Solutions and Notes:

String Theory

John Graham Reynolds

Outline, Motivation, and a Virtual Handshake for the Reader

This document houses notes and solutions to various advanced texts on string theory and mathematical physics related to the study of quantum gravity. **I solved and typed these notes independently while working full-time as a software engineer for Vanderbilt University Medical Central.** My study began in November 2023 and **continues currently, with a plan to return to a first class graduate school for the completion of a PhD in string theory and mathematical physics.**

A numerical summary of the solutions is as such: I have solved ? of the !fraction it out by chapter! (182 total) problems in the text *String Theory and M-Theory* written by Becker, Becker, and Schwarz (BBS) while also reading and making reference to Polchinki's masterpiece *String Theory: Volumes 1 and 2*. Zwiebach's *A First Course in String Theory*, despite being less rigorous, is filled with a plethora of enlightening problems. A selection of these solutions is presented here as well.

I completed my B.S. in Physics, Mathematics at Johns Hopkins. During that time I conducted extensive theoretical research on the black hole information problem (advised by David Kaplan). Prior to having the mathematical tool set for theory, I did two years of observational cosmology research on the CLASS telescope, designed to detect signatures of an inflationary period on the cosmic microwave background (advised by Tobi Marriage and Chuck Bennett). I am well versed in both general relativity and quantum field theory. Texts used for learning these subjects were Carroll's *An Introduction to General Relativity: Spacetime and Geometry* and Schwartz's *Quantum Field Theory and the Standard Model*. Solutions to these topics were never typed.

New string theory and related mathematical physics solutions are set to appear as I incrementally update this document. The possibility of typos is quite likely given the length and detailed nature of this document. Notes and corrections are gladly accepted at the author's email address. [1]

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Solutions and Notes:

String Theory and M-Theory

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Preface to BBS' String Theory and M-Theory

Contained within are solutions and notes to a variety of problems from the most recent graduate text on string theory, published in 2007. The BBS text is a rigorous and thorough introduction to areas of modern string theory research, beginning with the bosonic string and culiminating in ... to be expanded and revised to give the reader a fair overview. We present solutions to chapters 2 - ?. Each chapter's solutions are preceded with a small introduction to the material discussed within as is relevant to the homework problems that are solved.

Preface to Chapter 2 Problems

First, we derive the full form of the quantum Virasoro algebra, the central extension of the Witt (or classical Virasoro) algebra, following along with the closely related problems 2.13, 2.14, 2.15. We then solve a number of other problems from the chapter, adding notes and remarks for a bit of exposition where beneficial.

In this preliminary section of the book, we study the unphysical (or, rather, *physically incomplete*) theory of bosonic strings. There are a number of equivalent actions for this theory, but we take the most useful (most easily quantized) *string sigma model action* as our starting point. With an auxiliary world-sheet metric $h_{\alpha\beta}$, string spacetime embedding functions $X^\mu(\tau,\sigma)$, and string tension T, related to the string scale l_s by $T=(\pi l_s^2)^{-1}$, the string sigma model action reads:

$$S_\sigma = \frac{T}{2}\int d\sigma d\tau \sqrt{-h}h^{\alpha\beta}g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \tag{1}$$

To semi-simplify matters we make the (as it turns out, incomplete) gauge choice $h_{\alpha\beta}=\eta_{\alpha\beta}$ and furthermore choose our spacetime to be Minkowskian with $g_{\alpha\beta}=\eta_{\alpha\beta}$. Gauge fixing the worldsheet metric like so is only possible if there are no topological obstructions. That is, the worldsheet requires vanishing Euler characteristic.

Up to choice of boundary conditions, we have two possibilities: open and closed strings. One finds that the general closed string mode expansion satisfying the wave equation $\Box X^\mu =$

0 has two sets of excitation modes, those of *right-movers* and *left-movers*. The general solution is

$$X^\mu = X_L^\mu + X_R^\mu \tag{2}$$

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s p^\mu(\tau - \sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-2in(\tau-\sigma)} \tag{3}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s p^\mu(\tau + \sigma) + \frac{i}{2}l_s \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \tag{4}$$

In the case of the open string mode expansion we find that the left-movers and right-movers combine into standing waves, leaving only a single set of modes. With Neumann boundary conditions the mode expansion for an open string is:

$$X^\mu = x^\mu + l_s p^\mu \tau + i l_s \sum_{n\neq 0} \frac{1}{n}\alpha_n^\mu e^{-in\tau}\cos(m\sigma) \tag{5}$$

In light-cone coordinates, the energy-momentum tensor's mode expansion takes the form:

$$T_{--} = 2l_s^2 \sum_{m=-\infty}^{+\infty} L_m e^{-2im(\tau-\sigma)} \tag{6}$$

$$T_{++} = 2l_s^2 \sum_{m=-\infty}^{+\infty} \tilde{L}_m e^{-2im(\tau+\sigma)} \tag{7}$$

where the Fourier coefficients L_m and $\tilde{L}_m$ are the generators of the Virasoro algebra. They are defined:

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \alpha_{m-n}\cdot\alpha_n \tag{8}$$

$$\tilde{L}_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty} \tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n \tag{9}$$

In the classical theory, the Virasoro mode operators obey the Poisson bracket relations:

$$[\alpha_m^\mu, \alpha_n^\nu]_{P.B.} = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu]_{P.B.} = im\eta^{\mu\nu}\delta_{m+n,0} \tag{10}$$

and

$$[\alpha^\mu_m, \tilde{\alpha}^\nu_n]_{P.B.} = 0 \tag{11}$$

The full *classical* Virasoro algebra (also known as the Witt algebra) is then determined entirely by its Poisson bracket relation on the generators:

$$[L_m, L_n]_{P.B.} = i(m-n)L_{m+n} \tag{12}$$

The quantum theory can be constructed from the classical theory with the method of canonical quantization. After making the replacement $[\cdot,\cdot]_{P.B.} \to i[\cdot,\cdot]$, that is, Poisson brackets become canonical commutators, we find that the mode operators satisfy

$$[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\eta^{\mu\nu}_{m+n,0} \tag{13}$$

and

$$[\alpha^\mu_m, \tilde{\alpha}^\nu_n] = 0 \tag{14}$$

In the the full quantum theory the Virasoro algebra reads

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{15}$$

where c is defined as the *central charge*.

Our task in the next three problems is to derive this explicitly. Henceforth, we take the case of an open string and reduce ourselves to a single set of modes and generators for the sake of calculation.

BBS Problem 2.13

In the quantum theory, the Virasoro mode operators must be normal-ordered. Letting $a^\mu_m = \frac{1}{\sqrt{m}}\alpha^\mu_m$ and $a^{\mu\dagger}_m = \frac{1}{\sqrt{m}}\alpha^\mu_{-m}$ for $m > 0$ be the usual lowering and raising operators for quantum harmonic oscillators, we need all lowering operators to the left of all raising operators according to the normal-ordering prescription. The normal-ordered generators read

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty} :\alpha_{m-n}\cdot\alpha_n: \tag{16}$$

Since normal-ordering issues only arise when $m+n=0$, we expect the central extension of the Virasoro algebra to take the form

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0} \tag{17}$$

In the event that $A(1) \neq 0$, one can redefine L_0 by adding a constant to fix $A(1) = 0$. Letting $A(1) = k \neq 0$ we have

$$[L_1, L_{-1}] = 2L_0 + A(1)\delta_{0,0} = 2L_0 + k \tag{18}$$

Then choosing $L_0^{'} = L_0 + \frac{1}{2}k$ we can fix the condition that $A(1) = 0$ seeing that

$$[L_1, L_{-1}] = 2L_0 + A(1)\delta_{0,0} = 2L_0^{'} \tag{19}$$

Dropping the prime on L_0, it is easy to verify that $L_{\pm 1}, L_0$ form a closed subalgebra by (19) and the following commutation relations:

$$[L_1, L_0] = L_1 + A(1)\delta_{1,0} = L_1 \tag{20}$$

$$[L_{-1}, L_0] = -L_{-1} + A(-1)\delta_{-1,0} = -L_{-1} \tag{21}$$

$$[L_m, L_n] = 0, \text{ when } m = n \tag{22}$$

BBS Problem 2.14

A general equation for the coefficients of the central extension term $A(m)$ can be dervied by using the Jacobi identity:

$$[[L_m, L_n], L_p] + [[L_n, L_p], L_m] + [[L_p, L_m], L_n] = 0 \tag{23}$$

Expanding we find that the first term gives

$$(m-n)[L_{m+n}, L_p] = (m-n)(m+n-p)L_{m+n+p} + (m-n)A(m+n)\delta_{m+n+p,0} \tag{24}$$

The second and third term are expanded in the same manner and the whole equation becomes two terms after grouping

$$\begin{aligned} &L_{m+n+p}((m-n)(m+n-p) + (n-p)(n+p-m) + (p-m)(p+m-n)) \\ &+ \delta_{m+n+p,0}((m-n)A(m+n) + (n-p)A(n+p) + (p-m)A(p+m)) = 0 \end{aligned} \tag{25}$$

The first term goes to zero after expanding the terms in m, n, and p. Were left with the need for the second term to vanish and this gives us a set of two equations

$$(m-n)A(m+n)+(n-p)A(n+p)+(p-m)A(p+m)=0 \tag{26}$$

$$m+n+p=0 \tag{27}$$

With these we can now prove the general form of the central extension's coefficients. We make the assumption that they take the form

$$A(m)=\frac{m(m^2-1)}{6}A(2) \tag{28}$$

Such an assumption is consistent with our result from Problem 2.13 that $A(1)$ can always be fixed to vanish by rescaling L_0 by a constant. Returning to the constraint equations for the coefficients, we can, without loss of generality, choose $p=0$ giving $m=-n$. In terms of m, the coefficient equation becomes

$$-mA(-m)-mA(m)+2mA(0)=0 \tag{29}$$

Using the assumption for $A(m)$ we have that $A(0)=0$ and this equation reduces to tell us that $A(m)=-A(-m)$ is satisfied for all coefficients $A(m)$. Returning to the general case and reducing the coefficient constraint equation once more by removing p with $p=-m-n$ we have:

$$(m-n)A(m+n)+(2n-m)A(-m)+(-2m-n)A(-n)=0 \tag{30}$$

Again without loss of generality we can choose that $n=1$ to find a relation between $A(m)$ and its successor $A(m+1)$

$$\begin{aligned}
&(m-1)A(m+1)+(2-m)A(-m)+(-2m-1)A(-1)=0\\
&(m-1)A(m+1)-(2-m)A(m)=0\\
&A(m+1)=\frac{(m+2)}{(m-1)}A(m)
\end{aligned} \tag{31}$$

From this we can now use mathematical induction to show indeed that $A(m)$ takes the assumed form for all $m\in\mathbb{Z}$. We have shown the base cases of $A(0)=A(1)=0$. Now we take the relation between $A(m)$ and $A(m+1)$ to give:

$$\begin{aligned}
A(m+1) &= \frac{(m+2)m(m^2-1)}{(m-1)}\frac{A(2)}{6} \\
&= \frac{(m^4+2m^3-m^2-2m)}{(m-1)}\frac{A(2)}{6} \\
&= (m^3+3m^2+2m)\frac{A(2)}{6} \\
&= (m+1)((m+1)^2-1)\frac{A(2)}{6}
\end{aligned} \tag{32}$$

Thus we have proven using induction that $A(m) = \frac{m(m^2-1)}{6}A(2)$ for all $m > 0$. However, remembering the relation $-A(m) = A(-m)$ that we previously derived we see that indeed it is true for all $m \in \mathbb{Z}$. Lastly, from the definition of the full quantum Virasoro algebra (15) the unique value for the central charge can be quoted as $c = 2A(2)$.

