VFEA PART 2: Finite Element Analysis

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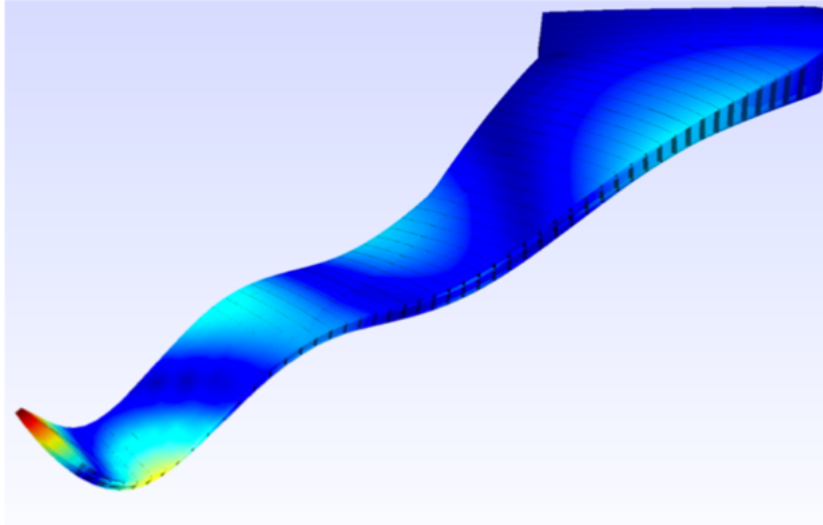
Contents

Introduction

> **How to use the book?**
>
> - This book is really complementary to the live lessons and contains also exercises to program live during the course (in Matlab).
>
> - Largely inspired by www's lecture note available on internet but also from Professors: xx or yy.

> **IN BLUE**
>
> This is a HOMEWORK

> **IN RED**
>
> This is a HANDWRITTEN exercice (during the lesson).

> **IN GREEN**
>
> This is a MATLAB's code to fill (during the lesson). Please download the skeleton's codes at `https://github.com/jomorlier/VFEA`.

Discrete systems

Why starting with Discrete systems? Probably you already seen this concept in electrical engineering for RLC system or express Equation of Motion (EoM) in General Mechanics through Newton's law or Lagrange equations.
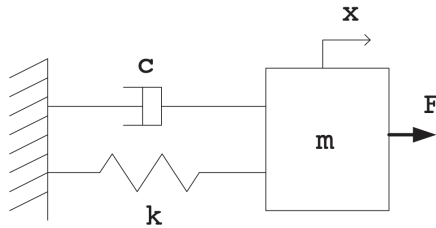
Figure 1: A single-degree-of-freedom system with mass m, spring constant k, and viscous damping c, the system undergoes a dynamic displacement x(t) measured from the static equilibrium position of the mass under the external force F(t).

   In this chapter we will study the responses of systems with a single degree of freedom. It is important topic to master, since the complicated multiple-degree-of-freedom systems (MDOF) can often treated as if they are simple collections of several single-degree-of-freedom (SDOF) systems. Once the responses of SDOF are understood, the study of complicated MDOF becomes relatively easy.

Single-Degree-of-Freedom Systems

Free Vibration.

For the free vibration of a single-degree-of-freedom system with mass m, spring constant k, and viscous damping c, the system undergoes a dynamic displacement x(t) measured from the static equilibrium position of the mass. Applying Newton's law, the equation of motion (EoM) of the system is represented by:

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{1}$$

   subject to the initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$. Note that the RHS is equal to zero which means we express here the intrinsic dynamic properties of the system.
   If we divide equation 1 by m we can reexpress it in terms as

Start Simple ... What does it mean to have RHS equal to ZERO? This equation will help you to compute the normal mode sol103 in Nastran for example with c equal to zero i.e. the intrinsic dynamic behaviour of the system.

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{2}$$

   where $\omega_n = \sqrt{k/m}$ is natural angular frequency and $\zeta = c/(2\sqrt{km})$ is the damping ratio. To solve the damped system of equation 2, assuming

$$x = Ae^{st} \tag{3}$$

Substituting equation 3 into equation 2 yields an algebraic equation in the form

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad (4)$$

The solutions of equation 4 are given by

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{(\zeta^2 - 1)} \qquad (5)$$

- (a) Overdamped Motion In this case, the damping ratio is greater than $1 (\zeta > 1)$. The discriminant of equation 5 is positive, resulting in a pair of distinct real roots. The solution of equation (2.1.1) then becomes

$$x(t) = e^{-\zeta\omega_n t}\left(Ae^{-\omega_n\sqrt{\zeta^2-1}t} + Be^{\omega_n\sqrt{\zeta^2-1}t}\right)$$

which represents that the vibration will not occur since the damping force is so large that the restoration force from the spring is not sufficient to overcome the damping force.

- (b) Underdamped Motion In this case the damping ratio is less than $1 (0 < \zeta < 1)$ and the discriminant of equation 5 is negative, resulting in a complex conjugate pair of roots. The solutions for this case can be expressed as

$$\begin{aligned} x(t) &= e^{-\zeta\omega_n t}\left(Ae^{j\omega_n\sqrt{1-\zeta^2}t} + Be^{-j\omega_n\sqrt{1-\zeta^2}t}\right) \\ &= e^{-\zeta\omega_n t}\left(Ae^{j\omega_d t} + Be^{-j\omega_d t}\right) \\ &= e^{-\zeta\omega_n t}\left(C\cos\omega_d t + D\sin\omega_d t\right) \\ &= Xe^{-\zeta\omega_n t}\sin\left(\omega_d t + \phi\right) \end{aligned}$$

where $j = \sqrt{-1}$, X and ϕ are constants. The the damped natural frequency is denoted by:

$$\omega_d = \omega_n\sqrt{1 - \zeta^2}$$

- (c) Critically Damped Motion In this last case, the damping ratio is exactly $1 (\zeta = 1)$. The solution takes the form

$$x(t) = (A + Bt)e^{-\omega_n t} \qquad (6)$$

The constants A and B are determined by the initial conditions.

> **HOMEWORK 1**
>
> - Review the symbolic computation by reading Exo.mlx
>
> - Describe the definition of logarithmic decrement in free vibration.

How many possibilities? In fact There are three possible cases! But case (b) often happens in physical systems such as structures.

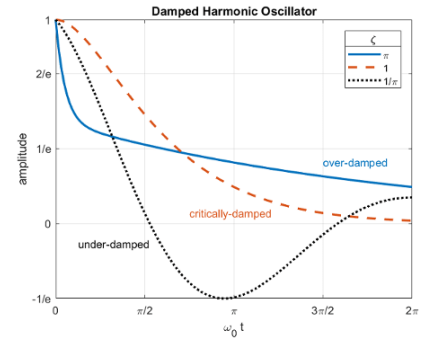Figure 2: The Physics of the Damped Harmonic Oscillator.

What happens when ζ^2 is small compared to 1?

For a single-degree-of-freedom system with viscous damping and subjected to a forcing function $F(t)$ as shown in figure 2.1 , the equation of motion can be written as:

$$m\ddot{x} + c\dot{x} + kx = f(t) \tag{7}$$

The complete solution to equation 7 consists of two parts, the homogenous solution (the complementary solution) and the particular solution. The homogenous solution is the same as the free vibration which was described in last section. It is often common to ignore the transient part of the total solution and focus only on the steady-state response. Taking Laplace transformation of a second order differential equation with zero initial conditions, the transfer function is

> Write the answer here
>
> $$\frac{X(s)}{F(s)} = \frac{...}{...} \tag{8}$$

where $\omega_n = \sqrt{k/m}, \zeta = c/2m\omega_n$. Substituting $j\omega$ for s to calculate the frequency response, where j is the imaginary operator:

$$\frac{X(j\omega)}{F(j\omega)} = \frac{1/m\omega^2}{\left[(\omega_n/\omega)^2 - 1\right] + 2j\zeta\left(\omega_n/\omega\right)}$$

> MATLAB CODE Ex1.m
>
> Plot the amplitude and phase angle of the single degree of freedom system with ζ as a parameter. Complete the following MATLAB Code

```
clf; clear all; close all;
m = 1;
zeta = 0.1:0.1:1; k = 1;
wn = sqrt(k/m);
w = logspace(-1,1,400);
rad2deg = 180/pi;
s = j*w;
for cnt = 1:length(zeta)
xfer(cnt,:)=...; %TODO
mag(cnt,:) = abs(xfer(cnt,:));
phs(cnt,:) = angle(xfer(cnt,:))*rad2deg;

figure(1)
```

Forced Vibration

Let's discuss about Direct Force Excitation. Check the RHS here !

