Reconstructing Propositional Proofs in Type Theory

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Research

Goal

Formalization in type theory, classical propositional derivations generated by the `Metis` theorem prover.

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Formalization in type theory, classical propositional derivations generated by the `Metis` theorem prover.

Topics

- Automatic reasoning using automatic theorem provers (ATPs) (e. g., `Metis`, `EProver`)
- Interactive proving using proof-assistants (e. g., `Agda`, `Coq`)
- Formal methods to verify outputs of ATPs in proof-assistants

Outcomes of the Research

Academic result: paper (work in progress)
Software related results:

- `Athena`[1]: a translator tool for `Metis` proofs to `Agda` in `Haskell`
- `Agda` libraries:
  - `Agda-Metis`[2]: `Metis` prover reasoning for propositional logic
  - `Agda-Prop`[3]: intuitionistic propositional logic with PEM
- Bugs found in `Metis`: see issues No. 2, No. 4, and commit `8a3f11e` in `Metis` official repository[4]

In parallel, we develop:

- `Online-ATPs`[5]: a client for the `TPTP` world in `Haskell` This tool allowed us to use `Metis` without installing it
- `Prop-Pack`[6]: compendium of `TPTP` problems in classical propositional logic used to test `Athena`

[1] `https://github.com/jonaprieto/athena`.

[2] `https://github.com/jonaprieto/agda-metis`.

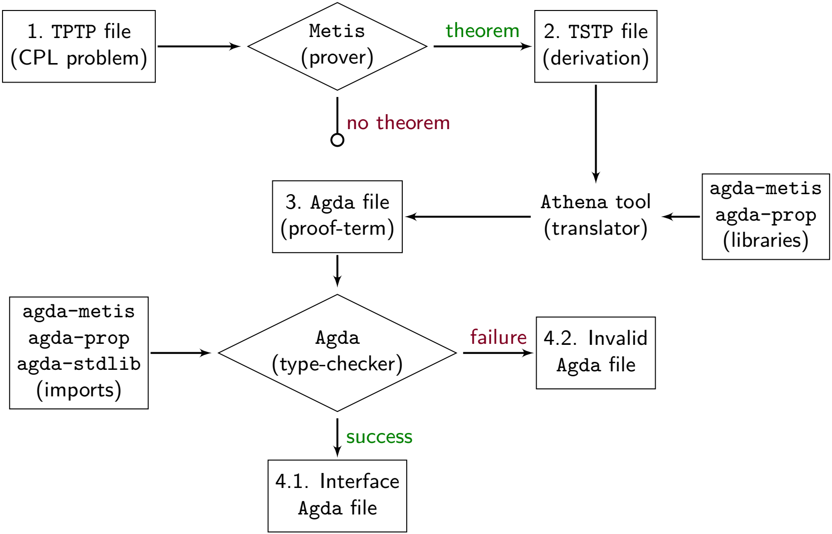
[3] `https://github.com/jonaprieto/agda-prop`.

[4] `https://github.com/gilith/metis`.

[5] `https://github.com/jonaprieto/online-atps`.

[6] `https://github.com/jonaprieto/prop-pack`.

Proof Reconstruction: Overview



Inference Rules of `Metis`

`TSTP` derivations by `Metis` exhibit the following inferences:

`Metis` **rule**	**Purpose**
`strip`	Strip a goal into subgoals
`conjunct`	Takes a formula from a conjunction
`resolve`	A general form of the resolution theorem
`canonicalize`	Normalization of the formula
`clausify`	Performs clausification
`simplify`	Simplify definitions and theorems

Proposition Type

- A data type for formulas

```
data Prop : Set where
  Var : Fin n → Prop
  ⊤    : Prop
  ⊥    : Prop
  _∧_ : (φ ψ : Prop) → Prop
  _∨_ : (φ ψ : Prop) → Prop
  _⇒_  : (φ ψ : Prop) → Prop
  _⇔_  : (φ ψ : Prop) → Prop
  ¬_   : (φ : Prop)   → Prop
```

- Intuitionistic Propositional Logic + PEM ($\Gamma \vdash \varphi \vee \neg\varphi$)

