \Phi(A \odot B; \alpha, \beta, \gamma) \le (1 - \Phi(A; \alpha, \beta, \gamma))(1 - \Phi(B; \alpha, \beta, \gamma))\,.$$

For recent studies on some Ky Fan's results about M-matrices and the Ky Fan product see [66]. The importance of the Ky Fan product is that the set of M-matrices is closed under this product.

9. Ky Fan generalization of Szász's inequality

Suppose that $A \in \mathbb{M}_n$ is a positive-definite matrix and P_k $(1 \leq k)$ is the product of all its $k \times k$ principal minors. The Szász inequality [64], as a generalization of Hadamard's determinantal inequality, says that $\prod_{i=1}^n a_{ii} = P_1 \geq \cdots \geq P_k^{1/\binom{n-1}{k-1}} \geq \cdots \geq P_n = \det(A)$. Ky Fan [31, 1967] strengthened the inequality for a certain class of matrices, named as GKK, including positive-definite matrices, totally positive matrices and M-matrices. By a GKK-matrix we mean one that all its principal minors are positive and the product of any two symmetrically situated almost principal minors is real and nonnegative. His result reads as follows.

Let $A \in \mathbb{M}_n$ be GKK and let $R_0 = I$, $R_k = Q_k^{1/\binom{p}{k}}$, $(1 \leq k \leq k)$, where $Q_k = \prod_{i_1 < \cdots < i_k} A(\alpha_{i_1} \cup \cdots \cup \alpha_{i_k})$, in which α_i $(1 \leq k \leq p)$ are $p(\geq 2)$ pairwise disjoint possibly empty subsets of $\{1, \ldots, n\}$. Then $R_k^2 \geq R_{k-1}R_{k+1}$ and $Q_k^{1/\binom{p-1}{k-1}} \geq Q_{k+1}^{1/\binom{p-1}{k}}$ $(1 \leq k \leq p-1)$, which gives rise to the Szász inequality for $p = n$ and $\alpha_i = \{i\}$.

Ky Fan [41, 1992] generalized its former result by removing the assumption of the pairwise disjointness of the α_i, and showing that it remains true if $A - I$ is either positive-definite or an M-matrix.

10. Ky Fan extension of Kantorovich inequality

The Kantorovich inequality asserts that

$$\langle Hx, x\rangle\langle H^{-1}x, x\rangle \leq (\lambda_1(H) + \lambda_n(H))^2/4\lambda_1(H)\lambda_n(H)$$

where x is a unit vector in $\mathbf{C}^n$ and H is an $n \times n$ positive-definite matrix. Ky Fan [29, 1966] generalized the inequality above by showing that if $0 < mI \leq H \leq MI$, $x_1, \ldots, x_m$ are vectors in $\mathbb{C}^n$ such that $\sum_{i=1}^n \|x_i\|^2 = 1$, then

$$\sum_{j=1}^m \langle H^p x_j, x_j\rangle \left[\sum_{j=1}^m \langle H x_j, x_j\rangle\right]^{-p} \leq (p-1)^{p-1}p^{-p}\left(b^p - a^p)^p((b-a)(ab^p - ba^p)^{p-1}\right)^{-1},$$

where p is any integer different from 0 and 1. In 1997 Mond and Pečarić [59] gave an operator version of Ky Fan's inequality above. Another extension of

Kantorovich inequality was given by Furuta [47]. Ky Fan proved that if H, K are positive operators on a Hilbert space, $H \geq K > 0$ and $MI \geq K \geq mI > 0$, then

$$\left(\frac{M}{m}\right)^{p-1} H^p \geq \frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}} H^p \geq K^p$$

holds for all $p \geq 1$. The constant $\kappa_+(m, M, p) = \frac{(p-1)^{p-1}}{p^p} \frac{(M^p-m^p)^p}{(M-m)(mM^p-Mm^p)^{p-1}}$ is called the Ky Fan–Furuta constant in the literature; cf. [48].

11. Ky Fan minimax inequality

In his seminal work [32, 1972], Ky Fan presented the following result:

Let S be a nonempty compact convex subset of a Hausdorff topological vector space X, and let $f(x, y)$ be a function from $S \times S$ to the real numbers that is a lower semicontinuous function of y for each fixed $x \in S$ and is a quasi-concave function of x for each fixed $y \in S$. Then the following minimax inequality holds:

$$\min_{y \in S} \max_{x \in S} f(x, y) \leq \sup_{x \in S} f(x, x) \qquad \text{(Ky Fan minimax inequality)}.$$

It is equivalent to the Brouwer–Kakutani fixed point theorem and plays an essential role in several fields such as game theory, mathematical economics, variational inequalities, fixed point theory, control theory as well as nonlinear and convex analysis.

This inequality has evoked the interest of many mathematicians: Ferro [46] got a minimax inequality for vector-valued functions. Ding [16] obtained a minimax inequality by a generalized relatively Knaster–Kuratowski–Mazurkiewicz (R-KKM) mapping. Kuwano, Tanaka and Yamada [53] obtained a minimax inequality by using a unified scalarization. Also Hussain, Khan and Agarwal [51] used common fixed point theorem for a condensing map to obtain some Ky Fan type approximation theorems.

12. Sharpening of the von Neumann inequality

The well-known von Neumann inequality asserts that if A is a proper contraction (i.e, $\|A\| < 1$) acting on a Hilbert space, then $\|f(A)\| \leq 1$, where $f(A)$ is defined by applying the Riesz–Dunford functional calculus to a suitable analytic function f. In several papers Ky Fan sharpened von Neumann's inequality by finding a bound of modulus less than 1 for $\|f(A)\|$; see [39, 1987].

(i) Ky Fan [35, 1979] proved the following inequality related to Julia's Lemma (for other results see [7, 1979]):

$$\left[I - \overline{f(z)}f(A)\right][I - f(A)^*f(A)]^{-1}[I - f(z)f(A)^*] \\ \leq \frac{1 - |f(z)|^2}{1 - |z|^2}(I - \bar{z}A)(I - A^*A)^{-1}(I - zA^*)\,.$$

(ii) Ky Fan [38, 1987] established that

$$\|f(A)\| \leq [\|\mu_W(A)\| + |f(w)|]\,[1 + \|\mu_W(A)\|\,|f(w)|]^{-1},$$

in which $|w| < 1$ and μ_W is the Möbius transformation $\mu_W(z) = \frac{z-w}{1-\overline{w}z}$.
For recent develeopments concerning a multivariable von Neumann inequality see [49].

13. Operator Harnack inequalities

Ky Fan [40, 1988] provided two operator Harnack inequalities:
If $F(z)$ is a Hilbert space operator-valued analytic function on the open unit disk such that $F(0) = I$ and $\mathrm{Re}(F(z)) > 0$, then

$$(1 - |z|)/(1 + |z|)I \leq \mathrm{Re}(F(z)) \leq (1 + |z|)/(1 - |z|)I$$

and

$$-2|z|/(1 - |z|^2)I \leq \mathrm{Im}(F(z)) \leq 2|z|/(1 - |z|^2)I.$$

See also similar results of Ky Fan in [36, 1980].

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