HINTS from the prof: vary ζ from 0.1 to 1, see the effect on the amplitude and phase (loglog)
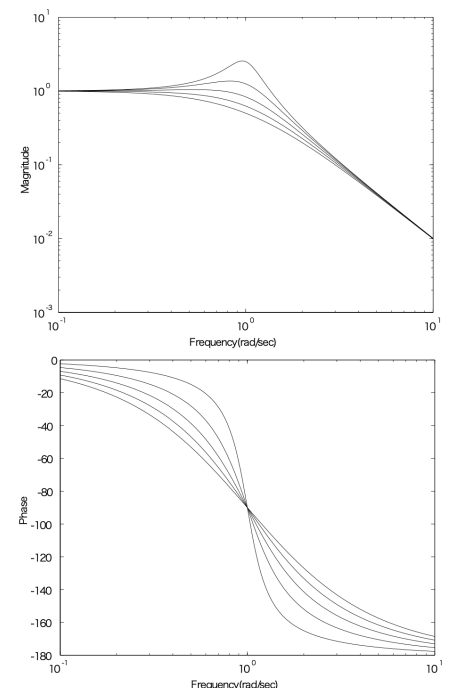
```
loglog(w,mag(cnt,:),'k-')
title('SDOF frequency response magnitudes for zeta = 0.2 to
    1.0 in steps of 0.2')
xlabel('Frequency(rad/sec)')
ylabel('Magnitude')
grid
hold on
figure(2)
semilogx(w,phs(cnt,:),'k-')
title('SDOF frequency response phases for zeta = 0.2 to 1.0
    in steps of 0.2')
xlabel('Frequency(rad/sec)')
ylabel('Phase')
grid
hold on
end
hold off
```

Every linear time invariant (LTI) systems such as for example the one from equation 8 can be specified by transfer functions. The corresponding command is : $sys = tf(num, den)$ The output sys is a model-specific data structure.

> MATLAB CODE Ex2a.m
> -
> Plot the amplitude and phase angle of the single degree of freedom system. Complete the following MATLAB Code

HINTS from the prof. Use the function tf to describe the system using equation 8 Plot the bode function.

```
clf; clear all;
m=1
zeta=0.1
k=1
wn=sqrt(k/m)
den=[...] %TODO
num=[...] %TODO
sys=tf(num,den)
bode(sys)
```
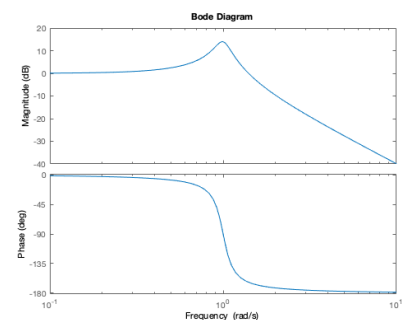

Figure 4: bode plot at fixed $\zeta = 0.1$

The function lsim simulates the response to more general classes of inputs.

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

s=p=j*omega
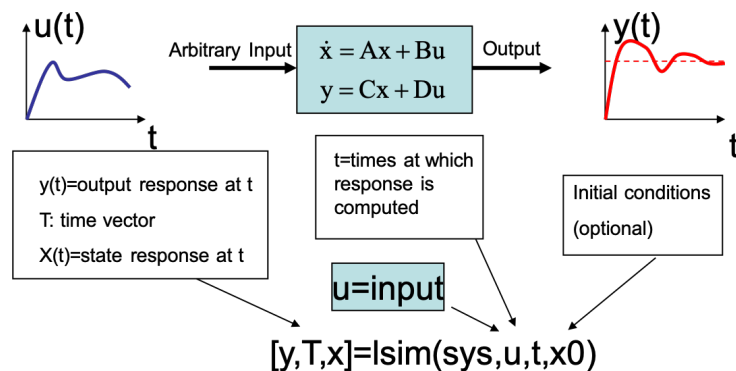
$$X(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}BU(s)$$

u(t)

Arbitrary Input → $\dot{x} = Ax + Bu$, $y = Cx + Du$ → Output

y(t)

t=times at which response is computed

Initial conditions (optional)

y(t)=output response at t
T: time vector
X(t)=state response at t

u=input

[y,T,x]=lsim(sys,u,t,x0)

MATLAB CODE EX2B.m

Try to simulate the response of the linear system sys to a sine wave (ω=1) for a duration of 80 seconds. Complete the following MATLAB Code

```
%2B
t=...; %TODO
u=...; %TODO
figure;
lsim(sys,u,t)
```

HINTS from the prof. Type help lsim. Sys is the transfer function given explicitely. From LTI system we can write $y = sys.u$

Figure 6: Simulation of the response y to a sine input u. By default x_0 is set to 0. Note that as the input is a sine; the ouptut will be a sine (LTI system)
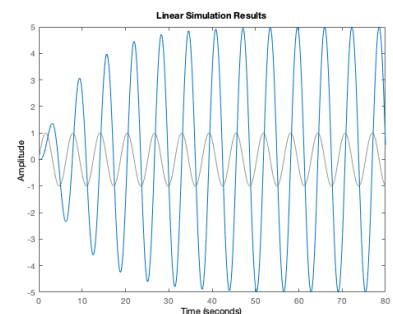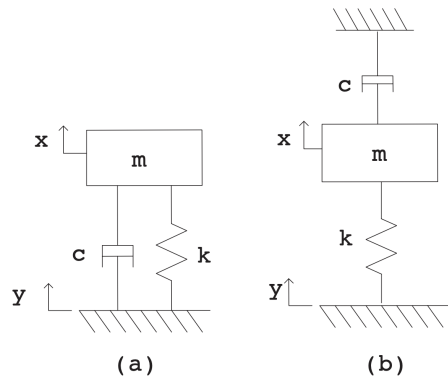
Consider the single degree of freedom system in Figure 7(a).

Figure 7: Free diagram of base excited single degree of freedom system.

The structure with mass m is connected to the base by stiffness, k, and damping with viscous damping coefficient c.

The equation of motion is

Often, machines are harmonically excited through elastic mounting, which may be modeled by springs and dashpots. For example, an automobile suspension system is excited by road surface. Please have a look also to satellites or building modal analysis on youtube

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

(a)Derive the displacement transmissibility, X/Y and plot the magnitude and phase. (b) The transmitted force by the base excitation to the structure is $F_T = k(x - y) + c(\dot{x} - \dot{y})$. The force transmissibility, F_T/kY is defined as the dimensionless relation between maximum base displacement Y and the transmitted force magnitude F_T. Derive the force transmissibility and plot as function of frequency ratio.

> ### HOMEWORK 2
>
> Consider the single degree of freedom system in Figure 7(b). The structure with mass m is connected to the base by stiffness, k. Let's suppose that the viscous damping with viscous damping coefficient c is connected to the Skyhook. (a) Derive the displacement transmissibility, X/Y and plot the magnitude and phase. (b) Derive the force transmissibility, $F_T = k(x - y)$, and plot as function of frequency ratio.

It is desirable to change the system equation for an n DOFs system with n second order differential equation to 2n first order differential equations. The first order form of equations for the system is called as state space form. Start by solving equation second order differential equations.

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t) \longrightarrow m\ddot{x} + c\dot{x} + kx = F \tag{9}$$

we define the state vector as

F so x is depending on t. Whatever F is, we can decompose it into Fourier Series. And of course use LTI system properties

$$\mathbf{x}(t) = [x(t) \quad \dot{x}(t)]^T \longrightarrow \mathbf{x} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$$

Then, adding the identity $\dot{x} = \dot{x}$, equation 9 can be written in the state form as

Please note the capital $\mathbf{x}$ in bold (LHS)

$$\begin{cases} \dot{x} = \dot{x} \\ \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x - \frac{F}{m} = 0 \end{cases} \tag{10}$$

$$\begin{cases} \dot{x} = \dot{x} \\ \ddot{x} = -\frac{k}{m}x - \frac{c}{m}\dot{x} + \frac{F}{m} \end{cases} \tag{11}$$

Reminder: $\dot{\mathbf{x}} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix}$

Write equation 11 in compact form 12 to identify $\mathbf{A}$ and $\mathbf{B}$

- -

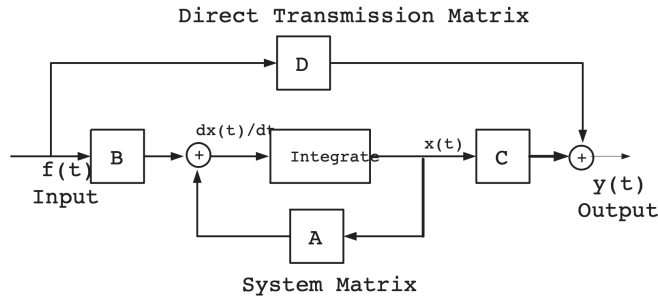$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}F \tag{12}$$

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} \cdot \\ \cdot \end{bmatrix} F \tag{13}$$

where the system matrix is $\mathbf{A} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$ and the input matrix

is $\mathbf{B} = \begin{bmatrix} \cdot \\ \cdot \end{bmatrix}$.

Schematically, a Single Input Single Output (SISO) state space system is represented as shown in figure 8.

Note: To account for the case where the desired output is not just the states but is some linear combination of the states, and output matrix $\mathbf{C}$ is defined to relate the outputs to the states. Also, a matrix $\mathbf{D}$, know as the direct transmission matrix, is multiplied by the input $F(t)$ to account for outputs that are related to the inputs but that bypass the states.

