Real Numbers As Rational Betweenness Relations

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August 7, 2025

Abstract

Irrational numbers cannot be known in a precise way in the same fashion that rational numbers can be. But one can be precise about the uncertainty. The assertion is that if one knows whether or not a real number is in any given rational interval, then one can claim to know the real number. This insight leads to a two tiered definition of real number. The top tier is the idealized rational betweenness relation which details what properties a relation on rational numbers must satisfy to qualify as a real number. The other tier is the practical approach to real numbers. These are non-unique procedures that give definite pathways to bringing forth the rational betweenness relations. Through these procedures, it will be established that the rational betweenness relations satisfy the axioms of the real numbers.

Real numbers are a fundamental part of mathematics. There are many different definitions and they each have their own inadequacies. This paper will present a new definition based on intervals. Intervals have been used before, but the definitions here should be more flexible and hopefully give a clearer structure in dealing with the tension of a real number as an infinite entity that practically is often only glimpsed at with finite means.

Decimals are the first presentation of real numbers that most people are introduced to and, for non-mathematicians, often the only one. They seem nice at first, giving a good intuitive sense as to where the real number is roughly located on a number line. But the issue is that computing out a decimal to n places of known accuracy may not be possible in finite time. A primary example is that of adding two numbers whose decimals happen to add up to 9 for many places. Until a non-9 is found or a proof is made that they will always be 9, there is uncertainty in all of those decimal places. Multiplication compounds this difficulty. A nice example is computing $1/9 * 1/9$ purely in terms of decimals; Norm Wildberger has a video entitled "Fractions and repeating decimals" which has a section, about 40 minutes in, on this computation. The multiplication has arbitrarily large carries forming a pattern which is hard to deduce from the decimal version alone. In producing the decimals, even as the uncertainty in the n-th decimal

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place may continue for awhile, the further computation is narrowing on an interval that contains the real number. This naturally leads one to thinking of a sequence of nested intervals being generated. This is a valid way of defining a real number as Bachmann essentially did in 1892 and as reviewed in the survey of real number definitions [Wei15]. This is one generalization of the decimal approach; it introduces an extreme abundance of different representations for a given real number.

Another approach using sequences is that of Cauchy sequences. This is the idea of a sequence of rational numbers with some notion of the terms eventually being arbitrarily close; they are taken to converge to the real number which is itself considered to be the sequence and its equivalents. One of the issues is the immense non-uniqueness of Cauchy sequences. In particular, any Cauchy sequence can have any finite sequence prepended without changing the tail. This highlights that the sequence of numbers is useless unless one also has a shrinking distance for which the following elements of the sequence are within that distance of the given element of the sequence. That is, the useful version of a Cauchy sequence is to see it as a sequence of intervals given by specifying the center of the interval and its radius. The sequence of intervals has the property that eventually the radii get arbitrarily close to 0 and all of the intervals overlap. Generalizing this to a set of intervals instead of a sequence is what this paper calls fonsis and is an approach to defining real numbers given by constructivists [Bri06].

A third approach is that of Dedekind cuts. One version has a real number be the set of all rational numbers less than itself. This works surprising well formally. But it is a little unclear what is being computed. In particular, to get a sense of what the number actually is, there has to be some computation of an upper bound on that set. Pairing an element of this set with an upper bound then provides an interval which ought to contain the real number. By pairing all the lower bounds with all the upper bounds leads to the idea of the set of all intervals that contain the real number. This is what this paper calls rational betweenness relations and is what is asserted to be the singular entity of what a real number is. Its equivalence to Dedekind cuts is detailed in [Tay24]. A related construct, which includes many more objects than just intervals, is that of minimal Cauchy filters as detailed in [Wei15].

In all of the approaches above, the object taken to be the real number is something of infinite scope presented as if it is completely known. That is, it is an object which exists in some fully realized sense. This is not possible for many real numbers of interest. The main approach of this paper is to use an auxiliary object which is, at its heart, a computational structure that allows one to refine the real number presentation as much as one wants, but there is not a notion of having a finished state. The procedure itself, when satisfying the required properties, is taken to be sufficient information to say that one knows the real number.

As a guide in contemplating different notions, it is useful to consider how equality and arithmetic are handled, particularly on how an irrational minus something that may be itself is computationally known to be 0 or not.

The first step towards this idea is to think of a real number x as providing a betweenness relation on rational numbers by a and b being x-related if x is, inclusively, between a and b. This almost works. But there are many situations in which one cannot definitively conclude that a given real number is between two particular rational numbers. The resolution of this is to have an underlying object managing the information. It allows for progressive refinements to be made.

These procedures have the property that given any interval of rational numbers, the procedure can answer whether there is a small interval containing the real number that intersects the interval in question. That small interval can then be used for further computations. The term oracle will be used for such procedures to indicate that it produces a kind of self-fulfilling prophecy. The setup allows for the computation of intervals, but does not require the computations be done in order to define the real number. The main downside is that multiple procedures can represent the same real number, but the ambiguity is nonconstructively removed at the higher level presentation of the x-relation viewpoint.

This paper will first establish some basic notations for working with rational intervals before defining both the rational betweenness relations and the oracles. A brief section on some examples will be given which will be light on details. See [Tay23a] for a more in-depth discussion of examples, uses, and comparisons to other definitions of real numbers. Equality, inequality, completeness, and arithmetic are all explored to fully establish the real number properties of these objects.

One aspect of this paper is to highlight when something is constructively known or not. For this to be relevant, it is necessary that the given computational objects, the oracles, can be computed in a predictable amount of finite time. This will not always be the case, but that is not the concern here. The main concern is to distinguish whether the claimed knowledge of the statements could be constructive if the oracle was constructive. The viewpoint of constructivity that is adopted here is whether or not one can say "this knowledge will be known after this amount of finite work."

1 Interval Notations

This section is in common to the other papers on this subject by this author, such as [Tay24].

The set of all rationals q such that q is between a and b, including the possibility that $q = a$ or $q = b$, is a **rational interval** denoted by $a \vdots b$. A rational interval can be thought of as an inclusive rational interval as it is inclusive of the endpoints, but as the default interval in this work is inclusive rational intervals, the term inclusive shall generally be omitted. If $a = b$, then this is a **rational singleton** denoted by $a \vdots a$. A singleton is a set of exactly one rational number, namely, a. To indicate $a \leq b$, the notation for the interval can be $a \iddots b$ as well as $b \ddots a$. If $a < b$ and the interval is being presented, then the notation $a \overset{\circ}{\iddots} b$ may be used as well as $b \overset{\circ}{\ddots} a$. To indicate that $a \neq b$ but without knowing the order, the symbol $a \overset{\circ}{\vdots} b$ may be used; these are the **neighborly intervals**. The symbol $\mathbb{I}$ will represent the set of all rational intervals.

It is also useful at times to consider rational intervals whose endpoints are excluded. The notation for those intervals is $(a \vdots b)$ which is the set of all rationals between a and

b not including a or b. The term for this is an **exclusive rational interval**; when rational interval is used without an adjective, it is assumed to be closed. If a is to be included, but not b, then this can be written as $a \vdots b)$ and is a half-exclusive rational interval.

The notations above will also be used to indicate betweenness. For example, if $a \leq b \leq c$ or $c \leq b \leq a$, then $a \vdots b \vdots c$ will be used to denote that. By definition, $c \vdots b \vdots a$ and $a \vdots b \vdots c$ represent the same betweenness assertion. This can be extended to any number of betweenness relations, such as $a \vdots b \vdots c \vdots d$ implying either $a \leq b \leq c \leq d$ or $d \leq c \leq b \leq a$. There are also some trivial ways to extend given betweenness chains. For example, if $a \vdots b \vdots c$, then $a \vdots a \vdots b \vdots c$ holds as well. Another example is that if $a \vdots b \vdots c$ and $b \vdots c \vdots d$, then $a \vdots b \vdots c \vdots d$ holds true. This all follows from standard inequality rules for rational numbers.

If b and c are between a and d, but it is not clear whether b is between a and c or between c and d, then the notation $a \vdots \{b, c\} \vdots d$ can be used. This can also be extended to have, for example, $a \vdots b \vdots \{c, d\}$ which would be shorthand for saying that both $a \vdots b \vdots c$ and $a \vdots b \vdots d$ hold true. In addition, the notation $\cancel{a \vdots b \vdots c}$ will be used to indicate that b is not between a and c. One could also indicate this by $b \vdots \{a, c\}$ which could be said in words that a and c are on the same side of b.

Rational numbers satisfy the fact that, given three distinct rational numbers, a, b, c, exactly one of the following holds true: $a \vdots b \vdots c$, $a \vdots c \vdots b$, or $b \vdots a \vdots c$. That is, exactly one of them is between the other two. It can be written in notation as $a \vdots b \vdots c$ holds true if and only if both $\cancel{a \vdots c \vdots b}$ and $\cancel{b \vdots a \vdots c}$ hold true. This follows from the pairwise ordering of each of them as provided by the Trichotomy property for rational numbers along with the Transitive property.

In this paper, often a potential relabeling will be invoked. This is to indicate that there are certain assumptions that are needed to be made, but they are notational assumptions and, in fact, some arrangement of the labels of that kind must hold. For example, if $\{a, d\} \vdots b \vdots c$ holds true, then either $a \vdots d \vdots b \vdots c$ or $d \vdots a \vdots b \vdots c$ holds true. If these are generic elements, then relabeling could be used to have $a \vdots d \vdots b \vdots c$ be true for definiteness, avoiding breaking the argument into separate, but identical, cases. If a and d were distinguished in some other way, such as being produced by different processes, then relabeling would not be appropriate to use.

If $a \vdots b \vdots c \vdots d$, then the union of $a \vdots c$ with $b \vdots d$ is the interval $a \vdots d$. If $b \neq c$, then the union of $a \vdots b$ and $c \vdots d$ as a set is not an interval. One can still consider the **intervalized union** $a \vdots d$ as the shortest interval that contains both intervals.

An interval $b \vdots c$ **strictly contains** a if a is between b and c, but is not equal to either of them. If a is contained in an exclusive interval $(b \vdots c)$ then it is automatically strictly contained in it. An interval $c \vdots d$ is strictly contained in $a \vdots b$ if all elements of $c \vdots d$ are strictly contained in $a \vdots b$. Any interval contained in an exclusive interval is strictly contained in it. An exclusive interval $(c \vdots d)$ is strictly contained in $a \vdots b$ if there exists an interval $e \vdots f$ containing $(c \vdots d)$ and that $e \vdots f$ i s strictly contained in $a \vdots b$.

An a**-rooted** interval is an interval who has an endpoint that is a, that is, they are of the form $a \vdots b$ for some b. An a**-neighborly** interval is an interval that strictly contains a. The set of all a-rooted intervals will be denoted $\mathbb{I}_a$ while the set of all a-neighborly intervals will be denoted by $\mathbb{I}_{(a)}$

For rational $\delta > 0$, the notation $(a)_\delta$ represents the **δ-halo of** a which is defined as $(a-\delta \vdots a+\delta)$. An interval is $(a)_\delta$ **compatible** if it is strictly contained in $(a)_\delta$ and is an a-neighborly interval. That is, the interval $c \odot d$ is $(a)_\delta$ compatible if $a-\delta < c < a < d < a+\delta$. The halo notation can be extended to any interval. The notation $(a \vdots b)_\delta$ will refer to $(a)_\delta \cup a \vdots b \cup b_\delta$. If $a \leq b$, this is the same as the interval $(a-\delta \vdots b+\delta)$. The notation $[a \vdots b]_\delta$ will be the **inclusive δ-halo** of $a \vdots b$ and it includes the δ shifted endpoints which for $a \odot b$ would be any rational q that satisfies $a - \delta \leq q \leq b + \delta$.

The notation $b \vdots a \text{¦}_\delta$ will indicate that the interval goes from b to the farthest endpoint of $(a)_\delta$ from b. If, for example, $b < a$, then $b \vdots a \text{¦}_\delta$ is the same as $b \vdots a+\delta$ while ${}_\delta\text{¦} b \vdots a$ would be the interval $b - \delta \vdots a$. Also, ${}_\delta\text{¦} a \vdots b$ is the same as $b \vdots a \text{¦}_\delta$. This notation makes the most sense for b outside of $(a)_\delta$. The intent of this notation is to avoid having to specify the inequality relation of a and b while still being able to expand outside of $a \vdots b$ in the specified direction. The potential notation of ${}_\delta\text{¦} a \vdots b \text{¦}_\delta$ would have the same meaning as $[a \vdots b]_\delta$; this paper will generally use the halo notation. Outside of an interval, a similar notation will be used to indicate a stand-alone number that is a δ-step from one number to another. For example, $a \text{¦}_{\delta,c}$ would represent the number $a + \frac{c-a}{|c-a|}\delta$. For smooth reading, it might also be written as ${}_{c,\delta}\text{¦} a$ in case $c \vdots a$ was used earlier.

Given m in $a \vdots b$, a **subwidth** δ shall mean a positive rational number such that $(m)_\delta$ is strictly contained in $a \vdots b$. For a subwidth to exist, it must be case that m is strictly contained in $a \vdots b$. A notation that can express this is $a \underset{\cdot}{\circ} (m)_\delta \underset{\cdot}{\circ} b$ which is asserting that if $p \in (m)_\delta$, then $p \in a \vdots b$, $p \neq a$ and $p \neq b$.

The term subinterval of $a \vdots b$ will include $a \vdots b$ as a subinterval but it does not include the empty set.

The length of $a \vdots b$ will be denoted by $|a \vdots b|$ and is equal to $|b - a|$. If $a \underset{\cdot}{\circ} b \vdots c$, then the distance between a and the interval $b \vdots c$ is $|a - b|$; this will be denoted by $|a; b \vdots c|$. If $b \vdots a \vdots c$, then $|a; b \vdots c|$ is defined to be 0. If $a \vdots b \underset{\cdot}{\circ} c \vdots d$, then the distance between the intervals $a \vdots b$ and $c \vdots d$ is $|b - c|$ which will be denoted by $|a \vdots b; c \vdots d|$. That is, the distance uses the closest endpoints for its computation if the intervals are disjoint. If the intervals overlap, then the distance is taken to be 0.

Throughout the paper, unless noted otherwise, the letters a through w will represent rational numbers, x, y, z, α, β will represent real numbers, and δ, ε will represent positive rational numbers. Primes on symbols will be assumed to be of the same type as the unprimed version.

2 Rational Betweenness Relations

The definition of a **rational betweenness relation**, with relational symbol $\vdots\underline{x}\vdots$, is that it is a symmetric relation on rational numbers which satisfies the properties listed below. If $a \vdots\underline{x}\vdots b$, then the interval $a \vdots b$ is said to be an **x-interval** and a and b are **x-related**.

1. Existence. There exists a and b such that $a \vdots\underline{x}\vdots b$.

2. Separation. If $a \vdots\underline{x}\vdots b$ and c is strictly between a and b, then exactly one of the following holds: 1) $a \vdots\underline{x}\vdots c$ and $\cancel{c \vdots\underline{x}\vdots b}$, 2) $c \vdots\underline{x}\vdots b$ and $\cancel{a \vdots\underline{x}\vdots c}$, or 3) $c \vdots\underline{x}\vdots c$.

3. Consistency. If $c{\vdots}a{\vdots}b{\vdots}d$ and $a \vdots\underline{x}\vdots b$, then $c \vdots\underline{x}\vdots d$.

4. Singular. If $c \vdots\underline{x}\vdots c$ and $d \vdots\underline{x}\vdots d$, then $c = d$.

5. Closed. If c is a rational number such that c is included in every x-interval $a{\vdots}b$, then $c \vdots\underline{x}\vdots c$.

The set of all such relations when coupled with appropriate ordered field operations satisfies the axioms of the complete field of real numbers. This definition was shown to work in [Tay24] by demonstrating that they are equivalent to Dedekind cuts. The focus of this paper is to establish these as a model of the real numbers using the practical approach of the oracle procedures defined in the next section.

For a given relation $\vdots\underline{x}\vdots$, it can be convenient to use the term Yes interval for intervals $a{\vdots}b$ that satisfy $a \vdots\underline{x}\vdots b$ and the term No interval if $\cancel{a \vdots\underline{x}\vdots b}$. These terms will be reused in other sections to refer to intervals using oracle procedures though it will eventually be established that the Yes intervals of an oracle representing x coincide with the Yes intervals of $\vdots\underline{x}\vdots$.

It can be useful to recast the properties above in the Yes/No interval language. Also of use is the term a **root of the relation** which is a rational number c such that $c \vdots\underline{x}\vdots c$.

1. Existence. There exists a Yes interval. Without this, one could have the null relation as a rational betweenness relation. Existence gives a starting point for further computations using Separation.

2. Separation. Any rational number c strictly contained in a Yes interval will divide that Yes interval into two new intervals one of which will be Yes and the other will be No unless c is a root of the relation in which case the two intervals and the singleton $c{\vdots}c$ are Yes intervals. Separation is inspired by the Intermediate Value Theorem and is the computational property allowing for a narrowing down of intervals.

3. Consistency. If an interval contains a Yes interval, then it is a Yes interval. This also implies that an interval contained in a No interval is a No interval. Consistency allows Separation to rule out two disjoint separated Yes intervals.

4. Singular. There is at most one root of the relation. This prevents the possibility of an interval of roots. The Separation property does not rule this out. Separation with consistency would rule out a finite number of roots, but not an interval of roots.

5. Closed. If a rational number c is in every Yes interval, then c is a root of the relation. This establishes a unique representative relation for rational numbers. Without this, one could have, for example, the relation consisting of neighborly

intervals of 0 along with 0-rooted intervals of the form $0 \dot{\odot} b$. Another variant would instead have the rooted intervals just be $0 \dot{\odot} b$. Both would represent 0 and be distinct. The Closed property, along with Consistency, rules out this multiplicity.

These properties ensure that two different rational betweenness relations cannot represent the same real number.

To get a little familiarity with these properties, it can be instructive to see how, given an interval $c \vdots d$, the properties give a pathway for determining whether $c \vdots \underline{x} \vdots d$ or $\cancel{c \vdots \underline{x} \vdots d}$. It starts with existence yielding an x-Yes interval $a \vdots b$. If $a \vdots b$ does not strictly contain $c \vdots d$, one can assume by relabeling that $a \vdots \{b, c\} \vdots d$. Then $a \vdots d \rvert_1$ is an interval that strictly contains $c \vdots d$ and contains $a \vdots b$. Thus, by Consistency $a \vdots d \rvert_1$ is an x-Yes interval that strictly contains $c \vdots d$. Relabel $a \vdots d \rvert_1$ as $a \vdots b$ such that $a \vdots c \vdots d \vdots b$. Given that, Separation can be used, using c and d. If either $c \vdots \underline{x} \vdots c$ or $d \vdots \underline{x} \vdots d$, then $c \vdots \underline{x} \vdots d$ by Consistency. Assuming not, Separation applied to c leads to either $\cancel{c \vdots \underline{x} \vdots b}$ or $c \vdots \underline{x} \vdots b$. In the first instance, Consistency implies $\cancel{c \vdots \underline{x} \vdots d}$. In the second, use d in Separation of $c \vdots \underline{x} \vdots b$. Then either $c \vdots \underline{x} \vdots d$ or $\cancel{c \vdots \underline{x} \vdots d}$. Thus, the properties of Existence, Separation, and Consistency lead to a conclusion being able to be drawn about every pairing of rational numbers.

