# Preferential/Random Attachment and the Barabási-Albert Network Model

Complexity and Networks

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**Abstract:** The Barabási-Albert (BA) model generates scale-free random networks, in contrast to models such as the Erdős–Rényi random graph model which has a sharper cutoff in the degree distribution. Numerical simulations of the BA model were produced and tested against the theory comparing preferential attachment, random attachment and mixed attachment mechanisms to find how degree distribution and largest degree are affected.

# 1 Introduction

There is a class of networks which exhibit no characteristic scale, where there is a significant probability of having a highly connected hub. The Barabási-Albert (BA) model generates such scale-free random networks utilising a "preferential attachment" mechanism, where new vertices added to the graph attach preferentially to vertices with a larger degree. The probability of attaching to an existing vertex is proportional to that vertex's degree. This follows the principle of "the rich get richer" or "cumulative advantage". Many networks in the real world are thought to be scale-free networks which have a power-law distribution, with examples being the world wide web and citation networks. This is in contrast to other random graph models such as the Erdős–Rényi model which has a binomial distribution.

In the BA model, an initial graph is taken and for each time step, a new vertex is added and connected to $m$ existing vertices with a probability $\frac{k}{\sum_k k}$. This process is repeated until the graph reaches a desired number of vertices $N$. In this report, numerical simulations of the preferential attachment of the BA model, random attachment and mixed attachment are studied to observe the models. A vertex in the network has degree $k$, corresponding to the number of edges attached to it.

Of particular interest is the degree distribution, $p(k)$ which denotes the probability of the degree in the network. We are also interested in the largest degree in the network, and the limitations of the numerical model in terms of finite size effects and how well they approximate the analytical model. In some situations an analytical description and a numerical method differ, but should converge in the large graph limits of interest.

# 2 Method

The Barabási-Albert (BA) model was implemented numerically, to produce random scale-free simple (undirected, unweighted and without self-loops) networks, based on the concept of preferential attachment. It was then modified to study random and mixed random-and-preferential attachment mechanisms.

For preferential attachment, an initial graph is taken or, if a graph is not specified, a complete graph is produced where the $m$ vertices are all connected to each other. This is historically denoted $m_0$ in the literature. The algorithm then adds new vertices to the network one by one until it reaches size $N$, creating $m$ edges between each new vertex and existing vertices with each addition; the probability $\Pi$ of an existing vertex being connected to the new one depends on the degree of the existing vertex. The exact initial configuration of the graph given is not mentioned in the BA model algorithm, and this potentially affects the initial network dynamics, as the degree distribution will evolve differently according to the configuration. Compared to a complete graph, a graph with one vertex of high degree will have new vertices more likely to attach to it.

A network has $N$ vertices and $E$ edges at time $t$ and $m$ edges are added for each new vertex added. $n(k)$ represents the number of vertices with degree $k$. For the addition of a new vertex, $n$ increases by 1 for each existing vertex with degree $k-1$ which is connected, and decreases by 1 for existing vertices with degree $k$ which increases to $k+1$. It also increases by 1 when $k = m$, as the new vertex will have a degree of $k$ also. This gives Equation 1.

$$n(k, t+1) = n(k, t) + m\Pi(k-1, t)n(k-1, t) - m\Pi(k, t)n(k, t) + \delta_{k,m} \tag{1}$$

The numerical method involves storing a list with the index of vertices in the network, where the indices are repeated $k_i$ times; $k_i$ is the degree of the vertex with index $i$. A random selection of $m$ unique vertices are chosen and for each increment of time, $t$, a new vertex is added and edges connected between it and the randomly chosen vertices. The repeated list of vertices is then updated and the process is iterated until the network has $N$ vertices. This works as it picks the vertices randomly with the correct probability $\frac{k}{\sum_k k}$ according to vertex degree.

To test the programme, a few simple cases of the model were produced and inspected as the network was grown, comparing the result to how the model would change if grown manually. The main parameters for the preferential attachment model are $N$, the number of vertices to grow the network to, and $m$, the number of edges between existing vertices and the new vertex.