Inference Rules For Propositional Logic I

$$\frac{}{\Gamma, \varphi \vdash \varphi}\ \textsf{assume} \qquad \frac{}{\Gamma \vdash \top}\ \top\textsf{-intro} \qquad \frac{}{\Gamma \vdash \varphi \lor \neg\, \varphi}\ \textsf{PEM}$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\ \bot\textsf{-elim} \qquad \frac{\Gamma, \varphi \vdash \bot}{\Gamma \vdash \neg\, \varphi}\ \neg\textsf{-intro} \qquad \frac{\Gamma \vdash \neg\, \varphi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \bot}\ \neg\textsf{-elim}$$

$$\frac{\Gamma \vdash \varphi \qquad \Gamma \vdash \psi}{\Gamma \vdash \varphi \land \psi}\ \land\textsf{-intro} \qquad \frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \varphi}\ \land\textsf{-proj}_1 \qquad \frac{\Gamma \vdash \varphi \land \psi}{\Gamma \vdash \psi}\ \land\textsf{-proj}_2$$

Inference Rules For Propositional Logic II

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ \vee\text{-intro}_1 \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi}\ \vee\text{-intro}_2 \qquad \frac{\Gamma, \varphi \vdash \gamma \qquad \Gamma, \psi \vdash \gamma}{\Gamma, \varphi \vee \psi \vdash \gamma}\ \vee\text{-elim}$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \Rightarrow \psi}\ \Rightarrow\text{-intro} \qquad \frac{\Gamma \vdash \varphi \Rightarrow \psi \qquad \Gamma \vdash \varphi}{\Gamma \vdash \psi}\ \Rightarrow\text{-elim}$$

Useful rules:

$$\frac{\Gamma \vdash \varphi}{\Gamma, \psi \vdash \varphi}\ \text{weaken} \qquad \frac{\Gamma, \neg\, \varphi \vdash \bot}{\Gamma \vdash \varphi}\ \text{RAA}$$

Syntactical Consequence Relation

- Inductive family $\_ \vdash \_$ with two indexes: a set of propositions Γ (the premises) and a proposition φ (the conclusion)

```
data _⊢_ : (Γ : Ctxt)(φ : Prop) → Set
```

- Constructors (inference rules)

```
assume, axiom, weaken, ⊤-intro, ⊥-elim, ¬-intro,
¬-elim, ∧-intro, ∧-proj₁, ∧-proj₂, ∨-intro₁,
∨-intro₂, ∨-elim, ⇒-intro, ⇒-elim, ⇔-intro,
⇔-elim₁, ⇔-elim₂.
```

- Natural deduction proofs for more than 90 theorems

```
⇔-equiv, ⇔-assoc, ⇔-comm, ⇒-⇔-¬∨, ⇔-¬-to-¬,
¬⇔-to-¬, ¬¬-equiv, ⇒⇒-⇔-∧⇒, ⇔-trans, ∧-assoc,
∧-comm, ∧-dist, ¬∧-to-¬∨¬, ¬∨¬-to-¬∧, ¬∨¬-⇔-¬∧,
subst⊢∧₁, subst⊢∧₂, ∨-assoc, ∨-comm, ∨-dist,
∨-equiv, ¬∨-to-¬∧¬, ¬∧¬-to-¬∨, ∨-dmorgan,
¬¬∨¬¬-to-∨, cnf, nnf, dnf, ...
```

Reconstructing Metis Rules in Type Theory

Let $\mathrm{metisRule}$ be a `Metis` inference rule. We define in `Agda` the function metisRule which has the following pattern[7]:

$$\text{metisRule} : \textsc{Premise} \to \textsc{Conclusion} \to \textsc{Prop}$$

$$\text{metisRule}\ \varphi\ \psi \quad = \begin{cases} \psi, & \text{if metisRule built } \psi \text{ by applying inference} \\ & \text{rules to } \varphi; \\ \varphi, & \text{otherwise;} \end{cases}$$

To justify all transformations done by the metisRule rule, we prove its soundness with a theorem like the following:

$$\text{If } \Gamma \vdash \varphi \text{ then } \Gamma \vdash \text{metisRule}\ \varphi\ \psi, \text{ where } \psi : \textsc{Conclusion}.$$

[7] $\textsc{Premise}$ and $\textsc{Conclusion}$ as synonyms of the $\textsc{Prop}$ type to describe in the function types the role of the arguments

Reconstructing Example

The `clausify` rule transforms a formula into its clausal normal form.