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}F$$

Direct Transmission Matrix
D
dx(t)/dt
f(t) B + Integrate x(t) C + y(t)
Input Output
A
System Matrix

If, for instance, we are interested in controlling the position of the mass, then the output equation **C** is:

$$ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} \tag{14} $$

The output matrix **C** has as many rows as outputs required and as many columns as states. The direct transmission matrix **D** has the same number of columns as the input matrix **B** and as many rows as the output matrix **C**.

> MATLAB CODE EX3.m
> -
> Numerically compute the free vibration of mass-spring-damper system by defining a state space *ss* and using *initial* function in MATLAB. $x_0 = 1$ and $\dot{x}_0 = 0$

HINTS from the prof: use equation of **A** and $\mathbf{C} = [1,0]T$ from equation 14 with function ss. Plot the response to x_0 with the funtion initial.

```
clf; clear all;
m=1;c=0.1;k=1;
A=...; %TODO
C=...; %TODO
sys=ss(A,[],C,[]);
x0=[1,0];
figure;
initial(sys,x0)
```
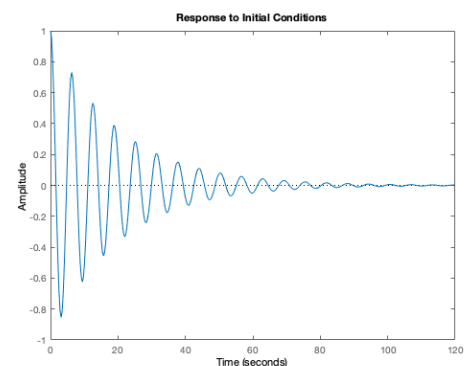

Figure 9: Initial response to $x0 = [1,0]$ of the state space defined by matrix A and C. Remind the vector dimension of $\mathbf{X} = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}$

> **MATLAB CODE EX3b.m**
>
> -
>
> Numerically compute the free vibration of mass-spring-damper system using ODE45 function in MATLAB. $x_0 = 1$ and $\dot{x}_0 = 0$

```matlab
clf; clear all;
t0=0;tf=50;
x0=[1,0];
figure;
[t,x]=ode45('sdof',[t0 tf],x0);
plot(t,x)


function xdot=sdof(t,x)
m=1;c=0.1;k =1;
A=...; %TODO
xdot=...; %TODO
end
```

HINTS from the prof: Help ode45. But what is the size of x ?

Figure 10: Initial response to x_0 via numerical integration of $xdot = A * x$

When RHS is not null: we can rewrite the ODE with a forced term:

$$m\ddot{x} + c\dot{x} + kx = F_o \cos \omega t$$
$$\ddot{x} + 2\varsigma\omega_n\dot{x} + \omega_n^2 x = f_o \cos \omega t$$

The state vectors become:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2\varsigma\omega_n\dot{x} - \omega_n^2 x + f_o \cos \omega t$$
$$\dot{x} = Ax(t) + f(t), f(t) = \begin{bmatrix} 0 \\ f_0 \cos \omega t \end{bmatrix}$$

Numerically here: $\mathbf{x} = \begin{bmatrix} x = x1 \\ \dot{x} = x2 \end{bmatrix}$

> **MATLAB CODE EX3c.m**
>
> -
>
> Numerically compute the forced vibration of mass-spring-damper system using ODE45 function in MATLAB for a sinusoidal input of $\omega = 2.5$ rad/s. m=100;k=1000;c=25;

```matlab
clf; clear all; close all;

m=100;k=1000;c=25;
w=2.5;F=1000;f_o=F;
t=[0:0.1:10];
```

HINTS from the prof: Help ode45. But what is the size of x ?
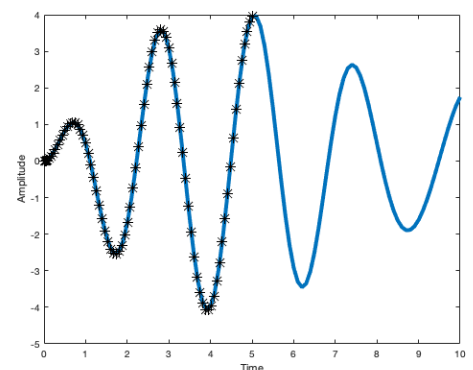
Figure 11: Forced response to x_0 via

```matlab
u=f_o*cos(w*t);
A=[0 1;-k/m -c/m];
%excitation of dof2
B=[0;1/m];
C=[1 0];
D=[0];
sys=ss(A,B,C,D);
x0=[0,0];
figure;
x=lsim(sys,u,t,x0);
plot(t,x);xlabel('Time');ylabel('Amplitude')
hold on;
TSPAN=[0 5];
Y0=[0;0];
[t,x] =ode45('sdof_forced',TSPAN,Y0);
plot(t,x(:,1),'k*')

function Xdot=sdof_forced(t,X)
m=100;k=1000;c=25;
ze=c/(2*sqrt(k*m));
wn=sqrt(k/m);
w=2.5;F=1000;f_o=F/m;
%excitation of dof2
f=[0 ;f_o*cos(w*t)];
%A=[0 1;-wn*wn -2*ze*wn];
A=[0 1;-k/m -c/m]
Xdot=A*X+f;
end
```

HOMEWORK 3

One of the common excitation in vibration is a constant force that is applied for a short period of time and then removed. Numerically calculate the response of mass-spring-dashpot system to this excilation in MATLAB.

$$m\ddot{x} + c\dot{x} + kx = F_o\left[1 - H\left(t - t_1\right)\right]$$

where H is Heaviside function. stepfun is useful command to solve this problem.

Multiple Degree of Freedom Systems

In the previous chapters a single degree of freedom system with a single mass, damper and spring was considered. Real systems have multiple degrees of freedom and their analysis is complicated by the large number of equations involved. To deal with them, matrix are used. The equation of motion for n-degree of freedom equation can be written as

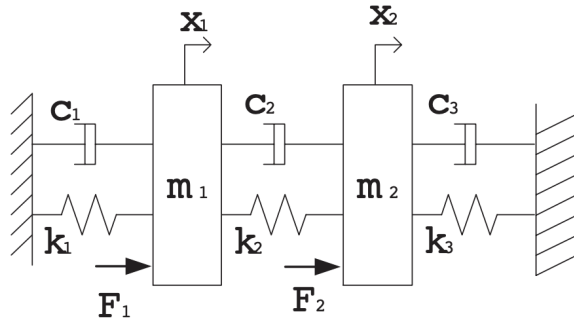$$[m]\{\ddot{\mathbf{x}}\} + [c]\{\dot{\mathbf{x}}\} + [k]\{\mathbf{x}\} = \left[b_f\right]\{\mathbf{f}\} \qquad (15)$$

Please note that m, c and k are not scalar (sdof) but now a matrix noted [m] or capital $\mathbf{M}$ and respectively $\mathbf{C}$ and $\mathbf{K}$

where the mass $[m]$, damping $[c]$, and stiffness $[k]$ matrices are symmetric.

Consider the system with two masses represented in figure 12.

Figure 12: A 2 DOFs system. k_i are in parallel to c_i. It means their associated matrices will have the same topology.

The equations of motion for 16 become

EoM for figure 12. Please note the isotopology betwenn C and K.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_1 \end{bmatrix} \left\{ \begin{array}{c} \ddot{x}_1 \\ \ddot{x}_2 \end{array} \right\} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \left\{ \begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array} \right\} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} = \left\{ \begin{array}{c} F_1(t) \\ F_2(t) \end{array} \right\}$$

First we need only to consider undamped vibration without excitation force:

$$[m]\{\ddot{\mathbf{x}}\} + [k]\{\mathbf{x}\} = 0 \qquad (16)$$

The system 16 can be solved by assuming a harmonic solution of the form

$$\mathbf{x} = \mathbf{u}e^{j\omega t}$$

Here, $\mathbf{u}$ is a vector of constants to be determined, ω is a constant to be determined. Substitution of this assumed form of the solution into the equation of motion yields

$$\left(-\omega^2 M + K\right) \mathbf{u}e^{j\omega t} = 0 \qquad (17)$$

This is a standard Eigenvalue problem with $\lambda = \omega^2$: $(K - \lambda M)\mathbf{u} = 0$

Note that the scalar $e^{j\omega t} \neq$ for any value of t and hence equation 17 yields the fact that ω and $\mathbf{u}$ must satisfy the vector equation

$$\left(-\omega^2 M + K\right) \mathbf{u} = \mathbf{0} \qquad (18)$$

Note that this represents two algebraic equations in the three unknowns; ω, u_1, u_2 where $\mathbf{u} = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$. The equation 18 is satisfied for any $\mathbf{u}$ if the determinant of the above equation is zero.

$$\left| -\omega^2 M + K \right| = 0 \tag{19}$$

The simultaneous solution of equation 19 results in the values of parameter ω^2. The ω is called as eigenvalues of the problem.

Once the value of ω is established, the value of the constant vector $\mathbf{u}$ can be found by solving equation 17.

