Theory of Cones

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ABSTRACT

This survey deals with the aspects of archimedian partially ordered finite-dimensional real vector spaces and order preserving linear maps which do not involve spectral theory. The first section sketches some of the background of entrywise nonnegative matrices and of systems of inequalities which motivate much of the current investigations. The study of inequalities resulted in the definition of a polyhedral cone K and its face lattice $F(K)$. In Section II.A the face lattice of a not necessarily polyhedral cone K in a vector space V is investigated. In particular the interplay between the lattice properties of $\mathcal{F}(K)$ and geometric properties of K is emphasized. Section II.B turns to the cones $\Pi(K)$ in the space of linear maps on V. Recall that $\Pi(K)$ is the cone of all order preserving linear maps. Of particular interest are the algebraic structure of $\Pi(K)$ as a semiring and the nature of the group $\mathrm{Aut}(K)$ of nonsingular elements $A\in\Pi(K)$ for which $A^{-1}\in\Pi(K)$ as well. In a short final section the cone P_n of $n\times n$ positive semidefinite matrices is discussed. A characterization of the set of completely positive linear maps is stated. The proofs will appear in a forthcoming paper.

I. HISTORICAL NOTES

In 1907 Perron [56] initiated the study of matrices $A=(a_{ij})$ all of whose entries are positive, and Frobenius [32, 33] carried this study further. If $\rho=\rho(A)=\max\{|\lambda|:\lambda \text{ is an eigenvalue of } A\}$, then their results showed that ρ is an eigenvalue of A which is a simple root of the characteristic polynomial and that the eigenvector belonging to ρ may be taken to have all entries positive. In 1912 Frobenius [34] showed that these results hold for a broader class of matrices with nonnegative entries. These are the irreducible matrices.

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A matrix A is called *reducible* iff there is a permutation matrix P such that

$$P^TAP=\begin{bmatrix} A_1 & 0 \\ B & A_2 \end{bmatrix},$$

where A_1 and A_2 are square matrices.

The study of nonnegative matrices remained dormant until 1950. Of course, certain special types such as stochastic matrices were investigated during this period, but Wieldandt's paper [75] seems to have attracted attention to this area again.

Kantorovich in 1935 published an often overlooked paper [44] concerning operators on a partially ordered Banach space. Krein and Rutman [45] picked up this thread. At this point we require a definition.

DEFINITION 1.1. Let V be a real topological vector space. A *cone* K in V is a subset of V which satisfies

(i) if $\alpha,\beta\geqslant 0$ and $x,y\in K$, then $\alpha x+\beta y\in K$,
(ii) $K\cap(-K)=\{0\}$.

We shall also require that K be a closed set. The cone K is *full* iff int $K\neq\varnothing$, and K is *reproducing* iff $K-K=V$.

REMARK 1.2. In the case of a finite-dimensional V (the situation with which we shall be principally concerned) a cone K is full iff it is reproducing.

Krein and Rutman considered operators which leave invariant a closed cone in a Banach space. With certain restrictions on the operators, Krein and Rutman obtained extensions of the Perron-Frobenius theorems. Birkhoff [22] gave a proof of the finite-dimensional case of this result. Vandergraft [72] also investigated the problem with an eye to the applications to the convergence of certain iterative processes. These applications have been fruitful, and one can consult Schröder [62], Trottenberg and Winter [70], and Vandergraft [73] for the developments along these lines. The spectral theory of cone preserving operators is now fairly complete and has been summarized in Barker and Schneider [14] and Berman and Plemmons [21, Chapter 1]. Since the subject of spectral theory is well summarized, we shall not consider it in detail in the main body of this paper. However, in order to have a complete extension of the Perron-Frobenius results it is necessary to have for cone preserving maps an analog of irreducibility. This extension is based on certain distinguished subsets of K, namely its faces. The following definition is due to Hans Schneider.

Definition 1.3. Let K be a closed cone in the finite-dimensional real space V. A cone $F\subset K$ is a *face* of K iff

$$x\in K,\ y-x\in K, \text{ and } y\in F \quad \text{imply} \quad x\in F.$$

The collection of all faces of K is denoted by $\mathfrak{F}(K)$ or simply $\mathfrak{F}$.

The trivial faces are $\{0\}$ and K. If $F\in\mathfrak{F}$ we write $F\lhd K$, and if $F\lhd K$, $G\lhd K$, and $F\subset G$, then $F\lhd G$ (cf. [4]).

Definition 1.4. A linear map A which leaves K invariant is *irreducible* iff A leaves invariant no nontrivial face of K.

The study of linear inequalities has led to the study of a special class of cones, the polyhedral cones (cf. Definition 1.5). This study, which dates at least from Farkas's paper of 1902 (see chapters 1 and 2 of Stoer and Witzgall [63] and their bibliography), has been applied to econometrics and mathematical programming. Before discussing this we need some notation.

Let K be a cone in V. If $x\in K$, we write $x\geqslant 0$. Then $x\geqslant y$ means $x-y\geqslant 0$. Also $x>0$ and $x\gg 0$ mean $x\neq 0$, $x\in K$, and $x\in \operatorname{int} K$ respectively. Thus K defines a partial order on V. Let K be a closed full cone, and let V' and Hom (V) be the dual space of V and the space of linear transformation on V respectively.

Set

$$K'=\{f\in V':fx\geqslant 0\ \forall x\in K\},$$

$$\Pi(K)=\{A\in \operatorname{Hom}(V):AK\subset K\}.$$

Then K' and $\Pi(K)$ are closed full cones in V' and Hom (V) respectively when K is a closed full cone. Thus $f\geqslant 0$ means $f\in K'$, and $A\geqslant 0$ means $A\in\Pi(K)$. The dimension of a face F, $\dim F$, is the dimension of the linear span $F-F$ of F. If $S\subset K$, then $\Phi(S)$ will denote the least face of K containing S, viz.,

$$\Phi(S)=\bigcap\{F:S\subset F, F\lhd K\}.$$

If $S=\{x\}$, we write $\Phi(x)$. If $\dim F=1$, F is called an extreme ray of K, and if $x\geqslant 0$ is such that $\dim\Phi(x)=1$, then x is called an extremal. One can show that if x is an extremal and $0\leqslant y\leqslant x$, then there is an $\alpha>0$ such that $\alpha y=x$. Also one can show that K is the convex hull of its extreme rays. (See Barker [2, 3] and Glazman and Ljubič [36]).

Let f be a linear functional on $\mathbb{R}^n$, and consider the homogeneous inequality $f(x) \geqslant 0$. This defines a half space in $\mathbb{R}^n$, and a finite set of inequalities is satisfied by any point in the intersection of the corresponding half spaces. Such an intersection need not be a cone, for it need not be pointed, nor is such an intersection necessarily full. However, we shall be interested only in those which are.

DEFINITION 1.5. A *polyhedral* cone is a cone which has a finite set of extreme rays.

The literature on polyhedral cones is extensive. The interested reader can begin with the following references: Gerstenhaber [35], Goldman and Tucker [37], Stoer and Witzgall [63], and Uzawa [71]. We state without proof two of the basic results for polyhedral cones (Uzawa [71, p. 28]).

THEOREM 1.6 (Representation theorem). *A cone K is polyhedral iff it is the intersection of a finite set of half spaces.*

THEOREM 1.7 (Duality theorem). *For any polyhedral cone K, the dual cone K' is also polyhedral, and $K''=K$.*

These results answer questions which arise naturally in studying linear inequalities. However, when considering cone preserving maps it is reasonable to inquire about the relation of K to $\Pi(K)$. We shall investigate this in some detail in Section II.B, but for the moment let us note one result.