Arithmetic is defined using interval arithmetic, as will be discussed later, but there is a closure step that may be needed. For example, if x is irrational, then $x - x = 0$ is still a statement that is desired, but the arithmetic of intervals will never produce any Yes intervals with an endpoint of 0, let alone $0 \vdots 0$. When all the intervals of x are subtracted from one another, every resulting interval will contain 0, but not all intervals that contain 0 are included. Thus, the arithmetic does not directly produce rational betweenness relations from rational betweenness relations. It does seem reasonable, however, to have a mechanism that would take from this process that the relation under discussion is the one that includes all intervals that contain 0. Since the mechanism to do so is also a convenient mechanism for how real numbers are actually used, it seems reasonable to focus on it. That mechanism consists of the oracles and their prophecies.

3 Oracles

A **real number oracle** is a procedure R, satisfying the properties listed below, whose core input is a rational interval along with a positive rational number. The core output, which is not necessarily exclusively defined by the core input, should either be an ordered pair where the first entry is a boolean of the form 0 or 1 and the second entry is a rational interval. It is also possible for the procedure to return a single number, either (0) or (-1). An oracle is trying to answer the question of whether the real number under consideration is contained in the given interval. A 0 would say No and a 1 is almost a Yes, where the almost is where the positive rational number is used. A -1 indicates an inability to answer either Yes or No. In this paper, the word oracle will always refer to a real number oracle and the word core will generally be implied for the inputs and outputs of an oracle.

The idea for the returned intervals is that they are considered to contain the real number. Any interval $c{:}d$ that appears as the second entry of a core output is a **prophecy** of the oracle and this is denoted as $c{:}d \in R$.

Writing $R(a{:}b, \delta) = (k, A)$ implies that one of the possible core outputs for that core input is (k, A). The expression $R(a{:}b, \delta) = c{:}d$ will be shorthand to imply that one of the outputs for that pair of inputs is $(1, c{:}d)$. The expression $R(a{:}b, \delta) = 1$ implies that there exists some output pair for that input such that 1 is the first entry of the pair. Returning $R(a{:}b, \delta) = 0$ implies either it is an output pair with 0 as the first entry or it is the singleton (0). The expression $R(a{:}b, \delta) \neq k$ means that none of the outputs for that input pair has k as its first entry; it will also mean that it is not -1. Writing $R(a{:}b) = k$ is stating that $R(a{:}b, \delta) = k$ for all δ. Writing $R(a{:}b) \neq k$ is stating that $R(a{:}b, \delta) \neq k$ for every δ.

The notation $\mathbb{I}_R$ represents the set of all intervals that contain a prophecy. The property of Consistency is based on the idea that since prophecies ought to contain x then the intervals that contain prophecies, namely those in $\mathbb{I}_R$, also ought to contain x.

The term core input is used to allow for the procedure R to have additional inputs, such as a starting guess for Newton's method. The core output is used to allow for additional outputs, such as a computed next guess in the application of Newton's method. Thus, a slightly more suggestive version of R might be presented as $R(a{:}b, \delta, \ldots) = (k, c{:}d, \ldots)$. In a Newton's method example, $R(a{:}b, \delta, x_n) = (k, c{:}d, x_m)$ could be used. The interval $c{:}d$ could be an interval centered on x_m whose extent is given by an error estimate, such as from the Newton-Kantorovich theorem. The boolean k would then be determined based on comparing $c{:}d$ and $a{:}b$. The use of m indicates that there might be multiple iterations used before the interval $c{:}d$ has enough resolution to decide on the value of k. In the theoretical development of oracles, these additional inputs and outputs are not generally relevant and will generally be ignored for convenience.

To be an oracle, the procedure must satisfy the following properties. The properties are:

1. Range. $R(a{:}b, \delta)$ always gives an answer and is one of the following forms:
   - $(1, c{:}d)$. $c{:}d$ is a subinterval of $(a{:}b)_\delta$ and intersects $a{:}b$.
   - $(0, c{:}d)$ is disjoint from $a{:}b$.
   - (0). There exists a δ' such that no prophecy both intersects $a{:}b$ and is contained in $(a{:}b)_{\delta'}$.
   - (-1). There should be no other core output for this core input.
2. Existence. There exists $a{:}b$ and δ such that $R(a{:}b, \delta) = (1, c{:}d)$.
3. Separation. If $a{:}b$ is a prophecy of R, m is contained in $a{:}b$, and $\delta > 0$, then one or both of $R(a{:}m, \delta) = 1$ and $R(m{:}b, \delta) = 1$ holds true.
4. Disjointness. If $c{:}d$ is a prophecy of R and $a{:}b$ is disjoint from $c{:}d$, and $\delta < |c{:}d; a{:}b|$, then $R(a{:}b, \delta)$ is either 0 or $(0, e{:}f)$ for some prophecy $e{:}f$.

5. Consistency. If $a \vdots b$ contains a prophecy of R, then $R(a \vdots b) \neq 0$.
6. Closed. If for each $\delta > 0$, $(a)_\delta$ contains a prophecy, then, for all b, $R(a \vdots b) \neq 0$. Such an a is called the **root** of the oracle.
7. Reasonableness. If $R(a \vdots b, \delta) \neq -1$, then $R(a \vdots b, \delta') \neq -1$ for any $\delta' > \delta$.

If multiple oracles representing distinct real numbers are being discussed, such as x and y, then R_x and R_y will represent their respective oracles. If multiple oracles that represent the same real number are used, then a prime will be used such as R'_x representing a different oracle than R_x but both are representing x. A later section will explore establishing when two oracles represent the same real number or not.

The expression $R(a \vdots b, \delta) \neq 0$ implies that no intervals disjoint from $a \vdots b$ are part of any output of R for that input. The expression $R(a \vdots b, \delta) = 0$ is asserting that there is an output of R for that input that has an interval disjoint from $a \vdots b$, but there may be other outputs for that input which could have overlapping intervals with $a \vdots b$.

For example, if approximating the square root of 2 and the interval is $1.3 \vdots 1.4$ with a δ of 0.1, then the procedure might generate $(1, 1.39 \vdots 1.42)$, but another computation with the procedure might generate $(0, 1.41 \vdots 1.42)$. Both $R(1.3 \vdots 1.4, 0.1) = 1.39 \vdots 1.42$ and $R(1.3 \vdots 1.4, 0.1) = 0$ would be valid statements given these computations. Only the latter would be definitive in excluding $1.3 \vdots 1.4$ from containing the square root. The interval $1.3 \vdots 1.4$ would not be in $\mathbb{I}_R$ while $1.39 \vdots 1.43$ would be as it contains $1.39 \vdots 1.42$ which was a returned output of R.

A rational interval $a \vdots b$ is a **Yes interval** of the oracle R if $R(a \vdots b) \neq 0$; this is a check across all δ. By Consistency, this includes all intervals that contain prophecies, but, by the Closed property, it also includes the a**-rooted** intervals where a is the root of the oracle. A rational interval $a \vdots b$ is a **No interval** if $R(a \vdots b, \delta) = 0$ for some δ. In a nonconstructive sense, every interval is a Yes or No interval, but determining that in a particular case may not be possible via a finite set of computations.

If an interval $a \vdots b$ is a Yes interval for R and R is to represent the real number x, then this can be expressed with the notation $a \vdots \widehat{x} \vdots b$. For No intervals, the notation is $\cancel{a \vdots \widehat{x} \vdots b}$. The notation $a \vdots c\,\big\rvert_{\delta}$ extends to the Yes intervals as $a \vdots \widehat{x} \vdots c\,\big\rvert_{\delta}$. These notations reflect that the Yes / No interval designation has created an x-betweenness relation on the rational numbers as will be established later.

It may be helpful to expand a little on what the properties mean. The basic idea is that a Yes interval ought to contain the real number; since this is defining the real number, this becomes more of a guiding idea rather than a deduction. The concept of the oracle is that a possible Yes interval is given along with a little error tolerance. The oracle should ideally respond with a prophecy which helps move the process along in ascertaining what the real number is and will determine if the fuzzy version of the given interval contains the real number or not. If it happens to not be able to respond with a prophecy, then the given interval is a No interval.

Here is a bit of explanation for the properties:

1. Range. The idea of the oracle is that its first entry indicates success (1) or failure (0). The outputs with the first entry being 1 are called Yes answers while if it is 0,

then it is a No answer. A 1 is less definitive than 0. As will be shown, Separation allows for the computation of a prophecy smaller than δ. Such an interval is either going to be disjoint from $a \vdots b$ or contained in $(a \vdots b)_\delta$. This is what allows for a non-constructive deduction that the ordered pair result is always possible.

The result of just 0 arises in oracles based on direct testing of whether the interval contains the number or not. It does not necessarily produce a prophecy disjoint from that interval in the testing. Thus, it would be an additional burden to compute out a prophecy disjoint from it.

The result of -1 is a null result. This should be taken as not being able to answer the question. It can be argued that the oracle properties combine so that an oracle never has a null result.

Procedures in this paper should be assumed to have this Range property unless otherwise noted.

2. Existence. This prevents the procedure of $R(a \vdots b, \delta) = (0)$ for all $a \vdots b$ and δ. It was extremely difficult to get the right formulation of Ranged to avoid the an implicit assumption of existence.

3. Separation. This property is inspired by the Intermediate Value Theorem. The inspiration is that a Yes interval should be able to be continually narrowed down by selecting a rational number m inside of it and then testing which of the two created intervals is a Yes interval and the other one would then be a No interval. Because of the possibility of the number of interest being near or at m, it is possible for both to be Yes. In that case, $R(m \vdots m, \delta) = 1$ holds true due to Consistency and Disjointness.

   It can also be the case that while $a \vdots b$ is a prophecy, none of its subintervals are. For example, let $R(a \therefore b, \delta) = (1, 2 \vdots b + \delta)$ if $a \leq 2$ and $b + \delta > 2$, $R(1 \vdots 1.5, 0.5) = (1, 1 \vdots 2)$, and is $(0, 2 \vdots 2 + \epsilon)$ otherwise where ϵ is either $\frac{2+a}{2}$ if $a > 2$ or is, say, δ if $b + \delta < 2$. Then the prophecies of R are $1 \vdots 2$ along with intervals of the form $2 \mathrel{\overset{\circ}{\vdots}} c$. Does the prophecy $1 \vdots 2$ satisfy the Separation property? Given $m \in 1 \vdots 2$ and δ, $R(m \vdots 2, \delta) = 2 \vdots 2 + \delta$ and thus it does satisfy Separation. The Yes intervals of this oracle are all the intervals that contain 2. That claim and that this is an oracle is straightforward to check.

4. Disjointness. Disjointness ensures that a single real number is being discussed. Without an assertion of negativity, one could have multiple disjoint regions. As an example, imagine a procedure which returns small intervals around 2 and small intervals around 5. Done correctly, this will be able to satisfy the other properties, including the Separation property because that property only applies to prophecies. Consistency does not help as that is about intervals containing prophecies, but does not demand that there is a prophecy containing other prophecies. Since rational betweenness relations do not make this distinction between prophecies and Yes intervals, it does not need this postulated as it can be deduced via Consistency and the exclusionary aspect of Separation.

5. Consistency. Consistency asserts that the oracle never contradicts itself. Since a prophecy is supposed to contain the real number, if another interval contains it, then that containing interval should also contain the real number and ought to be a Yes interval. This property says that if an interval contains a prophecy then every core output, regardless of δ, is a Yes answer.

6. Closed. This ensures that if there is a narrowing in to a single rational number, then the intervals with that rational number as endpoint are Yes intervals. It does not require that the interval in question be a prophecy nor contain one. The assumption that $(a)_\delta$ is in $\mathbb{I}_R$ for all $\delta > 0$ implies that R can always return a subinterval of $(a)_\delta$ for $R(a{:}b, \delta)$. Disjointness forces a to be in those prophecies in this case.

7. Reasonableness. In the case of Yes intervals, the properties of Consistency and Closed demand the oracle always answers Yes. For No intervals, however, it could be the case that larger fuzziness incorporates the real number and the option of saying Yes becomes possible though not required. This is an attempt to say that if there is a more narrow fuzziness with an answer than the larger fuzziness still gives an answer though it may be different. It is not a property that is essential to have, but it feels mostly harmless and convenient.

The term procedure is being used instead of function to suggest a more constructive approach. While it is perfectly fine for an R to be given as an explicit function, the more typical case is that R is computed out as needed and may not return the same result for the same core inputs. For example, if a procedure is based on Newton's method restricted to a region of unique convergence but no specific starting point is given, then different choices of starting points could lead to different returned intervals. The design here is to smoothly allow for that variation.

The core computational properties are Ranged, Existence, Separation, and Disjointness. These properties guide the creation of a set of prophecies that are informational about the real number. Any procedure satisfying these properties will be called a **proto-oracle**. Conceptually, adding Consistency, Reasonableness, and Closed is about filling in the answers after doing the core computations. It will be shown, non-constructively, that oracles cannot return a -1 as an answer, but proto-oracles are free to do so.

4 Examples

A few examples can be helpful in understanding oracles. Rational numbers as oracles allows one to explore a variety of oracle notions in a concrete way. After exploring rationals, then the oracle exploration is roughly divided into oracles in which an interval is being tested directly versus oracles in which an iterative process is creating small enough intervals to definitively answer the question that the oracle is being asked. This section concludes with a discussion of non-computable numbers in relation to oracles with the conclusion that it is largely the same conversation as with other real number representations.

4.1 Rational Oracles

Given a rational number q, the rational betweenness relation representing q is defined by $a \vdots \underline{q} \vdots b$ exactly when $a \vdots q \vdots b$. This condition is straightforward to check given explicit a, q, b. It is also quick to verify the rational betweenness relation properties. This is the only rational betweenness relation that represents q. The oracles, however, have a number of different options for representing q. Below are some of the more illuminating ones to ponder.

- Singular Oracle of q. The procedure R is defined as $R(a \vdots b, \delta) = (1, q \vdots q)$ if $a \vdots q \vdots b$ and $(0, q \vdots q)$ otherwise. All of the properties are easily verified: 1) Ranged is by definition; 2) Existence is $R(q \vdots q, 1) = (1, q \vdots q)$; 3) Separation follows from the fact that q is in the prophecy and thus wherever q lands in $a \vdots m \vdots b$ decides which of them have a Yes answer; 4) Disjointness follows from q not being in an interval disjoint from a prophecy; 5) An interval that contains q will pass that property onto anything that contains it; 6) Since q is either in $a \vdots b$ or not, if there is an output for some δ, then all δ will have that result; 7) If $a \neq q$, then there is a separation distance and $(a)_\delta$ cannot contain q for smaller fuzziness which implies the Closed condition only applies to $a = q$ for which the conclusion does hold.

- Reflexive Oracle of q. The procedure R is defined as $R(a \vdots b, \delta) = (1, a \vdots b)$ if $a \vdots q \vdots b$ and $(0, q \vdots q)$ otherwise. The prophecies are exactly those intervals that contain q and they coincide with the Yes intervals. The properties of an oracle are easily verified as before, amounting to a translation of whether q is in an interval and noting that δ plays no role again.

- Fuzzy Reflexive Oracle of q. The procedure R is defined as $R(a \vdots b, \delta) = (1, (a \vdots b)_\delta)$ if $q \in (a \vdots b)_\delta$ and (0) otherwise. An alternate version which fails to be an oracle because of the Reasonable property would be to have 1 if $q \in a \vdots b$, 0 if $q \notin (a \vdots b)_\delta$, and -1 otherwise. This would thus give an answer of -1 for large enough δ for any interval that does not include q.

- Halo Oracle of q. The procedure R is defined as $R(a \vdots b, 2\delta) = (1, (q)_\delta)$ if $(q)_\delta$ intersects $a \vdots b$ and returns $(0, (q)_\delta)$ otherwise. The length of $(q)_\delta$ is 2δ and thus if any part of it intersects $a \vdots b$, then it is contained in $(a \vdots b)_{2\delta}$. The prophecies are the intervals $(q)_\delta$; this does not include any intervals rooted at q though they all contain q and are centered on q. If $a \vdots b$ contains q, then $(a \vdots b)_\delta$ contains $(q)_\delta$. Thus, if $q \in a \vdots b$, $R(a \vdots b, 2\delta) = (1, (q)_\delta)$ for all δ. These are the Yes intervals. If $a \vdots b$ does not contain q, then take δ such that $0 < \delta < |q; a \vdots b|$. Then $(q)_\delta$ will be disjoint from $a \vdots b$ and $R(a \vdots b, 2\delta) = (0, (q)_\delta)$. Note that $q \in (a \vdots b)_{2\delta}$ is quite possible and yet the return is a 0. This is perfectly fine as the goal is to determine whether $q \in a \vdots b$

which it would not be if the return is 0. The oracle properties follow from the Yes and No interval characterization of whether the interval contains q.

- Random Halo Oracle of q. This is an oracle with an extra input of a function f that, when given a positive rational number δ, it generates a random rational r such that $0 < r < \delta$. The procedure R is then defined as $R(a \vdots b, 2\delta, f) = (1, (q)_{f(\delta)})$ if $(f(q))_\delta$ intersects $a \vdots b$ and returns $(0, (q)_{f(\delta)})$ otherwise. The considerations of the Fuzzy Oracle of q apply here equally well. This allows for the potential of both 1 and 0 being returned.

- Bisection Oracle of q. The procedure R includes an extra input of r_0 and an extra output of r_i. Let $a \vdots b$ and δ be given. The procedure then does an iterative procedure. At each step, R checks $q \vdots r_i$ in relation to $a \vdots b$ and δ. If $q \vdots r_i$ intersects $a \vdots b$ and is contained in $(a \vdots b)_\delta$, then $R(a \vdots b, \delta, r_0) = (1, q \vdots r_i, r_i)$. If $q \vdots r_i$ and $a \vdots b$ are disjoint, then R returns $(0, q \vdots r_i, r_i)$. Otherwise, R computes $r_{i+1} = \frac{q+r_i}{2}$ and then runs the check again. This will have a definite answer when $|q \vdots r_i| < \delta$ which will happen after enough bisections. The prophecies are the rooted intervals of q, including $q \vdots q$ which would occur only if $r_0 = q$ is chosen. Note that any interval in $\mathbb{I}_R$ contains $q \vdots r$ for some r and thus contains q. For the Closed property, the only rational to which all its halos contain a prophecy is q and all of the intervals of the form $q \vdots a$ do return a value of 1 as required. All of the properties are easily verified and the Yes intervals are again exactly those intervals that contain q.

