Reconstructing Propositional Proofs in Type Theory

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November 16, 2017

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Research

Goal

Formalization in type theory, classical propositional derivations generated by the `Metis` theorem prover.

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Formalization in type theory, classical propositional derivations generated by the `Metis` theorem prover.

Topics

- Automatic reasoning using automatic theorem provers (ATPs) (e. g., `Metis`, `EProver`)
- Interactive proving using proof-assistants (e. g., `Agda`, `Coq`)
- Formal methods to verify outputs of ATPs in proof-assistants

Outcomes of the Research

Academic result: paper (work in progress)
Software related results:

- Athena[1]: a translator tool for Metis proofs to Agda in Haskell
- Agda libraries:
  - Agda-Metis[2]: Metis prover reasoning for propositional logic
  - Agda-Prop[3]: intuitionistic propositional logic with PEM
- Bugs found in Metis: see issues No. 2, No. 4, and commit 8a3f11e in Metis official repository[4]

In parallel, we develop:

- Online-ATPs[5]: a client for the TPTP world in Haskell This tool allowed us to use Metis without installing it
- Prop-Pack[6]: compendium of TPTP problems in classical propositional logic used to test Athena

[1] https://github.com/jonaprieto/athena.
[2] https://github.com/jonaprieto/agda-metis.
[3] https://github.com/jonaprieto/agda-prop.
[4] https://github.com/gilith/metis.
[5] https://github.com/jonaprieto/online-atps.
[6] https://github.com/jonaprieto/prop-pack.

Bug in the Printing of the Proof

Fixed in `Metis` v2.3 (release 20161108)

$$\varphi := \neg p \wedge (\neg q \Leftrightarrow \neg r) \wedge (\neg p \Leftrightarrow (\neg q \Leftrightarrow \neg r))$$

$$\dfrac{\dfrac{\dfrac{\vdots}{\varphi}\,\text{canonicalize}}{\neg p \Leftrightarrow (\neg q \Leftrightarrow \neg r)}\,\text{conjunct} \qquad \dfrac{\dfrac{\vdots}{\varphi}\,\text{canonicalize}}{\neg q \Leftrightarrow \neg r}\,\text{conjunct} \qquad \dfrac{\dfrac{\vdots}{\varphi}\,\text{canonicalize}}{\neg p}\,\text{conjunct}}{\bot}\,\text{simplify}$$

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The bug was caused by the conversion of `Xor` sets to `Iff` lists. After reporting this, `Metis` developer fixed the printing of `canonicalize` inference rule

$$\varphi := \neg p \wedge (\neg q \Leftrightarrow \neg r) \wedge (\neg p \Leftrightarrow (\neg q \Leftrightarrow \textcolor{red}{r}))$$

Soundness Bug in Splitting goals

Fixed in `Metis` v2.3 (release 20170810)

Consider this `TPTP` problem

```
$ cat issue.tptp
fof(goal, conjecture, (~ (p <=> q)) <=> ((p => ~ q) & (q => ~p))).
```

`Metis` found a proof when other ATPs do not. Indeed, the problem is not a tautology.

```
$ metis issue.tptp
SZS status Theorem for issue.tptp
```

Testing with `EProver` with a client for `SystemOnTPTP` (`Online-ATPs`).

```
$ online-atps --atp=e issue.tptp
...
# No proof found!
# SZS status CounterSatisfiable
...
```

Soundness Bug in Splitting goals

Fixed in `Metis` v2.3 (release 20170810)