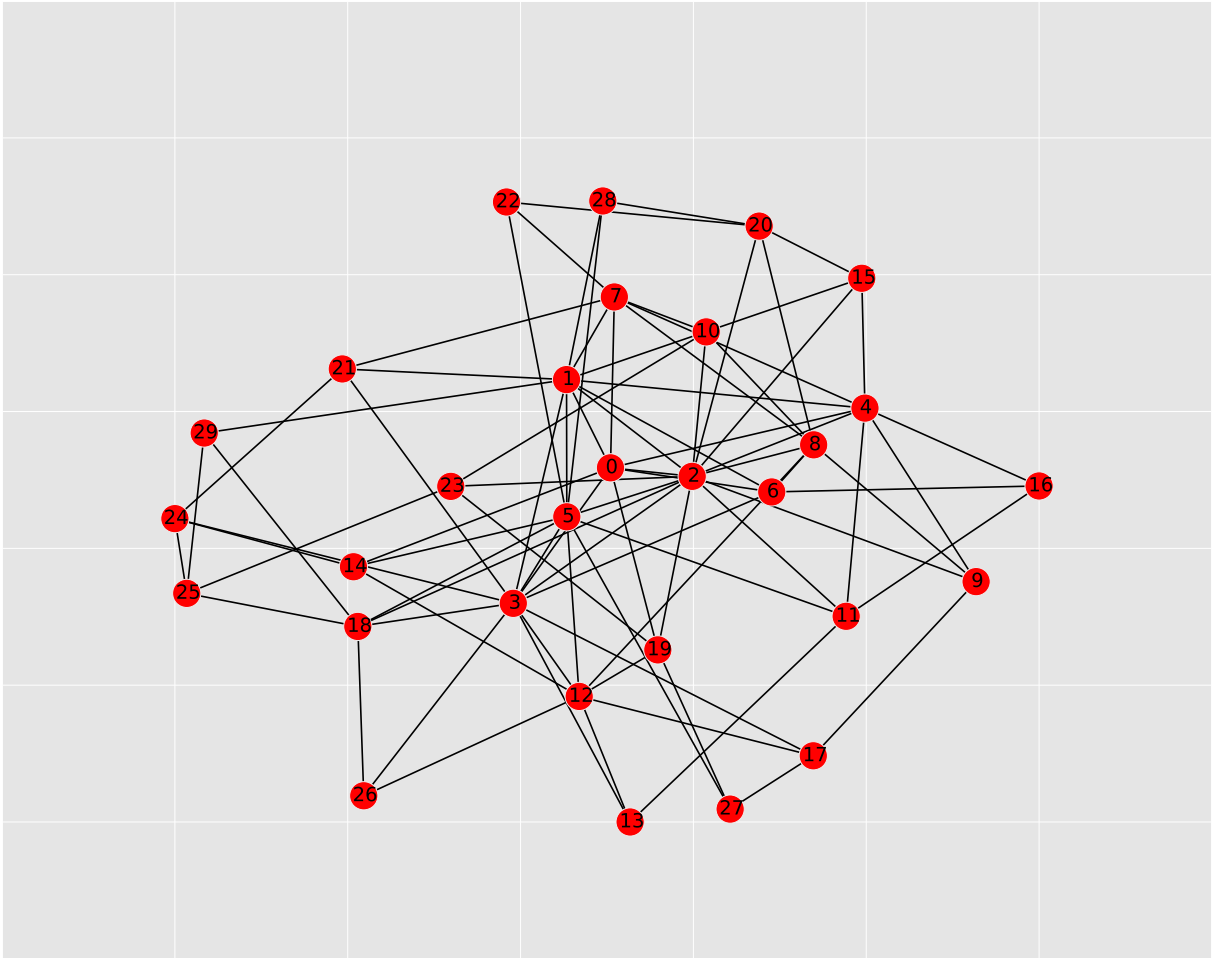


**Figure 1:** The BA model was tested by comparing the network at each time step to the expected behaviour produced manually. This figure shows the vertices added earlier tend to have a higher degree.

It was necessary to add restrictions to the parameters to prevent, for example, graphs with $m = 0$ being specified, as the algorithm is unable to attach preferentially when no edges would be created. Similarly, when no initial graph is specified, $m$ must be greater than one, otherwise the complete graph produced cannot preferentially attach if the degree of the only vertex is zero. This was done rather than allowing an exceptional case where an edge would be connected to a new vertex anyway, as the behaviour of interest is in larger $m$, rather than the tree-like structure with $m = 1$. $m$ was restricted to being

smaller or equal to $N$, as a new vertex would not be able to make $m$ unique links to existing vertices otherwise.

In these simple tests, the network was plotted as it was grown, to see that the graph was produced correctly and there was no undesired behaviour from invalid $N$ or $m$ values. For each iteration of the algorithm, the new vertex index, the repeated list of vertices and the random choice of existing vertices from the list were chosen and printed. Drawing the graph for each iteration showed the changes to the network visually were completed successfully with each addition.

# 3 Results and Analysis

## 3.1 Pure Preferential Attachment $\Pi_{\text{pa}}$

### 3.1.1 Theoretical Form of the Degree Distribution

The theoretical form of the degree distribution can be derived from the master equation (Equation 1), which specifies how the number of vertices with degree $k$, $n(k,t)$ evolves with time.

The sum of all degrees will be $2E$, so the probability of attaching to vertex with degree $k$, $\Pi(k) = k/2E$ and assuming all vertices are added with $m$ edges (in reality it might differ according to initial graph specified, however will tend to be true in the long time limit), $E = mN(t)$ so:

$$n(k,t+1) = n(k,t) + \frac{\cancel{m}(k-1)}{2\cancel{m}N} n(k-1,t) - \frac{\cancel{m}k}{2\cancel{m}N} n(k,t) + \delta_{k,m} \tag{2}$$

Rewriting as probabilities using $p = n/N$, and $N(t) = t$ from our definition of time:

$$p(k,t+1)N(t+1) = p(k,t)N(t) + \frac{k-1}{2} p(k-1,t) - \frac{k}{2} p(k,t) + \delta_{k,m} \tag{3}$$

$$(t+1)p(k,t+1) - tp(k,t) = \frac{k-1}{2} p(k-1,t) - \frac{k}{2} p(k,t) + \delta_{k,m} \tag{4}$$

In the long time limit, the time stable case gives

$$p_\infty(k) = \frac{1}{2}\left[(k-1)p_\infty(k-1) - kp_\infty(k)\right] + \delta_{k,m} \tag{5}$$

Solving this difference equation for $k > m$, $\delta_{k,m} = 0$ so:

$$2p_\infty(k) = (k-1)p_\infty(k-1) - kp_\infty(k) + \cancel{2\delta_{k,m}} \tag{6}$$

which gives

$$\frac{p_\infty(k)}{p_\infty(k-1)} = \frac{k-1}{k+2} \tag{7}$$

Comparing the form of Equation 7 with

$$\frac{f(z)}{f(z-1)} = \frac{z+a}{z+b}$$

which has the solution

$$f(z) = A\frac{\Gamma(z+1+a)}{\Gamma(z+1+b)}$$

leads to

$$p_\infty(k) = A\frac{\Gamma(k-1+1)}{\Gamma(k+2+1)} = A\frac{(k-1)!}{(k+2)!} = \frac{A}{k(k+1)(k+2)} \tag{8}$$

Considering the boundary case $k = m$, with Equation 5 allows the constant $A$ to be found as for $k < m$, $p_\infty(k) = 0$:

$$2p_\infty(m) = \cancel{(m-1)p_\infty(m-1)} - mp_\infty(m) + 2 \tag{9}$$

$$p_\infty(m) = \frac{2}{m+2} = \frac{2m(m+1)}{m(m+1)(m+2)} \tag{10}$$

Hence, $A = 2m(m+1)$ and so

$$p_\infty(k) = \frac{2m(m+1)}{k(k+1)(k+2)} \tag{11}$$

which tends to $p_\infty(k) \propto k^{-3}$, a power law distribution, in the large $k$ limit. Splitting the fraction from Equation 8 using partial fractions, and cancelling out terms:

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)} &= \frac{1}{2}\sum_{k=m}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1} + \frac{1}{k+2} - \frac{1}{k+1}\right) \\ &= \frac{1}{2}\left(\frac{1}{m} - \frac{1}{m+1}\right) \\ &= \frac{1}{2m(m+1)}\end{aligned} \tag{12}$$

$$\implies \sum_{k=1}^{\infty} p_\infty(k) = 1 \tag{13}$$

as expected for a normalised probability distribution. The $\infty$ subscript will be dropped later on for brevity, so that $p(k)$ denotes the long time limit degree distribution.

