Reconstructing Propositional Proofs in Type Theory

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Research

Goal

Formalization in type theory of the classical propositional derivations generated by the `Metis` theorem prover

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Topics

- Automatic reasoning using automatic theorem provers (ATPs) (e. g., `Metis`, `EProver`)
- Interactive proving using Proof-assistants (e. g., `Agda`, `Coq`)
- Formal methods to verify outputs of ATPs in Proof-assistants

Related Work

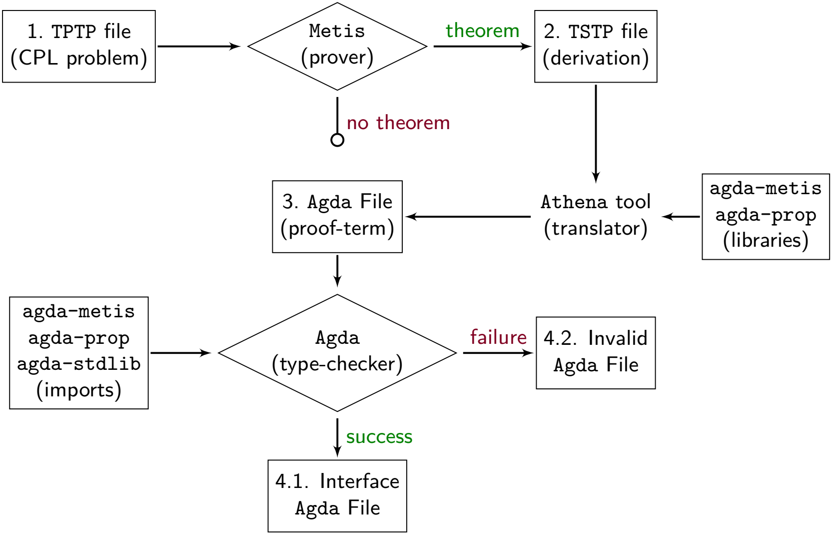
In type theory:

- Kanso in [5] reconstructs in `Agda` propositional proofs generated by `EProver` and `Z3`
- Foster and Struth in [2] describe proof-reconstruction in `Agda` for equational logic of `Waldmeister` prover
- Bezem, Hendriks, and Nivelle in [1] transform a proof produced by the first-order prover `Bliksem` in a `Coq` proof-term

In classical logic:

- Paulson and Susanto in [6] introduce `SledgeHammer`, a tool ables to reconstructs proofs of well-known ATPs: `EProver`, `Vampire`, among others using `SystemOnTPTP` server
- Hurd in [3] integrates the first-order resolution prover `Gandalf` prover for `HOL` proof-assistant
- Kaliszyk and Urban in [4] reconstruct proofs of different ATPs for `HOL Light`

Proof Reconstruction: Overview



Metis Theorem Prover

Metis is an automatic theorem prover for first-order logic with equality.

- Open source implemented
- Reads problems in TPTP format
- Outputs *detailed* proofs in TSTP format
- For the propositional logic, Metis has only three inference rules:

$$\frac{}{\Gamma \vdash \varphi_1 \vee \cdots \vee \varphi_n} \text{ axiom } \varphi_1, \cdots, \varphi_n$$

$$\frac{}{\Gamma \vdash \varphi \vee \neg\varphi} \text{ assume } \varphi$$

$$\frac{\Gamma \vdash \varphi_1 \vee \cdots \vee l \vee \cdots \vee \varphi_n \qquad \Gamma \vdash \psi_1 \vee \cdots \vee \neg\, l \vee \cdots \vee \psi_m}{\Gamma \vdash \varphi_1 \vee \cdots \vee \varphi_n \vee \psi_1 \vee \cdots \vee \psi_m} \text{ resolve } l$$

Inference Rules of Metis

TSTP derivations by Metis exhibit the following inferences:

Metis rule	Purpose
strip	Strip a goal into subgoals
conjunct	Takes a formula from a conjunction
resolve	A general form of the resolution theorem
canonicalize	Normalization of the formula
clausify	Performs clausification
simplify	Simplify definitions and theorems

The intuition behind the `Metis` Algorithm

Algorithm 1 `Metis` refutation strategy

procedure METIS
input: the goal and a set of *premises* a_i
output: maybe a derivation when $\{a_i\} \vdash \mathrm{goal}$, otherwise nothing.
  Strip the goal into a list of *subgoals*.
  for each subgoal s_i **do**
    Try to find by a refutation for $\neg s_i$:
      Apply clausification for the negated subgoal $\neg\, s_i$
    if a premise a_j is relevant **then**
      Apply clausification to a_j
    end if
      Application of `Metis` rules to get $\bot$ using $\neg\, s_i$ and a_j
    if $\bot$ can be derived **then**
      Keep the refutation and continue with the others subgoals
    else
      Exit. It's not a theorem.
    end if
  end for
end procedure