4.2 Testing Oracles

Some oracles are based on being able to directly test whether an interval ought to contain x. The Singular and Reflexive Oracles of q are clearly of this nature. The Fuzzy Oracles of q could also be seen as doing this, but those examples use an auxiliary interval and its intersection to test.

The classic non-rational example of this is also the primary example used in Dedekind cuts, namely, n-th roots. The x to represent is the positive real number such that x^n ought to be the positive rational q. The natural oracle R for this is that $R(a \therefore b, \delta) = (1, a \vdots b)$ whenever $b > 0$ and $\max(a, 0)^n \vdots q \vdots b^n$; it returns 0 otherwise.

A more general framework which covers a variety of cases of interest, can be cast in the language of a rational function f. Define $a \vdots b$ to be a **f-zero interval** whenever $f(a) \vdots 0 \vdots f(b)$ holds true. Assume there is an interval $A \vdots B$ for f that has the following properties: 1) $A \vdots B$ is an f-zero interval; 2) given any two disjoint intervals in $A \vdots B$, at most one of them is an f-zero interval; 3) any interval in $A \vdots B$ containing an f-zero interval is an f-zero interval. Given that, one can find a rational betweenness relation x such that for any f-zero interval $r \vdots s$ contained in $A \vdots B$, it will be the case that $r \vdots \underline{x} \vdots s$ holds true. This is basically the definition of x as a rational betweenness relation, but it has an issue with if x is rational and f actually jumps at x. That is, if $x = q$ and $f(q) \neq 0$ because of the jump, then it is not clear how $q \vdots \underline{x} \vdots q$ might get defined.

The oracle approach allows for a fuzzier definition that allows for practical progress to be made. Define the oracle in the following way. Given $a{:}b$ and δ, let $a'{:}b'$ be the intersection of $a{:}b$ with $A{:}B$. If there is no such intersection, then $R(a{:}b, \delta) = 0$. With $a'{:}b'$, let $c{:}d$ be $[a'{:}b']_{\delta/2}$ intersected with $A{:}B$. If $c{:}d$ is not an f-zero interval, then $R(a{:}b, \delta) = 0$. If $c{:}d$ is an f-zero interval, then $R(a{:}b, \delta) = (1, c{:}d)$.

In what follows, given an interval $a{:}b$, the interval $a'{:}b'$ will be $a{:}b$ restricted to $A{:}B$. The expression $[a'{:}b']'_{\delta/2}$ will be the closed $\delta/2$-halo of $a'{:}b'$ intersected with $A{:}B$

The properties are straightforward to verify:

1. Range. The only two types of returns are 0 and $(1, c{:}d)$.

2. Existence. By assumption, $A{:}B$ is an f-zero interval and $[A'{:}B']'_{\delta/2} = A{:}B$ for any δ. Thus, $R(A{:}B, \delta) = (1, A{:}B)$ for all δ.

3. Separation. Given a prophecy $a{:}b$, either $f(a)$ and $f(b)$ are of different sign or one of them is zero. Let m and $\delta > 0$ be given. By relabeling, if $f(a) = 0$, then $[a{:}m]'_{\delta/2}$ will contain the f-zero interval $f(c){:}f(c)$ and is therefore an f-zero interval, leading to $R(c{:}m$ and δ, compute $f(m)$. If neither are zero, then, by relabeling, it can be assumed that $f(a) < 0 < f(b)$. Compute $f(m)$. If $f(m) = 0$, then both $a{:}m$ and $b{:}m$ will be f-zero intervals. If $f(m) < 0$, then $b{:}m$ is an f-zero interval while if $f(m) > 0$ then $a{:}m$ is an f-zero interval. In all of the cases, either $R(a{:}m, \delta) = 1$ or $R(m{:}b, \delta) = 1$

   This works well and is independent of δ assuming that $f(m)$ can be precisely computed. Since that is often not practically possible to determine, the idea would to compute $f((m)_\delta)$ in terms of whether it contains 0, is in $f(c){:}0$, or is in $f(d){:}0$. One of these is true allowing for precise enough computation.

4. Disjointness. Let $a{:}b$ be disjoint from a prophecy $c{:}d$ and $\delta < |c{:}d, a{:}b|$ be given. Then $[a'{:}b']'_{\delta/2}$ is disjoint from the f-zero interval $c{:}d$ in $A{:}B$ which, by assumption, means it is not an f-zero interval. Thus, $R(a{:}b, \delta) = 0$

5. Consistency. A prophecy $c{:}d$ is an f-zero interval contained in $A{:}B$. If $a{:}b$ contains $c{:}d$, then $[a'{:}b']_{\delta/2}$ does as well. By assumption, this means it is an f-zero interval. Thus $R(a{:}b, \delta) = (1, [a'{:}b']_{\delta/2})$.

6. Reasonable. Every interval and δ has an answer of 0 or $(1, c{:}d)$.

7. Closed. Assume $(a)_\delta$ always contains a prophecy. This would mean that for each $(a)_\delta$ there exists an f-zero interval $c{:}d$ inside of it. That in turn implies that $A{:}a{:}B$ since otherwise there would be a distance between a and $A{:}B$ and a halo whose size was less than that would be disjoint from $A{:}B$ prohibiting a prophecy from being in it. Let $R(a{:}b, \delta)$ be given. The closed halo $[a{:}b']'_{\delta/2}$ contains $(a)_{\delta/5}$ which contains a prophecy by assumption. As any interval in $A{:}B$ that contains an f-zero interval is an f-zero interval, this implies the close halo is an f-zero interval. Hence, $R(a{:}b, \delta) = (1, [a{:}b']'_{\delta/2})$. This holds for all b and all δ.

One particular application is any rational function that satisfies being monotonic on an f-zero interval $A \vdots B$. To conclude that $f(x) = 0$, a rationalized version of continuity of f is required. For real functions, this approach can be suitably adapted as will be explored in [Tay23b]. One change is to have a bit of fuzziness for how $f(m)$ is computed.

For roots, taking $f(x) = x^n - q$ on the interval $0 \vdots \max(q, 1)$ satisfies the conditions as can be easily checked.

Given this framework, the bisection algorithm established later establishes that there exists arbitrarily small prophecies for these f-zero oracles. This allows one to have arbitrary precision available for as long as the Separation step can be completed.

There generally is no need for non-core inputs or outputs for f-zero oracles, but they do require some level of non-local knowledge of f.

Another example of a test oracle is that of computing a least upper bound of a bounded set of rationals E. The phrase "q does not exceed E" shall mean that there is some element $r \in E$ such that $q \leq r$. The phrase "An interval $a \therefore b$ straddles the top of E" if b is an upper bound of E and a does not exceed E. The basic computational ability assumed is that given a rational q and a positive rational ε, at least one of the following holds: 1) q is an upper bound of E, 2) q does not exceed E, or 3) $(q)_\varepsilon$ straddles the top of E. Then the least upper bound oracle of E can be defined as follows. Given an interval $a \vdots b$, if it straddles the top of E, then $R(a \vdots b, \delta) = (1, a \vdots b)$. If q is an endpoint of $a \vdots b$ and $[q]_{\delta/2}$ straddles the top of E, then $R(a \vdots b, \delta) = (1, [q]_{\delta/2})$. Otherwise, it must be the case that either both a and b are upper bounds of E or both a and b does not exceed E. In either of those cases, $R(a \vdots b, \delta) = 0$. It can be shown that this defines an oracle and it is the least upper bound of E, once inequality is defined for oracles. The basic claim to being the least upper bound is based on given another oracle, compute the distance between and look at a prophecy smaller than that distance. Then that interval contains both an element of E and an upper bound showing that the given oracle must be above another upper bound or below an element of E.

4.3 Algorithmic Oracles

A common situation for real number estimates is that the real number is computed as a sequence of intervals. For example, the n-th root of the rational q can be computed using Newton's method. Let $a_0 > 0$ be some positive rational. Given a_m, define $b_m = q/a_m^{n-1}$. It can be shown that $a_m^n \vdots q \vdots b_m^n$ will hold true. The iteration is defined as $a_{m+1} = a_m + (b_m - a_m)/n$. This is what results from applying Newton's method to $x^n - q$. The oracle procedure R would then be to compute m, a_m, b_m such that $|a_m \vdots b_m| < \delta$. Then $R(a \vdots b, \delta) = (k, a_m \vdots b_m)$ where $k = 1$ if $a_m \vdots b_m$ intersects $a \vdots b$ and $k = 0$ otherwise. Due to the length, $a_m \vdots b_m$ will be in $(a \vdots b)_\delta$ if it intersects $a \vdots b$.

This particular way of using Newton's method leads to a family of procedures. Given a different a_0, the intervals computed will differ and will lead to some computed intervals overlapping the endpoints of some intervals that other starting points would not overlap with. While this may generate different core outputs for the same core input, the ultimate determination of whether any given interval $a \vdots b$ is a Yes interval or a No interval is not dependent on these difference. If the root is the rational p, that is,

$p^n = q$, then intervals of the form $p\vdots b$ will never be returned by this procedure unless $a_0 = p = b_0$. Nevertheless, the procedure will always return 1 for any interval that contains p. The Yes intervals in this case will be exactly those that contain p.

More general uses of Newton's method can be discussed, but it is more efficient to discuss a general construct. Let $F(\varepsilon)$ be a function that given an ε, it returns an interval whose length is less than ε. It may be the case that F returns various different intervals for the same ε. Further suppose that any pair of intervals returned from F must intersect. Then define an oracle $R(a\vdots b, 2\delta)$ by computing $F(\delta) = I$ and if I intersects $a\vdots b$, then $R(a\vdots b, \delta) = (1, I)$. Otherwise, $R(a\vdots b, \delta) = (0, I)$. Note that if F is multi-valued, then R is multi-valued in the same fashion, which is fine. It is easy to verify that this is an oracle. Range and Reasonableness are immediate. Existence The rest can be handled as:

1. Range. By definition this is satisfied. If an interval $c\vdots d$ of length less than δ intersects $a\vdots b$, then $c\vdots d$ must be contained in $(a\vdots b)_\delta$.

2. Existence. This follows from letting $I = f(1)$ in which case $R(I, 1) = (1, I)$, in at least one instance.

3. Separation. Let $a\vdots b$ be a prophecy of R, m such that $a\vdots m\vdots b$, and $2\delta > 0$. Let $F(\delta) = c\vdots d$. By assumption, $c\vdots d$ intersects $a\vdots b$. If $c\vdots m\vdots d$, then $c\vdots d$ intersects $a\vdots m$ and $b\vdots m$. Thus, $R(a\vdots m, 2\delta) = R(b\vdots m, 2\delta) = (1, c\vdots d)$. The other case, after relabeling, is that $\{a, c\}\vdots d\vdots m\vdots b$. Thus, $c\vdots d$ intersects $a\vdots m$ and $R(a\vdots m, \delta) = 1$.

4. Disjointness. Let I be a prophecy of R and $a\vdots b$ be disjoint from I with $|I, a\vdots b| = \varepsilon$. Then let $\delta < \varepsilon$. Let $F(\delta) = J$. Since J intersects I, it cannot intersect $a\vdots b$ due to its length being less than the distance from I to $a\vdots b$. Therefore, $R(a\vdots b, \delta) = (0, J)$.

5. Consistency. Given $a\vdots b$ and δ with I being a prophecy contained in $a\vdots b$, let $F(\delta) = J$. By assumption, I and J intersect. Since I is contained in $a\vdots b$, J must intersect $a\vdots b$. Thus, $R(a\vdots b, \delta) = (1, J)$. As this is true for any response from F for δ, $R(a\vdots b, \delta) \neq 0$. As this was true for any δ, $R(a\vdots b, \delta) \neq 0$.

6. Closed. Assume that for each $\delta > 0$, $(a)_\delta$ contains a prophecy. The approach is to show that a is contained in every $F(\delta)$. Given $F(\delta) = u\vdots v$ let $|a; u\vdots v| = 2d$. If $d = 0$, this means that $u\vdots a\vdots v$. If $d > 0$, then this means, after relabeling, that $u\vdots v \mathbin{\overset{\cdot}{\underset{\cdot}{\circ}}} a$. Let $e\vdots f$ be a prophecy in $(a)_d$ whose existence is assumed. But then $u\vdots v \overset{\cdot}{\underset{\cdot}{\circ}}_d \mathbin{¦} a\vdots\{e, f, a\}$. Thus, there can be no rational in common between $u\vdots v$ and $e\vdots f$ which contradicts the assumption that all of the outputs of F intersect. Hence, $d = 0$ and $R(a\vdots b, \delta) = (1, u\vdots v)$. This applies for every δ and a-rooted interval.

7. Reasonableness. No answer of -1 is given.

A simple but common example is that of Cauchy sequences. A Cauchy sequence can be viewed as pairs of rationals a_n, ε_n where a_n is the n-th element of the sequence and ε_n is a bound for all future elements to be within ε_n of a_n. This leads to the sequence of intervals $a_n - \varepsilon_n \mathrel{⁝} a_n + \varepsilon_n = (a_n)_{\varepsilon_n}$. The Cauchy criterion ensures that the ε_n exists and that they can be taken to approach 0. To construct an F as above, let $F(\delta)$ be the interval associated with n where n is selected arbitrarily subject to the constraint that that $\varepsilon_n < \delta$. The F defined oracle above is then the limit of the Cauchy sequence. This will be referred to as the Cauchy oracle.

4.4 Non-computable Oracles

As is well-known, most of the reals are non-computable. For decimal expansions, this means there is no finite state machine algorithm for determining the digit expansion up to an arbitrary n. For Dedekind cuts, the non-computable task is given a rational q, determine whether $q < x$, $q = x$, or $q > x$. For Cauchy sequences, the task is computing out an arbitrary element of the sequence.

For oracles, the analog would be whether R is computable or not. R is computable if, given any rational interval $a \mathrel{⁝} b$ and rational $\delta > 0$, then there exists a finite state machine which yields, in finite time, an output of $R(a \mathrel{⁝} b, \delta)$. A non-computable oracle is one which does not have this property.

Not being able to produce an output would seem to violate the purpose and spirit of the oracle definition. But it is not any more severe of a problem than not being able to produce a digit expansion or determining set membership. It is a matter of mathematical taste as to what extent to embrace the existence of non-computable real numbers. The oracle approach is not particularly illuminating on this question.

The main question is whether an oracle is well-defined or not. As a general setup, let there may be a family of statements $P(n)$, each one ought to be either true or false. Define $a_n = (-1)^n \frac{1}{n^2}$ if $P(n)$ is false and $a_n = 0$ if $P(n)$ is true. Let the partial sums s_n be the sum from a_1 to a_n. The oracle is then the Cauchy oracle related to s_n.

If $P(n)$ is that Collatz process starting with n ends at 1, then the above oracle is currently known to be Yes on the interval $-10^-21 \mathrel{⁝} 10^-21$ or so. But until either it is proved true for all n or a counterexample is given, it is impossible to say if it is negative, zero, or positive. This is a real number which is well-specified, but potentially not computable.

5 Overlapping and Notionally Shrinking

Before establishing the field of rational betweenness relations using oracles, it is necessary to prove a few useful general statements about oracles. In particular, there are arbitrarily small Yes intervals and all Yes intervals intersect. Those two properties are critical in describing a real number as will be highlighted in discussing Families of Overlapping, Notionally Shrinking Intervals (fonsis).

The two crucial results are discussed here. Their implications are then explored in the rest of the section.

Proposition 5.1 (Bisection Algorithm)**.** Given a rational $\varepsilon > 0$ and a Ranged procedure

R satisfying both Existence and Separation, there exists a prophecy whose length is less than ε.

Proof. By Existence, there exists a prophecy $a_0 \vdots b_0$. If $|a_0 \vdots b_0| < \varepsilon$, then that interval is sufficient. Otherwise, the desired interval is found via iteration. The iterative step for determining a_{i+1} and b_{i+1} given $a_i \vdots b_i$ begins with letting m_i be the average of a_i and b_i. Apply Separation to $a_i \vdots b_i$ with m_i the dividing point and choose a subwidth $\delta < |a_i \vdots b_i|/12$. By Separation, there either exists a prophecy in $(a_i \vdots m_i)_\delta$ or there is one in $(m_i \vdots b_i)_\delta$; there may be one in both, but just one is needed. The length of those haloed intervals is $|a_i \vdots b_i|/2 + 2\delta < 2|a_i \vdots b_i|/3$. Thus, the length of $a_n \vdots b_n$ will be at most $(2/3)^n |a_0 \vdots b_0|$. To find the n in which this procedure has definitively achieved the goal of $|a_n \vdots b_n| < \varepsilon$, take, in reduced form, $p/q = \varepsilon/|a_0 \vdots b_0|$ and let D be the number of digits in q expressed in base 10. Then $n \geq 6D$ leads to $(2/3)^n \leq (2/3)^{6D} < 10^{-D} < 1/q \leq p/q$. Thus, $n \geq 6D$ would have $|a_n \vdots b_n| \leq (2/3)^n |a_0 \vdots b_0| < \varepsilon$ as desired. □

The above algorithm gives a finite bound on the number of times an oracle is called to achieve its goal once one prophecy is given. Further information about the oracle computational requirements would be needed to know what amount of time and resources would be required to actually obtain the result. There are no assumptions on R from the perspective of this paper allowing for the possibility of infinite time or resources needed.

Proposition 5.2 (Prophetic Intersection)**.** Given a Ranged procedure R satisfying Disjointness and Consistency, any pair of prophecies $a \vdots b$ and $c \vdots d$ will intersect.

Proof. Let two pair of prophecies $a \therefore b$ and $c \therefore d$ be given; by relabeling, it can be assumed $a \leq c$. There are two possibilities. Either $a \vdots c \vdots b \vdots d$ in which case they intersect or $a \vdots b \odot c \vdots d$ in which case they do not. What needs to be shown is that the second is not possible for two prophecies of the same oracle. If it were, then consider $\delta < |c - b|$. By Disjointness, $R(a \vdots b, \delta) = 0$. But by Consistency, $R(a \vdots b) \neq 0$. This contradiction leads to the conclusion that the prophecies intersect. □

5.1 Notionally Shrinking

This explores some implications of the Bisection Algorithm.

Since every prophecy is a Yes interval, the Bisection Algorithm immediately implies:

Corollary 5.3. Given an oracle R, there are arbitrarily small Yes intervals.

