VFEA PART 1: Vibration

Professor Joseph Morlier

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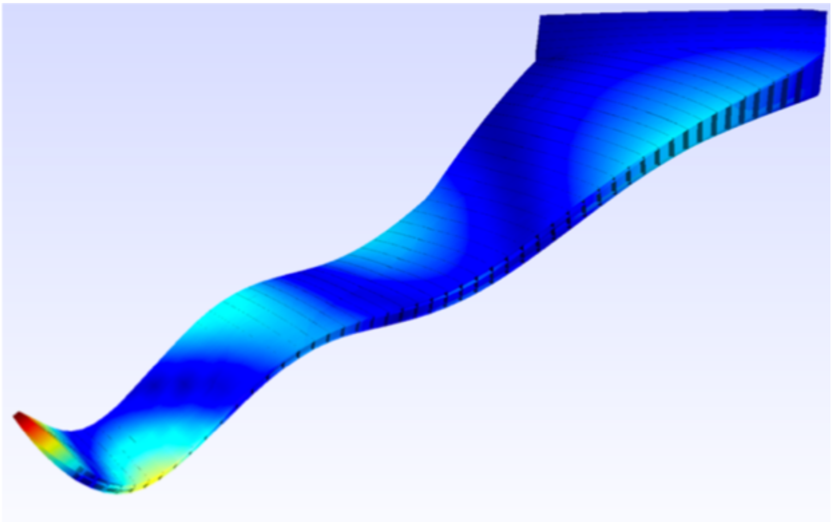


Contents

Introduction

How to use the book?

- This DRAFT book is really complementary to the live lessons and contains also exercises to program live during the course (in Matlab).
- Largely inspired by Park's lecture note available on internet but also from Professors: Inman, Blevins, Rao and Champaney.
- The first part is dedicated to vibration, the second to finite element analysis applied to Structural Mechanics.
- Python's version is coming (for Winter)
- Students are really welcome to send typos or a general feedback on the book to `joseph.morlier@isae-supaero.fr`

IN BLUE

This is a HOMEWORK

IN RED

This is a HANDWRITTEN exercice (during the lesson).

IN GREEN

This is a MATLAB's code to fill (during the lesson). Please download the skeleton's codes at `https://github.com/jomorlier/VFEA`.

Discrete systems

Why starting with Discrete systems? Probably you already seen this concept in electrical engineering for RLC system or in General Mechanics through Newton's law or Lagrange equations.

A very intuitive online course is available at UoM. Some nice videos are available at Purdue

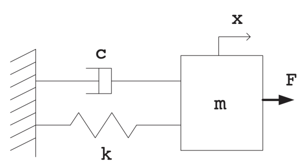


Figure 1: A single-degree-of-freedom system with mass m, spring constant k, and viscous damping c, the system undergoes a dynamic displacement x(t) measured from the static equilibrium position of the mass under the external force F(t).

In this chapter we will study the responses of systems with a single degree of freedom. It is important topic to master, since the complicated multiple-degree-of-freedom systems (MDOF) can often treated as if they are simple collections of several single-degree-of-freedom (SDOF) systems. Once the responses of SDOF are understood, the study of complicated MDOF becomes relatively easy.

Single-Degree-of-Freedom Systems

Free Vibration.

For the free vibration of a single-degree-of-freedom system with mass m, spring constant k, and viscous damping c, the system undergoes a dynamic displacement x(t) measured from the static equilibrium position of the mass. Applying Newton's law, the equation of motion (EoM) of the system is represented by:

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{1}$$

subject to the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$. Note that the RHS is equal to zero which means we express here the intrinsic dynamic properties of the system.

Start Simple ... What does it mean to have RHS equal to ZERO? This equation will help you to compute the normal mode sol103 in Nastran for example with c equal to zero i.e. the intrinsic dynamic behaviour of the system.

If we divide equation 1 by m we can reexpress it in terms as

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{2}$$

where $\omega_n = \sqrt{k/m}$ is natural angular frequency and $\zeta = c/(2\sqrt{km})$ is the damping ratio. To solve the damped system of equation 2, assuming

$$x = Ae^{st} \tag{3}$$

Substituting equation 3 into equation 2 yields an algebraic equation in the form

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{4}$$

The solutions of equation 4 are given by

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{(\zeta^2 - 1)} \tag{5}$$

Let's write $\delta = (\zeta^2 - 1)$. Depending on the sign of δ we get:

How many possibilities? In fact There are three possible cases! But case (b) often happens in physical systems such as structures.

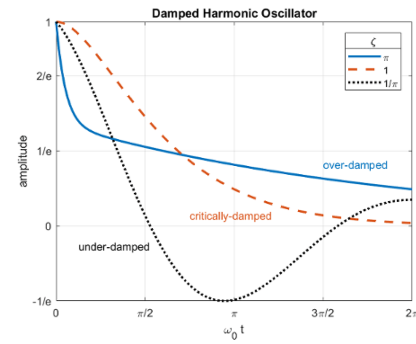


Figure 2: The Physics of the Damped Harmonic Oscillator.

- (a) Overdamped Motion ($\zeta > 1$). In this case, the damping ratio is greater than 1. The discriminant of equation 5 is positive, resulting in a pair of distinct real roots $s_{1,2} = \omega_n\left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)$. The solution becomes:

$$x(t) = Ae^{\left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} + Be^{\left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}$$

  which represents that the vibration will not occur since the damping force is so large that the restoration force from the spring is not sufficient to overcome the damping force.

- (b) Critically Damped Motion ($\zeta = 1$): In this last case, the damping ratio is exactly 1 ($s_1 = s_2 = -\omega_n$). The solution (aperiodic) takes the form:

$$x(t) = e^{-\zeta\omega_n t}(A + Bt) \tag{6}$$

  The constants A and B are determined by the initial conditions.

- (c) Underdamped Motion. In this case the damping ratio is less than $1 (0 < \zeta < 1)$ and the discriminant of equation 5 is negative, resulting in a complex conjugate pair of roots. The solutions for this case can be expressed as

$$\begin{aligned} x(t) &= e^{-\zeta\omega_n t}\left(Ae^{j\omega_n\sqrt{1-\zeta^2}t} + Be^{-j\omega_n\sqrt{1-\zeta^2}t}\right) \\ &= e^{-\zeta\omega_n t}\left(Ae^{j\omega_d t} + Be^{-j\omega_d t}\right) \\ &= e^{-\zeta\omega_n t}\left(C\cos\omega_d t + D\sin\omega_d t\right) \\ &= Xe^{-\zeta\omega_n t}\sin\left(\omega_d t + \phi\right) \end{aligned}$$

  where $j = \sqrt{-1}$, X and ϕ are constants. The damped natural frequency is denoted by:

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

What happens when ζ^2 is small compared to 1? Which value of ζ affects ω_d?

HOMEWORK 1

- Review the symbolic computation by reading Exo.mlx
- Describe the definition of logarithmic decrement in free vibration.

Frequency Response Function (FRF)

For a single-degree-of-freedom system with viscous damping and subjected to a forcing function $F(t)$ as shown in figure 2.1 , the equation of motion can be written as:

Start SIMPLE PROF! Check the RHS here !

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{7}$$

The complete solution to equation 7 consists of two parts, the homogenous solution (the complementary solution) and the particular solution. The homogenous solution is the same as the free vibration which was described in last section. It is often common to ignore the transient part of the total solution and focus only on the steady-state response. Taking Laplace transformation of a second order differential equation with zero initial conditions, the transfer function is

Handwritten exercice H1

$$H(s) = \frac{X(s)}{F(s)} = \underline{\qquad\qquad} \tag{8}$$

where $\omega_n = \sqrt{k/m}, \zeta = c/2m\omega_n$. Substituting $j\omega$ for s to calculate the frequency response, where j is the imaginary operator:

Note: a convolution in time is a multiplication in frequency domain ! It explains the everyday used of frequency approach (or Laplace) by engineers

$$\frac{X(j\omega)}{F(j\omega)} = \frac{1/m\omega^2}{\left[(\omega_n/\omega)^2 - 1\right] + 2j\zeta\,(\omega_n/\omega)}$$

$H(j\omega) = \frac{X(j\omega)}{F(j\omega)}$ is the Fourier transform of h(t) from equation 17

MATLAB CODE Ex1.m

Plot the amplitude and phase angle of the single degree of freedom system with ζ as a parameter. Complete the following MATLAB Code

```
clf; clear all; close all;
m = 1;
zeta = 0.1:0.1:1; k = 1;
wn = sqrt(k/m);
w = logspace(-1,1,400);
rad2deg = 180/pi;
s = j*w;
for cnt = 1:length(zeta)
xfer(cnt,:)=...; %TODO
mag(cnt,:) = abs(xfer(cnt,:));
phs(cnt,:) = angle(xfer(cnt,:))*rad2deg;
```

HINTS from the prof: ζ varies from 0.1 to 1. Check the changes in amplitude and phase (loglog) !

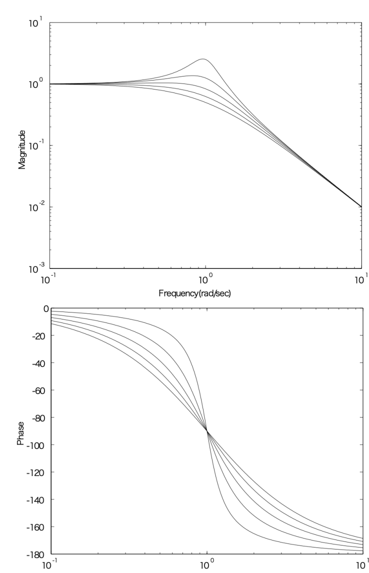


Figure 3: Note the local variation of amplitude and phase when ζ varies

```
figure(1)
loglog(w,mag(cnt,:),'k-')
title('SDOF frequency response magnitudes for zeta = 0.2 to
    1.0 in steps of 0.2')
xlabel('Frequency(rad/sec)')
ylabel('Magnitude')
grid
hold on
figure(2)
semilogx(w,phs(cnt,:),'k-')
title('SDOF frequency response phases for zeta = 0.2 to 1.0
    in steps of 0.2')
xlabel('Frequency(rad/sec)')
ylabel('Phase')
grid
hold on
end
hold off
```

Simulation with MATLAB

Every linear time invariant (LTI) systems such as for example the one from equation 8 can be specified by transfer functions. The corresponding command is : $sys = tf(num, den)$ The output sys is a model-specific data structure.

MATLAB CODE Ex2a.m

Plot the amplitude and phase angle of the single degree of freedom system. Complete the following MATLAB Code

```
clf; clear all;
m=1
zeta=0.1
k=1
wn=sqrt(k/m)
den=[...] %TODO
num=[...] %TODO
sys=tf(num,den)
bode(sys)
```

HINTS from the prof. Use the function tf to describe the system using equation 8 Plot the bode function.

