Defining Real Numbers as Oracles, an Overview

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Abstract

A new definition of a real number is that it is a rule which says Yes or No based on whether the real number ought to be in a given rational interval. Since this is defining real numbers, this cannot be the actual definition, but it is the motivation. Instead, the definition specifies properties that a rule must satisfy to qualify as an oracle. The key property is that given a Yes interval and a rational number in that interval, either the number divides the interval into one Yes and one No interval, or the real number is that rational number. This is a summary paper for formalizing, exploring, and generalizing this definition. The full exploration is given in the paper "Defining Real Numbers as Oracles".

1 What is a real number?

The question of how to define a real number has a long history in mathematics. There is a notion that it was settled 150 years ago with the introduction of Dedekind cuts and Cauchy sequences as real numbers. The former has the issue of not being very helpful in computations while the latter is rather too involved in the computations. Neither definition seems to have had much influence on how people perceive a real number or has had much impact on practical uses of it from a definitional point of view. Indeed, it is quite common to take real numbers as axiomatically given rather than delve into these definitions.

The first, and perhaps most lasting, notion of what a real number is may very well be its decimal expansion. It is problematic as the full decimal expansion cannot be written down and yet the definition seems to suggest that is what it is. The expansion is infinite and it is generally not easy to specify a pattern for the digits. As a compromise, a sufficient number of digits is written down to do the computations given one's constraints. One difficulty is that the arithmetic operations tend to expand the uncertainty while the naive computation methods produce more digits in the decimal expansion, falsely suggesting an increase in precision to the uninitiated.

Upon reflection, one way of viewing what is actually needed, given finite limitations, is an interval which the real number is guaranteed to be in and the interval is sufficiently

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small to ensure small enough intervals from future calculations. The core task then seems to be able to produce intervals as small as desired. Arithmetic operations will expand the length of the resulting intervals, but the loss of precision is communicated in the interval presentation itself. When the final interval is too large, one can tighten up the input intervals. On a practical level, this was already done long ago with interval analysis; see [MR09]. This paper intends to give a definition of real numbers that fits that practical use perfectly.

One approach might be to specifically view a real number as something that produces these intervals. The problem is that the production of an interval representative requires a choice which is external to the real number. The approach here is different. The approach involves no arbitrary choice in defining a real number. It does, however, make central a mechanism for producing a useful interval of any length once an initial interval is given.[1]

The idea is that a real number is an oracle that, when given a rational interval, says Yes or No. If it says Yes, then the real number is to be considered to be in that interval. If it says No, then the real number is to be considered to not be in that interval. The definition will formalize what ought to be true about the oracle's answers given that it is describing a single number. The main paper establishes that this definition does lead to the real number field. This paper gives the definition and then highlights the results with some examples of how to use it. This definition balances the need to be computationally friendly while defining an existence independent of computation.

This is not the first time that a definition of real numbers as rational intervals has been entertained. For example, both [Bri06] and [BV06] use the notion of a family of intersecting and arbitrarily fine intervals to define a real number; this is what this paper refers to as a fonsi. They roughly use equivalence relations to handle different families representing the same real number while the approach here, for a fonsi, would be to take the maximal fonsi as the unique definition of a real number. The fonsi definition of real number fits well with constructivist ideas. It was apparently motivated by the idea of measuring physical quantities and noting that an interval is generally returned from that procedure.

This is an overview of the main paper, "Defining Real Numbers as Oracles" [Tay23a]. The main paper contains the full details of what is glossed over in this paper as well as some numerical examples of oracles in practice.

2 Defining the oracles

An oracle is a rule, satisfying a few properties below, that takes in an inclusive rational interval and returns a 1 or a 0. The 1 should be returned when an interval ought to con-

[1] Even if one were to want to define the real numbers as being a function that produced approximations given a starting interval, there would still be the requirement to say that the starting interval contains the real number.

tain the real number.[2] This includes the inclusive rational intervals consisting of exactly one point which may be called a singleton. The main paper uses the unconventional, but useful, notation $a \mathrel{\vdots} b$ to represent the inclusive rational interval since the notation $[a, b]$ suggests a real-filled interval. For the notation $a \mathrel{\vdots} b$, there is no implication of a specific ordering or strict inequality. Indeed, the interval $a \mathrel{\vdots} a$ represents the interval consisting of just a and is a singleton. The notation $a \mathrel{\iddots} b$ is used to indicate $a \leqslant b$ when knowing which one is the lower bound is important.

In what follows, Yes represents a 1 and No represents a 0 result for the interval. The following are the properties that a rule R must satisfy to be an oracle.

1. Consistency. If an interval contains a Yes interval, then it is a Yes interval. This maintains the illusion of the real number being in the Yes intervals and also leads to this being a unique representative of a real number.

2. Existence. There should be a Yes interval. All other properties are conditional so this is the only one that says there is something there.

3. Closed. If a rational number is in every Yes interval, then its singleton should also be a Yes interval. This is the other part in making sure that there is only one representative of the real number. It is a primary reason for using inclusive intervals. This also makes arithmetic with rationals have an easy version.

4. Rooted. There is at most one Yes singleton. This helps ensure that the oracle is narrowed in on a single number.

5. Interval Separation. This is the key property. For a given Yes interval, a point c inside that interval creates two subintervals in addition to its own singleton. The requirement is that exactly one of those intervals is Yes unless $c \mathrel{\vdots} c$ is a Yes singleton, in which case all three are Yes since c is in all three.

If q is contained in every Yes interval, then q is the **root of the oracle**. An oracle with a root may be called a rooted oracle or a singleton oracle. Rooted oracles are the rational real numbers. If an oracle is not rooted, then it may be called a neighborly oracle. Neighborly oracles are the irrational real numbers.

While the first three properties above seem to be basic requirements, for maximality, there is flexibility on the last two. They can be replaced with equivalent properties.

The first equivalent replacement would be the two properties: 1) Two Point Separation, which states that given any two rational numbers in a Yes interval, there exists a Yes interval that does not contain at least one of the two numbers, and 2) Disjointness, which states that if two intervals are disjoint, then at most one of them can be a Yes interval. These two properties are more useful in generalizing to metric spaces, where

[2] Set theoretically, an oracle is a function $R\colon \mathbb{Q}^2 \to \{0, 1\}$ where each pair (a, b) is identified with the inclusive rational interval $a \mathrel{\vdots} b$. This viewpoint is suppressed here as the goal is to provide a framework for just-in-time determination. Another way of viewing the rule is that it is the characteristic function of the set of intervals that contain the real number, a viewpoint which is also not adopted here.

inclusive balls are considered instead of inclusive intervals. The interval separation does not generalize to that context, but two point separation does. The paper [Tay23d] gives more details about metric completion using oracles as well as other generlizations to other spaces and purposes.