Handwritten exercise

- -

Let's consider a conservative system and its energy. The Lagrange's Method for deriving equations of motion is now explained. It can be shown that if the Lagrangian L is defined as

$$L = T - V$$

Then the equations of motion can be calculated from

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

Which becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \frac{\partial T}{\partial q} + \frac{\partial V}{\partial q} = 0$$

Here q is a generalized coordinate
BTW, this is an important reminder: Let X and Y two vectors, and $\mathbf{A}$ a matrix. Thus we have:

$$\frac{\partial}{\partial Y}\left(Y^t \mathbf{A} X\right) = \mathbf{A} X$$

with $Y^t \mathbf{A} X$ a quadratic form.

Eigenvalue Problem:= undamped EoM

To determine the natural frequencies and natural mode shapes of the system, the undamped free vibration of the system is first considered. First we need to establish 3 quantities:

- Write $2T$ (Kinetic Energy) in a compact form. Deduce $\mathbf{M}$.

- Write $2V$ (Potential Energy) in a compact form. Deduce $\mathbf{K}$.

- Write Lagrangian Equation. Deduce the EoM

Thus the EoM for 17 is reduced to

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{20}$$

Let $c_1 = c_2 = c_3 = 0, m_1 = 5$ kg, $m_2 = 10$ kg, $k_1 = 2$ N/m, $k_2 = 2$ N/m, $k_3 = 4$ N/m. Substituting in equation 20 yields

$$\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \left\{ \begin{matrix} \ddot{x}_1 \\ \ddot{x}_2 \end{matrix} \right\} + \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} \left\{ \begin{matrix} x_1 \\ x_2 \end{matrix} \right\} = \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\}$$

Assume harmonic responses of the of the form $x_1 = u_1 \exp(i\omega t)$ and $x_2 = u_2 \exp(i\omega t)$. Equation 20 becomes

$$\omega^2 \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \left\{ \begin{matrix} u_1 \\ u_2 \end{matrix} \right\} = \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} \left\{ \begin{matrix} u_1 \\ u_2 \end{matrix} \right\}$$

.

> **Numerical Example**

> Interesting Properties: In physics ω is belonging to $\mathbb{R}^+$. Each eigen vector proportionnal (colinear) to an eigen vector is an eigen vector. It means mode shapes are defined up to a constant

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Handwritten exercise

- -

A 3-step approach:

- Solve equation 19: $\left| -\omega^2 M + K \right| = 0 \longrightarrow \omega_i$

- Solve equation 18 with $\omega = \omega_i$: $\left(-\omega^2 M + K \right) \mathbf{u} = \mathbf{0} \longrightarrow u_i$

- Physical sorting of natural frequencies (with corresponding mode shapes).

> **Solving Eigenvalue Problem with MATLAB**

The eigenvalue problem of a matrix is defined as

$$Au = \lambda u$$

and generalized eigenvalue problem is

$$Ku = \lambda M u$$

Matlab's *eig* subroutine is used for computing the eigenvalues and the eigenvectors of the matrix A or (K/M). The eigenvalues of system are stored as the diagonal entries of the diagonal matrix D and the associated eigenvectors are stored in columns of the matrix V.

MATLAB CODE EX4.m

- -

Complete the MATLAB Code to display the natural pulsations and the mode shapes.

```
clear all; close all;
M=[5 0 ;0 10];
```

> HINTS from the prof: check MATLAB's help
> $>> [V, D] = eig(A)$
> $>> [V, D] = eig(K, M)$
> A 2 by 2 A matrix has 2 eigenvalues and thus 2 mode shapes. The function eigsort helps you to sort eigenvalues (each is associated to its own eigenvector)

```
K=[4 -2;-2 6];
[v,d]=eig(K,M)
%The function eig in MATLAB gives unsorted eigenvalues,
% so it will be help to make sorting the eigenvalues of the
    system.
[u,wn]=eigsort(K,M);

function [u,wn]=eigsort(k,m);
wn=...%TODO
u=...%TODO
end

disp('The natural frequencies are (rad/sec)')
disp(' ')
wn
disp(' ')
disp('The eigenvectors of the system are')
u
end
```

The two natural frequencies are $\omega_1 = 0.6325\text{rad/s}$, $\omega_2 = 1\text{rad/s}$.
The eigenvectors are $u_1 = \left\{\begin{array}{cc} 1 & 1 \end{array}\right\}^T, u_2 = \left\{\begin{array}{cc} 1 & -0.5 \end{array}\right\}^T$
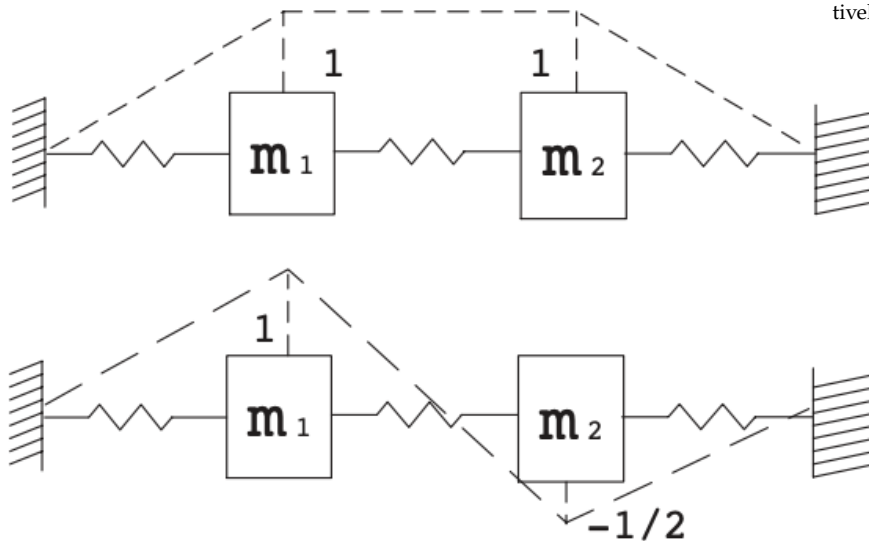
Figure 13: Drawing of u_1 and u_2 respectively associated with ω_1 and ω_2

The modes are orthogonal with respect to the mass matrix and stiffness matrix.

$$\{u\}_2^T[m]\{u\}_1 = 0$$
$$\{u\}_2^T[k]\{u\}_1 = 0 \tag{21}$$

Mass normalizing equation 21, we can get the general relations as

$$\{u\}_i^T[m]\{u\}_i = m_i, \quad i = 1,2$$
$$\{u\}_i^T[k]\{u\}_i = m_i\omega_i^2 = k_i, \quad i = 1,2 \tag{22}$$

where m_i and k_i is called modal mass and modal stiffness for the i-th modal vector of vibration.

The numerical values of the mode shape will be used to determine the modal mass and modal stiffness. The mode shapes were found to be $u_1 = \left\{ \begin{array}{cc} 1 & 1 \end{array} \right\}^T$ for $\omega_1 = \sqrt{2/5}$rad/s, and $u_2 = \left\{ \begin{array}{cc} 1 & -0.5 \end{array} \right\}^T$ for $\omega_1 = 1$rad/s

---

MATLAB CODE EX4B.m

- -

Verify with MATLAB Code that the modes are orthogonal. Compute modal mass and modal stiffness.

HINTS from the prof: It is just a verification. Eigenvectors are orthogonal !!

```
%4B
u1 = u(:, 1);
u2 = u(:, 2);
isitzero=u1'*m*u2
%modal mass
m_1 = ...%TODO
m_2 = ...%TODO
%modal stiffness
k_1 = ...%TODO
k_2 = ...%TODO
```

While above relations are related to the mass and stiffness of the modal space, it is important to remember that the magnitude of these quantities depends upon the normalization of the modal vectors. Therefore, only the combination of a modal vector together with the associated modal mass and stiffness represent a unique absolute characteristic concerning the system being described. When we scaled the eigenvector such that $m_i = 1$, the equation 22 becomes

$$\{u\}_i^T[m]\{u\}_i = 1, \quad i = 1,2$$
$$\{u\}_i^T[k]\{u\}_i = \omega_i^2, \quad i = 1,2 \tag{23}$$

This means that m_i is not unique. There are several ways to normalize the mode shapes. (1) The mode shapes can be normalized such that the modal mass m_i is set to unity. (2) The largest element of the mode shape is set to unity. (3) A particular element of the mode shape is set to unity. (4) The norm of the mode vector is set to unity.

Using the previous two degree of freedom example, normalize the modal vectors such that $\{u\}_i^T[m]\{u\}_i = 1,\quad i = 1,2$ The mass normalization of the first and second natural modes are

$$\{u\}_1 = \frac{1}{\sqrt{m_1}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{15}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\{u\}_2 = \frac{1}{\sqrt{m_2}}\begin{pmatrix}1\\-1/2\end{pmatrix} = \frac{1}{\sqrt{15/2}}\begin{pmatrix}1\\-1/2\end{pmatrix}$$

The orthogonality of modes permit us to transform the coupled equations of motion defined in physical coordinate to uncoupled system in the modal coordinate.

In solving the equations of motion for an undamped system 20, the major obstacle encountered when trying to solve for the system response x for a particular set of exciting forces and initial conditions, is the coupling between the equations. The coupling is seen in terms of non-zero off diagonal elements.