There is also another version of Separation which is very useful.

Proposition 5.4 (Tri-Separation)**.** Given an oracle R, a prophecy $a \vdots b$ of R, m contained in $a \vdots b$, and $\delta > 0$, then there exists an $(m)_\delta$ compatible interval $e \vdots f$ such that exactly one of the following holds true: ${}_{\delta}\text{¦} a \vdots e \in \mathbb{I}_R$, $e \vdots f \in \mathbb{I}_R$, or $f \vdots b \text{¦}_{\delta} \in \mathbb{I}_R$.

Proof. By the Bisection Algorithm, there exists a prophecy $c \vdots d$ whose length is less than $\delta/2$. It must intersect $a \vdots b$. If $c \vdots m \vdots d$, then $c \vdots d$ is strictly contained in $(m)_\delta$ due to the length of $c \vdots d$. Taking $e \vdots f$ to be $[m]_{2\delta/3}$, $c \vdots d$ is strictly contained in $e \vdots f$, $e \vdots f$ is a $(m)_\delta$-compatible interval that contains a prophecy, and both ${}_{\delta}{\mid}a \vdots e \notin \mathbb{I}_R$ and $f \vdots b{\mid}_{\delta} \notin \mathbb{I}_R$ by Disjointness.

If $c \vdots d$ does not contain m, then either $\{a, c\} \vdots d \mathbin{\overset{\cdot}{\underset{\cdot}{\circ}}} m$ or $m \mathbin{\overset{\cdot}{\underset{\cdot}{\circ}}} c \vdots \{b, d\}$. By relabeling, assume the first is the situation. Then, let e be the average of d and m and $f = m{\mid}_{b,\delta/2}$. Due to the length, ${}_{\delta}{\mid}a \mathbin{\overset{\cdot}{\underset{\cdot}{\circ}}} \{c, a\}$. Thus, $c \vdots d$ is a prophecy strictly contained in ${}_{\delta}{\mid}a \vdots e \in \mathbb{I}_R$ implying $e \vdots f \notin \mathbb{I}_R$ and $f \vdots b{\mid}_{\delta} \notin \mathbb{I}_R$ by Disjointness. □

It should be immediately clear that Tri-Separation also implies the Bisection Algorithm via the same argument as before. The Bisection Algorithm can also be used to imply the Separation proposition which will follow easily from the following proposition.

Proposition 5.5. Given an oracle R, let $a \vdots b$ be a prophecy and $a \vdots m \vdots b$. Then either $a \vdots m$ strictly contains a prophecy, $b \vdots m$ strictly contains a prophecy, or one of a, b, or m is a root.

Proof. For any given δ, the Bisection Algorithm yields a prophecy $c \vdots d$ whose length is less than δ. By Prophetic Intersection, $c \vdots d$ and $a \vdots b$ must intersect. If $c \vdots d$ is strictly contained in $a \vdots m$ or $b \vdots m$, then the desired result holds. Assume not. Then $c \vdots d$ contains a, m, or b which implies $c \vdots d$ is contained in $(a)_\delta$, $(m)_\delta$, or $(b)_\delta$, respectively. To the extent that a, m, and b are distinct, their halos will be disjoint for any δ less than half the distance between those points. Let q represent the point whose halo contains the prophecy disjoint from the other halos. The non-constructive is to argue that either there is a prophecy strictly contained in either $a \vdots m$ or $b \vdots m$, whichever one contains q, or every halo of q contains a prophecy. In the latter case, Closed says that q is a root of the oracles. □

Corollary 5.6. Given a procedure that satisfies all of the properties of an oracle except Separation, then Separation, Tri-Separation, and the Bisection Algorithm are all equivalent.

Proof. Separation has already been show to imply the Bisection Algorithm and the Bisection Algorithm has been shown to imply Tri-Separation.

Using the previous proposition, Separation holds by Consistency if either $a \vdots m$ or $b \vdots m$ contains a prophecy or Separation holds by the Closed property if it is a root that occurs.

The proof of the Bisection Algorithm works almost without alteration for the Tri-Separation property. Indeed, after applying Tri-Separation, take the generated prophecy and it will be contained in the same halo interval as given by the Separation property giving the same length shortening. □

A very similar proposition concerns Yes intervals.

Proposition 5.7. Let an oracle R be given. If $a \vdots b$ is a Yes interval, then either $a \vdots b$ strictly contains a prophecy of R or an endpoint of $a \vdots b$ is a root of R.

Proof. By definition of $a \,\vdots\widehat{x}\vdots\, b$, $R(a{:}b, \delta) \neq 0$ for all $\delta > 0$. This implies that, by Disjointness, all prophecies must intersect $a{:}b$. If any of them are strictly contained in $a{:}b$, then that satisfies what was to be shown. Thus, for the rest of this proof, assume all prophecies are not strictly contained in $a{:}b$.

Let $L = |a{:}b|$. By the Bisection Algorithm, there exists $c{:}d \in R$ such that $|c{:}d| < L/2$. Due to its length, $c{:}d$ cannot contain both a and b. As $c{:}d$ is a prophecy, it must intersect $a{:}b$. For it not to be contained in it, it must overlap one of the endpoints. By relabeling, assume $a \in c{:}d$.

Let $\delta < L/2$ be given. The task is to show that $(a)_\delta \in \mathbb{I}_R$. Let $e{:}f \in R$ be given such that $|e{:}f| < \delta$. By the same logic as before, the prophecy $e{:}f$ must contain one of the endpoints of $a{:}b$. Since it must intersect $c{:}d$ and the length of both intervals combined is less than L, the endpoint contained in $e{:}f$ must be a as well. Since the length of $e{:}f$ is less than δ, this implies that $e{:}f \subset (a)_\delta$.

Thus, as this holds for arbitrarily small δ and prophecies $e{:}f$, a is a root of R as was to be shown in this case. □

Proposition 5.8. If an oracle R has a root q, then that root is unique and an interval is a Yes interval exactly when q is in the interval.

Proof. There are three possible cases for how an interval $a{:}b$ can be related to q:

1. q is an endpoint of $a{:}b$. Since q is a root, any interval of the form $q{:}r$ is a Yes interval by the Closed property.

2. q is strictly contained in $a{:}b$. If $q \in a{:}b$ but is not an endpoint, then let L be the distance from q to the closest endpoint of $a{:}b$. Any interval $(q)_\delta$ with $\delta < L$ will then be contained in $a{:}b$ and hence $a{:}b \in \mathbb{I}_R$ since $(q)_\delta$ is.

3. q is outside of $a{:}b$. Let L be the distance from q to the closest endpoint. Then let $c{:}d$ be a prophecy contained in $(q)_\delta$ with $\delta < L$. This exists as q is a root. Due to the length, $c{:}d$ is disjoint from $a{:}b$. Thus, Disjointness yields a δ' such that $R(a{:}b, \delta') = 0$. Hence, $a{:}b$ is a No interval.

As for the uniqueness of the root, let b be any rational not equal to q and $\delta = \frac{|b-q|}{2}$. Then $(b)_\delta$ does not contain q and hence is a No interval. This means it does not contain any prophecy and b is not a root of the oracle. □

Proposition 5.9. If q is contained in arbitrarily small prophecies of an oracle, then q is the root of the oracle.

Proof. Given δ, let $a{:}b$ be a prophecy whose length is less than δ and contains q; this exists by assumption. Since q is contained in $a{:}b$, $a{:}b$ is contained in $(q)_\delta$. Thus, $(q)_\delta \in \mathbb{I}_R$. As this was an arbitrary δ, the Closed property applies to conclude q is the root of the oracle. □

Corollary 5.10. If q is contained in every prophecy of an oracle, then q is the root of that oracle.

Proof. By the Bisection Algorithm, there exists arbitrarily small prophecies. As q is contained in them, the proposition applies. $\square$

Corollary 5.11. Given an oracle, q is a root of the oracle if and only if q is an element of every Yes interval.

Proposition 5.12. Given an oracle, q is a root of the oracle if and only if q is in arbitrarily small Yes intervals.

Proof. The one direction immediately follows from the previous statements. For the other direction, assume that q is in arbitrarily small Yes intervals. Let δ be given and let $a{:}b$ be a Yes interval whose length is less than δ and contains q. Then $a{:}b$ is strictly contained in $(q)_\delta$. From above, either $a{:}b \in \mathbb{I}_R$ in which case $(q)_\delta$ is as well, or one of the endpoints is a root. In the latter case, by relabeling, let a be the endpoint that is the root. The distance from a to q is less than δ; call that L. Let $\delta' = \delta - L > 0$. As a is a root, $(a)_{\delta'} \in \mathbb{I}_R$ and it is contained in $(q)_\delta$ as $\delta' + L = \delta$. Thus, $(q)_\delta$ is in $\mathbb{I}_R$ in all cases and for all δ. $\square$

5.2 Overlapping

This explores the implication of all the prophecies intersecting.

Proposition 5.13. Let R be an oracle and $a{:}b$ be a given interval. Assume that there are arbitrarily small δ such that for each of the δ there exists a prophecy $e{:}f$ such that $R(a{:}b, \delta) = (1, e{:}f)$. Then $a{:}b$ intersects all prophecies.

Proof. Let $c{:}d$ be any interval disjoint from $a{:}b$. The task is to show it is not a prophecy. Let δ be less than the distance between $c{:}d$ and $a{:}b$ which disjointness gives as non-zero. Then consider $R(a{:}b, \delta)$. By assumption, there exists a $\delta' < \delta$ and a prophecy $e{:}f$ such that $R(a{:}b, \delta') = (1, e{:}f)$. By Ranged, $e{:}f$ intersects $a{:}b$ and is contained in $(a{:}b)_{\delta'}$. As that halo is disjoint from $c{:}d$ by choice of δ, $e{:}f$ is disjoint from $c{:}d$ as well. As shown above, all prophecies must intersect. Thus, $c{:}d$ cannot be a prophecy. As $c{:}d$ was an arbitrary interval other than being disjoint from $a{:}b$, it is the case that no prophecy can be disjoint from $a{:}b$. Thus, $a{:}b$ must intersect all prophecies. $\square$

Corollary 5.14. For an oracle R, $R(a{:}b) = 1$ if and only if $R(a{:}b) \neq 0$. The conditions of the proposition above are also sufficient to have either of these conclusions hold.

Proof. If $R(a{:}b) \neq 0$, then all of its core outputs have first entry of 1 which implies $R(a{:}b, \delta) = 1$ for all δ which is what $R(a{:}b) = 1$ means. Given $R(a{:}b) = 1$, then for every δ, there exists a prophecy $c{:}d$ such that $R(a{:}b, \delta) = (1, c{:}d)$. Thus, $a{:}b$ intersects every prophecy by the above proposition. This immediately implies that $R(a{:}b) \neq 0$ since, by the Ranged property, the only way to have $R(a{:}b, \delta) = (0, c{:}d)$ is to have the prophecy $c{:}d$ disjoint from $a{:}b$. The argument here just relied on the conclusion of the proposition so any interval $a{:}b$ satisfying its hypotheses would have this conclusion. $\square$

From now on, $R(a{\vdots}b) = 1$ will be freely used where $R(a{\vdots}b) \neq 0$ was used before.

Given any interval $a{\vdots}b$ and oracle R, there always exists a Yes interval $c{\vdots}d$ that intersects $a{\vdots}b$. This follows from taking a prophecy $e{\vdots}f$, which exists by Existence, and making a unionized interval, namely $\min(a, b, e, f){\vdots}\max(a, b, e, f)$ and calling that $c{\vdots}d$. The interval $c{\vdots}d$ contains a prophecy and thus $R(c{\vdots}d) = 1$, making it a Yes interval. It contains $a{\vdots}b$ and thus intersects it.

If an interval $a{\vdots}b$ has the property that $R(a{\vdots}b, \delta) = (0, c{\vdots}d)$ for some δ and some core output, then $a{\vdots}b$ cannot contain a prophecy and $a{\vdots}b \notin \mathbb{I}_R$.

The Yes intervals are united by supposedly containing the real number being represented. Thus, intersecting two such intervals should lead to a further narrowing towards the real number. This is indeed what happens.

Proposition 5.15. Given an oracle R, all Yes intervals intersect.

Proof. If R is rooted at q, then q is common to all Yes intervals. Hence, they intersect.

If R is not rooted, then all Yes intervals contain at least one prophecy. Those prophecies intersect implying the intervals that contain them will intersect as well. □

Proposition 5.16. If $a{\vdots}b$ is an interval that intersects every prophecy of an oracle, then, nonconstructively, $a{\vdots}b$ is a Yes interval.

Proof. Let δ be less than half the length of $a{\vdots}b$. By the Bisection Algorithm, there exists a prophecy $c{\vdots}d$ whose length is less than δ. By assumption, $c{\vdots}d$ intersects $a{\vdots}b$. This implies that either $c{\vdots}d$ is contained in $a{\vdots}b$, contains a but not b, or contains b but not a. If it is the first, then $a{\vdots}b \in \mathbb{I}_R$ and is therefore a Yes interval. If not, by relabeling, take a to be the endpoint that is contained in $c{\vdots}d$. By the same logic, all prophecies whose lengths are less than δ will contain either a or be contained in $a{\vdots}b$; they cannot contain the other endpoint due to it needing to intersect $c{\vdots}d$ and the two combined lengths are less than the length of $a{\vdots}b$. If there is no prophecy that is a subinterval of $a{\vdots}b$, then, nonconstructively, a is contained in arbitrarily small prophecies. This in turn implies a is a root of the oracle and, thus, $a{\vdots}b$ is a Yes interval by the Closed property. □

A useful idea is that every pair of disjoint intervals has at least one No interval in it.

Proposition 5.17. Given an oracle and two disjoint intervals $a{\vdots}b$ and $c{\vdots}d$, at least one of them is constructively known to be a No interval of that oracle.

Proof. As the intervals are disjoint, by potentially relabeling, it can be assumed $a{\vdots}b{\mathrel{.\kern-1pt^{\cdot}}}c{\vdots}d$. Then define $L = c - b > 0$. By the Bisection Algorithm, which is a finite constructive procedure, there is a Yes interval $e{\vdots}f$ whose length is less than L. The interval $e{\vdots}f$ must be disjoint from at least one of the intervals because of the length. The interval it is disjoint from is then a No interval by Disjointness. □

Proposition 5.18. If $a{\vdots}b$ and $c{\vdots}d$ are Yes intervals for the same oracle, then their intersection is a Yes interval.

Proof. All Yes intervals intersect. By relabeling, it can be assumed that $a{:}c{:}\{b, d\}$. Every prophecy intersects every Yes interval. Let $e{:}f$ be any prophecy. Then the only way for $e{:}f$ to intersect both $c{:}d$ and $a{:}b$ is for $e{:}f$ to intersect $c{:}b$. But this means that $c{:}b$ intersects every prophecy. It was established above that this implies $c{:}b$ is a Yes interval. □

Corollary 5.19. If $a \,{:}\widehat{x}{:}\, b$, $b \,{:}\widehat{x}{:}\, c$ and $a{:}b{:}c$, then $b \,{:}\widehat{x}{:}\, b$.

Proof. Apply the above noting that $b{:}b$ is the intersection of $a{:}b$ and $b{:}c$. □

A common trick is to look at small enough Yes intervals, or prophecies, such that they are all contained within a known interval. This is an immediate consequence of them all intersecting.

Proposition 5.20. Let $a{:}b$ be a Yes interval of a given oracle. Then any Yes interval of that oracle whose length is less than a given δ will be contained in $(a{:}b)_\delta$.

Proof. Given any Yes interval $c{:}d$ whose length is less than δ, it is the case that $c{:}d$ intersects $a{:}b$ as they are both Yes intervals of the same oracle. Since the maximum distance an element of $c{:}d$ can be from another element in $c{:}d$ is less than δ, the maximum distance an element can be from $a{:}b$ is less than δ. Thus, $c{:}d$ is contained in $(a{:}b)_\delta$. □

Proposition 5.21. Given an oracle R and a No interval $a{:}b$, then there exists a prophecy $c{:}d$ disjoint from $a{:}b$. Furthermore, there exists an $\varepsilon > 0$ such that any prophecy whose length is less than ε will be disjoint from $a{:}b$.

Proof. Being a No interval is saying that there is a δ such that either 1) $R(a{:}b, \delta) = (0, c{:}d)$ for some prophecy $c{:}d$ or that 2) $R(a{:}b, \delta) = (0)$ implying that there does not exist a prophecy contained in $(a{:}b)_\delta$ that also intersects $a{:}b$.

For 1), $c{:}d$ is the required prophecy that is disjoint from $a{:}b$. Take $\varepsilon = |a{:}b; c{:}d|$. Then any prophecy whose length is less than ε will be disjoint from $a{:}b$. This follows because all prophecies must intersect $c{:}d$ and if the prophecy's length is less than ε, then it is strictly contained in $(c{:}d)_\varepsilon$.

For 2), by the Bisection Algorithm, let $c{:}d$ be a prophecy whose length is less than δ. Then, if $c{:}d$ intersected $a{:}b$, $c{:}d$ would be contained in $(a{:}b)_\delta$ due to the length of $c{:}d$. Thus, $c{:}d$ must be disjoint from $a{:}b$. The argument applies to any prophecy whose length is less than δ. Thus, $\varepsilon = \delta$ satisfies the requirement. □

5.3 Fonsis

A **Family of Overlapping, Notionally Shrinking Intervals** (fonsi, pronounced faan-zee,) is a set of rational intervals such that any pair of rational intervals in the set intersect and, given a rational $\varepsilon > 0$, there exists at least one interval in the fonsi such that its length is less than ε. A sequence of nested intervals whose length approaches 0 would be an examples of a fonsi. In constructivist works, such as [Bri06], these objects, called fine families, are often taken as the definition of a real number and are likened to a set of measurements.

One can think of a fonsi as a function that takes in positive rational numbers and returns an interval whose length is less than the given number. Different requests may lead to different intervals. The requirements on the output is that every returned interval must intersect every other returned interval.

Proposition 5.22. A set of intervals $\mathcal{F}$ is a fonsi if and only if there is an oracle R whose set of prophecies equals $\mathcal{F}$.

Proof. Given an oracle R, its prophecies are notionally shrinking by the Bisection Algorithm. They are overlapping by Prophetic Intersection. Thus, the set of prophecies is a fonsi.

For the other direction, let $\mathcal{F}$ be a fonsi. The task is to define an oracle whose prophecies are exactly the elements of $\mathcal{F}$.

Let $a{:}b$ and rational $\delta > 0$ be given. For any interval $c{:}d$ from the fonsi such that $|c{:}d| < \delta$, define $R(a{:}b, \delta) = (k, c{:}d)$ where $k = 1$ if $c{:}d$ intersects $a{:}b$ and $k = 0$ if $c{:}d$ is disjoint from $a{:}b$. The potential multi-valued nature of R is convenient here.

