Theoretical, Mathematical Physics
Solutions and Notes

String Theory and M-Theory

K. Becker, M. Becker, J. H. Schwarz

John Graham Reynolds

Preface to Chapter 2 Problems

First, we derive the full form of the quantum Virasoro algebra, the central extension of the Witt (or classical Virasoro) algebra, following along with the closely related problems 2.13, 2.14, 2.15. We then solve a number of other problems from the chapter, adding notes and remarks for a bit of exposition where beneficial.

In this preliminary section of the book, we study the unphysical (or, rather, *physically incomplete*) theory of bosonic strings. There are a number of equivalent actions for this theory, but we take the most useful (most easily quantized) *string sigma model action* as our starting point. With an auxiliary world-sheet metric $h_{\alpha\beta}$, string spacetime embedding functions $X^\mu(\tau,\sigma)$, and string tension T, related to the string scale l_s by $T=(\pi l_s^2)^{-1}$, the string sigma model action reads:

$$S_\sigma = \frac{T}{2}\int d\sigma d\tau \sqrt{-h} h^{\alpha\beta} g_{\mu\nu}\partial_\alpha X^\mu \partial_\beta X^\nu \tag{1}$$

To semi-simplify matters we make the (as it turns out, incomplete) gauge choice $h_{\alpha\beta}=\eta_{\alpha\beta}$ and furthermore choose our spacetime to be Minkowskian with $g_{\alpha\beta}=\eta_{\alpha\beta}$. Gauge fixing the worldsheet metric like so is only possible if there are no topological obstructions. That is, the worldsheet requires vanishing Euler characteristic.

Up to choice of boundary conditions, we have two possibilities: open and closed strings. One finds that the general closed string mode expansion satisfying the wave equation $\Box X^\mu = 0$ has two sets of excitation modes, those of *right-movers* and *left-movers*. The general solution is

$$X^\mu = X_L^\mu + X_R^\mu \tag{2}$$

$$X_R^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s p^\mu(\tau-\sigma) + \frac{i}{2}l_s \sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu e^{-2in(\tau-\sigma)} \tag{3}$$

$$X_L^\mu = \frac{1}{2}x^\mu + \frac{1}{2}l_s p^\mu(\tau+\sigma) + \frac{i}{2}l_s \sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^\mu e^{-2in(\tau+\sigma)} \tag{4}$$

In the case of the open string mode expansion we find that the left-movers and right-movers combine into standing waves, leaving only a single set of modes. With Neumann boundary conditions the mode expansion for an open string is:

$$X^\mu = x^\mu + l_s p^\mu \tau + i l_s \sum_{n \neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(m\sigma) \tag{5}$$

In light-cone coordinates, the energy-momentum tensor's mode expansion takes the form:

$$T_{--} = 2l_s^2 \sum_{m=-\infty}^{+\infty} L_m e^{-2im(\tau-\sigma)} \tag{6}$$

$$T_{++} = 2l_s^2 \sum_{m=-\infty}^{+\infty} \tilde{L}_m e^{-2im(\tau+\sigma)} \tag{7}$$

where the Fourier coefficients L_m and $\tilde{L}_m$ are the generators of the Virasoro algebra. They are defined:

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \alpha_{m-n} \cdot \alpha_n \tag{8}$$

$$\tilde{L}_m = \frac{1}{2} \sum_{n=-\infty}^{+\infty} \tilde{\alpha}_{m-n} \cdot \tilde{\alpha}_n \tag{9}$$

In the classical theory, the Virasoro mode operators obey the Poisson bracket relations:

$$[\alpha_m^\mu, \alpha_n^\nu]_{P.B.} = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu]_{P.B.} = im\eta^{\mu\nu}\delta_{m+n,0} \tag{10}$$

and

$$[\alpha_m^\mu, \tilde{\alpha}_n^\nu]_{P.B.} = 0 \tag{11}$$

The full *classical* Virasoro algebra (also known as the Witt algebra) is then determined entirely by its Poisson bracket relation on the generators:

$$[L_m, L_n]_{P.B.} = i(m-n)L_{m+n} \tag{12}$$

The quantum theory can be constructed from the classical theory with the method of canonical quantization. After making the replacement $[\cdot,\cdot]_{P.B.} \to i[\cdot,\cdot]$, that is, Poisson brackets become canonical commutators, we find that the mode operators satisfy

$$[\alpha^\mu_m, \alpha^\nu_n] = [\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n] = m\eta^{\mu\nu}_{m+n,0} \tag{13}$$

and

$$[\alpha^\mu_m, \tilde{\alpha}^\nu_n] = 0 \tag{14}$$

In the the full quantum theory the Virasoro algebra reads

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \tag{15}$$

where c is defined as the *central charge*.

Our task in the next three problems is to derive this explicitly. Henceforth, we take the case of an open string and reduce ourselves to a single set of modes and generators for the sake of calculation.