THEOREM 1.8 (Schneider and Vidyasagar [61], Tam [67]). *The cone K is polyhedral iff $\Pi(K)$ is polyhedral.*

Not surprisingly, the theory of polyhedral cones is intimately related to the theory of convex polyhedra. Given a cone K and a hyperplane $H=\{x: fx=1$ for some fixed $f \in V'\}$ which meets the interior of K, the intersection $H \cap K$ is a cross section of K which is a convex polyhedron. This process can be reversed to obtain a cone in $\mathbb{R}^n$ from a convex body in $\mathbb{R}^{n-1}$ (cf. Stoer and Witzgall [63, pp. 73–74]). Inspection shows that this process does not depend upon the cone's being polyhedral. Thus for any cone K the set $\mathfrak{F}(K)$ becomes a lattice if for any $F, G \in \mathfrak{F}$ we set

$$F \wedge G = F \cap G,$$

and

$$F \vee G = \Phi(F \cup G).$$

The face lattice of a convex polyhedron is well known (cf. Grünbaum [38] or McMullen and Shephard [48]). Using the process previously described, we may translate a theorem about the face lattice of a convex polyhedron into a theorem about $\mathcal{F}(K)$ for a polyhedral cone K and vice versa. We shall say a great deal more about the lattice $\mathcal{F}(K)$ in the next section. For the present we note one result.

THEOREM 1.9 (Barker [4], Stoer and Witzgall [63]). *If K is a cone, then $\mathcal{F}(K)$ is a complemented lattice. If K is polyhedral, then $\mathcal{F}(K)$ is relatively complemented.*

If K is not polyhedral, then $\mathcal{F}(K)$ is not in general relatively complemented. The reader may refer to Birkhoff [23] for various terms from lattice theory.

II. STRUCTURE THEORY

This section consists of two parts. In the first we consider the structure of K and K', and in the second that of $\Pi(K)$. However, when it seems appropriate to do so we shall consider the relationships of K and $\Pi(K)$ in subsection A. Throughout this section we assume that K is a full closed pointed cone in V.

A. *The Theory for K*

As has been noted in Sec. I, $\mathcal{F}(K)$ is a lattice. The next proposition contains the basic arithmetic of faces.

PROPOSITION 2.A.1 (Barker [4], Barker and Carlson [10]). *Let S, T be nonempty subsets of K and $x_1, \ldots, x_r$ vectors in K. Then*

(i) $\Phi(S+T)=\Phi(S\cup T)=\Phi(\Phi(S)+\Phi(T))=\Phi(S)\vee\Phi(T)$,
(ii) $\Phi(x_1+\cdots+x_r)=\Phi(x_1)\vee\cdots\vee\Phi(x_r)$.

We started with the cone $\mathbb{R}^n_+$ (which we call the nonnegative orthant) of all vectors $[x_1,\ldots,x_n]^T$ for which $x_i \geq 0$. This cone is not only polyhedral, but it has n extreme rays, where n is the dimension of the ambient vector space. Since in general a polyhedral cone $K\subset V$ will have $k>n=\dim V$ extreme rays, we single out this special feature of $\mathbb{R}^n_+$.

DEFINITION 2.A.2. If K is a cone in an n-dimensional vector space, we call K *simplicial* iff K has exactly n extreme rays.

If K is simplicial, and if we take a representation of V as $\mathbb{R}^n$, then K is the image of $\mathbb{R}^n_+$ under a nonsingular linear transformation. It is easily checked that $\mathcal{F}(\mathbb{R}^n_+)$ and hence $\mathcal{F}(K)$ is distributive. This is the easy direction of the next result.

THEOREM 2.A.3 (Barker [4]). *$\mathcal{F}(K)$ is distributive iff K is simplicial.*

The proof utilizes a technique due to Hans Schneider.

Proof. For $n=1,2$, all cones K are simplicial and $\mathcal{F}(K)$ is distributive. Thus let $n\geqslant 3$ and suppose $\mathcal{F}(K)$ is distributive. We show that there are n distinct extreme rays. Call two extremals v_1, v_2 distinct if $\Phi(v_1)\neq\Phi(v_2)$. Suppose to the contrary that K has at least $n+1$ distinct extremals $\{x_1,\ldots,x_{n+1}\}$. Since $n=\dim K$, this set is linearly dependent, so we have a linear relation

$$\lambda_1 x_1+\cdots+\lambda_{n+1}x_{n+1}=0.$$

But K is pointed, so $\lambda_i>0$ for some i and $\lambda_j<0$ for some j. Drop the zero terms and let

$$z_i=\begin{cases}\lambda_i x_i & \text{if } \lambda_i>0,\\ -\lambda_j x_j & \text{if } \lambda_j<0.\end{cases}$$

Then the z_k are distinct extremals, and renumbering if necessary, we have

$$z_1+\cdots+z_q=z_{q+1}+\cdots+z_r.$$

By Proposition 2.A.1 (ii) we have

$$\Phi(z_1)\vee\cdots\vee\Phi(z_q)=\Phi\left(\sum_{i=1}^{q}z_i\right)=\Phi\left(\sum_{i=q+1}^{r}z_i\right)=\Phi(z_{q+1})\vee\cdots\vee\Phi(z_r).$$

If z_i, z_j are distinct extremals, then $\Phi(z_i)\wedge\Phi(z_j)=0$. Using the distributivity of $\mathcal{F}(K)$, we have

$$\begin{aligned}\Phi(z_1)&=[\Phi(z_1)\wedge\Phi(z_1)]\vee\cdots\vee[\Phi(z_1)\wedge\Phi(z_q)]\\&=\Phi(z_1)\wedge[\Phi(z_1)\vee\cdots\vee\Phi(z_q)]=\Phi(z_1)\wedge[\Phi(z_{q+1})\vee\cdots\vee\Phi(z_r)]\\&=[\Phi(z_1)\wedge\Phi(z_{q+1})]\vee\cdots\vee[\Phi(z_1)\wedge\Phi(z_r)]=\{0\}.\end{aligned}$$

This contradiction implies that there can be at most n extremals. However, K is full so there must be at least n extremals. Thus K is simplicial. ∎

A leitmotif has been conditions on $\mathcal{F}(K)$, $\Pi(K)$, or some other set related to K which are equivalent to K simplicial. We shall see a number of examples of this.

Perhaps the next strongest requirement to impose on $\mathcal{F}(K)$ is that of modularity. That is, if $F, G, H \in \mathcal{F}(K)$ and if $F \lhd H$, then

$$(F \vee G) \wedge H = F \vee (G \wedge H).$$

Distributive lattices are of course modular. For a simple example of a modular but not distributive face lattice, consider the following example. Let $\dim V \geqslant 3$, and let

$$K = \left\{ x \in V : x_1 \geqslant \left(x_2^2 + \cdots + x_n^2 \right)^{1/2} \right\}.$$

For $n=4$ this is the Lorentz cone of causal vectors of special relativity. It is easily seen that every nonzero $x \in \partial K$ is an extremal and consequently $\mathcal{F}(K)$ is modular. We can isolate this feature.

Definition 2.A.4. Let K be a cone in V. K is *strictly convex* iff for any nonzero noncollinear $x_1, x_2 \in \partial K$ we have $\Phi(x_1) \vee \Phi(x_2) = K$.

It is easily checked that for any strictly convex K, $\mathcal{F}(K)$ is modular. As we shall see later, the cone $m \times m$ of positive semidefinite matrices is a full closed pointed cone in the real space of $m \times m$ hermitian matrices and it has a modular face lattice. But for $m > 2$ this cone is not strictly convex. When $n = \dim V = 3$ we have the following converse.

Theorem 2.A.5 (Barker [4]). *If* $\dim K = 3$ *and if* $\mathcal{F}(K)$ *is modular but not distributive, then* K *is strictly convex.*

Problem. Classify those cones whose face lattices are modular.

The general problem is in some sense still open, although we can deal with some special cases. In another sense the problem is close to finished, as we shall see later. It is reasonable to consider the polyhedral case first, and in this situation we have a strong result. Note that for any K the lattice $\mathcal{F}(K)$ is of finite length, so we may take as the definition of semimodularity Corollary 1 of Birkhoff [23, p. 81].

DEFINITION 2.A.6. A lattice L of finite length is *semimodular* iff for all elements $a, b \in L$, a covers $a \wedge b$ only if $a \vee b$ covers b.