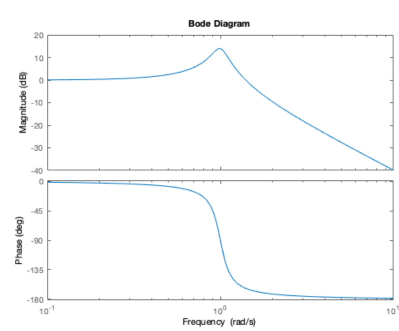


Figure 4: bode plot at fixed $\zeta = 0.1$

Sinusoidal force

Consider the usual spring mass damper system with applied force $F(t) = F_0 \cos \omega t$. The solution is the sum of homogenous and particular solution. The particular solution assumes form of forcing function: $x_p(t) = X \cos(\omega t)$. Let's take $c = 0$ to start in equation 1:

$$m\ddot{x}(t) = -kx(t) + F_0 \cos(\omega t)$$
$$\ddot{x}(t) + \omega_n^2 x(t) = f_0 \cos(\omega t)$$

where $f_0 = F_0/m, \quad \omega_n = \sqrt{k/m}$

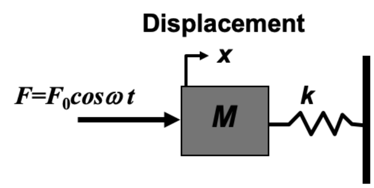


Figure 5: Harmonic Excitation of Undamped Systems. ω is called the driving frequency and F_0 is the magnitude of the applied force.

We can substitute the particular solution into the equation of motion:

$$\overbrace{-\omega^2 X \cos \omega t}^{\ddot{x}_p} + \overbrace{\omega_n^2 X \cos \omega t}^{\omega_n^2 x_p} = f_0 \cos \omega t$$

$$\text{solving yields: } X = \frac{f_0}{\omega_n^2 - \omega^2}$$

Thus the particular solution has the form:

$$x_p(t) = \frac{f_0}{\omega_n^2 - \omega^2} \cos(\omega t) \tag{9}$$

Physically the input wins! it drives completly the system !!

If we add the particular and the homogeneous solutions we can get the general solution:

$$x(t) =$$

$$\underbrace{A_1 \sin \omega_n t + A_2 \cos \omega_n t}_{\text{homogeneous}} + \overbrace{\frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t}^{\text{particular}}$$

A_1 and A_2 are constants of integration. By Applying the initial conditions we can evaluate the constants:

$$x(0) = A_1 \sin 0 + A_2 \cos 0 + \frac{f_0}{\omega_n^2 - \omega^2} \cos 0 = A_2 + \frac{f_0}{\omega_n^2 - \omega^2} = x_0$$
$$\Rightarrow A_2 = x_0 - \frac{f_0}{\omega_n^2 - \omega^2} \tag{10}$$

$$\dot{x}(0) = \omega_n (A_1 \cos 0 - A_2 \sin 0) - \frac{f_0}{\omega_n^2 - \omega^2} \sin 0 = \omega_n A_1 = v_0$$
$$\Rightarrow A_1 = \frac{v_0}{\omega_n} \Rightarrow \tag{11}$$

$$x(t) = \frac{v_0}{\omega_n} \sin \omega_n t + \left(x_0 - \frac{f_0}{\omega_n^2 - \omega^2} \right) \cos \omega_n t + \frac{f_0}{\omega_n^2 - \omega^2} \cos \omega t \tag{12}$$

What happens when ω is ω_n ? $x_p(t) = tX\sin(\omega t)$ substitute into equation and solve for $X = \frac{f_0}{2\omega}$. When the drive frequency and natural frequency are the same the amplitude of the vibration grows without bounds. This is known as a resonance condition. The most important concept in this course!

Comparison of free and forced response:

- Sum of two harmonic terms of different frequency
- Free response has amplitude and phase effected by forcing function
- Our solution is not defined for $\omega_n = \omega$ because it produces division by 0 .
- If forcing frequency is close to natural frequency the amplitude of particular solution is very large

$$x(t) = A_1 \sin\omega t + A_2 \cos\omega t + \overbrace{\frac{f_0}{2\omega} t \sin(\omega t)}^{\text{grows with out bound}}$$

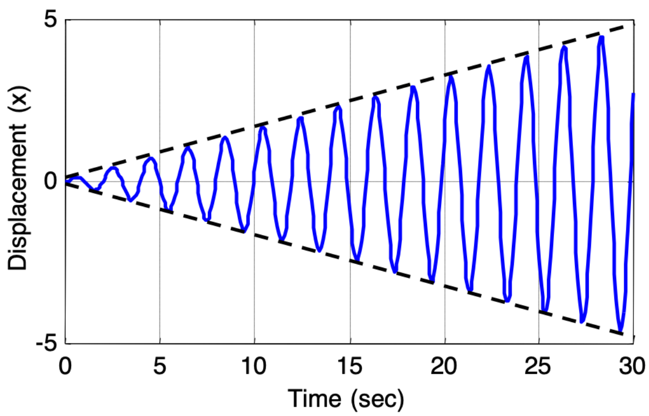


Figure 6: The resonance phenomenum. When the drive frequency and natural frequency are the same the amplitude of the vibration grows without bounds.

On the use of Laplace transform

The Laplace transform and its inverse can be used to find the solution of initial value problems for ordinary differential equations. Suppose that the function $f(t)$ is defined for all $t \geq 0$.

Then its Laplace transform is the function $F(s)$ as given by:

$$F(s) = \mathbf{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}dt$$

where $\mathbf{L}$ is the symbol used for the Laplace transform operator and s is a complex variable such that $s = i\omega[1/time]$

An important property of the Laplace transform is that it turns a derivative into an algebraic operation. For example,

$$\mathbf{L}\{\dot{f}\} = s\mathbf{L}\{f\} - f(0)$$

$$\mathbf{L}\{\ddot{f}\} = s^2\mathbf{L}\{f\} - sf(0) - f'(0)$$

The Laplace transform of a function can be easily computed in Matlab (Symbolic Maths and Signal Processing toolboxes required) as shown by the following examples.

Handwritten exercice H2

From the examples of laplaceExo.m. Fill the next table. The function f and variables t and s are all declared as symbolic variables.

$f(t)$	$F(s)$
a	
$A\sin(bt)$	
$A\cos(\mathrm{bt})$	
t^n	
$A\mathrm{e}^{-at}$	

Take the Laplace transform of the equation of motion with a sinusoidal excitation 9:

$$m\ddot{x} + kx = F_0 \cos\omega t \Rightarrow$$
$$\left(ms^2 + k\right) X(s) = \frac{F_0 s}{s^2 + \omega^2}$$

Now solve algebraic equation in s for $X(s)$

$$X(s) = \frac{F_0 s}{(ms^2 + k)(s^2 + \omega^2)} \tag{13}$$

To get the time response this must be "inverse transformed" !

Use *ilaplace* MATLAB function. See laplaceEx1.m

By inverting the equation 13 you should find the equation 9.

DOING PHYSICS with MATLAB: Use laplaceEx.m template

- Simple harmonic motion of a SDOF with m=1, c=0, k=4. $x(0) = 1$ and $v(0) = 0$, $f = 0$. Can you comment?
- System driven at its natural frequency of vibration with no damping. SDOF with m=1, c=0, k=4. $x(0) = 0$ and $v(0) = 0$, $\omega = 2$. Can you comment?
- System driven at its natural frequency of vibration with damping. SDOF with m=1, c=0.5, k=4. $x(0) = 0$ and $v(0) = 0$, $\omega = 2$. Can you comment?
- System driven far from natural frequency of vibration with damping. SDOF with m=1, c=0.5, k=4. $x(0) = 0$ and $v(0) = 0$, $\omega = 3$. Can you comment?

Systems Analysis and Duhamel's Principle

Consider a physical system in which the output or response $x(t)$ to the input function $f(t)$ is described by the differential equation

$$ax'' + bx' + cx = f(t) \tag{14}$$

where the constant coefficients a, b, and c are determined by the physical parameters of the system and are independent of $f(t)$.

For simplicity we assume that the system is initially passive: $x(0) = x'(0) = 0$. Then the transform of equation 14 is

$$as^2X(s) + bsX(s) + cX(s) = F(s),$$

so

$$X(s) = \frac{F(s)}{as^2 + bs + c} = H(s)F(s) \tag{15}$$

The function

$$H(s) = \frac{1}{as^2 + bs + c}$$

is called the transfer function of the system. Thus the transform of the response to the input $f(t)$ is the product of $H(s)$ and the transform $F(s)$. The function

$$h(t) = \mathcal{L}^{-1}\{H(s)\}$$

is called the impulse response function of the system. From equation 15 we see by convolution that

$$x(t) = \int_0^t h(\tau)f(t-\tau)d\tau \tag{16}$$

This formula is Duhamel's principle for the system. What is important is that the function $h(t)$ is determined completely by the parameters of the system. Once $h(t)$ has been determined, the integral in equation 16 gives the response of the system to an arbitrary input function $f(t)$.

Linear Superposition allows us to break up complicated forces into sums of simpler forces, compute the response and add to get the total solution. If x_1, x_2 are solutions of a linear homogeneous equation, then $x = a_1x_1 + a_2x_2$ is also a solution. If x_1 is the particular solution of $\ddot{x} + \omega_n^2 x = f_1$ and x_2 the particular solution of $\ddot{x} + \omega_n^2 x = f_2$

$\Rightarrow ax_1 + bx_2$ solves $\ddot{x} + \omega_n^2 x = af_1 + bf_2$

For example, the convolution integral (or Duhamel integral) for underdamped SDOF systems is:

$$\begin{aligned} x(t) &= \frac{1}{m\omega_d}e^{-\zeta\omega_n t}\int_0^t \left[F(\tau)e^{\zeta\omega_n\tau}\sin\omega_d(t-\tau)\right]d\tau \\ &= \frac{1}{m\omega_d}\int_0^t F(t-\tau)e^{-\zeta\omega_n\tau}\sin\omega_d\tau d\tau \end{aligned}$$

The response to any integrable force can be computed with either of these forms.

BTW Which form to use depends on which is easiest to compute?