If the system of equations could be uncoupled, so that we obtained diagonal mass and stiffness matrices, then each equation would be similar to that of a single degree of freedom system, and could be solved independent of each other. The process of deriving the system response by transforming the equations of motion into an independent set of equations is known as modal analysis

Thus the coordinated transformation we are seeking, is one that decouples the system. The new coordinate system can be found referring to orthogonal properties of the mode shapes discussed in equation 22 and 23.

$$\{x(t)\} = \sum_{i=1}^{n}\{u\}_i q_i(t) \tag{24}$$

where the physical coordinate, $\{x(t)\}$ are related with the normal modes, $\{u\}_i$ and the normal decoupled coordinate, q_i. Equation 24) may be written in matrix form as

$$\{x(t)\} = [P]\{q(t)\} \tag{25}$$

where $[P]$ is called the modal matrix. Thus, the modal matrix for a 2-DOF system can appear as

$$[P] = \begin{bmatrix}\{u\}_1 & \{u\}_2\end{bmatrix}$$

Substituting equation 25 into the general equation 15, we obtain as

$$[m][P]\{\ddot{q}\} + [c][P]\{\dot{q}\} + [k][P]\{q\} = \{f\}$$

Multiplying on the left by $[P]^T$,

$$[P]^T[m][P]\{\ddot{q}\} + [P]^T[c][P]\{\dot{q}\} + [P]^T[k][P]\{q\} = [P]^T\{f\}$$

We know that orthogonality of the modes with respect to mass and stiffness matrices. Assuming that the viscous damping can be decoupled by modal matrix, we obtain

$$\ddot{q}_i(t) + 2\zeta_i\omega_i\dot{q}(t) + \omega_i^2 q_i(t) = N_i(t), \quad i = 1, 2, \cdots$$

where $N_i(t)$ is

$$N_i(t) = \frac{\{u\}_i^T\{f(t)\}}{\{u\}_i^T[m]\{u\}_i} = \frac{\{u\}_i^T\{f(t)\}}{m_i} \tag{26}$$

The ratio in equation 26 $\frac{\{u\}_i^T}{\{u\}_i^T[m]\{u\}_i}$ is called modal participation factor. The displacement can be expressed as:

$$x = \sum_{i=1}^{\infty}\{u\}_i q_i = \sum_{i=1}^{\infty}\frac{\{u\}_i\{u\}_i^T\{f(t)\}}{m_i\left[(\omega_i^2 - \omega^2) + 2i\zeta_i\omega_i\right]}$$

where ω_i is the natural frequency in the i-th mode.

If eigenvector $\{u\}_i$ is mass normalized, $\{u\}_i^T[m]\{u\}_i = 1$.

> MATLAB CODE EX4C.m
> -
> Compute P and check that modal stiffness and mass is diagonal.

HINTS from the prof: It is just a verification. Eigenvectors are orthogonal !! which leads to projected diagonal matrices

```
%4C
P=...%TODO
...%TODO
...%TODO
```

$$[P]^T * [m] * [P] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[P]^T * [k] * [P] = \begin{pmatrix} 0.4 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[P]^T = \begin{pmatrix} 0.2582 & 0.2582 \\ 0.3651 & -0.1826 \end{pmatrix}$$

Numerical Example

Thus the equations of motion in modal coordinate are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{Bmatrix} + \begin{bmatrix} 0.4 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix} = \begin{Bmatrix} 0.2582f_1 + 0.2582f_2 \\ 0.3651f_1 - 0.1826f_2 \end{Bmatrix} = \begin{Bmatrix} f_1' \\ f_2' \end{Bmatrix} \tag{27}$$

The matrix equation 27 can be written in terms of algebraic differential equations

$$\ddot{q}_1 + 0.4q_1 = f_1'$$
$$\ddot{q}_2 + q_2 = f_2'$$

Hence, the system equations have been uncoupled by using the modal matrix as a coordinate transformation.
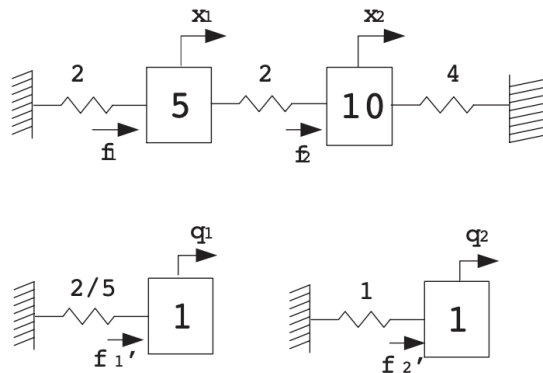

Figure 14: The undamped two degree of freedom system and broken down to two single degree of freedom systems.

These uncoupled (meands simpler) equations in the modal basis (compared to the physical equations) are resumed in the figure 14

MATLAB CODE EX4D.m

- -

Calculate the response of the system illustrated in figure 14 to the initial displacement $x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ with $\dot{x}(0) = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$ using modal analysis.

HINTS from the prof: retreave physical displacement by x=P*q

```
%4D
x0= [1 1]';
xdot0=[0 0]';
q0=P'*x0
qdot0=P'*xdot0
t=0:0.1:100;
q1=q0(1)*cos(wn(1)*t);
q2=q0(2)*cos(wn(2)*t);
q=[q1;q2];
x=P*q;
figure
plot(t,x);xlabel('Time');ylabel('Amplitude displacement of
    the 2 masses')
```
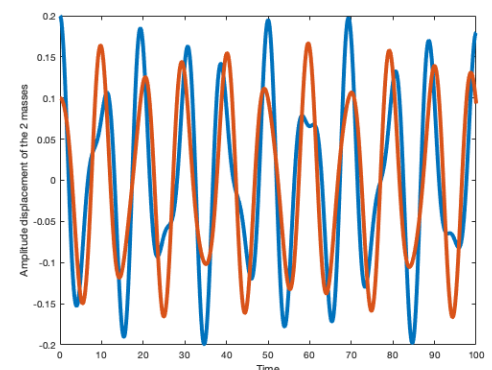

Figure 15: Real Displacement computed for the 2 masses

The initial conditions in modal space become

$$\{q(0)\} = [P]^T * \{x(0)\} = \begin{bmatrix} 0.5164 & 0.1826 \end{bmatrix}^T$$

$$\{\dot{q}(0)\} = [P]^T * \{\dot{x}(0)\} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T$$

The modal solution of equation is

$$q_1(t) = q_1(0)\cos(\omega_1 t) = 0.5164\cos(0.6325t)$$

$$q_2(t) = q_2(0)\cos(\omega_2 t) = 0.1826\cos(t)$$

Using the transformation $x(t) = Pq(t)$ yields that the solution in physical coordinates is

$$\mathbf{x}(t) = \begin{pmatrix} 0.1333\cos(0.6325t) + 0.0667\cos(t) \\ 0.1333\cos(0.6325t) - 0.0333\cos(t) \end{pmatrix}$$

Proportional Damping

Damping is present in all oscillatory systems. As there are several types of damping, viscous, hysteretic, coulomb etc., it is generally difficult to ascertain which type of damping is represented in a particular structure. In fact a structure may have damping characteristics resulting from a combination of all types. In many cases, however, the damping is small and certain simplifying assumptions can be made. The most common model for damping is proportional damping defined as

$$[c] = \alpha[m] + \beta[k]$$

where $[c]$ is damping matrix and α, β are constants. For the purposes of most practical problems, the simpler relationship will be sufficient.

Caughey [1] showed that there exists a necessary and sufficient condition for system 15 to be completely uncoupled is that $[m]^{-1}[c]$ commute with $[m]^{-1}[k]$.

$$\left([m]^{-1}[c]\right)\left([m]^{-1}[k]\right) = \left([m]^{-1}[k]\right)\left([m]^{-1}[c]\right)$$

or

$$[c][m]^{-1}[k] = [k][m]^{-1}[c]$$

[1] T.K. Caughey, "Classical Normal Modes in Damped Linear Systems", Journal of Applied Mechanics, Vol 27, Trans. ASME, pp. $269 - 271, 1960$

The forced response of a multiple-degree-of-freedom system can also be calculated by use of modal analysis.

> MATLAB CODE EX4E.m
>
> -
>
> For this example, let $m_1 = 9$ kg, $m_2 = 1$ kg, $k_1 = 27$ N/m, and $k_2 = 3$ kg. Assume that the damping is proportional with $\alpha = 0$ and $\beta = 0.1$, so that $c_1 = 2.4$Ns/m, and $c_1 = 0.3$Ns/m. Also assume that $F_1 = 0$, and $F_2 = 3\cos 2t$. Calculate the steady-state response.