THEOREM 2.A.7 (Barker [4]). *If K is polyhedral and $\mathcal{F}(K)$ is semimodular, then $\mathcal{F}(K)$ is distributive.*

Proof. We shall establish that K is simplicial. If $H \lhd K$, then $\mathcal{F}(H) = \{F \epsilon \mathcal{F}(K): F \lhd H\}$. Clearly $\mathcal{F}(H)$ is a sublattice of $\mathcal{F}(K)$ and is semimodular when $\mathcal{F}(K)$ is semimodular. We induct on $\dim K$. For $\dim K = 2$, K is simplicial. Assume the result holds for all polyhedral cones of dimension $<n$, and let $\dim K = n$. Let $x_1, \ldots, x_{n-1}$ be any $n-1$ distinct extremals of K. Then $\Phi(x_1) \vee \cdots \vee \Phi(x_{n-1}) \neq K$. For if equality were to hold, then (by renumbering the x_i) we could find an $m \leqslant n-1$ such that $H = \Phi(x_1) \vee \cdots \vee \Phi(x_{m-1})$ is a complement in the lattice $\mathcal{F}(K)$ to $\Phi(x_m)$. But $\dim H < n$, and $\mathcal{F}(H)$ is semimodular, so by the induction hypothesis $\mathcal{F}(H)$ is distributive, that is, H is the convex hull of the rays $\Phi(x_1), \ldots, \Phi(x_{m-1})$ and $\dim H = m-1$. But $\Phi(x_m)$ covers $\Phi(x_m) \wedge H = \{0\}$, so $\Phi(x_m) \vee H$ covers H. However, K is polyhedral, so $\dim(\Phi(x_m) \vee H) = 1 + \dim H = m \leqslant n-1$, while $\dim K = n$. This contradicts $\Phi(x_m) \vee H = K$. Thus $F = \Phi(x_1) \vee \cdots \vee \Phi(x_{n-1}) \neq K$. Therefore F is simplicial. Consequently the cone K is $n-1$ neighborly and must itself be simplicial (cf Grünbaum [38, pp. 122–123]). ■

When K is polyhedral it is well known that $\mathcal{F}(K)$ and $\mathcal{F}(K')$ are anti-isomorphic (McMullen and Shephard [48]). However, when K is not polyhedral the geometry is more complicated. To explore this matter we wish to associate with each face of K a face of K'. For any set $S \subset V$ let

$$S^{\perp} = \{f \in V' : fs = 0 \ \forall s \in S\}.$$

DEFINITION 2.A.8. If $F \lhd K$, we define the dual F^D of F to be

$$F^D = \{f \in K' : fx = 0 \ \forall x \in F\}.$$

Note that $F^D = K' \cap F^{\perp}$. If $G \in \mathcal{F}(K')$, we shall use the notation G^{δ} to denote the dual of G in $\mathcal{F}(K)$.

Barker [7] and Tam [69] independently introduced the duality operators. In an unpublished summary entitled "On the structure of the cone of positive operators" Tam states without proof a number of results not only for the operators between K and K' but also for the operators between $\Pi(K)$ and $\Pi(K)'$. Some proofs can be found in Tam's paper [69] previously mentioned, and others are in his doctoral thesis submitted to the University of Hong Kong.

For $F \lhd K$ where K is polyhedral we have $F^{D\delta}=F$, but for general cones K this equality does not hold. The next definition singles out this property.

DEFINITION 2.A.9. Let K be a cone and $F \lhd K$. F is *exposed* iff $F^{D\delta}=F$.

A face F is exposed, as one may easily verify, iff F is the intersection of a hyperplane with K. That is, there is some $f \in K'$ such that $F=\ker f \cap K$. We always have $F^D=F^{D\delta D}$, and the operation $\mathcal{G}F=F^{D\delta}$ is a closure operation. Thus the correspondences $F \to F^D$ and $G \to G^\delta$ define a Galois connection between $\mathcal{F}(K)$ and $\mathcal{F}(K')$. This is a generalization of the earlier remark that for polyhedral K, $\mathcal{F}(K)$ and $\mathcal{F}(K')$ are anti-isomorphic.

The duality operator is reasonably well behaved with respect to the lattice operations.

LEMMA 2.A.10. *If* $F_1, F_2 \lhd K$, *then*

(i) $(F_1 \vee F_2)^D = F_1^D \wedge F_2^D$,
(ii) $(F_1 \wedge F_2)^D \lhd F_1^D \vee F_2^D$.

The proof is not difficult and can be found in Barker [7], where examples are given to show that equality need not hold in (ii). Not surprisingly, equality in property (ii) relates to whether faces are exposed. In particular we have the following result.

THEOREM 2.A.11. *Every face in both* $\mathcal{F}(K)$ *and* $\mathcal{F}(K')$ *is exposed iff both*

(i) $(F_1 \wedge F_2)^D = F_1^D \vee F_2^D$, $F_1, F_2 \in \mathcal{F}(K)$, *and*
(ii) $(G_1 \wedge G_2)^\delta = G_1^\delta \vee G_2^\delta$, $G_1, G_2 \in \mathcal{F}(K')$.

As with the polyhedral cones, every face of the Lorentz cone is exposed. For the Lorentz cone K in $\mathbb{R}^n$, if we represent the dual space by the same $\mathbb{R}^n$, where linear functionals act via the usual inner product, then K and K' are comparable. In fact, $K=K'$. More generally, if V is an inner-product space, we may identify V' with V where $fx=\langle f, x\rangle$. Whenever we consider $\mathbb{R}^n$, we shall assume the usual inner product so that the cones K and K' lie in the same ambient space.

DEFINITION 2.A.12. Let V be an inner-product space. The cone $K \subset V$ is *self-dual* iff $K=K'$.

PROPOSITION 2.A.13. *Let* K *be self-dual. Then* $\mathcal{F}(K)$ *is modular iff it is semimodular and every face of* K *is exposed.*

This proposition is rather weak, but there are examples of self-dual cones for which $\mathcal{F}(K)$ is semimodular but not modular and of self-dual polyhedral cones which are not simplicial. The interesting part of the proof is the following fact.

LEMMA 2.A.14. *Let K be self-dual. If $F \triangleleft K$, then F^D is a complement of F.*

In the remainder of subsection A, with a few exceptions, we are concerned only with self-dual K. Consequently, the two duality operators coincide, and we use the notation F^D for the dual face. Further, if $F \triangleleft K$, then we let F^v denote the cone dual of F in span F. That is,

$$F^v = \{y \in \operatorname{span} F : (y, x) \geqslant 0 \ \forall x \in F\}.$$

It is useful to recall that

$$\operatorname{span} F = F - F = \{f_1 - f_2 : f_i \in F,\ i = 1,2\}.$$

The next definition is due to Raphael Loewy, in a private communication.

DEFINITION 2.A.15. Let K be a self-dual cone. K is *perfect* iff $\forall F \in \mathcal{F}(K)$, $F^v = F$.

THEOREM 2.A.16 (Barker [8]). *Let K be a self-dual cone. Then K is perfect iff $\mathcal{F}(K)$ is orthomodular under the duality operation $F \to F^D$.*

If K is perfect, a routine argument shows that $\mathcal{F}(K)$ is orthomodular. For the proof of the converse we need a technical lemma from Barker and Schneider [14], whose proof is omitted.

LEMMA 2.A.17. *Let K be a cone. Let $u \gg 0$, and let $-v \notin K$. Put $\varepsilon = \sup\{\alpha : u - \alpha v \geqslant 0\}$. Then $0 < \varepsilon < \infty$ and $u - \varepsilon v \in \partial K$.*

For the proof of the theorem let K be a self-dual cone whose face lattice $\mathcal{F}(K)$ is orthomodular. Let $F \in \mathcal{F}(K)$ with $F = \Phi(x)$, $x \geqslant 0$. Clearly, $F \subseteq F^v$. Suppose there is a $y \in F^v \setminus F$. Then by Lemma 2.A.17 there is an $\varepsilon > 0$ such that $x + \varepsilon y$ is in the relative boundary of F. If $\varepsilon = (1-\beta)/\beta$ for $\beta \in (0,1)$, then $z = \beta x + (1-\beta) y = \beta(x + \varepsilon y)$ is also in the relative boundary of F. Since $z \neq 0$, then $\Phi(z) \triangleleft F$. By assumption

$$\Phi(z) \vee \left[\Phi(z)^D \wedge F\right] = F,$$

whence there exists a $u \in \Phi(z)^D \wedge F$, $u \neq 0$. Thus $\langle u, z \rangle = 0$, so

$$0 = \langle z, u \rangle = \beta\langle x, u \rangle + (1-\beta)\langle y, u \rangle \geqslant \beta\langle x, u \rangle > 0,$$

a contradiction. Hence $F^{v} = F$, and K is perfect.

