Proof Reconstruction in Classical Propositional Logic

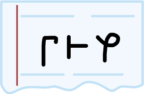
(Work in Progress)

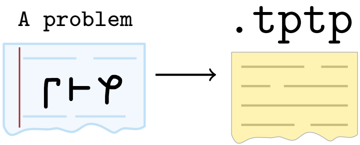
Jonathan Prieto-Cubides
Joint work with Andrés Sicard-Ramírez

Master in Applied Mathematics
Universidad EAFIT
Medellín, Colombia

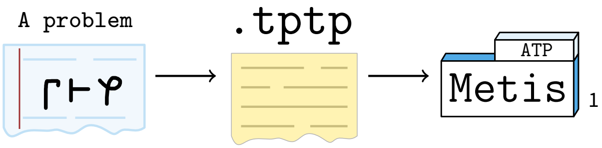
Agda Implementors' Meeting XXV
May 9-15th 2017

A problem



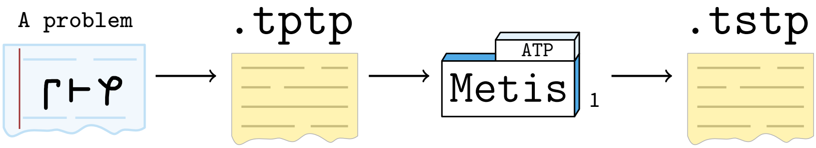

A problem
$\Gamma \vdash \varphi$
.tptp

Proof Reconstruction: Overview



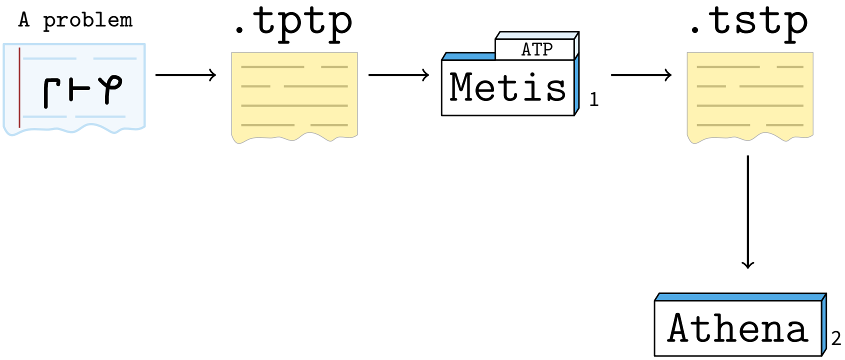
[1] http://www.gilith.com/software/metis