Proof Reconstruction

Stripping a Goal

Splitting a Conjunct

Resolution

Canonicalize

Clausification

Simplification

Formalization Challenges

- Terminating of functions to reconstruct inference rules
- Intuitionistic logic implementation

Complete Example

The problem[1]:

$$(p \Rightarrow q) \wedge (q \Rightarrow p) \vdash (p \vee q) \Rightarrow (p \wedge q)$$

In TPTP syntax:

$\texttt{fof(a}_1\texttt{, axiom, (p} \Rightarrow \texttt{q)} \wedge \texttt{(q} \Rightarrow \texttt{p)).}$
$\texttt{fof(goal, conjecture, (p} \vee \texttt{q)} \Rightarrow \texttt{(p} \wedge \texttt{q)).}$

Its TSTP solution using Metis:

$\texttt{fof(a}_1\texttt{, axiom, (p} \Rightarrow \texttt{q)} \wedge \texttt{(q} \Rightarrow \texttt{p)).}$
$\texttt{fof(goal, conjecture, (p} \vee \texttt{q)} \Rightarrow \texttt{(p} \wedge \texttt{q))).}$
$\texttt{fof(s}_1\texttt{, (p} \vee \texttt{q)} \Rightarrow \texttt{p, inf(strip, goal)).}$
$\texttt{fof(s}_2\texttt{, ((p} \vee \texttt{q)} \wedge \texttt{p)} \Rightarrow \texttt{q, inf(strip, goal)).}$
...

[1] Problem No. 13 in Disjunction Section in [7]

```
fof(s₁, (p ∨ q) ⇒ p, inf(strip, goal)).
fof(s₂, ((p ∨ q) ∧ p) ⇒ q, inf(strip, goal)).
fof(neg₁, ¬ ((p ∨ q) ⇒ p), inf(negate, s₁)).
fof(n00, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, a₁)).
fof(n01, ¬ q ∨ p, inf(conjunct, n00)).
fof(n02, ¬ p ∧ (p ∨ q), inf(canonicalize, neg₁)).
fof(n03, p ∨ q, inf(conjunct, n02)).
fof(n04, ¬ p, inf(conjunct, n02)).
fof(n05, q, inf(simplify,[n03, n04])).
cnf(r00, ¬ q ∨ p, inf(canonicalize, n01)).
cnf(r01, q, inf(canonicalize, n05)).
cnf(r02, p, inf(resolve, q, [r01, r00])).
cnf(r03, ¬ p, inf(canonicalize, n04)).
cnf(r04, ⊥, inf(resolve, p, [r02, r03])).
fof(neg₂, ¬ (((p ∨ q) ∧ p) ⇒ q), inf(negate, s2)).
fof(n10, ¬ q ∧ p ∧ (p ∨ q), inf(canonicalize, neg₂)).
fof(n11, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, a₁)).
fof(n12, ¬ p ∨ q, inf(conjunct, n11)).
fof(n13, ⊥, inf(simplify,[n10, n12])).
cnf(r10, ⊥, inf(canonicalize, n13)).
```

TSTP Refutation of Subgoal No. 1

```
fof(s_1, (p ∨ q) ⇒ p, inf(strip, goal)).
fof(neg_1, ¬ ((p ∨ q) ⇒ p), inf(negate, s_1)).
fof(n00, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, a_1)).
fof(n01, ¬ q ∨ p, inf(conjunct, n00)).
fof(n02, ¬ p ∧ (p ∨ q), inf(canonicalize, neg_1)).
fof(n03, p ∨ q, inf(conjunct, n02)).
fof(n04, ¬ p, inf(conjunct, n02)).
fof(n05, q, inf(simplify,[n03, n04])).
cnf(r00, ¬ q ∨ p, inf(canonicalize, n01)).
cnf(r01, q, inf(canonicalize, n05)).
cnf(r02, p, inf(resolve, q, [r01, r00])).
cnf(r03, ¬ p, inf(canonicalize, n04)).
cnf(r04, ⊥, inf(resolve, p, [r02, r03])).
```