### 3.1.2 Comparison of Theory and Data for Fixed $N$,

A numerical simulation of the BA model was initially carried out for fixed $N$, varying $m$. $N$ was chosen to be sufficiently large enough to exhibit the behaviour in the long time limit desired, in this case $N = 10^5$, and $m$ chosen from 3 to 18. The simulations were repeated 25 times in order to obtain better statistics across several random graphs. These values were restricted by the limitations of computer memory and computation time.

A fat tailed distribution was found, and with finite system sizes means that there are few vertices with large degrees, and many large $k$ values will have zero or few occurrences, for example if the probability for a large $k$ was $k^{-3}$ and $k^3$ was larger than $N$. Some of the $k$ values would have examples, but others would not. This makes it difficult to see the underlying shape of the degree distribution, with the raw points like Figure 3.

In the numerical simulation, log binning was used, where the probability distribution was produced using exponentially increasing bin widths for the degrees to find the shape of the distribution with the appropriate normalisation.

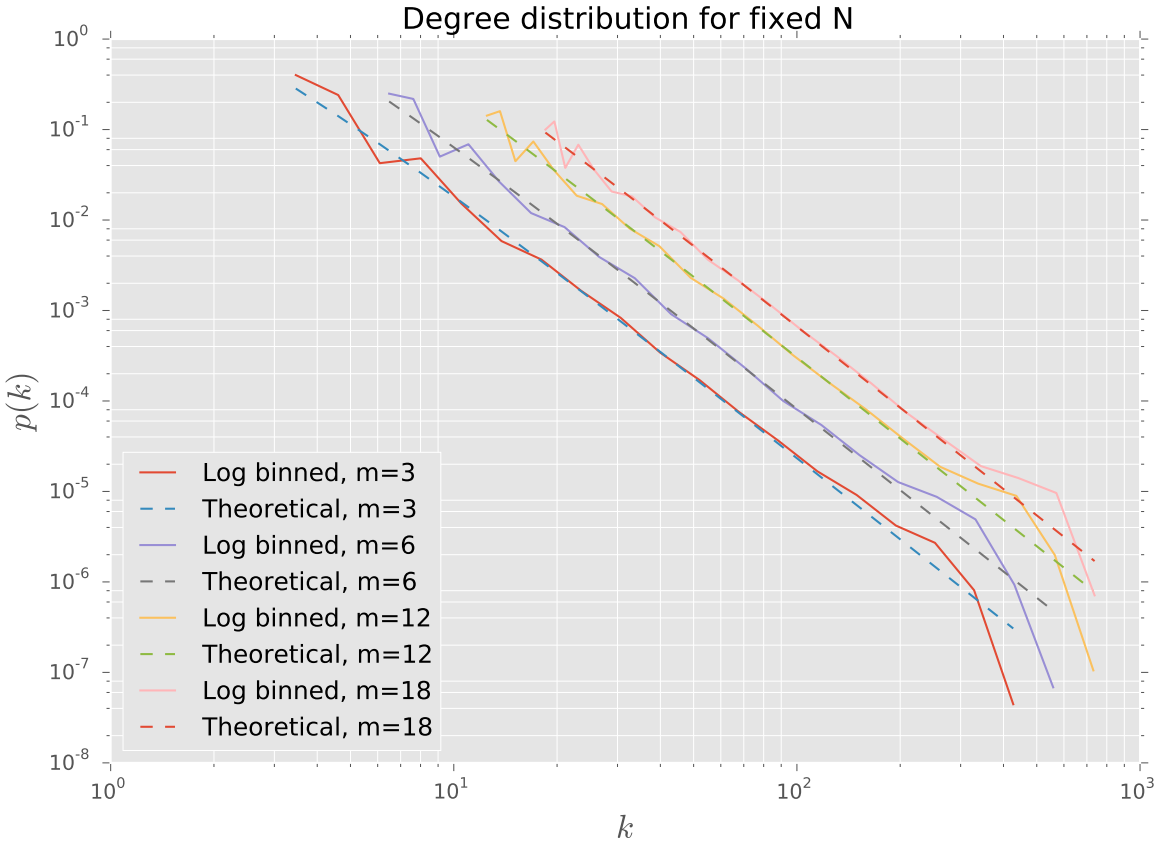


**Figure 2:** Degree distribution for N=$10^5$ comparing the numerical data (solid lines) to the theoretical distribution (dashed lines). The numerical data was produced by processing the raw data points to account for the low counts of high degree in the tail. The divergence of the end of the tail is due to finite size effects which shift depending on the parameters of the network.

Visually, there is a good fit between the numerical and the theoretical degree distribution of Equation 11 for much of the data, which represents the scaling region, approximately following the power law $p(k) \propto k^{-3}$, according to a power law fit to the graph, over different $m$ of -2.987 $\pm$ 0.098 (see Figure 2).

However, there is a noticeable deviation from the theory at the tail which shows a characteristic bump and drop off which occurs due to finite size effects, as the $k$ is limited by the number of vertices and edges in the network. The deviation of the smaller degrees from the theoretical is due to the log-binning only; plotting the raw data at these points shows that they actually lie very precisely on the theoretical line.