Proof Reconstruction: Overview



[1] http://www.gilith.com/software/metis

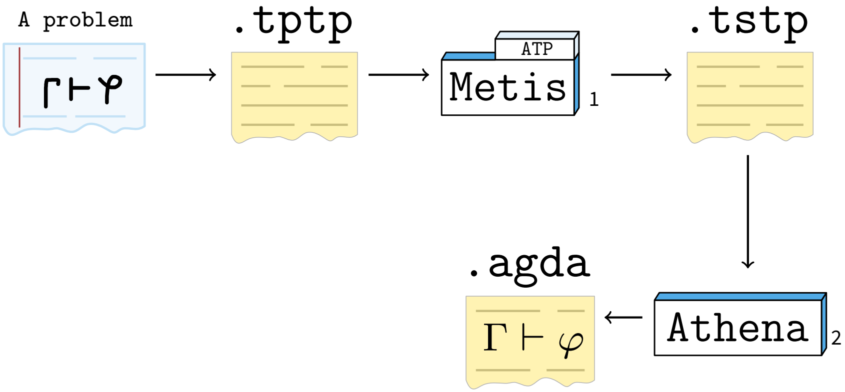
Proof Reconstruction: Overview



[1] http://www.gilith.com/software/metis

[2] http://github.com/jonaprieto/athena

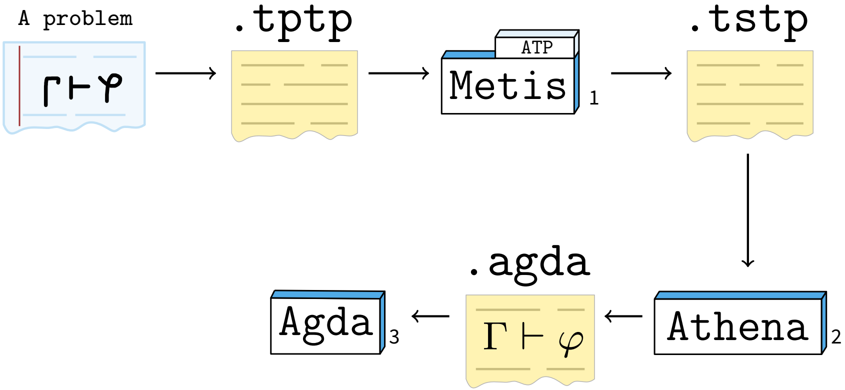
Proof Reconstruction: Overview



[1] http://www.gilith.com/software/metis

[2] http://github.com/jonaprieto/athena

Proof Reconstruction: Overview

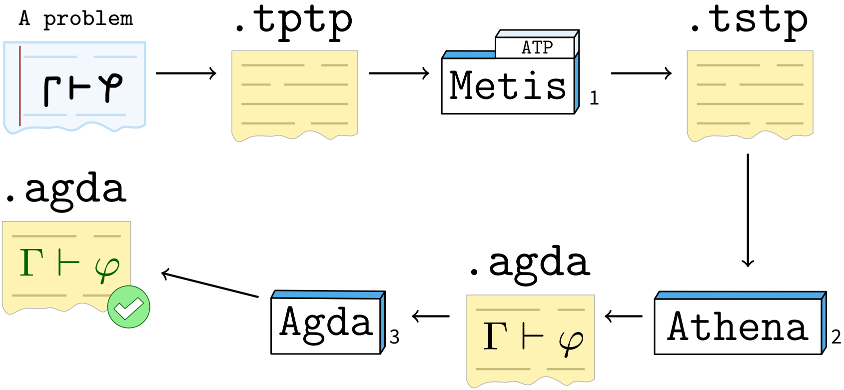


[1] http://www.gilith.com/software/metis

[2] http://github.com/jonaprieto/athena

[3] http://github.com/agda/agda

Proof Reconstruction: Overview

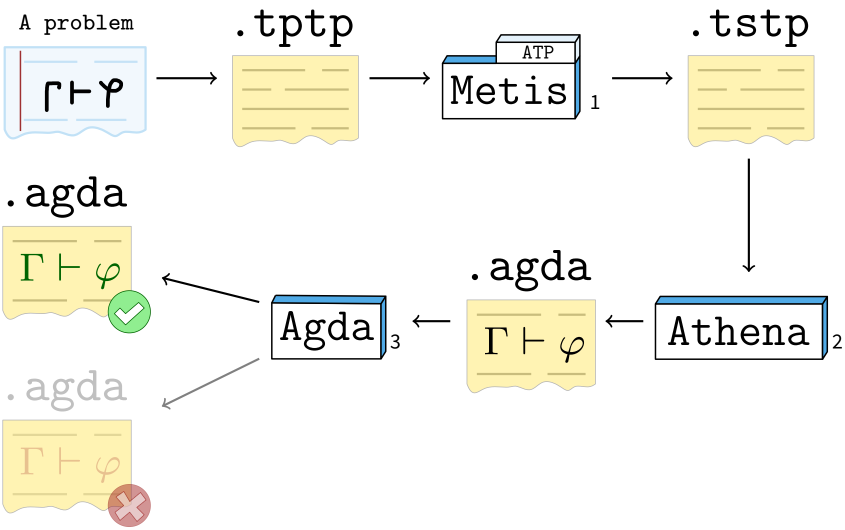


[1] http://www.gilith.com/software/metis

[2] http://github.com/jonaprieto/athena

[3] http://github.com/agda/agda

Proof Reconstruction: Overview

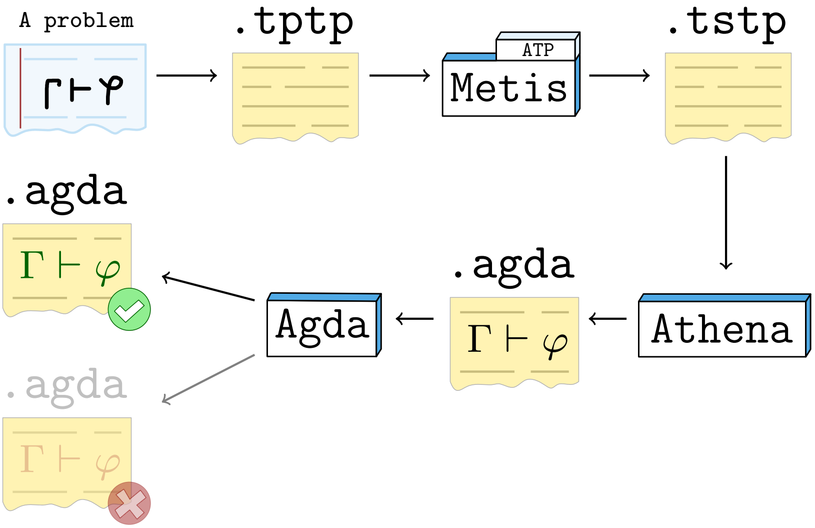


[1] http://www.gilith.com/software/metis

[2] http://github.com/jonaprieto/athena

[3] http://github.com/agda/agda

Proof Reconstruction: Overview



TPTP Syntax

Thousands of Problems for Theorem Provers

.tptp

- Is a language[4] to encode problems (Sutcliffe, 2009)
- Is the input of the ATPs
- Annotated formulas with the form

```
language(name, role, formula).
```

`language`	FOF or CNF
`name`	to identify the formula within the problem
`role`	axiom, definition, hypothesis, conjecture
`formula`	version in TPTP format

[4] `http://www.cs.miami.edu/~tptp/TPTP/SyntaxBNF.html`