Numerical Example

$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} \ddot{x}_1 \\ \ddot{x}_2 \end{array} \right\} + \begin{bmatrix} 2.7 & -0.3 \\ -0.3 & 0.3 \end{bmatrix} \left\{ \begin{array}{c} \dot{x}_1 \\ \dot{x}_2 \end{array} \right\} + \begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix} \left\{ \begin{array}{c} x_1 \\ x_2 \end{array} \right\} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{array}{c} F_1 \\ F_2 \end{array} \right\}$$

$$[P] = \begin{pmatrix} -0.2357 & -0.2357 \\ -0.7071 & 0.7071 \end{pmatrix}$$

$$[P]^T * [m] * [P] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$[P]^T * [c] * [P] = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.4 \end{pmatrix}$$

$$[P]^T * [k] * [P] = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

$$[P]^T [B] = \begin{pmatrix} 0 & -0.7071 \\ 0 & 0.7071 \end{pmatrix}$$

HINTS from the prof: That's the most complex case you will treat in this course

```
%4E
M=[9 0 ;0 1];
K=[27 -3;-3 3];
alpha=0
beta=0.1
C=...%TODO
[v,d]=eig(K,M);
%The function eig in MATLAB gives unsorted eigenvalues,
% so it will be help to make sorting the eigenvalues of the
    system.
[u,wn]=eigsort(K,M);
%4B
u1 = u(:, 1);
u2 = u(:, 2);
%4C
```

```
P=[u1 u2]

P'*M*P
P'*C*P
P'*K*P
B=[0 0; 0 1]
P'*B
```

Hence the decoupled modal equations become

$$\ddot{q}_1 + 0.2\dot{q} + 2q_1 = -0.7071 * 3 * \cos 2t$$
$$\ddot{q}_2 + 0.4\dot{q} + 4q_2 = 0.7071 * 3 * \cos 2t$$

(28)

Comparing the coefficient of $\dot{q}_i$ to $2\zeta_i \omega_i$ yields

$$\zeta_1 = \frac{0.2}{2\sqrt{2}}$$
$$\zeta_2 = \frac{0.2}{2*2}$$

Thus the damped natural frequencies become

$$\omega_{d1} = \omega_1 \sqrt{1 - \zeta_1^2} \simeq 1.41$$
$$\omega_{d2} = \omega_2 \sqrt{1 - \zeta_2^2} \simeq 1.99$$

Note that while the force F_2 is applied only to mass m_2, it becomes applied to both coordinate when transformed to modal coordinates. Let the particular solutions of equations 28 be q_{1p} and q_{2p}.

The steady state solution in the physical coordinate system is

$$\mathbf{x}_{ss}(t) = [P]\mathbf{q}_p(t) = \begin{pmatrix} -0.2357 q_{1p}(t) - 0.2357 q_{2p}(t) \\ -0.7071 q_{1p}(t) + 0.7071 q_{2p}(t) \end{pmatrix}$$

Continuous systems

Rods

Consider the vibation of an elastic rod (or bar) of length L and of varying cross-sectional (general case when it varies slowly) area shown in the left part of figure 16.
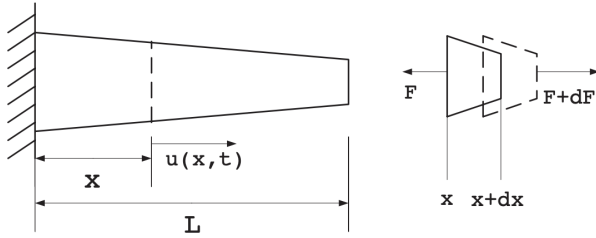

Figure 16: Cantilever rod in longitudinal vibration along x.

The forces on the infinitesimal element summed in the x direction are depicted in the right part of Figure 16:

$$F + dF - F = \rho A(x) dx \frac{\partial u(x,t)}{\partial t^2} \tag{29}$$

where $u(x,t)$ is the deflection of the rod in the x direction. From the solid mechanics,

$$F = EA(x) \frac{\partial u(x,t)}{\partial x} \tag{30}$$

where E is the Young's modulus. The differential form of F becomes

$$dF = \frac{\partial F}{\partial x} dx \tag{31}$$

from the chain rule for partial derivatives. Substitution of equation 30 and 31 into 29 and dividing by dx yields

$$\rho A(x) \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x} \left(EA(x) \frac{\partial u(x,t)}{\partial x} \right)$$

When $A(x)$ is a constant this equation becomes

$$\frac{\partial^2 u(x,t)}{\partial t^2} = \left(\frac{E}{\rho} \right) \frac{\partial^2 u(x,t)}{\partial x^2}$$

The quantity $c = \sqrt{E/\rho}$ defines the velocity of propagation of the displacement (or stress wave) in the rod.

HINTS from the prof: $f(t) = A \sin \omega t + B \cos \omega t$

> **Handwritten exercice**
> -
> Derive the solution by decoupling time and space in the solution writing $u(x,t) = f(t)g(x)$

beams

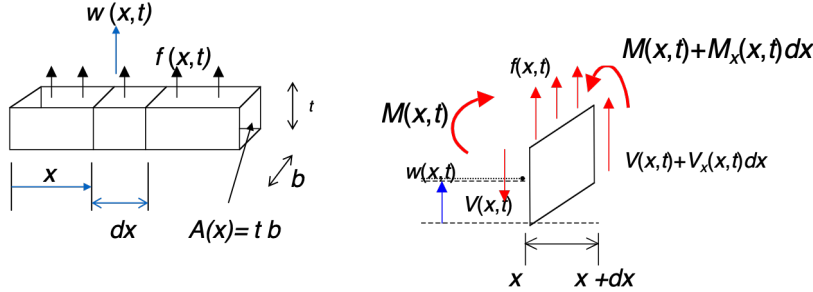Let's review Equation of Motion of bending beam depicted in figure 17.

Figure 17: prismatic beam of cross-section t*b oriented longitudinaly along x. Its lenght is L.

The equation of motion of Euler-Bernoulli Beam is

$$m(x)\frac{\partial^2 w}{\partial t^2} + c\frac{\partial w}{\partial t} + EI\frac{\partial^4 w}{\partial x^4} = f(x,t)$$

where, m is mass per unit length of beam defined as $m = \rho A$. If no damping and no external force is applied so that $c = 0$, $f(x,t) = 0$, and $EI(x)$ and $m(x)$ are assumed to be constant, the following equation simplifies the previous.

$$\frac{\partial^2 w}{\partial t^2} + \frac{EI}{m}\frac{\partial^4 w}{\partial x^4} = 0 \tag{32}$$

For the eigenvalue problem, assume the product solution as

$$w(x,t) = W(x)F(t) \tag{33}$$

where $W(x)$ depends on the spatial position alone and $F(t)$ depends on time alone. Introducing equation 33 into equation 32, we can obtain the following equation as

$$\frac{d^4 W(x)}{dx^4} - \beta^4 W(x) = 0 \tag{34}$$

where $\beta^4 = \frac{\omega^2 m}{EI}, 0 < x < L$.

Note that the free vibration equation 32 contains four spatial derivatives and hence requires four boundary conditions. The two time derivatives requires that two initial conditions, one for the displacement and one for the velocity.

HINTS form the prof: for solving the Eigenvalue Problem use the same approach than with the bar i.e. decouple space and time

Need to solve 34 first to solve then in ω and thus identify A and B in $F(t) = A\sin\omega t + B\cos\omega t$

The boundary conditions for the clamped-free case is depicted in figure 18.
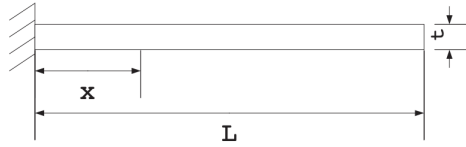
We can write:

$$W(0) = 0$$

$$\left.\frac{dW(x)}{dx}\right|_{x=0} = 0$$

$$\left.\frac{d^2W(x)}{dx^2}\right|_{x=L} = 0$$

$$\left.\frac{d^3W(x)}{dx^3}\right|_{x=L} = 0$$

The solution of equation (6.2.2) is

$$W(x) = C_1 \sin\beta x + C_2 \cos\beta x + C_3 \sinh\beta x + C_4 \cosh\beta x$$

Applying the boundary conditions for $x = 0$, we find

$$C_2 + C_4 = 0$$
$$C_1 + C_3 = 0$$

so that the eigenfunction is reduced to $x = L$, we get

$$C_1(\sin\beta L + \sinh\beta L) + C_2(\cos\beta L + \cosh\beta L) = 0$$
$$C_1(\cos\beta L + \cosh\beta L) - C_2(\sin\beta L - \sinh\beta L) = 0$$

Equating the determinant of the coefficients to zero, we obtain the characteristic equation

$$\begin{bmatrix} (\sin\beta L + \sinh\beta L) & (\cos\beta L + \cosh\beta L) \\ (\cos\beta L + \cosh\beta L) & -(\sin\beta L - \sinh\beta L) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

> **Handwritten exercice**
> -
> Compute the determinant of the system to obtain the characteristic equation ($C_1=C_2=0$ is a trivial solution i.e. need to avoid this one !)