Problem 2.13

In the quantum theory, the Virasoro mode operators must be normal-ordered. Letting $a^\mu_m = \frac{1}{\sqrt{m}}\alpha^\mu_m$ and $a^{\mu\dagger}_m = \frac{1}{\sqrt{m}}\alpha^\mu_{-m}$ for $m > 0$ be the usual lowering and raising operators for quantum harmonic oscillators, we need all lowering operators to the left of all raising operators according to the normal-ordering prescription. The normal-ordered generators read

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{+\infty} :\alpha_{m-n}\cdot\alpha_n: \tag{16}$$

Since normal-ordering issues only arise when $m+n=0$, we expect the central extension of the Virasoro algebra to take the form

$$[L_m, L_n] = (m-n)L_{m+n} + A(m)\delta_{m+n,0} \tag{17}$$

In the event that $A(1) \neq 0$, one can redefine L_0 by adding a constant to fix $A(1) = 0$. Letting $A(1) = k \neq 0$ we have

$$[L_1, L_{-1}] = 2L_0 + A(1)\delta_{0,0} = 2L_0 + k \tag{18}$$

Then choosing $L_0^{'} = L_0 + \frac{1}{2}k$ we can fix the condition that $A(1) = 0$ seeing that

$$[L_1, L_{-1}] = 2L_0 + A(1)\delta_{0,0} = 2L_0^{'} \tag{19}$$

Dropping the prime on L_0, it is easy to verify that $L_{\pm 1}, L_0$ form a closed subalgebra by (19) and the following commutation relations:

$$[L_1, L_0] = L_1 + A(1)\delta_{1,0} = L_1 \tag{20}$$

$$[L_{-1}, L_0] = -L_{-1} + A(-1)\delta_{-1,0} = -L_{-1} \tag{21}$$

$$[L_m, L_n] = 0, \text{ when } m = n \tag{22}$$

Problem 2.14

A general equation for the coefficients of the central extension term $A(m)$ can be dervied by using the Jacobi identity:

$$[[L_m, L_n], L_p] + [[L_n, L_p], L_m] + [[L_p, L_m], L_n] = 0 \tag{23}$$

Expanding we find that the first term gives

$$(m-n)[L_{m+n}, L_p] = (m-n)(m+n-p)L_{m+n+p} + (m-n)A(m+n)\delta_{m+n+p,0} \tag{24}$$

The second and third term are expanded in the same manner and the whole equation becomes two terms after grouping

$$\begin{aligned} &L_{m+n+p}((m-n)(m+n-p) + (n-p)(n+p-m) + (p-m)(p+m-n)) \\ &+ \delta_{m+n+p,0}((m-n)A(m+n) + (n-p)A(n+p) + (p-m)A(p+m)) = 0 \end{aligned} \tag{25}$$

The first term goes to zero after expanding the terms in m, n, and p. Were left with the need for the second term to vanish and this gives us a set of two equations

$$(m-n)A(m+n) + (n-p)A(n+p) + (p-m)A(p+m) = 0 \tag{26}$$

$$m + n + p = 0 \tag{27}$$

With these we can now prove the general form of the central extension's coefficients. We make the assumption that they take the form

$$A(m) = \frac{m(m^2-1)}{6}A(2) \tag{28}$$

Such an assumption is consistent with our result from Problem 2.13 that $A(1)$ can always be fixed to vanish by rescaling L_0 by a constant. Returning to the constraint equations for the coefficients, we can, without loss of generality, choose $p = 0$ giving $m = -n$. In terms of m, the coefficient equation becomes

$$-mA(-m) - mA(m) + 2mA(0) = 0 \tag{29}$$

Using the assumption for $A(m)$ we have that $A(0) = 0$ and this equation reduces to tell us that $A(m) = -A(-m)$ is satisfied for all coefficients $A(m)$. Returning to the general case and reducing the coefficient constraint equation once more by removing p with $p = -m - n$ we have:

$$(m - n)A(m + n) + (2n - m)A(-m) + (-2m - n)A(-n) = 0 \tag{30}$$

Again without loss of generality we can choose that $n = 1$ to find a relation between $A(m)$ and its successor $A(m + 1)$

$$\begin{aligned}
&(m - 1)A(m + 1) + (2 - m)A(-m) + (-2m - 1)A(-1) = 0 \\
&(m - 1)A(m + 1) - (2 - m)A(m) = 0 \\
&A(m + 1) = \frac{(m + 2)}{(m - 1)}A(m)
\end{aligned} \tag{31}$$

From this we can now use mathematical induction to show indeed that $A(m)$ takes the assumed form for all $m \in \mathbb{Z}$. We have shown the base cases of $A(0) = A(1) = 0$. Now we take the relation between $A(m)$ and $A(m + 1)$ to give:

$$\begin{aligned}
A(m + 1) &= \frac{(m + 2)m(m^2 - 1)}{(m - 1)}\frac{A(2)}{6} \\
&= \frac{(m^4 + 2m^3 - m^2 - 2m)}{(m - 1)}\frac{A(2)}{6} \\
&= (m^3 + 3m^2 + 2m)\frac{A(2)}{6} \\
&= (m + 1)((m + 1)^2 - 1)\frac{A(2)}{6}
\end{aligned} \tag{32}$$

Thus we have proven using induction that $A(m) = \frac{m(m^2-1)}{6}A(2)$ for all $m > 0$. However, remembering the relation $-A(m) = A(-m)$ that we previously derived we see that indeed it is true for all $m \in \mathbb{Z}$. Lastly, from the definition of the full quantum Virasoro algebra (15) the unique value for the central charge can be quoted as $c = 2A(2)$.

□

Problem 2.15

We can verify the quantum Virasoro algebra (15) directly by plugging in the generators (16) and using the commutation relations for the mode operators. It is difficult to verify the central charge term in full generality with this method, so we begin with the cases of $m = -n = 1$, $m = -n = 2$ and make use of the previous results above.