Impulse Response Function (IRF)

Another way of explaining the IRF is to consider the system with a specific force f called Impulse

An equivalent to initial velocity at Δt is:

$$\dot{x_0}|_{\Delta t} = \frac{F\Delta t}{m}$$

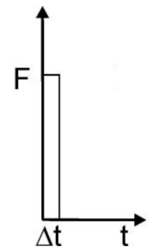


Figure 7: Impulse as an input

For an initial velocity, the response of the system is: $x(t) = \frac{e^{-\xi\omega_n t}\dot{x_0}}{\omega_d}\sin(\omega_d t)$ with $\dot{x_0} = \frac{F\delta t}{m}$

For a unit impulse $F\delta t = 1$, we define the impulse response $h(t)$ as:

$$h(t) = \frac{e^{-\xi\omega_n t}}{m\omega_d}\sin(\omega_d t) \tag{17}$$

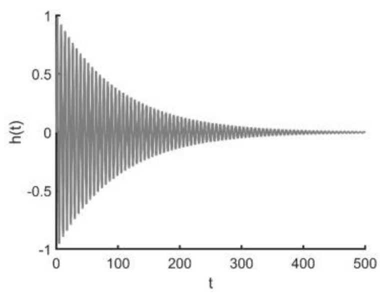


Figure 8: Impulse Response function h(t) with $\omega_n = 1, \xi = 0.01$

Duhamel's integral

$f(t)$ is decomposed into a series of short impulses at time t. The contribution of one impulse $f(\tau)d\tau$ to the response of the system is given by:

$$f(\tau).d\tau.h(t-\tau)$$

The total contribution is therefore:

$$x(t) = \int_{-\infty}^{\infty} f(\tau)h(t-\tau)d\tau = f(t) * h(t)$$

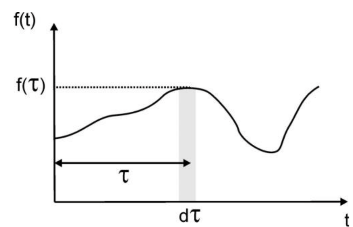


Figure 9: Decomposition of f(t) with the Dirac function

We have $h(t) = 0$ and $f(t) = 0$ for $t < 0$ so that we can write (* is the convolution operator)

$$x(t) = \int_0^t f(\tau)h(t-\tau)d\tau$$

h characterize the system. f(t) is the input x(t) is the output.

MATLAB CODE EX2B.m

Try to simulate the response of the linear system sys to a sine wave (ω=1) for a duration of 80 seconds. Complete the following MATLAB Code

HINTS from the prof. Type help lsim. Sys is the transfer function given explicitely by h(t) here. From LTI system we can write $x = h * f$ with * the convolution operator

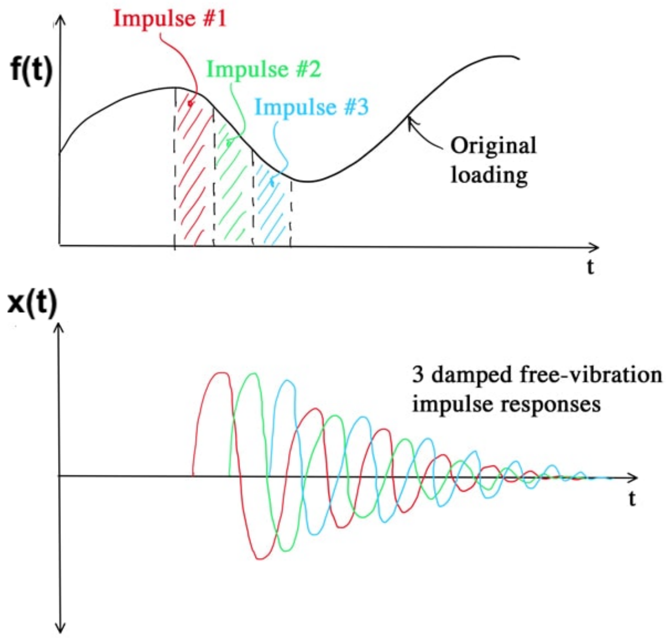


Figure 10: Principle of Duhamel's integration (or Convolution)

```
%2B
t=...; %TODO
f=...; %TODO
figure;
lsim(sys,f,t)
```

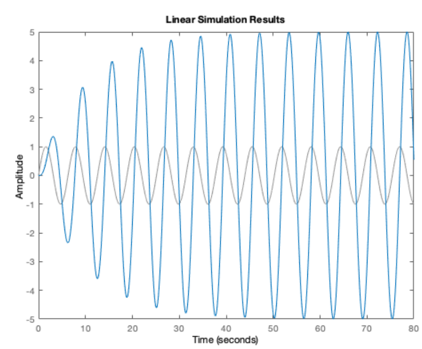


Figure 11: Simulation of the response y to a sine input f. By default x_0 is set to 0. Note that as the input is a sine; the ouptut will be a sine (LTI system)

Base excitation

Consider the single degree of freedom system in Figure 12(a).

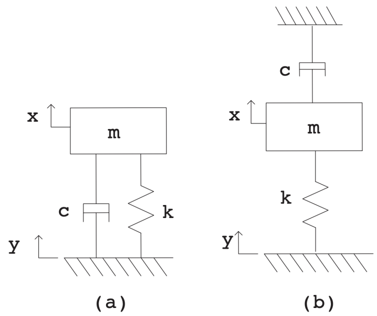


Figure 12: Free diagram of base excited single degree of freedom system.

The structure with mass m is connected to the base by stiffness, k, and damping with viscous damping coefficient c.

The equation of motion is

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

Often, machines are harmonically excited through elastic mounting, which may be modeled by springs and dashpots. For example, an automobile suspension system is excited by road surface. Please have a look also to satellites or building modal analysis on youtube

(a)Derive the displacement transmissibility, X/Y and plot the magnitude and phase. (b) The transmitted force by the base excitation to the structure is $F_T = k(x - y) + c(\dot{x} - \dot{y})$. The force transmissibility, F_T/kY is defined as the dimensionless relation between maximum base displacement Y and the transmitted force magnitude F_T. Derive the force transmissibility and plot as function of frequency ratio.

Vibration suppression

A Dynamic Vibration Absorber (DVA) is a device consisting of a reaction mass, a spring element with appropriate damping that is attached to a structure in order to reduce the dynamic response of the structure. The frequency of dynamic absorber is tuned to a particular structural frequency so that when that frequency is excited external force. The concept of DVA was first applied by Frahm in 1909 to reduce the rolling motion of ships as well as hull vibrations. A theroy for the DVA was presented later by Ormondroyd and Den Hartog (1928)[1]. The detailed study of optimal tuning and damping parameters was discussed in Den Hartog's on Mechanical Vibration (1940) book [2].

A good introduction can be found on youtube DVA

[1] J.Ormondroyd, and J.P.Den Hartog, "The theory of the dynamic vibration absorber", Trans. ASME, 50, 1928, pp. $9-15$ [2] J.P.Den Hartog, Mechanical vibration, Dover, 4th ed. Reprint, 1984

It is desirable to change the system equation for an n DOFs system with n second order differential equation to 2n first order differential equations. The first order form of equations for the system is called as state space form. Start by solving equation second order differential equations.

State Space Analysis. That's new? Check your control theory course?

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t) \longrightarrow m\ddot{x} + c\dot{x} + kx = F \tag{18}$$

f and so x is depending on t. Whatever f is, we can decompose it into Fourier Series. And of course use LTI system properties

we define the state vector as

$$\mathbf{x}(t) = [x(t) \quad \dot{x}(t)]^T \longrightarrow \mathbf{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

Please note the capital $\mathbf{x}$ in bold (LHS)

Then, adding the identity $\dot{x} = \dot{x}$, equation 18 can be written in the state form as

$$\begin{cases} \dot{x} = \dot{x} \\ \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x - \frac{F}{m} = 0 \end{cases} \tag{19}$$

$$\begin{cases} \dot{x} = \dot{x} \\ \ddot{x} = -\frac{k}{m}x - \frac{c}{m}\dot{x} + \frac{F}{m} \end{cases} \tag{20}$$

Reminder: $\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}$

Write equation 20 in compact form 21 to identify **A** and **B**

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}F \tag{21}$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} . & . \\ . & . \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} . \\ . \end{bmatrix} F \tag{22}$$

where the system matrix is $\mathbf{A} = \begin{bmatrix} . & . \\ . & . \end{bmatrix}$ and the input matrix is $\mathbf{B} = \begin{bmatrix} . \\ . \end{bmatrix}$.

Schematically, a Single Input Single Output (SISO) state space system is represented as shown in figure 13.

Note: To account for the case where the desired output is not just the states but is some linear combination of the states, and output matrix **C** is defined to relate the outputs to the states. Also, a matrix **D**, know as the direct transmission matrix, is multiplied by the input $F(t)$ to account for outputs that are related to the inputs but that bypass the states.

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}F$$

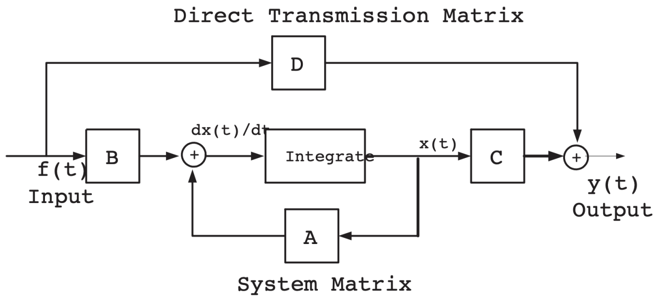


Figure 13: State space system block diagram. The scalar input $u(t)$ is fed into both the input matrix **B** and the direct transmission matrix **D**. The output of the input matrix is a $n \times 1$ vector, where n is the number of states. The output is fed into a summing junction to be added to the output of the **C** matrix.

The output of the **B** matrix is added to the feedback term coming from the system matrix and is fed intro an integrator block. The output matrix has as many rows as outputs, and has as many columns as states, n.

If, for instance, we are interested in controlling the position of the mass, then the output equation **C** is:

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{23}$$

The output matrix **C** has as many rows as outputs required and as many columns as states. The direct transmission matrix **D** has the same number of columns as the input matrix **B** and as many rows as the output matrix **C**.

MATLAB CODE EX3.m

Numerically compute the free vibration of mass-spring-damper system by defining a state space *ss* and using *initial* function in MATLAB. $x_0 = 1$ and $\dot{x}_0 = 0$

```
clf; clear all;
m=1;c=0.1;k=1;
A=...; %TODO
C=...; %TODO
sys=ss(A,[],C,[]);
x0=[1,0];
figure;
initial(sys,x0)
```

HINTS from the prof: use equation of **A** and $\mathbf{C} = [1, 0]T$ from equation 23 with function ss. Plot the response to x_0 with the funtion initial.