TPTP Examples

- $p \vdash p$

```
fof(a, axiom, p).
fof(goal, conjecture, p).
```

- $\vdash \neg(p \wedge \neg p) \vee (q \wedge \neg q)$

```
fof(goal, conjecture, ~ ((p & ~ p) | (q & ~ q))).
```

Go Back

`Metis` Theorem Prover [5]

`Metis` is an automatic theorem prover for First-Order Logic with Equality (Hurd, 2003)

Why Metis?

- Open source implemented in `Standard ML`
- Each refutation step is one of *6 rules*
- Reads problem in TPTP format
- Outputs detailed proofs in TSTP format

[5] `http://www.gilith.com/software/metis/`

Go Back

TSTP derivations by `Metis` exhibit these inferences [6]

Rule	Purpose
`canonicalize`	transforms formulas to CNF, DNF or NNF
`clausify`	performs clausification
`conjunct`	takes a formula from a conjunction
`negate`	applies negation to the formula
`resolve`	applies theorems of resolution
`simplify`	applies over a list of formula to simplify them
`strip`	splits a formula into subgoals

[6] Inference rules found in proofs of Propositional Logic theorems

Go Back

TSTP Syntax

.tstp

A TSTP derivation [7]

- Is a **D**irected **A**cyclic **G**raph where

  `leaf` is a formula from the TPTP input

  `node` is a formula inferred from parent formula

  `root` the final derived formula

- Is a list of annotated formulas with the form

```
language(name, role, formula, source [,useful info]).
```

where `source` typically is an inference record

```
inference(rule, useful info, parents)
```