First, we derive a useful relation:

$$\begin{aligned}
[L_m, \alpha_p^\mu] &= -[\alpha_p^\mu, \frac{1}{2}\sum_{k=-\infty}^{\infty} \alpha_{m-k}\cdot\alpha_k] \\
&= -\frac{1}{2}\sum_{k=-\infty}^{\infty}[\alpha_p^\mu, \alpha_{m-k}^\phi \alpha_k^\lambda \eta_{\phi\lambda}] \\
&= -\frac{1}{2}\sum_{k=-\infty}^{\infty}[\alpha_p^\mu, \alpha_{m-k}^\phi]\alpha_k^\lambda \eta_{\phi\lambda} + \frac{1}{2}\sum_{k=-\infty}^{\infty}\alpha_{m-k}^\phi[\alpha_k^\lambda, \alpha_p^\mu]\eta_{\phi\lambda} \\
&= -\frac{1}{2}\sum_{k=-\infty}^{\infty}(\alpha_{m-k}^\phi\alpha_p^\mu + (p)\delta_{p+m-k,0}\eta^{\phi\mu})\alpha_k^\lambda\eta_{\phi\lambda} + \frac{1}{2}\sum_{k=-\infty}^{\infty}\alpha_{m-k}^\phi\eta_{\phi\lambda}(\alpha_p^\mu\alpha_k^\lambda + (k)\delta_{k+p,0}\eta^{\mu\lambda}) \\
&= -\frac{1}{2}\sum_{k=-\infty}^{\infty}(p)\delta_{p+m-k,0}\eta^{\phi\mu}\alpha_k^\lambda\eta_{\phi\lambda} + \frac{1}{2}\sum_{k=-\infty}^{\infty}(k)\alpha_{m-k}^\phi\eta_{\phi\lambda}\delta_{p+k,0}\eta^{\mu\lambda} \\
&= -\frac{1}{2}(p)\alpha_{p+m}^\lambda\eta^{\phi\mu}\eta_{\phi\lambda} + \frac{1}{2}(-p)\alpha_{p+m}^\phi\eta_{\phi\lambda}\eta^{\mu\lambda} \\
&= (-p)\alpha_{p+m}^\mu
\end{aligned} \tag{33}$$

This relation helps expedite the cases of $m = -n = 1$ and $m = -n = 2$. We begin with the former and find

$$
\begin{aligned}
[L_1, L_{-1}] &= [L_1, \frac{1}{2}\sum_{p=-\infty}^{\infty} {:}\alpha_{-1-p}\cdot\alpha_p{:}] = [L_1, \sum_{p=0}^{\infty}\alpha_{-1-p}\cdot\alpha_p] \\
&= \sum_{p=0}^{\infty}[L_1, \alpha^\mu_{-1-p}]\alpha^\nu_p\eta_{\mu\nu} + \sum_{p=0}^{\infty}\alpha^\mu_{-1-p}[L_1,\alpha^\nu_p]\eta_{\mu\nu} \\
&= \sum_{p=0}^{\infty}(1+p)\alpha^\mu_{-p}\alpha^\nu_p\eta_{\mu\nu} + \sum_{p=0}^{\infty}(-p)\alpha^\mu_{-1-p}\alpha^\nu_{1+p}\eta_{\mu\nu} \\
&= \sum_{p=0}^{\infty}(1+p)\alpha^\mu_{-p}\alpha^\nu_p\eta_{\mu\nu} + \sum_{p=1}^{\infty}(-p+1)\alpha^\mu_{-p}\alpha^\nu_p\eta_{\mu\nu} \\
&= \alpha^\mu_0\alpha^\nu_0\eta_{\mu\nu} + 2\sum_{p=1}^{\infty}\alpha^\mu_{-p}\alpha^\nu_p\eta_{\mu\nu} \\
&= 2(\frac{1}{2}\alpha_0^2 + \sum_{p=1}^{\infty}\alpha_{-p}\cdot\alpha_p) = 2L_0
\end{aligned} \tag{34}
$$

which is what we expect given that $A(1) = 0$. In the next case of $m = -n = 2$ we expect to find the value for $A(2)$ which all $A(m)$ are proportional to with $c = 2A(2)$. For this case we compute:

$$
\begin{aligned}
[L_2, L_{-2}] &= [L_2, \frac{1}{2}\sum_{p=-\infty}^{\infty} {:}\alpha_{-2-p}\cdot\alpha_p{:}] \\
&= [L_2, \frac{1}{2}\alpha_{-1}^2 + \sum_{p=0}^{\infty}\alpha_{-2-p}\cdot\alpha_p] \\
&= [L_2, \frac{1}{2}\alpha_{-1}^2] + [L_2, \sum_{p=0}^{\infty}\alpha_{-2-p}\cdot\alpha_p]
\end{aligned} \tag{35}
$$