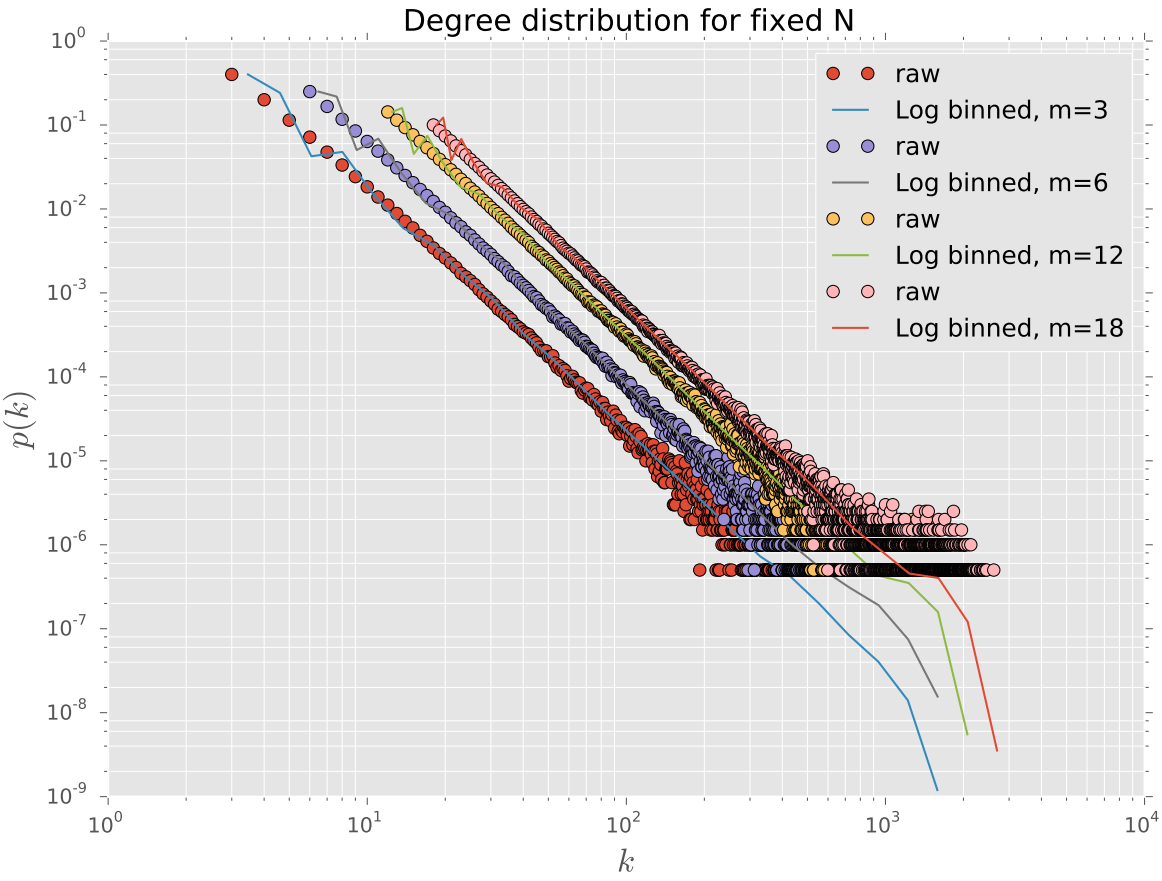


**Figure 3:** The degree distribution was calculated by counting the number of vertices with certain degrees within bins of exponentially increasing width, in order to deal with the difficulty that arises from the fat-tailed distribution. Each bin is 1.3x larger than the previous bin, with an initial bin width of 1, starting at $k = m$.

The numerical data was also compared to the theory statistically, through several measures. Originally, the coefficient of determination, $R^2$ and $\chi^2$ tests were considered. The coefficient of determination is given by

$$R^2 = 1 - \frac{\sum_i (y_i - f_i)^2}{\sum_i (y_i - \bar{y})^2} \tag{14}$$

where $y$ are the observed data with mean $\bar{y}$ and the theoretical values are $f$. The $\chi^2$ statistic is calculated from:

$$\chi^2 = \sum_i \frac{(O_i - E_i)^2}{E_i} \tag{15}$$

where $O$ denotes observed data and $E$ denotes expected (i.e. theoretical) data.

Even with a tail that diverges from the theoretical values, the tests show a very good fit, with $R^2$ very close to 1 and a small $\chi^2$. The p-value is the probability that a result is more extreme than what was observed and is useful for determining the ability to reject the null hypothesis that the two sets of data fit.

The tail of our degree distribution is orders of magnitude smaller than the scaling region, thus the deviation barely contributes to $R^2$ or $\chi^2$, making them poor tests of fit for the power law. Another factor to consider is the appropriateness of the $\chi^2$ with low

counts in the bins and the fact that the central limit theorem may not hold for fat tailed distributions.

Instead, the two-sample Kolmogorov-Smirnov (KS) test was used as a goodness of fit test which compares the distribution functions of two samples; the null hypothesis is that the two samples are drawn from the same distribution. The Anderson-Darling test is a similar test which is also useful.

The numerical data was compared against the theoretical distribution and other distributions as shown in Table 1. It shows a large p value comparing the simulation and theory, implying with statistical significance a high confidence that the null hypothesis should be accepted. In comparison, one can infer that the hypothesis that the data comes from the normal distribution should be rejected under even strict confidence levels.

| **Comparison with** | **Kolmogorov-Smirnov p** |
|---|---|
| Theoretical dist. | 0.999942127353 |
| Normal dist. | 0.000924010182 |

**Table 1**

The statistical comparison with the data produced measures very close to that of the theoretical distribution.

### 3.1.3 Finite Size Effect and Largest Degree

Using a heuristic argument that $k_1$ is at the scale where the order of one vertex should be found with a degree higher than $k_1$,

$$\sum_k n(k) = \sum_k p(k)N = 1 \tag{16}$$

$$\sum_{k=k_1}^{\infty} \frac{2m(m+1)}{k(k+1)(k+2)} = \frac{1}{N} \tag{17}$$

Taking a similar approach as Equation 12, splitting the sum into partial fractions which cancel yields a quadratic equation.

$$\left(\frac{1}{k_1} - \frac{1}{k_1+1}\right) m(m+1) = \frac{1}{N} \tag{18}$$

$$k_1(k_1+1) = Nm(m+1) \tag{19}$$

This can then be solved, for example, by using the quadratic formula:

$$k_1 = \frac{-1 + \sqrt{1 + 4Nm(m+1)}}{2} \tag{20}$$

For any reasonably large $N, m$ network, this reduces to a square-root dependence on $N$ for $k_1$.