[7] http://www.cs.miami.edu/~tptp/TPTP/QuickGuide/Derivations.html

Go Back

- Proof found by Metis Prover for the problem $p \vdash p$

ATP
Metis

```
$ metis --show proof problem.tptp
fof(a, axiom, p).
fof(goal, conjecture, p).
fof(subgoal_0, plain, p),
  inference(strip, [], [goal])).
fof(negate_0_0, plain, ~ p,
  inference(negate, [], [subgoal_0])).
fof(normalize_0_0, plain, ~ p,
  inference(canonicalize, [], [negate_0_0])).
fof(normalize_0_1, plain, p,
  inference(canonicalize, [], [a])).
fof(normalize_0_2, plain, $false,
  inference(simplify, [],
    [normalize_0_0, normalize_0_1])).
cnf(refute_0_0, plain, $false,
    inference(canonicalize, [], [normalize_0_2])).
```

Go Back

DAG Example

By refutation, we proved $p \vdash p$:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p}{p}\;\texttt{assume}}{p}\;\texttt{strip}}{\neg p}\;\texttt{negate} \qquad \dfrac{p}{p}\;\texttt{canonicalize}}{\bot}\;\texttt{simplify}}{\bot}\;\texttt{canonicalize}
$$

Is a Haskell program that translates proofs given by Metis Prover in TSTP format to Agda code

- Parsing of TSTP language
- Creation and analysis of **DAG** derivations
- Analysis of inference rules used in the TSTP derivation
- Agda code generation

Library	**Purpose**
Agda-Prop	axioms and theorems of Classical Propositional Logic
Agda-Metis	versions of the inference rules used by Metis

[8] http://github.com/jonaprieto/athena

Go Back

- Intuitionistic Propositional Logic + PEM ($\Gamma \vdash \phi \vee \neg\phi$)
- A data type for formulas

```
data Prop : Set where
  Var   : Fin n → Prop              -- Variables.
  ⊤     : Prop                      -- Top (truth).
  ⊥     : Prop                      -- Bottom (falsum).
  _∧_   : (φ ψ : Prop) → Prop       -- Conjunction.
  _∨_   : (φ ψ : Prop) → Prop       -- Disjunction.
  _⇒_   : (φ ψ : Prop) → Prop       -- Implication.
  _⇔_   : (φ ψ : Prop) → Prop       -- Biimplication.
  ¬_    : (φ : Prop) → Prop         -- Negation.
```

[9] https://github.com/jonaprieto/agda-prop

- A data type for theorems

```
data _⊢_ : (Γ : Ctxt)(φ : Prop) → Set
```

- Constructors

```
assume, axiom, weaken, ⊤-intro, ⊥-elim, ¬-intro,
¬-elim, ∧-intro, ∧-proj₁, ∧-proj₂, ∨-intro₁,
∨-intro₂, ∨-elim, ⇒-intro, ⇒-elim, ⇔-intro,
⇔-elim₁, ⇔-elim₂.
```

- Natural deduction proofs for more than 71 theorems

```
⇔-equiv, ⇔-assoc, ⇔-comm, ⇒-⇔-¬∨, ⇔-¬-to-¬,
¬⇔-to-¬, ¬¬-equiv, ⇒⇒-⇔-∧⇒, ⇔-trans, ∧-assoc,
∧-comm, ∧-dist, ¬∧-to-¬∨¬, ¬∨¬-to-¬∧, ¬∨¬-⇔-¬∧,
subst⊢∧₁, subst⊢∧₂, ∨-assoc, ∨-comm, ∨-dist,
∨-equiv, ¬∨-to-¬∧¬, ¬∧¬-to-¬∨, ∨-dmorgan,
¬¬∨¬¬-to-∨, cnf, nnf, dnf, RAA, …
```

[10] https://github.com/jonaprieto/agda-prop

Agda-Metis Library [11]

Rule	Purpose	Theorem
`canonicalize`	transforms formulas to CNF, DNF or NNF	`atp-canonicalize`
`clausify`	performs clausification	`atp-clausify`
`conjunct`	takes a formula from a conjunction	`atp-conjunct`
`negate`	applies negation to the formula	`atp-negate`
`resolve`	applies theorems of resolution	`atp-resolve`
`simplify`	applies over a list of formula to simplify them	`atp-simplify`
`strip`	splits a formula into subgoals	`atp-strip`