Given the that length is less than δ, if $c{:}d$ intersects $a{:}b$, then $c{:}d$ is contained in $(a{:}b)_\delta$.

Whenever it is required to show that $R(a{:}b, \delta) \neq 0$, it will be necessary to demonstrate that every element of the fonsi whose length is less than δ intersects $a{:}b$.

The prophecies of R are precisely the elements of the fonsi. That every prophecy is contained in the fonsi is clear from the definition. That every element of the fonsi is a prophecy follows by considering $R(c{:}d, |c{:}d| + 1)$ for a given interval $c{:}d$ of the fonsi. Since $|c{:}d| < |c{:}d| + 1$, $c{:}d$ itself matches the procedure constraint and is therefore a prophecy of R.

The properties are verified as follows:

1. Range. Satisfied by definition.

2. Existence. Let $c{:}d$ be an element of the fonsi whose length is less than, say, 1. Then $R(c{:}d, 2) = (1, c{:}d)$.

3. Separation. Let $a{:}b \in R$, that is, it is an element of the fonsi. Let m and δ be given such that $a{:}m{:}b$. Let $c{:}d$ be any element of the fonsi whose length is less than δ. Since $a{:}b$ and $c{:}d$ are in the fonsi, they intersect. By relabeling, assume $c{:}d$ intersects at least $a{:}m$. Then by definition, $R(a{:}m) = (1, c{:}d)$ as was to be shown.

4. Disjointness. Let $c{:}d \in R$ and $a{:}b$ be disjoint from $c{:}d$. Let $\delta < |c{:}d; a{:}b|$. Let $e{:}f$ be any element of the fonsi whose length is less than δ. Since $e{:}f$ must intersect $c{:}d$, it cannot intersect $a{:}b$ given the lengths. Thus, by definition, $R(c{:}d, \delta) = (0, e{:}f)$.

5. Consistency. Let $a{:}b \in \mathbb{I}_R$ which implies there exists an element $c{:}d$ of the fonsi which is a subinterval of $a{:}b$. Let δ be given. The task is to show $R(a{:}b, \delta) \neq 0$.

Let $e\colon f$ be any element of the fonsi whose length is less than δ. Since $e\colon f$ and $c\colon d$ intersect as they are both in the fonsi and as $c\colon d$ is contained in $a\colon b$, it is the case that $e\colon f$ intersects $a\colon b$. Therefore, the procedure returns $R(a\colon b, \delta) = (1, e\colon f)$ in this instance. As $e\colon f$ was arbitrary given the length, $R(a\colon b, \delta) \neq 0$.

6. Closed. Assume that a is given such that $(a)_\delta \in \mathbb{I}_R$ for all δ. Given $a\colon b$ and δ, the task is to show $R(a\colon b, \delta) \neq 0$. Let $c\colon d$ be an element of the fonsi whose length is less than δ. The question is whether $c\colon d$ intersects $a\colon b$ or not. Suppose $a\colon b$ does not intersect $c\colon d$, then let δ' be less than the length from $c\colon d$ to $a\colon b$. This implies that $(a)_{\delta'}$ is disjoint from $c\colon d$. Since $(a)_{\delta'} \in \mathbb{I}_R$, there exists an interval $e\colon f$ in the fonsi which is contained in $(a)_{\delta'}$. But being an element of the fonsi, it must intersect $c\colon d$. As this is not possible if $(a)_{\delta'}$ is disjoint from $c\colon d$, it must be the case that $(a)_{\delta'}$ and $c\colon d$ intersect. As this is a contradiction of the choice of δ' whose existence is based on $c\colon d$ being disjoint from $a\colon b$, it must be the case that $c\colon d$ intersects $a\colon b$. Thus, $R(a\colon b, \delta) = (1, c\colon d)$. As a side note, since this is true for all b and elements $c\colon d$ of the fonsi, it must be the case that a is contained in every element of the fonsi.

7. Reasonableness. Nothing is defined with a -1 response.

$\square$

Corollary 5.23. The Yes intervals form a fonsi.

Proof. The containment of the prophecies handles the notionally shrinking. The intersection of the Yes intervals is the claim of Proposition 5.15. $\square$

Corollary 5.24. Given an oracle, the complete set of all Yes intervals forms a maximal fonsi.

Proof. A maximal fonsi is one which is not contained in any other fonsi. That is equivalent to the fonsi having the property that every interval not in the fonsi is disjoint from at least one element of the fonsi. If an interval is not a Yes interval, then by assumption this is a No interval. For an interval to be a No interval, it must be disjoint from a prophecy. Thus, no interval can be added to the fonsi. $\square$

6 Oracles and Rational Betweenness Relations

Each oracle is associated with a single rational betweenness relation, but a rational betweenness relation is associated with multiple oracles. Two oracles represent the same relation if all of their respective prophecies intersect each other. This section will establish these claims.

6.1 Oracles Give Rise to Rational Betweenness Relations

The complete set of Yes intervals for an oracle is the rational betweenness relation associated with that oracle. The notation $a \,\vdots \widehat{x} \vdots\, b$ is used to indicate that $a{\vdots}b$ is a Yes interval for the oracle R_x while the notation $a \,\vdots \underline{x} \vdots\, b$ is used for the Yes intervals of the rational betweenness relation x. Complete means in this context that the complement of the set of Yes intervals is the set of No intervals; there is no interval that is considered unknown. Practically speaking, this cannot necessarily be achieved for the set of oracle Yes intervals. The assumption essentially requires being able to ascertain for every interval whether or not it intersects every prophecy of the oracle or not.

In what follows, properties from the oracle may be prepended with the term Oracle for clarity while those of the rational betweenness relations may have RBR prepended.

Lemma 6.1 (Consistency)**.** Let an oracle R be given representing x. Assume $c{\vdots}a{\vdots}b{\vdots}d$. If $a \,\vdots \widehat{x} \vdots\, b$, then $c \,\vdots \widehat{x} \vdots\, d$. If $\cancel{c \,\vdots \widehat{x} \vdots\, d}$, then $\cancel{a \,\vdots \widehat{x} \vdots\, b}$.

Proof. If $a \,\vdots \widehat{x} \vdots\, b$, then this means that either $a{\vdots}b \in \mathbb{I}_R$ or, potentially by relabeling, R is rooted at a. In the first instance, there is a prophecy $e{\vdots}f$ contained in $a{\vdots}b$. This implies $c{\vdots}d$ contains $e{\vdots}f$ since $c{\vdots}d$ contains every interval contained in $a{\vdots}b$. Thus, $c{\vdots}d \in \mathbb{I}_R$ and hence $c \,\vdots \widehat{x} \vdots\, d$.

If R is rooted at a, there are two cases. In the first case, a is strictly contained in $c{\vdots}d$. Then there is a positive distance L from a to the closest endpoint of $c{\vdots}d$. Let $\delta < L$. Then $(a)_\delta$ is contained in $c{\vdots}d$ and hence $c{\vdots}d \in \mathbb{I}_R$. The other case is that a is an endpoint of $c{\vdots}d$, say, $a = c$. Then $R(a{\vdots}d, \delta) \neq 0$ for all δ by the Closed property. Thus, $c \,\vdots \widehat{x} \vdots\, d$ in both cases.

If $\cancel{c \,\vdots \widehat{x} \vdots\, d}$, then, by the definition of a No interval, there exists a prophecy $e{\vdots}f$ which is disjoint from $c{\vdots}d$. Since it is disjoint from $c{\vdots}d$, it is also disjoint from any of its subintervals, such as $a{\vdots}b$. Thus, $\cancel{a \,\vdots \widehat{x} \vdots\, b}$. □

Tri-Separation can be extended to all Yes intervals.

Lemma 6.2 (Yes Interval Separation)**.** If $a{\vdots}b$ is a Yes interval for an oracle R, m is a rational number strictly between a and b, and a subwidth $\delta > 0$ is given, then there exists an $(m)_\delta$ compatible interval $e{\vdots}f$ such that one of the following holds true:

1. $a \,\vdots \widehat{x} \vdots\, e$, $\cancel{e \,\vdots \widehat{x} \vdots\, f}$, $\cancel{f \,\vdots \widehat{x} \vdots\, b}$;
2. $e \,\vdots \widehat{x} \vdots\, f$, $\cancel{a \,\vdots \widehat{x} \vdots\, e}$, $\cancel{f \,\vdots \widehat{x} \vdots\, b}$;
3. $f \,\vdots \widehat{x} \vdots\, b$, $\cancel{a \,\vdots \widehat{x} \vdots\, e}$, $\cancel{e \,\vdots \widehat{x} \vdots\, f}$.

Proof. Yes intervals are either in $\mathbb{I}_R$ or they are rooted. The subwidth term implies $(m)_\delta$ is strictly contained in $a{\vdots}b$.

If $a{\vdots}b$ is a rooted interval, then relabeling allows for the root to be a which implies $(a)_{\delta'} \in \mathbb{I}_R$ for all rational $\delta' > 0$. With $a{\vdots}e{\vdots}b$, it is immediately the case that $a{\vdots}e$ is an a-Rooted interval and hence is a Yes interval by the Closed property. Any $(m)_\delta$

compatible interval $e{\vdots}f$ will then have the property that $a \,{\vdots}\widehat{x}{\vdots}\, e$ while $\cancel{e \,{\vdots}\widehat{x}{\vdots}\, f}$ and $\cancel{f \,{\vdots}\widehat{x}{\vdots}\, b}$ because $e{\vdots}f$ and $f{\vdots}b$ are disjoint from the prophecies in $(a)_\delta$ for $\delta < |a{\vdots}e|$.

If $a{\vdots}b$ is not a rooted interval, then by Proposition 5.7, there exists a prophecy $c{\vdots}d$ strictly contained in $a{\vdots}b$.

If m is not in $c{\vdots}d$, then, by possibly relabeling, $a \odot m \odot c{\vdots}d{\vdots}b$. Let $e{\vdots}f$ be an $(m)_\delta$ compatible interval such that $a{\vdots}e \odot m \odot f \odot c{\vdots}d{\vdots}b$. Then $a{\vdots}e$ and $e{\vdots}f$ are disjoint from $c{\vdots}d$ while $f{\vdots}b$ contains $c{\vdots}d$. Thus, $f \,{\vdots}\widehat{x}{\vdots}\, b$, $\cancel{a \,{\vdots}\widehat{x}{\vdots}\, e}$ and $\cancel{e \,{\vdots}\widehat{x}{\vdots}\, f}$.

If m is in $c{\vdots}d$, then apply Tri-Separation using m and $\delta' < \min(|a;c|, |b;d|)$. If the produced $e{\vdots}f$ contains a prophecy, then the Tri-Separation asserts that $e \,{\vdots}\widehat{x}{\vdots}\, f$, $\cancel{a \,{\vdots}\widehat{x}{\vdots}\, e}$, and $\cancel{f \,{\vdots}\widehat{x}{\vdots}\, b}$.

If the produced $e{\vdots}f$ does not contain a prophecy, then it can be assumed by relabeling that ${}_{\delta}{\mid} c{\vdots}e \in \mathbb{I}_R$. Tri-Separation already gives $\cancel{e \,{\vdots}\widehat{x}{\vdots}\, f}$ and $\cancel{f \,{\vdots}\widehat{x}{\vdots}\, b}$. As δ' was chosen so that $a \odot {}_{a,\delta'}{\mid} c \odot c{\vdots}e$, the prophecy is strictly contained in $a{\vdots}e$ and thus $a \,{\vdots}\widehat{x}{\vdots}\, e$.

□

Lemma 6.3 (RBR Separation)**.** If $a \,{\vdots}\widehat{x}{\vdots}\, b$, then exactly one of the following holds:

1. $a \,{\vdots}\widehat{x}{\vdots}\, m$, $\cancel{m \,{\vdots}\widehat{x}{\vdots}\, b}$;
2. $m \,{\vdots}\widehat{x}{\vdots}\, m$;
3. $b \,{\vdots}\widehat{x}{\vdots}\, m$, $\cancel{m \,{\vdots}\widehat{x}{\vdots}\, a}$.

Proof. Given any δ, if $e{\vdots}f$, as in the previous proof, is not the returned Yes interval, then Consistency and Disjointness leads to either the first or third outcome, depending on which one contains the Yes interval.

If $e{\vdots}f$ is the Yes interval for each δ, then $(m)_\delta \in \mathbb{I}_R$ for all δ. This is a nonconstructive step. The Closed property yields that m is the root of the oracle and thus, $m \,{\vdots}\widehat{x}{\vdots}\, m$. □

Theorem 6.1. Given an oracle R_x, then ${\vdots}\widehat{x}{\vdots}$ is a rational betweenness relation.

Proof. As every interval is either Yes or No, ${\vdots}\widehat{x}{\vdots}$ is a relation on all pairs of rational numbers. It is symmetrical as the outputs of $R(a{\vdots}b, \delta)$ are the same as the outputs of $R(b{\vdots}a, \delta)$; this holds for all intervals $a{\vdots}b$ and positive rationals δ.

The properties are established as follows:

1. RBR Existence. By Oracle Existence, there exists $a{\vdots}b$ and a δ such that $R(a{\vdots}b, \delta) = (1, c{\vdots}d)$. As $c{\vdots}d$ is a prophecy, Oracle Consistency implies $R(c{\vdots}d, \delta) \neq 0$ for all δ. Thus, $c \,{\vdots}\widehat{x}{\vdots}\, d$.
2. RBR Separation. This is Lemma 6.3.
3. RBR Consistency. This is Lemma 6.1.
4. RBR Singular. Since $c \,{\vdots}\widehat{x}{\vdots}\, c$ is equivalent to saying that c is a root of the oracle, Proposition 5.8 states that c is unique. Thus, if $d \,{\vdots}\widehat{x}{\vdots}\, d$, then d must be c.

5. RBR Closed. Assume c is such that $a \mathrel{\vdots\widehat{x}\vdots} b$ implies $a \vdots c \vdots b$. This is equivalent to saying that c is in every Yes interval. By Corollary 5.10, c is a root of the oracle. By the Oracle Closed property, $R(c \vdots c, \delta) \neq 0$ for all δ. Thus, $c \mathrel{\vdots\widehat{x}\vdots} c$.

$\square$

6.2 Equivalent Oracles

Having established that a given oracle does have an associated rational betweenness relation, at least nonconstructively, this allows an equivalence between oracles based on having the same rational betweenness relation. Specifically, two oracles are **equivalent**, denoted $R_x \equiv R_y$, exactly when $\vdots\widehat{x}\vdots$ and $\vdots\widehat{y}\vdots$ are equal as completed relations. This is the same statement as saying the set of Yes intervals for each of them is exactly the same. Since the rational betweenness relation is unique for a given oracle, a relation between oracles based on equality of the rational betweenness relation will automatically be reflexive, symmetric, and transitive.

Proposition 6.4. Let R_x and R_y be two oracles. All of the prophecies of R_x will intersect all of the prophecies of R_y if and only if $R_x \equiv R_y$.

This proof does rely on non-constructive results which is in line with equality of real numbers having to be proven out rather than computed out.

Proof. If $R_x \equiv R_y$, then they have the same Yes intervals, which will include both sets of prophecies. Since all Yes intervals of a given oracle intersect each other, the two sets of prophecies intersect as well.

The other direction is to assume that all of the prophecies of R_x intersect all of the prophecies of R_y. The proof will proceed to establish that the Yes intervals of R_x are Yes intervals of R_y and that the No intervals of R_x are No intervals of R_y. Switching x and y then gives both directions.

Assume $a \mathrel{\vdots\widehat{x}\vdots} b$. By the non-constructive Proposition 5.7, either there exists a prophecy $c \vdots d$ strictly contained in $a \vdots b$ or one of the endpoints, say a, is a root of the oracle. In the first case, consider a prophecy $e \vdots f$ of R_y whose length is less than the distance from $c \vdots d$ to $a \vdots b$ (this is the smallest distance between the endpoints). By assumption, $e \vdots f$ intersects $c \vdots d$ and by choice of length, $e \vdots f$ must be strictly contained in $a \vdots b$. Thus, $a \vdots b$ is a Yes interval of R_y.

In the case that a is the root of R_x, this implies, by Proposition 5.8, that every prophecy of R_x contains a. Assume that there is a prophecy $c \vdots d$ of R_y which does not contain a. Then by letting δ be less than the distance from $c \vdots d$ to a, there exists, by a being a root, a prophecy $e \vdots f$ of R_x in $(a)_\delta$. It is disjoint from $c \vdots d$ due to the choice of δ. But it should intersect $c \vdots d$ by the assumption. This contradiction leads to the conclusion that all of the prophecies of R_y must contain a. By Corollary 5.9, R_y is then also rooted at a and the Yes intervals of both R_x and R_y are exactly the intervals that contain a.

As for the No intervals, let $a \vdots b$ be a No interval of R_x. By Proposition 5.21, there is a prophecy $c \vdots d$ of R_x such that $a \vdots b$ and $c \vdots d$ are disjoint. Let L be the distance between

$a \vdots b$ and $c \vdots d$. Let $e \vdots f$ be a prophecy of R_y whose length is less than L. This must intersect $c \vdots d$ by assumption. Thus, all elements of $e \vdots f$ are strictly contained in $(c \vdots d)_L$. This means that $e \vdots f$ is disjoint from $a \vdots b$ and, hence, $a \vdots b$ is a No interval of R_y as was to be shown. $\square$

Because of having to test whether infinitely many intervals intersect another set of infinitely many intervals, it may be impossible to establish that two oracles are equivalent. Instead, one can say that they are $a \vdots b$ **compatible** for a given interval $a \vdots b$ if $a \vdots b$ is a Yes interval for both oracles. All oracles share Yes intervals as can be seen by intervalizing the union of two of their separate Yes intervals. For example, if $a \vdots b$ is a Yes interval of x, $c \vdots d$ is a Yes interval of y, and the betweenness relation of $a \vdots \{b, c\} \vdots d$ holds, then the two oracles are $a \vdots d$ compatible.

Given a rational number q, the various Oracles of q are all equivalent as oracles. This is easy to see by considering their prophecies as being a subset of the intervals that contain q.

Indeed, the prophecies of the Singular Oracle of q are all of the form $q \vdots q$. The prophecies of the Halo Oracle of q are of the form $(q)_\delta$ and does not include $q \vdots q$. The prophecies of the Reflexive Oracle of q are of the form $a \vdots b$ where $q \in a \vdots b$. For those three oracles, every prophecy contains q. Thus, all the prophecies intersect and these are equivalent oracles. The associated rational betweenness relation is $a \vdots \widehat{q} \vdots b$ if and only if $q \in a \vdots b$.