Example

In the following `TSTP` derivation by `Metis`, we see how `clausify` transforms the $\mathtt{norm}_0$ formula to get $\mathtt{norm}_1$ formula.

$\mathtt{fof(norm_0,\ \neg\ p \lor (q \land r)\ ...}$
$\mathtt{fof(norm_1,\ (\neg\ p \lor q) \land (\neg\ p \lor r),\ inf(clausify,\ norm_0)).}$

Theorem

Let $\psi : \textsc{Conclusion}$*. If* $\Gamma \vdash \varphi$ *then* $\Gamma \vdash \mathsf{clausify}\ \varphi\ \psi$*, where*

$$\mathsf{clausify} : \textsc{Premise} \to \textsc{Conclusion} \to \textsc{Prop}$$

$$\mathsf{clausify}\ \varphi\ \psi \quad = \begin{cases} \psi, & \textit{if } \varphi \equiv \psi; \\ \mathsf{reorder}_{\land\lor}\ (\mathsf{cnf}\ \varphi)\ \psi, & \textit{otherwise.} \end{cases}$$

The Intuition behind the `Metis` Algorithm

Algorithm 1 `Metis` refutation strategy

procedure METIS
input: the goal and a set of *premises* $a_1, \cdots, a_n$
output: maybe a derivation when $a_1, \cdots, a_n \vdash \mathrm{goal}$, otherwise nothing.

strip the goal into a list of *subgoals* s_i
for each subgoal s_i **do**
try to find by a refutation for $\neg s_i$:
apply clausification for the negated subgoal $\neg\, s_i$
if a premise a_j is relevant **then**
apply clausification to a_j
end if
application of `Metis` inference rules
if a contradiction can be derived the assumptions **then**
keep the refutation and continue with the others subgoals
else
exit without a proof. The conjecture can not be derived from the premises
end if
end for
print the conjecture and the premises
print each refutation for each negated subgoal
end procedure

Challenges

- Formalization
  - Understanding the `Metis` reasoning without a proper documentation or description from the `Metis` author
  - Terminating of functions that reconstruct `Metis` inference rules
  - Intuitionistic logic implementation
- Software related
  - Parsing of `TSTP` derivations
  - Printing valid `Agda` files
  - Testing

Complete Example

The problem[8]:

$$(p \Rightarrow q) \wedge (q \Rightarrow p) \vdash (p \vee q) \Rightarrow (p \wedge q)$$

In TPTP syntax:

$\texttt{fof(a}_1\texttt{, axiom, (p} \Rightarrow \texttt{q)} \wedge \texttt{(q} \Rightarrow \texttt{p)).}$
$\texttt{fof(goal, conjecture, (p} \vee \texttt{q)} \Rightarrow \texttt{(p} \wedge \texttt{q)).}$

Its TSTP solution using Metis:

$\texttt{fof(a}_1\texttt{, axiom, (p} \Rightarrow \texttt{q)} \wedge \texttt{(q} \Rightarrow \texttt{p)).}$
$\texttt{fof(goal, conjecture, (p} \vee \texttt{q)} \Rightarrow \texttt{(p} \wedge \texttt{q))).}$
$\texttt{fof(s}_1\texttt{, (p} \vee \texttt{q)} \Rightarrow \texttt{p, inf(strip, goal)).}$
$\texttt{fof(s}_2\texttt{, ((p} \vee \texttt{q)} \wedge \texttt{p)} \Rightarrow \texttt{q, inf(strip, goal)).}$
...

[8] Problem No. 13 in Disjunction Section in [**Prieto-Cubides2017**]

fof(s_1, $(p \vee q) \Rightarrow p$, inf(strip, goal)).

fof(s_2, $((p \vee q) \wedge p) \Rightarrow q$, inf(strip, goal)).

fof(neg_1, $\neg\ ((p \vee q) \Rightarrow p)$, inf(negate, s_1)).

fof(n00, $(\neg\ p \vee q) \wedge (\neg\ q \vee p)$, inf(canonicalize, a_1)).

fof(n01, $\neg\ q \vee p$, inf(conjunct, n00)).

fof(n02, $\neg\ p \wedge (p \vee q)$, inf(canonicalize, neg_1)).

fof(n03, $p \vee q$, inf(conjunct, n02)).

fof(n04, $\neg\ p$, inf(conjunct, n02)).

fof(n05, q, inf(simplify,[n03, n04])).