So far we have not considered the fine structure of cones; specifically, whether there is any construction analogous to the direct-sum vector spaces. Haynsworth [39] considered an external direct sum, but Loewy and Schneider [46] formulated the extremely useful notion of decomposability, which is analogous to an internal direct sum. Here the cone K need not be self-dual.

DEFINITION 2.A.18. Let K be a cone in the vector space V. Let K_1 and K_2 be subsets of K. We say K is the *direct sum* of K_1 and K_2 (and write $K = K_1 \oplus K_2$) iff

(i) span $K_1 \cap$ span $K_2 = \{0\}$,
(ii) $K = K_1 + K_2$.

The cone K is called *decomposable* if there exist nonzero subsets K_1 and K_2 such that $K = K_1 \oplus K_2$, and *indecomposable* otherwise.

In Lemma 3.2 of the aforementioned paper, Loewy and Schneider show that if $K = K_1 \oplus K_2$, then $K_1, K_2 \in \mathfrak{F}(K)$. However, more can be said.

LEMMA 2.A.19. *Let $K = K_1 \oplus K_2$. If $x \in K_1$, $y \in K_2$ implies $\langle x, y \rangle = 0$, then $(K_1 \oplus K_2)' = K_1^{v} \oplus K_2^{v}$. If K is self-dual, then $K_1 = K_2^{D}$.*

The proof of the first assertion can be found in Berman [19] or Barker and Foran [12]. The second assertion is discussed in Barker [8]. This lemma is used in the characterization of polyhedral perfect cones by Barker and Foran [12].

THEOREM 2.A.20. *If K is a self-dual polyhedral cone such that every proper maximal face is self-dual in its span, then K is the image of the nonnegative orthant under an orthogonal transformation.*

Summary of the proof. A self-dual simplicial cone is the image of the nonnegative orthant under an orthogonal transformation, whence it is enough to show that K is simplicial. First establish by induction that K is perfect, that is, that every face of K is self-dual in its span. This is the longer portion of the proof. Now let us show again by induction that K is simplicial. Suppose that whenever $\dim K \leqslant n-1$ (and K is polyhedral) we have K perfect implies K

simplicial. Let $\dim K = n$. Let x be an extremal of K, and put $F = \Phi(x)^D$. Then F is a simplicial cone of dimension $n-1$. We claim that $K = F \oplus \Phi(x)$. Let $K_1 = F \oplus \Phi(x)$. By Lemma 2.A.19

$$K_1' = (F \oplus \Phi(x))' = F^v \oplus \Phi(x)^v = F \oplus \Phi(x) = K_1.$$

Thus $K_1 \subseteq K$ and $K_1 = K_1' \supseteq K' = K$. Hence $K = K_1$ and the theorem is proved. ■

Theorem (5.3) of Barker [7] is now a corollary of Theorems 2.A.16 and 2.A.20.

Corollary 2.A.21. *Let K be a self-dual polyhedral cone. Then $\mathfrak{F}(K)$ is orthomodular iff K is simplicial.*

It should be noted that there are self-dual polyhedral cones which are not simplicial in any V with $\dim V > 2$. Examples and a method of construction are given in [12].

Another corollary of Theorem 2.A.16, this time together with Proposition 2.A.13, is the following result.

Corollary 2.A.22. *If K is self-dual and $\mathfrak{F}(K)$ is modular, then K is perfect.*

The converse is appealing, since it would classify the self-dual cones with modular face lattices. The converse is true for $n = 1,2,3$, but false for $n = 4$. An example can be found in [9]. To classify modular face lattices, even for self-dual cones K, we would like to reduce the problem. Since sublattices of modular lattices are again modular, the next result proves useful.

Theorem 2.A.23 (Barker [8]). *Let K be a full closed pointed cone in V. If $K = K_1 \oplus K_2$, then $\mathfrak{F}(K) = \mathfrak{F}(K_1) \oplus \mathfrak{F}(K_2)$. Conversely, if $\mathfrak{F}(K) = \mathfrak{F}_1 \oplus \mathfrak{F}_2$, there are faces K_1, K_2 such that $\mathfrak{F}_i = \mathfrak{F}(K_i)$, $i = 1,2$, and $K = K_1 \oplus K_2$.*

Proof. The proof of the first statement is a routine verification, so let us consider the second. If $\mathfrak{F}(K) = \mathfrak{F}_1 \oplus \mathfrak{F}_2$, then every extremal belongs to either $\mathfrak{F}_1$ or $\mathfrak{F}_2$. Let K_i be the join of all extreme rays in $\mathfrak{F}_i$. Since $\mathfrak{F}(K)$ is atomic, so is $\mathfrak{F}_i$, and thus $\mathfrak{F}_i = \mathfrak{F}(K_i)$. Similarly $K_1 + K_2 = K$. To finish we must show that $\operatorname{span} K_1 \cap \operatorname{span} K_2 = \{0\}$. Obviously, $K_1 \cap K_2 = \{0\}$. Suppose $z \in \operatorname{span} K_1 \cap \operatorname{span} K_2$. For any $w_i \in K_1$, $u_i \in K_2$, $i = 1,2$, for which

$$z = w_1 - w_2 = u_1 - u_2,$$

we have

$$\Phi(w_1)\vee\Phi(u_2)=\Phi(u_1)\vee\Phi(w_2).$$

By the uniqueness of the representations in $\mathfrak{F}=\mathfrak{F}_1\oplus\mathfrak{F}_2$ it follows that

$$\Phi(w_1)=\Phi(w_2) \quad \text{and} \quad \Phi(u_1)=\Phi(u_2). \tag{*}$$

If $z\neq 0$, then $-z\in K_2$; otherwise $-z\in K_2\cap\operatorname{span}K_1=K_2\cap K_1=\{0\}$. So if we take u_1 in the relative interior of K_2, there is by Lemma 2.A.17 an $\varepsilon>0$ such that $u_1-\varepsilon z=u_2$ is in the relative boundary of K_2. Then

$$z=\varepsilon^{-1}(u_1-u_2)$$

and by (*) $\Phi(u_1)=\Phi(u_2)$. This is impossible unless $\dim\Phi(u_1)=\dim K_2=1$. In this case $-z\in K_2$ and we have a contradiction. Hence $z=0$ and $K=K_1\oplus K_2$. ■

This result shows that a classification of indecomposable cones with modular face lattices solves the problem. Let K be a cone whose face lattice $\mathfrak{F}(K)$ is modular. It is well known that every maximal chain (from $\{0\}$ to K) has the same length (Birkhoff [23]). This length, which is one less than the number of faces in the chain, is frequently called the dimension of the lattice. We use the term *height* for this length, and reserve dimension for the algebraic dimension of the vector space spanned by a face. Thus the Lorentz cone (of any dimension >1) has height 2. A useful property of lattices is subdirect irreducibility (cf. Crawley and Dilworth [27]). Its relevance here is expressed in the next lemma, from [9].

LEMMA 2.A.24. *Suppose $\mathfrak{F}(K)$ is modular. Then $\mathfrak{F}(K)$ is subdirectly irreducible iff K is indecomposable.*

This point is important because Crawley and Dilworth [27] present coordinatization theorems (Theorems 13.4 and 13.5) which imply that if K is an indecomposable cone for which $\mathfrak{F}(K)$ is a modular lattice of height $\geqslant 4$, then $\mathfrak{F}(K)$ is isomorphic qua lattice to the lattice of subspaces of a vector space over a division ring. The example of the cone of positive semidefinite matrices in the real space of $n\times n$ hermitian matrices over the quaternions shows that the division ring need not be a field even in our restricted setting. The problem of classifying the modular $\mathfrak{F}(K)$ of height 3 remains wide open. The next result, from [9], is a modest beginning.