An Oracle of q will refer to any oracle which is equivalent to the Singular Oracle of q.

A particular example to explore is $(\sqrt{2})^2$. While arithmetic will be defined later, what follows is a taste of it in this particular case. Let R be the oracle of this number. It should be an Oracle of 2. The arithmetic procedure leads to the following. Given $a \vdots b$ and δ, consider prophecies $c \therefore d$ and $e \therefore f$ of the $\sqrt{2}$ oracle that satisfy $|ce \vdots df| < \delta$ and $0 \therefore \{c, e\}$. It is the case that $ce \vdots 2 \vdots df$. Then $R(a \vdots b, \delta) = (k, ce \vdots df)$ where $k = 1$ if $ce \vdots df$ intersects $a \vdots b$ and $k = 0$ otherwise. After showing that this is an oracle, it is trivial to observe that all of the prophecies contain 2. Thus, this is an Oracle of 2.

One can also ponder comparing $(\sqrt{2})^2$ to $(\sqrt{1.9\ldots93})^2$ where the number of 9s is, say, 10^{23}. Perhaps the context is solving $f(x) = 0$ for some function f where one can use Newton's method to find the root, but not be able to explicitly produce it. It would be impractical to distinguish these computationally. This is an issue for all versions of real numbers though an interval approach has the advantage that it naturally gives the range of compatible numbers. Indeed, one can at least say that they are, for example, $1.9999999999 \vdots 2.000000001$ compatible. The previous paragraph's argument fails to apply in that it is not clear that all the prophecies of this uncertain x when multiplied together will contain 2 or simply contain a number very close to 2.

There is also a slightly different approach that can be used to establish equivalence of oracles. It is essentially that they agree on the No intervals.

Proposition 6.5. Let R_x and R_y be oracles such that whenever there is a prophecy of R_x that is disjoint from an interval $a \vdots b$, then there exists a prophecy of R_y that is also disjoint from $a \vdots b$. Then R_x and R_y are equivalent as oracles.

Proof. It is necessary to show that all the prophecies intersect. Let $a{\vdots}b$ be any prophecy of R_x and $c{\vdots}d$ be any prophecy of R_y. Either $a{\vdots}b$ and $c{\vdots}d$ intersect or they are disjoint. The unwelcome case is that of being disjoint. If they are disjoint, then, by the hypothesis, there is a prophecy of R_y disjoint from $c{\vdots}d$. But all prophecies of the oracle R_y must intersect each other. Thus, $a{\vdots}b$ and $c{\vdots}d$ cannot be disjoint. As these were arbitrary prophecies of their respective oracles, all of the prophecies must intersect. □

6.3 Rational Betweenness Relations Give Rise to the Reflexive Oracle

The above has established that given an oracle, there is a rational betweenness relation associated with it. The other direction needs to be established as well. While there is not a unique oracle given a rational betweenness relation, there is a canonical one.

Given a rational betweenness relation ${\vdots}\underline{x}{\vdots}$, the **Reflexive Oracle of** x is defined by the procedure $R(a{\vdots}b, \delta)$ is $(1, a{\vdots}b)$ if $a \,{\vdots}\underline{x}{\vdots}\, b$ and is (0) otherwise. This procedure is single-valued and the result is independent of δ. As all the prophecies of R are the Yes intervals of the relation, it should be clear that the oracle does generate this rational betweenness relation.

It does have to be shown that the procedure defined here is an oracle. A few facts need to be established first before proving the procedure is an oracle.

Proposition 6.6. Given a rational betweenness relation ${\vdots}\underline{x}{\vdots}$, it is the case that if $a \,{\vdots}\underline{x}{\vdots}\, b$ and $c \,{\vdots}\underline{x}{\vdots}\, d$, then $a{\vdots}b$ and $c{\vdots}d$ intersect and their intersection is an x-interval.

While this is basically a postulate for oracles, it is a deduction for rational betweenness relations.

Proof. The only case, after relabeling, of the two intervals possibly being disjoint is $a{\vdots}b{\vdots}m{\vdots}c{\vdots}d$. It will be shown that $b = m = c$. This will then imply that the intervals intersect.

By RBR Consistency, $a \,{\vdots}\underline{x}{\vdots}\, d$. By RBR Separation using m, exactly one of the following holds true: 1) $a \,{\vdots}\underline{x}{\vdots}\, m$ with $\cancel{m \,{\vdots}\underline{x}{\vdots}\, d}$, 2) $d \,{\vdots}\underline{x}{\vdots}\, m$ with $\cancel{m \,{\vdots}\underline{x}{\vdots}\, a}$, or 3) $m \,{\vdots}\underline{x}{\vdots}\, m$. By RBR Consistency, since $c \,{\vdots}\underline{x}{\vdots}\, d$, it is the case that $m \,{\vdots}\underline{x}{\vdots}\, d$. This rules out case 1. Also since $a \,{\vdots}\underline{x}{\vdots}\, b$, it is also the case by RBR Consistency that $a \,{\vdots}\underline{x}{\vdots}\, m$. This rules out case 2. Thus, case 3 must hold which is $m \,{\vdots}\underline{x}{\vdots}\, m$.

Next use RBR Separation on $a \,{\vdots}\underline{x}{\vdots}\, m$ with b as the Separation point. Since $a \,{\vdots}\underline{x}{\vdots}\, b$ by assumption and $b \,{\vdots}\underline{x}{\vdots}\, m$ by RBR Consistency using $m \,{\vdots}\underline{x}{\vdots}\, m$, it must be the case that $b \,{\vdots}\underline{x}{\vdots}\, b$. But the RBR Singular property would then assert that $m = b$. The same argument applies for c. Therefore, the intervals intersect at $b = m = c$.

Having established that they intersect, the task is to show the intersection is an x-interval. If one interval is contained in the other, then the intersection is that given x-interval. By relabeling, the other case is that of $a{\vdots}c{\vdots}b{\vdots}d$. From the given information and RBR Consistency, $a \,{\vdots}\underline{x}{\vdots}\, d$. Apply RBR Separation to $a{\vdots}c{\vdots}d$. As $c \,{\vdots}\underline{x}{\vdots}\, d$, either $c \,{\vdots}\underline{x}{\vdots}\, c$ or $\cancel{a \,{\vdots}\underline{x}{\vdots}\, c}$. If $c \,{\vdots}\underline{x}{\vdots}\, c$, then $c \,{\vdots}\underline{x}{\vdots}\, b$ by RBR Consistency. In the case of $\cancel{a \,{\vdots}\underline{x}{\vdots}\, c}$, apply RBR Separation to $a{\vdots}c{\vdots}b$ which immediately yields $c \,{\vdots}\underline{x}{\vdots}\, b$ as was to be shown.

□

Another proposition that is needed is that No intervals are disjoint from some Yes interval.

Proposition 6.7. Given a rational betweenness relation $\vdots\underline{x}\vdots$ and $\cancel{a\vdots b}$, then there exists $c \vdots\underline{x}\vdots d$ disjoint from $a\vdots b$.

Proof. By RBR Existence, there exists an RBR x-interval. By RBR Consistency, since $\cancel{a\vdots b}$, every subinterval of $a\vdots b$ cannot be an x-interval. In particular, both $a\vdots a$ and $b\vdots b$ are not x-intervals. By the RBR Closed property, there exists x-intervals $e\vdots f$ and $e'\vdots f'$ such that a and b are excluded from those intervals, respectively. From above, the two x-intervals must intersect in an x-interval. That intersection will either be disjoint from $a\vdots b$ or contained in it since it must exclude both a and b. Since the hypothesis prevents being contained, it must be disjoint. □

Proposition 6.8. Given the rational betweenness relation $\vdots\underline{x}\vdots$, the associated Reflexive procedure is an oracle.

Proof. The first step is to describe the set $\mathbb{I}_R$. If $a\vdots b \in \mathbb{I}_R$, then there must be an x-related interval $c\vdots d$ contained in $a\vdots b$. But by RBR Consistency, this implies $a \vdots\underline{x}\vdots b$. Thus, $a\vdots b \in \mathbb{I}_R$ if and only if $a \vdots\underline{x}\vdots b$ if and only if $R(a\vdots b, \delta) = (1, a\vdots b)$ for all δ if and only if $R(a\vdots b, \delta) \neq 0$ for all δ. This implies the set $\mathbb{I}_R$ is the set of prophecies itself and these are the Yes intervals of the relation $\vdots\underline{x}\vdots$.

1. Oracle Range. Given $a\vdots b$ and δ, $R(a\vdots b, \delta)$ is either 0 or $(1, a\vdots b)$ which certainly intersects $a\vdots b$ and is a subinterval of $(a\vdots b)_\delta$. If the procedure yields 0, this implies that $\cancel{a \vdots\underline{x}\vdots b}$. As shown above, there does exist an x-interval disjoint from $a\vdots b$ in this case. Taking the required δ' to be the distance between the two intervals leads to no x-interval whose length is less than that distance intersecting $a \vdots\underline{x}\vdots b$ as the x-intervals have to intersect.

2. Oracle Existence. RBR existence yields an interval $a\vdots b$ such that $a \vdots\underline{x}\vdots b$. Thus, $R(a\vdots b, 1) = (1, a\vdots b)$. As this is the only output for this input, $R(a\vdots b, 1) \neq 0$.

3. Oracle Separation. Let $a\vdots b$ be a prophecy of R, m contained in $a\vdots b$, and $\delta > 0$ be given. By RBR Separation, one of the following holds true: 1) $a \vdots\underline{x}\vdots m$ with $\cancel{m \vdots\underline{x}\vdots b}$, 2) $b \vdots\underline{x}\vdots m$ with $\cancel{m \vdots\underline{x}\vdots a}$, or 3) $m \vdots\underline{x}\vdots m$. If 1) holds true, then $R(a\vdots m, \delta) = (1, a\vdots m)$. If 2) holds true, then $R(b\vdots m, \delta) = (1, b\vdots m)$. If 3) holds true, then both $a \vdots\underline{x}\vdots m$ and $m \vdots\underline{x}\vdots b$ holds true. Thus, $R(a\vdots m, \delta) = (1, a\vdots m)$ and $R(b\vdots m, \delta) = (1, b\vdots m)$.

4. Oracle Disjointness. Assume $a\vdots b \in \mathbb{I}_R$, i.e, $a \vdots\underline{x}\vdots b$, and that $c\vdots d$ is disjoint from it. Since all Yes intervals of the relation intersect as shown above, $c\vdots d$ is a No interval. By definition of the procedure, $R(c\vdots d, \delta) = \emptyset$ for all δ.

5. Oracle Consistency. If $a\vdots b \in \mathbb{I}_R$, then by definition of the procedure, $R(a\vdots b, \delta) = (1, a\vdots b)$ for all $\delta > 0$ and thus $R(a\vdots b) \neq 0$.

6. Oracle Closed. Assume $(c)_\delta \in \mathbb{I}_R$ for all δ which implies $(c)_\delta$ is an x-related interval of the relation. Let $a \mathrel{\vdots\underline{x}\vdots} b$. If c is an endpoint, then $c \in a{\vdots}b$. Otherwise, let L be the distance from c to the closest endpoint. Consider $(c)_\delta$ for $\delta < L$. It cannot contain either of the endpoints. Thus, it is either contained in $a{\vdots}b$ or disjoint from it. But since $(c)_\delta$ is an x-related interval, it must intersect the x-related interval $a{\vdots}b$. Therefore, it is contained in it. The conclusion is that c is in every x-related interval implying by RBR Closed that $c \mathrel{\vdots\underline{x}\vdots} c$. This in turn implies by RBR Consistency that every interval that contains c is a an RBR Yes interval and hence a Yes interval for the oracle. In particular, $R(c{\vdots}b, \delta) \neq 0$ for all b and δ.

7. Oracle Reasonableness. Nothing ever replies with -1.

$\square$

It has now been established that every rational betweenness relation arises from at least one oracle and every oracle gives rise to exactly one rational betweenness relation.

7 Establishing the Real Number Field

In this section, the task is to establish that the rational betweenness relations with the appropriate inequality relations and field operations are a model for the complete real number field. The approach is to define these relations and operations on oracles and establish that they respect the equivalence relation on the oracles. While in a theoretical sense it seems as if this filters the operations from the RBR to the oracles and then back to the RBRs, in practice, it is oracles that are generally at hand to use.

As a note, equality of RBR is as relations and equality of oracles is via the equivalence relation as already established. In this section, the oracle R_x will often be implicitly linked to the associated relation x, with notation switching as the need arises. Thus, $x = y$ or $R_x \equiv R_y$ may be used to convey equality. Both $a \mathrel{\vdots\widehat{x}\vdots} b$ and $a \mathrel{\vdots\underline{x}\vdots} b$ may be used interchangeably and will alter depending on whether oracles or relations are being emphasized.

7.1 Inequality

Two oracles R_x and R_y are inequivalent if there are disjoint intervals $a{\vdots}b$ and $c{\vdots}d$ such that $a \mathrel{\vdots\widehat{x}\vdots} b$ and $c \mathrel{\vdots\widehat{y}\vdots} d$. Disointness then implies $\cancel{a \mathrel{\vdots\widehat{y}\vdots} b}$ and $\cancel{c \mathrel{\vdots\widehat{x}\vdots} d}$. It is clear that the oracles are not equivalent as they do not have the same Yes/No intervals. These disjoint intervals then allow for a comparison of oracles.

On the interval level, $a{\vdots}b < c{\vdots}d$ means that for every $p \in a{\vdots}b$ and every $q \in c{\vdots}d$, it is the case that $p < q$. Necessarily, the intervals must be disjoint to have this be possible and, for any two disjoint intervals, they will be related by one inequality.

We define the inequality relations as:

1. $R_x < R_y$ and $x < y$, if there exists $a \mathrel{\vdots\widehat{x}\vdots} b$ and $c \mathrel{\vdots\widehat{y}\vdots} d$ such that $a{\vdots}b < c{\vdots}d$.

2. $R_x > R_y$ and $x > y$, if there exists $a \vdots\widehat{x}\vdots\, b$ and $c \vdots\widehat{y}\vdots\, d$ such that $a \vdots b > c \vdots d$.

If the intervals $a \vdots b$ and $c \vdots d$ were not disjoint, then oracle equivalence is possible. If it can be shown that all Yes intervals of x are not greater than any of the Yes intervals of y, then that is noted as $x \leq y$. This would translate to the interval level as given any x-Yes interval $a \iddots b$ and any y-Yes interval $c \iddots d$ that it must be the case that $a \leq d$.

The oracle inequality operation satisfies transitivity, relying on the transitivity of inequality for rational intervals which in turn relies on the transitivity of rational numbers.

Proposition 7.1 (Transitivity)**.** If $x < y$ and $y < z$, then $x < z$.

Proof. By the hypothesis, let $a \vdots\widehat{x}\vdots\, b$, $c \vdots\widehat{y}\vdots\, d$, $e \vdots\widehat{y}\vdots\, f$, and $g \vdots\widehat{z}\vdots\, h$ be such that $a \vdots b < c \vdots d$ and $e \vdots f < g \vdots h$. The goal is to show that $a \vdots b < g \vdots h$. Let r be in the intersection of $c \vdots d$ and $e \vdots f$. This exists as $c \vdots d$ and $e \vdots f$ are Yes intervals of the same oracle. Let $p \in a \vdots b$ and $q \in g \vdots h$. As $r \in c \vdots d$, $p < r$; as $r \in e \vdots f$, $r < q$. By the transitivity of the inequality of rational numbers, $p < q$. As this holds for all $p \in a \vdots b$ and $q \in g \vdots h$, it is the case that $a \vdots b < g \vdots h$. □

It does need to be shown that there can be no contradiction.

Proposition 7.2. If $x < y$, then it is not true that $x > y$ and it is also not true that $x = y$.

Proof. The issue is that the comparison is based on two particular Yes intervals. It needs to be shown that two other Yes intervals would not contradict this statement.

Let $a \iddots b < c \iddots d$ be given that exemplifies $x < y$. Then $a \vdots b$ is disjoint from $c \vdots d$. This implies that $a \vdots b$ is a No interval for y while $c \vdots d$ is a No interval for x. Thus, the two oracles are not equivalent as their rational betweenness relations differ.

The other task is to show there are no Yes intervals that yield the opposite inequality. Let $p \iddots q$ be a Yes interval for x and $r \iddots s$ be a Yes interval for y. If $p \vdots q$ and $r \vdots s$ overlap, then there is no contradictory information.

Assume, therefore, that they are disjoint. There are exactly two configurations for disjoint intervals: $p \iddots q < r \iddots s$ or $r \iddots s < p \iddots q$. Since $p \vdots q$ must intersect $a \vdots b$ due to both being Yes intervals of the same oracle, let m be in their intersection. Similarly, let n be in the intersection of $r \vdots s$ and $c \vdots d$. Since $a \vdots b < c \vdots d$, it is the case that $m < n$. This is only compatible with $p \iddots q < r \iddots s$ which was to be shown.

□

The classical story is that the real numbers satisfy the Trichotomy property:

Proposition 7.3. Let x and y be two oracles. Then, nonconstructively, exactly one of the following holds true: $x < y$, $x > y$, or $x = y$.

Proof. If there exists two disjoint intervals, one an x-Yes interval and one y-Yes interval, then exactly one of the inequalities holds. The other case is that every Yes interval of x intersects every Yes interval of y. Proposition 6.4 establishes that $x = y$ in that situation. □

The above was non-constructive as, generically, it requires potentially checking infinitely many intervals in the case of equality. The constructivists use a property called ε-Trichotomy. This allows a definite determination with a finite, predictable amount of calls to the oracles.

Proposition 7.4 (ε-Trichotomy)**.** Given oracles x and y and a positive rational ε, exactly one of the following holds: $x < y$, $x > y$, or there exists an interval $a \vdots b$ of length no more than ε such that $a \vdots b$ is a Yes interval for both x and y.

Proof. By the Bisection algorithm, there exists an x-Yes interval $c \vdots d$ and a y-Yes interval $e \vdots f$ such that both intervals have length less than half of ε. If $c \vdots d$ and $e \vdots f$ are disjoint, then the oracles are unequal with the inequality being that of the intervals. If $c \vdots d$ and $e \vdots f$ overlap, then their union is a Yes interval for both x and y. That interval has length less than ε. $\square$

It is also the case that given $x < y$, there exists a length such that all Yes intervals of x and y of that length are disjoint.

Proposition 7.5. If $x < y$, there exists δ such that if $a \vdots \widehat{x} \vdots b$, $c \vdots \widehat{y} \vdots d$, $|a \vdots b| < \delta$ and $|c \vdots d| < \delta$, then $a \vdots b < c \vdots d$

Proof. Assume $e \vdots \widehat{x} \vdots f$ and $g \vdots \widehat{y} \vdots h$ are such that $e \iddots f < g \iddots h$. Let $L = g - f$. Then $\delta < L/2$ will satisfy the requirements of the statement.