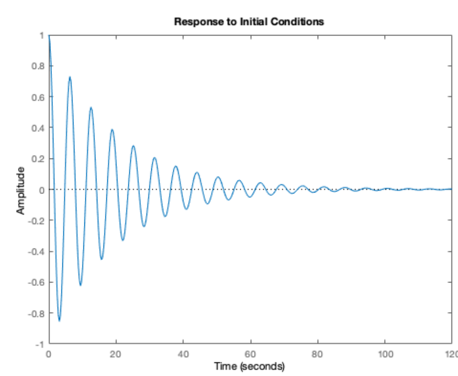


Figure 14: Initial response to $x0 = [1, 0]$ of the state space defined by matrix A and C. Remind the vector dimension of $\mathbf{X} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

Ordinary Differential Equation. That's new? Check your maths course?

MATLAB CODE EX3b.m

Numerically compute the free vibration of mass-spring-damper system using ODE45 in MATLAB. $x_0 = 1$ and $\dot{x_0} = 0$

HINTS from the prof: Help ode45. But what is the size of x ?

```
clf; clear all;
t0=0;tf=50;
x0=[1,0];
figure;
[t,x]=ode45('sdof',[t0 tf],x0);
plot(t,x)

function xdot=sdof(t,x)
m=1;c=0.1;k =1;
A=...; %TODO
xdot=...; %TODO
end
```

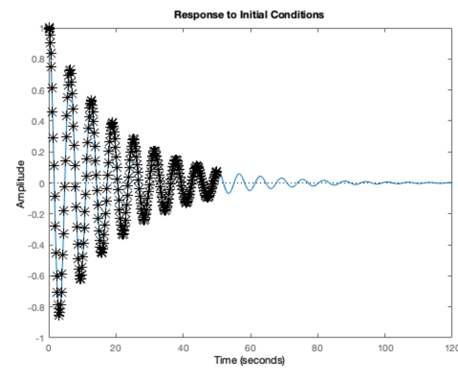


Figure 15: Initial response to x_0 via numerical integration of $\dot{x} = A.x$

When RHS is not null: we can rewrite the ODE with a forced term:

$$m\ddot{x} + c\dot{x} + kx = F_o \cos \omega t$$
$$\ddot{x} + 2\varsigma\omega_n \dot{x} + \omega_n^2 x = f_o \cos \omega t$$

The state vectors become:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2\varsigma\omega_n \dot{x} - \omega_n^2 x + f_o \cos \omega t$$
$$\dot{x} = Ax(t) + f(t), f(t) = \begin{bmatrix} 0 \\ f_0 \cos \omega t \end{bmatrix}$$

Numerically here: $\mathbf{x} = \begin{bmatrix} x = x1 \\ \dot{x} = x2 \end{bmatrix}$

MATLAB CODE EX3c.m

Numerically compute the forced vibration of mass-spring-damper system using ODE45 function in MATLAB for a sinusoidal input of $\omega = 2.5$ rad/s. m=100;k=1000;c=25;

But what is the size of x ?

```
clf; clear all; close all;
m=100;k=1000;c=25;
w=2.5;F=1000;f_o=F;
t=[0:0.1:10];
u=f_o*cos(w*t);
A=[0 1;-k/m -c/m];
```

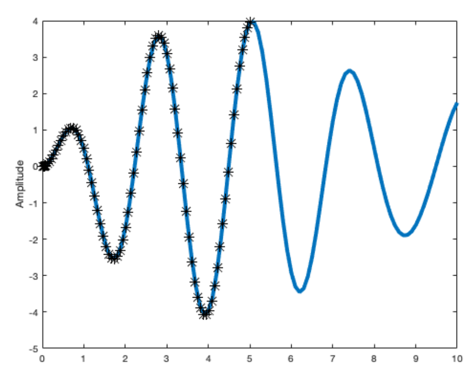


Figure 16: Forced response to x_0 via numerical integration of $\dot{x} = Ax(t) + f(t)$ and Matlab's *lsim*

```
%excitation of dof2
B=[0;1/m];
C=[1 0];
D=[0];
sys=ss(A,B,C,D);
x0=[0,0];
figure;
x=lsim(sys,u,t,x0);
plot(t,x);xlabel('Time');ylabel('Amplitude')
hold on;
TSPAN=[0 5];
Y0=[0;0];
[t,x] =ode45('sdof_forced',TSPAN,Y0);
plot(t,x(:,1),'k*')

function Xdot=sdof_forced(t,X)
m=100;k=1000;c=25;
ze=c/(2*sqrt(k*m));
wn=sqrt(k/m);
w=2.5;F=1000;f_o=F/m;
%excitation of dof2
f= ... % TODO
%A=[0 1;-wn*wn -2*ze*wn];
A=[0 1;-k/m -c/m]
Xdot=... % TODO;
end
```

HOMEWORK 3

One of the common excitation in vibration is a constant force that is applied for a short period of time and then removed. Numerically calculate the response of mass-spring-dashpot system to this excilation in MATLAB.

$$m\ddot{x} + c\dot{x} + kx = F_o \left[1 - H\left(t - t_1\right)\right]$$

where H is Heaviside function. stepfun is useful command to solve this problem.

Multiple Degree of Freedom Systems

Linear Vibrating System and Eigenvalue Problem

In the previous chapters a single degree of freedom system with a single mass, damper and spring was considered. Real systems have multiple degrees of freedom and their analysis is complicated by the large number of equations involved. To deal with them, matrix are used. The equation of motion for n-degree of freedom equation can be written as

$$[m]\{\ddot{\mathbf{x}}\} + [c]\{\dot{\mathbf{x}}\} + [k]\{\mathbf{x}\} = \left[b_f\right]\{\mathbf{f}\} \tag{24}$$

where the mass $[m]$, damping $[c]$, and stiffness $[k]$ matrices are symmetric.

Please note that m, c and k are not scalar (sdof) but now a matrix noted [m] or capital **M** and respectively **C** and **K**

Consider the system with two masses represented in figure 17.

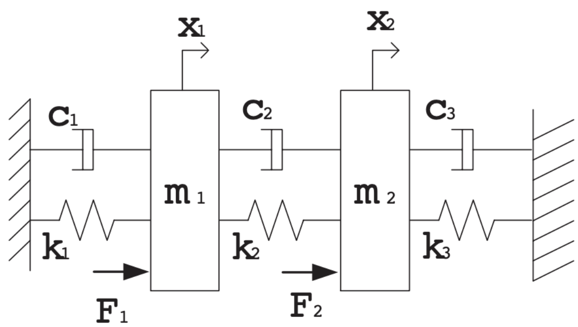


Figure 17: A 2 DOFs system. k_i are in parallel to c_i. It means their associated matrices will have the same topology.

The equations of motion for 25 become

EoM for figure 17. Please note the isotopology betwenn C and K.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

First we need only to consider undamped vibration without excitation force:

$$[m]\{\ddot{\mathbf{x}}\} + [k]\{\mathbf{x}\} = 0 \tag{25}$$

The system 25 can be solved by assuming a harmonic solution of the form

$$\mathbf{x} = \mathbf{u}e^{j\omega t}$$

Here, $\mathbf{u}$ is a vector of constants to be determined, ω is a constant to be determined. Substitution of this assumed form of the solution into the equation of motion yields

$$\left(-\omega^2 M + K\right)\mathbf{u}e^{j\omega t} = 0 \tag{26}$$

Note that the scalar $e^{j\omega t} \neq$ for any value of t and hence equation 26 yields the fact that ω and $\mathbf{u}$ must satisfy the vector equation

This is a standard Eigenvalue problem with $\lambda = \omega^2$: $(K - \lambda M)\,\mathbf{u} = 0$

$$\left(-\omega^2 M + K\right)\mathbf{u} = \mathbf{0} \tag{27}$$

Note that this represents two algebraic equations in the three unknowns; ω, u_1, u_2 where $\mathbf{u} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$. The equation 27 is satisfied for any $\mathbf{u}$ if the determinant of the above equation is zero.

$$\left| -\omega^2 M + K \right| = 0 \tag{28}$$

The simultaneous solution of equation 28 results in the values of parameter ω^2. The ω is called as eigenvalues of the problem.

Once the value of ω is established, the value of the constant vector $\mathbf{u}$ can be found by solving equation 26.

BTW, this is an important reminder: Let X and Y two vectors, and $\mathbf{A}$ a matrix. Thus we have:

$$\frac{\partial}{\partial Y}\left(Y^t \mathbf{A} X\right) = \mathbf{A} X$$

with $Y^t \mathbf{A} X$ a quadratic form.

Handwritten exercise

Let's consider a conservative system and its energy. The Lagrange' s Method for deriving equations of motion is now explained. It can be shown that if the Lagrangian L is defined as

$$L = T - V$$

Then the equations of motion can be calculated from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Which becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = 0$$

Here q is a generalized coordinate

To determine the natural frequencies and natural mode shapes of the system, the undamped free vibration of the system is first considered. First we need to establish 3 quantities:

- Write $2T$ (Kinetic Energy) in a compact form. Deduce $\mathbf{M}$.
- Write $2V$ (Potential Energy) in a compact form. Deduce $\mathbf{K}$.
- Write Lagrangian Equation. Deduce the EoM

Eigenvalue Problem:= undamped EoM

Thus the EoM for 26 is reduced to

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{29}$$

Numerical Example

Let $c_1 = c_2 = c_3 = 0, m_1 = 5$ kg, $m_2 = 10$ kg, $k_1 = 2$ N/m, $k_2 = 2$ N/m, $k_3 = 4$ N/m. Substituting in equation 29 yields

$$\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Assume harmonic responses of the of the form $x_1 = u_1 \exp(i\omega t)$ and $x_2 = u_2 \exp(i\omega t)$. Equation 29 becomes

$$\omega^2 \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

.

Interesting Properties: In physics ω is belonging to $\mathbb{R}^+$. Each eigen vector proportionnal (colinear) to an eigen vector is an eigen vector. It means mode shapes are defined up to a constant

Handwritten exercise

A 3-step approach:

- Solve equation 28: $\left|-\omega^2 M + K\right| = 0 \longrightarrow \omega_i$ (eigenvalues or natural pulsations)
- Solve equation 27 with $\omega = \omega_i$: $\left(-\omega^2 M + K\right) \mathbf{u} = \mathbf{0} \longrightarrow u_i$ (eigenvectors or modeshapes)
- Physical sorting of natural frequencies (with corresponding mode shapes).

Solving Eigenvalue Problem with MATLAB

The eigenvalue problem of a matrix is defined as

$$Au = \lambda u$$

and generalized eigenvalue problem is

$$Ku = \lambda M u$$

HINTS from the prof: check MATLAB's help $[V, D] = eig(A)$ $[V, D] = eig(K, M)$

A N by N A matrix has N eigenvalues and thus N eigenvectors. The function eigsort helps you to sort eigenvalues (each is associated to its own eigenvector).

Matlab's *eig* subroutine is used for computing the eigenvalues and the eigenvectors of the matrix A or (K/M). The eigenvalues of system are stored as the diagonal entries of the diagonal matrix D and the associated eigenvectors are stored in columns of the matrix V.