The first term can be easily computed:

$$
\begin{aligned}
\frac{1}{2}[L_2, \alpha^\mu_{-1}\alpha^\nu_{-1}\eta_{\mu\nu}] &= \frac{1}{2}[L_2,\alpha^\mu_{-1}]\alpha^\nu_{-1}\eta_{\mu\nu} + \frac{1}{2}\alpha^\mu_{-1}[L_2,\alpha^\nu_{-1}]\eta_{\mu\nu} \\
&= \frac{1}{2}\alpha^\mu_1\alpha^\nu_{-1}\eta_{\mu\nu} + \frac{1}{2}\alpha^\mu_{-1}\alpha^\nu_1\eta_{\mu\nu} \\
&= \frac{1}{2}(\alpha^\mu_{-1}\alpha^\nu_1 + (1)\eta^{\mu\nu}\delta_{0,0})\eta_{\mu\nu} + \frac{1}{2}\alpha^\mu_{-1}\alpha^\nu_1\eta_{\mu\nu} \\
&= \alpha_{-1}\cdot\alpha_1 + \frac{\eta^\mu_\mu}{2}
\end{aligned} \tag{36}
$$

The more extensive calculation for the second term gives:

$$
\begin{aligned}
[L_2, \sum_{p=0}^{\infty} \alpha^{\mu}_{-2-p}\alpha^{\nu}_{p}\eta_{\mu\nu}] &= \sum_{p=0}^{\infty}[L_2, \alpha^{\mu}_{-2-p}]\alpha^{\nu}_{p}\eta_{\mu\nu} + \sum_{p=0}^{\infty}\alpha^{\mu}_{-2-p}[L_2, \alpha^{\nu}_{p}]\eta_{\mu\nu} \\
&= \sum_{p=0}^{\infty}(2+p)\alpha^{\mu}_{-p}\alpha^{\nu}_{p}\eta_{\mu\nu} + \sum_{p=0}^{\infty}(-p)\alpha^{\mu}_{-2-p}\alpha^{\nu}_{2+p}\eta_{\mu\nu} \\
&= \sum_{p=0}^{\infty}(2+p)\alpha^{\mu}_{-p}\alpha^{\nu}_{p}\eta_{\mu\nu} + \sum_{p=2}^{\infty}(2-p)\alpha^{\mu}_{-p}\alpha^{\nu}_{p}\eta_{\mu\nu} \\
&= 2\alpha_0^2 + 3\alpha_{-1}\cdot\alpha_1 + 4\sum_{p=2}^{\infty}\alpha_{-p}\cdot\alpha_p \\
&= 4\sum_{p=0}^{\infty}\alpha_{-p}\cdot\alpha_p - \alpha_{-1}\cdot\alpha_1 \\
&= 4L_0 - \alpha_{-1}\cdot\alpha_1
\end{aligned} \tag{37}
$$

Finally, combining these terms we find:

$$
[L_2, L_{-2}] = 4L_0 + \frac{\eta^{\mu}_{\mu}}{2}, \tag{38}
$$

which tells us that the central charge c is exactly the spacetime dimension $c = D = \eta^{\mu}_{\mu}$. And as such, we have proven explicitly that the full quantum Virasoro algebra for any D dimensional spacetime takes the general form:

$$
[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m(m^2-1)\delta_{m+n,0} \tag{39}
$$

□

Problem 2.4

We preface this problem by quoting two results for the bosonic string in light-cone gauge. In the cases of the open and closed string, respectively, we have the following mass-shell conditions:

$$
\alpha^{'} M^2 = N - a \tag{40}
$$

and

$$
\alpha^{'} M^2 = 4(N-a) = 4(\tilde{N}-a) \tag{41}
$$

where N and $\tilde{N}$ are the number operators and the constant a is added to account for normal ordering ambiguities on the Virasoro zero mode L_0, which in the quantum theory is required to satisfy $(L_0 - a)|\phi\rangle = 0$, which can be thought of as the mass-shell condition. The text derives in various ways that $a = 1$ is the necessary value. The open string has only a single set of modes and thus only a single N. The N and $\tilde{N}$ for the closed string correspond to the oppositely moving sets of modes. They obey the so-called *level-matching condition* given by $N = \tilde{N}$, seen easily through the required difference $(L_0 - \tilde{L}_0)|\phi\rangle = 0$.

Given the following open string states:

$$|\phi_1\rangle = \alpha^i_{-1}|0;k\rangle\,, \qquad |\phi_2\rangle = \alpha^i_{-1}\alpha^j_{-1}|0;k\rangle$$
$$|\phi_3\rangle = \alpha^i_{-3}|0;k\rangle\,, \quad |\phi_4\rangle = \alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-2}|0;k\rangle$$

we can immediately see that the ith state above ($i \in \{1,2,3,4\}$) is excited i times and thus has $N = i$. The mass squared of each of these open-string states is

$$\alpha' M^2_{\phi_1} = 0, \quad \alpha' M^2_{\phi_2} = 1$$
$$\alpha' M^2_{\phi_3} = 2 \quad \alpha' M^2_{\phi_4} = 3$$

Now, the closed string states

$$|\phi_1\rangle = \alpha^i_{-1}\tilde{\alpha}^j_{-1}|0;k\rangle\,, \quad |\phi_2\rangle = \alpha^i_{-1}\alpha^j_{-1}\tilde{\alpha}^k_{-2}|0;k\rangle$$

clearly open the level matching condition. Meanwhile, a state such as

$$|\phi_3\rangle = \alpha^i_{-1}\tilde{\alpha}^j_{-2}|0;k\rangle$$

with $1 = N \neq \tilde{N} = 2$ violates the level-matching condition. This third state is unphysical while the mass squared of the first two closed-string states is

$$\alpha' M^2_{\phi_1} = 0, \quad \alpha' M^2_{\phi_2} = 4$$

Problem 2.6

The Lorentz generators of the open-string world-sheet are

$$J^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu - i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n) \tag{42}$$