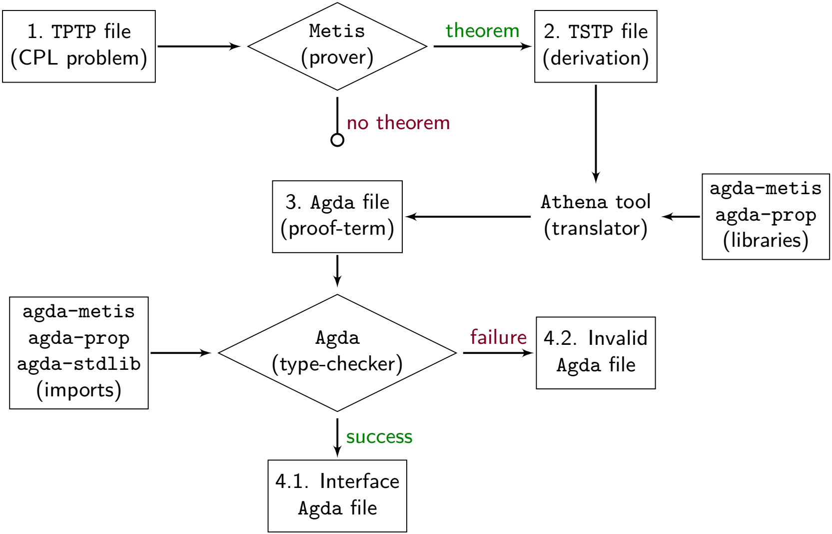
The bug was in the `strip` inference rule:

$$\neg\,(p \Leftrightarrow q) \Leftrightarrow ((p \Rightarrow \neg\, q) \wedge (q \Rightarrow \neg\, p))$$

Solved with:

$$\neg\,(p \Leftrightarrow q) \Leftrightarrow ((p \Rightarrow \neg\, q) \wedge (\neg\, q \Rightarrow p))$$

Proof Reconstruction: Overview



Inference Rules of `Metis`

`TSTP` derivations by `Metis` exhibit the following inferences:

`Metis` **rule**	**Purpose**
`strip`	Strip a goal into subgoals
`conjunct`	Takes a formula from a conjunction
`resolve`	A general form of the resolution theorem
`canonicalize`	Normalization of the formula
`clausify`	Performs clausification
`simplify`	Simplify definitions and theorems

Proposition Type in Agda

A data type for formulas

```
data Prop : Set where
  Var : Fin n → Prop
  ⊤   : Prop
  ⊥   : Prop
  _∧_ : (φ ψ : Prop) → Prop
  _∨_ : (φ ψ : Prop) → Prop
  _⇒_ : (φ ψ : Prop) → Prop
  _⇔_ : (φ ψ : Prop) → Prop
  ¬_  : (φ : Prop)   → Prop
```

Inference Rules For Propositional Logic I

Intuitionistic Propositional Logic + PEM ($\Gamma \vdash \varphi \vee \neg\varphi$)

$$\frac{}{\Gamma, \varphi \vdash \varphi}\ \text{assume} \qquad \frac{}{\Gamma \vdash \top}\ \top\text{-intro} \qquad \frac{}{\Gamma \vdash \varphi \vee \neg\, \varphi}\ \text{PEM}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\ \bot\text{-elim} \qquad \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\, \varphi}\ \neg\text{-intro} \qquad \frac{\Gamma \vdash \neg\, \varphi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \bot}\ \neg\text{-elim}$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}\ \wedge\text{-intro} \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}\ \wedge\text{-proj}_1 \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}\ \wedge\text{-proj}_2$$

Inference Rules For Propositional Logic II

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ \vee\text{-intro}_1 \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ \vee\text{-intro}_2$$

$$\frac{\Gamma, \varphi \vdash \gamma \qquad \Gamma, \psi \vdash \gamma}{\Gamma, \varphi \vee \psi \vdash \gamma}\ \vee\text{-elim}$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \Rightarrow \psi}\ \Rightarrow\text{-intro} \qquad \frac{\Gamma \vdash \varphi \Rightarrow \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}\ \Rightarrow\text{-elim}$$

Other Rules

- Weakening: to extend the hypotheses with additional formulas

$$\frac{\Gamma \vdash \varphi}{\Gamma, \psi \vdash \varphi}\ \text{weaken}$$

- The RAA rule is the formulation of the principle of proof by contradiction:

$$\frac{\Gamma, \neg\, \varphi \vdash \bot}{\Gamma \vdash \varphi}\ \text{RAA}$$

Syntactical Consequence Relation in Agda

- Inductive family $\_ \vdash \_$ with two indexes: a set of propositions Γ (the premises) and a proposition φ (the conclusion)