The second equivalent replacement would be the two properties: 1) Narrowing, which states that given a rational positive length, there is a Yes interval that is shorter than that length, and 2) Intersection, which is that all Yes intervals intersect. If a set of intervals satisfy these two properties, then they are called a Family of Overlapping, Notionally Shrinking Intervals, or a **fonsi**. Given a fonsi, there is a unique oracle whose set of Yes intervals contain the fonsi. The oracle is defined via the rule that an interval is considered to be a Yes interval if it contains an intersection of elements of the fonsi. This intersection can be finite or infinite. The infinite intersection is mainly of interest for rooted oracles as the singleton may not be in the fonsi nor included in finite intersections. This is a very practical tool in establishing oracles in a variety of contexts, such as a converging sum. A maximal fonsi is a fonsi in which all intersections of elements in the fonsi that result in rational intervals are in the fonsi and all intervals that contain an element of the fonsi is in the fonsi. A maximal fonsi is exactly the set of Yes intervals for the associated oracle.

2.1 Basic properties of an oracle

There are a variety of properties that can be deduced from these definitions. Proofs can be found in the main paper though in general these are fairly easy to check. In what follows below, there is a fixed oracle which the Yes intervals are referring to.

1. The intersection of two Yes intervals is non-empty. The intersection is a Yes interval.

2. The union of Yes intervals is a Yes interval.

3. Any interval contained in a No interval is a No interval.

4. The union of two intersecting No intervals is a No interval.

5. If two intervals are disjoint, then no more than one can be a Yes interval.

6. If an interval is disjoint from a Yes interval, then it is a No interval.

7. The separation property extends to a finite partition of a Yes interval. There will be exactly one subinterval that is a Yes interval unless one of the partition elements is a root of the oracle.

8. Given a positive length, there is a Yes interval that is shorter than that length. This follows by using the Separation property applied to successive bisection points to a given Yes interval. The existence property is essential in being able to start this process.

9. The Two Point Separation property holds by applying the Separation property to a number in between the given numbers to separate.
10. We can view the set of Yes intervals as an ultrafilter using the partial ordering on rational intervals given by interval inclusion.

2.2 Ordering of Oracles

The relation of two oracles can be seen by the ordering of sufficiently narrow Yes-intervals. An interval $a \therefore b$ is less than the interval $c \therefore d$ exactly when $b < c$. If α and β are two oracles, then:

1. $\alpha < \beta$ if there exists an α-Yes interval which is less than a β-Yes interval.
2. $\alpha > \beta$ if there exists an α-Yes interval which is greater than a β-Yes interval.
3. $\alpha = \beta$ if α and β have the same answer on every interval.
4. $\alpha ? \beta$ if α and β have the same answer on every interval that they have returned answers on. The intersection of all such Yes intervals is the current Resolution of Compatibility.

One notation that is helpful is to use a bracket around an oracle in a rational interval to indicate that there is a Yes interval that is between the two. Namely, $a \vdots [\alpha] \vdots b$ could be used to indicate that there exists an α-Yes interval whose endpoints are between a and b, potentially inclusive. This could be extended to have betweenness of oracles as well. The notation $[\alpha] \vdots [\beta] \vdots [\gamma]$ means that either $\alpha \leqslant \beta \leqslant \gamma$ or $\alpha \geqslant \beta \geqslant \gamma$.

The first three are what one would expect. The last one is a capitulation to the reality that it is often not possible to fully differentiate an oracle from a nearby one. The main paper gives an example of an oracle that is based on the Collatz conjecture. The resolution of compatibility with the Oracle of 0 is $0 \vdots 2^{-68}$ at this time. This example is not of practical significance, but it does illustrate potential limitations.

It is a merit of the oracle approach that this can be said clearly and naturally.

The main paper does establish that this is an ordering with the usual properties.

One key tool in determining that two oracles are equal is that they are equal if their Yes intervals can be shown to always overlap. This can be further refined to them being equal if every Yes interval of one of them can always be shown to contain an interval of the other. This is used in establishing the distributive property.

The distance between two oracles can be computed from computing the distances between the Yes-intervals and taking the greatest lower bound of the distances. This exists since the distances are bounded below by 0. The distance of two intervals is the maximal distance of any two numbers in the interval which is the difference of the two farthest endpoints of the intervals. The distance of an interval from itself is its length.

With the ordering established, closed and open intervals can be defined either as usual or one can define an oracle as being in the interval if certain intervals are Yes intervals for the oracle. Indeed, let γ be an oracle and the closed interval $[\alpha, \beta]$ be given. For γ to be in the interval, every rational interval $a \mathrel{\vdots} b$ must be a γ-Yes interval whenever a is a lower endpoint of an α-Yes interval and b is an upper endpoint of a β-Yes interval. In other words, if an interval is both α-Yes and β-Yes, then it must also be a γ-Yes interval. For an open interval, γ is in that interval if there exists a rational interval $a \mathrel{\vdots} b$ that is γ-Yes with a being an upper endpoint of an α-Yes interval and b a lower endpoint of a β-Yes interval. Another way of phrasing that is to say that γ is in the open interval if there exists a γ-Yes interval that is between, and disjoint from, two intervals, one of which is α-Yes and the other is β-Yes.

2.3 Rationals, roots, and zeros

The first examples to explore with oracles are the rationals and n-th roots.

The oracle of a rational number q is defined as the rule that $a \mathrel{\vdots} b$ is a Yes interval exactly when $a \leqslant q \leqslant b$. It is easy to check that the properties hold. It has a canonical representative interval in the form of the singleton $q \mathrel{\vdots} q$. This works because the rational numbers independently exist outside of this definition.

The n-th root of a positive rational number q is defined as the rule that $a \mathrel{\vdots} b$ is a Yes interval exactly when $\max(a, 0)^n \leqslant q \leqslant \max(b, 0)^n$. If $b > a > 0$, this is more simply stated as $a \mathrel{\vdots} b$ is a Yes interval exactly when q is between the n-th powers of a and b. That this defines an oracle can be established with the help of the monotonicity of x^n. The Closed Property takes a little work to show. Namely, one needs to show that if $p^n \neq q$, then there is a Yes interval that excludes p.