cnf(r00, $\neg\ q \vee p$, inf(canonicalize, n01)).

cnf(r01, q, inf(canonicalize, n05)).

cnf(r02, p, inf(resolve, q, [r01, r00])).

cnf(r03, $\neg\ p$, inf(canonicalize, n04)).

cnf(r04, $\perp$, inf(resolve, p, [r02, r03])).

fof(neg_2, $\neg\ (((p \vee q) \wedge p) \Rightarrow q)$, inf(negate, s2)).

fof(n10, $\neg\ q \wedge p \wedge (p \vee q)$, inf(canonicalize, neg_2)).

fof(n11, $(\neg\ p \vee q) \wedge (\neg\ q \vee p)$, inf(canonicalize, a_1)).

fof(n12, $\neg\ p \vee q$, inf(conjunct, n11)).

fof(n13, $\perp$, inf(simplify,[n10, n12])).

cnf(r10, $\perp$, inf(canonicalize, n13)).

TSTP Refutation of Subgoal No. 1

fof(s_1, (p ∨ q) ⇒ p, inf(strip, goal)).
fof(neg_1, ¬ ((p ∨ q) ⇒ p), inf(negate, s_1)).
fof(n00, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, a_1)).
fof(n01, ¬ q ∨ p, inf(conjunct, n00)).
fof(n02, ¬ p ∧ (p ∨ q), inf(canonicalize, neg_1)).
fof(n03, p ∨ q, inf(conjunct, n02)).
fof(n04, ¬ p, inf(conjunct, n02)).
fof(n05, q, inf(simplify,[n03, n04])).
cnf(r00, ¬ q ∨ p, inf(canonicalize, n01)).
cnf(r01, q, inf(canonicalize, n05)).
cnf(r02, p, inf(resolve, q, [r01, r00])).
cnf(r03, ¬ p, inf(canonicalize, n04)).
cnf(r04, ⊥, inf(resolve, p, [r02, r03])).

Tree for the Subgoal No. 1: $(p \vee q) \Rightarrow p$

```
fof(a₁, axiom, (p ⇒ q) ∧ (q ⇒ p)).
...
fof(n00, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, a₁)).
fof(n01, ¬ q ∨ p, inf(conjunct, n00)).
...
```

$$(\mathcal{D}_1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\ \text{axiom } a_1}{\Gamma, \neg s_1 \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)}\ \text{weaken}}{\Gamma, \neg s_1 \vdash (\neg p \vee q) \wedge (\neg q \vee p)}\ \text{canonicalize}}{\Gamma, \neg s_1 \vdash \neg q \vee p}\ \text{conjunct}$$

```
...
fof(s₁, (p ∨ q) ⇒ p, inf(strip, goal)).
fof(neg₁, ¬ ((p ∨ q) ⇒ p), inf(negate, s₁)).
...
fof(n02, ¬ p ∧ (p ∨ q), inf(canonicalize, neg₁)).
fof(n03, p ∨ q, inf(conjunct, n02)).
fof(n04, ¬ p, inf(conjunct, n02)).
...
```

$$(\mathcal{D}_2) \qquad \cfrac{\cfrac{\cfrac{}{\Gamma, \neg s_1 \vdash \neg s_1}\ \text{assume}}{\Gamma, \neg s_1 \vdash \neg p \wedge (p \vee q)}\ \text{canonicalize}}{\Gamma, \neg s_1 \vdash p \vee q}\ \text{conjunct}$$

$$(\mathcal{D}_3) \qquad \cfrac{\cfrac{\cfrac{}{\Gamma, \neg s_1 \vdash \neg s_1}\ \text{assume } \neg s_1}{\Gamma, \neg s_1 \vdash \neg p \wedge (p \vee q)}\ \text{canonicalize}}{\Gamma, \neg s_1 \vdash \neg p}\ \text{conjunct}$$

$$(\mathcal{D}_4) \qquad \dfrac{\dfrac{\mathcal{D}_2}{\Gamma, \neg s_1 \vdash p \vee q} \qquad \dfrac{\mathcal{D}_3}{\Gamma, \neg s_1 \vdash \neg p}}{\Gamma, \neg s_1 \vdash q}\ \textsf{simplify}$$

$$(\mathcal{R}_1) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma, \neg s_1 \vdash \neg q \vee p} \qquad \dfrac{\mathcal{D}_4}{\Gamma, \neg s_1 \vdash q}}{\Gamma, \neg s_1 \vdash p}\ \textsf{resolve } q \qquad \dfrac{\mathcal{D}_3}{\Gamma, \neg s_1 \vdash \neg p}}{\Gamma, \neg s_1 \vdash \bot}\ \textsf{resolve } p}{\Gamma \vdash s_1}\ \textsf{RAA}$$