$$k_1 \propto \sqrt{m(m+1)}\sqrt{N} \tag{21}$$

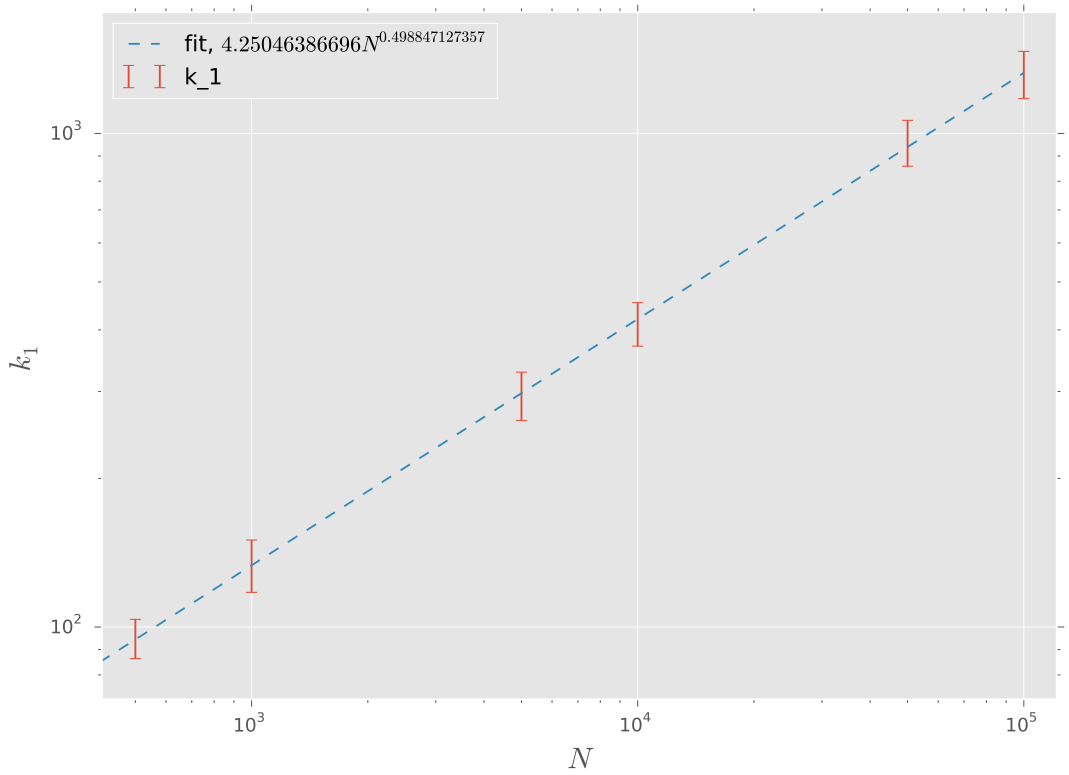


**Figure 4:** Plotting the $k_1$ values at various network sizes $N$ produced a power-law relation of $k_1 \propto N^{0.498}$. The $k_1$ values and their errors were found by repeating the random graphs several times and using their mean and standard deviation to use an optimisation fit to a power law.

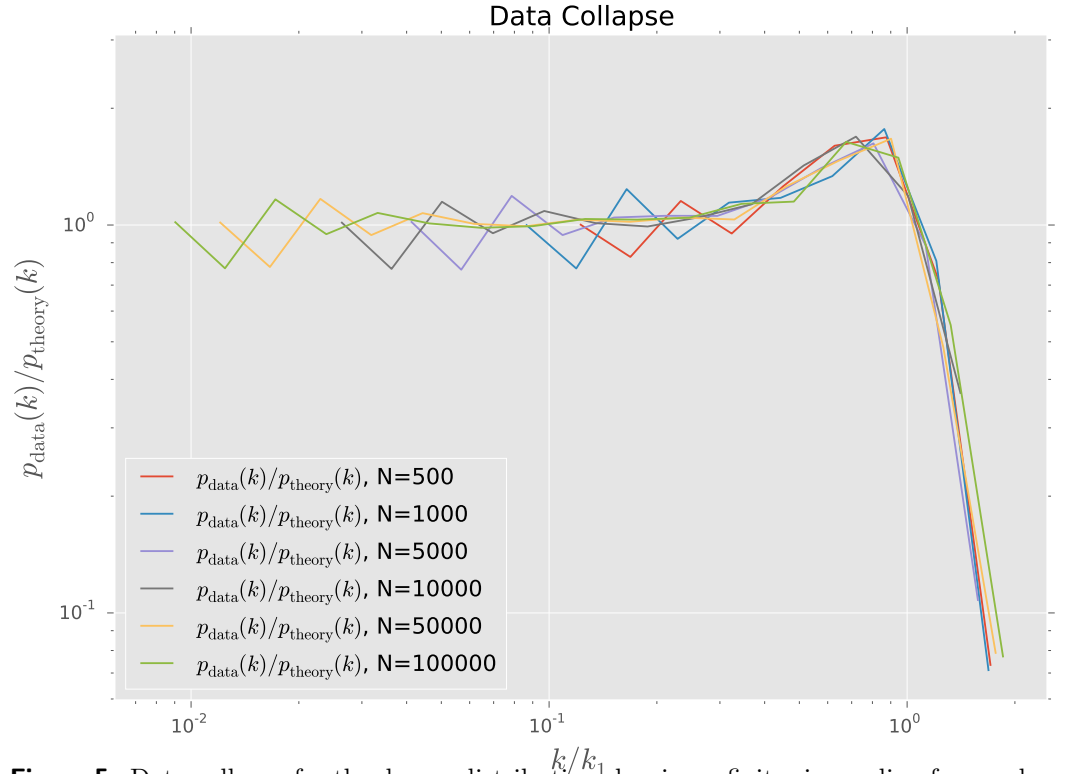


**Figure 5:** Data collapse for the degree distribution showing a finite size scaling form, where $k$ is rescaled by $k_1$ to align the characteristic bumps seen as finite size effects. As there is a shorter scaling region for smaller $N$, this is apparent as the extent on the x-axis is smaller than for large $N$.

Using several numerical runs, a numerical power law fit between $k_1$ and $N$ was found, with the exponent as $0.499 \pm 0.003$, in agreement with the theoretical data prediction of

scaling with the square root of the number of nodes $N$. As $k_1$ is the maximum degree of the distribution, it represents a scale at which the distribution drops off for our numerical simulations. Hence, the finite size effects are seen around this area, and so a data collapse can be produced, by rescaling the data with the theoretical distribution and rescaling the $k$ values with $k_1$ (or equivalently $N^{0.5}$), see Figure 5.

## 3.2 Pure Random Attachment $\Pi_{\text{rnd}}$

### 3.2.1 Theoretical Form of the Degree Distribution

Taking the master equation (Equation 1), with the probability of random attachment $\Pi_{\text{rnd}} = \frac{1}{m_0+t-1}$ required for normalisation and $n = pN$, $N = t$ in the long time limit gives:

$$(t+1)p(k,t+1) = tp(k,t) + \frac{mt}{m_0+t+1}p(k-1,t) - \frac{mt}{m_0+t}p(k,t) + \delta_{k,m} \tag{22}$$

$$p(k,t) = \frac{mt}{m_0+t+1}p(k-1,t) - \frac{mt}{m_0+t}p(k,t) + \delta_{k,m} \tag{23}$$

$$p(k)\left(1 + \frac{mt}{m_0+t}\right) = \frac{mt}{m_0+t-1}p(k-1) \tag{24}$$

In the long time limit, $t \gg m_0$ and 1, this simplifies to

$$p(k) = \frac{m}{m+1}p(k-1) \tag{25}$$

and hence the recurrence relation produces

$$p(k) = \left(\frac{m}{m+1}\right)^{k-m} p(m) \tag{26}$$

Using $k = m$,

$$\begin{aligned} p(m) &= \cancel{\frac{mt}{m_0+t-1}p(m-1)} - \frac{mt}{m_0+t}p(m) + 1 \\ &= \frac{m_0+t}{m_0+t+mt} = \frac{m_0/t+1}{m_0/t+1+m} \\ &= \frac{1}{m+1} \end{aligned} \tag{27}$$