MATLAB CODE EX4.m

Complete the MATLAB Code to display the natural pulsations and the mode shapes.

```
clear all; close all;
M=[5 0 ;0 10];
K=[4 -2;-2 6];
[v,d]=eig(K,M)
%The function eig in MATLAB gives unsorted eigenvalues,
% so it will be help to make sorting the eigenvalues of the
    system.
[u,wn]=eigsort(K,M);

function [u,wn]=eigsort(k,m);
wn=...%TODO
u=...%TODO
end

disp('The natural frequencies are (rad/sec)')
disp(' ')
wn
disp(' ')
disp('The eigenvectors of the system are')
u
end
```

The two natural frequencies are $\omega_1 = 0.6325\text{rad/s}$, $\omega_2 = 1\text{rad/s}$. The eigenvectors are $u_1 = \left\{ \begin{array}{cc} 1 & 1 \end{array} \right\}^T, u_2 = \left\{ \begin{array}{cc} 1 & -0.5 \end{array} \right\}^T$

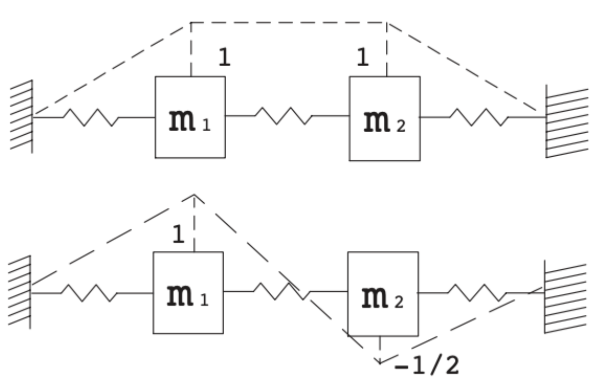


Figure 18: Drawing of u_1 (in phase) and u_2 (in opposite phase) respectively associated with ω_1 and ω_2

Orthogonality of normal modes

The modes are orthogonal with respect to the mass matrix and stiffness matrix.

$$\begin{aligned} \{u\}_2^T[m]\{u\}_1 = 0 \\ \{u\}_2^T[k]\{u\}_1 = 0 \end{aligned} \tag{30}$$

Mass normalizing equation 30, we can get the general relations as

$$\begin{aligned} \{u\}_i^T[m]\{u\}_i = m_i, \quad i = 1,2 \\ \{u\}_i^T[k]\{u\}_i = m_i\omega_i^2 = k_i, \quad i = 1,2 \end{aligned} \tag{31}$$

where m_i and k_i is called modal mass and modal stiffness for the i-th modal vector of vibration.

The numerical values of the mode shape will be used to determine the modal mass and modal stiffness. The mode shapes were found to be $u_1 = \left\{ \begin{array}{cc} 1 & 1 \end{array} \right\}^T$ for $\omega_1 = \sqrt{2/5}\text{rad/s}$, and $u_2 = \left\{ \begin{array}{cc} 1 & -0.5 \end{array} \right\}^T$ for $\omega_1 = 1\text{rad/s}$

MATLAB CODE EX4B.m

Verify with MATLAB Code that the modes are orthogonal. Compute modal mass and modal stiffness.

HINTS from the prof: It is just a verification. Eigenvectors are orthogonal !!

```
%4B
u1 = u(:, 1);
u2 = u(:, 2);
isitzero=u1'*m*u2
%modal mass
m_1 = ...%TODO
m_2 = ...%TODO
%modal stiffness
k_1 = ...%TODO
k_2 = ...%TODO
```

Normalization of Mode Shapes

While above relations are related to the mass and stiffness of the modal space, it is important to remember that the magnitude of these quantities depends upon the normalization of the modal vectors. Therefore, only the combination of a modal vector together with the associated modal mass and stiffness represent a unique absolute characteristic concerning the system being described. When we scaled the eigenvector such that $m_i = 1$, the equation 31 becomes

$$\begin{aligned} \{u\}_i^T[m]\{u\}_i = 1, \quad i = 1,2 \\ \{u\}_i^T[k]\{u\}_i = \omega_i^2, \quad i = 1,2 \end{aligned} \tag{32}$$

This means that m_i is not unique. There are several ways to normalize the mode shapes. (1) The mode shapes can be normalized such that the modal mass m_i is set to unity. (2) The largest element of the mode shape is set to unity. (3) A particular element of the mode shape is set to unity. (4) The norm of the mode vector is set to unity.

Numerical Example

Using the previous two degree of freedom example, normalize the modal vectors such that $\{u\}_i^T[m]\{u\}_i = 1, \quad i = 1,2$ The mass normalization of the first and second natural modes are

$$\{u\}_1 = \frac{1}{\sqrt{m_1}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{15}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
$$\{u\}_2 = \frac{1}{\sqrt{m_2}}\begin{pmatrix} 1 \\ -1/2 \end{pmatrix} = \frac{1}{\sqrt{15/2}}\begin{pmatrix} 1 \\ -1/2 \end{pmatrix}$$

The orthogonality of modes permit us to transform the coupled equations of motion defined in physical coordinate to uncoupled system in the modal coordinate.

Modal coordinates. This is the most tricky part !

In solving the equations of motion for an undamped system 29, the major obstacle encountered when trying to solve for the system response x for a particular set of exciting forces and initial conditions, is the coupling between the equations. The coupling is seen in terms of non-zero off diagonal elements.

If the system of equations could be uncoupled, so that we obtained diagonal mass and stiffness matrices, then each equation would be similar to that of a single degree of freedom system, and could be solved independent of each other. The process of deriving the system response by transforming the equations of motion into an independent set of equations is known as modal analysis

Thus the coordinated transformation we are seeking, is one that decouples the system. The new coordinate system can be found referring to orthogonal properties of the mode shapes discussed in equation 31 and 32.

$$\{x(t)\} = \sum_{i=1}^{n} \{u\}_i q_i(t) \tag{33}$$

where the physical coordinate, $\{x(t)\}$ are related with the normal modes, $\{u\}_i$ and the normal decoupled coordinate, q_i. Equation 33) may be written in matrix form as

$$\{x(t)\} = [P]\{q(t)\} \tag{34}$$

where $[P]$ is called the modal matrix. Thus, the modal matrix for a 2-DOF system can appear as

$$[P] = \begin{bmatrix} \{u\}_1 & \{u\}_2 \end{bmatrix}$$

Substituting equation 34 into the general equation 24, we obtain as

$$[m][P]\{\ddot{q}\} + [c][P]\{\dot{q}\} + [k][P]\{q\} = \{f\}$$

Multiplying on the left by $[P]^T$,

$$[P]^T[m][P]\{\ddot{q}\} + [P]^T[c][P]\{\dot{q}\} + [P]^T[k][P]\{q\} = [P]^T\{f\}$$

We know that orthogonality of the modes with respect to mass and stiffness matrices. Assuming that the viscous damping can be decoupled by modal matrix, we obtain

$$\ddot{q}_i(t) + 2\zeta_i\omega_i\dot{q}(t) + \omega_i^2 q_i(t) = N_i(t), \quad i = 1, 2, \cdots$$

where $N_i(t)$ is

$$N_i(t) = \frac{\{u\}_i^T\{f(t)\}}{\{u\}_i^T[m]\{u\}_i} = \frac{\{u\}_i^T\{f(t)\}}{m_i} \tag{35}$$

The ratio in equation 35 $\frac{\{u\}_i^T}{\{u\}_i^T[m]\{u\}_i}$ is called modal participation factor.

The displacement can be expressed as:

$$x = \sum_{i=1}^{\infty}\{u\}_i q_i = \sum_{i=1}^{\infty}\frac{\{u\}_i\{u\}_i^T\{f(t)\}}{m_i\left[\left(\omega_i^2 - \omega^2\right) + 2i\zeta_i\omega_i\right]}$$

where ω_i is the natural frequency in the i-th mode.

If eigenvector $\{u\}_i$ is mass normalized, $\{u\}_i^T[m]\{u\}_i = 1$.

MATLAB CODE EX4C.m

Compute P and check that the matrix modal stiffness and mass are both diagonal.

HINTS from the prof: It is just a verification. Eigenvectors are orthogonal !! which leads to projected diagonal matrices

```
%4C
P=...%TODO
...%TODO
...%TODO
```

$$[P]^T * [m] * [P] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[P]^T * [k] * [P] = \begin{pmatrix} 0.4 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[P]^T = \begin{pmatrix} 0.2582 & 0.2582 \\ 0.3651 & -0.1826 \end{pmatrix}$$

Numerical Example

Thus the equations of motion in modal coordinate are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 0.4 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0.2582 f_1 + 0.2582 f_2 \\ 0.3651 f_1 - 0.1826 f_2 \end{Bmatrix} = \begin{Bmatrix} f_1' \\ f_2' \end{Bmatrix} \tag{36}$$

The matrix equation 36 can be written in terms of algebraic differential equations

$$\ddot{q}_1 + 0.4q_1 = f_1{}'$$
$$\ddot{q}_2 + q_2 = f_2{}'$$

Hence, the system equations have been uncoupled by using the modal matrix as a coordinate transformation.

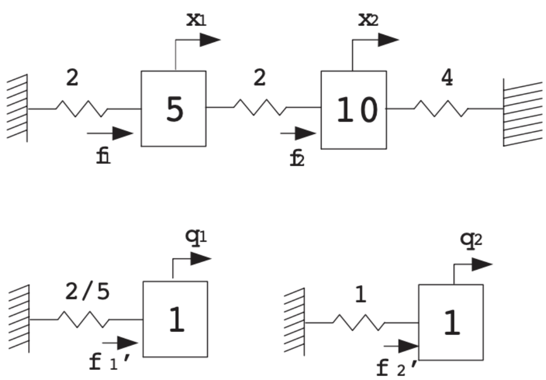


Figure 19: The undamped two degree of freedom system and broken down to two single degree of freedom systems.

These uncoupled (meands simpler) equations in the modal basis (compared to the physical equations) are resumed in the figure 19

MATLAB CODE EX4D.m

Calculate the response of the system illustrated in figure 19 to the initial displacement $x_0 = [1 \quad 1]^T$ and $\dot{x}_0 = [0 \quad 0]^T$ using modal analysis.