□

BBS Problem 2.15

We can verify the quantum Virasoro algebra (15) directly by plugging in the generators (16) and using the commutation relations for the mode operators. It is difficult to verify the central charge term in full generality with this method, so we begin with the cases of $m = -n = 1$, $m = -n = 2$ and make use of the previous results above.

First, we derive a useful relation:

$$\begin{aligned}
[L_m, \alpha_p^\mu] &= -[\alpha_p^\mu, \frac{1}{2}\sum_{k=-\infty}^{\infty}\alpha_{m-k}\cdot\alpha_k] \\
&= -\frac{1}{2}\sum_{k=-\infty}^{\infty}[\alpha_p^\mu, \alpha_{m-k}^\phi\alpha_k^\lambda\eta_{\phi\lambda}] \\
&= -\frac{1}{2}\sum_{k=-\infty}^{\infty}[\alpha_p^\mu, \alpha_{m-k}^\phi]\alpha_k^\lambda\eta_{\phi\lambda} + \frac{1}{2}\sum_{k=-\infty}^{\infty}\alpha_{m-k}^\phi[\alpha_k^\lambda, \alpha_p^\mu]\eta_{\phi\lambda} \\
&= -\frac{1}{2}\sum_{k=-\infty}^{\infty}(\alpha_{m-k}^\phi\alpha_p^\mu + (p)\delta_{p+m-k,0}\eta^{\phi\mu})\alpha_k^\lambda\eta_{\phi\lambda} + \frac{1}{2}\sum_{k=-\infty}^{\infty}\alpha_{m-k}^\phi\eta_{\phi\lambda}(\alpha_p^\mu\alpha_k^\lambda + (k)\delta_{k+p,0}\eta^{\mu\lambda}) \\
&= -\frac{1}{2}\sum_{k=-\infty}^{\infty}(p)\delta_{p+m-k,0}\eta^{\phi\mu}\alpha_k^\lambda\eta_{\phi\lambda} + \frac{1}{2}\sum_{k=-\infty}^{\infty}(k)\alpha_{m-k}^\phi\eta_{\phi\lambda}\delta_{p+k,0}\eta^{\mu\lambda} \\
&= -\frac{1}{2}(p)\alpha_{p+m}^\lambda\eta^{\phi\mu}\eta_{\phi\lambda} + \frac{1}{2}(-p)\alpha_{p+m}^\phi\eta_{\phi\lambda}\eta^{\mu\lambda} \\
&= (-p)\alpha_{p+m}^\mu
\end{aligned} \tag{33}$$

This relation helps expedite the cases of $m = -n = 1$ and $m = -n = 2$. We begin with the former and find

$$
\begin{aligned}
[L_1, L_{-1}] &= [L_1, \frac{1}{2}\sum_{p=-\infty}^{\infty} :\alpha_{-1-p}\cdot\alpha_p:] = [L_1, \sum_{p=0}^{\infty}\alpha_{-1-p}\cdot\alpha_p] \\
&= \sum_{p=0}^{\infty}[L_1, \alpha^{\mu}_{-1-p}]\alpha^{\nu}_p\eta_{\mu\nu} + \sum_{p=0}^{\infty}\alpha^{\mu}_{-1-p}[L_1, \alpha^{\nu}_p]\eta_{\mu\nu} \\
&= \sum_{p=0}^{\infty}(1+p)\alpha^{\mu}_{-p}\alpha^{\nu}_p\eta_{\mu\nu} + \sum_{p=0}^{\infty}(-p)\alpha^{\mu}_{-1-p}\alpha^{\nu}_{1+p}\eta_{\mu\nu} \\
&= \sum_{p=0}^{\infty}(1+p)\alpha^{\mu}_{-p}\alpha^{\nu}_p\eta_{\mu\nu} + \sum_{p=1}^{\infty}(-p+1)\alpha^{\mu}_{-p}\alpha^{\nu}_p\eta_{\mu\nu} \\
&= \alpha^{\mu}_0\alpha^{\nu}_0\eta_{\mu\nu} + 2\sum_{p=1}^{\infty}\alpha^{\mu}_{-p}\alpha^{\nu}_p\eta_{\mu\nu} \\
&= 2(\frac{1}{2}\alpha_0^2 + \sum_{p=1}^{\infty}\alpha_{-p}\cdot\alpha_p) = 2L_0
\end{aligned}
\tag{34}
$$

which is what we expect given that $A(1) = 0$. In the next case of $m = -n = 2$ we expect to find the value for $A(2)$ which all $A(m)$ are proportional to with $c = 2A(2)$. For this case we compute:

$$
\begin{aligned}
[L_2, L_{-2}] &= [L_2, \frac{1}{2}\sum_{p=-\infty}^{\infty} :\alpha_{-2-p}\cdot\alpha_p:] \\
&= [L_2, \frac{1}{2}\alpha^2_{-1} + \sum_{p=0}^{\infty}\alpha_{-2-p}\cdot\alpha_p] \\
&= [L_2, \frac{1}{2}\alpha^2_{-1}] + [L_2, \sum_{p=0}^{\infty}\alpha_{-2-p}\cdot\alpha_p]
\end{aligned}
\tag{35}
$$

The first term can be easily computed:

$$
\begin{aligned}
\frac{1}{2}[L_2, \alpha^{\mu}_{-1}\alpha^{\nu}_{-1}\eta_{\mu\nu}] &= \frac{1}{2}[L_2, \alpha^{\mu}_{-1}]\alpha^{\nu}_{-1}\eta_{\mu\nu} + \frac{1}{2}\alpha^{\mu}_{-1}[L_2, \alpha^{\nu}_{-1}]\eta_{\mu\nu} \\
&= \frac{1}{2}\alpha^{\mu}_1\alpha^{\nu}_{-1}\eta_{\mu\nu} + \frac{1}{2}\alpha^{\mu}_{-1}\alpha^{\nu}_1\eta_{\mu\nu} \\
&= \frac{1}{2}(\alpha^{\mu}_{-1}\alpha^{\nu}_1 + (1)\eta^{\mu\nu}\delta_{0,0})\eta_{\mu\nu} + \frac{1}{2}\alpha^{\mu}_{-1}\alpha^{\nu}_1\eta_{\mu\nu} \\
&= \alpha_{-1}\cdot\alpha_1 + \frac{\eta^{\mu}_{\mu}}{2}
\end{aligned}
\tag{36}
$$

The more extensive calculation for the second term gives:

$$
\begin{aligned}
[L_2, \sum_{p=0}^{\infty} \alpha^{\mu}_{-2-p}\alpha^{\nu}_{p}\eta_{\mu\nu}] &= \sum_{p=0}^{\infty}[L_2, \alpha^{\mu}_{-2-p}]\alpha^{\nu}_{p}\eta_{\mu\nu} + \sum_{p=0}^{\infty}\alpha^{\mu}_{-2-p}[L_2, \alpha^{\nu}_{p}]\eta_{\mu\nu} \\
&= \sum_{p=0}^{\infty}(2+p)\alpha^{\mu}_{-p}\alpha^{\nu}_{p}\eta_{\mu\nu} + \sum_{p=0}^{\infty}(-p)\alpha^{\mu}_{-2-p}\alpha^{\nu}_{2+p}\eta_{\mu\nu} \\
&= \sum_{p=0}^{\infty}(2+p)\alpha^{\mu}_{-p}\alpha^{\nu}_{p}\eta_{\mu\nu} + \sum_{p=2}^{\infty}(2-p)\alpha^{\mu}_{-p}\alpha^{\nu}_{p}\eta_{\mu\nu} \\
&= 2\alpha_0^2 + 3\alpha_{-1}\cdot\alpha_1 + 4\sum_{p=2}^{\infty}\alpha_{-p}\cdot\alpha_p \\
&= 4\sum_{p=0}^{\infty}\alpha_{-p}\cdot\alpha_p - \alpha_{-1}\cdot\alpha_1 \\
&= 4L_0 - \alpha_{-1}\cdot\alpha_1
\end{aligned}
\tag{37}
$$

Finally, combining these terms we find:

$$
[L_2, L_{-2}] = 4L_0 + \frac{\eta^{\mu}_{\mu}}{2}, \tag{38}
$$

which tells us that the central charge c is exactly the spacetime dimension $c = D = \eta^{\mu}_{\mu}$. And as such, we have proven explicitly that the full quantum Virasoro algebra for any D dimensional spacetime takes the general form:

$$
[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m+n,0} \tag{39}
$$

$\square$

BBS Problem 2.4

We preface this problem by quoting two results for the bosonic string in light-cone gauge. In the cases of the open and closed string, respectively, we have the following mass-shell conditions:

$$
\alpha^{'}M^2 = N - a \tag{40}
$$

and

$$
\alpha^{'}M^2 = 4(N-a) = 4(\tilde{N}-a) \tag{41}
$$

where N and $\tilde{N}$ are the number operators and the constant a is added to account for normal ordering ambiguities on the Virasoro zero mode L_0, which in the quantum theory is required to satisfy $(L_0 - a)|\phi\rangle = 0$, which can be thought of as the mass-shell condition. The text derives in various ways that $a = 1$ is the necessary value. The open string has only a single set of modes and thus only a single N. The N and $\tilde{N}$ for the closed string correspond to the oppositely moving sets of modes. They obey the so-called *level-matching condition* given by $N = \tilde{N}$, seen easily through the required difference $(L_0 - \tilde{L}_0)|\phi\rangle = 0$.