Let $a \vdots \widehat{x} \vdots b$ with $|a \vdots b| < \delta$ and $c \vdots \widehat{x} \vdots d$ with $|c \vdots d| < \delta$. By Proposition 5.20, $a \vdots b$ is contained in $(e \vdots f)_\delta$ and $c \vdots d$ is contained in $(g \vdots h)_\delta$. By choice of δ, $(e \vdots f)_\delta$ and $(g \vdots h)_\delta$ are disjoint and thus obey the same inequality as $e \vdots f$ and $g \vdots h$. Their subintervals are therefore disjoint and related in the same fashion. Thus, $a \vdots b < c \vdots d$. $\square$

In this section, the focus has been on the Yes intervals. It is also the case that if two oracles have Yes intervals that are disjoint, then there are prophecies of each that are disjoint from each other.

Corollary 7.6. If $x < y$, then there exists a prophecy $a \vdots b$ of R_x and a prophecy $c \vdots d$ of R_y such that $a \vdots b < c \vdots d$.

Proof. The Bisection Algorithm gives arbitrarily small prophecies and the previous proposition states that for small enough Yes intervals, they satisfy the same inequality and, therefore, are disjoint. $\square$

Thus, all the computations about inequalities can be done solely with the prophecies which may be a more concrete set of intervals to inspect. For example, this immediately establishes that if $q < r$, then the Oracle of q is less than the Oracle of r using the fact that the Singular Oracle version of these oracles have the sole prophecies of $q \vdots q$ and $r \vdots r$, respectively.

It is also important to establish that equivalent oracles have the same relation with non-equivalent oracles.

Proposition 7.7. Let $R_x \equiv R'_x$, $R_y \equiv R'_y$, and $R_x < R_y$. Then $R'_x < R'_y$.

Proof. Take $a \vdots\widehat{x}\vdots\, b$ and $c \vdots\widehat{y}\vdots\, d$ such that $a \vdots b < c \vdots d$; this exists by assumption of the inequality of R_x and R_y. It can also be assumed these are prophecies by the above corollary. Let δ be less than half the distance between $a \vdots b$ and $c \vdots d$. By the Bisection Algorithm, let $a' \vdots b'$ be a prophecy of R'_x whose length is less than δ and let $c' \vdots d'$ be a prophecy of R'_y whose length is less than δ. Since all prophecies of equivalent oracles intersect, it is the case that $a' \vdots b'$ intersects $a \vdots b$ and $c' \vdots d'$ intersects $c \vdots d$. Thus, by the length of the intervals, $a' \vdots b' \subset (a \vdots b)_\delta$ and $c' \vdots d' \subset (c \vdots d)_\delta$. By choice of δ, the halos are disjoint and satisfy $(a \vdots b)_\delta < (c \vdots d)_\delta$. Thus, $a' \vdots b' < c' \vdots d'$ and $R'_x < R'_y$. $\square$

With the oracle inequalities explored, the rational betweenness relations can be ordered in the following way. The relation $\vdots\underline{x}\vdots$ is less than $\vdots\underline{y}\vdots$ if an oracle representative R_x of $\vdots\underline{x}\vdots$ is less than an oracle representative R_y of $\vdots\underline{y}\vdots$. Due to the equivalence proposition, different representatives will lead to the same conclusion. By considering the Reflexive Oracle of the relations, it is clear that this inequality is the same one as obtained by saying that $\vdots\underline{x}\vdots < \vdots\underline{y}\vdots$ if and only if there exists $a \vdots\underline{x}\vdots\, b$ and $c \vdots\underline{y}\vdots\, d$ such that $a \vdots b < c \vdots d$.

Letting x, y, and z represent relations, the statement $x \vdots y \vdots z$ will mean that either $x \leq y \leq z$ or $z \leq y \leq x$. That is, y is between x and z. This notation can be extended as was done with rational numbers.

Proposition 7.8. If $a \vdots\underline{x}\vdots\, b$, then $\vdots\underline{a}\vdots \,\vdots\, \vdots\underline{x}\vdots \,\vdots\, \vdots\underline{b}\vdots$.

That is, if $a \vdots b$ is a x-Yes interval, then x is between a and b.

Proof. By Proposition 5.7, either $a \vdots b$ strictly contains a prophecy of R_x or an endpoint is a root of the oracle. If the latter, say a by relabeling, then $x = a$ as relation.

If $a \vdots b$ strictly contains a prophecy $c \vdots d$, then $a \vdots a$, $c \vdots d$, and $b \vdots b$ are disjoint. This implies either $a \vdots a < c \vdots d < b \vdots b$ or $a \vdots a > c \vdots d > b \vdots b$. Either way, the conclusion follows. $\square$

Notationally, if $a \iddots b$ is an x-Yes interval, then $a \leq x \leq b$ is understood as being the true statement $\vdots\underline{a}\vdots \leq \vdots\underline{x}\vdots \leq \vdots\underline{b}\vdots$.

7.2 Completeness

Completeness is a defining feature of real numbers. It can come in a variety of guises as wonderfully detailed by James Propp in [Pro13]. While any of the equivalent versions could be used, this paper will go with the choice Propp suggests as a good foundation: the Cut property. It is a simplified and symmetrized version of the least upper bound property.

For this section, the letters x and y will represent relations. That is, x is used instead of $\vdots\underline{x}\vdots$. For rational numbers, the symbols such as q will also represent the relation version $\vdots\underline{q}\vdots$. Thus, rational $q \in A$ means the relation $\vdots\underline{q}\vdots$ is an element of the set of relations A.

Theorem 7.1 (The Cut Property)**.** Let A and B be two disjoint, nonempty sets of rational betweenness relations such that $A \cup B$ is the whole set of such relations. In addition, for all $x \in A$ and for all $y \in B$, it is the case that $x < y$. Then there exists a rational betweenness relation κ such that whenever $x < \kappa < y$, it will be the case that $x \in A$ and $y \in B$.

Note that if $a \leq b$ and $a - \delta \in B$, then $(a⁝b)_\delta$ is entirely contained in B. Similarly, if $b+\delta \in A$, then $(a⁝b)_\delta$ is entirely contained in A. If neither of these hold, then $a-\delta \in A$, $b+\delta \in B$, and $(a⁝b)_\delta$ has elements of both A and B in it. It is also useful to note that if $a \in A$, then $a - \delta \in A$ and if $b \in B$, then $b + \delta \in B$ for all δ.

Proof. This property can be established for the rational betweenness relations by using a single-valued procedure which returns intervals that intersect both A and B. In particular, given $a ⋰ b$ and 2δ, if $a - \delta \in A$ and $b + \delta \in B$, then $R(a⁝b, 2\delta) = (1, [a⁝b]_\delta)$. Otherwise, $R(a⁝b, 2\delta) = (0)$. The expansion beyond $a⁝b$ facilitates dealing with rational cut points as detailed in the Closed property.

To establish this is an oracle, the properties follow by:

1. Range. By definition, the prophecy for a Yes answer is contained in the appropriate halo and intersects the given interval. As for a No answer, this happens if either $a - \delta \in B$ or $b + \delta \in A$. In either case, choosing $\delta' = \delta$ leads to $(a⁝b)_{\delta'}$ being strictly contained in B or A, respectively. Thus, no prophecy can be contained in it.

2. Existence. Let $a \in A$ and $b \in B$. These exist by assumption. Thus, $R(a⁝b, 2) = (1, [a⁝b]_1)$.

3. Separation. Let a prophecy $a ⋰ b$ be given. This implies $a \in A$, $b \in B$, and, due to the disjointness of A and B, $a \neq b$. Let m in $a⁝b$ and 2δ be given.

   If $m \in A$, then $m-\delta \in A$ and $b+\delta \in B$. Thus, $R(m⁝b, 2\delta) = (1, [m⁝b]_\delta)$. Similarly, if $m \in B$, then $m + \delta \in B$, $a - \delta \in A$ and $R(a⁝m, 2\delta) = (1, [a⁝m]_\delta)$.

   This is what was to be shown.

4. Disjointness. If $a ⋰ b$ is a prophecy, then $a \in A$ and $b \in B$. If $c⁝d$ is disjoint from $a⁝b$, then let 2δ be less than the distance from $a⁝b$ to $c⁝d$. If $a⁝b < c⁝d$, then $(c⁝d)_\delta$ is entirely contained in B; if $a⁝b > c⁝d$, then $(c⁝d)_\delta$ is entirely contained in A. In either case, $R(a⁝b, 2\delta) = 0$.

5. Consistency. Assume $a ⋰ b$ contains a prophecy $c ⋰ d$. This implies $a \in A$ and $b \in B$ as $a ⋰ c ⋰ d ⋰ b$. Thus, $a-\delta \in A$ and $b+\delta \in B$ for any δ. Hence, $R(a⁝b, 2\delta) = (1, [a⁝b]_\delta)$ for all δ and $R(a⁝b) \neq 0$.

6. Closed. Assume $(a)_\delta$ contains a prophecy for all δ. Let b and 2δ be given. The task is to show that $R(a⁝b, \delta) \neq 0$. As $(a)_\delta$ contains a prophecy, $a - \delta \in A$ and $a + \delta \in B$. If $b \leq a$, then $b - \delta \leq a - \delta$ and so $b - \delta \in A$. Thus, the procedure returns $(1, [a⁝b]_\delta)$. If $b \geq a$, then $b + \delta \geq a + \delta$ and so $b + \delta \in B$. Thus, the

procedure returns $(1, [a \vdots b]_\delta)$. In all cases, 0 is not returned and a is a root of the oracle.

7. Reasonableness. Nothing gets returned with the -1.

Having established that κ is an oracle, it now has to be established that it satisfies being a cut point. Let $x < \kappa < y$. That statement implies the existence of the prophecies $a \vdots \underline{x} \vdots b$, $c \vdots \underline{\kappa} \vdots d$, and $e \vdots \underline{y} \vdots f$ such that $a \therefore b < c \therefore d < e \therefore f$. Being a prophecy of κ, $c \therefore d$ has the property that $c \in A$ and $d \in B$. Since $c > b$, $b \in A$ and since $x \leq b$, $x \in A$. Similarly, $d < e$ implies $e \in B$ and $y \geq e$ implies $y \in B$.

$\square$

Having shown that the Cut property holds, the rational betweenness relations satisfy all the equivalent completeness properties.

The above assumes that membership in the sets A and B can always be determined. It is possible to handle a situation in which the boundary between A and B is fuzzier. The oracles are particularly helpful here as they come with a bit of fuzziness built in. The assumption on the sets is that given a fuzziness tolerance, any number can be either determined to be in one of the sets or is in an interval whose length is less than the tolerance whose endpoints are in the opposing sets from each other.

In particular, the governing assumption on the fuzzy cut sets is that given x and $\varepsilon > 0$, then either $x \in A$, $x \in B$, or there exists an x-interval whose length is less than ε and straddles the sets, i.e., contains elements of both sets. For fuzzy cut sets, the procedure $R(a \therefore b, 2\delta)$ for the cut oracle is defined in the following way. By the nature of A and B, either 1) $a - \delta \in A$, 2) $a - \delta \in B$, or 3) there exists an $a - \delta$-interval whose length is less than δ which straddles the sets. Due to the lengths, in case 3, $a \in B$ implying there is a δ' such that $(a)_{\delta'}$ is entirely in B. Thus, for case 2 or 3, the procedure returns 0. Similarly, for b, the cases are 1) $b + \delta \in B$, 2) $b + \delta \in A$, or 3) there exists an $b + \delta$-interval whose length is less than δ which straddles the sets. Due to the lengths, in case 3, $b \in A$ implying there is a δ' such that $(b)_{\delta'}$ is entirely in A. Thus, for case 2 or 3, the procedure returns 0. Otherwise, case 1 holds for both and the procedure returns $(1, a - \delta \vdots b + \delta)$ just as before. The verification that this procedure is an oracle should be similar to the above. One key property of the prophecies is that it is still the case that if $a \therefore b$ is a prophecy, then $a \in A$ and $b \in B$.

7.3 Arithmetic

For arithmetic, the operators of addition and multiplication need to be defined. Then it needs to be shown that the usual properties hold including the existence of additive and multiplicative identities and inverses, as appropriate.

The idea of arithmetic with oracles is to apply the operations to the intervals. The ideal approach would be that the Yes intervals of $x + y$ would be the intervals that result from adding Yes intervals of x to those of y. This almost works, but it fails with when two irrationals combine to give a ration. For example, consider $\sqrt{2} - \sqrt{2}$. This ought to be 0. All the Yes intervals of $\sqrt{2}$ added to those of $-\sqrt{2}$ do lead to intervals that contain 0. But there is no interval of the form $0 \vdots b$ that is a result of subtracting two Yes intervals of $\sqrt{2}$. This is where having the oracles becomes useful. The prophecies of R_{x+y} will consist of the result of combining the prophecies of R_x and R_y with the arithmetic operator, but with that scenario, the omission of $0 \vdots b$ intervals is not an issue as they can be deduced in the step going from the prophecies of R to the Yes intervals. The Yes step is theoretically doable, but may not always be actionable. For example, if solving $f(x) = 0$ with Newton's method to something that looks like $\sqrt{2}$ but cannot be proven as such, then $x - \sqrt{2}$ would be seen as compatible with 0, but could not be proven to be so via any finite computation.

Interval arithmetic is largely that of doing the operation on the endpoints. For addition, $a \iddots b \oplus c \iddots d = (a+c) \iddots (b+d)$. The length of the new interval is $|a \vdots b \oplus c \vdots d| = |(b+d) - (a+c)| = |b - a + c - d| = |a \vdots b| + |c \vdots b|$.

For multiplication, $a \vdots b \otimes c \vdots d = \min(ac, ad, bc, bd) \iddots \max(ac, ad, bc, bd)$. For $0 \iddots a \iddots b$ and $0 \iddots c \iddots d$, the interval multiplication becomes $ac \iddots bd$. The length computation is a bit more complicated than for arithmetic. Let M be an absolute bound for $a \vdots b$ and $c \vdots d$; an absolute bound on an interval is a number such that for any p in the interval, $|p| \leq M$. Then $|a \vdots b \otimes c \vdots d| \leq M(|a \vdots b| + |c \vdots d|)$. This follows by considering cases. The first case is that of all four endpoints being involved. By relabeling in this case, $ac \iddots bd$ can be taken to be the multiplication interval leading to $|bd - ac| = |bd - cb + cb - ac| = |b(d-c) + c(b-a)| \leq M(|a \vdots b| + |c \vdots d|)$. The second case is that of three of the endpoints being involved which implies one of them is repeated. This may happen, for example, if an interval contains 0. By relabeling, $ac \iddots ad$ can be taken to be the multiplication interval leading to $|ad - ac| = |a(d-c)| \leq M(|c \vdots d|) \leq M(|c \vdots d| + |a \vdots b|)$. The third case is that of only two endpoints being used. By relabeling, $ac \vdots ac$ can be taken to be the interval with a length of 0 which is less than or equal to any non-negative quantity. That last case is the scenario of multiplying two singletons.

Operating on subintervals of $a \vdots b$ and $c \vdots d$ yields a subinterval of the resulting interval operation on $a \vdots b$ and $c \vdots d$. This is because if $p \in a \vdots b$ and $q \in c \vdots d$, then $p + q \in a \vdots b \oplus c \vdots d$ and $p \times q \in a \vdots b \otimes c \vdots d$.

Most of the arithmetic properties hold for interval arithmetic, but the distributive property does not nor are there any additive and multiplicative inverses. The additive identity is $0 \vdots 0$ while the multiplicative identity is $1 \vdots 1$.

The negation operator is $\ominus a \vdots b = -a \vdots -b$; this is not the additive inverse as $a \vdots b \oplus (\ominus(a \vdots b)) = (a-b) \vdots (b-a)$ which does contain 0, but it is not $0 \vdots 0$ unless $a = b$.

The reciprocity operator is $1 \oslash a \vdots b = 1/a \vdots 1/b$ though this only applies to intervals excluding 0. If 0 was included, with $a < 0$, the resulting set would be $-\infty \vdots 1/a \cup 1/b \vdots \infty$ which is not an interval as used here. This is not the multiplicative inverse, but when multiplied with the original interval, it does contain 1. To see this, first consider the case of $0 \mathrel{\circ\!\!\!\iddots} a \iddots b$ which implies $0 \mathrel{\circ\!\!\!\iddots} 1/b \iddots 1/a$ Then $a \vdots b \otimes (1 \oslash a \vdots b) = a/b \iddots b/a$; since $a \leq b$, $a/b \leq b/b = 1$ and $b/a \geq a/a = 1$. Thus, the interval contains 1. In the case of $a \iddots b \mathrel{\circ\!\!\!\iddots} 0$, it is the case that $0 \mathrel{\circ\!\!\!\iddots} |b| \iddots |a|$. As $a/b = |a|/|b|$ and $b/a = |b|/|a|$, the interval $a \vdots b \otimes (1 \oslash a \vdots b) = |b| \iddots |a| \otimes (1 \oslash |b| \vdots |a|)$. Applying the first case, that interval includes 1. Neither case yields the singleton $1 \vdots 1$ unless $a = b$.

The distributive property is replaced with $a\vdots b \otimes (c\vdots d \oplus e\vdots f) \subset (a\vdots b \otimes c\vdots d) \oplus (a\vdots b \otimes e\vdots f)$. To see this, note that the left-side has boundaries chosen from $\{a(c+e), a(d+f), b(c+e), b(d+f)\}$ while the second has a boundary of the form $(\min(ac, ad, bc, bd) + \min(ae, af, be, bf))\vdots(\max(ac, ad, bc, bd) + \max(ae, af, be, bf))$. To demonstrate that they are indeed not equal for some examples, consider $2\vdots 3 \otimes (4\vdots 7 \oplus -6\vdots -3) = 2\vdots 3 \otimes -2\vdots 4 = -6\vdots 12$ compared to $(2\vdots 3 \otimes 4\vdots 7) \oplus (2\vdots 3 \otimes -6\vdots -3) = 8\vdots 21 \oplus -18\vdots -6 = -10\vdots 15$.

For more on interval analysis, see, for example, [MR09] or, in this context, [Tay23a]. In [MR09], there is also a discussion of the difference between $a\vdots b \otimes a\vdots b$ and a potential understanding of $(a\vdots b)^2 = \{c^2 | c \in a\vdots b\}$. The former will include all products which for intervals containing 0 will include negatives. The latter will only include nonnegative results. Oracle arithmetic will only use the former, but it ultimately does not matter as the intervals of main concern are arbitrarily small intervals which will either be away from 0 or one is squaring the oracle of 0 which comes out to 0 in any event.