The oracle of the n-th root therefore exists. The next step is to start finding some Yes intervals. A great algorithm is the one from Newton's Method. Start with a given positive number x for the n-th root of q, ideally somewhat close to the root though one can always use 1. The guess x has the associated Yes interval $x \mathrel{\vdots} \frac{q}{x^{n-1}}$ which satisfies $x^n \mathrel{\vdots} q \mathrel{\vdots} (\frac{q}{x^{n-1}})^n$. The next interval is obtained by taking the weighted average, specifically $\frac{1}{n}((n-1)x + \frac{q}{x^{n-1}})$ becomes the next x. Continue to iterate. At each stage, there is a Yes interval for $\sqrt[n]{q}$. This sequence of intervals is nested and the lengths are eventually shrinking quadratically to zero.

There are, of course, many techniques for obtaining the roots. One technique given in the main paper is to experiment with using right triangles to get an interval trapping of $\sqrt{2}$. This helps illustrate the natural perspective of interval containment.

Perfectly motivating examples for oracles are those produced by the foundation of the Intermediate Value Theorem. In particular, given a function f and a value y, the oracle α of interest is the solution to $f(\alpha) = y$. Assume that there is an interval $[a, b]$, a and b rational, on which the function f is continuous, strictly monotonic, and y is between $f(a)$ and $f(b)$, that is, $[f(a)] \mathrel{\vdots} [y] \mathrel{\vdots} [f(b)]$. Note that these outputs could be oracles. Then for $c \mathrel{\vdots} d$ in $a \mathrel{\vdots} b$, the interval is a Yes interval if $[f(c)] \mathrel{\vdots} [y] \mathrel{\vdots} [f(d)]$. The Yes intervals form a fonsi whose associated oracle is the solution. If f is strictly monotonic and continuous on $(-\infty, \infty)$, then the rule is immediately an oracle. Note that a singleton is included automatically if $f(q) = y$ for some rationals q and y.[3]

Some examples to explore include: π as the zero of $\sin(x)$ on the interval $3 \mathrel{\vdots} 4$, e as the solution to $\ln(x) = 1$ on the interval $2 \mathrel{\vdots} 3$, and $\sqrt{2}$ as the solution of $x^2 = 2$ on the interval $1 \mathrel{\vdots} 2$.[4] The requirement for these real-valued functions to be used is that the function evaluated on rationals can indicate whether the ultimate value is above, below, or the same as the target value.

This method provides more than just the existence of a solution, which is often insufficient for practical applications. Oracles in general come with the ability to construct a sequence of narrowing intervals. To do so, take the bisection point of $a \mathrel{\vdots} b$, namely $m = \frac{a+b}{2}$, and test it out. For an IVT oracle, compute $f(m)$ and decide how it relates to y, $f(a)$, and $f(b)$. Replace one of the input endpoints with m and then $f(m)$ replaces the corresponding output endpoint. The replacement should be chosen to maintain y being in the resulting interval, something which can always be done. Iterate this until sufficient precision is obtained. The midpoint strategy will halve the interval at each step which is convenient for estimating how many steps to take to achieve a desired precision.

Another useful choice is to use mediants. For the fractions $\frac{a}{b}$ and $\frac{c}{d}$, the mediant is $\frac{a+c}{b+d}$ and is always a number in between them. The main paper devotes a section to their use, the relation to continued fractions, the Farey process, and the Stern-Brocot tree. By using mediants and the above function evaluation method, the narrowing process generates the best rational approximations to the solution of the equation as well as generating the continued fraction representation of that solution.

If the solution of the equation is rational, then the mediant process will produce it exactly while the midpoint process will typically generate a sequence of approximations that never achieves the full solution. For example, solving $3x = 1$ starting in $0 \mathrel{\vdots} 1$ will produce the singleton solution of $\frac{1}{3} \mathrel{\vdots} \frac{1}{3}$ in two steps using mediant intervals ($\frac{0}{1} \mathrel{\vdots} \frac{1}{1}$, $\frac{0}{1} \mathrel{\vdots} \frac{1}{2}$, $\frac{1}{3} \mathrel{\vdots} \frac{1}{3}$), while the midpoint process produces $0 \mathrel{\vdots} \frac{1}{2}, \frac{1}{4} \mathrel{\vdots} \frac{1}{2}, \frac{1}{4} \mathrel{\vdots} \frac{3}{8}, \frac{5}{16} \mathrel{\vdots} \frac{3}{8}, \ldots$

The key point is that oracles provide a consistent framework for exploring these processes and for how the infinite can be handled by finite beings.

2.4 Being complete

There are a few tasks that real numbers handle that rationals are not capable of doing. These are the completion properties. Full details are in the main paper.

The first is the least upper bound property. Given a non-empty set E of rationals with an upper bound, the goal is to establish that there is an oracle which will serve as a least upper bound. Define the set of upper bounds to be U. Then define the oracle

[3]The IVT applies more generally than monotonic functions, of course. In that case, one can still define a procedure, but what oracle one ends up with as a solution may depend on the procedure for generating interval divisions. For the more general case and how to understand real-valued functions from the perspective of oracles, see [Tay23b].

[4]This assumes that the transcendental functions can be defined and sufficiently evaluated to be able to precisely determine the sign for a given input. For some discussion on these matters, see [Tay23b] on function oracles.

by the rule that $a \mathrel{\therefore} b$ is a Yes interval exactly when $a \leqslant y$ for all $y \in U$ and $b \geqslant x$ for all $x \in E$. Most of the proof is simply writing down the statements and working out the inequalities.

The main paper also augments this to any bounded set of oracles having a least upper bound. The details mainly differ by having to use the interval representation of the oracles in getting lower and upper endpoints for the intervals. Namely, the lower endpoints of the oracles in E and the upper endpoints of the oracles in U.

The second completion property is that Cauchy sequences should have limits. The basic definition of the limit oracle is that an interval is a Yes interval if it contains the tail of the sequence. The property of the narrowing of the Cauchy sequence establishes the No's of the Separation and Rooted property. For a Cauchy sequence converging to a rational number, it is required that the singleton be explicitly added since no singleton can contain a non-constant tail.

Related to this is the general notion of a fonsi as mentioned above. A primary example is that of nested, converging sequences of intervals which Cauchy sequences can be converted into. Absolutely converging series can be directly established to be oracles using a fonsi.

But a fonsi need not be sequential. As an example, consider a nesting function which takes in lengths and produces intervals shorter than that length that are nested within the larger intervals. A fonsi could also be a set that consists entirely of a single singleton. The main criterion is that every interval in the family overlaps and that it is possible to find intervals shorter than any given length. Converting them into oracles is largely filling in any missing intervals, particularly making sure consistency and the closed property hold.

