Gomory by column generation

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Based on joint works with Qi He and Angelike Wiegele

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- free software?!

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Our main tool was linear (not integer) programming. We needed to deal with an enormous matrix whose columns were all possible ways to cut up rolls of paper. Generalizing the Dantzig-Wolfe decomposition, we developed a technique that used the knapsack problem to generate implicitly all possible ways to cut paper and selected the best one using the shadow prices. This best cutting pattern was than added to the matrix and we then proceeded to the next step of the simplex method.

Paul C. Gilmore, 1925 – 2015

Some literature

- Ralph E. Gomory, An algorithm for the mixed integer problem, Technical Report RM-2597, The RAND Corporation, 1960.
- Ralph E. Gomory, An algorithm for integer solutions to linear programs, "Recent advances in mathematical programming", McGraw-Hill, New York, 1963, pp. 269–302.

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- Paul C. Gilmore, and Ralph E. Gomory, A linear programming approach to the cutting-stock problem, Operations Research, 9:849–859, 1961.

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If you are wondering, our framework works for both pure (with Q. He) and mixed (with A. Wiegele). But, because of time constraints, I will only do the pure case, and I will not remind you how classical Gomory cuts go.

Let's start

As is usual, $A \in \mathbb{Z}^{m \times n}$, $b \in \mathbb{Z}^m$, $c \in \mathbb{Z}^n$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$.

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Our starting point is

$$\max\{y'b \ : \ y'A \leq c',\ y_i \in \mathbb{Z} \text{ for } i \in \mathcal{I}\}, \tag{$D_{\mathcal{I}}$}$$

where $\mathcal{I} \subset \{1, 2, \ldots, m\}$. A nice feature of $(D_{\mathcal{I}})$ is that the dual of its continuous relaxation is the standard form linear-optimization problem

$$\min\{c'x \ : \ Ax = b,\ x \geq 0\}. \tag{P}$$

PAY ATTENTION: Our integer variables are on the "dual side".

Our pure cut

Let β be any basis for P. The associated dual basic solution (for the continuous relaxation (D)) is $\bar{y}' := c'_\beta A_\beta^{-1}$. Suppose that $\bar{y}_i$ is not an integer. Our goal is to derive a valid cut for $(\mathrm{D}_\mathcal{I})$ that is violated by $\bar{y}$. Let

$$\tilde{b} := \mathbf{e}^i + A_\beta r,$$

where $r \in \mathbb{Z}^m$, and $\mathbf{e}^i$ denotes the i-th standard unit vector in $\mathbb{R}^m$. Note that by construction, $\tilde{b} \in \mathbb{Z}^m$.

Theorem (He and Lee)

$\bar{y}'\tilde{b}$ is not an integer, and so $y'\tilde{b} \leq \lfloor \bar{y}'\tilde{b} \rfloor$ cuts off $\bar{y}$.

Proof.

$$\bar{y}'\tilde{b} = \bar{y}'(\mathbf{e}^i + A_\beta r) = \bar{y}_i + (c'_\beta A_\beta^{-1})A_\beta r = \underbrace{\bar{y}_i}_{\notin\mathbb{Z}} + \underbrace{c'_\beta r}_{\in\mathbb{Z}} \notin \mathbb{Z}.$$

□

Validity

At this point, we have an inequality $y'\tilde{b} \leq \lfloor \bar{y}'\tilde{b} \rfloor$ which cuts off $\bar{y}$, but we have not established its validity for $(\mathrm{D}_{\mathcal{I}})$.

Let $H_{\cdot i} := A_\beta^{-1}\mathbf{e}^i$, the i-th column of A_β^{-1}. Now let

$$w := H_{\cdot i} + r.$$

Clearly we can choose $r \in \mathbb{Z}^m$ so that $w \geq \mathbf{0}$; we simply choose $r \in \mathbb{Z}^m$ so that

$$r_k \geq -\lfloor h_{ki} \rfloor, \text{ for } k = 1, \ldots, m. \qquad (*_{\text{pure}})$$

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(It turns out that it might make sense to choose this at equality: (i) stronger cut, and (ii) used that way in the finite-convergence theory).

Theorem (He and Lee)

If $r \in \mathbb{Z}^m$ satisfies $(*_{\text{pure}})$, then $y'\tilde{b} \leq \lfloor \bar{y}'\tilde{b} \rfloor$ is valid for $(D_{\mathcal{I}})$.

Proof.

Because $w \geq 0$ and $y'A \leq c'$, we have the validity of

$$y'A_\beta(A_\beta^{-1}\mathbf{e}^i + r) \leq c'_\beta(A_\beta^{-1}\mathbf{e}^i + r),$$

even for the continuous relaxation (D) of $(\mathrm{D}_{\mathcal{I}})$. Simplifying this, we have

$$y'(\mathbf{e}^i + A_\beta r) \leq \bar{y}_i + c'_\beta r.$$

The left-hand side is clearly $y'\tilde{b}$, and the right-hand side is

$$\bar{y}_i + c'_\beta r = \bar{y}_i + \bar{y}'A_\beta r = \bar{y}'(\mathbf{e}^i + A_\beta r) = \bar{y}'\tilde{b}.$$

So we have that $y'\tilde{b} \leq \bar{y}'\tilde{b}$ is valid even for (D). Finally, observing that $\tilde{b} \in \mathbb{Z}^m$ and y is constrained to be in $\mathbb{Z}^m$ for $(\mathrm{D}_{\mathcal{I}})$, we can round down the right-hand side and get the result. □

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$$\bar{y}'\tilde{b} - \lfloor \bar{y}'\tilde{b} \rfloor < 0.$$

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In this way, we get a legitimate cutting-plane framework for $(\mathrm{D}_{\mathcal{I}})$ — though we emphasize that we do our computations as column generation with respect to (P).

Finite convergence

Fully inspired by Gomory's proofs, we have finitely converging cutting-plane algorithms based on our cuts. The somewhat technical and delicate proofs, following identical schemes (pure and mixed), rely on:

(i) assuming that the data is all integer and that the feasible region of the continuous relaxation of $(D_{\mathcal{I}})$ is bounded;

(ii) moving the objective of $(D_{\mathcal{I}})$ to the constraints via introduction of a new variable y_0;

(iii) assuming a technical condition to ensure that y_0 is integer at an optimum in the mixed case — so y_0 can also be integer restricted;

(iv) choosing the <u>minimum</u> i for which $\bar{y}_i \notin \mathbb{Z}$, among integer-restricted variables y_i;

(v) lexicographically minimizing $(y_0, y_1, \ldots, y_m)$ by applying the lexicographic primal simplex algorithm to (P);

(vi) choosing $r \in \mathbb{Z}^m$ so that $(*_{\text{pure}})$ is an equation; similarly for mixed.

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Let's go to the demonstration...

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$\emptyset \neq \mathcal{I} \subset \{1, 2, \ldots, m\}$ and $\bar{y}_i \notin$ for some $i \in \mathcal{I}$.

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Theorem (Lee and Wiegele)

Choosing $r \in \mathbb{Z}^m$ satisfying $(*_{\text{mixed}})$,

$$y' \left(A_\beta r - (\bar{y}_i - \lfloor \bar{y}_i \rfloor - 1)\, \mathbf{e}^i \right) \leq c'_\beta r - (\bar{y}_i - \lfloor \bar{y}_i \rfloor - 1)\, \lfloor \bar{y}_i \rfloor$$

is violated by $\bar{y}$ and valid for $(D_{\mathcal{I}})$.