Tree for the Subgoal No. 1: $(p \vee q) \Rightarrow p$

```
fof(a₁, axiom, (p ⇒ q) ∧ (q ⇒ p)).
...
fof(n00, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, a₁)).
fof(n01, ¬ q ∨ p, inf(conjunct, n00)).
...
```

$$(\mathcal{D}_1) \qquad \cfrac{\cfrac{\cfrac{\cfrac{}{\Gamma \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\,\text{axiom } a_1}{\neg s_1 \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\,\text{weaken}}{\neg s_1 \vdash (\neg p \vee q) \wedge (\neg q \vee p)}\,\text{canonicalize}}{\neg s_1 \vdash \neg q \vee p}\,\text{conjunct}$$

```
...
fof(s₁, (p ∨ q) ⇒ p, inf(strip, goal)).
fof(neg₁, ¬ ((p ∨ q) ⇒ p), inf(negate, s₁)).
...
fof(n02, ¬ p ∧ (p ∨ q), inf(canonicalize, neg₁)).
fof(n03, p ∨ q, inf(conjunct, n02)).
fof(n04, ¬ p, inf(conjunct, n02)).
...
```

$$(\mathcal{D}_2) \qquad \dfrac{\dfrac{\dfrac{}{\neg s_1 \vdash \neg s_1}\,\textsf{assume}}{\neg s_1 \vdash \neg p \wedge (p \vee q)}\,\textsf{canonicalize}}{\neg s_1 \vdash p \vee q}\,\textsf{conjunct}$$

$$(\mathcal{D}_3) \qquad \dfrac{\dfrac{\dfrac{}{\neg s_1 \vdash \neg s_1}\,\textsf{assume } \neg s_1}{\neg s_1 \vdash \neg p \wedge (p \vee q)}\,\textsf{canonicalize}}{\neg s_1 \vdash \neg p}\,\textsf{conjunct}$$

$$(\mathcal{D}_4) \qquad \dfrac{\dfrac{\mathcal{D}_2}{\neg s_1 \vdash p \vee q} \qquad \dfrac{\mathcal{D}_3}{\neg s_1 \vdash \neg p}}{\neg s_1 \vdash q}\ \textsf{simplify}$$

$$(\mathcal{R}_1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\neg s_1 \vdash \neg q \vee p} \qquad \dfrac{\mathcal{D}_4}{\neg s_1 \vdash q}}{\neg s_1 \vdash p}\ \textsf{resolve } q \qquad \dfrac{\mathcal{D}_3}{\neg s_1 \vdash \neg p}}{\neg s_1 \vdash \bot}\ \textsf{resolve } p}{\Gamma \vdash s_1}\ \textsf{RAA}$$

Tree for the Subgoal No. 2: $((p \vee q) \wedge p) \Rightarrow q$

```
fof(s2, ((p ∨ q) ∧ p) ⇒ q, inf(strip, goal)).
fof(neg2, ¬ (((p ∨ q) ∧ p) ⇒ q), inf(negate, s2)).
fof(n10, ¬ q ∧ p ∧ (p ∨ q), inf(canonicalize, neg2)).
fof(n11, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, a1)).
fof(n12, ¬ p ∨ q, inf(conjunct, n11)).
fof(n13, ⊥, inf(simplify,[n10, n12])).
cnf(r10, ⊥, inf(canonicalize, n13)).
```

$(\mathcal{R}_2)$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{assume $(\neg s_2)$}
\UnaryInfC{$\Gamma, \neg s_2 \vdash \neg s_2$}
\RightLabel{canonicalize}
\UnaryInfC{$\neg s_2 \vdash \neg q \wedge p \wedge (p \vee q)$}
\AxiomC{}
\RightLabel{axiom a_1}
\UnaryInfC{$\Gamma \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)$}
\RightLabel{weaken}
\UnaryInfC{$\neg s_2 \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)$}
\RightLabel{canonicalize}
\UnaryInfC{$\neg s_2 \vdash (\neg p \vee q) \wedge (\neg q \vee p)$}
\RightLabel{conjunct}
\UnaryInfC{$\neg s_2 \vdash \neg p \vee q$}
\RightLabel{simplify}
\BinaryInfC{$\neg s_2 \vdash \bot$}
\RightLabel{RAA}
\UnaryInfC{$\Gamma \vdash s_2$}
\end{prooftree}
$$

Summarizing the Example

The problem was:

$$(p \Rightarrow q) \wedge (q \Rightarrow p) \vdash (p \vee q) \Rightarrow (p \wedge q)$$