Two fonsis are associated with the same oracle if one can show that every element in one fonsi intersects with every interval of the other fonsi. An alternate statement is that the union of the two fonsis is a fonsi which only happens if they represent the same real number. They also represent the same oracle if, given any small length, there exist intervals in both fonsis which are smaller than that length and overlap each other. Two Cauchy sequences whose tails are arbitrarily close can thus be seen to have the same limit.

Fonsis are the objects that the constructivists, cited earlier, use to define real numbers. The concern for them is being explicit which a fonsi very much allows. The downside is that a real number is not defined uniquely. The approach here embraces having a unique definition of a real number which has the minor downside of having very long intervals that are not of any practical concern.

3 Arithmetic

Arithmetic is done with the intervals. Interval arithmetic is based on, but different than, the usual number arithmetic. Conceptually, combining intervals with arithmetic operators is to look at the smallest interval that contains the results of performing the indicated operation on every pairing of the elements from the interval. In practice, it means looking at the endpoint combinations. Division only works on same signed intervals. Addition is easy, multiplication with signed intervals is a little more involved. Computing the power of intervals is rather involved.

Many of the properties of ordinary arithmetic apply to intervals, but not all. There are no inverses. One side of the distributive property is included in, but not equal to, the other. Generically, combining intervals leads to longer intervals. All of this is gone into detail in the main paper, but it is straightforward to explore interval arithmetic.

Oracle arithmetic, while based on intervals, does create a single oracle answer instead of an interval. This is the result of oracles having notionally shrinking intervals and thus the combination of all of those intervals does produce an oracle with intervals as small as needed. The properties of usual arithmetic can then be established, including the distributive property as well as the existence of additive and multiplicative inverses.

Oracle ordering is compatible with arithmetic. Namely, adding the same oracle to both sides of an inequality preserves the inequality and two positive oracles multiplied together are positive. These are easy to show with sufficiently narrow Yes-intervals.

The main paper goes through all the different arithmetic operators, produces bounds for the interval arithmetic operators, and establishes the necessary results from that. This paper will simply illustrate arithmetic with an example.

3.1 An example of arithmetic

The task is to compute $x = \frac{e-\sqrt{2}}{\pi}$. This should be an oracle and so it should be able to answer questions about any interval given to it. For example, is x in the interval $Q = \frac{41}{100} \vdots \frac{42}{100}$?

The first step is to start computing the interval arithmetic with some narrow inputs. If it is sufficiently narrow, either the output interval will be contained in Q, in which case the oracle says Yes, or it will be disjoint from Q in which case the oracle says No. After testing this interval, the mediant process will be demonstrated also using this one computed interval.

To compute out a Yes interval, the inputs needed are interval representations for e, $\sqrt{2}$, and π. Using mediant approximations for these numbers is recommended:

1. e-Yes interval $A = \frac{106}{39} \iddots \frac{87}{32}$,
2. $\sqrt{2}$-Yes interval $B = \frac{41}{29} \iddots \frac{17}{12}$, and
3. π-Yes interval of $C = \frac{333}{106} \iddots \frac{22}{7}$.

The interval operations are then as follows:

1. Subtraction is $A \ominus B = (\frac{106}{39} - \frac{17}{12} \iddots \frac{87}{32} - \frac{41}{29}) = \frac{203}{156} \iddots \frac{1211}{928}$
2. Reciprocating, to convert division into multiplication, leads to $1 \oslash C = \frac{7}{22} \iddots \frac{106}{333}$
3. Multiply the reciprocal by the subtraction result to get $D = (A \ominus B) \oslash C = \frac{203}{156} * \frac{7}{22} \iddots \frac{1211}{928} * \frac{106}{333} = \frac{1421}{3432} \iddots \frac{64183}{154512}$

If that precision is insufficient, then the steps can be repeated with more precise input intervals. The main paper does have bounds that would allow for the computation of the precision ahead of time.

If Q contains D, then Q is a Yes interval by Consistency. To test whether Q contains D, the task is to compute whether the endpoints are greater than or less than one another. A useful observation is that $\frac{a}{b} < \frac{c}{d}$ exactly when $ad - bc < 0$. Equality happens if that quantity is 0. Computing $41 * 3432 - 100 * 1421 = -1388$, the conclusion is that $\frac{41}{100} < \frac{1421}{3432}$. For the other endpoint, the computation $64183 * 100 - 154512 * 42 = -71204$ yields $\frac{64183}{154512} < \frac{42}{100}$. The Yes interval D is therefore contained in the interval Q which, by consistency, forces the oracle to reply Yes for Q.

The rational intervals obtained from the arithmetic operators can have large denominated fractions as endpoints. The mediant process can reduce that complexity. The process starts with the unique Yes interval which is of length 1 with integral endpoints. For this oracle, that interval is $\frac{0}{1} \vdots \frac{1}{1}$ as D is clearly contained in it.

The approach is to compute the mediant of the interval and then see which of the two subintervals that it defines, still contains the interval D. This continues until the mediant is inside D and it ceases to be possible to determine from D which interval is a Yes interval.

1. Mediant is $\frac{1}{2}$. Computation with the the upper endpoint: $2 * 64183 - 1 * 154512 = -26146 < 0$. The mediant is therefore greater than the upper endpoint leading to D being in the interval $\frac{0}{1} \vdots \frac{1}{2}$. (L)

2. Mediant is $\frac{1}{3}$. This turns out to be below the lower endpoint: $3 * 1421 - 1 * 3432 = 831 > 0$. Thus, D is in the interval $\frac{1}{3} \vdots \frac{1}{2}$. (R)

3. Mediant is $\frac{2}{5}$. It is below the lower endpoint: $5 * 1421 - 2 * 3432 = 241 > 0$. Thus, D is in the interval $\frac{2}{5} \vdots \frac{1}{2}$. (R)

4. Mediant is $\frac{3}{7}$. It is above the upper endpoint: $7 * 64183 - 3 * 154512 = -14255 < 0$. Thus D is in the interval $\frac{2}{5} \vdots \frac{3}{7}$. (L)

5. Mediant is $\frac{5}{12}$. It is above the upper endpoint: $5 * 64183 - 12 * 154512 = -2364 < 0$. Thus D is in the interval $\frac{2}{5} \vdots \frac{5}{12}$. (L)

6. Mediant is $\frac{7}{17}$. It is below the lower endpoint: $17 * 1421 - 7 * 3432 = 133 > 0$. Thus D is in the interval $\frac{7}{17} \vdots \frac{5}{12}$. (R)

7. Mediant is $\frac{12}{29}$. It is below the lower endpoint: $29 * 1421 - 12 * 3432 = 25 > 0$. Thus D is in the interval $\frac{12}{29} \vdots \frac{5}{12}$. (R)

8. Mediant is $\frac{17}{41}$. It is not below the lower endpoint: $41 * 1421 - 17 * 3432 = -83$. It is not above the upper endpoint: $41 * 64183 - 17 * 154512 = 2368836 > 0$. D is not contained in either $\frac{12}{29} \vdots \frac{17}{41}$ or $\frac{12}{29} \vdots \frac{17}{41}$. To proceed further, a more narrow Yes interval must be computed to replace D.