THEOREM 2.A.25. *Let K be an indecomposable self-dual cone for which $\mathcal{F}(K)$ is modular. Let $h(K)$ denote the height of $\mathcal{F}(K)$. If* $\dim K=4$, *then K is strictly convex. If* $\dim K=5$, *then either $h(K)=2$ (so K is strictly convex) or $h(K)=3$. In the latter case every maximal face is of dimension 3 and is strictly convex.*

The proof is a tedious argument by cases. We close this section with the conjecture that the case $\dim K=5$, $h(K)=3$ is not possible.

B. The Theory for $\Pi(K)$

A given closed full cone K completely determines the cones K' and $\Pi(K)$. In particular, from a knowledge of the extreme rays of K one should be able to determine the extremals of K' and of $\Pi(K)$. Actually doing this seems to be quite difficult. Work in this direction was initiated by Loewy and Schneider [46].

NOTATION 2.B.1. Let K be a cone. The set of all extremals of K is denoted by $\operatorname{Ext} K$.

Thus the set of all extremals of $\Pi(K)$ is denoted by $\operatorname{Ext}\Pi(K)$.

THEOREM 2.B.2. *Let K be a full closed cone in V. The following are equivalent:*

(1) *K is indecomposable;*
(2) *if A is nonsingular and* $A(\operatorname{Ext} K)\subset \operatorname{Ext} K$, *then* $A\in\operatorname{Ext}\Pi(K)$;
(3) *if A is nonsingular and $A\,K=K$, then* $A\in\operatorname{Ext}\Pi(K)$;
(4) $I\in\operatorname{Ext}\Pi(K)$.

Before continuing with the discussion of $\operatorname{Ext}\Pi(K)$ let us give another interpretation of $\Pi(K)$. If we represent the action of V' on V by $\langle f, x\rangle$ and define an inner product in $\operatorname{Hom}(V)$ by $\langle A, B\rangle=\operatorname{tr}A^TB$, then we may identify $V\otimes V'$ with $\operatorname{Hom}(V)$, where $x\otimes f(y)=\langle f, y\rangle x$. If $K\otimes K'$ denotes the set of all bilinear functionals B such that $B(x, f)\geqslant 0$ $\forall x\in K$, $\forall f\in K'$, then $K\otimes K'$ can be identified with $\Pi(K)$. There is the obvious extension to tensor products of cones K_1 and K_2 in vector spaces V_1 and V_2. Here of course if x is a column vector and f is a row vector, we may identify $x\otimes f$ with the matrix product xf. Berman and Gaiha [20] noted that $\Pi(K)'$ is the closure of the convex hull of $\{f\otimes x: f\in K', x\in K\}$. Tam [67] strengthened this considerably.

THEOREM 2.B.3. *$\Pi(K)'$ is equal to the set of all nonnegative linear combinations of the form zy^T with $y\in K$, $z\in K'$.*

Now let $\Delta(K)=\Pi(K')'\subset\Pi(K)$. Thus $\Delta(K)$ is the set of all nonnegative linear combinations of rank-one matrices yz^T with $y\in K$, $z\in K'$. Barker and Loewy [13] established the following theorem.

THEOREM 2.B.4. *Let K be a closed full cone. The following are equivalent*:

(1) *K is indecomposable,*
(2) *K' is indecomposable,*
(3) *$\Pi(K)$ is indecomposable,*
(4) *$\Delta(K)$ is indecomposable.*

Let $x\in\operatorname{Ext}K$, $f\in\operatorname{Ext}K'$. Then $x\otimes f\in\operatorname{Ext}\Pi(K)$ and $f\otimes x\in\operatorname{Ext}\Pi(K')$. Following the arguments of Haynsworth, Fielder, and Pták [40], who presented this for polyhedral cones, one can show that every rank-one extremal is of this form. When are these all?

To phrase this question in terms of tensor products, consider the following slightly more general situation. Let K_1 and K_2 be cones in V_1 and V_2 respectively. As above let $K_1\otimes K_2$ denote the set of nonnegative bilinear functionals on $V_1'\otimes V_2'$. That is,

$$K_1\otimes K_2=\{B: B(f,g)\geqslant 0\ \forall f\in K_1', g\in K_2'\}.$$

Now define

$$K_1\otimes pK_2=\{\Sigma x_i\otimes y_i: x_i\in K_1, y_i\in K_2\},$$

where all sums are finite. Clearly $K_1\otimes pK_2\subseteq K_1\otimes K_2$. Barker [6] showed that if one of K_1 and K_2 is lattice ordered, then equality holds. Further it was conjectured in that paper that the converse is true. When $K_2=K_1'$ the statement can be rephrased as a statement about the self-duality of $\Pi(K)$. Tam [67] and Barker and Loewy [13] independently settled this question. We take the statements from the latter paper.

THEOREM 2.B.5. *Let I denote the identity matrix. $I\in\Delta(K)$ iff K is simplicial.*

THEOREM 2.B.6. *Let K be a cone in $\mathbb{R}^n$, and let P denote the nonnegative orthant. Then $\Pi(K)=\Pi(K)'$ iff $K=QP$ where Q is an orthogonal matrix.*

Thus even when K is polyhedral but not simplicial or when K is the Lorentz cone, we do not have $\Pi(K)=\Pi(K)'$. This implies that in general $\Pi(K)$ has many more extremals than just the set of rank-1 extremals.

For polyhedral cones Fiedler and Pták [30, 31] have given a penetrating analysis of extreme positive operators. Their main result says—roughly speaking—that the rank of an extreme positive operator may assume any of the possible values within certain natural boundaries which they spell out, except rank two.

If K is polyhedral with Ext K represented by $x_1,\dots,x_r$, set $P=[x_1 \ \cdots \ x_r]$, the $n\times r$ matrix with columns x_i. We may associate a similar matrix Q with K'. Burns, Fiedler, and Haynsworth [25] discuss the properties of K' and of $\Pi(K)$ in terms of these matrices.

The situation for the Lorentz cone is also far from trivial. Let C denote the Lorentz cone in $\mathbb{R}^n$, $n\geqslant 2$. Loewy and Schneider [47] established that Ext C consists of the rank-one extremals together with the set of maps of C onto itself. Nor is this situation typical. R. C. O'Brien [54] has constructed an indecomposable $K\subset\mathbb{R}^r$ and a nonsingular $A\in\operatorname{Ext}\Pi(K)$ which does not take Ext K into itself.

Clearly a great deal remains to be done in the analysis of $\Pi(K)$ for general K.

The collection of onto maps of K are interesting for another reason. They form a group called the automorphism group of K, Aut(K). Thus $A\in\operatorname{Aut}(K)$ iff $A^{-1}>0$. A special class of cones arise in the study of automorphic functions of several complex variables and also in Jordan algebras.

Definition 2.B.7. A self-dual cone $K\subset\mathbb{R}^n$ is *homogeneous* iff Aut(K) acts transitively on int K.

E. B. Vinberg [74] effected a classification of these cones using a correspondence with compact semisimple Jordan algebras. His main result is that every indecomposable homogeneous cone is unitarily equivalent to a cone of one of the following classes:

(I) the cone of $n\times n$ real positive semidefinite matrices;
(II) the cone of $n\times n$ positive semidefinite hermitian matrices;
(III) the cone of $n\times n$ positive semidefinite matrices over the quaternions;
(IV) the cone of 3×3 positive semidefinite matrices with elements from the Cayley numbers;
(V) the Lorentz cone.

(See also Loewy and Schneider [47] for a discussion of the Lorentz cone.) For a self-dual cone K in a (not necessarily finite-dimensional) real Hilbert space H there is a weaker notion of homogeneity. If $F\lhd K$, it is easy to check that F^D is well defined. Let P_F denote the orthogonal projection onto F and set

$$N_F=P_F-P_{F^\perp}.$$

Finally let $G_0(K)$ be the real Lie group generated by $\{N_F: F \triangleleft K\}$, and let $G_0(K)$ be the Lie group generated by $G_0(K)$.

DEFINITION 2.B.8. A cone K is *facially homogeneous* iff $G_0(K) \subseteq \mathrm{Aut}(K)$.

K homogeneous implies K is facially homogeneous, and as is often the case, the situation in finite-dimensional spaces is nicer. Bellissard, Iochum, and Lima [18] have established the following result.

THEOREM 2.B.9. *Let H be a finite-dimensional real Hilbert space. Then a self-dual cone K is homogeneous iff it is facially homogeneous.*

A discussion of the infinite-dimensional case can be found in a paper by Bellisard and Iochum [17].