We want to use this along with the canonical commutation relations to verify the Poincare algebra, given by

$$[p^\mu, p^\nu] = 0$$
$$[p^\mu, J^{\nu\sigma}] = -i\eta^{\mu\nu}p^\sigma + i\eta^{\mu\sigma}p^\nu$$
$$[J^{\mu\nu}, J^{\sigma\lambda}] = -i\eta^{\nu\sigma}J^{\mu\lambda} + i\eta^{\mu\sigma}J^{\nu\lambda} + i\eta^{\nu\lambda}J^{\mu\sigma} - i\eta^{\mu\lambda}J^{\nu\sigma}$$

The first commutation relation in the algebra relates the momentum operator as it is defined in momentum phase space. Since this commutation relation is equivalent in both momentum and position/configuration space, it can be verified by examining the position-space representation of the momentum operator p^μ, whose form can we found by using the Fourier transform. We find that $p^\mu \to -i\partial^\mu$ in position space. It follows that the first commutation relation can be written as $[p^\mu, p^\nu] = [-i\partial^\mu, -i\partial^\nu]$, which vanishes indeed because partials commute.

In the case of the second commutator, we first compute a helpful identity that relates the commutator of the momentum operator p^μ (henceforth we remain in position space) and the Virasoro mode operator α^ν_m. Noting that in the open string's mode expansion $\alpha^\mu_0 = l_s p^\mu$, we expand to see that

$$\begin{aligned}[p^\mu, \alpha^\nu_m] &= [l_s^{-1}\alpha^\mu_0, \alpha^\nu_m] \\ &= -l_s^{-1}[\alpha^\nu_m, \alpha^\mu_0] \\ &= -l_s^{-1}(m)\eta^{\mu\nu}\delta_{m+0,0} = 0\end{aligned} \tag{43}$$

which vanishes for all m. We begin expanding the second commutation relation in the algebra

$$\begin{aligned}[p^\mu, J^{\nu\sigma}] &= p^\mu x^\nu p^\sigma - p^\mu x^\sigma p^\nu - p^\mu i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\nu_{-n}\alpha^\sigma_n - \alpha^\sigma_{-n}\alpha^\nu_n) \\ &\quad - x^\nu p^\sigma p^\mu + x^\sigma p^\nu p^\mu + i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\nu_{-n}\alpha^\sigma_n - \alpha^\sigma_{-n}\alpha^\nu_n)p^\mu \\ &= [p^\mu, x^\nu p^\sigma] - [p^\mu, x^\sigma p^\nu] - [p^\mu, i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\nu_{-n}\alpha^\sigma_n - \alpha^\sigma_{-n}\alpha^\nu_n)]\end{aligned} \tag{44}$$

The third term vanishes after expanding and using the identity 43 above. Each of the following are useful in expanding the only two terms that now remain: take the commutator identity

$$[B_1, B_2B_3] = [B_1, B_2]B_3 + B_2[B_1, B_3],$$

the canonical commutation relations $[x^\mu, p^\nu] = i\eta^{\mu\nu}$, and the first commutation relation of the Poincare algebra proved above. Utilzing these we find

$$\begin{aligned}[p^\mu, J^{\nu\sigma}] &= [p^\mu, x^\nu]p^\sigma + x^\nu[p^\mu, p^\sigma] - [p^\mu, x^\sigma]p^\nu - x^\sigma[p^\mu, p^\nu] \\ &= [p^\mu, x^\nu]p^\sigma - [p^\mu, x^\sigma]p^\nu \\ &= -i\eta^{\mu\nu}p^\sigma + i\eta^{\mu\sigma}p^\nu\end{aligned} \tag{45}$$

All that's left to show is the last commutation relation of the algebra. Similarly, we begin by quoting a few useful identities. First, using again that $\alpha_0^\mu = l_s p^\mu$ for the open string, we conclude that:

$$[x^\mu, \alpha_n^\nu] = il_s\eta^{\mu\nu}\delta_{n,0} \tag{46}$$

This is useful for computing the following relation between the position operator and the Lorentz generators:

$$\begin{aligned}[J^{\mu\nu}, x^\lambda] &= [x^\mu p^\nu, x^\lambda] - [x^\nu p^\mu, x^\lambda] - [i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha_{-n}^\mu\alpha_n^\nu - \alpha_{-n}^\nu\alpha_n^\mu), x^\lambda] \\ &= -[x^\lambda, x^\mu]p^\nu - x^\mu[x^\lambda, p^\nu] + [x^\lambda, x^\nu]p^\mu + x^\nu[x^\lambda, p^\mu] \\ &= -ix^\mu\eta^{\lambda\nu} + ix^\nu\eta^{\lambda\mu}\end{aligned} \tag{47}$$