HINTS from the prof: retreave physical displacement by x=P*q

```
%4D
x0= [1 1]';
xdot0=[0 0]';
q0=P'*x0
qdot0=P'*xdot0
t=0:0.1:100;
q1=q0(1)*cos(wn(1)*t);
q2=q0(2)*cos(wn(2)*t);
q=[q1;q2];
x=P*q;
figure
plot(t,x);xlabel('Time');ylabel('Amplitude displacement of
    the 2 masses')
```

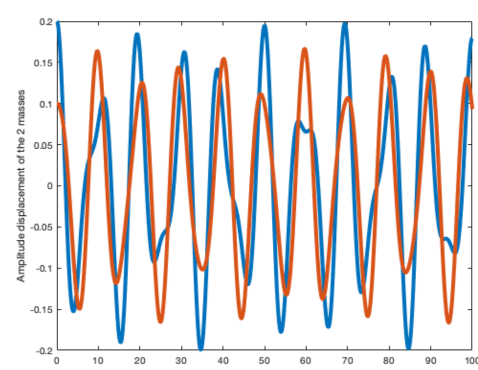


Figure 20: Displacement computed for the 2 masses from the modal to the physical basis using $x = P * q$

Numerical Example

The initial conditions in modal space become

$$\{q(0)\} = [P]^T * \{x(0)\} = \begin{bmatrix} 0.5164 & 0.1826 \end{bmatrix}^T$$
$$\{\dot{q}(0)\} = [P]^T * \{\dot{x}(0)\} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

The modal solution of equation is

$$q_1(t) = q_1(0)\cos(\omega_1 t) = 0.5164\cos(0.6325t)$$

$$q_2(t) = q_2(0)\cos(\omega_2 t) = 0.1826\cos(t)$$

Using the transformation $x(t) = Pq(t)$ yields that the solution in physical coordinates is

$$\mathbf{x}(t) = \begin{pmatrix} 0.1333\cos(0.6325t) + 0.0667\cos(t) \\ 0.1333\cos(0.6325t) - 0.0333\cos(t) \end{pmatrix}$$

Proportional Damping

Damping is present in all oscillatory systems. As there are several types of damping, viscous, hysteretic, coulomb etc., it is generally difficult to ascertain which type of damping is represented in a particular structure. In fact a structure may have damping characteristics resulting from a combination of all types. In many cases, however, the damping is small and certain simplifying assumptions can be made. The most common model for damping is proportional damping defined as

$$[c] = \alpha[m] + \beta[k]$$

where $[c]$ is damping matrix and α, β are constants. For the purposes of most practical problems, the simpler relationship will be sufficient.

Caughey [1] showed that there exists a necessary and sufficient condition for system 24 to be completely uncoupled is that $[m]^{-1}[c]$ commute with $[m]^{-1}[k]$.

$$\left([m]^{-1}[c]\right)\left([m]^{-1}[k]\right) = \left([m]^{-1}[k]\right)\left([m]^{-1}[c]\right)$$

or

$$[c][m]^{-1}[k] = [k][m]^{-1}[c]$$

[1] T.K. Caughey, "Classical Normal Modes in Damped Linear Systems", Journal of Applied Mechanics, Vol 27, Trans. ASME, pp. $269-271, 1960$

Modal Analysis of the Force Response

The forced response of a multiple-degree-of-freedom system can also be calculated by use of modal analysis.

Inman, D. J., Singh, R. C. (1994). Engineering vibration (Vol. 3). Englewood Cliffs, NJ: Prentice Hall. Example 4.6.1 in Inman, pp.296

MATLAB CODE EX4E.m

For this example, let $m_1 = 9$ kg, $m_2 = 1$ kg, $k_1 = 27$ N/m, and $k_2 = 3$ kg. Assume that the damping is proportional with $\alpha = 0$ and $\beta = 0.1$, so that $c_1 = 2.4$Ns/m, and $c_1 = 0.3$Ns/m. Also assume that $F_1 = 0$, and $F_2 = 3\cos 2t$. Calculate the steady-state response.

Numerical Example

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2.7 & -0.3 \\ -0.3 & 0.3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$[P] = \begin{pmatrix} -0.2357 & -0.2357 \\ -0.7071 & 0.7071 \end{pmatrix}$$

$$[P]^T * [m] * [P] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[P]^T * [c] * [P] = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.4 \end{pmatrix}$$

$$[P]^T * [k] * [P] = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

$$[P]^T [B] = \begin{pmatrix} 0 & -0.7071 \\ 0 & 0.7071 \end{pmatrix}$$

HINTS from the prof: That's the most complex case you will treat in this course

```
%4E
M=[9 0 ;0 1];
K=[27 -3;-3 3];
alpha=0
beta=0.1
C=...%TODO
[v,d]=eig(K,M);
%The function eig in MATLAB gives unsorted eigenvalues,
% so it will be help to make sorting the eigenvalues of the
    system.
[u,wn]=eigsort(K,M);
%4B
u1 = u(:, 1);
u2 = u(:, 2);
%4C
```

```
P=[u1 u2]

P'*M*P
P'*C*P
P'*K*P
B=[0 0; 0 1]
P'*B
```

Numerical Example

Hence the decoupled modal equations become

$$\begin{aligned} \ddot{q}_1 + 0.2\dot{q} + 2q_1 &= -0.7071 * 3 * \cos 2t \\ \ddot{q}_2 + 0.4\dot{q} + 4q_2 &= 0.7071 * 3 * \cos 2t \end{aligned} \tag{37}$$

Comparing the coefficient of $\dot{q}_i$ to $2\zeta_i\omega_i$ yields

$$\zeta_1 = \frac{0.2}{2\sqrt{2}}$$
$$\zeta_2 = \frac{0.2}{2 * 2}$$

Thus the damped natural frequencies become

$$\omega_{d1} = \omega_1\sqrt{1 - \zeta_1^2} \simeq 1.41$$
$$\omega_{d2} = \omega_2\sqrt{1 - \zeta_2^2} \simeq 1.99$$

Note that while the force F_2 is applied only to mass m_2, it becomes applied to both coordinate when transformed to modal coordinates. Let the particular solutions of equations 37 be q_{1p} and q_{2p}.

The steady state solution in the physical coordinate system is

$$\mathbf{x}_{ss}(t) = [P]\mathbf{q}_p(t) = \begin{pmatrix} -0.2357q_{1p}(t) - 0.2357q_{2p}(t) \\ -0.7071q_{1p}(t) + 0.7071q_{2p}(t) \end{pmatrix}$$

If you read this sentence, you probably got all the skills needed for solving vibration of discrete systems.

Continuous systems

Rods

Consider the vibation of an elastic rod (or bar) of length L and of varying cross-sectional (general case when it varies slowly) area shown in the left part of figure 21.

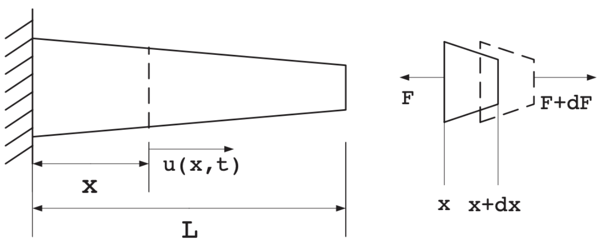


Figure 21: Cantilever rod in longitudinal vibration along x.

The forces on the infinitesimal element summed in the x direction are depicted in the right part of Figure 21:

$$F + dF - F = \rho A(x) dx \frac{\partial u(x,t)}{\partial t^2} \tag{38}$$

where $u(x,t)$ is the deflection of the rod in the x direction. From the solid mechanics,

$$F = EA(x) \frac{\partial u(x,t)}{\partial x} \tag{39}$$

where E is the Young's modulus. The differential form of F becomes

$$dF = \frac{\partial F}{\partial x} dx \tag{40}$$

from the chain rule for partial derivatives. Substitution of equation 39 and 40 into 38 and dividing by dx yields

$$\rho A(x) \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(EA(x) \frac{\partial u(x,t)}{\partial x} \right)$$

When $A(x)$ is a constant this equation becomes

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \left(\frac{E}{\rho} \right) \frac{\partial^2 u(x,t)}{\partial x^2}$$

The quantity $c = \sqrt{E/\rho}$ defines the velocity of propagation of the displacement (or stress wave) in the rod.

HINTS from the prof: $f(t) = A \sin \omega t + B \cos \omega t$

> Handwritten exercice
>
> Derive the solution by decoupling time and space in the solution writing $u(x,t) = f(t)g(x)$

Beams

Let's review Equation of Motion of bending beam depicted in figure 22.

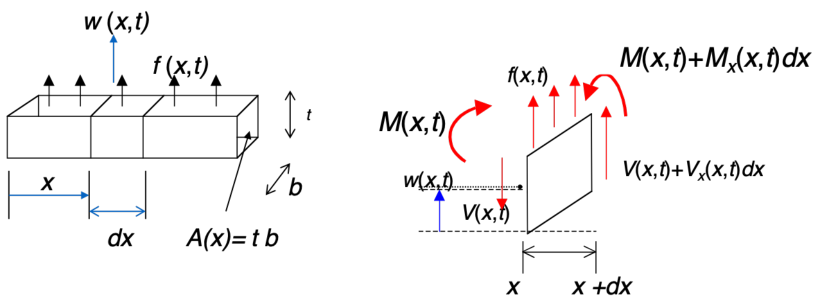


Figure 22: prismatic beam of cross-section t*b oriented longitudinaly along x. Its lenght is L.

The equation of motion of Euler-Bernoulli Beam is

$$m(x)\frac{\partial^2 w}{\partial t^2} + c\frac{\partial w}{\partial t} + EI\frac{\partial^4 w}{\partial x^4} = f(x,t)$$

where, m is mass per unit length of beam defined as $m = \rho A$. If no damping and no external force is applied so that $c = 0, f(x,t) = 0$, and $EI(x)$ and $m(x)$ are assumed to be constant, the following equation simplifies the previous.

$$\frac{\partial^2 w}{\partial t^2} + \frac{EI}{m}\frac{\partial^4 w}{\partial x^4} = 0 \qquad (41)$$

Note that the free vibration equation 41 contains four spatial derivatives and hence requires four boundary conditions. The two time derivatives requires that two initial conditions, one for the displacement and one for the velocity.

For the eigenvalue problem, assume the product solution as

$$w(x,t) = W(x)F(t) \qquad (42)$$

HINTS form the prof: for solving the Eigenvalue Problem use the same approach than with the bar i.e. decouple space and time

where $W(x)$ depends on the spatial position alone and $F(t)$ depends on time alone. Introducing equation 42 into equation 41, we can obtain the following equation as

$$\frac{d^4 W(x)}{dx^4} - \beta^4 W(x) = 0 \qquad (43)$$

where $\beta^4 = \frac{\omega^2 m}{EI}, 0 < x < L$.

Need to solve 43 first to obtain β_i then ω_i. For the time function, $F(t) = A\sin\omega t + B\cos\omega t$, two initial conditions, one for the displacement and one for the velocity are needed to obtain A and B

Start simple with BCs=CF

The boundary conditions for the clamped-free case is depicted in figure 23.