Since the group $\mathrm{Aut}(K)$ contains a copy of the positive reals (the positive scalar maps), it cannot be compact. Brown [24] considered compact groups.

THEOREM 2.B.10. *A compact group of (elementwise) nonnegative matrices is finite.*

This result does not extend to arbitrary K. For if we take a Lorentz cone in $\mathbb{R}^3$, say

$$K=\left\{x: x_1 \geqslant \left(x_2^2+x_3^2\right)^{1/2}\right\},$$

and let

$$A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix},$$

then A is a rotation of K around the x_1 axis. If θ is not a rational multiple of π, then A generates an infinite compact subgroup of $\mathrm{Aut}(K)$. In fact we have a more general result.

THEOREM 2.B.11 (Horne [43]). *If $A \in \mathrm{Aut}(K)$ and if A is irreducible and has a nonzero fixed point in K, then A belongs to a compact subgroup of* $\mathrm{Aut}(K)$.

Proof. If A is irreducible and has a fixed point in K, then its Perron root $\rho(A)=1$. Since $A \in \mathrm{Aut}(K)$, $\rho(A^{-1})=1$ also, as A has (up to scalar multiples) only one eigenvector in K. Thus all eigenvalues of A have modulus 1 and linear elementary divisors (cf. Barker and Schneider [14]). Thus the group $\{A^n : n=0,1,2,\dots\}$ is bounded and hence has compact closure. ■

This result leads to an extension and alternate proof of the result which states that a proper Lorentz has an eigenvector in the "light cone," that is, the boundary of a Lorentz cone in $\mathbb{R}^4$.

THEOREM 2.B.12. *Suppose K is a full cone in V, where* $\dim V = n$ *is even. Let $A \in \mathrm{Aut}(K)$, and assume that* $\det A > 0$. *Then A has an eigenvector in ∂K.*

Proof. By dividing A by its spectral radius we may assume that A has a nonzero fixed point in K. If A has no eigenvector in ∂K, it is irreducible (cf. Vandergraft [72]). So by Theorem 2.B.11, A belongs to a compact subgroup of $\mathrm{Aut}(K)$, whence all eigenvalues are of modulus 1 and are simple roots of the characteristic equation of A (Vandergraft [72] or Barker and Schneider [14]). Since $\det A > 0$, then $\det A = 1$. This cannot happen if n is even. ■

For polyhedral cones Brown's Theorem 2.B.10 still holds.

THEOREM 2.B.13. *Let K be a polyhedral cone. A compact group $G \subset \mathrm{Aut}(K)$ is finite.*

The proof of Theorem 2.B.13, which is found in [5], is modeled on Brown's proof of his result, and both rely on the spectral properties of nonnegative operators and some properties of Lie groups. Using Theorem 2.B.13, Horne [43] has obtained a very nice description of $\mathrm{Aut}(K)$ for polyhedral K.

THEOREM 2.B.14. *Let K be an indecomposable polyhedral cone. Then $\mathrm{Aut}(K)$ is isomorphic to the direct product of the group of positive reals (alias positive scalar maps) and a maximal finite subgroup.*

So far we have considered the structure of $\mathrm{Aut}(K) \subset \Pi(K)$. But $\Pi(K)$ itself can be considered a semiring; that is, $\Pi(K)$ is closed under addition, multiplication, and multiplication by positive scalars. Left, right, or two-sided ideals of $\Pi(K)$ are defined in the obvious way. Thus $\mathfrak{L} \subset \Pi(K)$ is a left ideal of $\Pi(K)$ iff $\mathfrak{L}$ is closed under addition and $AB \in \mathfrak{L}\ \forall A \in \Pi(K), \forall B \in \mathfrak{L}$. If one recalls Theorem 2.B.2 it is no surprise that there is a relation between the indecomposability of K and the noninvertible elements of $\Pi(K)$. Specifically, Horne [42] obtained this result.

THEOREM 2.B.15. *Let K be a full cone in V. Then K is indecomposable iff the only maximal right ideal of $\Pi(K)$ is the set of noninvertible elements.*

Proof. If K is indecomposable, then by Theorem 2.B.2 $I \in \operatorname{Ext}\Pi(K)$. Thus the set M of noninvertible elements of $\Pi(K)$ is closed under addition. For $I \in \operatorname{Ext}\Pi(K)$ iff some (and hence every) element of $\operatorname{Aut}(K)$ is extremal. Thus the sum of noninvertible elements is noninvertible. Therefore M is a proper (two-sided) ideal of $\Pi(K)$, and hence the only maximal ideal of any type. Now suppose K has a nontrivial decomposition $K = K_1 \oplus K_2$. Define $A_1 \in \Pi(K)$ by $A_1|\operatorname{span} K_1 = I$, $A_1|\operatorname{span} K_2 = 0$. Define A_2 analogously. Neither A_1 nor A_2 is invertible, and $A_1 + A_2 = I$. Then $\mathcal{R}(A_i) = A_i\Pi(K)$ are proper right ideals, and by a maximality argument each is contained in a maximal right ideal, say $\mathcal{R}(A_i) \subset \mathfrak{M}_i$ for $i = 1,2$. Since $I = A_1 + A_2$, we have $\mathfrak{M}_1 \neq \mathfrak{M}_2$, and $\Pi(K)$ contains at least two distinct maximal ideals. ∎

The cone $\Pi(K)$ is defined in terms of K, and a vector-space isomorphism of V clearly induces an isomorphism of $\Pi(K)$. To what extent is the converse true? Horne [42] considered polyhedral K.

THEOREM 2.B.16. *Let V_1 and V_2 be finite-dimensional real vector spaces with full polyhedral cones K_1 and K_2 respectively. Suppose T is an isomorphism of the semiring $\Pi(K_1)$ onto $\Pi(K_2)$. Then there exists a linear isomorphism $T^*: V_1 \to V_2$ mapping K_1 onto K_2 and satisfying*

$$T^*(A(x)) = T(A)T^*(x)$$

for all $A \in \Pi(K_1)$, $x \in K_1$.

Raphael Loewy in a private communication has indicated that he has established this result for nonpolyhedral cones as well. Tam [69] also extended Theorem 2.B.16, and we give his results here.

PROPOSITION 2.B.17. *$\Pi(K)$ has a unique minimal (nonzero) two-sided ideal, namely $\Pi(K')'$. Furthermore, $\Pi(K')'$ is a principal two-sided ideal generated by each of its nonzero elements. Also K is simplicial iff $\Pi(K)$ has no nontrivial two-sided ideals.*

Proof. It is easy to verify that $\Pi(K')'$ is a two-sided ideal. Let $\mathcal{I}$ be a nonzero two-sided ideal of $\Pi(K)$. Choose a nonzero $A \in \mathcal{I}$. Then there are $y' \in \operatorname{int} K$, $z' \in \operatorname{int} K'$ such that $(z')^T A y' = 1$. For any $y \in K$, $z \in K'$ we have $yz^T = (yz'^T)A(y'z^T) \in \mathcal{I}$. Thus $\mathcal{I} \supset \Pi(K')'$. The next statement is easy to prove. If $\Pi(K)$ contains no nontrivial two-sided ideal, then $\Pi(K')' = \Pi(K)$. The last statement now follows from Theorem 2.B.5, since $\Delta(K) = \Pi(K')'$. ∎

THEOREM 2.B.18. *If the unique minimal two-sided ideals of $\Pi(K_1)$ and $\Pi(K_2)$ are isomorphic as semirings, then K_1 and K_2 are linearly isomorphic.*

The ideal $\Pi(K')'$ also determines whether K is simplicial in another way.

THEOREM 2.B.19. *K is simplicial iff $\Pi(K')'$ is a prime two-sided ideal of $\Pi(K)$.*

Although $\Pi(K)$ is not commutative, the definition of prime ideal used here is this: If $\mathcal{I}$ is a two-sided ideal and if $AB \in \mathcal{I}$ and $A \notin \mathcal{I}$ implies $B \in \mathcal{I}$, then $\mathcal{I}$ is a prime ideal.