So,

$$p(k) = \frac{1}{m+1}\left(\frac{m}{m+1}\right)^{k-m} \tag{28}$$

which can be rewritten as:

$$p(k) = \frac{1}{m+1}\left[1 + \frac{1}{m}\right]^m \left[\left(1 + \frac{1}{m}\right)^m\right]^{-k/m} \tag{29}$$

which recovers the form of the continuous probability distribution in the limit of large $m$ as specified in [3].

$$p(k) = \frac{e}{m}e^{-k/m} \tag{30}$$

$$\sum_{k=1}^{\infty} p(k) = \sum_{k-m=0}^{\infty} \frac{1}{m+1}\left(\frac{m}{m+1}\right)^{k-m} = 1 \tag{31}$$

as $m/(m+1) < 1$, this forms a geometric series such that the sum of the probabilities is unity, so it is a normalised probability distribution.

### 3.2.2 Comparison of Theory and Data

The degree distribution for the random attachment matches well between theory and data, visually. This can be seen in Figure 6. $m$ acts as a characteristic length scale which modifies how the distribution decays, as can be seen most clearly in Equation 30.

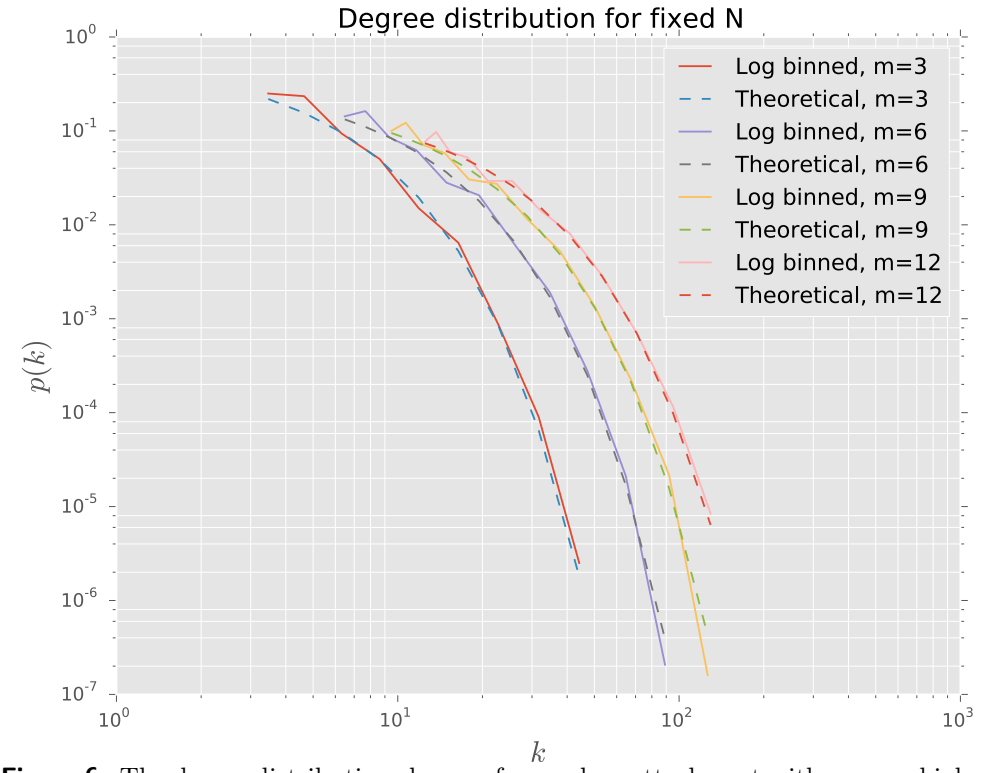


**Figure 6:** The degree distribution changes for random attachment with $m$, as a higher value of $m$ increases the average $k$ value. The distribution does not follow the power-law scaling as was seen in the BA model with pure preferential attachment.

For fixed m, the degree distribution is unchanged as $N$ changes, indicating that the largest degree is not dependent on the number of vertices in this case (Figure 7). This makes sense because the edges are added randomly. The distribution decays much faster than a power law and does not support hubs of large degree. The network is thus not scale-free, as it has a well defined scale at which it cuts off.

The fit of the numerical data to the theoretical is calculated with a high K-S p value, showing statistical evidence to support the fact that they are from the same distribution.

As the finite size effects are not pronounced with the random attachment, a data collapse does not show the same bump and drop off as in the preferential attachment model - the degree distribution already drops off and the probability of having a large degree is very small.

Once again, the theoretical and numerical results seem to fit very well, as can be seen in the figures and in Table 2, with a similar interpretation as above.

| Comparison with | Kolmogorov-Smirnov p |
|---|---|
| Theoretical dist. | 0.999999947134 |
| Normal dist. | 0.004564038189 |

