PSC 585 Dynamic and Computational Modeling

Problem Set 3
April 26, 2011

1. Consider the following variant of the dynamic bargaining game we considered in class. There are five players in set
$$N = \{1, 2, 3, 4, 5\}.$$
In each period $t = 1, 2, \ldots$ before an agreement has been reached, player i is recognized with probability p_i to offer a proposal $(C, x) \in \mathcal{D}_i \times \mathbb{R}^2$. Here $\mathcal{D}_i$ is the set of coalitions that the proposer is allowed to build. Then all five players vote yes or no, and if all players in C approve the proposal, then the proposal is implemented in all subsequent periods and the game ends. Otherwise, the game moves to the next period, a proposer is drawn according to the same probabilities p_i, etc., until a proposal is approved and implemented. Let
$$\mathcal{D}_i = \{\{j, h\} \mid j, h \neq i, j \neq h, j, h \in N\}.$$
Note that $\mathcal{D}_i$ has six elements for each i, i.e., six potential coalitions for i to form. We will index these coalitions by $m_i = 1, \ldots, 6$ and denote a particular coalition by $C_{m_i} \in \mathcal{D}_i$. If proposal (C_{m_i}, x) passes, then each player j derives utility $u_j(C_{m_i}, x) = u_j(x) + c^j_{m_i}$, where
$$u_j(x) = -(x_1 - \hat{x}_{j,1})^2 - (x_2 - \hat{x}_{j,2})^2 + K,$$
$c^j_{m_i} \in \mathbb{R}$ is a coalition/proposer specific shift in utility, and $K = 10$. Each period an agreement is not reached, players receive utility $\bar{u}_j = u_j(0, 0)$. Player j discounts the future by a factor $\delta_j = \delta = \frac{2}{3}$. Ideal points are given by $\hat{x}_1 = (-\frac{1}{4}, -\frac{1}{4})$, $\hat{x}_2 = (-\frac{1}{4}, \frac{3}{4})$, $\hat{x}_3 = (\frac{7}{16}, \frac{7}{16})$, $\hat{x}_4 = (\frac{7}{16}, -\frac{3}{16})$, and $\hat{x}_5 = (-\frac{1}{3}, -\frac{1}{2})$.
A no-delay equilibrium is a collection of:
   - voter continuation values v_i;
   - Optimal proposals $x^i_{m_i} \in \mathbb{R}^2$, for all i and all coalitions $C_m \in \mathcal{D}_i$, $m_i = 1, \ldots, 6$;
   - Mixing probabilities $\sigma^i_{m_i} \in [0, 1]$, for all i and all coalitions $C_m \in \mathcal{D}_i$, $m_i = 1, \ldots, 6$;
   - Multipliers $\lambda^i_{m_i} \in \mathbb{R}$, for all i and all coalitions $C_m \in \mathcal{D}_i$, $m_i = 1, \ldots, 6$;
   - Multipliers $\mu^{i,m_i}_j \in \mathbb{R}$, for all i, all coalitions $C_{m_i} \in \mathcal{D}_i$, $m_i = 1, \ldots, 6$, and all $j \in C_{m_i}$;

   such that:
$$v_i - \sum_{j=1}^{5} p_j \sum_{m_j=1}^{6} \sigma^j_{m_j}(u_i(x^j_{m_j}) + c^i_{m_j}) = 0, i = 1, \ldots, 5 \tag{1}$$
$$\sum_{m'_i=1}^{6} \sigma^i_{m'_i}(u_i(x^i_{m'_i}) + c^i_{m'_i}) - u_i(x^i_{m_i}) - c^i_{m_i} - max\{0, -\lambda^i_{m_i}\}^2 = 0, i = 1, \ldots, 5, m_i = 1, \ldots, 6 \tag{2}$$
$$\sum_{m_i=1}^{6} \sigma^i_{m_i} - 1 = 0, i = 1, \ldots, 5 \tag{3}$$
$$\sigma^i_{m_i} - max\{0, \lambda^i_{m_i}\}^2 = 0, i = 1, \ldots, 5, m_i = 1, \ldots, 6 \tag{4}$$
$$Du_i(x^i_{m_i}) + \sum_{j \in C_{m_i}} max\{0, \mu^{i,m_i}_j\}^2 Du_j(x^i_{m_i}) = 0, i = 1, \ldots, 5, m_i = 1, \ldots, 6 \tag{5}$$
$$u_j(x^i_{m_i}) + c^j_{m_i} - (1 - \delta)\bar{u}_i - \delta v_j - max\{0, -\mu^{i,m_i}_j\}^2 = 0, i = 1, \ldots, 5, m_i = 1, \ldots, 6, j \in C_{m_i} \tag{6}$$