Given the following open string states:

$$
\begin{aligned}
|\phi_1\rangle &= \alpha^i_{-1}|0;k\rangle\,, \quad & |\phi_2\rangle &= \alpha^i_{-1}\alpha^j_{-1}|0;k\rangle \\
|\phi_3\rangle &= \alpha^i_{-3}|0;k\rangle\,, \quad & |\phi_4\rangle &= \alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-2}|0;k\rangle
\end{aligned}
$$

we can immediately see that the ith state above ($i \in \{1,2,3,4\}$) is excited i times and thus has $N = i$. The mass squared of each of these open-string states is

$$
\begin{aligned}
\alpha' M^2_{\phi_1} &= 0, \quad & \alpha' M^2_{\phi_2} &= 1 \\
\alpha' M^2_{\phi_3} &= 2 \quad & \alpha' M^2_{\phi_4} &= 3
\end{aligned}
$$

Now, the closed string states

$$
|\phi_1\rangle = \alpha^i_{-1}\tilde{\alpha}^j_{-1}|0;k\rangle\,, \quad |\phi_2\rangle = \alpha^i_{-1}\alpha^j_{-1}\tilde{\alpha}^k_{-2}|0;k\rangle
$$

clearly obey the level matching condition. Meanwhile, a state such as

$$
|\phi_3\rangle = \alpha^i_{-1}\tilde{\alpha}^j_{-2}|0;k\rangle
$$

with $1 = N \neq \tilde{N} = 2$ violates the level-matching condition. This third state is unphysical while the mass squared of the first two closed-string states is

$$
\alpha' M^2_{\phi_1} = 0, \quad \alpha' M^2_{\phi_2} = 4
$$

BBS Problem 2.6

The Lorentz generators of the open-string world-sheet are

$$
J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n) \tag{42}
$$

We want to use this along with the canonical commutation relations to verify the Poincare algebra, given by

$$\begin{aligned}[][p^\mu, p^\nu] &= 0\\ [p^\mu, J^{\nu\sigma}] &= -i\eta^{\mu\nu}p^\sigma + i\eta^{\mu\sigma}p^\nu\\ [J^{\mu\nu}, J^{\sigma\lambda}] &= -i\eta^{\nu\sigma}J^{\mu\lambda} + i\eta^{\mu\sigma}J^{\nu\lambda} + i\eta^{\nu\lambda}J^{\mu\sigma} - i\eta^{\mu\lambda}J^{\nu\sigma}\end{aligned}$$

The first commutation relation in the algebra relates the momentum operator as it is defined in momentum phase space. Since this commutation relation is equivalent in both momentum and position/configuration space, it can be verified by examining the position-space representation of the momentum operator p^μ, whose form can we found by using the Fourier transform. We find that $p^\mu \to -i\partial^\mu$ in position space. It follows that the first commutation relation can be written as $[p^\mu, p^\nu] = [-i\partial^\mu, -i\partial^\nu]$, which vanishes indeed because partials commute.

In the case of the second commutator, we first compute a helpful identity that relates the commutator of the momentum operator p^μ (henceforth we remain in position space) and the Virasoro mode operator α_m^ν. Noting that in the open string's mode expansion $\alpha_0^\mu = l_s p^\mu$, we expand to see that

$$\begin{aligned}[][p^\mu, \alpha_m^\nu] &= [l_s^{-1}\alpha_0^\mu, \alpha_m^\nu]\\ &= -l_s^{-1}[\alpha_m^\nu, \alpha_0^\mu]\\ &= -l_s^{-1}(m)\eta^{\mu\nu}\delta_{m+0,0} = 0\end{aligned} \tag{43}$$

which vanishes for all m. We begin expanding the second commutation relation in the algebra

$$\begin{aligned}[][p^\mu, J^{\nu\sigma}] &= p^\mu x^\nu p^\sigma - p^\mu x^\sigma p^\nu - p^\mu i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^\nu\alpha_n^\sigma - \alpha_{-n}^\sigma\alpha_n^\nu)\\ &\quad - x^\nu p^\sigma p^\mu + x^\sigma p^\nu p^\mu + i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^\nu\alpha_n^\sigma - \alpha_{-n}^\sigma\alpha_n^\nu)p^\mu\\ &= [p^\mu, x^\nu p^\sigma] - [p^\mu, x^\sigma p^\nu] - [p^\mu, i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^\nu\alpha_n^\sigma - \alpha_{-n}^\sigma\alpha_n^\nu)]\end{aligned} \tag{44}$$

The third term vanishes after expanding and using the identity (43) above. Each of the following are useful in expanding the only two terms that now remain: take the commutator identity

$$[B_1, B_2B_3] = [B_1, B_2]B_3 + B_2[B_1, B_3],$$

the canonical commutation relations $[x^\mu, p^\nu] = i\eta^{\mu\nu}$, and the first commutation relation of the Poincare algebra proved above. Utilzing these we find

$$\begin{aligned}
[p^\mu, J^{\nu\sigma}] &= [p^\mu, x^\nu]p^\sigma + x^\nu[p^\mu, p^\sigma] - [p^\mu, x^\sigma]p^\nu - x^\sigma[p^\mu, p^\nu] \\
&= [p^\mu, x^\nu]p^\sigma - [p^\mu, x^\sigma]p^\nu \\
&= -i\eta^{\mu\nu}p^\sigma + i\eta^{\mu\sigma}p^\nu
\end{aligned} \tag{45}$$

All that's left to show is the last commutation relation of the algebra. Similarly, we begin by quoting a few useful identities. First, using again that $\alpha_0^\mu = l_s p^\mu$ for the open string, we conclude that:

$$[x^\mu, \alpha_n^\nu] = i l_s \eta^{\mu\nu}\delta_{n,0} \tag{46}$$

This is useful for computing the following relation between the position operator and the Lorentz generators:

$$\begin{aligned}
[J^{\mu\nu}, x^\lambda] &= [x^\mu p^\nu, x^\lambda] - [x^\nu p^\mu, x^\lambda] - [i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^\mu\alpha_n^\nu - \alpha_{-n}^\nu\alpha_n^\mu), x^\lambda] \\
&= -[x^\lambda, x^\mu]p^\nu - x^\mu[x^\lambda, p^\nu] + [x^\lambda, x^\nu]p^\mu + x^\nu[x^\lambda, p^\mu] \\
&= -ix^\mu\eta^{\lambda\nu} + ix^\nu\eta^{\lambda\mu}
\end{aligned} \tag{47}$$

Lastly, the most extensive part of the algebra for this commutation comes from the terms which include the Virasoro modes. As such, we compute the very helpful relation:

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}[\alpha_{-n}^\mu\alpha_n^\nu, \alpha_{-m}^\sigma\alpha_m^\lambda] &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}(\alpha_{-n}^\mu\alpha_n^\nu\alpha_{-m}^\sigma\alpha_m^\lambda - \alpha_{-m}^\sigma\alpha_m^\lambda\alpha_{-n}^\mu\alpha_n^\nu) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}(\alpha_{-n}^\mu\alpha_{-m}^\sigma\alpha_n^\nu\alpha_m^\lambda - \alpha_{-m}^\sigma\alpha_{-n}^\mu\alpha_m^\lambda\alpha_n^\nu \\
&\qquad + (n)\alpha_{-n}^\mu\alpha_m^\lambda\eta^{\sigma\nu}\delta_{n-m,0} - (m)\alpha_{-m}^\sigma\alpha_n^\nu\eta^{\mu\lambda}\delta_{m-n,0}) \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}((-n)\alpha_n^\nu\alpha_m^\lambda\eta^{\sigma\mu}\delta_{-m-n,0} - (m)\alpha_{-m}^\sigma\alpha_{-n}^\mu\eta^{\nu\lambda}\delta_{m+n,0} + \\
&\qquad + (n)\alpha_{-n}^\mu\alpha_m^\lambda\eta^{\sigma\nu}\delta_{n-m,0} - (m)\alpha_{-m}^\sigma\alpha_n^\nu\eta^{\mu\lambda}\delta_{m-n,0}) \\
&= \sum_{m=1}^{\infty}\frac{1}{m}(\alpha_{-m}^\nu\alpha_m^\lambda\eta^{\sigma\mu} + \alpha_{-m}^\sigma\alpha_m^\mu\eta^{\nu\lambda} + \alpha_{-m}^\mu\alpha_m^\lambda\eta^{\sigma\nu} - \alpha_{-m}^\sigma\alpha_m^\nu\eta^{\mu\lambda})
\end{aligned} \tag{48}$$

With these identities fleshed out we are able to work out the entirety of the third commutation relation. It goes as

$$
\begin{aligned}
[J^{\mu\nu}, J^{\sigma\lambda}] &= [x^\mu p^\nu, J^{\sigma\lambda}] - [x^\nu p^\mu, J^{\sigma\lambda}] - [i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n), J^{\sigma\lambda}] \\
&= -[J^{\sigma\lambda}, x^\mu]p^\nu - x^\mu[J^{\sigma\lambda}, p^\nu] + [J^{\sigma\lambda}, x^\nu]p^\mu + x^\nu[J^{\sigma\lambda}, p^\mu] - [i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n), J^{\sigma\lambda}] \\
&= (ix^\sigma\eta^{\lambda\mu} - ix^\lambda\eta^{\sigma\mu})p^\nu + x^\mu(-i\eta^{\sigma\nu}p^\lambda + i\eta^{\lambda\nu}p^\sigma) + (-ix^\sigma\eta^{\lambda\nu} + ix^\lambda\eta^{\sigma\nu})p^\mu - x^\nu(-i\eta^{\sigma\mu}p^\lambda + i\eta^{\mu\lambda}p^\sigma) \\
&- [i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n), J^{\sigma\lambda}] \\
&= i\eta^{\lambda\mu}(x^\sigma p^\nu - x^\nu p^\sigma) + i\eta^{\sigma\mu}(x^\nu p^\lambda - x^\lambda p^\nu) + i\eta^{\lambda\nu}(x^\mu p^\sigma - x^\sigma p^\mu) + i\eta^{\sigma\nu}(x^\lambda p^\mu - x^\mu p^\lambda) - \\
&- [i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n), J^{\sigma\lambda}]
\end{aligned}
\tag{49}
$$

We now expand the fifth term on its own. After expanding, we make use of the identity found in (48). The expansion and simplification of term 5 above is

$$
\begin{aligned}
[-i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n), J^{\sigma\lambda}] &= [-i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n), -i\sum_{m=1}^{\infty}\frac{1}{m}(\alpha^\sigma_{-m}\alpha^\lambda_m - \alpha^\lambda_{-m}\alpha^\sigma_m)] \\
&= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}([\alpha^\mu_{-n}\alpha^\nu_n, \alpha^\sigma_{-m}\alpha^\lambda_m] - [\alpha^\mu_{-n}\alpha^\nu_n, \alpha^\lambda_{-m}\alpha^\sigma_m] - \\
&\qquad - [\alpha^\nu_{-n}\alpha^\mu_n, \alpha^\sigma_{-m}\alpha^\lambda_m] + [\alpha^\nu_{-n}\alpha^\mu_n, \alpha^\lambda_{-m}\alpha^\sigma_m]) \\
&= \sum_{m=1}^{\infty}\frac{1}{m}(\eta^{\nu\sigma}(\alpha^\mu_{-m}\alpha^\lambda_m - \alpha^\lambda_{-m}\alpha^\mu_m) - \eta^{\mu\sigma}(\alpha^\nu_{-m}\alpha^\lambda_m - \alpha^\lambda_{-m}\alpha^\nu_m) - \\
&\qquad - \eta^{\nu\lambda}(\alpha^\mu_{-m}\alpha^\sigma_m - \alpha^\sigma_{-m}\alpha^\mu_m) + \eta^{\mu\lambda}(\alpha^\nu_{-m}\alpha^\sigma_m - \alpha^\sigma_{-m}\alpha^\nu_m))
\end{aligned}
\tag{50}
$$