The characteristic equation is

$$\cos\beta L \cosh\beta L = -1$$

Prof Can you solve this in Matlab? Complete Characteristic.m

- -

2 choices: visual solution

$$\cos\beta L = -1/\cosh\beta L$$

or nonlinear equation to solve using Matlab's fsolve

$$\cos\beta L \cosh\beta L + 1 = 0$$

Figure 19: Look at the 3 first solutions: βL=1.8751, 4.6941, 7.8548

```matlab
%solve nonlinear equation using fsolve
clear all; close all;
betaL=0:0.1:10;
plot...%TODO;
hold on;
plot...%TODO;
xlabel('betaL');
ylabel('solving visually the charactric equation')
hold on;

fun = @ (x) ...%TODO;
for x=2:2:8
%Test 2,4,8
x0 = x;
x = fzero(fun, x0)

stem(x,1,'filled','LineWidth',3);
hold on;
end
```

From the numerical analysis $\beta_1 L = 1.875, \beta_2 L = 4.694, \beta_3 L = 7.855$

If I know β (spatial frequency), I know directly the associated natural pulsation ω

$$\omega_1 = (1.875)^2 \sqrt{\frac{EI}{mL^4}} \quad \text{rad/sec}$$
$$\omega_2 = (4.694)^2 \sqrt{\frac{EI}{mL^4}} \quad \text{rad/sec}$$
$$\omega_3 = (7.855)^2 \sqrt{\frac{EI}{mL^4}} \quad \text{rad/sec}$$

We obtain the corresponding eigenfunctions

$$W_r(x) = C_r(\cos\beta x - \cosh\beta x) + C_r \frac{\sin\beta L - \sinh\beta L}{\cos\beta L + \cosh\beta L}(\sin\beta x - \sinh\beta x)$$

$$= A_r\left[(\sin\beta_r L - \sinh\beta_r L)(\sin\beta_r x - \sinh\beta_r x) + (\cos\beta_r L + \cosh\beta_r L)(\cos\beta_r x - \cosh\beta_r x)\right]$$

Numerical Example

The geometric and material properties are given in the table.

ρ (Density)	L (Length)	b (Width)	t (Thickness)	E
2750 kg/m^3	340 mm	22 mm	2 mm	7×10^{10} N/m^2

Given the data from next table, the natural frequencies are:

$$\omega_1 = 88.6 \text{rad/sec} = 14.1 \text{ Hz}$$
$$\omega_2 = 555.2 \text{rad/sec} = 88.4 \text{ Hz}$$
$$\omega_3 = 1554.7 \text{rad/sec} = 247.4 \text{ Hz}$$

Normalize the eigenfunction as $\int_0^L W_i^2 dx = 1$

$$A_1 = 0.56461, \quad W_1(x) = A_1[1.72[\cos(5.51x) - \cosh(5.51x)] - 1.26[\sin(5.51x) - \sinh(5.51x)]]$$
$$A_2 = 0.03139, \quad W_2(x) = A_2[1.72[\cos(13.81x) - \cosh(13.81x)] - 1.75[\sin(13.81x) - \sinh(13.81x)]]$$
$$A_3 = 0.00133, \quad W_3(x) = A_3[1.71[\cos(23.10x) - \cosh(23.10x)] - 1.71[\sin(23.1x) - \sinh(23.1x)]]$$

> Matlab exercice EX9.m
> -
> Plot the 3 first mode shapes of the given clamped-free beam
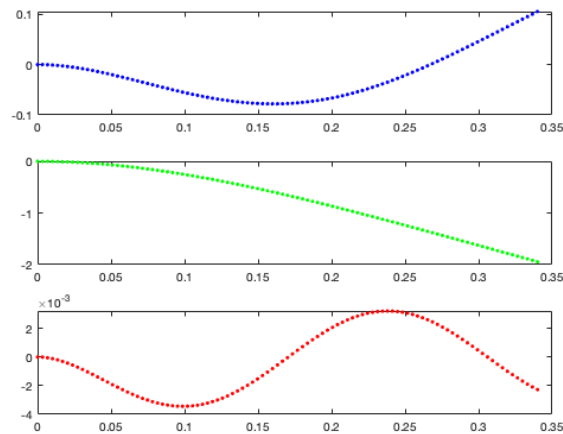

Figure 20: WHO'S WHO? Which graph is mode 1, mode 2, mode 3? Why?

OF COURSE changing the BCs will affect natural frequencies AND mode shapes !!! but engineers can sense the physics. Do not compute each characteristic equation and mode shapes please use the information from table 21.

CC, HH, CH, CF, FF, symmetry?

Boundary Conditions		$n = 1$	$n = 2$	$n = 3$	$n = 4$
Case 1 Clamped-clamped	$\Omega_n/\pi^{\ddagger}$	1.5056	2.4998	3.5	4.5 $(n + 0.5, n \geq 4)$
	Mode shapes				
	Node points*	None	0.5	0.3585, 0.642	0.279, 0.5, 0.721
Case 2 Hinged-hinged	$\Omega_n/\pi^{\ddagger}$	1	2	3	4 $(n, n \geq 4)$
	Mode shapes				
	Node points*	None	0.5	0.333, 0.667	0.25, 0.5, 0.75
Case 3 Clamped-hinged	$\Omega_n/\pi^{\ddagger}$	1.25	2.25	3.25	4.25 $(n + 0.25, n \geq 4)$
	Mode shapes				
	Node points*	None	0.5575	0.386, 0.692	0.295, 0.529, 0.768
Case 4 Clamped-free	$\Omega_n/\pi^{\ddagger}$	0.5969	1.4942	2.5002	3.500 $(n - 0.5, n \geq 4)$
	Mode shapes				
	Node points*	None	0.783	0.504, 0.868	0.358, 0.644, 0.906
Case 5 Free-free	$\Omega_n/\pi^{\ddagger}$	1.5056	2.4998	3.500	4.500 $(n + 0.5, n \geq 4)$
	Mode shapes				
	Node points*	0.224, 0.776	0.132, 0.5, 0.868	0.094, 0.356, 0.644, 0.906	0.0735, 0.277, 0.5, 0.723, 0.927

$\ddagger$ The natural frequency f_n in Hertz as a function of Ω_n is given by Eq. (9.94).
* Values of η not including the boundaries.

Figure 21: Visual explanation of mode shapes wrt standard BCs. What is a node point? check the slope at left and right BCs.

HOMEWORK 7

Calculate the natural frequency and plot first four mode shapes of beam with free-free boundary condition.
Solve the modeshape for a CF beam with a local mass $m = \rho AL$ on the free edge for 2 cases m tends to zero, m tends to infinity.

Engineering Tips (from linear algebra)

As a general conclusion I offer some tips from linear algebra.

A change of coordinates is introduced to capitalize on existing mathematics For a symmetric, positive definite matrix M :

$$M = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, M^{-1} = \begin{bmatrix} 1/m_1 & 0 \\ 0 & 1/m_2 \end{bmatrix}, M^{-1/2} = \begin{bmatrix} 1/\sqrt{m_1} & 0 \\ 0 & 1/\sqrt{m_2} \end{bmatrix}$$

Let $\mathbf{x}(t) = M^{-1/2}\mathbf{q}(t)$ and multiply by $M^{-1/2}$:

$$\underbrace{M^{-1/2}MM^{-1/2}}_{I \text{ identity}} \ddot{\mathbf{q}}(t) + \underbrace{M^{-1/2}KM^{-1/2}}_{\tilde{K} \text{ symmetric}} \mathbf{q}(t) = \mathbf{0}$$

or $\ddot{\mathbf{q}}(t) + \tilde{K}\mathbf{q}(t) = \mathbf{0}$ where $\tilde{K} = M^{-1/2}KM^{-1/2}$ $\tilde{K}$ is called the mass normalized stiffness and is similar to the scalar $\frac{k}{m}$ used extensively in single degree of freedom analysis. The key here is that $\tilde{K}$ is a SYMMETRIC matrix allowing the use of many nice properties and computational tools.

How the vibration problem relates to the real symmetric eigenvalue problem Assume $\mathbf{q}(t) = \mathbf{v}e^{j\omega t}$ in $\ddot{\mathbf{q}}(t) + \tilde{K}\mathbf{q}(t) = \mathbf{0}$

$$-\omega^2 \mathbf{v}e^{j\omega t} + \tilde{K}\mathbf{v}e^{j\omega t} = \mathbf{0}, \quad \mathbf{v} \neq \mathbf{0} \text{ or}$$

$$\underbrace{\tilde{K}\mathbf{v} = \omega^2\mathbf{v}}_{\substack{\text{vibration problem} \\ \text{(standard)}}} \quad \Leftrightarrow \quad \underbrace{\tilde{K}\mathbf{v} = \lambda\mathbf{v}}_{\substack{\text{real symmetric} \\ \text{eigenvalue problem} \\ (new)}} \quad \mathbf{v} \neq \mathbf{0}$$

Note that the matrix $\tilde{K}$ contains the same type of information as does ω_n^2 in the single degree of freedom case.