Lastly, the most extensive part of the algebra for this commutation comes from the terms which include the Virasoro modes. As such, we compute the very helpful relation:

$$\begin{aligned}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}[\alpha_{-n}^\mu\alpha_n^\nu, \alpha_{-m}^\sigma\alpha_m^\lambda] &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}(\alpha_{-n}^\mu\alpha_n^\nu\alpha_{-m}^\sigma\alpha_m^\lambda - \alpha_{-m}^\sigma\alpha_m^\lambda\alpha_{-n}^\mu\alpha_n^\nu) \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}(\alpha_{-n}^\mu\alpha_{-m}^\sigma\alpha_n^\nu\alpha_m^\lambda - \alpha_{-m}^\sigma\alpha_{-n}^\mu\alpha_m^\lambda\alpha_n^\nu \\ &\qquad + (n)\alpha_{-n}^\mu\alpha_m^\lambda\eta^{\sigma\nu}\delta_{n-m,0} - (m)\alpha_{-m}^\sigma\alpha_n^\nu\eta^{\mu\lambda}\delta_{m-n,0}) \\ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}((-n)\alpha_n^\nu\alpha_m^\lambda\eta^{\sigma\mu}\delta_{-m-n,0} - (m)\alpha_{-m}^\sigma\alpha_{-n}^\mu\eta^{\nu\lambda}\delta_{m+n,0} + \\ &\qquad + (n)\alpha_{-n}^\mu\alpha_m^\lambda\eta^{\sigma\nu}\delta_{n-m,0} - (m)\alpha_{-m}^\sigma\alpha_n^\nu\eta^{\mu\lambda}\delta_{m-n,0}) \\ &= \sum_{m=1}^{\infty}\frac{1}{m}(\alpha_{-m}^\nu\alpha_m^\lambda\eta^{\sigma\mu} + \alpha_{-m}^\sigma\alpha_m^\mu\eta^{\nu\lambda} + \alpha_{-m}^\mu\alpha_m^\lambda\eta^{\sigma\nu} - \alpha_{-m}^\sigma\alpha_m^\nu\eta^{\mu\lambda})\end{aligned} \tag{48}$$

With these identities fleshed out we are able to work out the entirety of the third commutation relation. It goes as

$$
\begin{aligned}
[J^{\mu\nu}, J^{\sigma\lambda}] &= [x^\mu p^\nu, J^{\sigma\lambda}] - [x^\nu p^\mu, J^{\sigma\lambda}] - [i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n), J^{\sigma\lambda}] \\
&= -[J^{\sigma\lambda}, x^\mu]p^\nu - x^\mu[J^{\sigma\lambda}, p^\nu] + [J^{\sigma\lambda}, x^\nu]p^\mu + x^\nu[J^{\sigma\lambda}, p^\mu] - [i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n), J^{\sigma\lambda}] \\
&= (ix^\sigma\eta^{\lambda\mu} - ix^\lambda\eta^{\sigma\mu})p^\nu + x^\mu(-i\eta^{\sigma\nu}p^\lambda + i\eta^{\lambda\nu}p^\sigma) + (-ix^\sigma\eta^{\lambda\nu} + ix^\lambda\eta^{\sigma\nu})p^\mu - x^\nu(-i\eta^{\sigma\mu}p^\lambda + i\eta^{\mu\lambda}p^\sigma) \\
&- [i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n), J^{\sigma\lambda}] \\
&= i\eta^{\lambda\mu}(x^\sigma p^\nu - x^\nu p^\sigma) + i\eta^{\sigma\mu}(x^\nu p^\lambda - x^\lambda p^\nu) + i\eta^{\lambda\nu}(x^\mu p^\sigma - x^\sigma p^\mu) + i\eta^{\sigma\nu}(x^\lambda p^\mu - x^\mu p^\lambda) - \\
&- [i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n), J^{\sigma\lambda}]
\end{aligned} \tag{49}
$$

We now expand the fifth term on its own. After expanding, we make use of the identity found in 48. The expansion and simplification of term 5 above is

$$
\begin{aligned}
[-i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n), J^{\sigma\lambda}] &= [-i\sum_{n=1}^{\infty}\frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n), -i\sum_{m=1}^{\infty}\frac{1}{m}(\alpha^\sigma_{-m}\alpha^\lambda_m - \alpha^\lambda_{-m}\alpha^\sigma_m)] \\
&= -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm}([\alpha^\mu_{-n}\alpha^\nu_n, \alpha^\sigma_{-m}\alpha^\lambda_m] - [\alpha^\mu_{-n}\alpha^\nu_n, \alpha^\lambda_{-m}\alpha^\sigma_m] - \\
&\qquad - [\alpha^\nu_{-n}\alpha^\mu_n, \alpha^\sigma_{-m}\alpha^\lambda_m] + [\alpha^\nu_{-n}\alpha^\mu_n, \alpha^\lambda_{-m}\alpha^\sigma_m]) \\
&= \sum_{m=1}^{\infty}\frac{1}{m}(\eta^{\nu\sigma}(\alpha^\mu_{-m}\alpha^\lambda_m - \alpha^\lambda_{-m}\alpha^\mu_m) - \eta^{\mu\sigma}(\alpha^\nu_{-m}\alpha^\lambda_m - \alpha^\lambda_{-m}\alpha^\nu_m) - \\
&\qquad - \eta^{\nu\lambda}(\alpha^\mu_{-m}\alpha^\sigma_m - \alpha^\sigma_{-m}\alpha^\mu_m) + \eta^{\mu\lambda}(\alpha^\nu_{-m}\alpha^\sigma_m - \alpha^\sigma_{-m}\alpha^\nu_m))
\end{aligned} \tag{50}
$$