The conclusion so far is that $\frac{12}{29} \vdots \frac{5}{12}$ is an $\frac{e-\sqrt{2}}{\pi}$-Yes interval. If an error of .0029 is sufficient, then the computation can stop here. In comparison, the error in interval D is .0013. Notice that the mediant representative interval has much simpler fractions to work with which is the advantage of using mediants when possible. To check on the initial interval, Q, the computations are $41 * 29 - 100 * 12 = -11$ and $5 * 100 - 12 * 42 = -4$ which implies $\frac{41}{100} \therefore \frac{12}{29} \therefore \frac{5}{12} \therefore \frac{42}{100}$. This leads to the same conclusion, but with simpler arithmetic involved.

It is useful to note that the mediant process above is computing out the continued fraction for the oracle. As mentioned in the main paper, the continued fraction is an accounting of whether the left or right subinterval is selected. The pattern was L,L,R,R,L,L,R,R.[5] Thus, doing a count, the continued fraction at this point is $[0; 2, 2, 2, 2]$ and the next one, representing the choice to be made with $\frac{17}{41}$, is either $[0; 2, 2, 2, 2, 1]$ or $[0; 2, 2, 2, 3]$. As a continued fraction, those two choices both represent $\frac{17}{41}$, but for the mediant process they represent either a left selection or a right selection, respectively. In fact, the next two selections will be a right selection before switching, leading to $[0; 2, 2, 2, 4, 1]$. The next three intervals are therefore, $\frac{17}{41} \vdots \frac{5}{12}$, $\frac{22}{53} \vdots \frac{5}{12}$, and then, with the switch, $\frac{22}{53} \vdots \frac{27}{65}$. That final interval has a length of about 0.00029.

This has been an example of oracle arithmetic. It is messy and it will rarely end with an exact production of a number. But it can produce intervals to arbitrary precision given sufficient computational resources and the ability to be arbitrarily precise on the input intervals. The mediant process allows for the reduction of complexity.

3.2 The decimal version

It is useful to compare the oracle computation with decimal computations. In what follows, the symbol = will stand in for approximately equal. The first difficulty is what exactly that means. There are four likely interpretations of this:

- Short Rounding. x is in the interval $a.d_1d_2d_3 \dots d_n \pm 0.5 * 10^{-n}$. $\pi = 3.14$ implies the interval $3.135 \vdots 3.145$.
- Truncation. x is in the interval $a.d_1d_2d_3 \dots d_n \vdots a.d_1d_2d_3 \dots d_n + 10^{-n}$. $\pi = 3.14$ implies the interval $3.14 \vdots 3.15$.
- Long Rounding. x is in the interval $a.d_1d_2d_3 \dots d_n \pm 10^{-n}$. $\pi = 3.14$ implies the interval $3.13 \vdots 3.15$.
- Big Uncertainty in Last Digit. x is in the interval $a.d_1d_2d_3 \dots d_n \pm 5 * 10^{-n}$. $\pi = 3.14$ implies the interval $3.09 \vdots 3.19$.

Redoing the computation of $x = \frac{e-\sqrt{2}}{\pi}$ using $\pi = 3.14$, $e = 2.72$, $\sqrt{2} = 1.41$, the single value computation yields roughly $u = 0.417195$. To about that precision, x is 0.4150978.

[5] The first L comes from selecting $\frac{0}{1} \vdots \frac{1}{0}$ from the actual starting "interval" of $\frac{0}{1} \vdots \frac{1}{0}$. The leading 0 in the continued fraction represents the number of initial right selections which is equal to the integer portion of the number.

This does suggest that the rough computation needs to be further rounded or truncated to avoid giving a false precision. In what follows, two intervals per each method will be produced. The first interval is the interval level that the default interpretation would contain both x and u. The second interval will be the interval that gets produced by doing interval arithmetic and then the appropriate representative to produce something that contains that interval will also be given.

- Short Rounding. 0.42 represents the interval $0.415 \mathrel{⁝} 0.425$ which contains both x and u. Computing the implied interval, the result is $\frac{2.715-1.415}{3.145} \mathrel{⁝} \frac{2.725-1.405}{3.135} = 0.4133 \mathrel{⁝} 0.4211$. This does include x, but to translate it into a single number with the implied spread covering this entire interval would yield 0.4 because 0.4133 is not contained in 0.42's interval under short rounding. The implied interval is $0.35 \mathrel{⁝} 0.45$.

- Truncation. 0.41 would work as the interval of $0.41 \mathrel{⁝} 0.42$ contains both x and u. The interval computation is $\frac{2.72-1.42}{3.15} \mathrel{⁝} \frac{2.73-1.41}{3.14} = 0.4126 \mathrel{⁝} 0.4204$. Again, while x is in this interval, to cover the edge, here 0.4204, one cannot write 0.41. Thus, as before, 0.4 is the single number representative; it implies $0.4 \mathrel{⁝} 0.5$ as the interval.

- Long Rounding. 0.42 implies $0.41 \mathrel{⁝} 0.43$ and 0.41 implies $0.40 \mathrel{⁝} 0.42$. So either version will contain both x and u. As for the interval computation, $\frac{2.71-1.42}{3.15} \mathrel{⁝} \frac{2.73-1.40}{3.13} = 0.4095 \mathrel{⁝} 0.4250$. Neither 0.42 nor 0.41 will have that whole interval contained in it. So 0.4 is to be chosen implying $0.3 \mathrel{⁝} 0.5$.

- Big Uncertainty in the Last Digit. There are several representatives that can cover both x and u. 0.41 leads to the interval $0.36 \mathrel{⁝} 0.46$. The interval computation is $\frac{2.67-1.46}{3.19} \mathrel{⁝} \frac{2.77-1.36}{3.09} = 0.3793 \mathrel{⁝} 0.4564$. The representative 0.42 leads to $0.37 \mathrel{⁝} 0.47$ which just barely covers the computed interval.