Other natural algebraic questions concerning ideals revolve around principal ideals and maximal ideals. In the full matrix algebra the ascending and descending chain conditions hold, since right and left ideals are subspaces. The semiring $\Pi(K)$ is not so well behaved. Horne [42] considered descending chains, and Tam [69] modified Horne's technique for ascending chains. We can summarize their results as follows.

THEOREM 2.B.20. *Let K be an arbitrary cone in a vector space V with $\dim V \geq 2$. Then $\Pi(K)$ contains both infinite strictly ascending and descending sequences of principal right or left ideals.*

For an indecomposable K Theorem 2.B.15 settles the question of maximal ideals. Horne considered the maximal left and right ideals in the semiring N_n of elementwise nonnegative matrices. Tam extended these considerations in the following way. Let K be decomposable, and let $K = K_1 \oplus \cdots \oplus K_r$ denote the unique representation of K as a direct sum of indecomposable subcones. Denote P_i the projection onto span K_i along the sum of the spans of the remaining faces. Then $P_i \in \Pi(K)$ for all i, and $P_1 + \cdots + P_r = I$.

THEOREM 2.B.21. *With K and P_i as previously described the semiring, $\Pi(K)$ has exactly r maximal right ideals, namely*

$$\mathcal{R}_i = \{A \in \Pi(K) : P_i \notin A\Pi(K)\}, \qquad i = 1, \ldots, r;$$

and exactly r maximal left ideals, namely

$$\mathcal{L}_i = \{A \in \Pi(K) : P_i \notin \Pi(K)A\}, \qquad i = 1, \ldots, r.$$

The proof proceeds along the expected lines except that the proof of closure under addition employs the fact that the P_i are extremals in $\Pi(K)$.

Tam also considers two-sided ideals in $\Pi(A)$. He shows that every maximal two-sided ideal is one of

$$\mathcal{M}_i = \{A \in \Pi(K) : \text{there do not exist } B, C \in \Pi(K) \text{ for which } BAC = P_i\}.$$

Although each $\mathfrak{M}_i$, $i=1,\ldots,r$, is a two-sided ideal, not all of these are in general maximal.

So far we have considered what might be called the arithmetic theory of ideals. However, $\Pi(K)$ has another algebraic property, namely the partial order. The general theory for the faces of $\Pi(K)$ is unsettled at the present. However, Tam in his thesis and in [69] introduced a simple kind of face.

DEFINITION 2.B.22. For $F,G\in\mathfrak{F}(K)$ set

$$\Pi(F,G)=\{A\in\Pi(K):AF\subset G\}.$$

Not all faces of $\Pi(K)$ are of this form, and in particular $\Phi(I)\lhd\Pi(K)$ is not. However, it can be shown that all maximal faces are of this type. Barker [8] considered the faces $\Pi(F,\{0\})$ and $\Pi(K,F)$ for a perfect cone (Definition 2.A.15). It is easily seen that $\Pi(F,\{0\})$ is a left ideal and $\Pi(K,F)$ is a right ideal.

DEFINITION 2.B.23. A face of $\Pi(K)$ which is also a right (left) ideal will be called a *right* (left) *facial* ideal.

Tam observed that there is a one-to-one correspondence between right facial ideals of $\Pi(K)$ and right ideals in the algebra Hom(V). A similar remark holds for left and two-sided facial ideals. Thus there are no proper two-sided facial ideals, although unless K is simplicial, K has proper two-sided ideals. The ideal structure suggests a classification of the right and left facial ideals. To verify the details we need some lemmas and notation.

NOTATION 2.B.24. If $S\subseteq\Pi(K)$, we define $S^T\subseteq\Pi(K')$ by

$$S^T=\{A^T:A\in S\}.$$

LEMMA 2.B.25. *If F, G are faces of K, then*

$$\Pi(F,G)^T\subset\Pi(G^D,F^D).$$

In addition equality holds if G is exposed.

Proof. Let $A\in\Pi(F,G)$, $y\in F$, $z\in G^D$. Then $0=\langle z,Ay\rangle=\langle A^Tz,y\rangle$, whence $A^T(G^D)\subset F^D$. The first statement follows. Assume further that G is exposed. Taking the transpose of the inclusion, we obtain

$$\Pi(F,G)\subseteq\Pi(G^D,F^D)^T.$$

On the other hand

$$\Pi(G^D,F^D)^T \subseteq \Pi(F^{D\delta},G^{D\delta}) = \Pi(F^{D\delta},G),$$

since G is exposed. Clearly, since $F \lhd F^{D\delta}$, we have

$$\Pi(F^{D\delta},\, G) \subseteq \Pi(F,G).$$

Equality follows. ∎

LEMMA 2.B.26. *Let* $F_1, F_2 \in \mathfrak{F}(K)$.

(1) $\Pi(K, F_1) \subset \Pi(K, F_2)$ *iff* $F_1 \lhd F_2$.
(2) *If every face of K is exposed, then*

$$\Pi\big(F_1, \{0\}\big) \supseteq \Pi\big(F_2, \{0\}\big) \quad \textit{iff} \quad F_1 \lhd F_2.$$

Proof. We prove (2), and the proof of (1) is similar. In (2) clearly $F_1 \lhd F_2$ implies $\Pi(F_1,\{0\}) \supseteq \Pi(F_2,\{0\})$. So assume the latter containment is true. Let y be in the relative interior of F_2^D, and let $x \in \operatorname{int} K$. Recall that $x \otimes y \in \Pi(K)$ is defined by $(x\otimes y)(z) = \langle y, z\rangle x$. Then $\ker(x \otimes y) \cap K = F_2$ from the discussion following Definition 2.A.9. Thus

$$x \otimes y \in \Pi\big(F_2, \{0\}\big) \subseteq \Pi\big(F_1, \{0\}\big),$$

whence

$$F_1 \lhd \ker(x \otimes y) \cap K = F_2.$$

∎

COROLLARY 2.B.27. *Let every face of K be exposed, and let* $F_1, F_2 \in \mathfrak{F}(K)$. *Then*

(1) $\Pi(F_1,\{0\}) + \Pi(F_2,\{0\}) \subseteq \Pi(F_1 \wedge F_2, \{0\})$,
(2) $\Pi(K, F_1) + \Pi(K, F_2) \subseteq \Pi(K, F_1 \vee F_2)$.

THEOREM 2.B.28. *A subset of* $\Pi(K)$ *is a right facial ideal iff it is of the form* $\Pi(K, F)$ *for some* $F \lhd K$.

Proof. Sufficiency is straightforward. Let $\Phi(A)$ be a right facial ideal of $\Pi(K)$. We show that $\Phi(A)=\Pi(K,\Phi(Ay))$ for any $y\in\operatorname{int} K$. Clearly $\Phi(A)\lhd\Pi(K,\Phi(Ay))$.Choose $z\in\operatorname{int} K'$. Since $\Phi(A)$ is a right ideal, $Ay\otimes z\in\Phi(A)$. But $Ay\otimes z(K\backslash\{0\})$ is contained in the relative interior of $\Phi(Ay)$, whence $Ay\otimes z$ is in the relative interior of $\Pi(K,\Phi(Ay))$. Hence $\Pi(K,\Phi(Ay))=\Phi(Ay\otimes z)\lhd\Phi(A)$, and equality follows. ∎

Theorem 2.B.29. *A subset of* $\Pi(K)$ *is left facial ideal iff it can be expressed in the form* $\Phi(y\otimes z)$ *with* $y\in\operatorname{int} K$, $z\in K'$. *If, in addition, the correspondence* $F\to F^D$ *is surjective, the left facial ideals of* $\Pi(K)$ *are of the form* $\Pi(F,\{0\})$ *with* $F\lhd K$.

Proof. To establish the first part note that if $\Phi(A)$ is a left facial ideal of $\Pi(K)$, then $\Phi(A^T)=\Phi(A)^T$ is a right facial ideal of $\Pi(K')$. Now modify the proof of Theorem 2.B.28.

Suppose in addition that $F\to F^D$ is surjective. Then every face of K' is exposed. So from the proof of Theorem 2.B.28 we have

$$\Phi(A^T)=\Pi\big(K',\Phi(A^Tz')\big)$$

for $x\in\operatorname{int} K'$. Hence for $F=(\Phi(A^Tz))^\delta\lhd K$ we have

$$\Phi(A)=\Pi\big(K',\Phi(A^Tz')\big)^T=\Pi(F,\{0\}),$$

and the proof is finished. ∎

Using these results we may also say something about zero divisors in $\Pi(A)$.