Example

In [8] we define $\_ \vdash \_$ as follows

```
data _⊢_ : (Γ : Ctxt)(φ : Prop) → Set
  ...
  ∧-intro
    : ∀ {Γ} {φ ψ}
    → Γ ⊢ φ → Γ ⊢ ψ
    → Γ ⊢ φ ∧ ψ

  ∧-proj₁
    : ∀ {Γ} {φ ψ}
    → Γ ⊢ φ ∧ ψ
    → Γ ⊢ φ

  ∧-proj₂
    : ∀ {Γ} {φ ψ}
    → Γ ⊢ φ ∧ ψ
    → Γ ⊢ ψ
  ...
```

Reconstructing `Metis` Rules in Type Theory

Let $\mathrm{metisRule}$ be a `Metis` inference rule. We define the function metisRule in type theory which has the following pattern[7]:

$$\mathsf{metisRule} : \textsc{Premise} \to \textsc{Conclusion} \to \textsc{Prop}$$

$$\mathsf{metisRule}\ \varphi\ \psi \quad = \begin{cases} \psi, & \text{if metisRule built } \psi \text{ by applying inference} \\ & \text{rules to } \varphi; \\ \varphi, & \text{otherwise;} \end{cases}$$

To justify all transformations done by the metisRule rule, we prove its soundness with a theorem like the following:

$$\text{If } \Gamma \vdash \varphi \text{ then } \Gamma \vdash \mathsf{metisRule}\ \varphi\ \psi, \text{ where } \psi : \textsc{Conclusion}.$$

[7] Premise and Conclusion as synonyms of the Prop type to describe in the function types the role of the arguments

Reconstructing Example

The `clausify` rule transforms a formula into its clausal normal form.

Example

In the following `TSTP` derivation by `Metis`, we see how `clausify` transforms the $\mathtt{norm}_0$ formula to get $\mathtt{norm}_1$ formula.

$\mathtt{fof(norm_0,\ \neg\ p \lor (q \land r)\ ...}$
$\mathtt{fof(norm_1,\ (\neg\ p \lor q) \land (\neg\ p \lor r),\ inf(clausify,\ norm_0)).}$

Theorem

Let $\psi : \textsc{Conclusion}$*. If* $\Gamma \vdash \varphi$ *then* $\Gamma \vdash \mathsf{clausify}\ \varphi\ \psi$*, where*

$$\mathsf{clausify} : \textsc{Premise} \to \textsc{Conclusion} \to \textsc{Prop}$$

$$\mathsf{clausify}\ \varphi\ \psi \quad = \begin{cases} \psi, & \textit{if } \varphi \equiv \psi; \\ \mathsf{reorder}_{\land\lor}\ (\mathsf{cnf}\ \varphi)\ \psi, & \textit{otherwise.} \end{cases}$$

The Intuition behind the `Metis` Algorithm

Algorithm 1 `Metis` refutation strategy

procedure METIS
input: the goal and a set of *premises* $a_1, \cdots, a_n$
output: maybe a derivation when $a_1, \cdots, a_n \vdash \mathrm{goal}$, otherwise nothing.

strip the goal into a list of *subgoals* s_i
for each subgoal s_i **do**
try to find by a refutation for $\neg s_i$:
apply clausification for the negated subgoal $\neg\, s_i$
if a premise a_j is relevant **then**
apply clausification to a_j
end if
application of `Metis` inference rules
if a contradiction can be derived from the assumptions **then**
keep the refutation and continue with the others subgoals
else
exit without a proof.
end if
end for
print the conjecture and the premises
print each refutation for each negated subgoal
end procedure

Some Challenges

- Formalization
  - Understanding the `Metis` reasoning without a proper documentation or description from the `Metis` author
  - Terminating of functions that reconstruct `Metis` inference rules
  - Intuitionistic logic implementation
- Software related
  - Parsing of `TSTP` derivations
  - Printing valid `Agda` files

Complete Example

The problem[8]:

$$(p \Rightarrow q) \wedge (q \Rightarrow p) \vdash (p \vee q) \Rightarrow (p \wedge q)$$

In TPTP syntax:

```
fof(a₁, axiom, (p ⇒ q) ∧ (q ⇒ p)).
fof(goal, conjecture, (p ∨ q) ⇒ (p ∧ q)).
```