[11] https://github.com/jonaprieto/agda-metis

- Definition

$$conjunct(\phi_1 \wedge \phi_2 \wedge \cdots \wedge \boldsymbol{\varphi_i} \wedge \cdots \wedge \phi_n, \boldsymbol{\varphi_i}) \longrightarrow \boldsymbol{\varphi_i}$$

- Function[12]:

```
conjunct : Prop → Prop → Prop
conjunct (φ ∧ ψ) ω with ⌊ eq φ ω ⌋ | ⌊ eq ψ ω ⌋
... | true  | _     = φ
... | false | true  = ψ
... | false | false = conjunct φ ω
conjunct φ ω        = φ
```

- Theorem[12]

```
atp-conjunct
  : ∀ {Γ} {φ}
  → (ω : Prop)
  → Γ ⊢ φ
  → Γ ⊢ conjunct φ ω
```

[12] Excerpt from the Agda-Metis library available in ATP.Metis.Rules.Conjunct module

[13] https://github.com/jonaprieto/agda-metis

```
atp-conjunct
  : ∀{Γ}{φ}
  → (ω : Prop)
  → Γ ⊢ φ
  → Γ ⊢ conjunct φ ω

atp-conjunct {Γ} {φ ∧ ψ} ω Γ⊢φ with ⌊ eq φ ω ⌋ | ⌊ eq ψ ω ⌋
... | true  | _     = ∧-proj₁ Γ⊢φ
... | false | true  = ∧-proj₂ Γ⊢φ
... | false | false =
  atp-conjunct {Γ = Γ} {φ = φ} ω (∧-proj₁ Γ⊢φ)
atp-conjunct {_} {Var x}     _ = id
atp-conjunct {_} {⊤}       _ = id
atp-conjunct {_} {⊥}       _ = id
atp-conjunct {_} {φ ∨ φ₁} _ = id
atp-conjunct {_} {φ ⇒ φ₁} _ = id
atp-conjunct {_} {φ ⇔ φ₁} _ = id
atp-conjunct {_} {¬ φ}    _ = id
```

Go Back

Agda Code Example

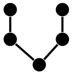
Generated by Athena Tool

- The problem is $p \wedge q \vdash q \wedge p$
- In TPTP format

```
fof(a, axiom, p & q).
fof(goal, conjecture, q & p).
```

- A natural deduction proof

$$\dfrac{\dfrac{\phi \wedge \psi}{\phi}\,\wedge\text{-proj}_1 \qquad \dfrac{\phi \wedge \psi}{\psi}\,\wedge\text{-proj}_2}{\psi \wedge \phi}\,\wedge\text{-intro}$$



TSTP Derivation I

```
fof(a, axiom, p & q).
fof(goal, conjecture, q & p).
fof(subgoal_0, plain, q,
    inference(strip, [], [goal])).
fof(subgoal_1, plain, q => p,
    inference(strip, [], [goal])).
fof(negate_0_0, plain, ~ q,
inference(negate, [], [subgoal_0])).
fof(normalize_0_0, plain, (~ q),
    inference(canonicalize, [], [negate_0_0])).
fof(normalize_0_1, plain, p & q,
    inference(canonicalize, [], [a])).
fof(normalize_0_2, plain, q,
    inference(conjunct, [], [normalize_0_1])).
fof(normalize_0_3, plain, $false,
    inference(simplify, [],
        [normalize_0_0, normalize_0_2])).
cnf(refute_0_0, plain, $false,
    inference(canonicalize, [], [normalize_0_3])).
```

TSTP Derivation II

```
fof(negate_1_0, plain, ~ (q => p),
    inference(negate, [], [subgoal_1])).
fof(normalize_1_0, plain, ~ p & q,
    inference(canonicalize, [], [negate_1_0])).
fof(normalize_1_1, plain, p & q,
     inference(canonicalize, [], [a])).
fof(normalize_1_2, plain, p,
    inference(conjunct, [], [normalize_1_1])).
fof(normalize_1_3, plain, q,
    inference(conjunct, [], [normalize_1_1])).
fof(normalize_1_4, plain, $false,
    inference(simplify, [],
      [normalize_1_0, normalize_1_2, normalize_1_3])).
cnf(refute_1_0, plain, ($false),
    inference(canonicalize, [], [normalize_1_4])).
```