Having discussed interval arithmetic, oracle arithmetic can now be defined based on that. The essential extra ability is that of being able to use narrower intervals. Much of the discussion will be the same for both arithmetic and multiplication. To facilitate that, the symbol $\odot$ will be used to represent either $\oplus$ or $\otimes$ operating on intervals and $\cdot$ will then be used for the corresponding operator operating on individual numbers.

Let oracles R_x and R_y be given. One way to define an oracle $R_{x\cdot y}$ is the following. Given $a\vdots b$ and δ, take an interval $c\vdots d = (e\vdots f) \odot (g\vdots h)$ such that $e \vdots \underline{x} \vdots f$, $g \vdots \underline{y} \vdots h$, and $|c\vdots d| < \delta$. Then $R_{x\cdot y}(a\vdots b, \delta) = (k, c\vdots d)$ where $k = 1$ if $c\vdots d$ intersects $a\vdots b$ and $k = 0$ if $c\vdots d$ does not intersect $a\vdots b$.

The existence of an interval of the form $c\vdots d$ follows from the computational bounds on the arithmetic operators as well as the Bisection Algorithm applied to R_x and R_y. For addition, that choice is ensured by choosing the lengths of each to be less than $\delta/2$. For multiplication, let M be a bound on some fattened Yes intervals of R_x and R_y such that they strictly contain other Yes intervals. Then use the Bisection Algorithm on R_x and R_y to obtain Yes intervals whose lengths are less than $\delta/(2M)$ and are strictly contained in the bounding Yes intervals.

The proof that the above is an oracle is straightforward, but longer than desired and not different than what has been done earlier. Instead, the path chosen below is to recognize that when the prophecies of two oracles are combined either under addition or multiplication, the newly formed set is a fonsi. The oracle and relation then follows from the fonsi.

Lemma 7.9. *Let $a\vdots b, a'\vdots b' \in R_x$, $c\vdots d, c'\vdots d' \in R_y$, $I = a\vdots b \odot c\vdots d$, $I' = a'\vdots b' \odot c'\vdots d'$. Then I and I' intersect. Furthermore, if $|I'| < \delta$, then $I' \subset (I)_\delta$.*

Proof. As prophecies of the same oracle intersect, let p be in the intersection of $a\vdots b$ and $a'\vdots b'$ and let q be in the intersection of $c\vdots d$ and $c'\vdots d'$. Then $p \cdot q = r$ is contained in both I and I'.

If the length of $|I'| < \delta$, then the distance of any element in I' is less than δ from r. As r is in I, any element of I' is within δ of an element of I. This means that $I' \subset (I)_\delta$. $\square$

Proposition 7.10. Given oracles R_x, R_y, the set $\mathcal{S} = \{a{:}b \oplus c{:}d | a{:}b \in R_x, c{:}d \in R_y\}$ is a fonsi.

Proof. Let $\delta > 0$ be given.

Choose δ' and δ'' such that $\delta' + \delta'' < \delta$, such as $\frac{\delta}{3}$ for both of them.

Choose intervals $a{:}b \in R_x$ and $c{:}d \in R_y$ such that $|a{:}b| < \delta'$ and $|c{:}d| < \delta''$. This is allowed by the Bisection Algorithm. Let $e{:}f = a{:}b \oplus c{:}d$.

Earlier, the interval length of a sum was computed: $|a{:}b \oplus c{:}d| \leq |b-a| + |c-d| < \delta' + \delta'' < \delta$. Therefore, there are arbitrarily small summed prophecies in $\mathcal{S}$.

By the lemma, any pair of summed prophecies in $\mathcal{S}$ intersect. Thus, it is a fonsi. $\square$

The associated oracle with this fonsi will be denoted as R_{x+y}; it is equivalent to the oracle sketched previously as can be seen by comparing prophecies. The associated rational betweenness relation will simply be written as $x + y$.

Proposition 7.11. Given oracles R_x, R_y, the set $\mathcal{P} = \{a{:}b \otimes c{:}d | a{:}b \in R_x, c{:}d \in R_y\}$ is a fonsi.

Proof. Let $\delta > 0$ be given. Let intervals $e'{:}f' \in R_x$ and $e''{:}f'' \in R_y$ be chosen as allowed by the existence property. Let $M = \max(|e'|, |f'|, |e''|, |f''|) + 1$.

Choose δ' and δ'' such that $M(\delta' + \delta'') < \delta$, such as $\frac{\delta}{3M}$ for both of them.

Choose intervals $a{:}b \in R_x$ and $c{:}d \in R_y$ such that $|a{:}b| < \min(\delta', 1)$ and $|c{:}d| < \min(\delta'', 1)$. This is allowed by the Bisection Algorithm. As all prophecies of length no greater than 1 are within the 1-halo of a given prophecy, it is the case that $|a|, |b|, |c|, |d| < M$.

Earlier, the interval length of a product interval was shown to have the bound $|a{:}b \otimes c{:}d| \leq M(|b-a| + |c-d|) < M(\delta' + \delta'') < \delta$. Therefore, there are arbitrarily small product prophecies in $\mathcal{P}$.

By the lemma, any pair of product prophecies in $\mathcal{P}$ intersect. Thus, it is a fonsi. $\square$

The associated oracle with this fonsi will be denoted as R_{xy}; it is equivalent to the oracle sketched previously as can be seen by comparing prophecies. The associated rational betweenness relation will simply be written as xy. In the case that the lack of a symbol is awkward, the notation $x * y$ may be used as well.

One also has to establish that different oracles for a real number lead to an equivalent oracle under these operations.

Proposition 7.12. Let $R_x \equiv R'_x$ and $R_y \equiv R'_y$, then $R_{x \cdot y} \equiv R'_{x \cdot y}$.

Proof. The task is to show that the prophecies of $R_{x \cdot y}$ intersect the prophecies of $R'_{x \cdot y}$. By the equivalence, all the prophecies of R_x intersect all the prophecies of R'_x and all the prophecies of R_y intersect the prophecies of R'_y. Let $a{:}b \in R_{x \cdot y}$ which means there exists $c{:}d \in R_x$ and $e{:}f \in R_y$ such that $a{:}b = c{:}d \odot e{:}f$. Similarly, let $a'{:}b' \in R'_{x \cdot y}$ with $c'{:}d' \in R'_x$, $e'{:}f' \in R'_y$ such that $a'{:}b' = c'{:}d' \odot e'{:}f'$. Then by assumption of the equivalences, there exists p that is in both $c{:}d$ and $c'{:}d'$ along with a q in both $e{:}f$ and $e'{:}f'$. Then $p \cdot q$ will be in both $a{:}b$ and $a'{:}b'$. Thus, they intersect. This shows that all prophecies of $R_{x \cdot y}$ intersect all the prophecies of $R_{x' \cdot y'}$ which implies they are equivalent. $\square$

The equivalence implies that they represent the same completed rational betweenness relation.

Having established that these are oracles, it is necessary to check the field properties. The key fact to use is that if two oracles have prophecies that are always intersecting, let alone being equal, then they are equivalent oracles and their corresponding rational betweenness relations are identical.

1. Commutativity. To show $x \cdot y = y \cdot x$, the interval fact that $a{:}b \odot c{:}d = c{:}d \odot a{:}b$ establishes that the two have the same set of prophecies and hence are equivalent. The interval equality follows from the commutativity of $\cdot$ on the rationals.

2. Associativity. To show $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, the interval fact to use is that $(a{:}b \odot c{:}d) \odot e{:}f = a{:}b \odot (c{:}d \odot e{:}f)$. This establishes that the two have the same set of prophecies and hence are equivalent. The interval equality follows from the associativity of $\cdot$ on the rationals.

3. Identities. Let R_0 be the Singular Oracle of 0 and R_1 be the Singular Oracle of 1. The task is to show $x + 0 = x$ and $1x = x$. Since $a{:}b \oplus 0{:}0 = a{:}b$, the prophecies of R_{x+0} and R_x are identical. As $a{:}b \otimes 1{:}1 = a{:}b$, the prophecies of R_{x*1} and R_x are identical. This establishes the additive identity as 0 and the multiplicative identity as 1.

4. Distributive Property. As previously mentioned, $(a{:}b \otimes (c{:}d \oplus e{:}f) \subset (a{:}b \otimes c{:}d) \oplus (a{:}b \otimes e{:}f)$. Thus, the prophecies of $R_{x(y+z)}$ are contained in the prophecies of R_{xy+yz}. Since that means they intersect, these are equivalent oracles and the Distributive Property holds.

5. Additive Inverse. The additive inverse for x can be given by the procedure $R_{-x}(a{:}b, \delta) = (1, \ominus(c{:}d))$ where $R_x(\ominus(a{:}b), \delta) = (1, c{:}d)$ and is (0) if R_x is 0 for those inputs. It can be shown that it is an oracle basically because the procedure returns 0 under the same conditions as the original procedure.

   Alternatively, given the fonsi that is the set of prophecies of R_x, define the additive inverse fonsi as $\{-a{:} -b | a{:}b \in R_x\}$. This is a fonsi as $|-a{:} -b| = |a{:}b|$ and $q \in a{:}b$ if and only if $-q \in -a{:} -b$ which implies that the intersection of two intervals is non-empty if and only if the intersection of the two negated intervals is.

   By either method, there is an oracle R_{-x} whose prophecies are the negated versions of the prophecies of R_x. The corresponding rational betweenness relation will be denoted by $-x$.

To verify that this is the additive inverse, compute $x+(-x)$ by looking at $a\vdots b \oplus c\vdots d$ where $a \therefore b \in R_x$ and $c \therefore d \in R_{-x}$. For $c\vdots d \in R_{-x}$, it must be the case that $-c\vdots -d \in R_x$. The intervals $-c\vdots -d$ and $a\vdots b$ must intersect as they are both prophecies of R_x. Let p be a point of that intersection which implies $a\vdots p\vdots b$ and $c\vdots -p\vdots d$. Then $a\vdots b \oplus c\vdots d$, which is $(a+c)\therefore(b+d)$, has $(a+c)\vdots(p+(-p)=0)\vdots(b+d)$ in its interval. Since $a\vdots b$ and $c\vdots d$ were random prophecies of their oracles, it must be true that all intervals in the range of the sum procedure must contain 0. This leads to the prophecies of R_{x-x} all containing 0 which means 0 is the root of the oracle R_{x-x} which in turn means $R_{x-x} \equiv R_0$. Thus, $-x$ is the additive inverse of x.

6. Multiplicative Inverse. For the multiplicative inverse of x, it is slightly more complicated. This only applies if $x \neq 0$. Being not 0, there exists at least one prophecy $u\vdots v$ of x that does not contain 0. Once one has that prophecy, then if L is the distance to 0 from $u\vdots v$, which would be $\min(|u|, |v|)$, any prophecy whose size is less than L also excludes 0.

   Define a fonsi to be $\mathcal{F} = \{1/a\vdots 1/b | a\vdots b \in R_x \wedge |a\vdots b| < L\}$. To show this is a fonsi, let $a \therefore b$ and $c \therefore d$ be two intervals in $\mathcal{F}$. It is required to show that they intersect. Since $1/a\vdots 1/b \in R_x$ and $1/c\vdots 1/d \in R_x$, those reciprocated intervals intersect. Let $\{1/a, 1/c\}\vdots p\vdots\{1/b, 1/d\}$. Then $\{a, c\}\vdots 1/p\vdots\{b, d\}$ as all of these are the same sign. Thus, $1/p$ is in both $a\vdots b$ and $c\vdots d$.

   The other aspect of a fonsi is to have arbitrarily small intervals in it. Let $\delta > 0$ be given. Let $M = L/2$. By the Bisection Algorithm, let $a\vdots b \in R_x$ be such that $|a\vdots b| < \min(M, \delta/M^2)$. This implies $a\vdots b \subset (u\vdots v)_M$ implying that $|a|, |b| > M$. By the length of $a\vdots b$, it does not contain 0, and thus its reciprocal is an interval in $\mathcal{F}$ and $ab > 0$. Then $|1/a\vdots 1/b| = |b-a|/ab$. As $|a|, |b| > M$, it is the case that $1/(ab) < 1/M^2$. Thus, $|1/a\vdots 1/b| = |a\vdots b|/ab < M^2|a\vdots b| < M^2\delta/M^2 = \delta$ demonstrating that there are arbitrarily small intervals in $\mathcal{F}$ and completing the proof that $\mathcal{F}$ is a fonsi.

   The fonsi generates an oracle $R_{1/x}$ which generates the betweenness relation denoted as $1/x$.

   To verify that this is the multiplicative inverse, compute $R_{x*1/x}$. As with the additive inverse, given $a\vdots b \in R_x$ and $c\vdots d \in R_{1/x}$, there exists p that is common to both $a\vdots b$ and $1 \oslash (c\vdots d)$. Thus, the multiplication of these intervals leads to $p * 1/p = 1$ being in the product interval and, as this is true for all pairings of prophecies of R_x with $R_{1/x}$, it is the case that $R_{x*1/x} \equiv R_1$. Thus, $1/x$ is the multiplicative inverse of x; this exists for all $x \neq 0$.

Having established that the field properties hold for oracles, these properties transfer to the rational betweenness relations which establishes that they are a field.

7.4 The Real Numbers

The rational betweenness relations are ordered, complete, and a field. It needs to be shown that it is an ordered field, that is, that the arithmetic operations respects the ordering. To do this, it is sufficient to show that 1) for all x, y, z in the field, if $x < y$, then $x + z < y + z$; and 2) for all x, y in the field, if $x > 0$ and $y > 0$, then $xy > 0$.

By definition of the inequality, there exists intervals $a \vdots \underline{x} \vdots b$ and $c \vdots \underline{y} \vdots d$ such that $a \iddots b < c \iddots d$. Thus, $c - b > 0$. By Bisection, there exists a z-interval $e \iddots f$ such that $|e \vdots f| < |c \vdots b|$. Then $a + e \vdots \underline{x + z} \vdots b + f$ and $c + e \vdots \underline{y + z} \vdots d + f$. Since $a + e \leq b + f$ and $c + e \leq d + f$, it suffices to show that $b + f < c + e$. This is equivalent to showing $f - e < c - b$, but that is true by choice of $e \vdots f$. Thus, $x + z < y + z$. As there was nothing special about x, y, or z, this holds for all rational betweenness relations.

The multiplication is a bit shorter. Let $x > 0$, $y > 0$, $a \vdots \underline{x} \vdots b$ such that $0 < a \iddots b$, $c \vdots \underline{y} \vdots d$ such that $0 < c \iddots d$. Then $ac \vdots \underline{xy} \vdots bd$ and $0 \vdots 0 < ac \iddots bd$. Hence, the rational betweenness relation xy is greater than $\underline{0}$.

The rationals are also dense in the rational betweenness relations. Let $x < y$ be two given relations. Let $a \iddots b < c \iddots d$ where $a \vdots \underline{x} \vdots b$ and $c \vdots \underline{y} \vdots d$; this is allowed by the meaning of the inequality. Let m be the average of b and c. Then the betweenness version of m, say $\vdots \underline{m} \vdots$, satisfies $x < \vdots \underline{m} \vdots < y$ as evidenced by $a \vdots b < m \vdots m < c \vdots d$.

Taken as a whole, all of the above translates into the following theorem.

Theorem 7.2. The set of rational betweenness relations equipped with the arithmetic and inequality operators as defined above satisfy all the axioms of the ordered, complete field of real numbers.

8 Concluding Thoughts

The idea was quite simple: a real number is best represented by the intervals that contain it. To do this, there are three distinct structures.

The initial structure is that of the rational betweenness relations. While Dedekind cuts emphasize, in some sense, being away from the real number, the relations suggest enclosing the real number. The idea is that if one knows which rational intervals contain the real number, then it is reasonable to say that one knows the real number. Unfortunately, this is not achievable for all real numbers by finite processes.

What is achievable is the structure of oracles. They are more computationally accessible. They are best thought of as a prescription as to what to compute and do rather than some given completed object.

But given some attempt to represent a particular real number, many oracles are possible. This suggests trying to find a structure that oracles are approximating. This is the betweenness relations. These are pure. They represent the intervals that contain the real number. In order to compute out all the Yes intervals for a given oracle, it may be necessary to check infinitely many intervals. This can mean that to state the relations is to delve into nonconstructivist territories.

Both versions have their trade offs. In terms of practical uses, the oracles are more compatible with using a computer program. The betweenness relations are very useful in theoretical uses.

The third structure that emerges is that of fonsis. While algorithms generally fit the oracles better, the fonsi notion naturally arises in arithmetic. The fonsis are the prophecies of the oracles and they are a partial, but functional, version of the rational betweenness relations.

A famous motivation for Dedekind was to prove that $\sqrt{2}\sqrt{3} = \sqrt{6}$. How might that go with the relations? The Yes intervals for $\sqrt{p}$ are those intervals $\{0, a\} \iddots b$ that satisfy $\max(0, a)^2 \vdots p \vdots b^2$. Let $0 \vdots 0 < a \iddots b$ be a $\sqrt{2}$-Yes interval and $0 \vdots 0 < c \iddots d$ be a $\sqrt{3}$-Yes interval. Multiplication of them leads to $ac \iddots bd$. Squaring the multiplication leads to $a^2c^2 \iddots b^2d^2$. Since $a^2 \iddots 2 \iddots b^2$ and $c^2 \iddots 3 \iddots d^2$, multiplication leads to $a^2c^2 \iddots 6 \iddots b^2d^2$. Thus, $\sqrt{2}\sqrt{3} = \sqrt{6}$. This can obviously be generalized to any n-th roots.

In general, using intervals is how arithmetic can be explored. The computation of $e + \pi$ can be explored with intervals. One would take Yes intervals of e and Yes intervals of π and then add them together. The intervals can be shrunk as small as one likes if one has the computational power to do it. There is nothing infinite in the computation up to a certain desired level of precision. Using fractions allows for accurate precision.

There are other works relevant to this idea being worked on by the author. The paper [Tay24] explores the betweenness relations and how they relate to Dedekind cuts. The paper [Tay23d] extends this idea to give a new completion of metric spaces. Basically, inclusive balls replace the inclusive rational intervals. The Separation property gets replaced by the property that given two points in a Yes ball, there exists a Yes ball that excludes at least one of them. There is also work, [Tay23c], to extend this idea to the novel topological spaces of linear structures from [Mau14].

Functions are also a topic of interest to extend oracles to. With the idea of a real number being based on rational intervals, it suggests that functions ought to respect that. Exploring the implications of that is the business of [Tay23b]. A comprehensive tome, [Tay23a], covers these various topics including comparing the other definitions of real numbers to this one and explaining how it suggests techniques such as using mediants to compute continued fractions.

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