Its TSTP solution using Metis:

```
fof(a₁, axiom, (p ⇒ q) ∧ (q ⇒ p)).
fof(goal, conjecture, (p ∨ q) ⇒ (p ∧ q))).
fof(s₁, (p ∨ q) ⇒ p, inf(strip, goal)).
fof(s₂, ((p ∨ q) ∧ p) ⇒ q, inf(strip, goal)).
...
```

[8] Problem No. 13 in Disjunction Section in [7]

```
fof(s1, (p ∨ q) ⇒ p, inf(strip, goal)).
fof(s2, ((p ∨ q) ∧ p) ⇒ q, inf(strip, goal)).
fof(neg1, ¬ ((p ∨ q) ⇒ p), inf(negate, s1)).
fof(n00, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, a1)).
fof(n01, ¬ q ∨ p, inf(conjunct, n00)).
fof(n02, ¬ p ∧ (p ∨ q), inf(canonicalize, neg1)).
fof(n03, p ∨ q, inf(conjunct, n02)).
fof(n04, ¬ p, inf(conjunct, n02)).
fof(n05, q, inf(simplify,[n03, n04])).
cnf(r00, ¬ q ∨ p, inf(canonicalize, n01)).
cnf(r01, q, inf(canonicalize, n05)).
cnf(r02, p, inf(resolve, q, [r01, r00])).
cnf(r03, ¬ p, inf(canonicalize, n04)).
cnf(r04, ⊥, inf(resolve, p, [r02, r03])).
fof(neg2, ¬ (((p ∨ q) ∧ p) ⇒ q), inf(negate, s2)).
fof(n10, ¬ q ∧ p ∧ (p ∨ q), inf(canonicalize, neg2)).
fof(n11, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, a1)).
fof(n12, ¬ p ∨ q, inf(conjunct, n11)).
fof(n13, ⊥, inf(simplify,[n10, n12])).
cnf(r10, ⊥, inf(canonicalize, n13)).
```

TSTP Refutation of Subgoal No. 1

```
fof(s₁, (p ∨ q) ⇒ p, inf(strip, goal)).
fof(neg₁, ¬ ((p ∨ q) ⇒ p), inf(negate, s₁)).
fof(n00, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, a₁)).
fof(n01, ¬ q ∨ p, inf(conjunct, n00)).
fof(n02, ¬ p ∧ (p ∨ q), inf(canonicalize, neg₁)).
fof(n03, p ∨ q, inf(conjunct, n02)).
fof(n04, ¬ p, inf(conjunct, n02)).
fof(n05, q, inf(simplify,[n03, n04])).
cnf(r00, ¬ q ∨ p, inf(canonicalize, n01)).
cnf(r01, q, inf(canonicalize, n05)).
cnf(r02, p, inf(resolve, q, [r01, r00])).
cnf(r03, ¬ p, inf(canonicalize, n04)).
cnf(r04, ⊥, inf(resolve, p, [r02, r03])).
```

Tree for the Subgoal No. 1: $(p \vee q) \Rightarrow p$

```
fof(a₁, axiom, (p ⇒ q) ∧ (q ⇒ p)).
...
fof(n00, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, a₁)).
fof(n01, ¬ q ∨ p, inf(conjunct, n00)).
...
```

$$(\mathcal{D}_1) \qquad \cfrac{\cfrac{\cfrac{\cfrac{}{\Gamma \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\ \text{axiom } a_1}{\Gamma, \neg s_1 \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\ \text{weaken}}{\Gamma, \neg s_1 \vdash (\neg p \vee q) \wedge (\neg q \vee p)}\ \text{canonicalize}}{\Gamma, \neg s_1 \vdash \neg q \vee p}\ \text{conjunct}$$

```
...
fof(s₁, (p ∨ q) ⇒ p, inf(strip, goal)).
fof(neg₁, ¬ ((p ∨ q) ⇒ p), inf(negate, s₁)).
...
fof(n02, ¬ p ∧ (p ∨ q), inf(canonicalize, neg₁)).
fof(n03, p ∨ q, inf(conjunct, n02)).
fof(n04, ¬ p, inf(conjunct, n02)).
...
```