After eliminating the mixing probabilities using equation (4) and eliminating (3) which are implied by (2) we get the system

$$v_i - \sum_{j=1}^{5} p_j \sum_{m_j=1}^{6} \max\{0, \lambda^j_{m_j}\}^2 (u_i(x^j_{m_j}) + c^i_{m_j}) = 0, i = 1, \ldots, 5 \tag{7}$$

$$\sum_{m'_i=1}^{6} \max\{0, \lambda^i_{m'_i}\}^2 (u_i(x^i_{m'_i}) + c^i_{m'_i}) - u_i(x^i_{m_i}) - c^i_{m_i} - \max\{0, -\lambda^i_{m_i}\}^2 = 0, i = 1, \ldots, 5, m_i = 1, \ldots, 6 \tag{8}$$

$$Du_i(x^i_{m_i}) + \sum_{j \in C_{m_i}} \max\{0, \mu^{i,m_i}_j\}^2 Du_j(x^i_{m_i}) = 0, i = 1, \ldots, 5, m_i = 1, \ldots, 6 \tag{9}$$

$$u_j(x^i_{m_i}) + c^j_{m_i} - (1-\delta)\bar{u}_i - \delta v_j - \max\{0, -\mu^{i,m_i}_j\}^2 = 0, i = 1, \ldots, 5, m_i = 1, \ldots, 6, j \in C_{m_i} \tag{10}$$

Verify that there are 5 unknown voter continuation values, $5 \times 6 = 30$ unknowns corresponding to multipliers λ, $5 \times 6 \times 2 = 60$ unknowns corresponding to optimal proposals, and $5 \times 6 \times 2 = 60$ unknown multipliers μ. Also, there are 5 equation in (7), 30 in (8), 30×2 in (9), and 60 in (10), that is the number of equations is equal to the number of unknowns equal to 155.

Stack the unknowns according to the order $y = (v, \lambda, x, \mu)$, where

$$v = (v_1, \ldots, v_5)$$

$$\lambda = (\lambda^1_1, \ldots, \lambda^1_6, \ldots, \lambda^5_1, \ldots, \lambda^5_6),$$
$$x = (x^1_{1,1}, x^1_{1,2}, \ldots, x^1_{6,1}, x^1_{6,2}, \ldots, x^5_{1,1}, x^5_{1,2}, \ldots, x^5_{6,1}, x^5_{6,2}),$$
$$\mu = (\mu^{1,1}_{j_1}, \mu^{1,1}_{j_2}, \ldots, \mu^{1,6}_{j_1}, \mu^{1,6}_{j_2}, \ldots, \mu^{5,1}_{j_1}, \mu^{5,1}_{j_2}, \ldots, \mu^{5,6}_{j_1}, \mu^{5,6}_{j_2}).$$

Also, enumerate the coalitions in lexicographic order so that, for each i and any $C_{m_i}, C_{m'_i} \in \mathcal{D}_i$, if $m_i < m'_i$ then

$$\min_j \{j \in C_{m_i}\} < \min_j \{j \in C_{m'_i}\},$$

or

$$\min_j \{j \in C_{m_i}\} = \min_j \{j \in C_{m'_i}\}, \max_j \{j \in C_{m_i}\} < \max_j \{j \in C_{m'_i}\}.$$

(a) Let $y = (v, \lambda, x, \mu)$. For any vector $\mathbf{c}$ containing coalition specific payoffs $c^j_{m_i}$, define the system of equations (7)-(11) (call the system $F(y; \mathbf{c}) = 0$) where the equations are stacked in the order they are listed (and according to the listing of the indices; e.g., in (8) list the equations in the order $i = 1, m_i = 1$, then $i = 1, m_i = 2$, …, $i = 1, m_i = 6$, $i = 2, m_i = 1$, etc.). Derive the Jacobian of F, $DF(y; \mathbf{c})$, and write code that implements Newton's method to solve the system $F(y; \mathbf{c}) = 0$:

- At y_t compute
$$dy_t = -[DF(y_t; \mathbf{c})]^{-1} F(y_t; \mathbf{c}).$$
- Set $y_t = y_t + dy_t$ and $t = t + 1$.
- Stop if $t > T$ or $||dy_t|| < \epsilon$.