Plugging this into the full expansion of (49), we can verify, at long last, the third and final commutation relation of the Poincare algebra. Appropriately grouping terms we have

$$
\begin{aligned}
[J^{\mu\nu}, J^{\sigma\lambda}] &= i\eta^{\lambda\mu}(x^\sigma p^\nu - x^\nu p^\sigma) + \eta^{\lambda\mu}\sum_{m=1}^{\infty}\frac{1}{m}(\alpha^\nu_{-m}\alpha^\sigma_m - \alpha^\sigma_{-m}\alpha^\nu_m) \\
&+ i\eta^{\sigma\mu}(x^\nu p^\lambda - x^\lambda p^\nu) - \eta^{\sigma\mu}\sum_{m=1}^{\infty}\frac{1}{m}(\alpha^\nu_{-m}\alpha^\lambda_m - \alpha^\lambda_{-m}\alpha^\nu_m) \\
&+ i\eta^{\lambda\nu}(x^\mu p^\sigma - x^\sigma p^\mu) - \eta^{\lambda\mu}\sum_{m=1}^{\infty}\frac{1}{m}(\alpha^\mu_{-m}\alpha^\sigma_m - \alpha^\sigma_{-m}\alpha^\mu_m) \\
&+ i\eta^{\sigma\nu}(x^\lambda p^\mu - x^\mu p^\lambda) + \eta^{\lambda\mu}\sum_{m=1}^{\infty}\frac{1}{m}(\alpha^\nu_{-m}\alpha^\sigma_m - \alpha^\sigma_{-m}\alpha^\nu_m) \\
&= -i\eta^{\nu\sigma}J^{\mu\lambda} + i\eta^{\mu\sigma}J^{\nu\lambda} + i\eta^{\nu\lambda}J^{\mu\sigma} - i\eta^{\mu\lambda}J^{\nu\sigma},
\end{aligned}
\tag{51}
$$

and the verification of the Poincare algebra is complete.

BBS Problem 2.7

The bosonic closed string's spacetime coordinates are described collectively by (2), (3), and (4). To compute the angular-momentum (or Lorentz) generators we integrate:

$$J^{\mu\nu} = \int_0^\pi J_0^{\mu\nu} d\sigma = T \int_0^\pi (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu) d\sigma \tag{52}$$

where

$$\begin{aligned} X^\mu \dot{X}^\nu = {} & x^\mu p^\nu l_s^2 + l_s^4 p^\mu p^\nu \\ & + x^\mu l_s \sum_{n\neq 0}^{\infty} (\alpha_n^\nu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\nu e^{-2in(\tau+\sigma)}) \\ & + l_s^3 p^\mu \tau \sum_{n\neq 0}^{\infty} (\alpha_n^\nu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\nu e^{-2in(\tau+\sigma)}) \\ & + \frac{i}{2} l_s^3 \sum_{m\neq 0}^{\infty} \frac{1}{m} (\alpha_m^\mu e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^\mu e^{-2im(\tau+\sigma)}) p^\nu \\ & + \frac{i}{2} l_s^2 \sum_{m\neq 0}^{\infty} \sum_{n\neq 0}^{\infty} \frac{1}{m} (\alpha_m^\mu e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^\mu e^{-2im(\tau+\sigma)})(\alpha_n^\nu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\nu e^{-2in(\tau+\sigma)}) \end{aligned} \tag{53}$$

and $X^\nu \dot{X}^\mu$ is identical with $\mu \leftrightarrow \nu$. Upon integration w.r.t. to σ, the terms in $X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu$ proportional to $e^{-2ik(\tau\pm\sigma)}$ will vanish. In the case of the last term, however, we can simplify to see that some terms lose their $e^{-2ik(\tau\pm\sigma)}$ factor when $n + m = 0$. We expand, apply the Virasoro mode commutation relations, and sum over only the non-vanishing terms where $n = -m$ as follows

$$\begin{aligned} (\text{last term of } X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu) = {} & \frac{i}{2} l_s^2 \sum_{m\neq 0}^{\infty} \sum_{n\neq 0}^{\infty} \frac{1}{m} (\alpha_m^\mu \alpha_n^\nu e^{-2i(n+m)(\tau-\sigma)} + \tilde{\alpha}_m^\mu \tilde{\alpha}_n^\nu e^{-2i(n+m)(\tau+\sigma)}) \\ & + \frac{i}{2} l_s^2 \sum_{m\neq 0}^{\infty} \sum_{n\neq 0}^{\infty} \frac{1}{n} (\alpha_n^\nu \alpha_m^\mu e^{-2i(n+m)(\tau-\sigma)} + \tilde{\alpha}_n^\nu \tilde{\alpha}_m^\mu e^{-2i(n+m)(\tau+\sigma)}) \\ = {} & -\frac{i}{2} l_s^2 \sum_{m\neq 0}^{\infty} \frac{1}{m} (\alpha_{-m}^\mu \alpha_m^\nu - \alpha_{-m}^\nu \alpha_m^\mu + \tilde{\alpha}_{-m}^\mu \tilde{\alpha}_m^\nu - \tilde{\alpha}_{-m}^\nu \tilde{\alpha}_m^\mu) \end{aligned} \tag{54}$$

Combining all this, and ignoring terms which integrate to zero, we have

$$
\begin{aligned}
J^{\mu\nu} &= T\int_0^\pi (l_s^2(x^\mu p^\nu - x^\nu p^\mu) - \frac{i}{2}l_s^2\sum_{m\neq 0}^{\infty}\frac{1}{m}(\alpha_{-m}^\mu\alpha_m^\nu - \alpha_{-m}^\nu\alpha_m^\mu + \tilde{\alpha}_{-m}^\mu\tilde{\alpha}_m^\nu - \tilde{\alpha}_{-m}^\nu\tilde{\alpha}_m^\mu))d\sigma \\
&= x^\mu p^\nu - x^\nu p^\mu - \frac{i}{2}l_s^2\sum_{m\neq 0}^{\infty}\frac{1}{m}(\alpha_{-m}^\mu\alpha_m^\nu - \alpha_{-m}^\nu\alpha_m^\mu + \tilde{\alpha}_{-m}^\mu\tilde{\alpha}_m^\nu - \tilde{\alpha}_{-m}^\nu\tilde{\alpha}_m^\mu)
\end{aligned}
\tag{55}
$$

which is identical to the generators of angular momentum for the open string but with a factor of a half while accounting for both the right- and left-moving modes.

BBS Problem 2.8

The angular-momentum generators of the open string, defined above in (42) are appropriate for covariant quantization. We're interested in their formulas in the case of light-cone gauge quantization, so we begin with some remarks on this gauge choice.

As we remarked in the beginning, we used the diffeomorphism invariance and Weyl rescaling symmetries available to the bosonic string theory in order to fix the auxiliary world-sheet metric as flat, $h_{\alpha\beta} = \eta_{\alpha\beta}$, but we have not gauged fixed the theory entirely. There still exist residual reparameterizations that *are themselves* Weyl resclaings. These residual reparameterizations satisfy

$$\partial^\alpha\epsilon^\beta + \partial^\beta\epsilon^\alpha = \Lambda\eta^{\alpha\beta} \tag{56}$$

where ϵ^α is an infinitesimal parameter for a reparameterization and Λ is an infinitesimal parameter for a Weyl rescaling. A solution to this is given by

$$\epsilon^+ = \epsilon^+(\sigma^+) \text{ and } \epsilon^- = \epsilon^-(\sigma^-) \tag{57}$$

where $\epsilon^\pm = \epsilon^0 \pm \epsilon^1$ and $\sigma^\pm$ are the world-sheet light-cone coordinates. The light-cone gauge makes two definitions:

1. We begin by reparameterizing the string world-sheet coordinates by using the residual symmetries above. That is, $\sigma^\pm \to \epsilon^\pm(\sigma^\pm)$, and this transformation can be written explicitly as

$$\tilde{\tau} = \frac{1}{2}[\epsilon^+(\sigma^+) + \epsilon^-(\sigma^-)] \tag{58}$$

$$\tilde{\sigma} = \frac{1}{2}[\epsilon^+(\sigma^+) - \epsilon^-(\sigma^-)], \tag{59}$$

and it is noted that $\tilde{\tau}$ itself can solve the free massless wave equation:

$$(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2})\tilde{\tau} = 0 \tag{60}$$

2. Next, we reparameterize two of the D available spacetime coordinates as follows

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}) \tag{61}$$

The light-cone gauge now uses the residual freedom under these changes to make the following choice

$$\alpha^+_m = 0 \text{ for } m \neq 0, \tag{62}$$

which allows us to write

$$X^+(\tilde{\tau}, \tilde{\sigma}) = x^+ + l_s^2 p^+ \tilde{\tau}, \tag{63}$$

thereby eliminating the oscillator modes of X^+. By using the Virasoro constraints, see page 49, one can similarly determine the oscillator modes of X^- as well. The constraints allow us to solve for the modes of X^- completely in terms of the traverse oscillations as follows

$$X^- = x^- + l_s^2 p^- \tilde{\tau} + il_s \sum_{n\neq 0}^{\infty} \frac{1}{n}\alpha^-_n e^{-in\tilde{\tau}} cos(n\tilde{\sigma}) \tag{64}$$

with

$$\alpha^-_n = \frac{1}{p^+ l_s}(\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty} :\alpha^i_{n-m}\alpha^i_m: -a\delta_{n,0}) \tag{65}$$

where D is again the spacetime dimension and a is the normal-ordering constant referenced in Problem 2.4. The remaining $D-2$ transverse mode expansions take the form given in (5).

Returning now to the problem at hand, we note that the Lorentz generators of the open string (42) are given by a tensor equation. Since tensors are coordinate-independent, we can simply use the nice properties of the light-cone gauge's mode expansions and quote the results. We have

$$J^{+-} = x^+ p^- - x^- p^+ \tag{66}$$

$$J^{+i} = x^+ p^i - x^i p^+ \tag{67}$$

$$J^{-i} = x^- p^i - x^i p^- - i\sum_{n=1}^{\infty} \frac{1}{n}(\alpha^-_{-n}\alpha^i_n - \alpha^i_{-n}\alpha^-_n) \tag{68}$$

and lastly,

$$J^{ij} = x^i p^j - x^j p^i - i \sum_{p=1}^{\infty} \frac{1}{n} (\alpha_{-n}^i \alpha_n^j - \alpha_{-n}^j \alpha_n^i), \tag{69}$$

where we remark again for significance that the modes α_n^- can be completely written in terms of α_n^i and the normal-ordering constant a.

Preface to Chapter 3 Problems

This chapter begins with the notion of performing a Wick rotation $\tau \to -i\tau$ on the world-sheet coordinates to introduce a Euclidean metric signature, thereby making the world-sheet metric $h_{\alpha\beta}$ positive definite. Next one can define complex coordinates $z, \bar{z}$ on local patches to treat the world-sheet as a Riemann surface. The residual symmetries in the conformal gauge described in chapter 2 of BBS (and its solutions above) become conformal mappings $z \to f(z), \bar{z} \to \bar{f}(\bar{z})$, such that f is biholomorphic. This lays the foundation for introducing two-dimensional conformal field theory.