Theorem 2.B.30. *A linear operator* $A\in\Pi(K)$ *is a right (or left) zero divisor iff* A *belongs to a proper right (or left) facial ideal.*

See Tam [69] for the proof.

We close on the refrain of conditions equivalent to K a simplicial cone. Recall that if K is perfect then every face is exposed.

Theorem 2.B.31. *Let* K *be a perfect cone. The sum of any two left facial ideals is a left facial ideal when and only when* K *is simplicial. The corresponding result also holds for right facial ideals.*

Proof. If K is simplicial, we may assume it to be the nonnegative orthant. It is easy to show that

$$\Pi(F_1,\{0\})+\Pi(F_2,\{0\})=\Pi(F_1\wedge F_2,\{0\}).$$

Suppose for any $F_1, F_2\in\mathcal{F}(K)$ we have that $\Pi(F_1,\{0\})+\Pi(F_2,\{0\})$ is a left facial ideal. In particular, if $F_2=F_1^D$ and $F_1\neq\{0\}$, $F_1\neq K$, then by Theorem 2.B.29 there is a $G\lhd K$ such that

$$\Pi(F_1,\{0\})+\Pi(F_1^D,\{0\})=\Pi(G,\{0\}).$$

But by Corollary 2.B.27 (1) and Lemma 2.B.26 (1), we have $G\lhd F_1\wedge F_1^D=\{0\}$, whence $\Pi(G,\{0\})=\Pi(K)$. Thus there are nonzero $A_i\in\Pi(F_i,\{0\})$ such that $I=A_1+A_2$ and K is decomposable. We check that $K=F_1\oplus F_1^D$. For any $x\in K$, $x=Ix=A_1^Tx+A_2^Tx$. Further, $A_1^Tx\in F_1^D$ and $A_2^Tx\in F_1$, by Lemma 2.B.25 and the fact that $F_1^{DD}=F$. Since it is clear that $(\operatorname{span} F)\cap(\operatorname{span} F^D)=\{0\}$, we have $K=F_1\oplus F_1^D$.

Now if F_1 is any nontrivial face and if $x_2\in F_1^D$ is an extremal, note that

$$F_1^D=\Phi(x_2)\otimes\left[\Phi(x_2)^D\wedge F_1^D\right]. \tag{$*$}$$

To verify $(*)$ let $y\in F_1^D$. Then $y=ax_2+x_3$, where $x_3\in\Phi(x_2)^D$, since $K=\Phi(x_2)\oplus\Phi(x_2)^D$. But then $y\geqslant x_3\geqslant 0$, so $x_3\in F_1^D$, and thus $x_3\in\Phi(x_2)^D\wedge F_1^D$. Now start with $F_1=\Phi(x_1)$ where x_1 is an extremal, and apply $(*)$ n times, where $n=\dim V$. We obtain $K=\Phi(x_1)\oplus\cdots\oplus\Phi(x_n)$, and K is simplicial.

The statement for right facial ideals follows from what has been proven and Lemma 2.B.25. ■

III. THE CONE OF POSITIVE SEMIDEFINITE MATRICES

Let $\mathcal{H}_n$ be the real space of $n\times n$ hermitian matrices, and let $\mathcal{P}_n$ be the subset of $n\times n$ positive semidefinite matrices. The common inner product on $n\times n$ matrices given by $\langle A,B\rangle=\operatorname{tr}(B^*A)$ when restricted to $\mathcal{H}_n$ makes $\mathcal{H}_n$ into a real inner product space. It is readily checked that $\mathcal{P}_n$ is a closed full pointed cone in $\mathcal{H}_n$. It is also well known (cf. Berman [19, p. 55]) that $\mathcal{P}_n$ is self-dual. An outstanding problem in this area is a description of $\Pi(\mathcal{P}_n)$ (cf. de Pillis [57]). In particular, what are the extremals in $\Pi(\mathcal{P}_n)$? As we know from Sec. II.A, we may identify $\Pi(\mathcal{P}_n)$ with $\mathcal{P}_n\otimes\mathcal{P}_n$, which is not self-dual, since $\mathcal{P}_n$ is not simplicial (Theorem 2.B.6). First we would like to know the extreme ray of $\mathcal{P}_n$ itself. The classification has been part folklore for a number of years. The version presented here is from Barker and Carlson [10], where there are some indications of other references. Let $\mathcal{O}_n$ be the set of orthogonal projections, that is, $A\in\mathcal{O}_n$ iff $A\in\mathcal{H}_n$ and $A^2=A$. The usual order on projections is $A\leqslant B$ iff range $A\subseteq$ range B. It is easily checked that this order coincides with the partial order $\mathcal{O}_n$ inherits as a subset of $\mathcal{P}_n$. $\mathcal{O}_n$ is a lattice if we define $A\vee B$ and $A\wedge B$ to be the hermitian projections onto, respectively, range $A+$ range B and range $A\cap$ range B. Thus $\mathcal{O}_n$ is isomorphic with the lattice of all subspaces $\mathbf{C}^n$. It is also isomorphic to the face lattice $\mathcal{F}(\mathcal{P}_n)$ of $\mathcal{P}_n$.

THEOREM 3.1. *The map from $\mathcal{O}_n$ to $\mathcal{F}(\mathcal{P}_n)$ given by $A\to\Phi(A)$ is an order preserving lattice isomorphism.*

We now consider the linear transformations on $\mathcal{H}_n$. There are two representations of the tensor product commonly used, and we must take some care in distinguishing them. For $A, B\in\mathcal{H}_n$ we have

$$(A\otimes_D B)(C)=\operatorname{tr}(BC)A \quad \text{(the dyad product)},$$
$$(A\otimes_k B)(C)=ACB^T \quad \text{(the Kronecker product)}.$$

The dyad product is the one which corresponds to the tensor products used in Sec. II. Thus in our present notation we have $\Pi(\mathcal{P}_n)=\mathcal{P}_n\otimes_D\mathcal{P}_n$. We also know from Theorem 2.B.3 that

$$\Pi(\mathcal{P}_n)'=\left\{\sum A_i\otimes_D B_i \middle| A_i, B_i\in\mathcal{P}_n\right\}.$$

Therefore, the extremals of $\Pi(\mathcal{P}_n)'$ can be represented as operators of the form $P\otimes_D Q$ where P and Q are rank one orthoprojectors. The extremal structure of $\Pi(\mathcal{P}_n)$ remains open.

The investigation of $\Pi(\mathcal{P}_n)$ has led to a study of the class of completely positive operators. A linear operator T on M_n is completely positive iff for all m the map on the set of $m\times m$ matrices whose entries are $n\times n$ matrices given by $[A_{ij}]\to[T(A_{ij})]$ preserves the positive semidefinite matrices of $M_m\otimes_K M_n$. Let $\mathcal{C}_n$ denote the set of restrictions of completely positive maps to $\mathcal{H}_n$. In a forthcoming paper with Richard Hill and Ray Haertel using both cone theoretic methods and techniques developed by Hill and Poluikis [41] we are able to describe $\mathcal{C}_n$.

PROPOSITION 3.2. $\Pi(\mathcal{P}_n)\supset\mathcal{C}_n\supset\Pi(\mathcal{P}_n)'$, *where for $n>2$ the inclusions are strict.*

In fact Choi [26] shows that $\Pi(\mathcal{P}_n)=\mathcal{C}_n$ for $n=2$ and briefly discusses why the result does not extend. Since $\mathcal{P}_2$ is not simplicial, then $\mathcal{C}_2$ strictly contains $\Pi(\mathcal{P}_2)'$.

THEOREM 3.3. *$\mathcal{C}_n$ is isometrically isomorphic with $\mathcal{P}_{n^2}$, the cone of $n^2\times n^2$ positive semidefinite matrices.*

Theorem (3.3) indicates why in many respects it is $\mathcal{C}_n$ and not $\Pi(\mathcal{P}_n)$ which is the natural notation of a positive operator for maps which preserve the $n\times n$ hermitian matrices.

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