> **Handwritten exercice**
>
> Use K and M from exercise Ex4 i.e. :
> $$M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } K = \begin{bmatrix} 2.7 & -0.3 \\ -0.3 & 0.3 \end{bmatrix}$$
>
> $$\tilde{K} = M^{-1/2}KM^{-1/2} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}\begin{bmatrix} 27 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$$

Hints from the prof: Easy to inverse a diagonal matrix such as M ?

so $\tilde{K} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ which is symmetric.

$$\det(\tilde{K} - \lambda I) = \det\begin{bmatrix} 3-\lambda & -1 \\ -1 & 3-\lambda \end{bmatrix} = \lambda^2 - 6\lambda + 8 = 0$$

which has roots: $\lambda_1 = 2 = \omega_1^2$ and $\lambda_2 = 4 = \omega_2^2$

$$\left(\tilde{K} - \lambda_1 I\right)\mathbf{v}_1 = \mathbf{0} \Rightarrow$$

$$\begin{bmatrix} 3-2 & -1 \\ -1 & 3-2 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$v_{11} - v_{12} = 0 \Rightarrow \mathbf{v}_1 = \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\|\mathbf{v}_1\| = \sqrt{\alpha^2(1+1)} = 1 \Rightarrow \alpha = 1/\sqrt{2}$$

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{The first normalized eigenvector}$$

Likewise the second normalized eigenvector is computed and shown to be orthogonal to the first, so that the set is orthonormal

$$\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_1^T\mathbf{v}_2 = \frac{1}{2}(1-1) = 0$$

$$\mathbf{v}_1^T\mathbf{v}_1 = \frac{1}{2}(1+1) = 1$$

$$\mathbf{v}_2^T\mathbf{v}_2 = \frac{1}{2}(1 + (-1)(-1)) = 1$$

$$\Rightarrow \mathbf{v}_i \text{ are orthonormal}$$

> Matlab exercice conclusion.m
> -
> Solve a 3DOFs CF system with k=m=1

```
clear all;close all;
%3DOFs BC left cantilever BC Right Free
%BC k m k m k m
%k=1=m

K=[2 -1 0; -1 2 -1; 0 -1 1]
M=[1 0 0; 0 1 0; 0 0 1]
%%
%YES I can Write this ! M is diagonal
Ktilde=...%TODO

%compare the results
[V,D] =eig(Ktilde)
%it is the same !!
[V,D] =eig(K,M)
```

The 2 approaches give exactly the same eigenvectors and eigenvalues: 0.1981, 1.5550, 3.2470

For a given $n \times n$ matrix, Gershgorin's theorem defines n discs in the complex plane whose union contains the eigenvalues of the matrix. The theorem can provide approximations to eigenvalues. It can also provide qualitative information, such as that all the eigenvalues lie in a particular half-plane. Theorem 1 (Gershgorin's theorem). The eigenvalues of $A \in \mathbb{C}^{n \times n}$ lie in the union of the n discs in the complex plane

$$D_i = \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}| \right\}, \quad i = 1 : n.$$

Proof. Let λ be an eigenvalue of A and x a corresponding eigenvector and let $|x_k| = \|x\|_\infty$. From the k th equation in $Ax = \lambda x$ we have

$$\sum_{\substack{j=1 \\ j \neq k}}^{n} a_{kj} x_j = (\lambda - a_{kk}) x_k.$$

Hence

$$|\lambda - a_{kk}| \leq \sum_{\substack{j=1 \\ j \neq k}}^{n} |a_{kj}| \, |x_j| \, / \, |x_k|,$$

and since $|x_j| \, / \, |x_k| \leq 1$ it follows that λ belongs to the k th disc, D_k.

> Matlab exercice conclusion.m
> -
> Plot the Gershgorin circles of the 3DOFs CF system with k=m=1. Can you discuss about Ktilde?

```
clear all;close all;
%3DOFs BC left cantilever BC Right Free
%BC k m k m k m
%k=1=m

K=[2 -1 0; -1 2 -1; 0 -1 1]
M=[1 0 0; 0 1 0; 0 0 1]
%%
Ktilde=inv(sqrt(M))*K*inv(sqrt(M))
%what can tell you these discs ?
%Just having a look to Ktilde
gershdisc(Ktilde)
```
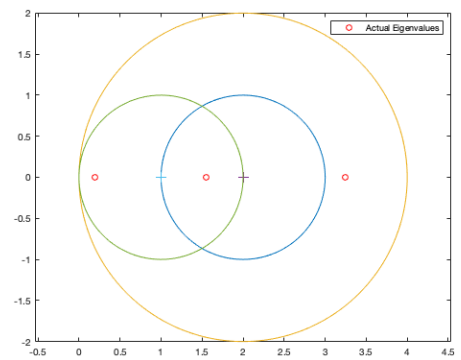

Figure 22: Look at Ktilde and directly approximate the natural frequencies!

The Rayleigh-Ritz method consists of seeking an approximation of the vibration modes in a space of dimension N generated by N functions ϕ_i chosen. That is to say, we are looking for solutions of the form:

$$W(x,y,z) = \sum_{i=1}^{N} q_i \phi_i(x,y,z) = \{q\}^T\{\phi\} \quad \text{with} \quad \{q\} = \left\{ \begin{array}{c} q_1 \\ \vdots \\ q_N \end{array} \right\} \quad \text{and} \quad \{\phi\} = \left\{ \begin{array}{c} \phi_1 \\ \vdots \\ \phi_N \end{array} \right\}$$

The functions ϕ_i being known, the Rayleigh quotient of such a form depends only on the parameters q_i and can be put in the form:

$$R(W) = R(q_1, \ldots, q_N) = \frac{\{q\}^T\mathbf{K}\{q\}}{\{q\}^T\mathbf{M}\{q\}}$$

where $\mathbf{K}$ is the Stiffness matrix and $\mathbf{M}$ the mass matrix. Minimizing the Rayleigh quotient with respect to the parameters q_i gives the system:

$$\left\{ \begin{array}{c} \frac{\partial R}{\partial q_1} = 0 \\ \vdots \\ \frac{\partial R}{\partial q_N} = 0 \end{array} \right. \quad \text{noted} \quad \frac{\partial R}{\partial \{q\}} = 0$$

which corresponds to :

$$\frac{\mathbf{K}\{q\}\{q\}^T\mathbf{M}\{q\} - \{q\}^T\mathbf{K}\{q\}\mathbf{M}\{q\}}{\left(\{q\}^T\mathbf{M}\{q\}\right)^2} = 0$$

Noting :

$$\omega^2 = \frac{\{q\}^T\mathbf{K}\{q\}}{\{q\}^T\mathbf{M}\{q\}},$$

The minimisation gives :

$$\frac{\mathbf{K}\{q\} - \omega^2\mathbf{M}\{q\}}{\{q\}^T\mathbf{M}\{q\}} = 0$$

The quantities ω_i which ensure the minimum of the Rayleigh quotient are therefore the terms which cancel the determinant:

$$\left| \mathbf{K} - \omega^2\mathbf{M} \right| = 0$$

We can show that ω_i is an excess approximation of the ith natural frequency of the system.

Matlab exercice conclusion.m

- -

Compute the approximation of the 2 first natural pulsations thanks to (Lord) Rayleigh's quotient using 3 different vector as initial guess ($X1 = [123]'$ or $[111]'$ or a vector close to u_1 $X1 = [-.32; -.59; -.73]$

```matlab
%Why RR
clear all;close all;

%3DOFs BC left cantilever BC Right Free
%BC k m k m k m
%k=1=m
K=[2 -1 0; -1 2 -1; 0 -1 1]
M=[1 0 0; 0 1 0; 0 0 1]
%%
%YES I can Write this ! M is diagonal
Ktilde=M^(-1/2)*K*M^(-1/2)
%Ktilde=inv(sqrt(M))*K*inv(sqrt(M))
%compare the results
[V,D] =eig(Ktilde)
%it is the same !!
[V,D] =eig(K,M)

%what can tell you these discs ? Just having a look to Ktilde
figure;
gershdisc(Ktilde)


% Rayleigh Ritz
% Let's try this x1 vector
X1=[1 2 3]';
X=X1;
%compare to D(1)
RR=...%TODO
%error with respect to D(1,1) in pc
error1=...%TODO
%is it better?
Xbest=[1 1 1]';
X=Xbest;
%compare to D(1)
RR=...%TODO
%error with respect to D(1,1) in pc
error1bis=...%TODO
%even worst

% ...compare to D(1)
Xbest=[-.32;-.59;-.73];
X=Xbest;
RR=...%TODO
%true value
```

```
RR==D(1,1)
%error with respect to D(1,1) in pc
error1ter=...%TODO

error = [error1 error1bis error1ter];
%[1 2 3],{'[1 2 3]','[1 1 1]','[-.32;-.59;-.73]'})
figure;
bar([1 2 3],error);xlabel('[1 2 3],[1 1 1],[-.32;-.59;-.73]')
    ; ylabel(' error on \omega_1 approximation');
%Oh I Can do that ...compare to D(2)
Xbest=V(:,2); %should try this one...
X=Xbest;
RR=...%TODO
RR==D(2,2)
%error in pc
error2=...%TODO
```
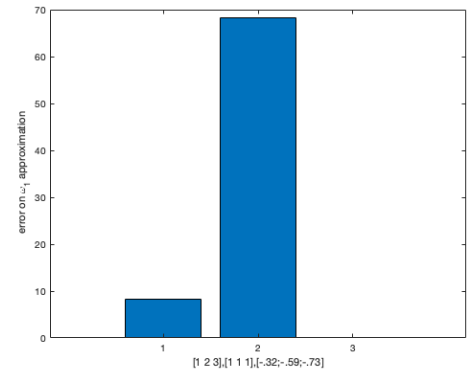

Figure 23: Look at the error! If you infere a good shape for the mode and you could approximate the first ith natural pulsation. That's also the spirit of Finite Element Analysis!