Its `TSTP` solution using `Metis` was:

```
fof(a₁, axiom, (p ⇒ q) ∧ (q ⇒ p)).
fof(goal, conjecture, (p ∨ q) ⇒ (p ∧ q))).
fof(s₁, (p ∨ q) ⇒ p, inf(strip, goal)).
fof(s₂, ((p ∨ q) ∧ p) ⇒ q, inf(strip, goal)).
...
```

The proof is:

$$\dfrac{\dfrac{}{\Gamma \vdash (s_1 \wedge s_2) \Rightarrow \text{goal}}\ \text{strip} \qquad \dfrac{\dfrac{\mathcal{R}_1}{\Gamma \vdash s_1} \qquad \dfrac{\mathcal{R}_2}{\Gamma \vdash s_2}}{\Gamma \vdash s_1 \wedge s_2}\ \wedge\text{-intro}}{\Gamma \vdash \text{goal}}\ \Rightarrow\text{-elim}$$

Results

Academic results: paper (work in progress)
Software related results:

- Athena[2]: a translator tool for Metis proofs to Agda in Haskell
- Agda libraries:
  - Agda-Metis[3]: Metis prover reasoning for propositional logic
  - Agda-Prop[4]: intuitionistic propositional logic with PEM
- Bugs found in Metis: see issues No. 2, No. 4, and commit 8a3f11e in Metis official repository[5]

In parallel, we develop:

- Online-ATPs[6]: a client for the TPTP world in Haskell This tool allowed us to use Metis without installing it
- Prop-Pack[7]: Compendium of TPTP problems in classical propositional logic used to test Athena

[2] https://github.com/jonaprieto/athena.
[3] https://github.com/jonaprieto/agda-metis.
[4] https://github.com/jonaprieto/agda-prop.
[5] https://github.com/gilith/metis.
[6] https://github.com/jonaprieto/online-atps.
[7] https://github.com/jonaprieto/prop-pack.

Future Work

Further research directions include, but are not limited to:

- improve the performance of the `canonicalize` rule
- extend the proof-reconstruction presented in this paper to
  - support the proposition logic with equality of `Metis`
  - support other ATPs for propositional logic like `EProver` or `Z3`. See Kanso's Ph.D. thesis [5]
  - support `Metis` first-order proofs

References I

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Lawrence C. Paulson and Kong Woei Susanto. Source-level Proof Reconstruction For Interactive Theorem Proving. In: TPHOLs. Vol. 4732. Springer. 2007, pp. 232–245 (cit. on p. 4).

Jonathan Prieto-Cubides. A Collection of Propositional Problems in TPTP Format. June 2017. DOI: 10.5281/ZENODO.817997 (cit. on p. 17).

- Is a language[8] to encode problems
- Is the input of the ATPs
- Annotated formulas with the form

```
language(name, role, formula).
```

language FOF or CNF
name to identify the formula within the problem
role axiom, definition, hypothesis, conjecture
formula formula in TPTP format

[8] http://www.cs.miami.edu/~tptp/TPTP/SyntaxBNF.html.

TSTP Syntax

A TSTP derivation[9]

- Is a **D**irected **A**cyclic **G**raph where

  `leaf` is a formula from the TPTP input
  `node` is a formula inferred from parent formula
  `root` the final derived formula

- Is a list of annotated formulas with the form

```
language(name, role, formula, source [,useful info]).
```

where `source` typically is an inference record

```
inference(rule, useful info, parents).
```

[9] http://www.cs.miami.edu/~tptp/TPTP/QuickGuide/Derivations.html.

TSTP Example

- Proof found by Metis for the problem $p \vdash p$

```
$ metis --show proof problem.tptp
fof(a, axiom, p).
fof(goal, conjecture, p).
fof(subgoal_0, plain, p),
  inference(strip, [], [goal])).
fof(negate_0_0, plain, ~ p,
  inference(negate, [], [subgoal_0])).
fof(normalize_0_0, plain, ~ p,
  inference(canonicalize, [], [negate_0_0])).
fof(normalize_0_1, plain, p,
  inference(canonicalize, [], [a])).
fof(normalize_0_2, plain, $false,
  inference(simplify, [],
    [normalize_0_0, normalize_0_1])).
cnf(refute_0_0, plain, $false,
    inference(canonicalize, [], [normalize_0_2])).
```

DAG Example

By refutation, we proved $p \vdash p$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg p}\,\text{assume}}{\neg p}\,\text{strip} \qquad \dfrac{\dfrac{}{p}\,\text{axiom}}{p}\,\text{canonicalize}}{\bot}\,\text{simplify}}{\bot}\,\text{canonicalize}$$