The problem with decimal arithmetic is that the imprecision grows making it difficult to use a single representative without sacrificing claims of precision. The last two options for interpretation also imply multiple reasonable representatives for a given decimal and level of precision.

How about calculators? The calculator used here will be a TI-84. The calculator reports $e = 2.718281828$, $\pi = 3.141592654$, and $\sqrt{2} = 1.414213562$. The calculator seems to do short rounding to produce decimals, for example, if expanding π just a little further, the result is 3.14159265358 which the calculator rounds up. Computing out the decimals, the calculator reports 0.4150978214 with the implication that the result should be in the interval $0.41509782135 \mathrel{⁝} 0.41509782145$. WolframAlpha reports it as 0.415097821354 so it is indeed in that interval, but just barely.

This success is because the calculator is computing more digits than is shown. If the directly displayed digits were used for the numbers, then the result is 0.4150978213 which implies the interval $0.41509782125 \mathrel{⁝} 0.41509782135$ and this is not an interval that contains x.

Computing an interval from the display version and using WolframAlpha's abilities to handle longer strings of digits, the result is roughly $0.41509782088 \vdots 0.41509782166$ which does contain x. The closest short round representative whose associated interval would include this computed interval, however, would be 0.41509782 as 0.415097821 would not imply including the numbers above 0.4150978215.

On a practical level, our tools have been engineered to produce sufficiently accurate numbers for any practical concerns, but it can be potentially misleading, particularly for those just learning how to use decimals. Intervals lead to precise statements that convey the limitations inherently and unambiguously at the cost of length and time.

3.3 Arithmetic with rationals

For rational oracles, arithmetic can be performed with their singletons. Doing the multiplication $2 * \pi$ using the π-Yes interval $\frac{333}{106} \vdots \frac{22}{7}$ leads to $\frac{333}{53} \vdots \frac{44}{7}$ being a 2π-Yes interval. If one then tried to recover 2 by dividing by π, using that same interval for π, the result is $\frac{333}{53} * \frac{7}{22} \vdots \frac{44}{7} * \frac{106}{333} = \frac{2331}{1166} \vdots \frac{4664}{2331} \approx 1.9991 \vdots 2.0009$. That is, while 2 is in that interval and will be in all such intervals, the interval arithmetic does not lead to an immediate conclusion of the singleton. This is a practical problem with the arithmetic of real numbers which seems to apply to all definitions of real numbers. This approach at least gives immediate bounds as part of the process.

The mediant process hits an immediate problem when attempting to apply it to simplifying 2π-Yes interval endpoints. The first step of the mediant process is to get the integer part. One needs to choose between the intervals $\frac{1}{1} \vdots \frac{2}{1}$ and $\frac{2}{1} \vdots \frac{1}{0}$. Since 2 is the solution but cannot be seen as such, this process will not be able to decide which interval. It cannot proceed as above. But it is possible to modify the other endpoints using mediants. That is, the process can consider choosing both subintervals and proceeding until the known Yes interval cannot distinguish any further. On the left, the k-th mediant subinterval will be $\frac{1+2k}{1+k} \vdots \frac{2}{1}$, that is, the process is repeatedly adding $\frac{2}{1}$ to the left endpoint. Each time, the comparison is $\frac{1+2k}{1+k}$ to $\frac{2331}{1166}$. Solving for k, the result is that $\frac{2331}{1166}$ is one of these mediants. The other side is adding $\frac{2}{1}$ to the right endpoint repeatedly. This is $\frac{2k+1}{k+0}$ and comparing it to $\frac{4664}{2331}$. Setting the k fraction to be less than the fixed endpoint and solving, the inequality is $4664k > 2 * 2331k + 2331$ leading to $k > 1165.5$. Testing 1165 and 1166 leads to finding $\frac{2333}{1166} < \frac{4664}{2331} < \frac{2331}{1165}$. Since this was for the upper bound, we choose the larger one. Thus, the best mediant-based interval enclosing the calculated interval would be $\frac{2331}{1166} \vdots \frac{2331}{1165}$. If it could be established that this pattern continues with finer initial input intervals, then the 2 could be established to be the oracle.

3.4 The reals

The oracles with the above relations and arithmetic operators do form an ordered field. The rationals are embedded as a subfield. The rationals are dense as given any two distinct oracles, there are disjoint Yes intervals for them and any rational between the two closest endpoints will root a rational oracle in between them.

The Archimedean property follows by writing down disjoint intervals of the two reals and then scaling the lesser one past the upper one using the property that rationals can be scaled past each other.

Another property of the real numbers is that they are uncountable. The proof of that with oracles is fairly simple. Given a list of oracles, one can use the mediant process, possibly applying it twice in one step, to choose a mediant interval which does not contain the next item on the list. Since the oracle answers questions about intervals and since the mediant process will go past the point where a fraction with a given denominator can appear in that interval, this oracle will always give an answer that satisfies the required properties. Hence, this is an oracle and it is not on the list by construction.

4 Relation to other definitions

The main paper has a lengthy discussion on how this definition relates to other ones. This will be a brief version of that. For a reference of an overview of different real number definitions, please see [Wei15]. For those curious about the deficiencies of other definitions in detail, please check out the videos by Norm Wildberger, such as the provocative "Real numbers as Cauchy sequences don't work!"[6]

Many definitions of real numbers consist of a particular numerical representation of a real number, such as summations or the wonderful continued fraction representations. Decimals are another example which includes the non-essential choice of base 10.

Cauchy sequences are different in that there is no particular specification of a single Cauchy sequence for a real number. In some sense, Cauchy sequences encompass all those particular representations in addition to many others. This arbitrariness in any single sequence leads to using equivalence classes of Cauchy sequences as the real number defined by the Cauchy sequence. While that is a unique representation, it has the unfortunate side effect of making every real number look the same. Indeed, fix a sequence of n rational numbers and choose any equivalence class of Cauchy sequences. Then there are infinitely many elements of that equivalence class with those numbers as the initial segment of the sequence. In fact, one can take any sequence in the equivalence class and prepend the given n sequence to its beginning. For comparing Cauchy sequences in an equivalence class, the control defined in the Cauchy sequence is lost across the equivalence class. For example, take a Cauchy sequence that converges to π, say the mediant approximations given by its continued fraction representation. That is a perfectly good Cauchy sequence for π. Consider a similar Cauchy sequence based on the continued fraction representation of $\sqrt{2}$. Now take the first 10^{1000} elements of the $\sqrt{2}$ mediant sequence and prepend them to the mediant Cauchy sequence for π. If one were to write out this representative of π, then one would surely be forgiven after the first million terms of thinking that this would represent $\sqrt{2}$. Cauchy sequences suffer

[6]https://www.youtube.com/watch?v=3cI7sFr707s

from the fact that humans can easily deal with an initial finite portion of the sequence while the mathematics only cares about the infinite tail behavior of it. The core part of a Cauchy sequence is the sequence of shrinking intervals that contain the tail of the sequence. The initial, visible, portion of the sequence is arbitrary and irrelevant.