Definitions

```
p, q, a, goal, subgoal₀, subgoal₁ : Prop

-- Axiom.
a = (p ∧ q)

-- Premise.
Γ : Ctxt
Γ = [ a ]

-- Conjecture.
goal = (q ∧ p)

-- Subgoals.

subgoal₀ = q

subgoal₁ = (q ⇒ p)
```

```
-- Axiom.
a : Prop
a = (p ∧ q)

-- Subgoal.
subgoal₀ : Prop
subgoal₀ = q

proof₀ : Γ ⊢ subgoal₀
proof₀ =
  (RAA
    (atp-canonicalize
      (atp-simplify
        (atp-canonicalize
          (atp-strip
            (assume {Γ = Γ} (atp-negate subgoal₀))))
        (atp-conjunct (q)
          (atp-canonicalize
            (weaken (atp-negate subgoal₀)
              (assume {Γ = ∅} a)))))))
```

```
subgoal₁ : Prop
subgoal₁ = (q ⇒ p)

proof₁ : Γ ⊢ subgoal₁
proof₁ =
  (RAA
    (atp-canonicalize
      (atp-simplify
        (atp-conjunct (q)
          (atp-canonicalize
            (weaken (atp-negate subgoal₁)
              (assume {Γ = ∅} a))))
        (atp-simplify
          (atp-canonicalize
            (atp-strip
              (assume {Γ = Γ} (atp-negate subgoal₁))))
          (atp-conjunct (p)
            (atp-canonicalize
              (weaken (atp-negate subgoal₁)
                (assume {Γ = ∅} a))))))))
```

```
proof : Γ ⊢ goal
proof =
  ⇒-elim
    atp-splitGoal
    (∧-intro proof₀ proof₁)
```

Go Back

```
$ metis --version
metis 2.3 (release 20161108)
$ metis --show proof problem.tptp
...
fof(normalize_2_0, plain,
  (~ p & (~ q <=> ~ r) & (~ p <=> (~ q <=> ~ r))),
  inference(canonicalize, [], [negate_2_0])).
fof(normalize_2_1, plain, ~ p <=> (~ q <=> ~ r),
    inference(conjunct, [], [normalize_2_0])).
fof(normalize_2_2, plain, ~ q <=> ~ r,
    inference(conjunct, [], [normalize_2_0])).
fof(normalize_2_3, plain, ~ p,
  inference(conjunct, [], [normalize_2_0])).
fof(normalize_2_4, plain, $false,
    inference(simplify, [],
      [normalize_2_1, normalize_2_2, normalize_2_3])).
...
```

SledgeHammer

(Paulson and Susanto, 2007)

- `Isabelle/HOL` mature tool
- `Metis` ported within `Isabelle`
- Reconstruct proofs of well-known ATPs: `EProver`, `Vampire`, among others using `SystemOnTPTP` server

Integrating `Waldmeister` into Agda

(Foster and Struth, 2011)

- Framework for a integration between Agda and ATPs
  - Equational Logic
  - Reflection Layers
- Source code is not available[14]

[14] `http://simon-foster.staff.shef.ac.uk/agdaatp`

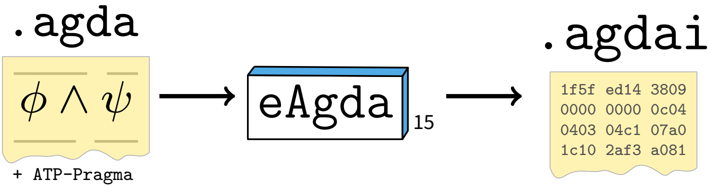
At the moment, the communication between Agda and the ATPs is unidirectional because the ATPs are being used as oracles (Sicard-Ramírez, Bove, and Dybjer, 2015)