**Table 2**

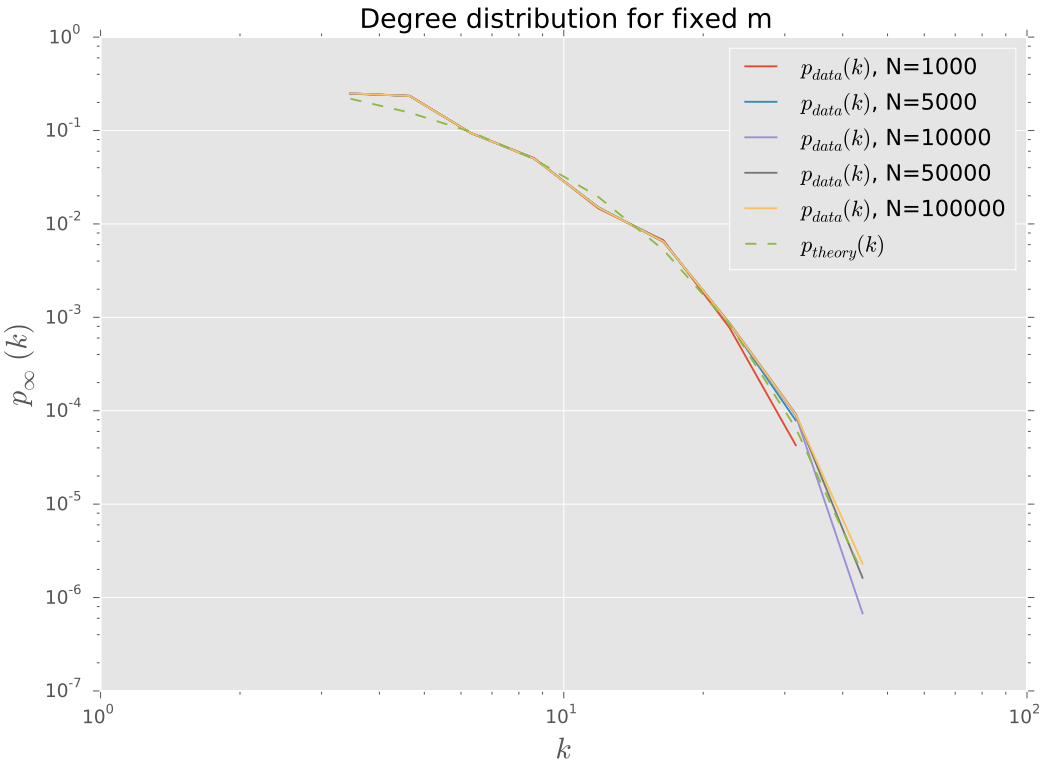


**Figure 7:** The degree distribution is relatively unchanged for different values of $N$; the data lies well on top of the theoretical distribution, and follows the cutoff.

### 3.2.3 Largest Degree

Using a similar approach as in Equation 16, a theoretical $k_1$ can be estimated:

$$\begin{aligned}\sum_{k=k_1}^{\infty} p(k) &= \sum_{k-m=k_1-m}^{\infty} \frac{1}{m+1}\left(\frac{m}{m+1}\right)^{k-m} = \frac{1}{N} \\ &= \sum_{k-m=0}^{\infty} \frac{1}{m+1}\left(\frac{m}{m+1}\right)^{k-m} - \sum_{k-m=0}^{k-m=k_1-m-1} \frac{1}{m+1}\left(\frac{m}{m+1}\right)^{k-m}\end{aligned} \tag{32}$$

is a geometric series.

$$\begin{aligned}\frac{1}{N} &= \frac{\frac{1}{m+1}}{1-\frac{m}{m+1}} - \frac{\frac{1}{m+1}}{1-\frac{m}{m+1}}\left(1 - \frac{m}{m+1}^{k_1-m-1}\right) \\ &= \left(\frac{m}{m+1}\right)^{k_1-m-1}\end{aligned} \tag{33}$$

Taking logarithms:

$$\log(\frac{1}{N}) = (k_1 - m - 1)\log(\frac{m}{m+1}) \tag{34}$$

$$k_1 = m + 1 - \frac{\log(N)}{\log(\frac{m}{m+1})} \tag{35}$$

As shown in Figure 8, $k_1$ seems to scale with the logarithm of $N$. In fact a simpler fit seems more accurate than the theoretical derivation obtained. However, at larger system sizes, they will converge. It is notable that $k_1$ is much smaller than in the preferential attachment mechanism, where "the rich get richer". In comparison, the $k_1$ here are orders of magnitude smaller and so the largest degree is small even for large networks.

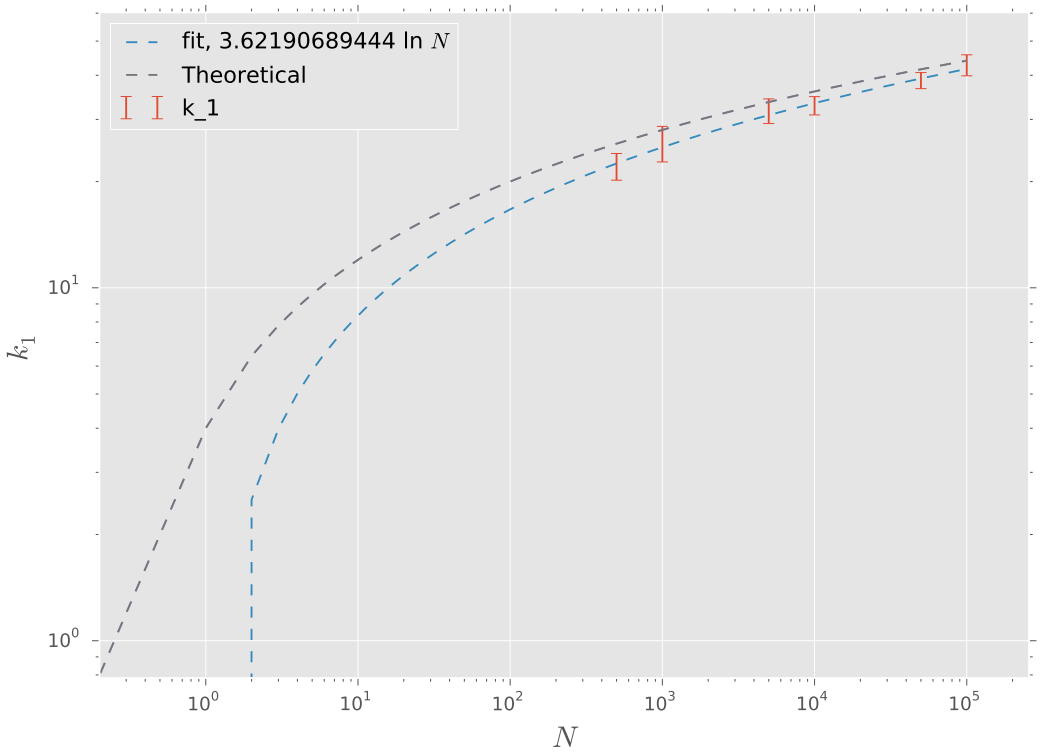


**Figure 8:** $k_1$ plotted against $N$ shows a $\log(N)$ dependence.

## 3.3 Mixed Preferential and Random Attachment

### 3.3.1 Theoretical Form of the Degree Distribution

For the mixed attachment, the probability,

$$\Pi_i = q\Pi_{pa} + (1-q)\Pi_{rnd} = q\frac{k_i}{2t-1} + (1-q)\frac{1}{m_0+t-1} \tag{36}$$