Well-specified Cauchy sequences are good tools to describe a particular real number in a particular way, but there is arbitrariness in the choices defining a sequence. They also essentially demand the computation be done in order to define the number. Oracles have no such arbitrariness in them nor does there need to be computations done just to define the number. Large, universe spanning intervals are present as Yes intervals in an oracle, but the burden of what is relevant is on the questioner, not the oracle.

Continued fraction representations are not that arbitrary. Finite continued fractions do have a non-uniqueness representing which subinterval to select, but infinite ones are unique. Arithmetic with continued fractions is an acquired taste[7] though ordering is very pleasant to work with. Many aspects of them do feel natural, but most of that is brought out even more from the perspective of the mediant approximation method which flows naturally from oracles.

There are other approaches involving a fonsi in one way or another. Bachmann's approach of 1892 was to use nested sequences of shrinking intervals. The constructivist texts of [Bri06] and [BV06] take a fonsi as a real number. The constructivists also prefer to think of equality of real numbers as being demonstrated by two fonsis having the same overlaps rather than thinking of an equivalence class defining a single version of a real number. From the oracle viewpoint, a fonsi is an infinite approximation scheme for the oracle.

An even larger approach, as discussed in [Wei15], is that of minimal Cauchy filters. This is roughly equivalent to defining a real number as the set of all sets that contain a given fonsi without including the singleton.

The approach taken here sits in between the fonsi and the Cauchy filter. There is a single version of the number, just like the minimal Cauchy filter and unlike the fonsi approaches. In terms of singletons, the only version which is guaranteed to have a rational in the real number representative is the oracle approach. The fonsi approaches may or may not include the rational. The filter approach specifically excludes it. A more substantial difference, and what is at the core of oracles, is that oracles are the only approach in which the interval separation property is guaranteed to work. The fonsi approach guarantees a smaller interval can be obtained, but not how and not conveniently. The Yes intervals are a part of the filter though there are many more general sets included in which separation feels irrelevant in addition to the problem if the singleton is chosen in the filter approach.

The final definition to discuss is that of Dedekind cuts. They are not arbitrary as many of the other representations are. Indeed, they are the lower bounds of the Yes-intervals, excluding singletons. They work, but they seem unhelpful. For example, in the intermediate value definition for a strictly increasing function, the solution by Dedekind

[7] See the main paper for an example arithmetic computation using continued fractions. It is an interesting experience.

cuts is defined as $\{x|f(x) < y\}$. This does not actually suggest a way of computation nor does this seem clearer or easier to handle than $f(x) = y$. The oracle perspective provides an easy definition and suggests how to compute a good representative. It is not even clear with Dedekind cuts what one ought to compute while for oracles, computing smaller intervals is strongly suggested by its very formulation.

It can be useful to consider how these definitions handle the completion properties. Cauchy sequences are their own limits? Dedekind cuts yield a kind of completed version of the set E itself as the least upper bound of the set E? For the least upper bound, Cauchy sequences lead to having to do the algorithm to define it. For Dedekind cuts, the Cauchy sequence limit definition of an x being less than a tail segment feels like just a partial specification.

Ultimately, all of these definitions of real numbers have their place as viewpoints and useful representations, but, as a definition, they all fall short. With oracles, there is no arbitrariness in the representative. Oracles suggest strongly how to compute and approach a number without demanding the computation be done to define the number. In many applications of trying to find a real number, the definition fits quite naturally in both assuring the existence of the solution in addition to yielding a computation of it with built in precision information. The arithmetic and ordering seem fairly natural in their definition and use.

The final criteria is how well does the definition generalize. The specific representations, such as continued fractions, are very particular to the real numbers. Dedekind cuts rely on the ordering and do not generalize to unordered spaces though do see [Mau14] for some generalizations. They are intimately connected to the 1-dimensionality of real numbers as is the Interval Separation property used here. But the oracle idea is easy to generalize using the Two Point Separation property. Cauchy sequences and the filters do generalize, but their problems carry over as well.

5 Completing a metric space

To complete a metric space, the rational inclusive intervals are replaced with inclusive balls, specified by a center c and a radius r. An oracle in that space is then a rule that says Yes or No when presented with an inclusive ball. Singletons are balls of radius 0. The properties the oracle needs to satisfy are

1. Consistency. If a ball contains a Yes ball, then it is a Yes ball.
2. Existence. There exists a Yes ball.
3. Intersecting. Two Yes balls must intersect and their intersection must contain a Yes ball.
4. Two Point Separating. Given a Yes ball and two points in it, then there is a different Yes ball strictly contained[8] inside the first one which does not contain at

[8] We use the term strictly contain to mean that the contained ball has a positive distance away from the boundary of the containing ball.

least one of the given points.

5. Closed. If a point q in the original space is contained in all Yes balls, then the singleton of q is a Yes ball.

In [Mau14], there is a new approach to topology using lines as the basis of what is open or closed. It leads to different conclusions and is rooted in topology for physical spaces, particularly allowing for a unified treatment of finite, discrete, and continuous spaces. [Tay23c] explores applying these ideas to this new topological framework. In particular, the concept of betweenness is a common crucial aspect to both the linear structure topologies and oracles.

As discussed in [Tay23d], such oracles form a new metric space where the distance is inherited through ball representatives, the triangle inequality, and the greatest lower bound. The original space's elements are represented by the balls that contain them, including their singleton ball; these are the rooted ones. Given a Cauchy sequence of points in these spaces, the balls that contain the tails are the Yes-balls of the limit oracle. The properties of the limit as an oracle can be easily established. For the closed property, it is necessary to add in the singleton as a Yes ball if the sequence converges to a point in the original space; it is easy to show that there are no difficulties with doing so.

The paper also investigates more general topological spaces, specifically using these techniques to compactify topological spaces that are amenable to that. Other topological spaces are investigated such as topological groups and linear structure topologies [Mau14].