(b) Let $\mathbf{c}^0$ be a vector containing distinct coalition specific payoffs $c^j_{m_i}$. Define $\mathbf{c}^0$ to be equal to $c^i_{m_i} = 0$ and $c^j_{m_i} = \frac{c}{1-p_j}$ for all i, all $j \neq i$ and all $m_i = i$, and $c^i_{m_i} = -c$ and $c^j_{m_i} = \frac{c}{1-p_j}$ for all i, all $j \neq i$ and all $m_i \neq i$. Verify that $y^0 = (v^0, \lambda^0, x^0, \mu^0)$ with

$$\begin{aligned}
v_i^0 &= \sum_{j=1}^{5} p_j u_i(\hat{x}_j) + c, i = 1, \ldots, 5, \\
\lambda_{m_i}^{0,i} &= 1, i = 1, \ldots, 5, m_i = i, \\
\lambda_{m_i}^{0,i} &= -\sqrt{c}, i = 1, \ldots, 5, m_i \neq i, \\
x_{m_i}^{0,i} &= \hat{x}_i, i = 1, \ldots, 5, m_i = 1, \ldots, 6, \\
\mu_j^{0,i,m_i} &= -\sqrt{u_j(\hat{x}_i) + \frac{c}{1-p_j} - (1-\delta)\bar{u}_j - \delta v_j^0}, i = 1, \ldots, 5, m_i = 1, \ldots, 6, j \in C_{m_i},
\end{aligned}$$

satisfies

$$F(y^0; \mathbf{c}^0) = 0$$

for some large enough $c > 0$.

(c) Your goal is to solve for an equilibrium for some $\mathbf{c}^1$ containing distinct coalition specific payoffs $c_{m_i}^j$. For that purpose define for any $t \in [0,1]$ the function $H : [0,1] \times \mathbb{R}^{155} \to \mathbb{R}^{155}$ as

$$H(t, y) = F(y; (1-t)\mathbf{c}^0 + t\mathbf{c}^1).$$

Let $y(t)$ be defined implicitly as the solution to

$$H(t, y(t)) = 0$$

for all $t \in [0,1]$. Using the implicit function theorem, we compute

$$\frac{\partial y(t)}{\partial t} = -[D_y H(t,y)]^{-1} D_t H(t,y).$$

The above along with the initial conditions $y(0) = y^0$ define an ordinary differential equation. One approach to solving for an equilibrium is then to use one of the *ode* solvers of Matlab. You may try such a method (or mere Newton's method for $t = 1$) but you are required to code and implement the following predictor corrector method.

- Specify $\bar{\gamma}$ and $\underline{\gamma}$ (with $\bar{\gamma} > \underline{\gamma} > 0$), T, and ϵ.
- Initialize the iterations by setting $t = 0$ and $y = y^0$ and $\gamma = \bar{\gamma}$.
- At each iteration:
  i. Let $t_{old} = t$ and $y_{old} = y$ and compute

  $$t = \min\{1, t + \gamma\},$$

  and

  $$y = y + \gamma \frac{\partial y(t)}{\partial t}.$$

  ii. Solve the equation $H(t, y^*) = 0$ for y^* using Newton's method (coded in (a)) with initial value y from step (i).
    - If Newton method fails to solve the equation, set $y = y_{old}$ and $t = t_{old}$ and $\gamma = \frac{\gamma}{2}$. If $\gamma < \underline{\gamma}$ terminate without a solution.
    - If Newton method succeeds, set $y = y^*$ and $\gamma = \min\{\bar{\gamma}, 2\gamma\}$.

The value of $\mathbf{c}^1$ is given in the 30×5 matrix Cs in $Cs.mat$, where $c_{m_i}^j$ is given in the $(i-1)+m_i$-th row, j-th column of Cs for all i, m_i, j.