$$(\mathcal{D}_2) \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \neg s_1 \vdash \neg s_1}\ \text{assume}}{\Gamma, \neg s_1 \vdash \neg p \wedge (p \vee q)}\ \text{canonicalize}}{\Gamma, \neg s_1 \vdash p \vee q}\ \text{conjunct}$$

$$(\mathcal{D}_3) \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \neg s_1 \vdash \neg s_1}\ \text{assume } \neg s_1}{\Gamma, \neg s_1 \vdash \neg p \wedge (p \vee q)}\ \text{canonicalize}}{\Gamma, \neg s_1 \vdash \neg p}\ \text{conjunct}$$

$$(\mathcal{D}_4) \qquad \dfrac{\dfrac{\mathcal{D}_2}{\Gamma, \neg s_1 \vdash p \vee q} \qquad \dfrac{\mathcal{D}_3}{\Gamma, \neg s_1 \vdash \neg p}}{\Gamma, \neg s_1 \vdash q}\ \textsf{simplify}$$

$$(\mathcal{R}_1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, \neg s_1 \vdash \neg q \vee p} \qquad \dfrac{\mathcal{D}_4}{\Gamma, \neg s_1 \vdash q}}{\Gamma, \neg s_1 \vdash p}\ \textsf{resolve } q \qquad \dfrac{\mathcal{D}_3}{\Gamma, \neg s_1 \vdash \neg p}}{\Gamma, \neg s_1 \vdash \bot}\ \textsf{resolve } p}{\Gamma \vdash s_1}\ \textsf{RAA}$$

Tree for the Subgoal No. 2: $((p \vee q) \wedge p) \Rightarrow q$

```
fof(s₂, ((p ∨ q) ∧ p) ⇒ q, inf(strip, goal)).
fof(neg₂, ¬ (((p ∨ q) ∧ p) ⇒ q), inf(negate, s2)).
fof(n10, ¬ q ∧ p ∧ (p ∨ q), inf(canonicalize, neg₂)).
fof(n11, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, a₁)).
fof(n12, ¬ p ∨ q, inf(conjunct, n11)).
fof(n13, ⊥, inf(simplify,[n10, n12])).
cnf(r10, ⊥, inf(canonicalize, n13)).
```

$$(\mathcal{R}_2) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, \neg s_2 \vdash \neg s_2}\ \text{assume } (\neg s_2)}{\Gamma, \neg s_2 \vdash \neg q \wedge p \wedge (p \vee q)}\ \text{canonicalize} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\ \text{axiom } a_1}{\Gamma, \neg s_2 \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\ \text{weaken}}{\Gamma, \neg s_2 \vdash (\neg p \vee q) \wedge (\neg q \vee p)}\ \text{canonicalize}}{\Gamma, \neg s_2 \vdash \neg p \vee q}\ \text{conjunct}}{\dfrac{\Gamma, \neg s_2 \vdash \bot}{\Gamma \vdash s_2}\ \text{RAA}}\ \text{simplify}$$

Summarizing the Example

The problem was:

$$(p \Rightarrow q) \wedge (q \Rightarrow p) \vdash (p \vee q) \Rightarrow (p \wedge q)$$

Its TSTP solution using Metis was:

```
fof(a₁, axiom, (p ⇒ q) ∧ (q ⇒ p)).
fof(goal, conjecture, (p ∨ q) ⇒ (p ∧ q))).
fof(s₁, (p ∨ q) ⇒ p, inf(strip, goal)).
fof(s₂, ((p ∨ q) ∧ p) ⇒ q, inf(strip, goal)).
...
```

The proof is:

$$\dfrac{\dfrac{}{\Gamma \vdash (s_1 \wedge s_2) \Rightarrow \text{goal}}\,\text{strip} \qquad \dfrac{\dfrac{\mathcal{R}_1}{\Gamma \vdash s_1} \qquad \dfrac{\mathcal{R}_2}{\Gamma \vdash s_2}}{\Gamma \vdash s_1 \wedge s_2}\,\wedge\text{-intro}}{\Gamma \vdash \text{goal}}\,\Rightarrow\text{-elim}$$

(Live example using Agda and Athena)

Future Work

Further research directions include, but are not limited to:

- improve the performance of the `canonicalize` rule
- extend the proof-reconstruction presented in this paper to
  - support the proposition logic with equality of `Metis`
  - support other ATPs for propositional logic like `EProver` or `Z3`. See Kanso's Ph.D. thesis [5]
  - support `Metis` first-order proofs

Related Work

In type theory:

- Kanso in [5] reconstructs in `Agda` propositional proofs generated by `EProver` and `Z3`
- Foster and Struth in [2] describe proof-reconstruction in `Agda` for equational logic of `Waldmeister` prover
- Bezem, Hendriks, and Nivelle in [1] transform a proof produced by the first-order prover `Bliksem` in a `Coq` proof-term

In classical logic:

- Paulson and Susanto in [6] introduce `SledgeHammer`, a tool ables to reconstructs proofs of well-known ATPs: `EProver`, `Vampire`, among others using `SystemOnTPTP` server
- Hurd in [3] integrates the first-order resolution prover `Gandalf` prover for `HOL` proof-assistant
- Kaliszyk and Urban in [4] reconstruct proofs of different ATPs for `HOL Light`

References I

- Marc Bezem, Dimitri Hendriks, and Hans de Nivelle. Automated Proof Construction in Type Theory Using Resolution. Journal of Automated Reasoning 29.3-4 (2002), pp. 253–275. DOI: 10.1023/A:1021939521172 (cit. on p. 29).
- Simon Foster and Georg Struth. Integrating an Automated Theorem Prover in Agda. In: NASA Formal Methods (NFM 2011). Ed. by Mihael Bobaru et al. Vol. 6617. Lecture Notes in Computer Science. Springer, 2011, pp. 116–130. DOI: 10.1007/978-3-642-20398-5_10 (cit. on p. 29).
- Joe Hurd. Integrating Gandalf and HOL. In: Theorem Proving in Higher Order Logics (TPHOLs 2001). Ed. by Yves Bertot, Gilles Dowek, Laurent Théry, and Christine Paulin. Vol. 1690. Lecture Notes in Computer Science. Springer, 2001, pp. 311–321. DOI: 10.1007/3-540-48256-3_21 (cit. on p. 29).

References II

- Cezary Kaliszyk and Josef Urban. PRocH: Proof Reconstruction for HOL Light. In: Automated Deduction (CADE-24). Ed. by Maria Paola Bonacina. Vol. 7898. Lecture Notes in Artifical Intellingence. Springer, 2013, pp. 267–274. DOI: 10.1007/978-3-642-38574-2_18 (cit. on p. 29).
- Karim Kanso. Agda as a Platform for the Development of Verified Railway Interlocking Systems. PhD thesis. Department of Computer Science. Swansea University, 2012. URL: http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.310.1502 (cit. on pp. 28, 29).
- Lawrence C. Paulson and Kong Woei Susanto. Source-level Proof Reconstruction For Interactive Theorem Proving. In: TPHOLs. Vol. 4732. Springer. 2007, pp. 232–245 (cit. on p. 29).
- Jonathan Prieto-Cubides. A Collection of Propositional Problems in TPTP Format. June 2017. DOI: 10.5281/ZENODO.817997 (cit. on p. 20).

References III

Jonathan Prieto-Cubides. A Library for Classical Propositional Logic in Agda. 2017. DOI: 10.5281/zenodo.398852 (cit. on p. 15).

BONUS SLIDES

- Is a language[9] to encode problems
- Is the input of the ATPs
- Annotated formulas with the form

  `language(name, role, formula).`

language FOF or CNF
name to identify the formula within the problem
role axiom, definition, hypothesis, conjecture
formula formula in TPTP format

[9] http://www.cs.miami.edu/~tptp/TPTP/SyntaxBNF.html.

`Metis` Theorem Prover

`Metis` is an automatic theorem prover for first-order logic with equality.