The generators of an infinitesimal conformal transformation of the form described above take the form

$$l_n = -z^{n+1}\partial \tag{70}$$

$$\bar{l}_n = -\bar{z}^{n+1}\bar{\partial} \tag{71}$$

defined for $n < -1$ on the punctured plane $\mathbb{C} - \{0\}$ and for $n > 1$ on the plane with the point at infinity removed $\mathbb{C} - \{\infty\}$. These generators satisfy the classical Virasoro algebras

$$[l_m, l_n] = (m-n)l_{m+n} \tag{72}$$

$$[\bar{l}_m, \bar{l}_n] = (m-n)\bar{l}_{m+n} \tag{73}$$

$$[l_m, \bar{l}_n] = 0 \tag{74}$$

The generators $l_0, l_{\pm 1}$ and $\bar{l}_0, \bar{l}_{\pm 1}$, all defined on the Riemann sphere, form a finite closed subalgebra of the entire infinite dimensional Virasoro algebra as discussed previously and proven in BBS Problem 2.13. In the quantum theory, the Virasoro algebra can acquire a *central extension*, as mathematicians would say, denoting that the central extension multiplies the identity element of the algebra. Physicists refer to this as a *conformal anomaly* given that it can be interpreted as the quantum mechanical breaking of the classical conformal symmetry. This leads to the familiar full form of the quantum Virasoro algebra (15) derived above in the the problems for BBS Chapter 2.

Of particular importance to the solutions below and the study of conformal field theory as a whole are a few other mathematical notions: conformal and/or primary fields of dimension $(h, \tilde{h})$, the operator product expansion, fermioninic/bosonic Faddeev-Popov ghosts fields, the BRST quantization, and more. We add notes and further detail as relevant per each problem.

BBS Problem 3.4

The Virasoro generators appear as the Fourier coefficients in the mode expansion of the energy momentum tensor $T_{\alpha\beta}$. By Weyl symmetry in the lightcone gauge we had that $T_{+-} = T_{-+} = 0$. After a Wick rotation, the lightcone indices $\pm$ are replaced by $(z, \bar{z})$. The conserved Noether current $\partial^\alpha T_{\alpha\beta} = 0$ in the complex positive definite metric signature world-sheet theory then gives the conservation conditions

$$\bar{\partial} T_{zz} = 0 \tag{75}$$

$$\partial T_{\bar{z}\bar{z}} = 0 \tag{76}$$

with $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial\bar{z}$. That is, the nonvanishing components of the energy-momentum tensor $T_{zz} = T(z)$, $T_{\bar{z}\bar{z}} = \tilde{T}(\bar{z})$ are holomophic and anti-holomorphic, respectively. Making the choice given by BBS as

$$z = e^{2(\tau - i\sigma)} \tag{77}$$

$$\bar{z} = e^{2(\tau + i\sigma)} \tag{78}$$

and choosing throughout this chapter to make the definition $l_s = \sqrt{2\alpha'} = 1$ unless otherwise made explicit, we have that the closed-string mode expansion takes the new form

$$X_R^\mu(z) = \frac{1}{2}x^\mu - \frac{i}{4}p^\mu \ln z + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu z^{-n} \tag{79}$$

$$X_L^\mu(\bar{z}) = \frac{1}{2}x^\mu - \frac{i}{4}p^\mu \ln \bar{z} + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu \bar{z}^{-n} \tag{80}$$

From this one can construct the (anti-)holomorphic derivatives

$$\partial X^{\mu}(z,\bar{z}) = -\frac{i}{2}\sum_{n=-\infty}^{\infty}\alpha_n^{\mu}z^{-n-1} \tag{81}$$

$$\bar{\partial} X^{\mu}(z,\bar{z}) = -\frac{i}{2}\sum_{n=-\infty}^{\infty}\tilde{\alpha}_n^{\mu}\bar{z}^{-n-1} \tag{82}$$

The (anti-)holomorphic components of the energy-momentum tensor follow immediately

$$T(z) = -\frac{1}{\alpha'} :\partial X\cdot\partial X: = \sum_{n=-\infty}^{\infty}\frac{L_n}{z^{n+2}} \tag{83}$$

$$\tilde{T}(\bar{z}) = -\frac{1}{\alpha'} :\bar{\partial} X\cdot\bar{\partial} X: = \sum_{n=-\infty}^{\infty}\frac{\tilde{L}_n}{\bar{z}^{n+2}} \tag{84}$$

Essential to this problem and others that follow is the notion of the operator product expansion (OPE). This is similar to the Taylor expansion of calculus, the polynomial expansion of a differentiable function about some fixed point which forms an approximation of the function up to arbitrary order. The OPE of quantum field theory is a similar mathematical notion related to the product of radially ordered field operators. For the sake of detailed exposition we quote the definition given by Polchinski in *String Theory: Volume 1.* Given two field operators $\mathcal{F}$ and $\mathcal{G}$, their normal ordered product can be expanded as such

$$:\mathcal{F}: :\mathcal{G}: = :\mathcal{F}\mathcal{G}: + \sum(\text{cross-contractions}) \tag{85}$$

where the sum over cross-contractions is given by replacing all possible field combinations (single pairs, double pairs, and so on) of pairs of fields in $\mathcal{F}$ and $\mathcal{G}$ with their individual operator expansions. Such a product is usually written to be ordered in a way that the most singular terms appear first and non-singular terms are left unspecified. After all, it is the singular points that are relevant for the sake of calculation, particularly in the interest of computing integrals in the complex plane. Of immediate relevance to Problem 3.4 is the complete operator expansion of the product of X^{μ} with itself at separate points z and w such that $|w| < |z|$:

$$:X^{\mu}(z,\bar{z}): :X^{\nu}(w,\bar{w}): = :X^{\mu}(z)X^{\nu}(w): -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(|z-w|^2) \tag{86}$$

Using the accepted prescription described above for writing this product one has

$$:X^{\mu}(z,\bar{z}): :X^{\nu}(w,\bar{w}): = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln(|z-w|^2) + \ldots \tag{87}$$

where the $+\dots$ denotes non-singular terms. Throughout we use the symbol $\sim$ to denote "equal up to singular terms", in which case the above becomes

$$:X^{\mu}(z,\bar{z}): :X^{\nu}(w,\bar{w}): \sim -\frac{\alpha^{'}}{2}\eta^{\mu\nu}\ln(|z-w|^2) \tag{88}$$

Under a conformal transformation that is a rescaling, one such that $z' = \zeta z$ for $\zeta \in \mathbb{C}$, a *conformal field* of dimension $(h, \tilde{h})$ is a field $\mathcal{A}$ that transforms according to

$$\mathcal{A}^{'}(z', \bar{z}^{'}) = \zeta^{-h}\bar{\zeta}^{-\tilde{h}}\mathcal{A}(z,\bar{z}) \tag{89}$$

The OPE of the energy-momentum tensor with such a field is given by

$$:T(z): :\mathcal{A}(w,\bar{w}):= \dots + \frac{h}{(z-w)^2}\mathcal{A}(w,\bar{w}) + \frac{1}{(z-w)}\partial\mathcal{A}(w,\bar{w}) + \dots \tag{90}$$

where the dots to the left denote terms more singular than $(z-w)^{-2}$ and the dots to the right denote nonsingular terms as discussed before. There is an analogous equation for the OPE of $\tilde{T}(\bar{z})\mathcal{A}(w,\bar{w})$ in terms of the weight $\tilde{h}$ and the conjugate coordinates $\bar{z}$ and $\bar{w}$.

A special case is when the OPE of the energy-momentum tensor with a conformal field operator, call it $\mathcal{O}$, has no terms more singular than $(z-w)^{-2}$. The OPE above then reads

$$:T(z): :\mathcal{O}(w,\bar{w}): = \frac{h}{(z-w)^2}\mathcal{O}(w,\bar{w}) + \frac{1}{(z-w)}\partial\mathcal{O}(w,\bar{w}) + \dots \tag{91}$$

Such a field operator is said to be a *tensor operator* or more commonly a *primary field* of dimension $(h, \tilde{h})$ as before. Under the conformal rescaling described above such a field operator transforms as

$$\mathcal{O}^{'}(z', \bar{z}^{'}) = \Big(\frac{\partial w}{\partial z}\Big)^{-h}\Big(\frac{\partial \bar{w}}{\partial \bar{z}}\Big)^{-\tilde{h}}\mathcal{O}(z,\bar{z}) \tag{92}$$

We make use of all this information to solve Problem 3.4 and its successors. For Problem 3.4, we use Polchinski's notation for the definition of the OPE (85) and the assignment $l_s = \sqrt{2\alpha'} = 1$ to write

$$\begin{aligned} :T(z): :X^{\mu}(w,\bar{w}):\ &= -2 :\partial X\cdot\partial X: :X^{\mu}(w,\bar{w}): \\ &= -2 :\partial X^{\nu}(z,\bar{z})\partial X_{\nu}(z,\bar{z})X^{\mu}(w,\bar{w}): + \\ &+ 2 :\partial X_{\nu}(z,\bar{z}): \big(\frac{1}{4}\eta^{\mu\nu}\partial\ln(|z-w|^2) + 2 :\partial X^{\nu}(z,\bar{z}): \big(\frac{1}{4}\eta^{\mu}_{\nu}\partial\ln(|z-w|^2\big) \\ &\sim \frac{1}{(z-w)} :\partial X^{\mu}(z,\bar{z}): = \frac{1}{(z-w)}\partial^{'}X^{\mu}(w,\bar{w}) \end{aligned} \tag{93}$$

where in third line we account for the cross-contractions and in the last line we normal order $\partial X^\mu(z,\bar{z})$ by Taylor expanding it to be expressed in terms of local operators at the point $z = w$. Comparing this OPE with the formula given in (91), we see immediately that $X^\mu(z,\bar{z})$ is a primary field of conformal dimension $(0,0)$.