Tree for the Subgoal No. 2: $((p \vee q) \wedge p) \Rightarrow q$

```
fof(s_2, ((p ∨ q) ∧ p) ⇒ q, inf(strip, goal)).
fof(neg_2, ¬ (((p ∨ q) ∧ p) ⇒ q), inf(negate, s2)).
fof(n10, ¬ q ∧ p ∧ (p ∨ q), inf(canonicalize, neg_2)).
fof(n11, (¬ p ∨ q) ∧ (¬ q ∨ p), inf(canonicalize, a_1)).
fof(n12, ¬ p ∨ q, inf(conjunct, n11)).
fof(n13, ⊥, inf(simplify,[n10, n12])).
cnf(r10, ⊥, inf(canonicalize, n13)).
```

$$
(\mathcal{R}_2) \qquad
\begin{prooftree}
\AxiomC{}
\RightLabel{assume $(\neg s_2)$}
\UnaryInfC{$\Gamma, \neg s_2 \vdash \neg s_2$}
\RightLabel{canonicalize}
\UnaryInfC{$\Gamma, \neg s_2 \vdash \neg q \wedge p \wedge (p \vee q)$}
\AxiomC{}
\RightLabel{axiom a_1}
\UnaryInfC{$\Gamma \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)$}
\RightLabel{weaken}
\UnaryInfC{$\Gamma, \neg s_2 \vdash (p \Rightarrow q) \wedge (q \Rightarrow p)$}
\RightLabel{canonicalize}
\UnaryInfC{$\Gamma, \neg s_2 \vdash (\neg p \vee q) \wedge (\neg q \vee p)$}
\RightLabel{conjunct}
\UnaryInfC{$\Gamma, \neg s_2 \vdash \neg p \vee q$}
\RightLabel{simplify}
\BinaryInfC{$\Gamma, \neg s_2 \vdash \bot$}
\RightLabel{RAA}
\UnaryInfC{$\Gamma \vdash s_2$}
\end{prooftree}
$$

Summarizing the Example

The problem was:

$$(p \Rightarrow q) \wedge (q \Rightarrow p) \vdash (p \vee q) \Rightarrow (p \wedge q)$$

Its `TSTP` solution using `Metis` was:

```
fof(a₁, axiom, (p ⇒ q) ∧ (q ⇒ p)).
fof(goal, conjecture, (p ∨ q) ⇒ (p ∧ q))).
fof(s₁, (p ∨ q) ⇒ p, inf(strip, goal)).
fof(s₂, ((p ∨ q) ∧ p) ⇒ q, inf(strip, goal)).
...
```

The proof is:

$$\dfrac{\dfrac{}{\Gamma \vdash (s_1 \wedge s_2) \Rightarrow \mathsf{goal}}\,\text{strip} \qquad \dfrac{\dfrac{\mathcal{R}_1}{\Gamma \vdash s_1} \quad \dfrac{\mathcal{R}_2}{\Gamma \vdash s_2}}{\Gamma \vdash s_1 \wedge s_2}\,\wedge\text{-intro}}{\Gamma \vdash \mathsf{goal}}\,\Rightarrow\text{-elim}$$

Future Work

Further research directions include, but are not limited to:

- improve the performance of the `canonicalize` rule
- extend the proof-reconstruction presented in this paper to
  - support the proposition logic with equality of `Metis`
  - support other ATPs for propositional logic like `EProver` or `Z3`. See Kanso's Ph.D. thesis [**Kanso2012**]
  - support `Metis` first-order proofs

Related Work

In type theory:

- **Kanso2012** in [**Kanso2012**] reconstructs in Agda propositional proofs generated by EProver and Z3
- **foster2011integrating** in [**foster2011integrating**] describe proof-reconstruction in Agda for equational logic of Waldmeister prover
- **Bezem2002** in [**Bezem2002**] transform a proof produced by the first-order prover Bliksem in a Coq proof-term

In classical logic:

- **paulson2007source** in [**paulson2007source**] introduce SledgeHammer, a tool ables to reconstructs proofs of well-known ATPs: EProver, Vampire, among others using SystemOnTPTP server
- **Hurd1999** in [**Hurd1999**] integrates the first-order resolution prover Gandalf prover for HOL proof-assistant
- **kaliszyk2013** in [**kaliszyk2013**] reconstruct proofs of different ATPs for HOL Light

References I

BONUS SLIDES

- Is a language[9] to encode problems
- Is the input of the ATPs
- Annotated formulas with the form

  `language(name, role, formula).`

`language` FOF or CNF
`name` to identify the formula within the problem
`role` axiom, definition, hypothesis, conjecture
`formula` formula in TPTP format

[9] `http://www.cs.miami.edu/~tptp/TPTP/SyntaxBNF.html`.

`Metis` Theorem Prover

`Metis` is an automatic theorem prover for first-order logic with equality.

- Open source implemented
- Reads problems in `TPTP` format
- Outputs *detailed* proofs in `TSTP` format
- For the propositional logic, `Metis` has only three inference rules:

$$\frac{}{\Gamma \vdash \varphi_1 \vee \cdots \vee \varphi_n}\ \text{axiom}\ \varphi_1, \cdots, \varphi_n$$

$$\frac{}{\Gamma \vdash \varphi \vee \neg\varphi}\ \text{assume}\ \varphi$$

$$\frac{\Gamma \vdash \varphi_1 \vee \cdots \vee l \vee \cdots \vee \varphi_n \qquad \Gamma \vdash \psi_1 \vee \cdots \vee \neg\, l \vee \cdots \vee \psi_m}{\Gamma \vdash \varphi_1 \vee \cdots \vee \varphi_n \vee \psi_1 \vee \cdots \vee \psi_m}\ \text{resolve}\ l$$

TSTP Syntax

A TSTP derivation[10]

- Is a **D**irected **A**cyclic **G**raph where

  `leaf` is a formula from the TPTP input
  `node` is a formula inferred from parent formula
  `root` the final derived formula

- Is a list of annotated formulas with the form

```
language(name, role, formula, source [,useful info]).
```

where `source` typically is an inference record

```
inference(rule, useful info, parents).
```

[10] http://www.cs.miami.edu/~tptp/TPTP/QuickGuide/Derivations.html.

Another TSTP Example

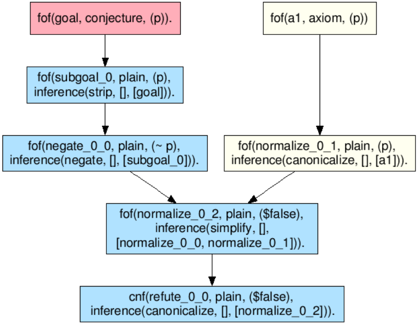
- Proof found by Metis for the problem $p \vdash p$

```
$ metis --show proof problem.tptp
fof(a, axiom, p).
fof(goal, conjecture, p).
fof(subgoal_0, plain, p),
  inference(strip, [], [goal])).
fof(negate_0_0, plain, ~ p,
  inference(negate, [], [subgoal_0])).
fof(normalize_0_0, plain, ~ p,
  inference(canonicalize, [], [negate_0_0])).
fof(normalize_0_1, plain, p,
  inference(canonicalize, [], [a])).
fof(normalize_0_2, plain, $false,
  inference(simplify, [],
    [normalize_0_0, normalize_0_1])).
cnf(refute_0_0, plain, $false,
    inference(canonicalize, [], [normalize_0_2])).
```

DAG Example

By refutation, we proved $p \vdash p$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\neg p}\,\text{assume}}{\neg p}\,\text{strip} \qquad \dfrac{\dfrac{}{p}\,\text{axiom}}{p}\,\text{canonicalize}}{\bot}\,\text{simplify}}{\bot}\,\text{canonicalize}$$



Athena tool

Is a Haskell program that translates proofs given by Metis in TSTP format to Agda code

- Parsing of TSTP language[11]
- Creation[??] and analysis of **DAG** derivations
- Analysis of inference rules used in the TSTP derivation
- Agda code generation

Library	**Purpose**
Agda-Prop	axioms and theorems of classical propositional logic
Agda-Metis	versions of the inference rules used by Metis

[11] https://github.com/agomezl/tstp2agda.