- Open source implemented
- Reads problems in `TPTP` format
- Outputs *detailed* proofs in `TSTP` format
- For the propositional logic, `Metis` has only three inference rules:

$$\frac{}{\Gamma \vdash \varphi_1 \vee \cdots \vee \varphi_n} \text{ axiom } \varphi_1, \cdots, \varphi_n$$

$$\frac{}{\Gamma \vdash \varphi \vee \neg\varphi} \text{ assume } \varphi$$

$$\frac{\Gamma \vdash \varphi_1 \vee \cdots \vee l \vee \cdots \vee \varphi_n \qquad \Gamma \vdash \psi_1 \vee \cdots \vee \neg\, l \vee \cdots \vee \psi_m}{\Gamma \vdash \varphi_1 \vee \cdots \vee \varphi_n \vee \psi_1 \vee \cdots \vee \psi_m} \text{ resolve } l$$

TSTP Syntax

A TSTP derivation[10]

- Is a **D**irected **A**cyclic **G**raph where

  `leaf` is a formula from the TPTP input
  `node` is a formula inferred from parent formula
  `root` the final derived formula

- Is a list of annotated formulas with the form

```
language(name, role, formula, source [,useful info]).
```

where `source` typically is an inference record

```
inference(rule, useful info, parents).
```

[10] http://www.cs.miami.edu/~tptp/TPTP/QuickGuide/Derivations.html.

Another TSTP Example

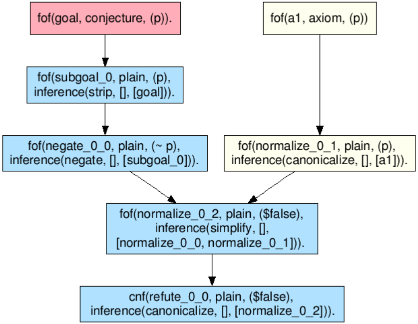
- Proof found by Metis for the problem $p \vdash p$

```
$ metis --show proof problem.tptp
fof(a, axiom, p).
fof(goal, conjecture, p).
fof(subgoal_0, plain, p),
  inference(strip, [], [goal])).
fof(negate_0_0, plain, ~ p,
  inference(negate, [], [subgoal_0])).
fof(normalize_0_0, plain, ~ p,
  inference(canonicalize, [], [negate_0_0])).
fof(normalize_0_1, plain, p,
  inference(canonicalize, [], [a])).
fof(normalize_0_2, plain, $false,
  inference(simplify, [],
    [normalize_0_0, normalize_0_1])).
cnf(refute_0_0, plain, $false,
    inference(canonicalize, [], [normalize_0_2])).
```

DAG Example

By refutation, we proved $p \vdash p$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg p}\,\text{assume}}{\neg p}\,\text{strip} \qquad \dfrac{\dfrac{}{p}\,\text{axiom}}{p}\,\text{canonicalize}}{\bot}\,\text{simplify}}{\bot}\,\text{canonicalize}$$



Athena tool

Is a Haskell program that translates proofs given by Metis in TSTP format to Agda code

- Parsing of TSTP language
- Creation and analysis of **DAG** derivations
- Analysis of inference rules used in the TSTP derivation
- Agda code generation

Library	Purpose
Agda-Prop	axioms and theorems of classical propositional logic
Agda-Metis	versions of the inference rules used by Metis

Agda-Metis: Conjunct Inference [11]

- Definition

$$\text{conjunct}(\overbrace{\varphi_1 \wedge \cdots \wedge \varphi_n}^{\varphi}, \psi) = \begin{cases} \varphi_i & \text{if } \psi \text{ is equal to some } \varphi_i \\ \varphi & \text{otherwise} \end{cases}$$

[11] https://github.com/jonaprieto/agda-metis.

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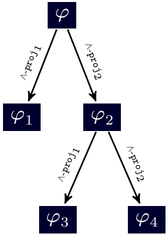
- Inference rules involved

$$\frac{\varphi_1 \wedge \varphi_2}{\varphi_1}\ \wedge\text{-}\mathtt{proj}_1 \qquad \frac{\varphi_1 \wedge \varphi_2}{\varphi_2}\ \wedge\text{-}\mathtt{proj}_2$$

- Example

$$\varphi := \varphi_1 \wedge \overbrace{(\varphi_3 \wedge \varphi_4)}^{\varphi_2}$$



  - conjunct $(\varphi, \varphi_3 \wedge \varphi_1) \equiv \varphi$
  - conjunct $(\varphi, \varphi_3) \equiv \varphi_3$
  - conjunct $(\varphi, \varphi_2) \equiv \varphi_2$

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