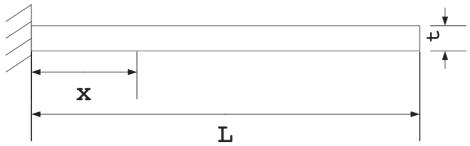


Figure 23: Clamped-Free transverse beam.

We can write:

$$W(0) = 0$$
$$\left.\frac{dW(x)}{dx}\right|_{x=0} = 0$$
$$\left.\frac{d^2W(x)}{dx^2}\right|_{x=L} = 0$$
$$\left.\frac{d^3W(x)}{dx^3}\right|_{x=L} = 0$$

The solution of equation (6.2.2) is

$$W(x) = C_1 \sin\beta x + C_2 \cos\beta x + C_3 \sinh\beta x + C_4 \cosh\beta x$$

Applying the boundary conditions for $x = 0$, we find

$$C_2 + C_4 = 0$$
$$C_1 + C_3 = 0$$

so that the eigenfunction is reduced to $x = L$, we get

$$C_1(\sin\beta L + \sinh\beta L) + C_2(\cos\beta L + \cosh\beta L) = 0$$
$$C_1(\cos\beta L + \cosh\beta L) - C_2(\sin\beta L - \sinh\beta L) = 0$$

Equating the determinant of the coefficients to zero, we obtain the characteristic equation

$$\begin{bmatrix} (\sin\beta L + \sinh\beta L) & (\cos\beta L + \cosh\beta L) \\ (\cos\beta L + \cosh\beta L) & -(\sin\beta L - \sinh\beta L) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Handwritten exercice

Compute the determinant of the system to obtain the characteristic equation (C1=C2=0 is a trivial solution)

Numerical Example

The characteristic equation is

$$\cos\beta L \cosh\beta L = -1$$

Prof Can you solve this in Matlab? Complete Characteristic.m

2 choices: visual solution

$$\cos \beta L = -1/\cosh \beta L$$

or nonlinear equation to solve using Matlab's fsolve

$$\cos \beta L \cosh \beta L + 1 = 0$$

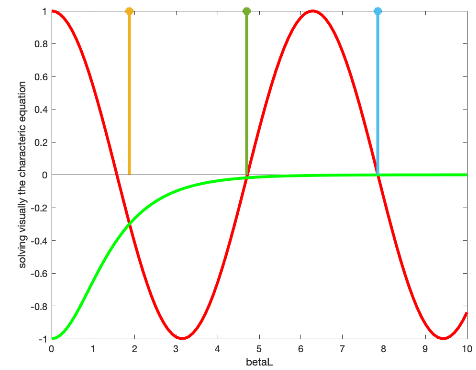


Figure 24: Look at the 3 first solutions: βL=1.8751, 4.6941, 7.8548

```
%solve nonlinear equation using fsolve
clear all; close all;
betaL=0:0.1:10;
plot...%TODO;
hold on;
plot...%TODO;
xlabel('betaL');
ylabel('solving visually the characteric equation')
hold on;

fun = @ (x) ...%TODO;
for x=2:2:8
%Test 2,4,8 but WHY??
x0 = x;
x = fzero(fun, x0)

stem(x,1,'filled','LineWidth',3);
hold on;
end
```

From the numerical analysis $\beta_1 L = 1.875, \beta_2 L = 4.694, \beta_3 L = 7.855$

If I know β (spatial frequency), I know directly the associated natural pulsation ω

$$\begin{aligned}\omega_1 &= (1.875)^2\sqrt{\frac{EI}{mL^4}} \quad \text{rad/sec}\\ \omega_2 &= (4.694)^2\sqrt{\frac{EI}{mL^4}} \quad \text{rad/sec}\\ \omega_3 &= (7.855)^2\sqrt{\frac{EI}{mL^4}} \quad \text{rad/sec}\end{aligned}$$

We obtain the corresponding eigenfunctions

$$W_r(x) = C_r(\cos\beta x - \cosh\beta x) + C_r\frac{\sin\beta L - \sinh\beta L}{\cos\beta L + \cosh\beta L}(\sin\beta x - \sinh\beta x)$$

$$= A_r\left[(\sin\beta_r L - \sinh\beta_r L)(\sin\beta_r x - \sinh\beta_r x) + (\cos\beta_r L + \cosh\beta_r L)(\cos\beta_r x - \cosh\beta_r x)\right]$$

This is a huge equation !

Numerical Example

The geometric and material properties are given in the table.

ρ (Density)	L (Length)	b (Width)	t (Thickness)	E
2750 kg/m^3	340 mm	22 mm	2 mm	7×10^{10} N/m^2

Given the data from next table, the natural frequencies are:

$$\omega_1 = 88.6\text{rad/sec}$$
$$\omega_2 = 555.2\text{rad/sec}$$
$$\omega_3 = 1554.7\text{rad/sec}$$

Normalize the eigenfunction as $\int_0^L W_i^2 dx = 1$

$$A_1 = 0.56461, \quad W_1(x) = A_1[1.72[\cos(5.51x) - \cosh(5.51x)] - 1.26[\sin(5.51x) - \sinh(5.51x)]]$$
$$A_2 = 0.03139, \quad W_2(x) = A_2[1.72[\cos(13.81x) - \cosh(13.81x)] - 1.75[\sin(13.81x) - \sinh(13.81x)]]$$
$$A_3 = 0.00133, \quad W_3(x) = A_3[1.71[\cos(23.10x) - \cosh(23.10x)] - 1.71[\sin(23.1x) - \sinh(23.1x)]]$$

Matlab exercice EX9.m

Plot the 3 first mode shapes of the given clamped-free beam

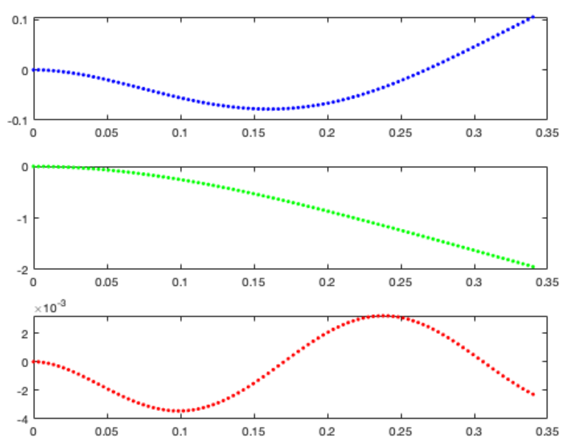
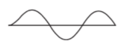
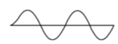
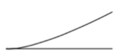
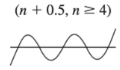


Figure 25: WHO'S WHO? Which graph is mode 1, mode 2, mode 3? Why?

Did you see the effect of the BCs ? Check the slope at $x = 0$ for every modeshape?

CC, HH, CH, CF, FF, symmetry?

OF COURSE changing the BCs will affect natural frequencies AND mode shapes !!! but engineers can sense the physics. Do not compute each characteristic equation and mode shapes please use the information from table 26.

Boundary Conditions		$n = 1$	$n = 2$	$n = 3$	$n = 4$
Case 1 Clamped-clamped	Ω_n/π ‡	1.5056	2.4998	3.5	4.5 ($n + 0.5, n \geq 4$)
	Mode shapes				
	Node points *	None	0.5	0.3585, 0.642	0.279, 0.5, 0.721
Case 2 Hinged-hinged	Ω_n/π ‡	1	2	3	4 ($n, n \geq 4$)
	Mode shapes				
	Node points *	None	0.5	0.333, 0.667	0.25, 0.5, 0.75
Case 3 Clamped-hinged	Ω_n/π ‡	1.25	2.25	3.25	4.25 ($n + 0.25, n \geq 4$)
	Mode shapes				
	Node points *	None	0.5575	0.386, 0.692	0.295, 0.529, 0.768
Case 4 Clamped-free	Ω_n/π ‡	0.5969	1.4942	2.5002	3.500 ($n - 0.5, n \geq 4$)
	Mode shapes				
	Node points *	None	0.783	0.504, 0.868	0.358, 0.644, 0.906
Case 5 Free-free	Ω_n/π ‡	1.5056	2.4998	3.500	4.500 ($n + 0.5, n \geq 4$)
	Mode shapes				
	Node points *	0.224, 0.776	0.132, 0.5, 0.868	0.094, 0.356, 0.644, 0.906	0.0735, 0.277, 0.5, 0.723, 0.927

‡ The natural frequency f_n in Hertz as a function of Ω_n is given by Eq. (9.94).
* Values of η not including the boundaries.

Figure 26: Visual explanation of mode shapes wrt standard BCs. What is a node point? check the slope at left and right BCs.

If you read this sentence, you probably got all the skills needed for solving vibration of 1D continuous systems.

HOMEWORK 7

Calculate the natural frequency and plot first four mode shapes of beam with free-free boundary condition.
Solve the modeshape for a CF beam with a local mass m=ρAL on the free edge for 2 cases: m tends to zero and m tends to infinity.

Conclusions (Engineering Tips)

As a general conclusion I offer some tips from linear algebra.

A change of coordinates is introduced to capitalize on existing mathematics For a symmetric, positive definite matrix M :

Properties of M

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, M^{-1} = \begin{bmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{bmatrix}, M^{-1/2} = \begin{bmatrix} 1/\sqrt{m_1} & 0 \\ 0 & 1/\sqrt{m_2} \end{bmatrix}$$

Let $\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t)$ and multiply by $M^{-1/2}$:

$$\underbrace{M^{-1/2}MM^{-1/2}}_{I \text{ identity}}\ddot{\mathbf{q}}(t) + \underbrace{M^{-1/2}KM^{-1/2}}_{\tilde{K} \text{ symmetric}}\mathbf{q}(t) = \mathbf{0}$$

or $\quad \ddot{\mathbf{q}}(t) + \tilde{K}\mathbf{q}(t) = \mathbf{0} \quad$ where $\tilde{K} = M^{-1/2}KM^{-1/2}$ $\tilde{K}$ is called the mass normalized stiffness and is similar to the scalar $\frac{k}{m}$ used extensively in single degree of freedom analysis. The key here is that $\tilde{K}$ is a SYMMETRIC matrix allowing the use of many nice properties and computational tools.

How the vibration problem relates to the real symmetric eigenvalue problem Assume $\mathbf{q}(t) = \mathbf{v}e^{j\omega t}$ in $\ddot{\mathbf{q}}(t) + \tilde{K}\mathbf{q}(t) = \mathbf{0}$

$$-\omega^2\mathbf{v}e^{j\omega t} + \tilde{K}\mathbf{v}e^{j\omega t} = \mathbf{0}, \quad \mathbf{v} \neq \mathbf{0} \text{ or}$$

$$\underbrace{\tilde{K}\mathbf{v} = \omega^2\mathbf{v}}_{\substack{\text{vibration problem} \\ \text{(standard)}}} \quad \Leftrightarrow \quad \underbrace{\tilde{K}\mathbf{v} = \lambda\mathbf{v}}_{\substack{\text{real symmetric} \\ \text{eigenvalue problem} \\ (new)}} \qquad \mathbf{v} \neq \mathbf{0}$$

Note that the matrix $\tilde{K}$ contains the same type of information as does ω_n^2 in the single degree of freedom case.