+ ATP-Pragma

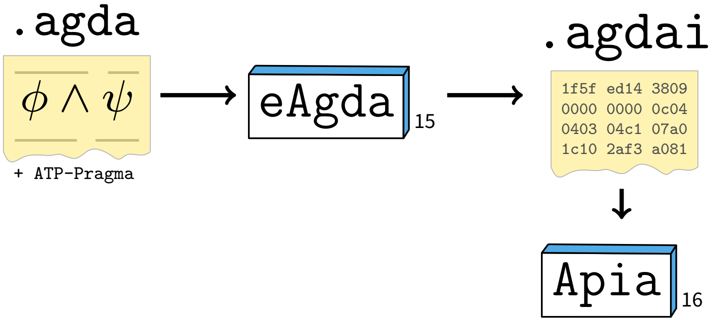
```
module Or where

data _∨_ (A B : Set) : Set where
  inj₁ : A → A ∨ B
  inj₂ : B → A ∨ B

postulate
  A B    : Set
  ∨-comm : A ∨ B → B ∨ A
{-# ATP prove ∨-comm #-}
```

Related Work: Apia

Proving First-Order theorems written in Agda using automatic theorem provers for First-Order Logic

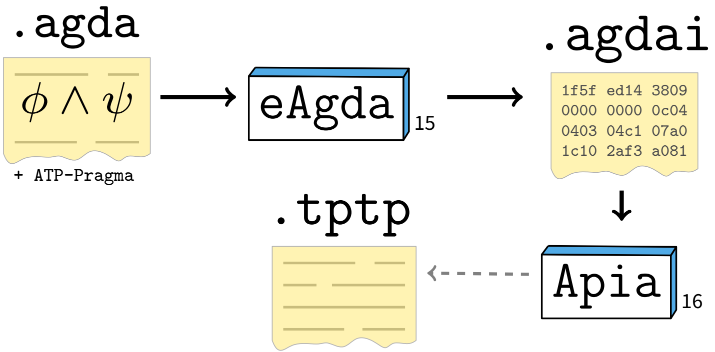


[15] https://github.com/asr/eagda



[15] https://github.com/asr/eagda

[16] https://github.com/asr/apia

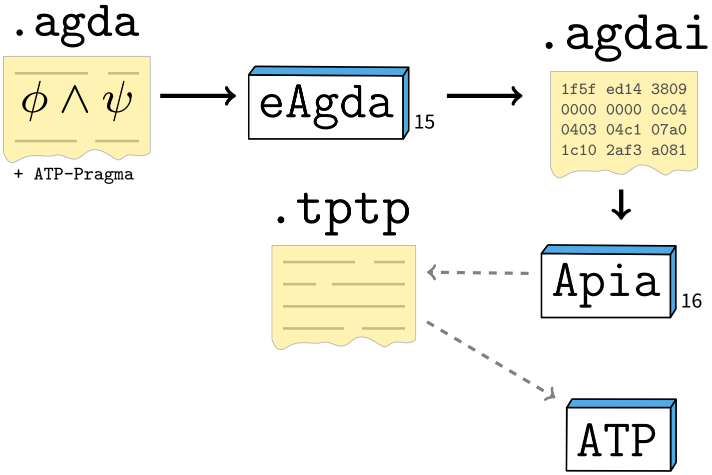


[15] https://github.com/asr/eagda

[16] https://github.com/asr/apia

Related Work: Apia

Proving First-Order theorems written in Agda using automatic theorem provers for First-Order Logic