Of the definitions of real numbers, most are particular to real numbers. Cauchy sequences are a notable exception, but using them usually contains the equivalence class approach which is a bit like looking for information in the Library of Babel.[9]

6 Function oracles

The next idea to tackle is that of functions. The definition of functions should respect the fact of dealing with oracles. In particular, it is clear that while the rationals can be precisely specified in a single assignment, the irrationals should not be. This translates into being able to assign rationals particular values while irrationals can only get a neighborly agreement on their values. Details can be found in [Tay23b].

The idea is to have Yes rectangles whose wall should contain the image of the base. The wall is allowed to be larger than the image. The core requirement is that there should exist smaller and smaller rectangles that shrink down to what would generally be understood as the value of the function at a given real number.

Singletons are allowed to be the base or wall of a rectangle. This is what allows rational numbers to be independent of their neighbors.

More explicitly, a function oracle f is a rule that answers Yes or No when provided with a rational rectangle and satisfies the following properties:

[9] `https://libraryofbabel.info/`

1. Elongating Consistency. If the wall of M contains all of the walls of a collection of Yes rectangles $\mathcal{N}$ with the union of the bases of $\mathcal{N}$ being equal to the base of M, then M is an f-Yes rectangle.

2. Narrowing Consistency. If the base of M contains the base of rectangle N with N and M sharing the same wall, then N is an f-Yes rectangle if M is.

3. Intersection. If two f-Yes rectangles have intersecting bases, then the two rectangles have non-empty intersection and that intersection is an f-Yes rectangle.

4. Separating. Given a Yes rectangle M, an oracle α contained in the base of M, and two y-values r and s contained in the wall of M, then there exists a Yes rectangle contained in M, not containing at least one of those values in its wall, and α is contained in its base.

5. Closed. If a rational pair (u, v) is contained in every f-Yes rectangle whose base contains u, then $u \vdots u \times v \vdots v$ is an f-Yes rectangle.

The first two properties allow for elongating and narrowing rectangles. In particular, rectangles can be narrowed as much as needed. Rectangles with overlapping bases must intersect on the wall as well. That is, there are no vertically disjoint rectangles.

The oracle is answering the question "Does the wall of the rectangle contain the image of the base under this function?" Just like real numbers, as this is defining the function, the answer to the question is on the aspirational side.

By working with the properties, it can be shown that for every oracle α in the base of a Yes rectangle, there is a unique oracle β whose Yes intervals are the walls of the Yes rectangles whose base contains α. The common notation of $f(\alpha) = \beta$ for this association is therefore justified.

It can be proven that oracle functions are equivalent to the set of usual real functions that are continuous everywhere except possibly at the rationals. Thomae's function is the classic example. Thomae's function qualifies as a function oracle because singletons can be the entire base of a rectangle. Any given rational can be assigned to any oracle. The family of rectangles for that rational would include rectangles whose base is the singleton and whose walls are the Yes intervals of the output oracle. The irrationals, however, cannot be picked out in such a way. The bases of their rectangles must include some neighbors and this is how continuity arises for them.

The rationals are not entirely independent of their neighbors in the aggregate. Any particular rational can be whatever value, but eventually the rationals must do some kind of alignment with the irrationals as in Thomae's function. As an example, the characteristic function of the rationals cannot arise from a function oracle. Indeed, any non-singleton rectangle will include both rational and irrational oracles which means the wall must include at least the interval $0 \vdots 1$. It fails to satisfy the Separating property.

There is an equivalent idea to fonsis with function oracles. These are foundational sets that can be narrowed, satisfy the intersection property, and whose wall values can be separated.

Given a function oracle f and a rationally continuous function g, another function oracle can be defined by composition, specifically, g-mapping the wall of a Yes rectangle. Specifically, if $a \,⁝\, b \times c \,⁝\, d$ is an f-Yes rectangle, then $a \,⁝\, b \times g(c \,⁝\, d)$ is a foundational Yes rectangle. As a special case, every continuous function can be represented by a function oracle composed with the function oracle x whose foundational rectangles are $a \,⁝\, b \times a \,⁝\, b$.

With the usual error estimates, Taylor polynomials can also be used to construct function oracles. Arithmetic of function oracles can be easily established using the arithmetic of oracles. Composition can be generalized but it is constrained. For example, consider the composition of $\frac{x}{\sqrt{2}}$ as the inner function into Thomae's function, namely the function that is 0 on the irrationals and is $\frac{1}{q}$ for rational inputs $\frac{p}{q}$ in reduced form. The composition function Yes rectangles whose bases contain $\sqrt{2}$ will always have an intermediate neighborly oracle interval containing 1 so that the output will contain the interval $0 \,⁝\, 1$. Thus, the composition is not a function oracle. This reflects the numerical impossibility of usefully working with this composition of functions.

Given an f-Yes rectangle, f is Riemann integrable over the base. Indeed, the function oracle setup is that of the bounding rectangles one could use to compute the areas.

A version of the intermediate value theorem can be proven. Given a function oracle f, an interval $a \,⁝\, b$ in the domain of the function oracle,[10] and an oracle y satisfying $[f(a)] \,⁝\, [y] \,⁝\, [f(b)]$, then there exists an oracle c such that all non-singleton rectangles containing c in the base will have y in the wall. If the oracle c is not a singleton, then $f(c) = y$. If the oracle c is a singleton, then it could be defined differently from y and the function oracle would be necessarily discontinuous at c.

Function oracles can be extended to metric spaces where the sides of the rectangles are balls in the metric space instead of intervals. Even more generally, we can think of the generalized containers as the base and walls of the rectangles.

7 Conclusion

Oracles give a new perspective on what a real number is. It is about determining which intervals contain it. It is about trying to locate the number by asking about where it is. It rejects the arbitrariness in other definitions of a real number, but embraces external arbitrariness in practical representation.

The oracle approach motivates how to approximate a real number. The mediant process gives the best rational representation and this framework naturally suggests using it. It elevates rational numbers as the key elements to understanding real numbers. It also highlights the differences between rational and irrational numbers in a clear fashion.

Oracles give a language for handling the imprecision that is inherent with real numbers. A Yes interval can express how much is known about a real number and that expression is reflective of the purpose of the definition. There is no trail of dots with

[10]An interval is in the domain if every oracle in the interval is in the base of a Yes rectangle. An oracle being in an interval means the interval is a Yes-interval for that oracle.

oracles.

The hope is that oracles become something that changes the way to teach and understand real numbers. There should be clear rules of the game for real numbers and those rules should be given early on. Oracles naturally lead to better manipulation of fractions, mediants, intervals, geometric bounds, and infinite processes. The computations with them can be easy to understand.

There is no need to pretend that the completed infinity exists. All that is needed is to have a mechanism that answers human questions, particularly those of practical concerns. That is what an oracle does.

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