Numerical Example

Handwritten exercise

Use K and M from exercise Ex4 i.e. :

$$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } K = \begin{bmatrix} 2.7 & -0.3 \\ -0.3 & 0.3 \end{bmatrix}$$

$$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} . & . \\ . & . \end{bmatrix}\begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} . & . \\ . & . \end{bmatrix}$$

so $\tilde{K} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ which is symmetric.

Hints from the prof: Easy to inverse a diagonal matrix such as M ?

$$\det(\tilde{K} - \lambda I) = \det\begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 8 = 0$$

which has roots: $\lambda_1 = 2 = \omega_1^2$ and $\lambda_2 = 4 = \omega_2^2$

$$\left(\tilde{K} - \lambda_1 I\right)\mathbf{v}_1 = \mathbf{0} \Rightarrow$$
$$\begin{bmatrix} 3-2 & -1 \\ -1 & 3-2 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$
$$v_{11} - v_{12} = 0 \Rightarrow \mathbf{v}_1 = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$\|\mathbf{v}_1\| = \sqrt{\alpha^2(1+1)} = 1 \Rightarrow \alpha = 1/\sqrt{2}$$
$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{The first normalized eigenvector}$$

Likewise the second normalized eigenvector is computed and shown to be orthogonal to the first, so that the set is orthonormal

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_1^T\mathbf{v}_2 = \frac{1}{2}(1-1) = 0$$
$$\mathbf{v}_1^T\mathbf{v}_1 = \frac{1}{2}(1+1) = 1$$
$$\mathbf{v}_2^T\mathbf{v}_2 = \frac{1}{2}(1+(-1)(-1)) = 1$$
$$\Rightarrow \mathbf{v}_i \text{ are orthonormal}$$

Matlab exercise conclusion.m

Solve a 3 DOFs CF system with k=m=1

```
clear all;close all;
%3DOFs BC left cantilever BC Right Free
%BC k m k m k m
%k=1=m

K=[2 -1 0; -1 2 -1; 0 -1 1]
M=[1 0 0; 0 1 0; 0 0 1]
%%
%YES I can Write this ! M is diagonal
Ktilde=...%TODO

%compare the results
[V,D] =eig(Ktilde)
%it is the same !!
[V,D] =eig(K,M)
```

The 2 approaches give exactly the same eigenvectors and of course the same eigenvalues: 0.1981, 1.5550, 3.2470

Gershgorin's theorem

For a given $n \times n$ matrix, Gershgorin's theorem defines n discs in the complex plane whose union contains the eigenvalues of the matrix. The theorem can provide approximations to eigenvalues. It can also provide qualitative information, such as that all the eigenvalues lie in a particular half-plane. Theorem 1 (Gershgorin's theorem). The eigenvalues of $A \in \mathbb{C}^{n\times n}$ lie in the union of the n discs in the complex plane

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j\neq i} |a_{ij}| \right\}, \quad i = 1 : n.$$

Proof. Let λ be an eigenvalue of A and x a corresponding eigenvector and let $|x_k| = \|x\|_\infty$. From the k th equation in $Ax = \lambda x$ we have

$$\sum_{\substack{j=1\\ j\neq k}}^{n} a_{kj}x_j = (\lambda - a_{kk})\, x_k.$$

Hence

$$|\lambda - a_{kk}| \leq \sum_{\substack{j=1\\ j\neq k}}^{n} \left|a_{kj}\right| \left|x_j\right| / \left|x_k\right|,$$

and since $\left|x_j\right| / |x_k| \leq 1$ it follows that λ belongs to the k th disc, D_k.

Matlab exercice conclusion.m

Plot the Gershgorin circles of the 3DOFs CF system with k=m=1. Can you discuss about Ktilde?

```
clear all;close all;
%3DOFs BC left cantilever BC Right Free
%BC k m k m k m
%k=1=m

K=[2 -1 0; -1 2 -1; 0 -1 1]
M=[1 0 0; 0 1 0; 0 0 1]
%%
Ktilde=inv(sqrt(M))*K*inv(sqrt(M))
%what can tell you these discs ?
%Just having a look to Ktilde
gershdisc(Ktilde)
```

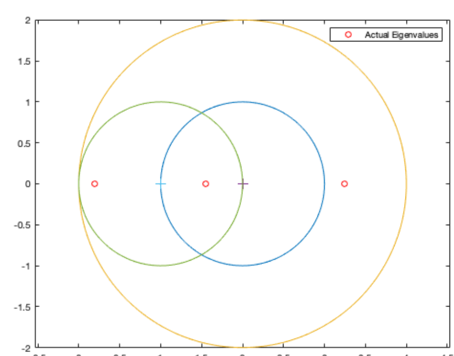


Figure 27: Look at Ktilde and directly approximate the natural frequencies!

On some approximation method

The Rayleigh-Ritz method consists of seeking an approximation of the vibration modes in a space of dimension N generated by N functions ϕ_i chosen. That is to say, we are looking for solutions of the form:

$$W(x,y,z) = \sum_{i=1}^{N} q_i \phi_i(x,y,z) = \{q\}^T\{\phi\} \quad \text{with} \quad \{q\} = \left\{ \begin{array}{c} q_1 \\ \vdots \\ q_N \end{array} \right\} \quad \text{and} \quad \{\phi\} = \left\{ \begin{array}{c} \phi_1 \\ \vdots \\ \phi_N \end{array} \right\}$$

The functions ϕ_i being known, the Rayleigh quotient of such a form depends only on the parameters q_i and can be put in the form:

$$R(W) = R(q_1, \ldots, q_N) = \frac{\{q\}^T \mathbf{K} \{q\}}{\{q\}^T \mathbf{M} \{q\}}$$

where $\mathbf{K}$ is the Stiffness matrix and $\mathbf{M}$ the mass matrix. Minimizing the Rayleigh quotient with respect to the parameters q_i gives the system:

$$\left\{ \begin{array}{c} \frac{\partial R}{\partial q_1} = 0 \\ \vdots \\ \frac{\partial R}{\partial q_N} = 0 \end{array} \right. \quad \text{noted} \quad \frac{\partial R}{\partial \{q\}} = 0$$

which corresponds to :

$$\frac{\mathbf{K}\{q\}\{q\}^T\mathbf{M}\{q\} - \{q\}^T\mathbf{K}\{q\}\mathbf{M}\{q\}}{\left(\{q\}^T\mathbf{M}\{q\}\right)^2} = 0$$

Noting :

$$\omega^2 = \frac{\{q\}^T\mathbf{K}\{q\}}{\{q\}^T\mathbf{M}\{q\}},$$

The minimisation gives :

$$\frac{\mathbf{K}\{q\} - \omega^2\mathbf{M}\{q\}}{\{q\}^T\mathbf{M}\{q\}} = 0$$

The quantities ω_i which ensure the minimum of the Rayleigh quotient are therefore the terms which cancel the determinant:

$$\left| \mathbf{K} - \omega^2 \mathbf{M} \right| = 0$$

We can show that ω_i is an excess approximation of the ith natural frequency of the system.

Matlab exercice conclusion.m

Compute the approximation of the 2 first natural pulsations thanks to (Lord) Rayleigh's quotient using 3 different vector as initial guess ($X1 = [123]'$ or $[111]'$ or a vector really close to u_1 $X1 = [-.32; -.59; -.73]$

```
%Why RR
clear all;close all;

%3DOFs BC left cantilever BC Right Free
%BC k m k m k m
%k=1=m
K=[2 -1 0; -1 2 -1; 0 -1 1]
M=[1 0 0; 0 1 0; 0 0 1]
%%
%YES I can Write this ! M is diagonal
Ktilde=M^(-1/2)*K*M^(-1/2)
%Ktilde=inv(sqrt(M))*K*inv(sqrt(M))
%compare the results
[V,D] =eig(Ktilde)
%it is the same !!
[V,D] =eig(K,M)

%what can tell you these discs ? Just having a look to Ktilde
figure;
gershdisc(Ktilde)


% Rayleigh Ritz
% Let's try this x1 vector
X1=[1 2 3]';
X=X1;
%compare to D(1)
RR=...%TODO
%error with respect to D(1,1) in pc
error1=...%TODO
%is it better?
Xbest=[1 1 1]';
X=Xbest;
%compare to D(1)
RR=...%TODO
%error with respect to D(1,1) in pc
error1bis=...%TODO
%even worst

% ...compare to D(1)
Xbest=[-.32;-.59;-.73];
X=Xbest;
RR=...%TODO
%true value
```

```
RR==D(1,1)
%error with respect to D(1,1) in pc
error1ter=...%TODO

error = [error1 error1bis error1ter];
%[1 2 3],{'[1 2 3]','[1 1 1]','[-.32;-.59;-.73]'})
figure;
bar([1 2 3],error);xlabel('[1 2 3],[1 1 1],[-.32;-.59;-.73]')
    ; ylabel(' error on \omega_1 approximation');
%Oh I Can do that ...compare to D(2)
Xbest=V(:,2); %should try this one...
X=Xbest;
RR=...%TODO
RR==D(2,2)
%error in pc
error2=...%TODO
```

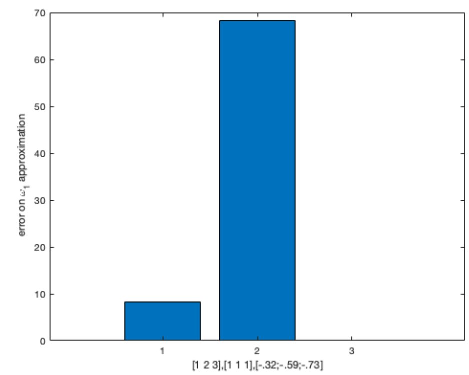


Figure 28: Look at the error!

If you read this sentence, you probably got all the skills needed for solving vibration problem as an engineer.

What can you conclude with Rayleigh-Ritz method?

⇒ If you infere a good shape for the mode and you could approximate the first ith natural pulsation. That's also the spirit of Finite Element Analysis that you will discover in VFEA Part 2!!

What is missing?

⇒ Probably the Experimental Modal Analysis and System Identification part. But Wait... you can find a really nice lecture note in french at:

`https://pagespro.isae-supaero.fr/IMG/pdf/LectureNote_.pdf`

All the best.

J.