In the long time limit, $t \gg m_0 - 1$ and $2t \gg 1$, hence

$$\frac{\partial k_i}{\partial t} = m\Pi_i = \frac{1}{t}\left(\frac{qk_i}{2} + m(1-q)\right) \tag{37}$$

which can be integrated to obtain:

$$k_i = \frac{2}{q}\left(\left(\frac{mq}{2} + m(1-q)\right)\left(\frac{t}{t_i}\right)^{q/2} - m(1-q)\right) \tag{38}$$

as $k_i < k$,

$$t_i > \left(\frac{mq/2 + m(1-q)}{kq/2 + m(1-q)}\right)^{2/q} t \tag{39}$$

Therefore the probability that the node $i$ has degree $k_i$ smaller than $k$ is equivalent to:

$$P(k_i(t) < k) = 1 - P\left(t_i \leq \left(\frac{mq/2 + m(1-q)}{kq/2 + m(1-q)}\right)^{2/q} t\right) \tag{40}$$

We assume that we disregard the initial graph, such that the probability density of $t_i$:

$$P_i(t_i) = \frac{1}{m_0 + t} \approx \frac{1}{t} \tag{41}$$

so that

$$P(k_i(t) < k) = 1 - \left( \frac{mq/2 + m(1-q)}{kq/2 + m(1-q)} \right)^{2/q} \tag{42}$$

The pdf is the derivative, which gives the degree distribution:

$$p(k) = \frac{\partial P(k_i(t) < k)}{\partial k} = \frac{2}{q} \left( \frac{(2-q)m}{q} \right)^{2/q} \left( k + \frac{2(1-q)m}{q} \right)^{-(2+q)/q} \tag{43}$$

### 3.3.2 Comparison of Theory and Data

The mixed preferential and random attachment mechanism produces a degree distribution somewhere between the pure preferential and random distributions, where the distribution does not decay as sharply as the random, however, does not maintain a very accurate power-law scaling region. This is to be expected as the distribution has characteristics of both distributions.

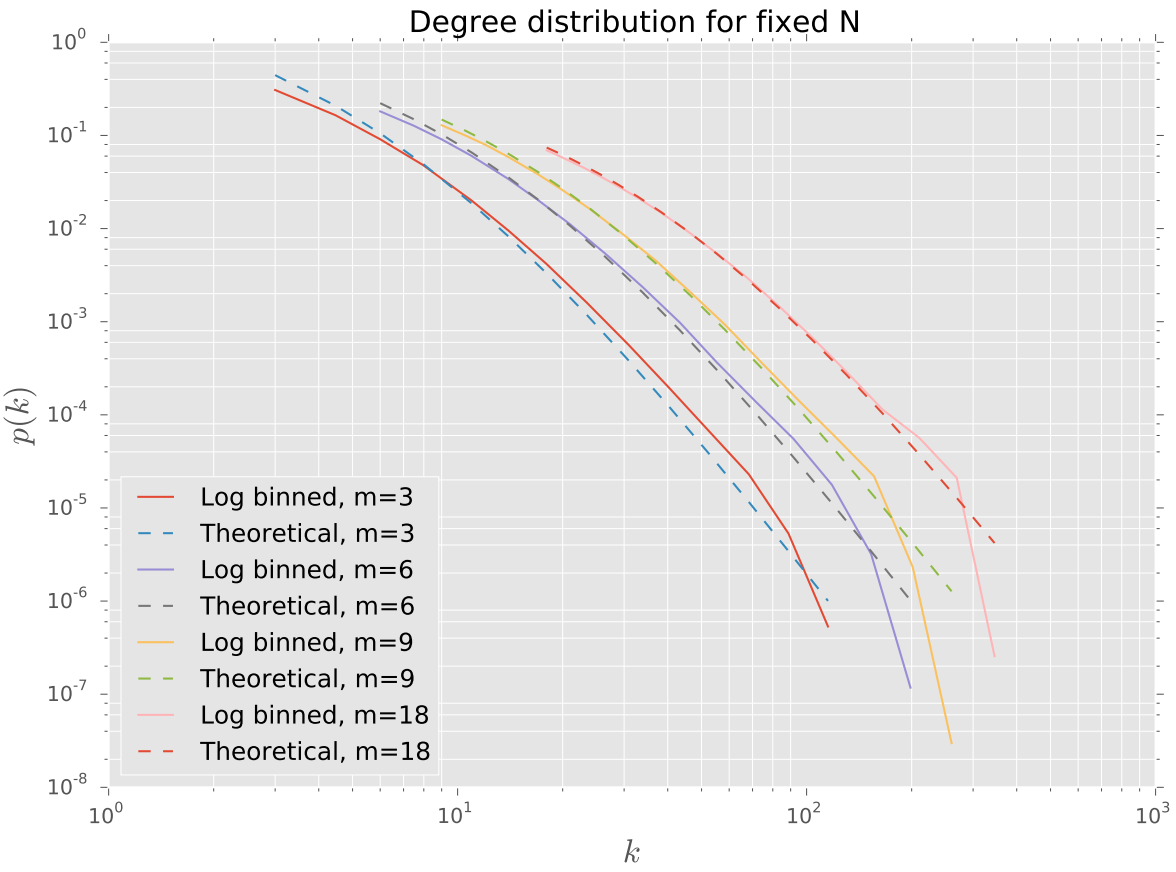


**Figure 9:** Mixed preferential and random attachment with $q = 0.5$ i.e. equal probability of attaching randomly or preferentially.

The fit of the numerical data to the theoretical is calculated with a high K-S p value, showing statistical evidence to support the fact that they are from the same distribution.

As with the preferential attachment model, the degree distribution displays finite size effects with the bump and cutoff at the end of the tail. Once again, the theoretical and numerical distributions seem to fit very well, as can be seen in the figures and in Table 3, with a statistical significance.

| **Comparison with** | **Kolmogorov-Smirnov p** |
|---|---|
| Theoretical dist. | 0.999999999415 |
| Normal dist. | 0.002680210732 |

**Table 3**

# 4 Conclusion

The Barabási-Albert model generates scale-free random networks which have a power-law scaling region. Numerical simulations of the BA model were produced and compared against the theory demonstrating, through the Kolmogorov-Smirnov test, that the preferential attachment mechanism leads to the power law scaling. The largest degree was found to scale with $N^{0.5}$ and is the scale at which finite size effects affect the numerical simulation. In contrast, the pure random attachment mechanism does not show scale-free behaviour, whereas the mixed model shows characteristics of both. An area for further analysis would compare the models to other random networks models such as the E-R model and test for statistical fits.

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