Plugging this into the full expansion of 49, we can verify, at long last, the third and final commutation relation of the Poincare algebra. Appropriately grouping terms we have

$$
\begin{aligned}
[J^{\mu\nu}, J^{\sigma\lambda}] &= i\eta^{\lambda\mu}(x^\sigma p^\nu - x^\nu p^\sigma) + \eta^{\lambda\mu}\sum_{m=1}^{\infty}\frac{1}{m}(\alpha^\nu_{-m}\alpha^\sigma_m - \alpha^\sigma_{-m}\alpha^\nu_m) \\
&+ i\eta^{\sigma\mu}(x^\nu p^\lambda - x^\lambda p^\nu) - \eta^{\sigma\mu}\sum_{m=1}^{\infty}\frac{1}{m}(\alpha^\nu_{-m}\alpha^\lambda_m - \alpha^\lambda_{-m}\alpha^\nu_m) \\
&+ i\eta^{\lambda\nu}(x^\mu p^\sigma - x^\sigma p^\mu) - \eta^{\lambda\mu}\sum_{m=1}^{\infty}\frac{1}{m}(\alpha^\mu_{-m}\alpha^\sigma_m - \alpha^\sigma_{-m}\alpha^\mu_m) \\
&+ i\eta^{\sigma\nu}(x^\lambda p^\mu - x^\mu p^\lambda) + \eta^{\lambda\mu}\sum_{m=1}^{\infty}\frac{1}{m}(\alpha^\nu_{-m}\alpha^\sigma_m - \alpha^\sigma_{-m}\alpha^\nu_m) \\
&= -i\eta^{\nu\sigma}J^{\mu\lambda} + i\eta^{\mu\sigma}J^{\nu\lambda} + i\eta^{\nu\lambda}J^{\mu\sigma} - i\eta^{\mu\lambda}J^{\nu\sigma},
\end{aligned} \tag{51}
$$

and the verification of the Poincare algebra is complete.

Problem 2.7

The bosonic closed string's spacetime coordinates are described collectively by 2, 3, and 4. To compute the angular-momentum (or Lorentz) generators we integrate:

$$J^{\mu\nu} = \int_0^\pi J_0^{\mu\nu} d\sigma = T \int_0^\pi (X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu) d\sigma \tag{52}$$

where

$$\begin{aligned} X^\mu \dot{X}^\nu = {} & x^\mu p^\nu l_s^2 + l_s^4 p^\mu p^\nu \\ & + x^\mu l_s \sum_{n \neq 0}^{\infty} (\alpha_n^\nu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\nu e^{-2in(\tau+\sigma)}) \\ & + l_s^3 p^\mu \tau \sum_{n \neq 0}^{\infty} (\alpha_n^\nu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\nu e^{-2in(\tau+\sigma)}) \\ & + \frac{i}{2} l_s^3 \sum_{m \neq 0}^{\infty} \frac{1}{m} (\alpha_m^\mu e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^\mu e^{-2im(\tau+\sigma)}) p^\nu \\ & + \frac{i}{2} l_s^2 \sum_{m \neq 0}^{\infty} \sum_{n \neq 0}^{\infty} \frac{1}{m} (\alpha_m^\mu e^{-2im(\tau-\sigma)} + \tilde{\alpha}_m^\mu e^{-2im(\tau+\sigma)})(\alpha_n^\nu e^{-2in(\tau-\sigma)} + \tilde{\alpha}_n^\nu e^{-2in(\tau+\sigma)}) \end{aligned} \tag{53}$$

and $X^\nu \dot{X}^\mu$ is identical with $\mu \leftrightarrow \nu$. Upon integration w.r.t. to σ, the terms in $X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu$ proportional to $e^{-2ik(\tau\pm\sigma)}$ will vanish. In the case of the last term, however, we can simplify to see that some terms lose their $e^{-2ik(\tau\pm\sigma)}$ factor when $n + m = 0$. We expand, apply the Virasoro mode commutation relations, and sum over only the non-vanishing terms where $n = -m$ as follows

$$\begin{aligned} \text{last term of } X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu = {} & \frac{i}{2} l_s^2 \sum_{m \neq 0}^{\infty} \sum_{n \neq 0}^{\infty} \frac{1}{m} (\alpha_m^\mu \alpha_n^\nu e^{-2i(n+m)(\tau-\sigma)} + \tilde{\alpha}_m^\mu \tilde{\alpha}_n^\nu e^{-2i(n+m)(\tau+\sigma)}) \\ & + \frac{i}{2} l_s^2 \sum_{m \neq 0}^{\infty} \sum_{n \neq 0}^{\infty} \frac{1}{n} (\alpha_n^\nu \alpha_m^\mu e^{-2i(n+m)(\tau-\sigma)} + \tilde{\alpha}_n^\nu \tilde{\alpha}_m^\mu e^{-2i(n+m)(\tau+\sigma)}) \\ = {} & -\frac{i}{2} l_s^2 \sum_{m \neq 0}^{\infty} \frac{1}{m} (\alpha_{-m}^\mu \alpha_m^\nu - \alpha_{-m}^\nu \alpha_m^\mu + \tilde{\alpha}_{-m}^\mu \tilde{\alpha}_m^\nu - \tilde{\alpha}_{-m}^\nu \tilde{\alpha}_m^\mu) \end{aligned} \tag{54}$$