BBS Problem 3.5

The OPE of the energy-momentum tensor with each of these field operators follows the same process as for Problem 3.4 but with added partial derivatives. Throughout we denote $\partial' = \partial/\partial w$ and $\bar{\partial}' = \partial/\partial\bar{w}$. For the field $\partial' X^\mu(w,\bar{w})$ we have

$$\begin{aligned} :T(z): :\partial' X^\mu(w,\bar{w}): &= -2 :\partial X \cdot \partial X: :\partial' X^\mu(w,\bar{w}): \\ &= -2 :\partial X(z,\bar{z}) \cdot \partial X(z,\bar{z}) \partial' X^\mu(w,\bar{w}): + \\ &+ 2 :\partial X^\mu(z,\bar{z}): (\frac{1}{4}\partial\partial' \ln(|z-w|^2) + 2 :\partial X^\mu(z,\bar{z}): (\frac{1}{4}\partial\partial' \ln(|z-w|^2) \\ &\sim -\partial \frac{1}{(z-w)} :\partial X^\mu(z,\bar{z}): = \frac{1}{(z-w)^2} :\partial X^\mu(z,\bar{z}): \\ &= \frac{1}{(z-w)^2}\Big(\partial' X^\mu(w,\bar{w}) + \partial'^2 X^\mu(w,\bar{w})(z-w)\Big) \\ &= \frac{1}{(z-w)^2}\partial' X^\mu(w,\bar{w}) + \frac{1}{(z-w)}\partial'^2 X^\mu(w,\bar{w}) \end{aligned} \tag{94}$$

which tells us that $\partial' X^\mu(w,\bar{w})$ is a primary field of conformal dimension $(1,0)$. Next, the OPE of $T(z)$ with the anti-holomorphic derivative of $X^\mu(w,\bar{w})$ goes as

$$\begin{aligned} :T(z): :\bar{\partial}' X^\mu(w,\bar{w}): &= -2 :\partial X \cdot \partial X: :\bar{\partial}' X^\mu(w,\bar{w}): \\ &= -2 :\partial X(z,\bar{z}) \cdot \partial X(z,\bar{z}) \bar{\partial}' X^\mu(w,\bar{w}): + \\ &+ 2 :\partial X^\mu(z,\bar{z}): (\frac{1}{4}\partial\bar{\partial}' \ln(|z-w|^2) + 2 :\partial X^\mu(z,\bar{z}): (\frac{1}{4}\partial\bar{\partial}' \ln(|z-w|^2) \\ &\sim \partial \frac{1}{(\bar{z}-\bar{w})} :\partial X^\mu(z,\bar{z}): = 0 \end{aligned} \tag{95}$$

which is entirely nonsingular, and thus, gives us no information about whether $\bar{\partial}' X^\mu(w,\bar{w})$ is a conformal field. However, we can compute the operator product expansion of the anti-holomorphic component of the energy-momentum tensor (84) with this field operator to give:

$$\begin{aligned}
:\tilde{T}(\bar{z})\colon :\bar{\partial}' X^\mu(w,\bar{w})\colon &= -2\,:\bar{\partial}X\cdot\bar{\partial}X\colon :\bar{\partial}' X^\mu(w,\bar{w})\colon \\
&= -2\,:\bar{\partial}X(z,\bar{z})\cdot\bar{\partial}X(z,\bar{z})\bar{\partial}' X^\mu(w,\bar{w})\colon\ + \\
&+ 2\,:\bar{\partial}X^\mu(z,\bar{z})\colon (\frac{1}{4}\bar{\partial}\bar{\partial}'\ln(|z-w|^2) + 2\,:\bar{\partial}X^\mu(z,\bar{z})\colon (\frac{1}{4}\bar{\partial}\bar{\partial}'\ln(|z-w|^2) \\
&\sim -\bar{\partial}\frac{1}{(\bar{z}-\bar{w})}\ :\partial X^\mu(z,\bar{z})\colon = \ \frac{1}{(\bar{z}-\bar{w})^2}\ :\partial X^\mu(z,\bar{z})\colon \\
&= \frac{1}{(\bar{z}-\bar{w})^2}\left(\partial' X^\mu(w,\bar{w}) + \partial'^2 X^\mu(w,\bar{w})(\bar{z}-\bar{w})\right) \\
&= \frac{1}{(\bar{z}-\bar{w})^2}\bar{\partial}' X^\mu(w,\bar{w}) + \frac{1}{(\bar{z}-\bar{w})}\bar{\partial}'^2 X^\mu(w,\bar{w})
\end{aligned} \tag{96}$$

This tells us indeed that $\bar{\partial}' X^\mu(w,\bar{w})$ is a primary field of conformal dimension $(0,1)$. For the second holomorphic derivative, $\partial'^2 X^\mu(w,\bar{w})$, we compute

$$\begin{aligned}
:T(z)\colon :\partial' X^\mu(w,\bar{w})\colon &= -2\,:\partial X\cdot\partial X\colon :\partial'^2 X^\mu(w,\bar{w})\colon \\
&= -2\,:\partial X(z,\bar{z})\cdot\partial X(z,\bar{z})\partial'^2 X^\mu(w,\bar{w})\colon\ + \\
&+ 2\,:\partial X^\mu(z,\bar{z})\colon (\frac{1}{4}\partial\partial'^2\ln(|z-w|^2) + 2\,:\partial X^\mu(z,\bar{z})\colon (\frac{1}{4}\partial\partial'^2\ln(|z-w|^2) \\
&\sim -\partial\partial'\frac{1}{(z-w)}\ :\partial X^\mu(z,\bar{z})\colon = -\partial\frac{1}{(z-w)^2}\ :\partial X^\mu(z,\bar{z})\colon \\
&= \frac{2}{(z-w)^3}\ :\partial X^\mu(z,\bar{z})\colon \\
&= \frac{2}{(z-w)^3}\left(\partial' X^\mu(w,\bar{w}) + \partial'^2 X^\mu(w,\bar{w})(z-w) + \frac{1}{2}\partial'^3 X^\mu(w,\bar{w})(z-w)^2\right) \\
&= \frac{2}{(z-w)^3}\partial' X^\mu(w,\bar{w}) + \frac{2}{(z-w)^2}\partial'^2 X^\mu(w,\bar{w}) + \frac{1}{(z-w)}\partial'^3 X^\mu(w,\bar{w})
\end{aligned} \tag{97}$$

which has the form given in (90) for a *nontensorial* conformal field of dimension $(2,0)$. Lastly, for the sake of clarity with a result given in the text, we derive the OPE of the energy-momentum tensor with itself. The calculation is

$$\begin{aligned}
:T(z)\colon :T(w)\colon &= 4\,:\partial X^\mu(z,\bar{z})\partial X_\mu(z,\bar{z})\colon :\partial' X^\nu(w,\bar{w})\partial' X_\nu(w,\bar{w})\colon \\
&= 4\,:\partial X^\mu(z,\bar{z})\partial X_\mu(z,\bar{z})\partial' X^\nu(w,\bar{w})\partial' X_\nu(w,\bar{w})\colon\ + \\
&(4)4\,:\partial X^\mu(z,\bar{z})\partial' X_\mu(w,\bar{w})\colon (-\frac{1}{4}\partial\partial'\ln(|z-w|))+ \\
&+ (2)4(-\frac{1}{4}\partial\partial'\ln(|z-w|))^2\eta^{\mu\nu}\eta_{\mu\nu} \\
&\sim \frac{D/2}{(z-w)^4} + 2(\frac{1}{(z-w)^2})\ :-2\partial X^\mu(z,\bar{z})\partial' X_\mu(w,\bar{w})\colon
\end{aligned} \tag{98}$$

where in the third line we've accounted for all four contributions of the single field pair contractions and in the fourth line we've accounted for both two field pair contractions. Finally, we simplify the normal ordered product of the energy-momentum tensor by Taylor expanding it so that all operators are multiplied locally at the point w. We make use of the reverse product rule which tells us that $\partial'^2 X^\mu(w,\bar{w})\partial' X_\mu(w,\bar{w}) = \frac{1}{2}\partial'\left(\partial' X^\mu(w,\bar{w})\partial' X_\mu(w,\bar{w})\right)$ under the relabeling of dummy indices. Under the normal ordered expansion the OPE becomes

$$
\begin{aligned}
&:T(z): :T(w): \sim \\
&\sim \frac{D/2}{(z-w)^4} + \frac{2}{(z-w)^2} :-2\partial' X^\mu(w,\bar{w})\partial' X_\mu(w,\bar{w}) - 2\partial'^2 X^\mu(w,\bar{w})\partial' X_\mu(w,\bar{w})(z-w): \\
&= \frac{D/2}{(z-w)^4} + \frac{2}{(z-w)^2}\left(:-2\partial' X^\mu(w,\bar{w})\partial' X_\mu(w,\bar{w}): + \frac{1}{2}\partial' :\big(-2X^\mu(w,\bar{w})\partial' X_\mu(w,\bar{w})\big): (z-w)\right) \\
&= \frac{D/2}{(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \frac{1}{(z-w)}\partial' T(w)
\end{aligned}
\tag{99}
$$

This tells us that the holomorphic component of the energy-momentum tensor does not transform as a primary field because of the contribution from the conformal anomaly proportional to the spacetime dimension $D = \eta^{\mu\nu}\eta_{\mu\nu}$. Nonetheless, it is a conformal field of dimension $(2,0)$. Likewise, the antiholomorphic component of the energy-momentum tensor $\tilde{T}(\bar{z})$ is a conformal field of dimension $(0,2)$. In the event that we can remove this conformal anomaly, the energy-momentum would transform as a tensor. We will see later on in superstring theory how adding Faddeev-Popov ghost fields to the action allows for the cancellation of the contribution to the conformal anomaly from the bosonic spacetime embedding fields X^μ and their fermionic superpartners ψ^μ in $D = 10$ dimensional spacetime.

BBS Problem 3.6

We previously derived the mode expansions for the (anti-)holomorphic derivatives of the spacetime embedding functions in (81) and (82). By using Cauchy's integral formula

$$
f(a) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-a} dz \tag{100}
$$

one can invert the mode expansions to give

$$
\alpha_m^\mu = \left(\frac{2}{\alpha'}\right)^{1/2} \frac{1}{2\pi}\oint dz\, z^m \partial X^\mu(z,\bar{z}) \tag{101}
$$

$$\tilde{\alpha}^{\mu}_{m} = (\frac{2}{\alpha'})^{1/2}\frac{1}{2\pi}\oint dz\, \bar{z}^{m}\bar{\partial}X^{\mu}(z,\bar{z}) \tag{102}$$

From this one can write the commutation relation $[\alpha^{\mu}_{m}, \alpha^{\nu}_{n}]$ as

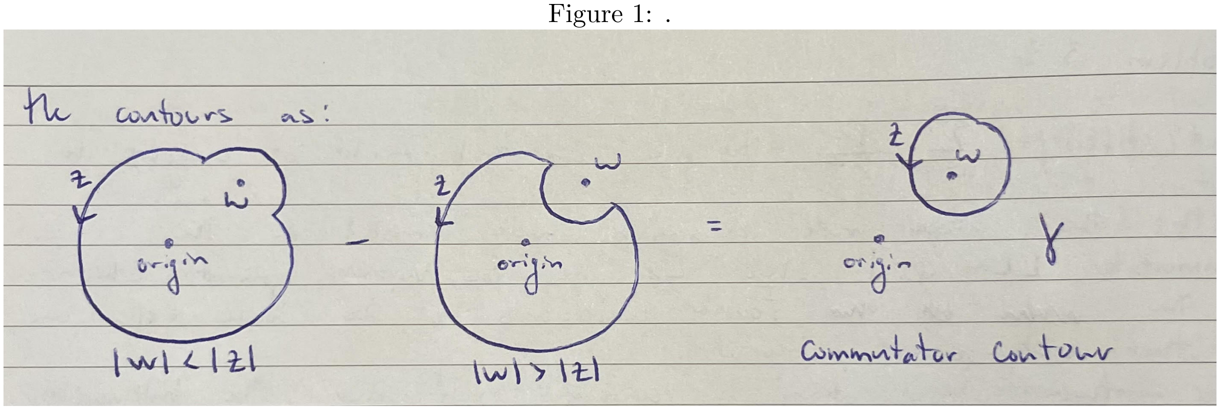
$$[(\frac{2}{\alpha'})^{1/2}\frac{1}{2\pi}\oint dz\, z^{m}\partial X^{\mu}(z,\bar{z}), (\frac{2}{\alpha'})^{1/2}\frac{1}{2\pi}\oint dw\, w^{n}\partial' X^{\nu}(w,\bar{w})] \tag{103}$$

Expanding we have

$$\frac{2}{\alpha'}\frac{1}{(2\pi)^2}\left(\oint dw\, w^{n}\oint dz\, z^{m}\partial X^{\mu}(z,\bar{z})\partial' X^{\nu}(w,\bar{w}) - \oint dw\, w^{n}\oint dz\, z^{m}\partial' X^{\nu}(w,\bar{w})\partial X^{\mu}(z,\bar{z})\right) \tag{104}$$

where we've chosen to compute the z integral first in each term. To compute these contour integrals more easily we can make use of the OPE $\partial X\partial' X$, but we must note that the OPE is defined to be radially ordered. As such, the first integral must be computed over a region of the complex plane such that $|w| < |z|$, and likewise over a region with $|z| < |w|$ for the second integral. We can construct such contours respectively as given in Figure 2, with the first integral's contour given as the first large perturbed circle on the left and the second's integral given as the second large perturbed circle on the right. By taking the difference between these two contours, we construct the z contour for the commutator encircling w given on the far right and denoted as γ.