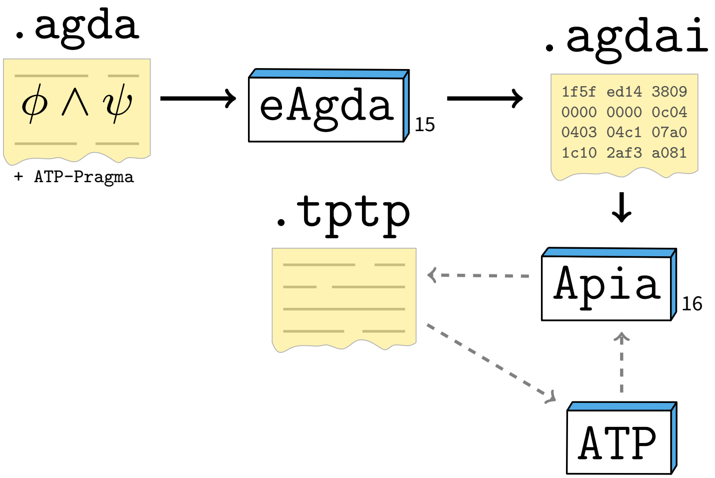


[15] https://github.com/asr/eagda

[16] https://github.com/asr/apia

Related Work: Apia

Proving First-Order theorems written in Agda using automatic theorem provers for First-Order Logic

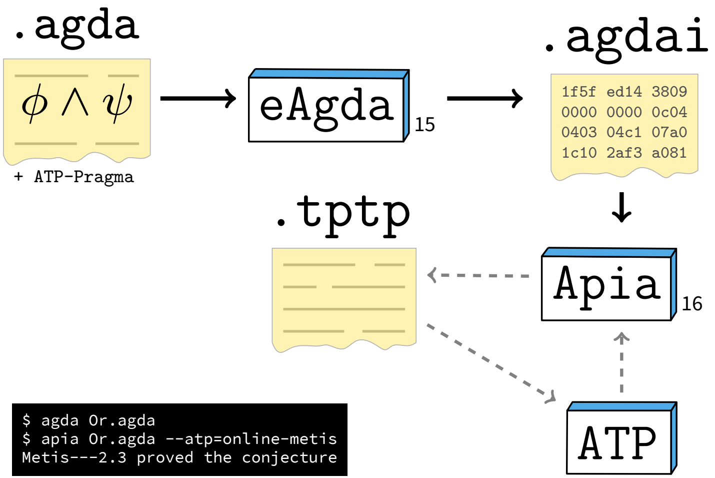


[15] https://github.com/asr/eagda

[16] https://github.com/asr/apia

Related Work: Apia

Proving First-Order theorems written in Agda using automatic theorem provers for First-Order Logic




```
$ agda Or.agda
$ apia Or.agda --atp=online-metis
Metis---2.3 proved the conjecture
```

[15] https://github.com/asr/eagda

[16] https://github.com/asr/apia

Pending Work

- There are missing cases with the `simplify` inference
- Is not clear, how `canonicalize` inference choose what normal form use to transform the formulas
- Splitting a goal in a list of subgoals is not verified yet

Future Work

- Integration with `Apia`
- Support First-Order Logic with Equality
- Support another prover like `EProver` or `Vampire`

References

Foster, Simon and Georg Struth (2011). "Integrating an Automated Theorem Prover into Agda". In: *NASA Formal Methods: Third International Symposium, NFM 2011, Pasadena, CA, USA, April 18-20, 2011. Proceedings*. Ed. by Mihaela Bobaru et al. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 116–130.

Hurd, Joe (2003). "First-order proof tactics in higher-order logic theorem provers". In: *Design and Application of Strategies/Tactics in Higher Order Logics, number NASA/CP-2003-212448 in NASA Technical Reports*, pp. 56–68.

Paulson, Lawrence C. and Kong Woei Susanto (2007). "Source-Level Proof Reconstruction for Interactive Theorem Proving". In: *Theorem Proving in Higher Order Logics: 20th International Conference, TPHOLs 2007, Kaiserslautern, Germany, September 10-13, 2007. Proceedings*. Ed. by Klaus Schneider and Jens Brandt. Berlin, Heidelberg: Springer Berlin Heidelberg, pp. 232–245.

Sicard-Ramírez, Andrés, Ana Bove, and Peter Dybjer (2015). *Reasoning about functional programs by combining interactive*