Combining all this, and ignoring terms which integrate to zero, we have

$$
\begin{aligned}
J^{\mu\nu} &= T\int_0^\pi (l_s^2(x^\mu p^\nu - x^\nu p^\mu) - \frac{i}{2}l_s^2\sum_{m\neq 0}^{\infty}\frac{1}{m}(\alpha_{-m}^\mu\alpha_m^\nu - \alpha_{-m}^\nu\alpha_m^\mu + \tilde{\alpha}_{-m}^\mu\tilde{\alpha}_m^\nu - \tilde{\alpha}_{-m}^\nu\tilde{\alpha}_m^\mu))d\sigma \\
&= x^\mu p^\nu - x^\nu p^\mu - \frac{i}{2}l_s^2\sum_{m\neq 0}^{\infty}\frac{1}{m}(\alpha_{-m}^\mu\alpha_m^\nu - \alpha_{-m}^\nu\alpha_m^\mu + \tilde{\alpha}_{-m}^\mu\tilde{\alpha}_m^\nu - \tilde{\alpha}_{-m}^\nu\tilde{\alpha}_m^\mu)
\end{aligned} \tag{55}
$$

which is identical to the generators of angular momentum for the open string but with a factor of a half while accounting for both the right- and left-moving modes.

Problem 2.8

The angular-momentum generators of the open string, defined above in 42 are appropriate for covariant quantization. We're interested in their formulas in the case of light-cone gauge quantization, so we begin with some remarks on this gauge choice.

As we remarked in the beginning, we used the diffeomorphism invariance and Weyl rescaling symmetries available to the bosonic string theory in order to fix the auxiliary world-sheet metric as flat, $h_{\alpha\beta} = \eta_{\alpha\beta}$, but we have not gauged fixed the theory entirely. There still exist residual reparameterizations that *are themselves* Weyl resclaings. These residual reparameterizations satisfy

$$\partial^\alpha\epsilon^\beta + \partial^\beta\epsilon^\alpha = \Lambda\eta^{\alpha\beta} \tag{56}$$

where ϵ^α is an infinitesimal parameter for a reparameterization and Λ is an infinitesimal parameter for a Weyl rescaling. A solution to this is given by

$$\epsilon^+ = \epsilon^+(\sigma^+) \text{ and } \epsilon^- = \epsilon^-(\sigma^-) \tag{57}$$

where $\epsilon^\pm = \epsilon^0 \pm \epsilon^1$ and $\sigma^\pm$ are the world-sheet light-cone coordinates. The light-cone gauge makes two definitions:

1. We begin by reparameterizing the string world-sheet coordinates by using the residual symmetries above. That is, $\sigma^\pm \to \epsilon^\pm(\sigma^\pm)$, and this transformation can be written explicitly as

$$\tilde{\tau} = \frac{1}{2}[\epsilon^+(\sigma^+) + \epsilon^-(\sigma^-)] \tag{58}$$

$$\tilde{\sigma} = \frac{1}{2}[\epsilon^+(\sigma^+) - \epsilon^-(\sigma^-)], \tag{59}$$

and it is noted that $\tilde{\tau}$ itself can solve the free massless wave equation:

$$(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2})\tilde{\tau} = 0 \tag{60}$$

2. Next, we reparameterize two of the D available spacetime coordinates as follows

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}) \tag{61}$$

The light-cone gauge now uses the residual freedom under these changes to make the following choice

$$\alpha_m^+ = 0 \text{ for } m \neq 0, \tag{62}$$

which allows us to write

$$X^+(\tilde{\tau}, \tilde{\sigma}) = x^+ + l_s^2 p^+ \tilde{\tau}, \tag{63}$$

thereby eliminating the oscillator modes of X^+. By using the Virasoro constraints, see page 49, one can similarly determine the oscillator modes of X^- as well. The constraints allow us to solve for the modes of X^- completely in terms of the traverse oscillations as follows

$$X^- = x^- + l_s^2 p^- \tilde{\tau} + i l_s \sum_{n\neq 0}^{\infty} \frac{1}{n} \alpha_n^- e^{-in\tilde{\tau}} cos(n\tilde{\sigma}) \tag{64}$$

with

$$\alpha_n^- = \frac{1}{p^+ l_s}(\frac{1}{2}\sum_{i=1}^{D-2}\sum_{m=-\infty}^{\infty} :\alpha_{n-m}^i \alpha_m^i: -a\delta_{n,0}) \tag{65}$$

where D is again the spacetime dimension and a is the normal-ordering constant referenced in Problem 2.4. The remaining $D-2$ transverse mode expansions take the form given in 5.

Returning now to the problem at hand, we note that the Lorentz generators of the open string 42 are given by a tensor equation. Since tensors are coordinate-independent, we can simply use the nice properties of the light-cone gauge's mode expansions and quote the results. We have

$$J^{+-} = x^+ p^- - x^- p^+ \tag{66}$$

$$J^{+i} = x^+ p^i - x^i p^+ \tag{67}$$

$$J^{-i} = x^- p^i - x^i p^- - i \sum_{n=1}^{\infty} \frac{1}{n}(\alpha_{-n}^- \alpha_n^i - \alpha_{-n}^i \alpha_n^-) \tag{68}$$

and lastly,

$$J^{ij} = x^i p^j - x^j p^i - i \sum_{p=1}^{\infty} \frac{1}{n} (\alpha^i_{-n} \alpha^j_n - \alpha^j_{-n} \alpha^i_n), \tag{69}$$

where we remark again for significance that the modes α^-_n can be completely written in terms of α^i_n and the normal-ordering constant a.