Figure 1: .



The OPE of $\partial X\partial' X$ can be easily computed as

$$:\partial X^{\mu}(z,\bar{z}): :\partial' X\nu(w,\bar{w}): \sim -\frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-w)^2} \tag{105}$$

Returning to the form of the commutator we have

$$\begin{aligned}
[\alpha^\mu_m, \alpha^\nu_n] &= \frac{2}{\alpha'}\frac{1}{(2\pi)^2}\oint dw\, w^n \oint_\gamma dz\, z^m \big(- \frac{\alpha'}{2}\frac{\eta^{\mu\nu}}{(z-w)^2}\big) \\
&= \frac{i}{(2\pi)}\oint dw\, w^{n+m-1}\big(- m\eta^{\mu\nu}\big) \\
&= m\delta_{m+n,0}\eta^{\mu\nu}
\end{aligned} \tag{106}$$

where the integral in the second line vanishes unless there is a singularity in w, thereby requiring that $m+n=0$. The case of $[\alpha^\mu_m, \alpha^\nu_n]$ gives an identical result.

There is a simple route to show the vanishing of the commutator of different Virasoro mode operators $[\alpha^\mu_m, \tilde{\alpha}^\nu_n]$. The same process above is conducted under the use of Cauchy's integral formula to compute a contour integral of the commutator $[\partial X, \bar{\partial}' X]$. We saw earlier in Problem 3.5 that the OPE $T(z)\bar{\partial}' X(w,\bar{w})$ gave a completely nonsingular result. Given the definition of the $T(z)$ in (83) and the aforementioned nonsingular expansion, we see that $\partial X(z,\bar{z})\bar{\partial}' X(w,\bar{w})$ must also be nonsingular. Thus, following the same procedure as before we have that the expression

$$\begin{aligned}
[\alpha^\mu_m, \tilde{\alpha}^\nu_n] &= \frac{2}{\alpha'}\frac{1}{(2\pi)^2}\bigg(\oint d\bar{w}\, \bar{w}^n \oint dz\, z^m \partial X^\mu(z,\bar{z})\bar{\partial}' X^\nu(w,\bar{w}) - \oint d\bar{w}\, \bar{w}^n \oint dz\, z^m \bar{\partial}' X^\nu(w,\bar{w})\partial X^\mu(z,\bar{z})\bigg) \\
&= \frac{2}{\alpha'}\frac{1}{(2\pi)^2}\oint d\bar{w}\, \bar{w}^n \oint_\gamma dz\, z^m \big(\text{nonsingular OPE of } \partial X(z,\bar{z})\bar{\partial}' X(w,\bar{w})\big)
\end{aligned} \tag{107}$$

must vanish everywhere by Cauchy's theorem. This verifies that $[\alpha^\mu_m, \tilde{\alpha}^\nu_n] = 0$.

BBS Problem 3.7

Of particular importance to this problem is the notion of *highest weight states* and *state-operator correspondence*. That is, given a holomorphic primary field $\Phi(z)$ of conformal dimension h, a highest weight state satisfies

$$L_0 \left|\Phi\right\rangle = h \left|\Phi\right\rangle \tag{108}$$

and

$$L_n \left|\Phi\right\rangle = 0 \;\; \text{for} \;\; n > 0 \tag{109}$$

The definition of the state-operator correspondence is

$$\left|\Phi\right\rangle = \lim_{z\to 0}\Phi(z)\left|0\right\rangle \tag{110}$$

The extension to the closed-string case for both of the above is straight forward. We can write the mode expansion of a primary field $\Phi(z)$ of conformal dimension h as

$$\Phi(z) = \sum_{n=-\infty}^{\infty} \frac{\Phi_n}{z^{n+h}} \tag{111}$$

In this problem we want to compute the commutator $[L_m, \Phi_n]$. We begin by using Cauchy's integral formula to invert the mode expansions for the primary field and the energy-momentum tensor. We have

$$L_m = \oint \frac{dz}{2\pi i} z^{m+1} T(z) \tag{112}$$

and

$$\Phi_n = \oint \frac{dw}{2\pi i} w^{n+h-1} \Phi(w) \tag{113}$$

Thus, we compute

$$\begin{aligned} [L_m, \Phi_n] &= [\oint \frac{dz}{2\pi i} z^{m+1} T(z), \oint \frac{dw}{2\pi i} w^{n+h-1} \Phi(w)] \\ &= \oint \frac{dw}{2\pi i} w^{n+h-1} \oint_\gamma \frac{dz}{2\pi i} \left(\frac{h}{(z-w)^2} \Phi(w) + \frac{1}{(z-w)} \partial \Phi(w) \right) \end{aligned} \tag{114}$$

where we have combined the integrals by constructing the contour γ given above in Figure 1 and used the OPE expansion of the energy-momentum tensor with a primary field of conformal dimension h to rewrite the integrand. This is the same procedure we used in Problem 3.6. Computing the z integral we have

$$\oint \frac{dw}{2\pi i} (m+1) w^{n+h-1} \tag{115}$$

Solutions and Notes:

A First Course in String Theory

Barton Zwiebach

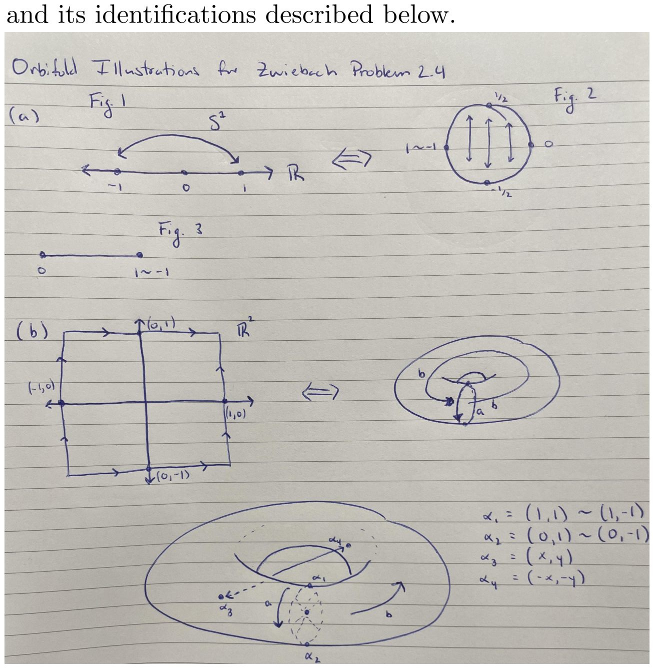
John Graham Reynolds

Preface to Zwiebach's A First Course in String Theory

Contained within are solutions and notes to a variety of problems from Zwiebach's introductory string theory text, geared towards advanced undergradutes at MIT. This text is not as rigorous as BBS or Polchinksi, but it is filled with wonderfully concise detail and many rewarding problems. I have primarily used this text as a third companion reference to the texts above while learning string theory, and yet, in that time, I have stumbled into a number of very interesting and exciting problems that caught my eye. They are solved as follows.

Zwiebach Problem 2.4

Figure 2: Part (a) shows the orbifold for $S^1/\mathbb{Z}_2$ and its identifications described below. Part (b) similarly shows the orbifold $T^2/\mathbb{Z}_2$ and its identifications described below.



We examine some simple topological structures known as orbifolds. In the first problem, part (a), we are given the circle S^1 defined on the real line $\mathbb{R}$ with the identication $x \sim x+2$ and the fundamental domain $-1 \leq x \leq 1$. We are interested in the orbifold given by the quotient $S^1/\mathbb{Z}_2$, which has a $\mathbb{Z}_2$ symmetry under the identification $x \sim -x$. One can imagine this identification as given in Figure 2 part (a), shown to be equivalent to the familiar circle. By imposing the $\mathbb{Z}_2$ identification, shown in the lines identifying opposite points within the circle, the result is simply a line segment of fundamental domain $(-1 \leq x \leq 0) \sim (0 \leq x \leq 1)$ where it is understood that $x \sim -x$ and either choice is valid for the fundamental domain. It is immediate that there are two fixed points $-1 \sim 1$ and $0 \sim 0$, fixing the endpoints of the resulting line segment orbifold.

The second problem, part (b), is more interesting. We begin with the torus, T^2, constructed from the real plane $\mathbb{R}^2$ and the identifications $x \sim x+2$ and $y \sim y+2$. We choose a fundamental domain $-1 < x, y \leq 1$. Figure 2 part (b) shows how to construct the torus from identifying opposite edges according to the orientation of the arrows on the square show in the (x, y) plane. The torus, being a genus-1 surface, has a canonical basis of 2 one-cycles around the surface, denoted by the arrows a and b in the equivalent picture to the right of the (x, y) plane. Taking the one-cycle a to run around the x coordinates and the one-cycle b to run around the y coordinates, one can label a number of points to see how the the ultimate $\mathbb{Z}_2$ identification $(x, y) \sim (-x, -y)$ will mold T^2 into a different space. The point α_1 in Figure 2 can be labeled as $(1, 1) \sim (1, -1)$ in correlation with the defined orientation of the one-cycles. As such, the point α_2, halfway around the one-cycle a, is $(0, 1) \sim (0, -1)$. Just as we traversed from α_1 to α_2, one can travel around the torus to any other point on the surface. Picking an arbitrary point, say $\alpha_3 = (x, y)$, we can traverse T^2 half way around each of the one-cycles to the point opposite this, $\alpha_4 = (-x, -y)$. The $\mathbb{Z}_2$ identification maps these two points together for all x, y. Such an identification leaves four (x, y) points fixed:

$$
\begin{gathered}
(0,0) \sim (0,0) \\
(0,1) \sim (0,-1) \\
(1,0) \sim (-1,0) \\
(1,1) \sim (1,-1) \sim (-1,1) \sim (-1,-1)
\end{gathered}
$$

To see the molding of the remaining points under the $\mathbb{Z}_2$ identification, one can visualize points along the interior of one half of T^2 connecting to points on the exterior on the other half of T^2, passing, in the process, through the opposite interior wall. The exception to this are the points of the form $(0, y) \sim (0, -y)$, $(1, y) \sim (-1, -y) \sim (1, -y) \sim (-1, y)$, $(x, 0) \sim (-x, 0)$, and $(x, 1) \sim (-x, -1) \sim (-x, 1), \sim (x, -1)$. This process can be thought of as turning the torus inside-out, removing its genus, and leaving behind 4 tight seams (tight in the sense that we have crossed smooth spaces over one another sharply here) along the lines of exception mentioned above which connect the fixed points as corners (vertices). The final space $T^2/\mathbb{Z}_2$ is a closed pillow with four corners (vertices), each connected by a tight seam to two other corners. The author is not nearly artistic enough to draw such a shape.

Zwiebach Problem 3.6

This problem is concerned with the analytic continuation of the gamma function. The gamma function is commonly interpreted as the extension of the factorial. With $n \in \mathbb{Z}^+$,

$$\Gamma(n) = (n-1)! \tag{116}$$

Bernoulli showed that there exists a convergent complex valued integral representation for the gamma function. Explicitly, for some complex argument z with $\Re(z) > 0$

$$\Gamma(z) = \int_0^\infty dt e^{-t} t^{z-1} \tag{117}$$

By breaking the integral into two separate integrals like so

$$\Gamma(z) = \int_0^1 dt e^{-t} t^{z-1} + \int_1^\infty dt e^{-t} t^{z-1} \tag{118}$$

and using the Taylor expansion for the exponential in the first integral, one can pull out the first N terms of the expansion. We demonstrate this by computing the first integral and letting $z = a + ib$ be a complex number such that $a > 0$, that is, $\Re(z) > 0$ as before

$$\int_0^1 dt(t^{z-1})e^{-t} = \int_0^1 dt(t^{z-1}) \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} = \sum_{n=0}^{\infty} \int_0^1 \frac{dt}{n!}(-1)^n (t^{n+a+ib-1}) \tag{119}$$

When evaluated, this gives

$$\int_0^1 dt(t^{z-1})e^{-t} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{z+n} \tag{120}$$

Now, removing the first N terms of the Taylor expansion in the first integral and adding their integrated value outside the integral allows one to completely rewrite the gamma function as

$$\Gamma(z) = \int_0^1 dt(t^{z-1})(e^{-t} - \sum_{n=0}^{N} \frac{(-t)^n}{n!}) + \sum_{n=0}^{N} \frac{(-1)^n}{n!} \frac{1}{z+n} + \int_1^\infty dt(t^{z-1})e^{-t} \tag{121}$$

The first integral can then be simplified to read as only the Taylor expansion of the exponential inclusive of terms higher than N

$$\Gamma(z) = \int_0^1 dt(t^{z-1})(\sum_{n=N+1}^{\infty} \frac{(-t)^n}{n!}) + \sum_{n=0}^{N} \frac{(-1)^n}{n!} \frac{1}{z+n} + \int_1^\infty dt(t^{z-1})e^{-t} \tag{122}$$

For a complex number $z = a + ib$, the first integral here can be written concisely as

$$\sum_{n=N+1}^{\infty}\int_0^1 dt\frac{(-1)^n}{n!}t^{a+ib+n-1} = \sum_{n=N+1}^{\infty}\frac{(-1)^n}{n!}\frac{t^{a+ib+n}}{(a+ib+n)}\bigg|_{t=0}^{t=1} \tag{123}$$

In order that this definite integral is well defined on the bounds, we need $a+n>0$. Since the sum begins with $n=N+1$, we need $a+N+1>0 \Rightarrow a>-N-1$. As such, this form of the gamma function (121) constitutes the analytic continuation of the function to the complex values z such that $\Re(z)>-N-1$. It is immediate based on this form that the analytic continuation of the gamma function has simple poles located at $z=0,-1,-2,...,-N$.

Lastly, we compute the residue at one of these simple poles. For simple poles at the points $z=\alpha$, we can use the formula

$$\mathrm{Res}(f,\alpha)=\lim_{z\to\alpha}(z-\alpha)f(z) \tag{124}$$

In our case, we have poles at $z=-n$ where $n\in\mathbb{Z}^+$, which implies:

$$\mathrm{Res}(\Gamma,-n)=\lim_{z\to -n}(z+n)\Gamma(z)=\frac{(-1)^n}{n!} \tag{125}$$

Zwiebach Problem 12.4

This problem is concerned with the analytic continuation of the Riemann zeta function, making use of the analytically continued gamma function in the problem above. The Riemann zeta function is traditionally defined for complex numbers s with $\Re(s)>1$ as

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s} \tag{126}$$

We begin by noting that the product of the gamma function and the zeta function can be written simply as

$$\zeta(s)\Gamma(s)=\int_0^{\infty}dt\frac{t^{s-1}}{e^t-1} \tag{127}$$

defined for $\Re(s)>1$. This can be seen by first beginning with Bernoulli's complex formula of the gamma function (defined for $\Re(s)>0$) given above in (117) and letting $t\to nt$

$$\Gamma(s)\to n\int_0^{\infty}dt\, e^{-nt}t^{s-1}n^{s-1}=\int_0^{\infty}dt\, e^{-nt}t^{s-1}n^s \tag{128}$$

Now we can multiply this by the zeta function and use the result

$$\sum_{n=1}^{\infty} e^{-nt} = \frac{1}{e^t - 1} \tag{129}$$

to give us the desired formula

$$\zeta(s)\Gamma(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \int_0^{\infty} dt\, e^{-nt} t^{s-1} n^s = \int_0^{\infty} dt (\sum_{n=1}^{\infty} e^{-nt}) t^{s-1} = \int_0^{\infty} dt \frac{t^{s-1}}{e^t - 1} \tag{130}$$

We further examine the convergent exponential sum given in (129) and examine its small t behavior. That is, when $t \approx 0$. We use the Taylor expansion to give:

$$f(t) = \frac{1}{e^t - 1} = \frac{1}{(\sum_{n=0}^{\infty} \frac{t^n}{n!}) - 1} = \frac{1}{(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots)} \tag{131}$$

Now we Taylor expand $f(t)$ around the point $t \approx 0$ to terms up to first order in t. For the zeroth order contribution to the Taylor expansion of (131) truncated to relevant order in t we have, after a partial fraction decomposition,

$$f(t)\Big|_{t\approx 0} = \frac{1}{t} + \frac{-3 - t}{6 + 3t + t^2} \approx \frac{1}{t} - \frac{1}{2} \tag{132}$$

Ignoring terms second order and beyond in t, the first order contribution to $f(t)$ after another partial fraction decomposition to relevant order is

$$f'(t)\Big|_{t\approx 0} = \frac{-1}{t^2} - \frac{3(1+t)}{(6+3t+t^2)^2} + \frac{1}{6+3t^2} \approx \frac{1}{12} \tag{133}$$

The resulting small t expansion is given as

$$f(t) = \frac{1}{e^t - 1} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} + \mathcal{O}(t^2) \tag{134}$$

To much higher order one can verify the Taylor expansion of (129) about $t \approx 0$ as

$$f(t) = \frac{1}{e^t - 1} = \frac{1}{t} - \frac{1}{2} + \frac{t}{12} - \frac{t^3}{720} + \frac{t^5}{30240} + \mathcal{O}(t^7) \tag{135}$$

which is clearly in agreement with the second order calculation above.

Using then the same process as in Zwiebach Problem 3.6 to separate the integral formula of the gamma function into two integrals of connected bound and "integrating out" the first few terms of Taylor expanded exponential function in the integrand, we can rewrite the product of the gamma and zeta functions in (127) as follows

$$\zeta(s)\Gamma(s) = \int_0^1 dt\, t^{s-1}(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} - \frac{t}{12}) + \frac{1}{s-1} - \frac{1}{2s} + \frac{1}{12(s+1)} + \int_1^{\infty} dt \frac{t^{s-1}}{e^t - 1} \tag{136}$$

The definite integrals of the leading order terms in the Taylor expansion of (129) given by

$$\int_0^\infty dt\, t^{s-1}(-\frac{1}{t}+\frac{1}{2}-\frac{t}{12}) = -\frac{1}{s-1}+\frac{1}{2s}-\frac{1}{12(s+1)} \tag{137}$$

require, respectively, that $\Re(s) > 1$, $\Re(s) > 0$, and $\Re(s) > -1$. Given that we are integrating out or removing these factors from the first integral, it comes as an immediate result that the product (136) is defined up to $\Re(s) > -2$, or in general up to $\Re(s) > -n$ where n is the order to which we expand the function $\frac{1}{e^t-1}$ and integrate out its terms from the first integral along the region $t \in [0,1]$.

Lastly, we can use the previously derived pole structure of the gamma function to determine the values assigned to the zeta function under its analytic continuation. $\Gamma(s)$ has simple poles at $s = 0, -1, -2, ...$, so it's inverse $\frac{1}{\Gamma(s)}$ has zeroes at these points. One can divide the product in (136) by $\Gamma(s)$ to give

$$\zeta(s) = \frac{1}{\Gamma(s)}\left(\int_0^1 dt\, t^{s-1}(\frac{1}{e^t-1}-\frac{1}{t}+\frac{1}{2}-\frac{t}{12}) + (\frac{1}{s-1}-\frac{1}{2s}+\frac{1}{12(s+1)}) + \int_1^\infty dt\frac{t^{s-1}}{e^t-1}\right) \tag{138}$$

Selecting s as one of the gamma function's poles causes all terms within the parentheses that aren't a pole at the same location to vanish. For terms in the parentheses that are poles at s, the gamma's zero and the parentheses' singular term cancel one another. As a remainder one is left with an inverse factor of the residue of the gamma function at that pole times the coefficient of the term within the parentheses. That is, for $s = 0$ and $s = -1$

$$\zeta(0) = \frac{0!x}{(-1)^0}(-\frac{1}{2}) = -\frac{1}{2} \tag{139}$$

$$\zeta(-1) = \frac{1!}{(-1)^1}(\frac{1}{12}) = -\frac{1}{12} \tag{140}$$

Rewriting these values assigned to the analytically continued zeta function at $s = 0$ and $s = -1$ in terms of the original definition of the zeta function (126) gives the mind-bending results

$$\zeta(0) = \sum_{n=1}^{\infty}\frac{1}{n^0} = 1+1+1+1+\cdots = -\frac{1}{2} \tag{141}$$

$$\zeta(-1) = \sum_{n=1}^{\infty} n = 1+2+3+4+\cdots = -\frac{1}{12} \tag{142}$$